

Markov Chains and Unemployment

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1 Markov Chains

A model of a stochastic process with discrete number of states.

1.1 Random Variable and Mathematical Expectation

Notation for discrete states:

- $n = 1, \dots, N$ represents for possible "states of the world" (e.g. individual unemployed, employed, ...)

- $\pi_n = \mathbb{P}(\text{state of the world is } n)$

$$\pi_n \geq 0, \sum_{n=1}^N \pi_n = 1, \text{ i.e. world must be in one of the states}$$

$$\text{Stack as a vector: } \pi \equiv \begin{bmatrix} \pi_1 & \dots & \pi_N \end{bmatrix}$$

- Random variable $Y \in \{y_1, \dots, y_N\}$

- Values mapping states of the world for r.v. Y : $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$

e.g. If event n is unemployed, then income if unemployed is y_n

- $\mathbb{E}[Y] = \sum_{n=1}^N \mathbb{P}(Y = y_n) y_n = \sum_{n=1}^N \pi_n y_n = \pi \cdot y$ (i.e., inner product)

– i.e. weight the realizations with the probabilities

- e.g., if the probability of unemployment is $\pi_1 = 0.1$, income from unemployment insurance is $y_1 = 15,000$;
probability of employment is $\pi_2 = 0.9$, income from employment is $y_2 = 40,000$.
Then expected income (or average across states of world):

$$\mathbb{E}[Y] = (0.1 \times 15,000) + (0.9 \times 40,000)$$

- We could use this to find an individual's expected income at some point in the future. Alternatively, we can use this to find averages for a continuum of population. A step towards aggregation.
- e.g., if 10 % of population is unemployed at \$15,000 and 90 % of population is employed at \$40,000. Then the average income is $\mathbb{E}[Y]$

1.2 Transitions

For example, Let ϕ = probability to become employed. Let State 1 $\leftrightarrow E$, State 2 $\leftrightarrow U$

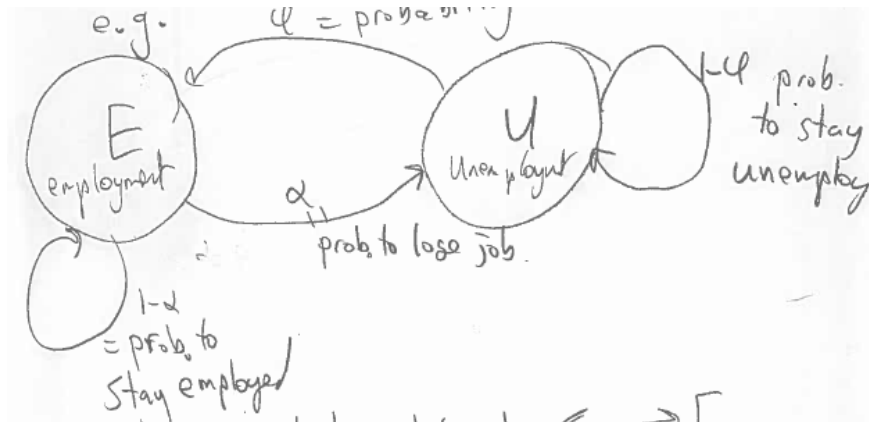


Figure 1: Markov Chain

Transition Matrix:

$$P = \begin{matrix} & \begin{matrix} state_{1,t+1} & state_{2,t+1} \end{matrix} \\ \begin{matrix} state_{1,t} \\ state_{2,t} \end{matrix} & \begin{pmatrix} 1 - \alpha & \alpha \\ \phi & 1 - \phi \end{pmatrix} \end{matrix} \quad (1)$$

Let π_t be the probability mass function (pmf) of a random variable of an agent's employment status at time t . This is a probability mass function (pmf) since the possible events is discrete.

- If employed at time 0, $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$
- If 50% chance of employment at time 3, $\pi_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

1.3 Evolution of Probability Distribution

Find the evolution of the probability mass function for the random variable with the transition matrix P . A property of markov chains:

$$\pi_{t+1} = \pi_t \cdot P \quad (2)$$

Careful with the order of the matrix product!

Iterate forward:

$$\pi_{t+j} = \pi_t \cdot P^j \quad (3)$$

Example:

- Started employed at $t = 0$, i.e. $\pi_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$
- Probability of unemployment/employment at $t = 1$:

$$\pi_1 = \pi_0 \cdot P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \end{bmatrix} \quad (4)$$

At time 2:

$$\pi_2 = \pi_1 \cdot P \quad (5)$$

$$= \begin{bmatrix} 1-\alpha & \alpha \end{bmatrix} \cdot \begin{bmatrix} 1-\alpha & \alpha \\ \phi & 1-\phi \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} (1-\alpha)^2 + \alpha\phi \\ (1-\alpha)\alpha + \alpha(1-\phi) \end{bmatrix}' \quad (7)$$

Interpret:

$$= \begin{bmatrix} \mathbb{P}(E \rightarrow E, E \rightarrow E) + \mathbb{P}(E \rightarrow U, U \rightarrow E) \\ \mathbb{P}(E \rightarrow E, E \rightarrow U) + \mathbb{P}(E \rightarrow U, U \rightarrow U) \end{bmatrix} \quad (8)$$

Iterating Forward:

$$\pi_{t+j} = \pi_t \cdot \underbrace{P \cdot P \dots P}_{j \text{ times}} = \pi_t \cdot P^j \quad (9)$$

Stationarity and asymptotics. One possibility:

$$\pi_\infty = \lim_{j \rightarrow \infty} \pi_{t+j} = \lim_{j \rightarrow \infty} \pi_t \cdot P^j \quad (10)$$

Another is to find a π_∞ which doesn't change, i.e.

$$\pi_\infty = \pi_\infty P \quad (11)$$

Questions :

- Does a unique limit exist? Is it independent of π_t ?
- Is there an absorbing state? (e.g., all end up unemployed forever)
- These answers depend on P .
- In some cases, we will refer to π_∞ as the stationary distribution.

2 Example

2.1 Non-Degenerate Stationary Distribution

What if:

$$P = \begin{bmatrix} 1 - \alpha & \alpha \\ \phi & 1 - \phi \end{bmatrix}, \quad 0 < \alpha < 1, \quad 0 < \phi < 1 \quad (12)$$

Definition of stationary random variable, π_∞ :

$$\pi_\infty = \pi_\infty \cdot P \quad (13)$$

- i.e. It's the R.V. associated with P such that it doesn't change between periods.

Remark: In linear algebra, the left eigenvector associated with the unit eigenvalue.

To find π_∞ :

- Use software to find the left eigenvector, or
- Solve system for simple examples:

Let $\bar{\pi}$ = prob of being employed; $\pi_\infty = [\bar{\pi} \quad 1 - \bar{\pi}]$.

Then:

$$\begin{bmatrix} \bar{\pi} & 1 - \bar{\pi} \end{bmatrix} = \begin{bmatrix} \bar{\pi} & 1 - \bar{\pi} \end{bmatrix} \cdot \begin{bmatrix} 1 - \alpha & \alpha \\ \phi & 1 - \phi \end{bmatrix} \quad (14)$$

$$\Rightarrow \begin{bmatrix} \bar{\pi} \\ 1 - \bar{\pi} \end{bmatrix}' = \begin{bmatrix} \bar{\pi}(1 - \alpha) + (1 - \bar{\pi})\phi \\ \bar{\pi} \cdot \alpha + (1 - \bar{\pi})(1 - \phi) \end{bmatrix}' \leftrightarrow \text{equation 1} \quad (15)$$

$$\leftrightarrow \text{equation 2}$$

1st Equation:

$$\bar{\pi} = (1 - \alpha)\bar{\pi} - \phi\bar{\pi} + \phi \quad (16)$$

$$\Rightarrow (1 - (1 - \alpha) + \phi)\bar{\pi} = \phi \quad (17)$$

$$\Rightarrow \bar{\pi} = \frac{\phi}{\alpha + \phi} \quad (18)$$

$$\Rightarrow \pi_{\infty} = \left[\frac{\phi}{\alpha + \phi} \right]' \quad (19)$$

2nd Equation: would find identical solution. (luckily, since there is only 1 variable and 2 equations)

2.2 Example: Unemployment

Assume: $P = \begin{bmatrix} 1 - \alpha & \alpha \\ \phi & 1 - \phi \end{bmatrix} \leftrightarrow E$
 $\leftrightarrow U$

Invariant Distribution (i.e. "long run")

$$\bar{\pi} = \mathbb{P}(E), 1 - \bar{\pi} = \mathbb{P}(U) \quad (20)$$

Solves:

$$\begin{bmatrix} \bar{\pi} & 1 - \bar{\pi} \end{bmatrix} \begin{bmatrix} 1 - \alpha & \alpha \\ \phi & 1 - \phi \end{bmatrix} = \begin{bmatrix} \bar{\pi} & 1 - \bar{\pi} \end{bmatrix} \quad (21)$$

Equation:

$$\bar{\pi}(1 - \alpha) + \phi(1 - \bar{\pi}) = \bar{\pi} \quad (22)$$

$$\Rightarrow \bar{\pi} = \frac{\phi}{\alpha + \phi} \quad (23)$$

$$1 - \bar{\pi} = \frac{\alpha}{\alpha + \phi} \quad (24)$$

Dividing top and bottom by $\phi\alpha$:

$$1 - \bar{\pi} = \frac{1/\phi}{1/\phi + 1/\alpha} \quad (25)$$

What is the average unemployment spell?

- In each period an unemployed person gets a job with probability $1 - \phi$. Otherwise, stays unemployed. - Let N be the random variable "length of time it takes to find a job". $N = 1$



Figure 2: Employment Chain

means 1 person.

- Let $p_j = \mathbb{P}(N = j)$.

Then:

$$p_1 = \phi, \quad (\text{success}) \quad (26)$$

$$p_2 = \phi(1 - \phi), \quad (\text{fail, success}) \quad (27)$$

$$p_3 = \phi(1 - \phi)^2, \quad (\text{fail, fail, success}) \quad (28)$$

$$\boxed{p_j = \phi(1 - \phi)^{j-1}} \quad (29)$$

$$\Rightarrow \sum_{j=1}^{\infty} p_j = \phi \sum_{j=1}^{\infty} (1 - \phi)^{j-1} = \phi \sum_{j=0}^{\infty} (1 - \phi)^j = \frac{\phi}{1 - (1 - \phi)}$$

i.e., a proper probability distribution.

Another Geometric Series Result:

$$\boxed{\sum_{j=1}^{\infty} ja^{j-1} = \frac{1}{(1 - a)^2} \text{ for } |a| < 1, \text{ (can derive from } Z\text{-transforms)}} \quad (30)$$

Back to the question:

$$p_j = \mathbb{P}(N = j) = \phi(1 - \phi)^{j-1} \quad (31)$$

$$\mathbb{E}[N] = \text{expected / mean time to find a job} \quad (32)$$

$$= \sum_{j=1}^{\infty} j \cdot p_j = \phi \sum_{j=1}^{\infty} j(1 - \phi)^{j-1} = \phi \cdot \frac{1}{(1 - (1 - \phi))^2} = \frac{1}{\phi} \quad (33)$$

- So the average # of periods in unemployment = $\frac{1}{\phi}$.
- More generally, this is the mean waiting time for a geometric distribution, i.e., if arrivals happen with probability a , then the expected wait time = $\frac{1}{a}$.

Summarizing Formula:

- $\bar{\pi}$ = proportion employed, $1 - \bar{\pi}$ = proportion unemployed.

$$- \bar{\pi} = \frac{\phi}{\alpha + \phi}, \quad 1 - \bar{\pi} = \frac{\alpha}{\alpha + \phi} = \frac{1/\phi}{1/\phi + 1/\alpha}.$$

- $\mathbb{E}[\# \text{ of periods to become employed} \mid \text{start unemployed}] = \frac{1}{\phi}$
- $\mathbb{E}[\# \text{ of periods to become unemployed} \mid \text{start employed}] = \frac{1}{\alpha}$

2.2.1 Example with Data (approx. 2007 US Data)

- Average unemployment duration = 16.8 weeks = 3.87 months.
- Civilian unemployment: 4.7%
- Employment / population: 63%
- Labor force / population: 66%
- Civilian population: 231 million
- Civilian labor force: 153 million = 231 \times 66% (not institutional military, etc.)
- Unemployment: 7 million = 153 million \times 4.7%

Stationary Distribution:

$$1 - \bar{\pi} = 0.47 \text{ (proportion unemployed)} \quad (34)$$

$$\frac{1}{\phi} = 3.87 \text{ (average unemployment length, in months)} \quad (35)$$

Equation for stationary distribution:

$$1 - \bar{\pi} = \frac{1/\phi}{1 + \phi + 1/\alpha} \quad (36)$$

$$\Rightarrow 0.047 = \frac{3.87}{3.87 + 1/\alpha} \quad (37)$$

Solve for $\frac{1}{\alpha}$:

$$\frac{1}{\alpha} = 78.8 \quad (38)$$

i.e., average job length is 78 months.

So, the transition matrix is:

$$P = \begin{bmatrix} 1 - \frac{1}{78.8} & \frac{1}{78.8} \\ \frac{1}{3.87} & 1 - \frac{1}{3.87} \end{bmatrix} \approx \begin{bmatrix} 0.987 & 0.013 \\ 0.258 & 0.742 \end{bmatrix} \quad (39)$$

Stationary:

$$\pi_{\infty} = \begin{bmatrix} 0.953 & 0.047 \end{bmatrix} \quad (40)$$

Question

1. Total Jobs Destroyed/Month: $0.013 \times 146 \text{ million} \approx 1.85 \text{ million}$
2. If employed worker today, what is the probability to be employed in j months?

$$\mathbb{P}(E \text{ at } j) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\substack{\text{need} \\ \text{employment} \\ \text{start}}} \cdot \left(\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\pi_0=E} P^j \right)' \quad (41)$$

What about as $j \rightarrow \infty$? $\mathbb{P}(E \text{ at } j \rightarrow \infty) = \bar{\pi}$

3. The economy is away from its stationary equilibrium: $\pi_0 \neq \pi_{\infty}$.

What is the predicted sequence of unemployment rates?

$$\pi_j = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot [\pi_0 P^j]'$$

A Degenerate Markov Chains

A.1 Example: Absorbing State of Unemployment

$$\text{Let } P = \begin{bmatrix} \alpha & 1 - \alpha \\ 0 & 1 \end{bmatrix},$$

i.e. α chance to stay employed, and in unemployment never get a job ("absorbing").

Let $\pi_0 = \begin{bmatrix} a & 1-a \end{bmatrix}$

$$\bullet \pi_1 = \pi_0 P \quad (\text{A.1})$$

$$= \begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \quad (\text{A.2})$$

$$= \begin{bmatrix} \alpha a \\ (1-\alpha)a + 1-a \end{bmatrix}' \quad (\text{A.3})$$

$$= \begin{bmatrix} \text{kept job} \\ \text{lost job or never had one} \end{bmatrix}' \quad (\text{A.4})$$

$$\bullet \pi_2 = \pi_1 P \quad (\text{A.5})$$

$$= \begin{bmatrix} \alpha a & (1-\alpha)a + 1-a \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \quad (\text{A.6})$$

$$= \begin{bmatrix} \alpha^2 a \\ (1-\alpha) \cdot \alpha a + (1-\alpha)a + (1-a) \end{bmatrix}' \quad (\text{A.7})$$

Note:

- $\alpha^2 a$ represents for kept job twice

- Nominator and denominator must sum to 1

Example continued

$$\pi_j = \pi_0 \cdot P^j \quad (\text{A.8})$$

$$= \begin{bmatrix} \alpha^j \cdot a \\ 1 - \alpha^j \cdot a \end{bmatrix} \quad (\text{A.9})$$

$$\lim_{j \rightarrow \infty} \pi_j = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ (i.e. all end up unemployed, independent of } \pi_0) \quad (\text{A.10})$$

$$\text{or: } P \cdot P = \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{bmatrix} \quad (\text{A.11})$$

$$= \begin{bmatrix} \alpha^2 & \alpha(1-\alpha) + (1-\alpha) \\ 0 & 1 \end{bmatrix} \quad (\text{A.12})$$

$$= \begin{bmatrix} \alpha^2 & 1-\alpha^2 \\ 0 & 1 \end{bmatrix} \quad (\text{A.13})$$

Generalize:

$$P^j = \begin{bmatrix} \alpha^j & 1 - \alpha^j \\ 0 & 1 \end{bmatrix} \quad (\text{A.14})$$

$$\lim_{j \rightarrow \infty} P^j = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad (\text{A.15})$$

$$\pi_\infty = \lim_{j \rightarrow \infty} \pi_0 P^j = \begin{bmatrix} a & 1 - a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad (\text{A.16})$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ (i.e. all unemployed independent of } \pi_0) \quad (\text{A.17})$$

Alternatively :

$$\pi_\infty = \begin{bmatrix} \bar{\pi} & 1 - \bar{\pi} \end{bmatrix} \quad (\text{A.18})$$

$$\Rightarrow \pi_\infty = \pi_\infty \cdot P \quad (\text{A.19})$$

$$= \begin{bmatrix} \bar{\pi} & 1 - \bar{\pi} \end{bmatrix} \begin{bmatrix} \alpha & 1 - \alpha \\ 0 & 1 \end{bmatrix} \quad (\text{A.20})$$

Equation:

$$\bar{\pi} = \alpha \cdot \bar{\pi} + 0 \quad (\text{A.21})$$

If $\alpha < 1$, then this

$$\Rightarrow \bar{\pi} = 0, \Rightarrow \pi_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (\text{A.22})$$

A.2 Example: No Ergodic Distribution

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ (i.e., switch from whatever you had)} \quad (\text{A.23})$$

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (\text{A.24})$$

$$P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (\text{A.25})$$

$$\dots P^j = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } j \text{ even} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \text{if } j \text{ odd} \end{cases} \quad (\text{A.26})$$

$$(\text{A.27})$$

$\lim_{j \rightarrow \infty} P^j$ doesn't exist in general.

Alternatively :

$$\begin{bmatrix} a & 1-a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 1-a \end{bmatrix} \quad (\text{A.28})$$

Equations:

$$\begin{bmatrix} 1-a \\ a \end{bmatrix}' = \begin{bmatrix} a \\ 1-a \end{bmatrix}' \Rightarrow \pi_\infty = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}' \quad (\text{A.29})$$

i.e. must start out with 50/50% probability.