Question 1

Consider two scenarios for a consumer planning consumption with income process $y_t = y_0\delta^t, \quad \forall t \geq 0$ and $F_0 = 0$.

Scenario 1 for Consumer The consumer maximizes the following welfare

$$U \equiv \max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \tag{1}$$

$$\text{s.t. } F_{t+1} = R(F_t + y_t - c_t), \quad \forall t \geq 0 \tag{2}$$

$$\text{(transversality condition)} \tag{3}$$

Scenario 2 for Consumer The consumer faces the same problem as Scenario 1, except with **no borrowing**: $F_{t+1} \geq 0$ for all $t \geq 0$, and the initial level of y_0 is potentially different (define it as y_0^{NB})

Define the PDV of utility (i.e., the welfare) of this as U^{NB}.

(a) Let $\beta = 0.95, R = 1.04, \delta = 1.02, y_0 = 1$, and $y_0^{NB} = 1$. Calculate U and U_{NB}.

(b) Let $y_0 = 1$. Now find a y_0^{NB} such that $U = U^{NB}$. The difference between y_0 and y_0^{NB} is the amount of sacrifice in terms a consumer with a borrowing constraint would pay to be free to borrow. A measure of the welfare loss of the no borrowing constraint.

(c) Maintain $y_0 = 1$. Now, let $\beta = .99, R = 1.04$, and $\delta = 1.01$. What is c_0 and F_1 here under Scenario 1? Repeat part (b) to find y_0^{NB} such that $U = U^{NB}$ with these new parameters. What can you conclude about the welfare cost of no borrowing in this case?

Question 2

Let the consumer have power utility,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 1$$

Given $F_0 = 0, B \geq 0, \beta R = 1$, and the deterministic income stream $y_t = \delta^t$, the consumer maximizes

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

$$\text{s.t. } F_{t+1} = R(F_t + y_t - c_t), \quad \forall t \geq 0 \tag{5}$$

$$F_{t+1} \geq -B \tag{6}$$

$$F_0 = 0 \tag{7}$$

$$\text{(transversality condition)} \tag{8}$$

(a) Derive the euler equation as an inequality, and the condition for it holding with equality.

(b) Let $\delta > 1$ and $B = \infty$. What is $\{c_t\}_{t=0}^{\infty}$?

(c) Let $\delta > 1$ and $B = 0$. What is $\{c_t\}_{t=0}^{\infty}$?

(d) Let $\delta < 1$ and $B = 0$. What is $\{c_t\}_{t=0}^{\infty}$?

(e) Assume that the consumer optimally eats their entire income each period, i.e., $c_t = y_t = \delta^t$ which implies $c_{t+1} = \delta c_t$. Setup, using dynamic programming, an equation to find the value $V(c)$ recursively.

(f) Guess that $V(c) = k_0 + k_1 c^{1-\gamma}$ for some undetermined k_0 and k_1.[1] Solve for k_0 and k_1 and evaluate $V(1)$ (i.e., the value of starting with $c_0 = 1$.

[1] Note that this equation deliberately is avoiding any t subscripts! This makes it a truly recursive expression.

Question 3

Take the search model we did in class with an endogenous choice of accepting a job, but add in the following elements:

- If you are unemployed and reject a wage, there is only a $\theta \in (0, 1)$ probability to get a wage offer. Otherwise, you can recall a wage you previously rejected last period. (hint: they would never choose to recall a wage in equilibrium, but it may help you write down the Bellman equation cleanly.)
- There is an exogenous probability $\alpha \in (0, 1)$ of being fired at the end of any period you are working. You then draw a new wage as an unemployed agent entering the next period with certainty (i.e., bypass the θ probability going into the next period.

To summarize the timing here: As in our example in class, let $v(w)$ be the value of coming a period unemployed with wage offer w and when they are about to choose to accept or reject.

If they reject an offer, they gain unemployment insurance c, and have the probability θ to gain the draw with expected value

$$Q = \int_0^B v(\hat{w}) f(\hat{w}) \mathrm{d}\hat{w}$$

If they accept they gain the wage w that period, and then have the α chance of being fired as they come into the next period—at which point they get the wage offer draw with certainty, as discussed. (Hint: if they are not fired, the value next period is $v(w)$, the same as if they were first offered w.)

(a) Draw a Markov chain with two states E and U. Let the probability of staying unemployed be λ which will end up endogenous. You will also need the α transition probability

(b) Write the value of a worker with wage offer w who chooses to reject the offer (hint: if they reject they don't necessarily gain a new draw, but could have $v(w)$ as their value next period since they can recall the rejected w).

(c) Write the value of a worker accepting the offer of w. (hint: may need to be recursive now, unlike what we did in class)

(d) Combine the values in the previous two parts to form a Bellman equation with $v(w)$ and the max for the choice.

(e) Write the equation for an indifference point $\bar{w}$, where they are at the threshold of accepting (or rejecting) the wage.[2]

(f) Assuming that you could numerically solve the previous equation to find a $\bar{w}$, what is the expression for the stationary proportion of unemployed workers as a function of $\alpha, \theta, f(\cdot)$, and $\bar{w}$. (Hint: derive the λ, then use older notes on unemployment. However, recall that the $U \to E$ transition only occurs if both a wage offer and endogenous acceptance occur).

[2] As in the case of class, the Bellman equation could be reorganized to eliminate the $v(\cdot)$ function to give an implicit equation in $\bar{w}$ and parameters. This is significantly trickier than what we did, so only try to simplify the indifference equation if you wish.

Question 4

Each period, a previously unemployed worker draws two offers to work forever at wage w from the cumulative distribution function ('cdf')

$$F(w) = \Big(w/B\Big)^{\frac{1}{2}}, \quad 0 \leq w \leq B$$

where $F(w) = \text{Prob}(\tilde{w} \leq w)$ where $\tilde{w}$ is a particular wage offer. Successive draws within a period and across periods are identically and independently distributed. The unemployed worker is free to inspect both offers in a period and, if he or she wants, accept the highest among offers he or she has drawn that period. Offers from past periods cannot be recalled. The offers are to work at the accepted wage forever. There is no option to quit after an offer has been accepted, and there is no prospect of being fired. The worker wants to maximize

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t y_t\right], \quad 0 < \beta < 1,$$

where

$$y_t = \begin{cases} w & \text{if employed at wage } w, \\ c & \text{if unemployed} \end{cases}$$

where w is the wage, c is unemployment compensation, and $\mathbb{E}[\ldots]$ is an expected value before the offers are drawn.

(a) Verify that $F(w) = \Big(w/B\Big)^{\frac{1}{2}}$ is a legitimate cdf. Find the cdf for the maximum of 2 draws (i.e., find the cdf of $z \equiv \max\{z_1, z_2\} \in \tilde{F}(z)$) and verify it is a cdf.

(b) Find the worker's optimal search strategy and show that it has a 'reservation wage' form.[3] Draw the value function for a worker given a particular w.

(c) Let $\bar{w}$ be the reservation wage. Find a formula for the reservation wage as a function of B, β, c.[4]

(d) Given this $\bar{w}$, find a formula for $\psi =$ probability that an unemployed worker leaves unemployment this period as a function of $\bar{w}, B, \beta, c$ (eliminating $\bar{w}$ if you found a closed form solution in part c).

[3] Hints: (1) setup the model recursively in a way isomorphic to the model from class, (2) For any n and independent draws, $z_1, \ldots, z_n$ from cdf $F(z)$, the cdf of the maximum of these is $z \equiv \max\{z_1, \ldots z_n\} \sim F(z)^n$

[4] Hint: While you could solve for $\bar{w}$ directly, feel free to leave it in an *implicit* form if you are finding the algebra difficult. However, you should be able to eliminate any recursive value functions