Question 1

A consumer chooses consumption and savings to maximize their welfare subject to a budget constraints.

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$
$$\text{s.t. } F_{t+1} = R_t \left(F_t + y_t - c_t\right), \quad \text{for all } t \geq 0 \tag{2}$$
$$c_t \geq 0, \quad \text{for all } t \geq 0 \tag{3}$$
$$\lim_{T \to \infty} \beta^T u'(c_T) F_{T+1} = 0 \quad \text{(Transversality Condition)} \tag{4}$$

Where $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $\beta \in (0,1)$, $\gamma > 0$, $F_0 = 0$, and $\{y_t\}_{t=0}^{\infty}$ is an exogenous, deterministic sequence of labor income with at least some positive y_t. $\{R_t\}_{t=0}^{\infty}$ are the gross interest rate on financial assets, and are an exogenous, deterministic sequence known to the consumer.

(a) Setup the Lagrangian for this problem, being clear on Lagrange Multipliers and equality/inequality constraints.[1]

(b) Show that the $c_t \geq 0$ constraint can never bind, then find the Euler equation for the consumer at all $t \geq 0$.[2]

(c) For some $\delta \geq 0$ and $\phi \geq 0$, let the labor income process be

$$y_t = \begin{cases} y_0 \delta^t & t = 0, \dots T \\ y_0 \delta^t \phi^{t-T} & t = T+1, \dots \infty \end{cases}$$

It just happens that $\{R_t\}_{t=0}^{\infty}$ is such that the consumer optimally sets $c_t = y_t$ and $F_{t+1} = 0$ for all t (i.e., this is a particular sequence of R_t which rationalizes this behavior). Find a formula for $\{R_t\}_{t=0}^{\infty}$ and justify your formula.[3]

(d) Interpret your formula for R_t in terms of (i) the consumer's impatience, and (ii) the consumer's income growth.

[1] Hint: You can be sloppy and skip the multiplier on the Transversality condition, as we have done in class. As always, I strongly suggesting using present-value Lagrange multipliers to simplify algebra.

[2] Hint: The only change from our standard problem is the time varying interest rate. You will need to be careful with the timing when taking first order conditions. To proof that $c_t > 0$, you will need to use the marginal utility and use the fact that there is at least some positive income.

[3] Hint: What does optimality mean? Also, be a little careful around T for the calculation of R_t

Question 2

There are two consumers ($i = 1, 2$) with potentially different consumption and income processes (c_t^i and y_t^i), initial financial wealth $F_0^i = 0$, and identical preferences subject to an intertemporal budget constraint,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{5}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i \tag{6}$$

where $u'(c) > 0, u''(c) < 0, \beta \in (0,1)$, and $\beta R = 1$. Assume that the two income processes are

$$y_t^1 = \{0, 1, 0, 1, \ldots\} \tag{7}$$

$$= \begin{cases} 0 & \text{if } t \text{ even} \\ 1 & \text{if } t \text{ odd} \end{cases} \tag{8}$$

$$y_t^2 = \{1, 0, 1, 0, \ldots\} \tag{9}$$

$$= \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases} \tag{10}$$

(a) Apply the permanent income result to find c_t^i for both agents.[4]

(b) For every t, compare $c_t^1 + c_t^2$ vs. $y_t^1 + y_t^2$. Would this comparison change if $\beta R \neq 1$? (no need to solve for the exact c_t^i in that case)

(c) Assuming that both agents start with no financial wealth, i.e. $F_0^1 = F_0^2 = 0$, compute the asset trades between consumer 1 and 2 to support the c_t^i where the period-by-period budget constraint for $i = 1, 2$ is

$$F_{t+1}^i = R(F_t^i + y_t^i - c_t^i)$$

[4] Hints: Note that if $a_t = \{1, 0, 1, 0 \ldots\}$ then $\sum_{t=0}^{\infty} \beta^t a_t = 1 + \beta^2 + \beta^4 + \ldots = \sum_{t=0}^{\infty} \left(\beta^2\right)^t$.

Question 3

Consider a markov chain with two states: U for unemployment and E for employment.

- With probability $\lambda \in (0, 1)$, a person unemployed today becomes employed tomorrow.
- With probability $\alpha \in (0, 1)$, a person employed today becomes unemployed tomorrow.

(a) Let $N \geq 1$ be the number of periods until a currently unemployed person becomes employed. Calculate $\mathbb{E}[N]$.

(b) Let $M \geq 1$ be the number of periods until a currently employed person becomes unemployed. Calculate $\mathbb{E}[M]$.

(c) Please compute the fraction of time an infinitely lived person can expect to be unemployed and the fraction of time they can expect to be employed.

Question 4

An economy has 3 states for workers:

- U: unemployment.
- V: if they have found a potential employer and are being verified to see if they are a good fit.
- E: if a worker has been verified and is employed.

The probabilities that they jump between these states each period is:

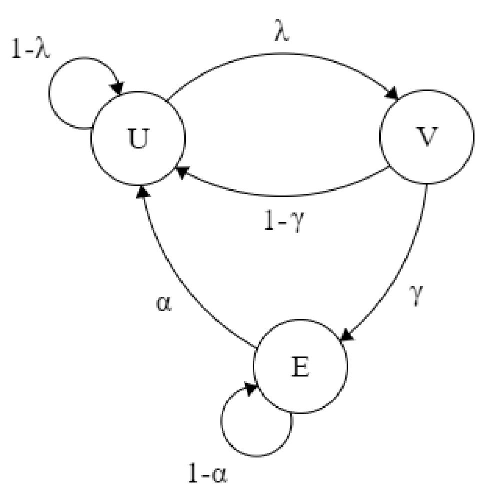

Figure 1: Markov Chain

i.e. probability γ they are a good fit, and the verification takes 1 period.

(a) Write a Markov transition matrix for this process, P.

(b) Write an expression for the stationary distribution across states in the economy, $\pi \in \mathbb{R}^3$ (You can leave in terms of P).

(c) If a worker is U today, write an expression for the probability they will be employed exactly j periods in the future (considering any possible transitions which end in employment at j periods).[5].

(d) Assume that $\alpha = 0, \lambda = 0$. Is the stationary distribution unique? If not, describe the sorts of distributions that could exist and the intuition from the perspective of the Markov chain.

[5]Note: This is only looking at j periods into the future. i.e. this is **not** the probability that they become at employed at least once during the j periods, which is a much more difficult calculation.