Growth Models

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1 Quick Review from General Equilibrium

- Homogeneous Functions/CRS (Inputs x, y)

    - If a function $F(x, y)$ has:

$$F(\gamma x, \gamma y) = \gamma \cdot F(x, y) \tag{1}$$

    it is called constant returns to scale or homogeneous of degree 1.

- We can also get from CRS:

$$F(x, y) = \underbrace{\overbrace{\underbrace{x}_{\text{quantity } x} \cdot \underbrace{\boldsymbol{\partial}_x F(x, y)}_{\substack{\text{marginal} \\ \text{product} \\ \text{of } x}}}^{\text{Also be the "competitive factor payments"}}} + \underbrace{y}_{\text{quantity } y} \cdot \underbrace{\boldsymbol{\partial}_y F(x, y)}_{\substack{\text{marginal} \\ \text{product} \\ \text{of } y}} \tag{2}$$

$$CRS \Rightarrow \begin{cases} \boldsymbol{\partial}_x F(\gamma x, \gamma y) = \boldsymbol{\partial}_x F(x, y) \\ \boldsymbol{\partial}_y F(\gamma x, \gamma y) = \boldsymbol{\partial}_y F(x, y) \end{cases} \text{(homogeneous of degree 0)} \tag{3}$$

- Define $f\left(\frac{x}{y}\right) \equiv F\left(\frac{x}{y}, 1\right) \Rightarrow F(x, y) = yf\left(\frac{x}{y}\right)$, i.e. let $\gamma = \frac{1}{y}$

    - Then from (3)

$$\boldsymbol{\partial}_x F = f'\left(\frac{x}{y}\right) \tag{4}$$

    - And we can also show that

$$\boldsymbol{\partial}_y F = f\left(\frac{x}{y}\right) - f'\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) \tag{5}$$

We will use these to express everything in the capital-labor <u>ratio</u> with CRS.

2 Malthusian Growth Model

- <u>Setup</u>:

$$Y_t = zF(\underbrace{L_t}_{\text{Land}}, \underbrace{N_t}_{\text{Labor}}) \text{ where } F \text{ is CRS (e.g. food products)} \tag{6}$$

  - Let $L_t = L$ be <u>fixed</u>, i.e. can't create land.
  - Let $C_t = Y_t$, consume all productions.
  - Also assume all population, N_t, works.

- <u>Population grows</u>:

  - Based on output (i.e. food) supply.

$$N_{t+1} = N_t\, G\left(\frac{C_t/N_t}{c^*}\right) \tag{7}$$

    where $G(\cdot)$ is growth function, c^* is a constant. And $c_t \equiv C_t/N_t$ is consumption per capita.

  - And growth factor: $G(1) = 1, G' > 0, G'' < 0$
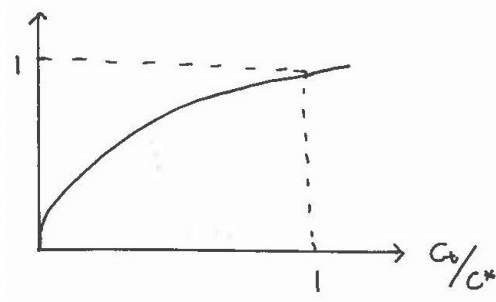 - Example Growth Factor: $G(\frac{c_t}{c^*}) = \left(\frac{c_t}{c^*}\right)^{\gamma}$ for $\gamma \in (0,1)$

Figure 1: Malthusian Growth Model

- <u>Interpretation</u>

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^{\gamma} \tag{8}$$

So we have:

$$\begin{cases} c_t > c^* & \Rightarrow \frac{N_{t+1}}{N_t} > 1 \quad , \text{ population grows} \\ c_t < c^* & \Rightarrow \frac{N_{t+1}}{N_t} < 1 \quad , \text{ population shrinks} \end{cases} \tag{9}$$

so c^* is a <u>subsistence level of consumption</u>

- <u>Production</u>:

$$C_t = zF(L, N_t) \tag{10}$$

$$\Rightarrow C_t = N_t z F\left(\frac{L}{N_t}, 1\right) \text{ by CRS} \tag{11}$$

$$\Rightarrow c_t \equiv zF(\ell_t, 1) \text{ where } \ell_t \equiv \frac{L}{N_t} \text{ (land-per-capita)} \tag{12}$$

For example, assume

$$F(\ell_t, 1) = \ell_t{}^\alpha, \ \alpha \in (0,1) \tag{13}$$

- <u>Equations</u>

$$c_t = z\ell_t{}^\alpha \tag{14}$$

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma \tag{15}$$

- <u>Substitute</u>:

$$\frac{N_{t+1}}{N_t} = \left(\frac{z(\frac{L}{N_t})^\alpha}{c^*}\right)^\gamma = \frac{z^\gamma (\frac{L}{N_t})^{\alpha\gamma}}{(c^*)^\gamma} \tag{16}$$

$$\Rightarrow \boxed{N_{t+1} = \left(\frac{z}{c^*}\right)^\gamma L^{\alpha\gamma} N_t^{1-\alpha\gamma}} \tag{17}$$

- <u>Steady State</u>:

$$\bar{N} = \left(\frac{z}{c^*}\right)^\gamma L^{\alpha\gamma} \bar{N}^{1-\alpha\gamma} \tag{18}$$

$$\Rightarrow 1 = \left(\frac{z}{c^*}\right)^\gamma \left(\frac{L}{\bar{N}}\right)^{\alpha\gamma} \tag{19}$$

$$\Rightarrow \bar{N} = \frac{L}{\left(\frac{c^*}{z}\right)^{\frac{1}{\alpha}}} \tag{20}$$

Substitute (20) into (14) to show that $c_t = c^*$. So only equilibrium is one of subsistence per-capita consumption <u>for all</u>. $\uparrow z \Rightarrow \bar{N} \uparrow$ but $c = c^*$. Pessimistic comment on

technological change?

- Dynamics

  - Take logs of (1):

  $$\log N_{t+1} = \underbrace{\gamma\left(\log z - \log c^* + \alpha \log L\right)}_{\phi_0} + \underbrace{(1 - \alpha\gamma)}_{\phi_1}\log N_t \tag{21}$$

  - Or of form:

  $$n_{t+1} = \phi_0 + \phi_1 n_t \text{ if } n_t \equiv \log N_t \tag{22}$$

  - Can use our old toolset of linear difference equations:

  $$x_t = \begin{pmatrix} 1 \\ n_t \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ \phi_0 & \phi_1 \end{pmatrix}, \boxed{x_t = A^t x_0} \tag{23}$$

3 Solow Growth Model

- Setup:

  - Let N_t be <u>exogenous</u>: $N_{t+1} = (1 + g)N_t$, N_0 given and g a constant population growth rate.

  - Other factor of production is capital, not land.

  $$\underbrace{K_{t+1}}_{\substack{\text{Next} \\ \text{periods} \\ \text{capital}}} = \underbrace{(1-\delta)}_{\substack{\text{depreciation} \\ \text{of capital}}} K_t + \underbrace{X_t}_{\substack{\text{investment} \\ \text{in new} \\ \text{capital}}}, \delta \in (0,1) \tag{24}$$

- Production:

  - We have:

  $$Y_t = zF(K_t, N_t) \text{ with C.R.S} \tag{25}$$

  which can be used for consumption or for capital.

  - And:

  $$C_t + X_t = zF(K_t, N_t) \tag{26}$$

  which is the <u>resource constraint for economy</u>.

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- Define the capital-labor ratio:

$$k_t \equiv \frac{K_t}{N_t} \qquad (27)$$

- And investment per capital:

$$x_t \equiv \frac{X_t}{N_t} \qquad (28)$$

- Consumption/capital:

$$c_t \equiv \frac{C_t}{N_t} \qquad (29)$$

- Converting equations, divide (24) by N_t:

$$\frac{K_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}, \quad \text{or:} \qquad (30)$$

$$\frac{N_{t+1}}{N_{t+1}}\frac{K_{t+1}}{N_t} = \left(\frac{N_{t+1}}{N_t}\right)\left(\frac{K_{t+1}}{N_{t+1}}\right) = (1-\delta)\frac{K_t}{N_t} + \frac{X_t}{N_t}, \quad \text{where } N_{t+1} \text{ is the growth of population} \qquad (31)$$

$$\Rightarrow \boxed{k_{t+1}(1+g) = (1-\delta)k_t + x_t} \qquad (32)$$

- Next, <u>assume</u> (not based on optimization!):

  - Agents save fraction $s \in (0,1)$ of output. i.e. by economy resource constraint.

$$C_t = (1-s)zF(K_t, N_t) \qquad (33)$$
$$\Rightarrow \text{dividing by } N_t \text{ , } c_t = (1-s)zF(k_t) \qquad (34)$$

  - Plug into equation (32):

$$\boxed{k_{t+1} = \frac{(1-\delta)k_t}{1+g} + \frac{sz}{1+g}f(k_t)} \qquad (35)$$

  - This is a non-linear difference equation in k_t

– Find a steady state solution:

$$\bar{k} = \frac{1-\delta}{1+g}\bar{k} + \frac{sz}{1+g}f(\bar{k}) \tag{36}$$

$$\Rightarrow \left(\frac{1+g}{1+g} - \frac{1-\delta}{1+g}\right)\bar{k} = \frac{sz}{1+g}f(\bar{k}) \tag{37}$$

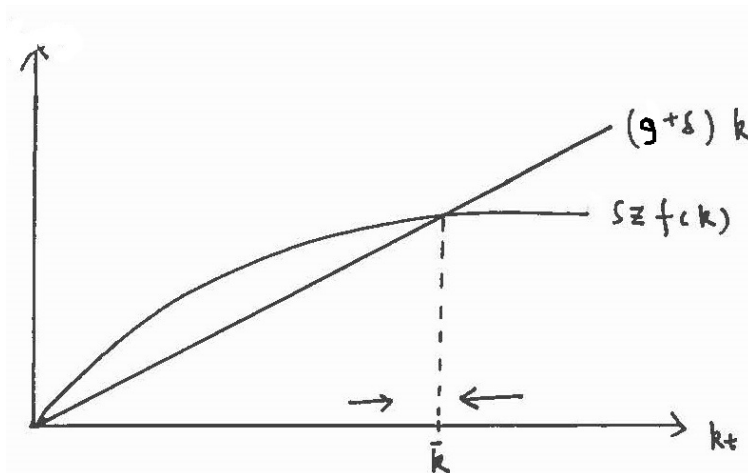$$\Rightarrow \boxed{(g+\delta)\bar{k} = szf(\bar{k})} \tag{38}$$

Figure 2: Convergence in a Solow Growth Model

- Use to understand effect on $\bar{k}$ of:

    – increase in z

    – increase in s

    – increase in g

4 Neoclassical growth

- Setup:

    – Solow took savings rate, s, as constant

    – Instead, make it choice of a representative consumer

    – First solve planner's problem, then competition equilibrium.

- Technology:

    – $Y = F(K, N)$ with CRS.

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- Let $k \equiv \frac{K}{N}$ then per capita like in Solow

- Let $f(k) \equiv F(k, 1)$, then $Nf(k) = F(K, N)$ from CRS:

$$c_t + \underbrace{x_t}_{\equiv k_{t+1} - (1-\delta)k_t} = f(k_t) \tag{39}$$

or,

$$\underbrace{c_t + k_{t+1}}_{\substack{\text{consume of} \\ \text{capital} \\ \text{next period}}} = \underbrace{f(k_t)}_{\text{production}} + \underbrace{(1-\delta)k_t}_{\substack{\text{last period's capital,} \\ \text{depreciated}}} \quad \text{i.e., the gross output per capita}$$

$$\tag{40}$$

$$\Leftrightarrow C_t + K_{t+1} = F(K_t, N_t) + (1-\delta)K_t \tag{41}$$

where $c_t = \frac{C_t}{N}$, replace K_{t+1} as k_{t+1}, $f(k) = \frac{1}{N}F(K, N)$

- Note: we are leaving out a possible irreversibility constraint on capital.

  * If you don't want the consumer to be able to "eat" their machines, then don't let capital contract more than the depreciation rate.

$$x_t \geq 0 \tag{42}$$

or,

$$K_{t+1} \geq (1-\delta)K_t \tag{43}$$

- Preferences:

$$u(c) \text{ (no disutility of labor)} \tag{44}$$

where $u'(c) > 0$, $u''(c) < 0$, $u'(0) = \infty$
Welfare:

$$\sum_{t=0}^{T} \beta^t u(c_t), \text{ where } T \leq \infty \tag{45}$$

- Planning Problem:

$$\max_{c_t,k_{t+1}} \sum_{t=0}^{T} \beta^t u(c_t) \tag{46}$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t, \text{ where } k_0 \text{ was given} \tag{47}$$

$$\text{Transversality or } k_{T+1} \geq 0 \tag{48}$$

- Lagrangian (without transversality):

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + \lambda_t \left(f(k_t) + (1 - \delta)k_t - c_t - k_{t+1} \right) \right) \tag{49}$$

- FONC:

$$[c_t] : u'(c_t) - \lambda_t = 0 \tag{50}$$

$$[k_{t+1}] : -\lambda_t + \beta\lambda_{t+1} \left[f'(k_{t+1}) + (1 - \delta) \right] = 0 \text{ for } t = 0, \cdots, T - 1 \tag{51}$$

$$[k_{T+1}] : -\beta^T \lambda_T \leq 0, = 0 \text{ if } k_{T+1} > 0 \tag{52}$$

$$\text{Or if } T = \infty, \lim_{T \to \infty} \beta^T \lambda_T k_{T+1} = 0 \tag{53}$$

- Summary of Equations:

  - Equations in c_t, λ_t, k_t given k_0:

$$\boxed{\begin{aligned} u'(c_t) &= \lambda_t \\ \lambda_t &= \beta\lambda_{t+1} \left[f'(k_{t+1}) + (1 - \delta) \right] \\ k_{t+1} + c_t &= f(k_t) + (1 - \delta)k_t \end{aligned}} \tag{54} \tag{55} \tag{56}$$

  - (54) shows the interpretation of the present-value Lagrange multiplier λ_t as the marginal utility of consumption on relaxing the resource constraint

  - Assume $T = \infty$, look for steady state:

$$\lambda_t = \bar{\lambda} \tag{57}$$

$$c_t = \bar{c} \tag{58}$$

$$k_t = \bar{k} \tag{59}$$

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– Also:

$$1 + \underbrace{\rho}_{\substack{\text{discount} \\ \text{rate}}} \equiv \frac{1}{\beta} = f'(\bar{k}) + (1 - \delta) \tag{60}$$

Or: $\boxed{f'(\bar{k}) = \rho + \delta}$ (Modified golden rule!) $\tag{61}$

– Note: $\bar{k}$ is independent of preference except β. Then:

$$\bar{k} + \bar{c} = f(\bar{k}) + (1 - \delta)\bar{k} \Rightarrow \boxed{\bar{c} = f(\bar{k}) - \delta\bar{k}} \tag{62}$$

- Dynamics: Shooting Method:

  – Finite horizon: $k_{T+1} = 0$

  – Equations:

$$u'(c_t) = \lambda_t \tag{63}$$
$$\lambda_t = \beta\lambda_{t+1}\left[f'(k_{t+1}) + (1 - \delta)\right] \tag{64}$$
$$k_{t+1} + c_t = f(k_t) + (1 - \delta)k_t \tag{65}$$

  – Boundaries: k_0 given, $k_{T+1} = 0$ if $T < \infty$.

  – Shooting method:

    * Guess λ_0
    * Use (63) to get c_0
    * Use (65) to get k_1 since k_0 given
    * Use (64) to get λ_1
    * Use (63) to get c_1
    * keep repeating until k_{T+1} calculated
    * Check that $k_{T+1} \approx 0$
    * If not, change λ_0 and repeat
      If $k_{T+1} > 0$, "eat more" by lowering λ_0

  – If infinite horizon, know that $k_t \to k_\infty$, where $f'(k_\infty) = \rho + \delta$

  – Can try to "shoot" at $k_\infty = k_{T+1}$ for very large T

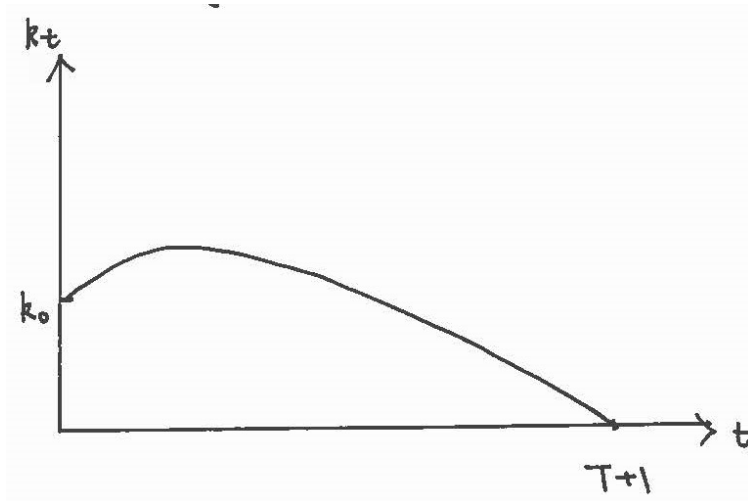- "Turnpike" Theorem For large T, will spend more of the time near k_∞
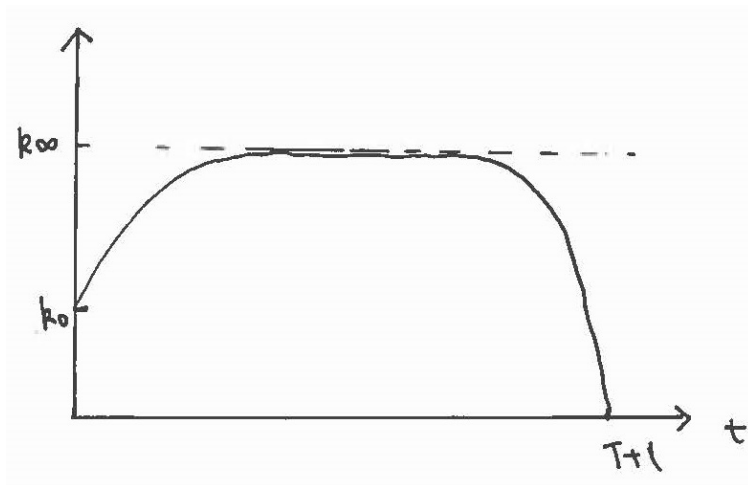
Figure 3: Shooting Method

Figure 4: Turnpike Theorem, with large T

5 Decentralization with a Competitive Equilibrium

5.1 Basic setup

- Take the same technology $F(K_t, N_t)$

- Add in potential for endogenous labor supply N_t

- Formulate with large number of households, firms.

- <u>Price system</u>: $\{q_t, \tilde{w}_t, \tilde{r}_t\}_{t=0}^{\infty}$ at date 0 (i.e. time 0 budget), where q_t is the price of good, $\tilde{w}_t$ is wage and $\tilde{r}_t$ is the capital rental.

5.2 Households

- Owns labor, capital. Rent both to firms.

- Purchases goods for consumption or capital investment.

- Include disutility of labor in utility: $u(c, n)$

- Solve problem:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \tag{66}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \underbrace{q_t}_{\substack{\text{Price of} \\ \text{homogeneous} \\ \text{good at } t \\ \text{in time 0}}} \left(c_t + \underbrace{k_{t+1} - (1-\delta)k_t}_{\equiv\, x_t,\ \text{net saving}} \right) \leq \sum_{t=0}^{\infty} (\underbrace{\tilde{w}_t}_{\substack{\text{wages} \\ \text{on labor} \\ \text{supplied}}} n_t + \underbrace{\tilde{r}_t}_{\substack{\text{rents} \\ \text{from} \\ \text{capital} \\ \text{supplied}}} k_t) \tag{67}$$

$$0 \leq n_t \tag{68}$$

$$n_t \leq 1 \quad \text{1 unit of time endowment} \tag{69}$$

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t, n_t) + \lambda_t(1 - n_t) + \phi_t n_t \right] \tag{70}$$

$$+ \mu \left[\sum_{t=0}^{\infty} (\tilde{w}_t n_t + \tilde{r}_t k_t) - \sum_{t=0}^{\infty} q_t \left(c_t + k_{t+1} - (1-\delta)k_t \right) \right] \tag{71}$$

where λ_t is LM on $n_t \leq 1$, ϕ_t is LM on $n_t \geq 0$ and μ is LM on time 0 budget.

- FONC:

$$[c_t] : \beta^t \boldsymbol{\partial}_c u(c_t, n_t) - \mu q_t = 0 \tag{72}$$

$$[n_t] : \beta^t \left(\boldsymbol{\partial}_n u(c_t, n_t) + \phi_t - \lambda_t \right) + \mu \tilde{w}_t = 0 \tag{73}$$

$$[k_{t+1}] : r_{t+1} + q_{t+1}(1 - \delta) - q_t = 0 \tag{74}$$

$$\text{Complimentary slackness conditions: } \begin{cases} \lambda_t(1 - n_t) = 0 \\ \phi_t n_t = 0 \end{cases} \tag{75}$$

Note: (75) can also be written as:

$$- \beta^t \boldsymbol{\partial}_n u(c_t, n_t) \begin{cases} = \mu \tilde{w}_t & \text{if } 0 < n_t < 1 \\ > \mu \tilde{w}_t & \text{if } n_t = 1 \\ < \mu \tilde{w}_t & \text{if } n_t = 0 \end{cases} \tag{76}$$

- <u>Summarizing Equations</u> for $\{c_t, k_t, n_t\}$ in terms of $\{q_t, \tilde{r}_t, \tilde{w}_t\}$:

$$q_t = q_{t+1}(1 - \delta) + r_{t+1} \tag{77}$$

$$\mu q_t = \beta^t \boldsymbol{\partial}_c u(c_t, n_t) \tag{78}$$

$$- \beta^t \boldsymbol{\partial}_n u(c_t, n_t) \begin{cases} = \mu \tilde{w}_t & \text{if } 0 < n_t < 1 \\ > \mu \tilde{w}_t & \text{if } n_t = 1 \\ < \mu \tilde{w}_t & \text{if } n_t = 0 \end{cases} \tag{79}$$

Combining (78) and (79) with interior solutions gives us the same as our labor supply decision in the static general equilibrium,

$$\frac{-\boldsymbol{\partial}_n u(c_t, n_t)}{\boldsymbol{\partial}_c u(c_t, n_t)} = \frac{\tilde{w}_t}{q_t} \equiv w_t \tag{80}$$

For now, assume that the agent gains no utility from leisure, so:

$$u(c, n) = u(c) \Rightarrow \boldsymbol{\partial}_n u = 0 \Rightarrow n_t = 1 \tag{81}$$

i.e. Inelastic supply of all labor

5.3 Firms (Price taker)

- Face prices : $\{q_t, \tilde{w}_t, \tilde{r}_t\}$, choose inputs for production.

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- Solve Problem:

$$\max_{K_t, N_t} \sum_{t=0}^{\infty} \left[\underbrace{q_t F(K_t, N_t)}_{\text{revenue}} \underbrace{- \tilde{r}_t K_t - \tilde{w}_t N_t}_{\text{input costs}} \right] \tag{82}$$

Use $F(K, N) = \boldsymbol{\partial}_K F(K, N) K + \boldsymbol{\partial}_N F(K, N) N$ with CRS:

$$= \max_{K_t, N_t} \sum_{t=0}^{\infty} \left[(q_t \boldsymbol{\partial}_K F(K_t, N_t) - \tilde{r}_t) K_t + (q_t \boldsymbol{\partial}_N F(K_t, N_t) - \tilde{w}_t) N_t \right] \tag{83}$$

- Since CRS, we have a 0-profit condition as before. Necessary conditions:

$$q_t \boldsymbol{\partial}_K F(K_t, N_t) = \tilde{r}_t \tag{84}$$

$$q_t \boldsymbol{\partial}_N F(K_t, N_t) = \tilde{w}_t \tag{85}$$

Since CRS, use $F\left(\frac{K}{N}, 1\right) = f(k)$ for $k \equiv \frac{K}{N}$, and other expressions from this sections:

$$\boldsymbol{\partial}_k F(K, N) = \boldsymbol{\partial}_k F\left(\frac{K}{N}, 1\right) = f'(k) \tag{86}$$

$$\boldsymbol{\partial}_N F(K, N) = f(k) - k f'(k) \tag{87}$$

So:

$$\frac{\tilde{r}_t}{q_t} = f'(k_t) \tag{88}$$

$$\frac{\tilde{w}_t}{q_t} = f(k_t) - k_t f'(k_t) \tag{89}$$

So workers paid marginal products of inputs and firms of indeterminate size, but 0 profits.

5.4 Solve for Competitive Equilibrium

- Definition

    A C.E. is a Price system $\{q_t . \tilde{w}_t, \tilde{r}_t\}_{t=0}^{\infty}$ and a Feasible allocation $\{k_{t+1}, c_t\}_{t=0}^{\infty}$, such that:

    – Given price system, allocations solve firm problem.

    – Given price system, allocations solve Households problem.

- Solving for the equilibrium:

    – Solve planning problems $\{c_t, k_{t+1}\}$

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– Guess $\mu = 1$ (indeterminate), plug in with c_t:

$$q_t = \beta^t u'(c_t) \quad \text{(from consumer's problem)} \tag{90}$$

$$\tilde{r}_t = q_t f'(k_t) \quad \text{(from firm's problem)} \tag{91}$$

$$\tilde{w}_t = q_t \left(f(k_t) - k_t f'(k_t) \right) \quad \text{(from firm's problem)} \tag{92}$$

– Verify that this price system and allocations fulfills all of the conditions of the Firm and HH problems (Budget constraints, etc).

- Conclusion:

  – For this economy, the planning problem and the competitive equilibrium have the same allocations.

  – The prices decentralize the planning problem.