Question 1

A consumer chooses consumption and savings to maximize their welfare subject to a budget constraints.

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$\text{s.t. } F_{t+1} = R_t \left(F_t + y_t - c_t\right), \quad \text{for all } t \geq 0 \tag{2}$$

$$c_t \geq 0, \quad \text{for all } t \geq 0 \tag{3}$$

$$\lim_{T\to\infty} \beta^T u'(c_T) F_{T+1} = 0 \quad \text{(Transversality Condition)} \tag{4}$$

Where $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, $\beta \in (0, 1)$, $\gamma > 0$, $F_0 = 0$, and $\{y_t\}_{t=0}^{\infty}$ is an exogenous, deterministic sequence of labor income with at least some positive y_t. $\{R_t\}_{t=0}^{\infty}$ are the gross interest rate on financial assets, and are an exogenous, deterministic sequence known to the consumer.

(a) Setup the Lagrangian for this problem, being clear on Lagrange Multipliers and equality/inequality constraints.[1]

(b) Show that the $c_t \geq 0$ constraint can never bind, then find the Euler equation for the consumer at all $t \geq 0$.[2]

(c) For some $\delta \geq 0$ and $\phi \geq 0$, let the labor income process be

$$y_t = \begin{cases} y_0 \delta^t & t = 0, \dots T \\ y_0 \delta^t \phi^{t-T} & t = T+1, \dots \infty \end{cases}$$

It just happens that $\{R_t\}_{t=0}^{\infty}$ is such that the consumer optimally sets $c_t = y_t$ and $F_{t+1} = 0$ for all t (i.e., this is a particular sequence of R_t which rationalizes this behavior). Find a formula for $\{R_t\}_{t=0}^{\infty}$ and justify your formula.[3]

(d) Interpret your formula for R_t in terms of (i) the consumer's impatience, and (ii) the consumer's income growth.

[1]Hint: You can be sloppy and skip the multiplier on the Transversality condition, as we have done in class. As always, I strongly suggesting using present-value Lagrange multipliers to simplify algebra.

[2]Hint: The only change from our standard problem is the time varying interest rate. You will need to be careful with the timing when taking first order conditions. To proof that $c_t > 0$, you will need to use the marginal utility and use the fact that there is at least some positive income.

[3]Hint: What does optimality mean? Also, be a little careful around T for the calculation of R_t

Question 2

There are two consumers ($i = 1, 2$) with potentially different consumption and income processes (c_t^i and y_t^i), initial financial wealth $F_0^i = 0$, and identical preferences subject to an intertemporal budget constraint,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{5}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i \tag{6}$$

where $u'(c) > 0, u''(c) < 0, \beta \in (0,1)$, and $\beta R = 1$. Assume that the two income processes are

$$y_t^1 = \{0, 1, 0, 1, \ldots\} \tag{7}$$

$$= \begin{cases} 0 & \text{if } t \text{ even} \\ 1 & \text{if } t \text{ odd} \end{cases} \tag{8}$$

$$y_t^2 = \{1, 0, 1, 0, \ldots\} \tag{9}$$

$$= \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases} \tag{10}$$

(a) Apply the permanent income result to find c_t^i for both agents.[4]

(b) For every t, compare $c_t^1 + c_t^2$ vs. $y_t^1 + y_t^2$. Would this comparison change if $\beta R \neq 1$? (no need to solve for the exact c_t^i in that case)

(c) Assuming that both agents start with no financial wealth, i.e. $F_0^1 = F_0^2 = 0$, compute the asset trades between consumer 1 and 2 to support the c_t^i where the period-by-period budget constraint for $i = 1, 2$ is

$$F_{t+1}^i = R(F_t^i + y_t^i - c_t^i)$$

[4]Hints: Note that if $a_t = \{1, 0, 1, 0 \ldots\}$ then $\sum_{t=0}^{\infty} \beta^t a_t = 1 + \beta^2 + \beta^4 + \ldots = \sum_{t=0}^{\infty} \left(\beta^2\right)^t$.

Question 3

A consumer chooses consumption and savings to maximize their welfare subject to a budget constraints.

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t c_t \tag{11}$$

$$\text{s.t. } F_{t+1} = R\left(F_t - c_t\right), \quad \text{for } t = 0, \dots T \tag{12}$$

$$c_t \geq 0, \quad \text{for } t = 0, \dots T \tag{13}$$

$$F_{T+1} = 0 \tag{14}$$

where $T < \infty, F_0 > 0, R > 0$, and $\beta \in (0, 1)$.[5] Note that there is positive initial wealth, but no labor income. They are choosing how to spend their wealth

(a) Check if the utility function is concave.

(b) Setup the Lagrangian and find the first-order necessary conditions.[6]

(c) Using the first-order necessary conditions, find the optimal path of $\{c_t, F_{t+1}\}_{t=0}^{T}$. Is the solution always unique, and if not, why?

(d) Interpret how does the optimal allocation depends on the relationship between R and β.

[5] Hint: This has **not** assumed any relationship between β and R. Consider that $\beta R \lesseqqgtr 1$ as potentially having different behaviors, which might require analyzing different cases since you don't have the luxury to pick a single R which is convenient. The $\beta R = 1$ case will be very familiar.

[6] Hint: You will have to be very careful with Lagrange multipliers here and cannot just directly use our formulas. The complementarity conditions will be important for the c_t constraints. And remember that linear objectives and linear constraints usually means corners, so the $c_t \geq 0$ constraint may actually be important.