1. (Sequential Formulation of Neoclassical Growth with Interest Rates)

A competitive (i.e., price taking) equilibrium exists for households and firms. This is identical to the competitive equilibrium of the deterministic neoclassical growth model we did in class except:

- Consumer's can smooth consumption by investing in capital, but also have access to financial assets (with prices determined in general equilibrium).
- There will not have a set of assets traded at time 0 which provide a claim to consumption at time t (i.e., our old complete set of assets with prices q_t^0 doesn't exist).
- Instead, household's can buy and sell (at each time t rather than time 0) claims to consumption at time $t+1$ (i.e., 1 period bonds). This single asset can be sold to each other(or to the government) on competitive markets that operate at each time period (i.e., spot markets).
- Instead of a lifetime budget constraint, households have sequential budget constraints.
- The price of the consumption good at time t is normalized to 1, so the price system will be $\{r_t, w_t, i_{t+1}\}$ where r_t and k_t are the real rental rates for capital and labor in time t goods, and i_{t+1} is the net interest rate on a bond purchased at time t.
- The gross interest rate of purchasing a unit of the bond is $1+i_{t+1}$. Consequently, buying a claim to 1 unit of the good at time t costs $\frac{1}{1+i_{t+1}}$.[1]
- The government also has a sequential budget constraint, and smooths in expenditures through the bond market and tax policy (i.e., it can also buy and sell bonds).
- Bond's are in 0 net supply. That is, if consumers hold $B_t > 0$ of the bonds, then the government would need to hold $-B_t < 0$ of the bonds.

To summarize the entire equilibrium for a representative consumer and firm,

Allocation: $\{c_t, k_t, B_t\}_{t=0}^{\infty}$. Bond holdings, B_t, are pieces of paper. k_0 and B_0 are given.

Price System: $\{r_t, w_t, i_{t+1}\}_{t=0}^{\infty}$

Government Policy: $\{\tau_{ct}, g_t, B_t^g\}_{t=0}^{\infty}$. That is, a consumption tax, government expenditures, and government bond holdings (which is negative if they owe the households money). Assume that the Government Policy is given exogenously, though it will need to be budget feasible.

Feasibility: The firm operates the same neoclassical production function (with $f'(\cdot) > 0$ and $f''(\cdot) < 0$ as before, and the bond markets clear,

$$c_t + k_{t+1} - (1-\delta)k_t + g_t = f(k_t) \tag{1}$$

$$B_t^g = -B_t, \text{ i.e. bonds are in 0 net supply} \tag{2}$$

[1] From our interest notes, consider a claim to a unit of consumption delivered at time $t+1$ but priced in time t good, q_{t+1}^t. Then the interest rate on this 1 period claim at time t is defined as $\rho_{t,t+1}$ by $q_{t+1}^t \equiv \frac{1}{1+\rho_{t,t+1}}$. This calculation is doing the same thing for these 1 period claims.

Government Budget: The government policy is given exogenously, but it must "balance" in the long-run (i.e., when government debt is taken into account with the endogenous prices).

$$\underbrace{g_t}_{\text{Expenditures}} + \underbrace{\frac{1}{1+i_{t+1}}}_{\text{With Interest}} \underbrace{B^g_{t+1}}_{\text{New Bonds}} \leq \underbrace{\tau_{ct} c_t}_{\text{Tax Income}} + \underbrace{B^g_t}_{\text{Previous Bonds}} \quad \text{for all } t \geq 0 \tag{3}$$

For example, if $B^g_t = B^g_{t+1}$ then the government is rolling over their bonds and paying (or getting) the interest. There will also be a no-ponzi scheme condition (e.g. $\lim_{t\to\infty} |B^g_t| < \infty$)

Households's Problem: A large number of identical consumers have a typical strictly concave utility function($u'(\cdot) > 0, u''(\cdot) < 0, u'(0) = \infty$), and provide 1 unit of labor inelastically. Taking B_0, k_0, prices, and government polices as given

$$\max_{\{c_t, k_{t+1}, B_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

$$\text{s.t. } (1+\tau_{ct})c_t + k_{t+1} + \frac{1}{1+i_{t+1}} B_{t+1} \leq B_t + (1-\delta)k_t + r_t k_t + w_t, \quad \text{for } t \geq 0 \tag{5}$$

$$(+\text{ a no-ponzi scheme transversality condition}) \tag{6}$$

Note that the period by period budget constraint is written with the price of the consumption good normalized to 1.

Firm's Problem: A large number of identical firms operate a constant returns to scale (CRS) production function $F(K, N)$ with the usual result that $F(\frac{K}{N}, 1) = \frac{1}{N} f(k)$ when $k \equiv \frac{K}{N}$.[2] Taking prices as given they maximize,

$$\max_{K_t, N_t} \{F(K_t, N_t) - w_t N_t - r_t K_t\} \tag{7}$$

With this complete specification of the equilibrium,

(a) Define a competitive equilibrium

(b) Solve for the first-order-necessary conditions (FONC) of the firms to get expressions for the real rental rate of capital in terms of production function $f(\cdot)$ and the aggregate capital k_t (which is also the aggregate capital to labor ratio here since $N_t = 1$). Why are we able to use a representative firm?

(c) Solve for the FONC of the household's choice of $\{c_t, k_{t+1}, B_{t+1}\}_{t=0}^{\infty}$. There will now be 2 Euler equations: one for capital investment, and another for bond investment. Substitute from the FONC of the firm's problems to express in terms of allocations where possible.

(d) Define the real return on capital, as in the class notes: $R_t \equiv 1 - \delta + f'(k_{t+1})$. What is i_{t+1} in terms of R_t? Interpret this relationship.[3]

[2] Note that if there is a representative firm, then $F(\frac{K}{N}) = f(k)$ and $k = K$.

[3] Hint: stare at the two Euler equations and combine. Is there a way to think of arbitrage between using these two different methods to smooth consumption?

(e) If the government policy has $B_t^g = 0$ for all t, then note that $B_t = 0$ from the feasibility condition. Is there any trading of bonds between consumers, and if not, why not? If so, then why is there still an interest rate?

(f) Let $B_t^g = B_t = 0$ forever. You can either assume that $g_t = \tau_{ct} = 0$, or that these are fixed such that $\bar{\tau}_c, \bar{g}$ balance the government budget. Calculate the steady state $\left\{\bar{k}, \bar{c}, \bar{B}, \bar{r}, \bar{i}\right\}$.

2. (Variations on Financing Government Expenditures)
The consumer values consumption, and provides 1 unit of labor inelastically. The period utility be $u(c) = \log(c)$.

Take our standard neoclassical growth model, with the possibility of consumption taxes, τ_{ct}, lump-sum taxes, τ_{ht}, and labor taxes, τ_{nt}. First, assume that taxes and government expenditures are 0 (i.e., $\bar{g} = \bar{\tau}_h, = \bar{\tau}_c$) and that the economy is in a steady state (i.e. $\bar{k}$ as the steady state capital, and $\bar{c} = f(\bar{k}) - \delta\bar{k}$. Let the capital at time 0 be this steady state capital, i.e. $k_0 = \bar{k}$.

There is a sudden announcement that $g_t = \bar{g}$ for all $t \geq 0$, where $\bar{g} = \frac{1}{4}(f(k_0) - \delta k_0)$, and the government expenditures are financed entirely through lump-sum taxes, $\bar{\tau}_h$.

(a) Calculate the new steady state $\bar{c}$ and $\bar{k}$.

(b) What is the transition path of c_t and k_t from the k_0 initial condition?

(c) What is the behavior of c_{t+1}/c_t and $R_{t+1} \equiv f'(k_{t+1}) + 1 - \delta$ along this transition path?

(d) Argue that the timing of the lump-sum taxes is irrelevant (i.e., any τ_{ht} fulfilling the long-run government budget constraint gives the same allocation).

(e) Now, consider the alternative policy that the government finances its expenditures entirely through consumption taxes. First assume that consumption taxes are constant(i.e., $\tau_{ct} = \bar{\tau}_c$ for all $t \geq 0$). Find the new steady state $\bar{c}$ and $\bar{k}$ and the transition path from k_0, .

(f) In this case, would the timing of the consumption tax matter (i.e., does any τ_{ct} fulfilling the long-run government budget constraint deliver the same allocations $\{c_t, k_{t+1}\}$ along the transition dynamics?) If not, why?

(g) Without solving the full model, would financing expenditures entirely through constant labor taxes $\bar{\tau}_n$ have the same steady state as that of lump-sum taxes? What about the transition dynamics?

3. (Special Permanent Income Model + Asset Pricing)
A consumer faces a time-invariant, risk-free gross interest rate of $R \equiv e^r$ with $r > 0$. The consumer can borrow or lend at this rate up to a "no-Ponzi scheme" condition. The savings (or debt) of the consumer is denoted F_t.

Let the discount factor be β, and define $\beta \equiv e^{-\rho}$ where ρ is the discount rate.[4]

Finally, assume that the consumer's income, Y_t, is a stochastic process following:

$$Y_{t+1} = Y_t \exp(\sigma \epsilon_{t+1})$$

where $\epsilon_{t+1} \sim N(0,1)$. The consumer chooses $\{C_t, F_{t+1}\}_{t=0}^{\infty}$ (which may now be stochastic) to maximize their expected utility, i.e.

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right] \tag{8}$$

$$\text{s.t.}\, C_t + F_{t+1} \leq RF_t + Y_t, \quad \text{for } t \geq 0 \tag{9}$$

With the LOM

$$Y_{t+1} = Y_t e^{\sigma \epsilon_{t+1}} \tag{10}$$

(a) Find the first-order necessary conditions for this problem.[5] Verify that they imply,

$$1 = \mathbb{E}_t \left[R\, M_{t+1} \right] \tag{11}$$

where

$$M_{t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

(b) Verify that

$$M_{t+1} = \exp(-(\rho + \gamma(c_{t+1} - c_t)))$$

where $c_t \equiv \log C_t$. Then take logs to define $\log M_{t+1} \equiv m_{t+1}$ and

$$m_{t+1} = -\rho - \gamma(c_{t+1} - c_t) \tag{12}$$

(c) Assume there is a representative consumer and the consumer is observed to set $C_t = Y_t$ and $F_{t+1} = 0$ for all $t \geq 0$ (thinking of a Lucas-style asset pricing model in general equilibrium). Use (12), the log $y_t \equiv \log Y_t$ in (10) with

$$y_{t+1} - y_t = \sigma \epsilon_{t+1}$$

To show that $m_{t+1} \sim N(\mu_m, \sigma_m^2)$, i.e. an iid random normal for some mean μ_m and variance σ_m^2 functions of parameters.

Find the μ and σ_m^2.

[4] From a Taylor Series approximation, this is approximately equal to $\beta \equiv \frac{1}{1+\rho}$ for small ρ.

[5] Be careful to keep expectations around when the information set only allows forecasts. As the law of motion in (10) is not a constraint on the choice, you don't put it in as a Lagrange Multiplier. Instead, you should solve the problem with the binding constraint in (9) and then apply (10) as the forecast after you have the Euler Equation.

(d) Use (11) and the previous parts to find the value of constant net interest rate, r, which rationalizes this behavior.[6]

(e) Interpret the role of aggregate uncertainty, σ, on interest rates, r. Why/when would γ matter? If $\sigma = 0$, why/when would γ matter?

[6] Hint:Use the optimality conditions from previous parts and deduce the r necessary to clear the markets. Also, for a normal random variable $z \sim N(\mu_z, \sigma_z^2)$ note that $\mathbb{E}\left[e^z\right] = e^{\mu_z + \frac{1}{2}\sigma_z^2}$.

4. (**Lack of Access to Banking**)
Let the consumer have log utility,

$$u(c) = \log(c)$$

Given $F_0 = 0, \beta R = 1$, and the deterministic income stream $y_t = \delta^t$, the consumer maximizes

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{13}$$
$$\text{s.t. } F_{t+1} = R(F_t + y_t - c_t), \quad \forall t \geq 0 \tag{14}$$
$$F_{t+1} \leq 0 \tag{15}$$
$$\text{(transversality condition)} \tag{16}$$

Note that the constraint is for consumers that are "unbanked', and do not have any access to savings accounts through a formal banking system (though they may borrow through informal at gross interest rate R through other sources)

(a) Derive, similar to what we did in class, the Euler equation (actually an inequality here), and the condition for it holding with equality.[7]

(b) Let $\delta > 1$. What is $\{c_t\}_{t=0}^{\infty}$? Briefly interpret from the perspective of lack of access to the banking system

(c) Let $0 < \delta < 1$. What is $\{c_t\}_{t=0}^{\infty}$? Briefly interpret from the perspective of lack of access to the banking system

(d) Assume that the consumer optimally eats their entire income each period, i.e., $c_t = y_t = \delta^t$ which implies $c_{t+1} = \delta c_t$. Setup, using dynamic programming, an equation to find the value $V(c)$ recursively (i.e., connecting to the above $V(c_0) = U_0$). (hint: you can write down this recursive Bellman equation directly rather than deriving from the sequential formula).

(e) Guess that $V(c) = k_0 + k_1 \log(c)$ for some undetermined k_0 and k_1.Solve for k_0 and k_1, and evaluate $V(1)$ (i.e., the value of starting with $c_0 = 1$). Hint: Note that this equation deliberately is avoiding any t subscripts! This makes it a truly recursive expression.

[7] Hint: It is very close to our problem in class, and the equivalent problem with our canonical constrained optimization approach has constraint $-F_{t+1} \geq 0$