Permanent Income with No Borrowing, and Dynamic Programming

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1 Basic setup

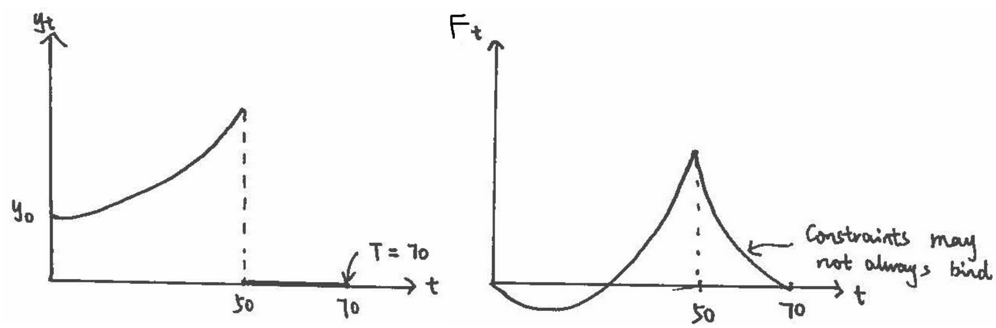

Figure 1: Income and Savings over Lifecycle with Borrowing

Recall: Previous example in Figure 1 with growing income and retirement. Assets may become negative early in lifecycle.

1.1 Add constraints on PIH

$$\max_{\{c_t, F_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

$$\text{s.t. } F_{t+1} = R(F_t + y_t - c_t) \tag{2}$$

$$\lim_{T\to\infty} \left(\beta^{T+1} F_{T+1}\right) \geq 0 \text{ (No ponzi)} \tag{3}$$

$$\left.\begin{matrix} F_{t+1} \geq 0 \\ c_t \geq 0 \end{matrix}\right\} \text{(Add constraints: no borrowing!)} \tag{4}$$

where F_0 is given, $\beta < 1$,$R > 1$; Assume $\lim_{c \to 0} u'(c) = \infty$ (Called an “Inada Condition”).

1.2 Set up Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(u(c_t) + \lambda_t [R(F_t + y_t - c_t) - F_{t+1}] + \nu_{t+1} \cdot F_{t+1} + \alpha_t \cdot c_t \right) \tag{5}$$

where λ_t is LM on inter-temporal budget; ν_{t+1} is LM on $F_{t+1} \geq 0$; α_t is LM on the $c_t \geq 0$

- First Order Necessary Condition:

$$[c_t] : \beta^t u'(c_t) - \beta^t \lambda_t R + \beta^t \cdot \alpha_t = 0, \forall t \geq 0 \tag{6}$$

$$[F_{t+1}] : -\lambda_t + \nu_{t+1} + \beta \lambda_{t+1} R = 0, \forall t \geq 0 \tag{7}$$

  – For constraints:

$$\nu_{t+1} \geq 0 \tag{8}$$

$$\alpha_t \geq 0 \tag{9}$$

$$\nu_{t+1} F_{t+1} = 0 \tag{10}$$

$$\alpha_t c_t = 0 \tag{11}$$

  – Reorganize (6)

$$\Rightarrow u'(c_t) + \alpha_t = \lambda_t R, \text{with } \alpha_t \geq 0 \tag{12}$$

$$\Rightarrow u'(c_t) \leq \lambda_t R, = \text{if } \alpha_t = 0 \tag{13}$$

From $\alpha_t = 0$ or $c_t = 0$, note if $c_t = 0$, use "Inada Condition" that $u'(0) = \infty$, which is a contradiction. So we have $\alpha_t = 0$,$c_t > 0$,implying:

$$\boxed{u'(c_t) = \lambda_t \cdot R} \tag{14}$$

  – Reorganize (7), multiply by R

$$\beta R R \lambda_{t+1} - R \lambda_t + R \nu_{t+1} = 0 \tag{15}$$

Use (14)

$$\beta R u'(c_{t+1}) + R \nu_{t+1} = u'(c_t) \tag{16}$$

Then use complementarity, since $\nu_{t+1} \geq 0$

$$\beta R u'(c_{t+1}) \leq u'(c_t), = \text{ if } F_{t+1} > 0, \forall t \geq 0 \tag{17}$$

- Note that if $F_{t+1} = 0$ and $\nu_{t+1} > 0$, then from budget constraint:

$$0 = R(F_t + y_t - c_t) \tag{18}$$

$$\Rightarrow c_t = F_t + y_t \text{ (eats all income and savings)} \tag{19}$$

- Summarizing results: under no-borrowing constraints:

$$\boxed{\begin{aligned} &u'(c_t) = \beta R u'(c_{t+1}); \text{or} \\ &u'(c_t) > \beta R u'(c_{t+1}) \ \underline{\text{and}}\ c_t = F_t + y_t \end{aligned}} \tag{20, 21}$$

1.3 Example on preferences

- We assume $u(c) = \log(c) \Rightarrow u'(c) = \frac{1}{c_0}, \beta R = 1$
  Then:

$$\frac{1}{c_t} = \frac{1}{c_{t+1}} \Rightarrow c_{t+1} = c_t, \ \underline{\text{or}} \tag{22}$$

$$\frac{1}{c_t} > \frac{1}{c_{t+1}} \Rightarrow c_{t+1} > c_t \ \underline{\text{and}}\ c_t = F_t + y_t \tag{23}$$

- <u>Example 1</u>:
  - Let $y_{t+1} = \delta y_t \Rightarrow y_t = \delta^t y_0$ s.t. $\delta > 1$ and $F_0 = 0$
  - Solution: (Guess always constrained, then verify:)

$$c_t = y_t, \forall t \tag{24}$$

$$F_t = 0, \forall t \tag{25}$$

  So the person is <u>always</u> borrowing constrained.
  Verify: $y_t > y_{t-1} \Rightarrow c_t > c_{t-1}, \forall t$; $F_t = 0 \Rightarrow c_t = y_t$

- <u>Example 2</u>:
  - Let $0 < \delta < 1, y_t = \delta^t y_0, F_0 = 0$
  - Solution:(Guess unconstrained and then verify)

Guess $c_t = \bar{c}$, such that

$$\underbrace{\bar{c} = (1-\beta) y_0 \sum_{t=0}^{\infty} \delta^t \beta^t}_{\text{unconstrained formula if } F_0 = 0} \quad \Rightarrow \tag{26}$$

$$\boxed{\bar{c} = (1-\beta) \frac{y_0}{1-\beta\delta}} \tag{27}$$

$$F_{t+1} = R(F_t + y_t - \bar{c}) \tag{28}$$

– Note: $c_0 = \frac{1-\beta}{1-\beta\delta} y_0 < \frac{1-\beta}{1-\beta} y_0 = y_0 \Rightarrow y_0 - c_0 > 0$, saves, not borrows
In the limit as $t \to \infty, y_t \to 0$ but $c_t = \bar{c}$. Put into budget to look for a steady state:

$$\bar{A} = R(\bar{A} + 0 - \bar{c}) \text{ if } F_t \approx F_{t+1} \text{ for large } t \tag{29}$$

$$\Rightarrow R\bar{c} = (R-1)\bar{A}, \text{ by } R = 1 + r \tag{30}$$

$$\bar{c} = \frac{R-1}{R} \bar{A} = \underbrace{\frac{r}{1+r}}_{\text{annuity value}} \bar{A} \tag{31}$$

– So lives off annuity value of savings eventually (Also $\frac{r}{1+r} = 1 - \beta$ if $\beta \equiv \frac{1}{1+r}$). E.g. Can by an annuity for a stream of income, they buy it and pay you a set amount forever (or until death).
See Figure 2 for example of decreasing income (not to scale).

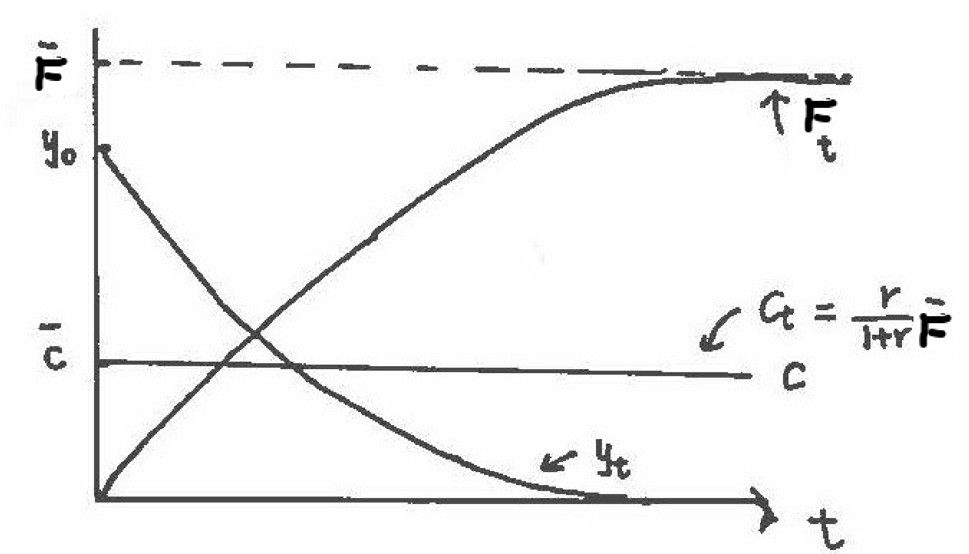

Figure 2: Asymptotic Behavior with Decreasing Income

2 Welfare cost of no-borrowing

- Setup:
  - Given a feasible $\{c_t\}$, the lifetime utility of an agent $U = \sum_{t=0}^{\infty} \beta^t u(c_t)$. This is their welfare, their objective function.
  - In general, adding constraints to the set of feasible $\{c_t\}$ weakly decreases welfare, as in Figure 3.

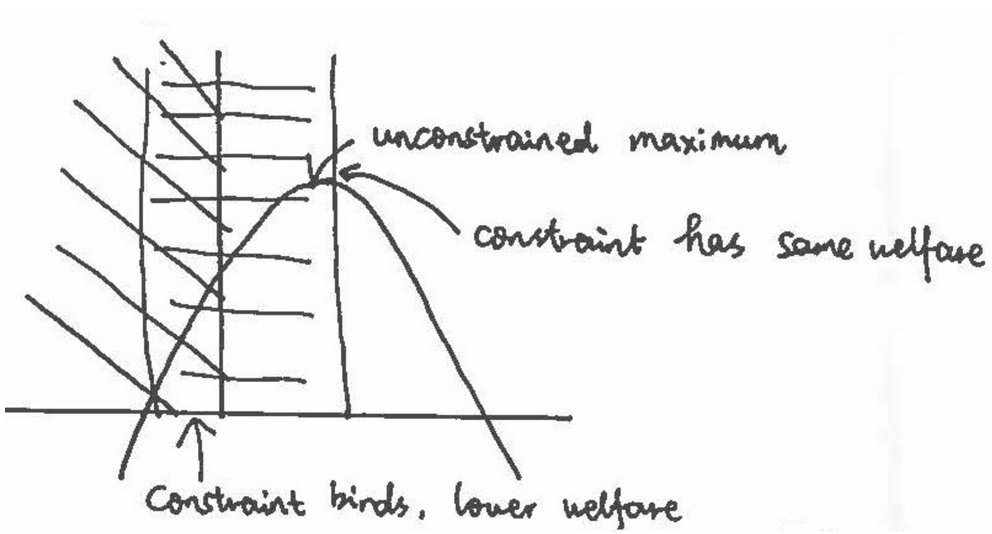

Figure 3: Constrained and First-Best

  - Unconstrained $T = \infty$
  - Assume $1 \leq \beta R \leq \delta, u(c) = \log(c), F_0 = 0$, subject to $\lim_{T\to\infty} F_{T+1}\beta^T$, no ponzi scheme.

- Solution:
  - Euler: $u'(c_t) = \beta R u'(c_{t+1}) \Rightarrow c_t = \beta \cdot Rc_{t+1} \Rightarrow c_t = (\beta R)^t c_0$
  - Lifetime budget:

$$0 = \sum_{j=0}^{\infty} R^{-j}(c_{t+j} - y_{t+j}) \tag{32}$$

$$\Rightarrow \sum_{t=0}^{\infty} R^{-t}\beta^t R^t c_0 = \sum_{t=0}^{\infty} R^{-t}\delta^t \cdot y_0 \tag{33}$$

$$\Rightarrow \underbrace{\frac{c_0}{1-\beta}}_{\text{PV of consumption}} = \underbrace{\frac{y_0}{1-\frac{\delta}{R}}}_{\text{PV of income}} \tag{34}$$

$$\Rightarrow c_0 = (1-\beta)\frac{y_0}{1-\frac{\delta}{R}} \tag{35}$$

- Consumer's Lifetime Utility:

$$
\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t \log(c_t) && (36)\\
&= \sum_{t=0}^{\infty} \beta^t \log\left((\beta R)^t \cdot \frac{1-\beta}{1-\frac{\delta}{R}} y_0 \right) && (37)\\
&= \sum_{t=0}^{\infty} \beta^t \left[\log(y_0) + \log(\frac{1-\beta}{1-\frac{\delta}{R}}) + t\log(\beta R) \right] && (38)\\
&= \frac{\log y_0 + \log(1-\beta) - \log(1-\frac{\delta}{R})}{1-\beta} + \log(\beta R) \underbrace{\sum_{t=0}^{\infty} t\beta^t}_{\substack{\text{Recall sum from}\\ \text{previous Markov}\\ \text{waiting time}}} && (39)
\end{aligned}
$$

$$
\boxed{U = \frac{1}{1-\beta}\left[\log(y_0) + \log(1-\beta) - \log(1-\frac{\delta}{R})\right] + \log(\beta R)\left[\frac{\beta}{(1-\beta)^2}\right]} \tag{40}
$$

- No borrowing, $T = \infty$
  - Same assumptions but $F_{t+1} \geq 0$
  - As solved before, consumes all income $y_t = y_0\delta^t \Rightarrow c_t = y_0\delta^t$

$$
\begin{aligned}
U^{NB} &= \sum_{t=0}^{\infty} \beta^t \log(y_0\delta^t) && (41)\\
&= \sum_{t=0} \beta^t \left[\log(y_0) + t\log(\delta)\right] && (42)\\
&= \frac{\log(y_0)}{1-\beta} + \log(\delta)\sum_{t=0}^{\infty} t\beta^t && (43)\\
&= \frac{\log(y_0)}{1-\beta} + \log(\delta)\frac{\beta}{(1-\beta)^2} \neq U && (44)
\end{aligned}
$$

3 Dynamic Programming Approach

- Suppose $c_t = c_0\delta^t$ for $t \geq 0$. We want to evaluate:$V(c_0) = \sum_{t=0}^{\infty} \beta^t \log(c_t)$
- Note: $V(c_0) = \log(c_0) + \beta \sum_{j=0}^{\infty} \beta^j \log(c_{1+j}) = \log(c_0) + \beta V(c_1)$, where $c_1 = \delta c_0$, Markov!
- Bellman Equation: $V(c) = \log(c) + \beta V(\delta c)$. We want to find $V(c)$ function, then evaluate at c_0
- Process:
  - Guess $V(c) = k_0 + k_1 \log(c)$, where k_0, k_1 are undetermined coefficients

– Plug in:

$$k_0 + k_1 \log(c) = \log(c) + \beta \left[k_0 + k_1 \log(\delta c)\right] \tag{45}$$

$$= \log(c) + \beta k_0 + \beta k_1 \log(\delta) + \beta k_1 \log(c) \tag{46}$$

$$= \underbrace{[1 + \beta k_1] \log(c)}_{k_1} + \underbrace{[\beta k_0 + \beta k_1 \log(\delta)]}_{k_0}, \text{ by using undetermined coefficients} \tag{47}$$

$$k_1 = (1 + \beta k_1) \Rightarrow k_1 = \frac{1}{1-\beta} \tag{48}$$

$$k_0 = \beta k_0 + \beta k_1 \log(\delta) = \beta k_0 + \beta \frac{\log(\delta)}{1-\beta} \Rightarrow k_0 = \frac{\beta}{(1-\beta)^2} \log(\delta) \Rightarrow \tag{49}$$

$$V(c) = \frac{1}{1-\beta} \log(c) + \frac{\beta}{(1-\beta)^2} \log(\delta), \text{ which agrees with our earlier solution.} \tag{50}$$