Question 1

Calculate the following matrices and matrix-vector multiplication.[1]

(a) $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 5 \\ 1 & 3 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 7 & 1 \\ 0 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Question 2

Being formal and explicit in the rules of matrix algebra (e.g. when things are commutative, distributive, etc.) solve the following equations for $x \in \mathbb{R}^N$, with vector $b \in \mathbb{R}^N$, matrices A, D, Q, R all $\mathbb{R}^{N \times N}$, and scalar $m \in \mathbb{R}$,

(a) $Ax + \left(x^T D\right)^T = b$

(b) $Q^{-1} (A\,x\,m + mb) = Rx$

Question 3

Transform the following linear equations into a linear system with matrices/vectors

(a) $\begin{cases} 2x + 3y = 2 \\ x - 2y = -1 \end{cases}$ where $\{x, y\}$ are variables

(b) $\begin{cases} 2x - 3y = 1 \\ 3x + my = -2 \end{cases}$ where $\{x, y\}$ are variables

(c) $\begin{cases} 2a + b = 1 \\ 3b + 4c = 2 \\ -2a + c = 0 \end{cases}$ where $\{a, b, c\}$ are variables

(d) $\begin{cases} a + b = -3 \\ c - 2 - 4b = 0 \end{cases}$ where $\{a, b, c\}$ are variables (this will not be of full rank)

Question 4

[1] No need to hand this question in. This is just for your own practice.

(a) Find a linear transformation $G \in \mathbb{R}^2$ such that $G \cdot \begin{bmatrix} a & b \end{bmatrix}^T$ always returns the second element, b.

Question 5

Use undetermined coefficients example to solve the following functional and difference equations.[2]

(a) Take a simple linear ODE: $\boldsymbol{\partial} f(z) = f(z)$. Guess that $f(z) = C_1 e^z + C_2$ and use undetermined coefficients to solve for C_1 and C_2.

(b) Take the functional equation $[f(z)]^2 = z^2 + 2z + 1$. Guess that the solution is of the form $f(z) = C_1 z + C_2$. Use undetermined coefficients to find C_1 and C_2.

(c) Take the difference equation $z_{t+1} = g z_t$. Guess $z_t = C_1 C_2^t + C_3$. Show that C_1 is indeterminate and find C_2 and C_3. What if we add subject to $z_0 = A$? Show how this pins down C_1?.

Question 6

Let X and Y be random variables such that $X \in \{0, 1\}$ and $Y \in \{1, 2\}$. These are correlated such that[3]

$$\mathbb{P}(X = 0 \text{ and } Y = 1) = .1 \quad (1)$$
$$\mathbb{P}(X = 0 \text{ and } Y = 2) = .3 \quad (2)$$
$$\mathbb{P}(X = 1 \text{ and } Y = 1) = .4 \quad (3)$$
$$\mathbb{P}(X = 1 \text{ and } Y = 2) = .2 \quad (4)$$

(a) Calculate the (conditional) $\mathbb{P}(X = 0 \,|Y = 1)$ and $\mathbb{P}(X = 1 \,|Y = 1)$

(b) Calculate the (unconditional) $\mathbb{E}[X]$, $\mathbb{E}[Y]$, and $\mathbb{E}[XY]$

(c) Calculate the (conditional) $\mathbb{E}[X \mid Y = 1], \mathbb{E}[X \mid Y = 2]$, and $\mathbb{E}[XY \mid Y = 1]$

(d) Calculate $\mathbb{E}[X \mid Y = 1 \text{ or } Y = 2]$

Question 7

Consider a worker who may be employed or unemployed (E or U), and an economy that may be good or bad (G or B). Let X be the random variable of the worker's employment status and Y be the random variable of the aggregate economy. Now assume we know the following probabilities

- $\mathbb{P}(X = E \text{ and } Y = G) = 0.5 + \gamma$

[2] Remember the notation $\boldsymbol{\partial} f(z) \equiv \frac{df(z)}{dz}$. You don't need to know anything about differential equations to do this problem.

[3] Make sure to show the correct setup with numbers in the equations, but I don't need to see intermediate steps in the calculation after that.

- $\mathbb{P}(X = U \text{ and } Y = G) = 0.1$
- $\mathbb{P}(X = E \text{ and } Y = B) = 0.3 - \gamma$
- $\mathbb{P}(X = U \text{ and } Y = B) = 0.1$

for some parameter $|\gamma| < 0.3$.

(a) Find conditions on γ for statistical independence of the individuals unemployment and the economies state, and interpret.

Question 8

Solve the following optimization problems. Please be explicit in your transformation to our canonical form of constrained optimization, and be formal with Lagrange multipliers, first order necessary conditions, inequalities, etc.

(a)

$$\max_x \left\{-x^2 + 2x + 3\right\} \tag{5}$$
$$\text{s.t. } x \geq 0 \tag{6}$$

(b)

$$\min_x \{2x + 3\} \tag{7}$$
$$\text{s.t. } x \leq 1 \tag{8}$$