

# Growth Models

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## 1 Quick Review from General Equilibrium

- Homogeneous Functions/CRS (Inputs  $x, y$ )

– If a function  $F(x, y)$  has:

$$F(\gamma x, \gamma y) = \gamma \cdot F(x, y) \quad (1)$$

it is called constant returns to scale or homogeneous of degree 1.

- We can also get from CRS:

$$F(x, y) = \overbrace{\underbrace{x}_{\text{quantity } x} \cdot \underbrace{\partial_x F(x, y)}_{\substack{\text{marginal} \\ \text{product} \\ \text{of } x}}}^{\text{Also be the "competitive factor payments"}} + \underbrace{y}_{\text{quantity } y} \cdot \underbrace{\partial_y F(x, y)}_{\substack{\text{marginal} \\ \text{product} \\ \text{of } y}} \quad (2)$$

$$CRS \Rightarrow \begin{cases} \partial_x F(\gamma x, \gamma y) = \partial_x F(x, y) \\ \partial_y F(\gamma x, \gamma y) = \partial_y F(x, y) \end{cases} \text{ (homogeneous of degree 0)} \quad (3)$$

- Define  $f\left(\frac{x}{y}\right) \equiv F\left(\frac{x}{y}, 1\right) \Rightarrow F(x, y) = y f\left(\frac{x}{y}\right)$ , i.e. let  $\gamma = \frac{1}{y}$

– Then from (3)

$$\partial_x F = f'\left(\frac{x}{y}\right) \quad (4)$$

– And we can also show that

$$\partial_y F = f\left(\frac{x}{y}\right) - f'\left(\frac{x}{y}\right) \left(\frac{x}{y}\right) \quad (5)$$

We will use these to express everything in the capital-labor ratio with CRS.

## 2 Malthusian Growth Model

- Setup:

$$Y_t = zF(\underbrace{L_t}_{\text{Land}}, \underbrace{N_t}_{\text{Labor}}) \text{ where } F \text{ is CRS (e.g. food products)} \quad (6)$$

- Let  $L_t = L$  be fixed, i.e. can't create land.
- Let  $C_t = Y_t$ , consume all productions.
- Also assume all population,  $N_t$ , works.

- Population grows:

- Based on output (i.e. food) supply.

$$N_{t+1} = N_t G\left(\frac{C_t/N_t}{c^*}\right) \quad (7)$$

where  $G(\cdot)$  is growth function,  $c^*$  is a constant. And  $c_t \equiv C_t/N_t$  is consumption per capita.

- And growth factor:  $G(1) = 1, G' > 0, G'' < 0$
- Example Growth Factor:  $G(\frac{c_t}{c^*}) = \left(\frac{c_t}{c^*}\right)^\gamma$  for  $\gamma \in (0, 1)$

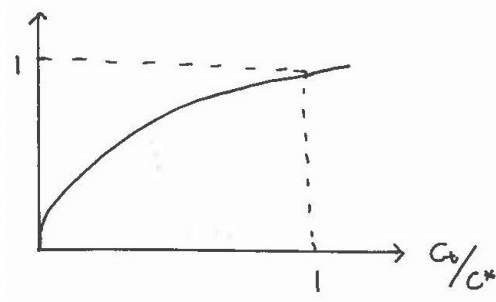


Figure 1: Malthusian Growth Model

- Interpretation

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma \quad (8)$$

So we have:

$$\begin{cases} c_t > c^* & \Rightarrow \frac{N_{t+1}}{N_t} > 1 \quad , \text{ population grows} \\ c_t < c^* & \Rightarrow \frac{N_{t+1}}{N_t} < 1 \quad , \text{ population shrinks} \end{cases} \quad (9)$$

so  $c^*$  is a subsistence level of consumption

- Production:

$$C_t = zF(L, N_t) \quad (10)$$

$$\Rightarrow C_t = N_t z F\left(\frac{L}{N_t}, 1\right) \text{ by CRS} \quad (11)$$

$$\Rightarrow c_t \equiv zF(\ell_t, 1) \text{ where } \ell_t \equiv \frac{L}{N_t} \text{ (land-per-capita)} \quad (12)$$

For example, assume

$$F(\ell_t, 1) = \ell_t^\alpha, \alpha \in (0, 1) \quad (13)$$

- Equations

$$c_t = z\ell_t^\alpha \quad (14)$$

$$\frac{N_{t+1}}{N_t} = \left(\frac{c_t}{c^*}\right)^\gamma \quad (15)$$

- Substitute:

$$\frac{N_{t+1}}{N_t} = \left(\frac{z(\frac{L}{N_t})^\alpha}{c^*}\right)^\gamma = \frac{z^\gamma (\frac{L}{N_t})^{\alpha\gamma}}{(c^*)^\gamma} \quad (16)$$

$$\Rightarrow \boxed{N_{t+1} = \left(\frac{z}{c^*}\right)^\gamma L^{\alpha\gamma} N_t^{1-\alpha\gamma}} \quad (17)$$

- Steady State:

$$\bar{N} = \left(\frac{z}{c^*}\right)^\gamma L^{\alpha\gamma} \bar{N}^{1-\alpha\gamma} \quad (18)$$

$$\Rightarrow 1 = \left(\frac{z}{c^*}\right)^\gamma \left(\frac{L}{\bar{N}}\right)^{\alpha\gamma} \quad (19)$$

$$\Rightarrow \bar{N} = \frac{L}{\left(\frac{c^*}{z}\right)^{\frac{1}{\alpha}}} \quad (20)$$

Substitute (20) into (14) to show that  $c_t = c^*$ . So only equilibrium is one of subsistence per-capita consumption for all.  $\uparrow z \Rightarrow \bar{N} \uparrow$  but  $c = c^*$ . Pessimistic comment on

technological change?

- Dynamics

- Take logs of (1):

$$\log N_{t+1} = \underbrace{\gamma (\log z - \log c^* + \alpha \log L)}_{\phi_0} + \underbrace{(1 - \alpha\gamma)}_{\phi_1} \log N_t \quad (21)$$

- Or of form:

$$n_{t+1} = \phi_0 + \phi_1 n_t \text{ if } n_t \equiv \log N_t \quad (22)$$

- Can use our old toolset of linear difference equations:

$$x_t = \begin{pmatrix} 1 \\ n_t \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ \phi_0 & \phi_1 \end{pmatrix}, \boxed{x_t = A^t x_0} \quad (23)$$

### 3 Solow Growth Model

- Setup:

- Let  $N_t$  be exogenous:  $N_{t+1} = (1 + g)N_t$ ,  $N_0$  given and  $g$  a constant population growth rate.
  - Other factor of production is capital, not land.

$$\underbrace{K_{t+1}}_{\substack{\text{Next} \\ \text{periods} \\ \text{capital}}} = \underbrace{(1 - \delta)}_{\substack{\text{depreciation} \\ \text{of capital}}} K_t + \underbrace{X_t}_{\substack{\text{investment} \\ \text{in new} \\ \text{capital}}}, \delta \in (0, 1) \quad (24)$$

- Production:

- We have:

$$Y_t = zF(K_t, N_t) \text{ with C.R.S} \quad (25)$$

which can be used for consumption or for capital.

- And:

$$C_t + X_t = zF(K_t, N_t) \quad (26)$$

which is the resource constraint for economy.

- Define the capital-labor ratio:

$$k_t \equiv \frac{K_t}{N_t} \quad (27)$$

- And investment per capital:

$$x_t \equiv \frac{X_t}{N_t} \quad (28)$$

- Consumption/capital:

$$c_t \equiv \frac{C_t}{N_t} \quad (29)$$

- Converting equations, divide (24) by  $N_t$ :

$$\frac{K_{t+1}}{N_t} = (1 - \delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}, \text{ or:} \quad (30)$$

$$\frac{N_{t+1}}{N_{t+1}} \frac{K_{t+1}}{N_t} = \left( \frac{N_{t+1}}{N_t} \right) \left( \frac{K_{t+1}}{N_{t+1}} \right) = (1 - \delta) \frac{K_t}{N_t} + \frac{X_t}{N_t}, \text{ where } N_{t+1} \text{ is the growth of population} \quad (31)$$

$$\Rightarrow \boxed{k_{t+1}(1 + g) = (1 - \delta)k_t + x_t} \quad (32)$$

- Next, assume (not based on optimization!):

- Agents save fraction  $s \in (0, 1)$  of output. i.e. by economy resource constraint.

$$C_t = (1 - s)zF(K_t, N_t) \quad (33)$$

$$\Rightarrow \text{dividing by } N_t, c_t = (1 - s)zF(k_t) \quad (34)$$

- Plug into equation (32):

$$\boxed{k_{t+1} = \frac{(1 - \delta)k_t}{1 + g} + \frac{sz}{1 + g}f(k_t)} \quad (35)$$

- This is a non-linear difference equation in  $k_t$

- Find a steady state solution:

$$\bar{k} = \frac{1-\delta}{1+g}\bar{k} + \frac{sz}{1+g}f(\bar{k}) \quad (36)$$

$$\Rightarrow \left( \frac{1+g}{1+g} - \frac{1-\delta}{1+g} \right) \bar{k} = \frac{sz}{1+g}f(\bar{k}) \quad (37)$$

$$\Rightarrow \boxed{(g+\delta)\bar{k} = szf(\bar{k})} \quad (38)$$

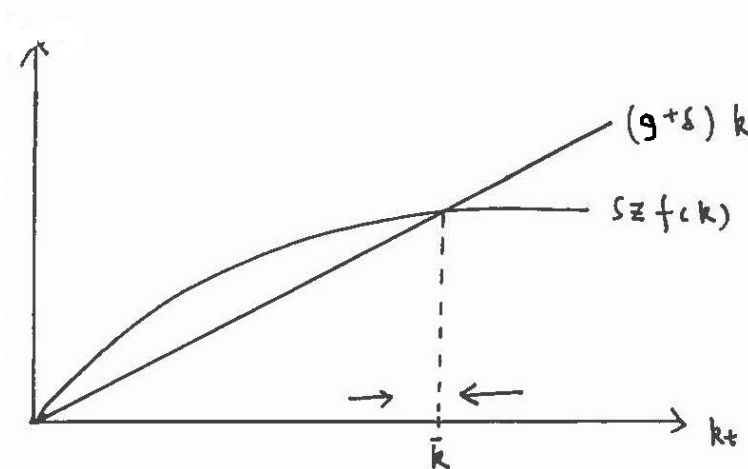


Figure 2: Convergence in a Solow Growth Model

- Use to understand effect on  $\bar{k}$  of:
  - increase in  $z$
  - increase in  $s$
  - increase in  $g$

## 4 Neoclassical growth

- Setup:
  - Solow took savings rate,  $s$ , as constant
  - Instead, make it choice of a representative consumer
  - First solve planner's problem, then competition equilibrium.
- Technology:
  - $Y = F(K, N)$  with CRS.

- Let  $k \equiv \frac{K}{N}$  then per capita like in Solow
- Let  $f(k) \equiv F(k, 1)$ , then  $Nf(k) = F(K, N)$  from CRS:

$$c_t + \underbrace{x_t}_{\equiv k_{t+1} - (1-\delta)k_t} = f(k_t) \quad (39)$$

or,

$$\underbrace{c_t + k_{t+1}}_{\substack{\text{consume of} \\ \text{capital} \\ \text{next period}}} = \underbrace{f(k_t)}_{\text{production}} + \underbrace{(1-\delta)k_t}_{\substack{\text{last period's capital,} \\ \text{depreciated}}} \quad \text{i.e., the gross output per capita} \quad (40)$$

$$\Leftrightarrow C_t + K_{t+1} = F(K_t, N_t) + (1-\delta)K_t \quad (41)$$

where  $c_t = \frac{C_t}{N}$ , replace  $K_{t+1}$  as  $k_{t+1}$ ,  $f(k) = \frac{1}{N}F(K, N)$

- Note: we are leaving out a possible irreversibility constraint on capital.
  - \* If you don't want the consumer to be able to “eat” their machines, then don't let capital contract more than the depreciation rate.

$$x_t \geq 0 \quad (42)$$

or,

$$K_{t+1} \geq (1-\delta)K_t \quad (43)$$

- Preferences:

$$u(c) \text{ (no disutility of labor)} \quad (44)$$

where  $u'(c) > 0$ ,  $u''(c) < 0$ ,  $u'(0) = \infty$

Welfare:

$$\sum_{t=0}^T \beta^t u(c_t), \text{ where } T \leq \infty \quad (45)$$

- Planning Problem:

$$\max_{c_t, k_{t+1}} \sum_{t=0}^T \beta^t u(c_t) \quad (46)$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t, \text{ where } k_0 \text{ was given} \quad (47)$$

$$\text{Transversality or } k_{T+1} \geq 0 \quad (48)$$

- Lagrangian (without transversality):

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (u(c_t) + \lambda_t (f(k_t) + (1 - \delta)k_t - c_t - k_{t+1})) \quad (49)$$

- FONC:

$$[c_t] : u'(c_t) - \lambda_t = 0 \quad (50)$$

$$[k_{t+1}] : -\lambda_t + \beta \lambda_{t+1} [f'(k_{t+1}) + (1 - \delta)] = 0 \text{ for } t = 0, \dots, T - 1 \quad (51)$$

$$[k_{T+1}] : -\beta^T \lambda_T \leq 0, = 0 \text{ if } k_{T+1} > 0 \quad (52)$$

$$\text{Or if } T = \infty, \lim_{T \rightarrow \infty} \beta^T \lambda_T k_{T+1} = 0 \quad (53)$$

- Summary of Equations:

- Equations in  $c_t, \lambda_t, k_t$  given  $k_0$ :

$$u'(c_t) = \lambda_t \quad (54)$$

$$\lambda_t = \beta \lambda_{t+1} [f'(k_{t+1}) + (1 - \delta)] \quad (55)$$

$$k_{t+1} + c_t = f(k_t) + (1 - \delta)k_t \quad (56)$$

- (54) shows the interpretation of the present-value Lagrange multiplier  $\lambda_t$  as the marginal utility of consumption on relaxing the resource constraint

- Assume  $T = \infty$ , look for steady state:

$$\lambda_t = \bar{\lambda} \quad (57)$$

$$c_t = \bar{c} \quad (58)$$

$$k_t = \bar{k} \quad (59)$$



– Also:

$$1 + \underbrace{\rho}_{\text{discount rate}} \equiv \frac{1}{\beta} = f'(\bar{k}) + (1 - \delta) \quad (60)$$

$$\text{Or: } \boxed{f'(\bar{k}) = \rho + \delta} \text{ (Modified golden rule!)} \quad (61)$$

– Note:  $\bar{k}$  is independent of preference except  $\beta$ . Then:

$$\bar{k} + \bar{c} = f(\bar{k}) + (1 - \delta)\bar{k} \Rightarrow \boxed{\bar{c} = f(\bar{k}) - \delta\bar{k}} \quad (62)$$

• Dynamics: Shooting Method:

– Finite horizon:  $k_{T+1} = 0$

– Equations:

$$u'(c_t) = \lambda_t \quad (63)$$

$$\lambda_t = \beta\lambda_{t+1} [f'(k_{t+1}) + (1 - \delta)] \quad (64)$$

$$k_{t+1} + c_t = f(k_t) + (1 - \delta)k_t \quad (65)$$

– Boundaries:  $k_0$  given,  $k_{T+1} = 0$  if  $T < \infty$ .

– Shooting method:

- \* Guess  $\lambda_0$
- \* Use (63) to get  $c_0$
- \* Use (65) to get  $k_1$  since  $k_0$  given
- \* Use (64) to get  $\lambda_1$
- \* Use (63) to get  $c_1$
- \* keep repeating until  $k_{T+1}$  calculated
- \* Check that  $k_{T+1} \approx 0$
- \* If not, change  $\lambda_0$  and repeat
- If  $k_{T+1} > 0$ , "eat more" by lowering  $\lambda_0$

– If infinite horizon, know that  $k_t \rightarrow k_\infty$ , where  $f'(k_\infty) = \rho + \delta$

– Can try to "shoot" at  $k_\infty = k_{T+1}$  for very large  $T$

- "Turnpike" Theorem For large  $T$ , will spend more of the time near  $k_\infty$

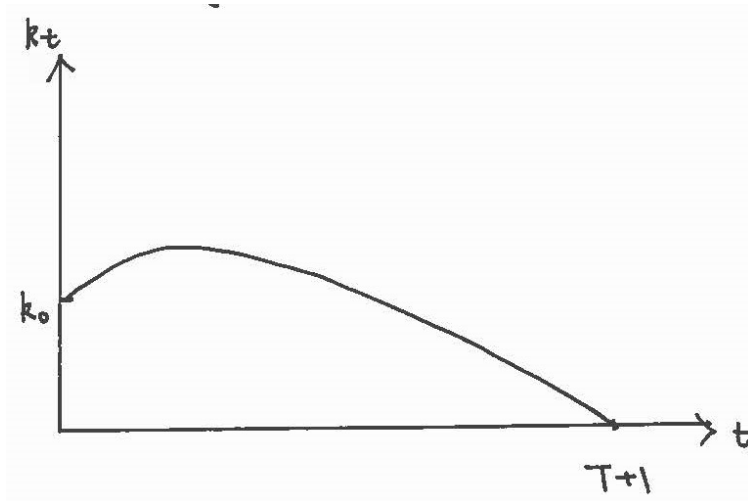


Figure 3: Shooting Method

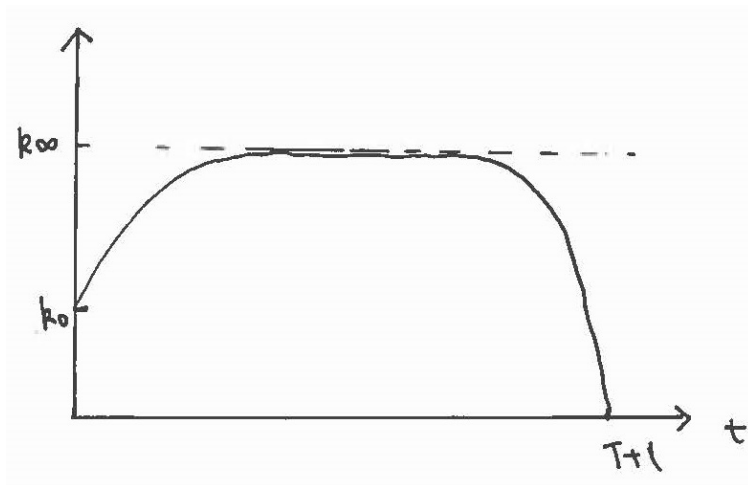


Figure 4: Turnpike Theorem, with large  $T$

## 5 Decentralization with a Competitive Equilibrium

### 5.1 Basic setup

- Take the same technology  $F(K_t, N_t)$
- Add in potential for endogenous labor supply  $N_t$
- Formulate with large number of households, firms.
- Price system:  $\{q_t, \tilde{w}_t, \tilde{r}_t\}_{t=0}^{\infty}$  at date 0 (i.e. time 0 budget), where  $q_t$  is the price of good,  $\tilde{w}_t$  is wage and  $\tilde{r}_t$  is the capital rental.

### 5.2 Households

- Owns labor, capital. Rent both to firms.
- Purchases goods for consumption or capital investment.
- Include disutility of labor in utility:  $u(c, n)$
- Solve problem:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) \quad (66)$$

$$\text{s.t. } \sum_{t=0}^{\infty} \underbrace{q_t}_{\substack{\text{Price of} \\ \text{homogeneous} \\ \text{good at } t \\ \text{in time 0}}} \left( c_t + \underbrace{k_{t+1} - (1 - \delta)k_t}_{\equiv x_t, \text{ net saving}} \right) \leq \sum_{t=0}^{\infty} \left( \underbrace{\tilde{w}_t}_{\substack{\text{wages} \\ \text{on labor} \\ \text{supplied}}} n_t + \underbrace{\tilde{r}_t}_{\substack{\text{rents} \\ \text{from} \\ \text{capital} \\ \text{supplied}}} k_t \right) \quad (67)$$

$$0 \leq n_t \quad (68)$$

$$n_t \leq 1 \quad 1 \text{ unit of time endowment} \quad (69)$$

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(c_t, n_t) + \lambda_t(1 - n_t) + \phi_t n_t] \quad (70)$$

$$+ \mu \left[ \sum_{t=0}^{\infty} (\tilde{w}_t n_t + \tilde{r}_t k_t) - \sum_{t=0}^{\infty} q_t (c_t + k_{t+1} - (1 - \delta)k_t) \right] \quad (71)$$

where  $\lambda_t$  is LM on  $n_t \leq 1$ ,  $\phi_t$  is LM on  $n_t \geq 0$  and  $\mu$  is LM on time 0 budget.

- FONC:

$$[c_t] : \beta^t \partial_c u(c_t, n_t) - \mu q_t = 0 \quad (72)$$

$$[n_t] : \beta^t (\partial_n u(c_t, n_t) + \phi_t - \lambda_t) + \mu \tilde{w}_t = 0 \quad (73)$$

$$[k_{t+1}] : r_{t+1} + q_{t+1}(1 - \delta) - q_t = 0 \quad (74)$$

$$\text{Complimentary slackness conditions: } \begin{cases} \lambda_t(1 - n_t) = 0 \\ \phi_t n_t = 0 \end{cases} \quad (75)$$

Note: (75) can also be written as:

$$-\beta^t \partial_n u(c_t, n_t) \begin{cases} = \mu \tilde{w}_t & \text{if } 0 < n_t < 1 \\ > \mu \tilde{w}_t & \text{if } n_t = 1 \\ < \mu \tilde{w}_t & \text{if } n_t = 0 \end{cases} \quad (76)$$

- Summarizing Equations for  $\{c_t, k_t, n_t\}$  in terms of  $\{q_t, \tilde{r}_t, \tilde{w}_t\}$ :

$$q_t = q_{t+1}(1 - \delta) + r_{t+1} \quad (77)$$

$$\mu q_t = \beta^t \partial_c u(c_t, n_t) \quad (78)$$

$$-\beta^t \partial_n u(c_t, n_t) \begin{cases} = \mu \tilde{w}_t & \text{if } 0 < n_t < 1 \\ > \mu \tilde{w}_t & \text{if } n_t = 1 \\ < \mu \tilde{w}_t & \text{if } n_t = 0 \end{cases} \quad (79)$$

Combining (78) and (79) with interior solutions gives us the same as our labor supply decision in the static general equilibrium,

$$\frac{-\partial_n u(c_t, n_t)}{\partial_c u(c_t, n_t)} = \frac{\tilde{w}_t}{q_t} \equiv w_t \quad (80)$$

For now, assume that the agent gains no utility from leisure, so:

$$u(c, n) = u(c) \Rightarrow \partial_n u = 0 \Rightarrow n_t = 1 \quad (81)$$

i.e. Inelastic supply of all labor

### 5.3 Firms (Price taker)

- Face prices :  $\{q_t, \tilde{w}_t, \tilde{r}_t\}$ , choose inputs for production.

- Solve Problem:

$$\max_{K_t, N_t} \sum_{t=0}^{\infty} \left[ \underbrace{q_t F(K_t, N_t)}_{\text{revenue}} - \underbrace{\tilde{r}_t K_t - \tilde{w}_t N_t}_{\text{input costs}} \right] \quad (82)$$

Use  $F(K, N) = \partial_K F(K, N)K + \partial_N F(K, N)N$  with CRS:

$$= \max_{K_t, N_t} \sum_{t=0}^{\infty} [(q_t \partial_K F(K_t, N_t) - \tilde{r}_t) K_t + (q_t \partial_N F(K_t, N_t) - \tilde{w}_t) N_t] \quad (83)$$

- Since CRS, we have a 0-profit condition as before. Necessary conditions:

$$q_t \partial_K F(K_t, N_t) = \tilde{r}_t \quad (84)$$

$$q_t \partial_N F(K_t, N_t) = \tilde{w}_t \quad (85)$$

Since CRS, use  $F\left(\frac{K}{N}, 1\right) = f(k)$  for  $k \equiv \frac{K}{N}$ , and other expressions from this sections:

$$\partial_K F(K, N) = \partial_k F\left(\frac{K}{N}, 1\right) = f'(k) \quad (86)$$

$$\partial_N F(K, N) = f(k) - k f'(k) \quad (87)$$

So:

$$\frac{\tilde{r}_t}{q_t} = f'(k_t) \quad (88)$$

$$\frac{\tilde{w}_t}{q_t} = f(k_t) - k_t f'(k_t) \quad (89)$$

So workers paid marginal products of inputs and firms of indeterminate size, but 0 profits.

## 5.4 Solve for Competitive Equilibrium

- Definition

A C.E. is a Price system  $\{q_t, \tilde{w}_t, \tilde{r}_t\}_{t=0}^{\infty}$  and a Feasible allocation  $\{k_{t+1}, c_t\}_{t=0}^{\infty}$ , such that:

- Given price system, allocations solve firm problem.
- Given price system, allocations solve Households problem.

- Solving for the equilibrium:

- Solve planning problems  $\{c_t, k_{t+1}\}$

- Guess  $\mu = 1$  (indeterminate), plug in with  $c_t$ :

$$q_t = \beta^t u'(c_t) \text{ (from consumer's problem)} \quad (90)$$

$$\tilde{r}_t = q_t f'(k_t) \text{ (from firm's problem)} \quad (91)$$

$$\tilde{w}_t = q_t (f(k_t) - k_t f'(k_t)) \text{ (from firm's problem)} \quad (92)$$

- Verify that this price system and allocations fulfills all of the conditions of the Firm and HH problems (Budget constraints, etc).

- Conclusion:

- For this economy, the planning problem and the competitive equilibrium have the same allocations.
- The prices decentralize the planning problem.