Question 1

There are two consumers ($i = 1, 2$) with potentially different consumption and income processes (c_t^i and y_t^i), initial financial wealth $F_0^i = 0$, and identical preferences subject to an intertemporal budget constraint,

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{1}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \beta^t c_t^i = \sum_{t=0}^{\infty} \beta^t y_t^i \tag{2}$$

where $u'(c) > 0, u''(c) < 0, \beta \in (0,1)$, and $\beta R = 1$. Assume that the two income processes are

$$y_t^1 = \{0, 1, 0, 1, \ldots\} \tag{3}$$

$$= \begin{cases} 0 & \text{if } t \text{ even} \\ 1 & \text{if } t \text{ odd} \end{cases} \tag{4}$$

$$y_t^2 = \{1, 0, 1, 0, \ldots\} \tag{5}$$

$$= \begin{cases} 1 & \text{if } t \text{ even} \\ 0 & \text{if } t \text{ odd} \end{cases} \tag{6}$$

(a) Apply the permanent income result to find c_t^i for both agents.[1]

(b) For every t, compare $c_t^1 + c_t^2$ vs. $y_t^1 + y_t^2$. Would this comparison change if $\beta R \neq 1$? (no need to solve for the exact c_t^i in that case)

(c) Assuming that both agents start with no financial wealth, i.e. $F_0^1 = F_0^2 = 0$, compute the asset trades between consumer 1 and 2 to support the c_t^i where the period-by-period budget constraint for $i = 1, 2$ is

$$F_{t+1}^i = R(F_t^i + y_t^i - c_t^i)$$

[1]Hints: Note that if $a_t = \{1, 0, 1, 0 \ldots\}$ then $\sum_{t=0}^{\infty} \beta^t a_t = 1 + \beta^2 + \beta^4 + \ldots = \sum_{t=0}^{\infty} \left(\beta^2\right)^t$.

Question 2

Let $y_t \in \mathbb{R}$ be a sequence of dividends such that

$$y_{t+1} = y_t + w_{t+1}$$

where $w_{t+1} \sim N(0, \sigma^2)$ for some $\sigma > 0$. i.e. $\mathbb{E}_t [w_{t+1}] = 0$ and $\mathbb{E}_t \left[w_{t+1}^2\right] = \sigma^2$. An agent prices an asset as the expected PDV of dividends,

$$p_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right]$$

Define the forecast error as the deviation in the expected price tomorrow calculated at time t from the actual price tomorrow. i.e.,

$$FE_{t+1|t} \equiv p_{t+1} - \mathbb{E}_t [p_{t+1}]$$

(a) Setup in our canonical Linear Gaussian State Space model.

(b) Solve for p_t in terms of y_t and model intrinsics.

(c) Find the <u>expected forecast error</u>: $\mathbb{E}_t \left[FE_{t+1|t}\right] = \mathbb{E}_t [p_{t+1} - \mathbb{E}_t [p_{t+1}]]$

(d) Find the <u>variance of forecast errors</u>:

$$\begin{aligned}
\mathbb{V}_t \left(FE_{t+1|t}\right) &\equiv \mathbb{E}_t \left[FE_{t+1|t}^2\right] - \left(\mathbb{E}_t \left[FE_{t+1|t}\right]\right)^2 \\
&= \mathbb{E}_t \left[(p_{t+1} - \mathbb{E}_t [p_{t+1}])^2\right] - (\mathbb{E}_t [p_{t+1} - \mathbb{E}_t [p_{t+1}]])^2
\end{aligned}$$

(e) Setup the problem recursively as p_t define in terms of p_{t+1}. Test, using your solution from early in the question, if the stochastic process p_t a martingale (i.e., just a random walk in this case)? Recall that a martingale is a process such that $\mathbb{E}_t [p_{t+1}] = p_t$. Give any intuition you can on this result.