

Permanent Income with No Borrowing, and Dynamic Programming

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1 Basic setup

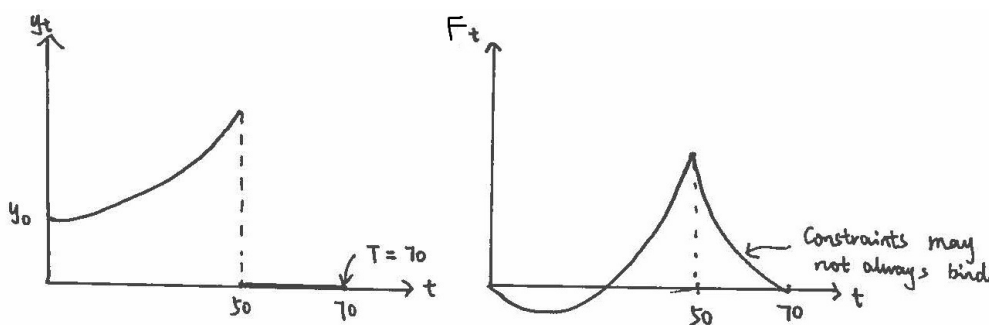


Figure 1: Income and Savings over Lifecycle with Borrowing

Recall: Previous example in Figure 1 with growing income and retirement. Assets may become negative early in lifecycle.

1.1 Add constraints on PIH

$$\max_{\{c_t, F_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad (1)$$

$$\text{s.t. } F_{t+1} = R(F_t + y_t - c_t) \quad (2)$$

$$\lim_{T \rightarrow \infty} (\beta^{T+1} F_{T+1}) \geq 0 \text{ (No ponzi)} \quad (3)$$

$$\left. \begin{array}{l} F_{t+1} \geq 0 \\ c_t \geq 0 \end{array} \right\} \text{ (Add constraints: no borrowing!)} \quad (4)$$

where F_0 is given, $\beta < 1, R > 1$; Assume $\lim_{c \rightarrow 0} u'(c) = \infty$ (Called an "Inada Condition").

1.2 Set up Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (u(c_t) + \lambda_t [R(F_t + y_t - c_t) - F_{t+1}] + \nu_{t+1} \cdot F_{t+1} + \alpha_t \cdot c_t) \quad (5)$$

where λ_t is LM on inter-temporal budget; ν_{t+1} is LM on $F_{t+1} \geq 0$; α_t is LM on the $c_t \geq 0$

- First Order Necessary Condition:

$$[c_t] : \beta^t u'(c_t) - \beta^t \lambda_t R + \beta^t \cdot \alpha_t = 0, \forall t \geq 0 \quad (6)$$

$$[F_{t+1}] : -\lambda_t + \nu_{t+1} + \beta \lambda_{t+1} R = 0, \forall t \geq 0 \quad (7)$$

- For constraints:

$$\nu_{t+1} \geq 0 \quad (8)$$

$$\alpha_t \geq 0 \quad (9)$$

$$\nu_{t+1} F_{t+1} = 0 \quad (10)$$

$$\alpha_t c_t = 0 \quad (11)$$

- Reorganize (6)

$$\Rightarrow u'(c_t) + \alpha_t = \lambda_t R, \text{ with } \alpha_t \geq 0 \quad (12)$$

$$\Rightarrow u'(c_t) \leq \lambda_t R, \text{ if } \alpha_t = 0 \quad (13)$$

From $\alpha_t = 0$ or $c_t = 0$, note if $c_t = 0$, use "Inada Condition" that $u'(0) = \infty$, which is a contradiction. So we have $\alpha_t = 0, c_t > 0$, implying:

$$\boxed{u'(c_t) = \lambda_t \cdot R} \quad (14)$$

- Reorganize (7), multiply by R

$$\beta R R \lambda_{t+1} - R \lambda_t + R \nu_{t+1} = 0 \quad (15)$$

Use (14)

$$\beta R u'(c_{t+1}) + R \nu_{t+1} = u'(c_t) \quad (16)$$

Then use complementarity, since $\nu_{t+1} \geq 0$

$$\beta R u'(c_{t+1}) \leq u'(c_t), = \text{if } F_{t+1} > 0, \forall t \geq 0 \quad (17)$$

- Note that if $F_{t+1} = 0$ and $\nu_{t+1} > 0$, then from budget constraint:

$$0 = R(F_t + y_t - c_t) \quad (18)$$

$$\Rightarrow c_t = F_t + y_t \text{ (eats all income and savings)} \quad (19)$$

- Summarizing results: under no-borrowing constraints:

$$u'(c_t) = \beta R u'(c_{t+1}); \text{ or} \quad (20)$$

$$u'(c_t) > \beta R u'(c_{t+1}) \text{ and } c_t = F_t + y_t \quad (21)$$

1.3 Example on preferences

- We assume $u(c) = \log(c) \Rightarrow u'(c) = \frac{1}{c}$, $\beta R = 1$

Then:

$$\frac{1}{c_t} = \frac{1}{c_{t+1}} \Rightarrow c_{t+1} = c_t, \text{ or} \quad (22)$$

$$\frac{1}{c_t} > \frac{1}{c_{t+1}} \Rightarrow c_{t+1} > c_t \text{ and } c_t = F_t + y_t \quad (23)$$

- Example 1:

- Let $y_{t+1} = \delta y_t \Rightarrow y_t = \delta^t y_0$ s.t. $\delta > 1$ and $F_0 = 0$
- Solution: (Guess always constrained, then verify:)

$$c_t = y_t, \forall t \quad (24)$$

$$F_t = 0, \forall t \quad (25)$$

So the person is always borrowing constrained.

Verify: $y_t > y_{t-1} \Rightarrow c_t > c_{t-1}, \forall t; F_t = 0 \Rightarrow c_t = y_t$

- Example 2:

- Let $0 < \delta < 1, y_t = \delta^t y_0, F_0 = 0$
- Solution: (Guess unconstrained and then verify)

Guess $c_t = \bar{c}$, such that

$$\bar{c} = (1 - \beta) y_0 \underbrace{\sum_{t=0}^{\infty} \delta^t \beta^t}_{\text{unconstrained formula if } F_0 = 0} \Rightarrow \quad (26)$$

$$\boxed{\bar{c} = (1 - \beta) \frac{y_0}{1 - \beta \delta}} \quad (27)$$

$$F_{t+1} = R(F_t + y_t - \bar{c}) \quad (28)$$

– Note: $c_0 = \frac{1-\beta}{1-\beta\delta} y_0 < \frac{1-\beta}{1-\beta} y_0 = y_0 \Rightarrow y_0 - c_0 > 0$, saves, not borrows

In the limit as $t \rightarrow \infty$, $y_t \rightarrow 0$ but $c_t = \bar{c}$. Put into budget to look for a steady state:

$$\bar{A} = R(\bar{A} + 0 - \bar{c}) \text{ if } F_t \approx F_{t+1} \text{ for large } t \quad (29)$$

$$\Rightarrow R\bar{c} = (R - 1)\bar{A}, \text{ by } R = 1 + r \quad (30)$$

$$\bar{c} = \frac{R - 1}{R} \bar{A} = \underbrace{\frac{r}{1 + r}}_{\text{annuity value}} \bar{A} \quad (31)$$

– So lives off annuity value of savings eventually (Also $\frac{r}{1+r} = 1 - \beta$ if $\beta \equiv \frac{1}{1+r}$).

E.g. Can buy an annuity for a stream of income, they buy it and pay you a set amount forever (or until death).

See Figure 2 for example of decreasing income (not to scale).

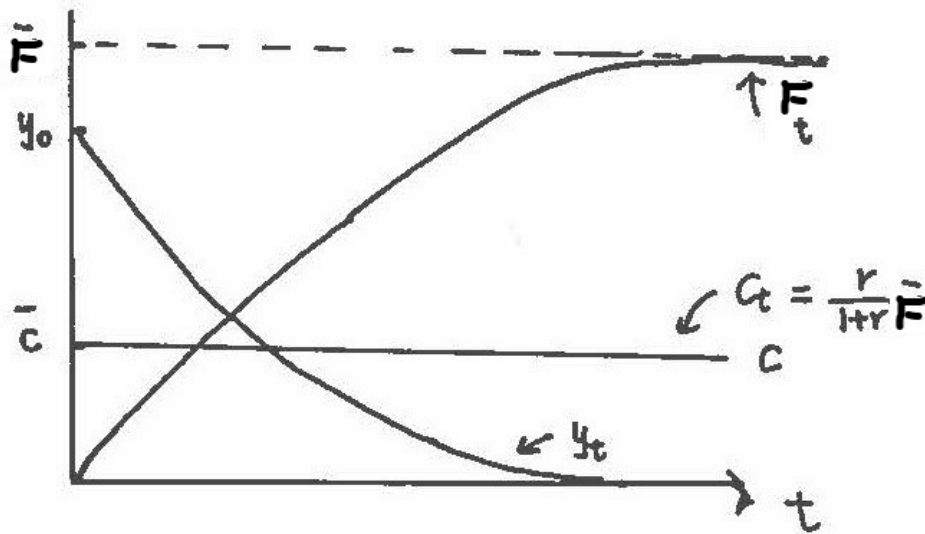


Figure 2: Asymptotic Behavior with Decreasing Income

2 Welfare cost of no-borrowing

- Setup:

- Given a feasible $\{c_t\}$, the lifetime utility of an agent $U = \sum_{t=0}^{\infty} \beta^t u(c_t)$. This is their welfare, their objective function.
- In general, adding constraints to the set of feasible $\{c_t\}$ weakly decreases welfare, as in Figure 3.

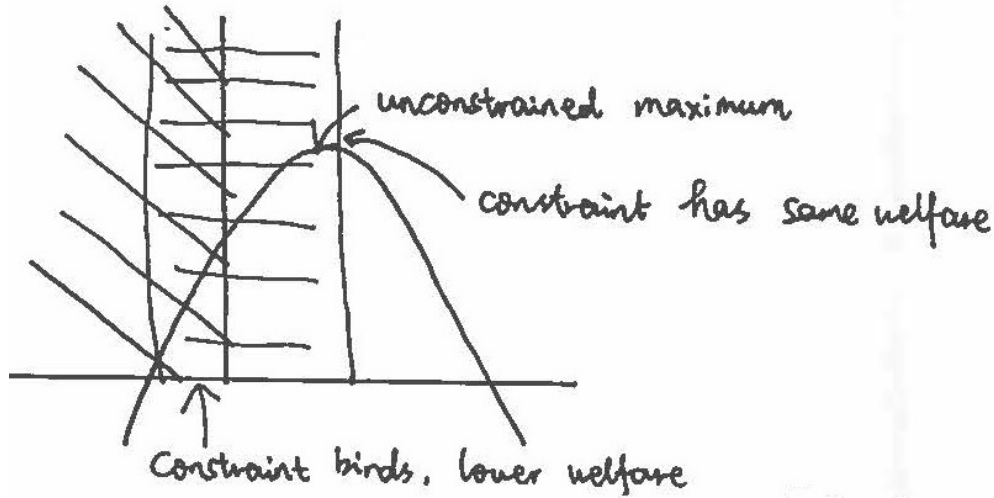


Figure 3: Constrained and First-Best

- Unconstrained $T = \infty$
- Assume $1 \leq \beta R \leq \delta$, $u(c) = \log(c)$, $F_0 = 0$, subject to $\lim_{T \rightarrow \infty} F_{T+1} \beta^T$, no ponzi scheme.

- Solution:

- Euler: $u'(c_t) = \beta R u'(c_{t+1}) \Rightarrow c_t = \beta \cdot R c_{t+1} \Rightarrow c_t = (\beta R)^t c_0$
- Lifetime budget:

$$0 = \sum_{j=0}^{\infty} R^{-j} (c_{t+j} - y_{t+j}) \quad (32)$$

$$\Rightarrow \sum_{t=0}^{\infty} R^{-t} \beta^t R^t c_0 = \sum_{t=0}^{\infty} R^{-t} \delta^t \cdot y_0 \quad (33)$$

$$\Rightarrow \underbrace{\frac{c_0}{1 - \beta}}_{\text{PV of consumption}} = \underbrace{\frac{y_0}{1 - \frac{\delta}{R}}}_{\text{PV of income}} \quad (34)$$

$$\Rightarrow c_0 = (1 - \beta) \frac{y_0}{1 - \frac{\delta}{R}} \quad (35)$$

- Consumer's Lifetime Utility:

$$U = \sum_{t=0}^{\infty} \beta^t \log(c_t) \quad (36)$$

$$= \sum_{t=0}^{\infty} \beta^t \log \left((\beta R)^t \cdot \frac{1-\beta}{1-\frac{\delta}{R}} y_0 \right) \quad (37)$$

$$= \sum_{t=0}^{\infty} \beta^t \left[\log(y_0) + \log\left(\frac{1-\beta}{1-\frac{\delta}{R}}\right) + t \log(\beta R) \right] \quad (38)$$

$$= \frac{\log y_0 + \log(1-\beta) - \log(1-\frac{\delta}{R})}{1-\beta} + \log(\beta R) \underbrace{\sum_{t=0}^{\infty} t \beta^t}_{\text{Recall sum from previous Markov waiting time}} \quad (39)$$

$$\boxed{U = \frac{1}{1-\beta} \left[\log(y_0) + \log(1-\beta) - \log(1-\frac{\delta}{R}) \right] + \log(\beta R) \left[\frac{\beta}{(1-\beta)^2} \right]} \quad (40)$$

- No borrowing, $T = \infty$
 - Same assumptions but $F_{t+1} \geq 0$
 - As solved before, consumes all income $y_t = y_0 \delta^t \Rightarrow c_t = y_0 \delta^t$

$$U^{NB} = \sum_{t=0}^{\infty} \beta^t \log(y_0 \delta^t) \quad (41)$$

$$= \sum_{t=0}^{\infty} \beta^t [\log(y_0) + t \log(\delta)] \quad (42)$$

$$= \frac{\log(y_0)}{1-\beta} + \log(\delta) \sum_{t=0}^{\infty} t \beta^t \quad (43)$$

$$= \frac{\log(y_0)}{1-\beta} + \log(\delta) \frac{\beta}{(1-\beta)^2} \neq U \quad (44)$$

3 Dynamic Programming Approach

- Suppose $c_t = c_0 \delta^t$ for $t \geq 0$. We want to evaluate: $V(c_0) = \sum_{t=0}^{\infty} \beta^t \log(c_t)$
- Note: $V(c_0) = \log(c_0) + \beta \sum_{j=0}^{\infty} \beta^j \log(c_{1+j}) = \log(c_0) + \beta V(c_1)$, where $c_1 = \delta c_0$, Markov!
- Bellman Equation: $V(c) = \log(c) + \beta V(\delta c)$. We want to find $V(c)$ function, then evaluate at c_0
- Process:
 - Guess $V(c) = k_0 + k_1 \log(c)$, where k_0, k_1 are undetermined coefficients

– Plug in:

$$k_0 + k_1 \log(c) = \log(c) + \beta [k_0 + k_1 \log(\delta c)] \quad (45)$$

$$= \log(c) + \beta k_0 + \beta k_1 \log(\delta) + \beta k_1 \log(c) \quad (46)$$

$$= \underbrace{[1 + \beta k_1] \log(c)}_{k_1} + \underbrace{[\beta k_0 + \beta k_1 \log(\delta)]}_{k_0}, \text{ by using undetermined coefficients} \quad (47)$$

$$k_1 = (1 + \beta k_1) \Rightarrow k_1 = \frac{1}{1 - \beta} \quad (48)$$

$$k_0 = \beta k_0 + \beta k_1 \log(\delta) = \beta k_0 + \beta \frac{\log(\delta)}{1 - \beta} \Rightarrow k_0 = \frac{\beta}{(1 - \beta)^2} \log(\delta) \Rightarrow \quad (49)$$

$$V(c) = \frac{1}{1 - \beta} \log(c) + \frac{\beta}{(1 - \beta)^2} \log(\delta), \text{ which agrees with our earlier solution.} \quad (50)$$