

Linear Difference Equations and Asset Pricing

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1 Solutions (and Uniqueness) of Difference Equations

From the previous lecture notes, pricing a sequence $\{y_{t+j}\}$ of payoffs:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j}, \text{ at time } t. \text{ (Sequential formulation)} \quad (1)$$

Can be written,

$$P_t = y_t + \beta P_{t+1} \quad (2)$$

1.1 Solving with Guess and Verify

How can we solve a difference equation?

Example: $y_t = \bar{y}$

$$P_t = \bar{y} + \beta P_{t+1} \quad (3)$$

A Guess: $P_t = \bar{P}$, independent of t . Plug in equation (3):

$$\bar{P} = \bar{y} + \beta \bar{P} \quad (4)$$

$$\Rightarrow \bar{P} = \frac{\bar{y}}{1 - \beta}, \text{ consistent with } P = \sum_{t=0}^{\infty} \beta^t y_t \quad (5)$$

Role of $|\beta| < 1$:

- Keep from “exploding”: stability
- Will have equivalent condition for more complicated difference equations

1.2 Rational Bubbles

Let $y_t = \bar{y}$ for all t .

Fundamental value:

$$P_t = \sum_{j=0}^{\infty} \beta^j \bar{y} \quad (6)$$

$$= \frac{\bar{y}}{1 - \beta}, \text{ (unique)} \quad (7)$$

Remember that this solves the recursive problems as well:

$$\frac{\bar{y}}{1 - \beta} = \bar{y} + \beta \left(\frac{\bar{y}}{1 - \beta} \right) \Rightarrow \text{true!} \quad (8)$$

Is $P_t = \frac{\bar{y}}{1 - \beta}$ the unique solution to $P_t = \bar{y} + \beta P_{t+1}$? **No!** Like the undetermined coefficient in differential equations.

Example:

$$P_t = \underbrace{\frac{\bar{y}}{1 - \beta}}_{\text{fundamental value}} + \underbrace{c\beta^{-t}}_{\text{bubble term}} \text{ for any } c \quad (9)$$

Check: $P_t = \bar{y} + \beta P_{t+1}$

$$\frac{\bar{y}}{1 - \beta} + c\beta^{-t} = \bar{y} + \beta \left[\frac{\bar{y}}{1 - \beta} + c\beta^{-(t+1)} \right] \quad (10)$$

$$= \bar{y} + \left(\frac{\beta}{1 - \beta} \right) \bar{y} + c\beta^{-t} \quad (11)$$

$$= \frac{\bar{y}}{1 - \beta} + c\beta^{-t} \quad (12)$$

So it fulfills the difference equation for any c, t , etc. *Rational* as every agent in the economy would agree on the price, no-one needs to be tricked or making a pricing mistake, and there is no arbitrage. An example of a self-fulfilling equilibrium.

1.2.1 Size of the “Rational Bubble”

$$\underbrace{P_0 - P_{fund}}_{\text{difference from fundamental}} = \frac{\bar{y}}{1 - \beta} - \frac{\bar{y}}{1 - \beta} + c\beta^0 = c \quad (13)$$

Expectations:

- Prices rise because they are expected to rise.
- Self fulfilling. Will depend on coordination of expectations.
- Is Fiat money a bubble?

2 Extending our Asset Pricing Model

We will generalize our results to include systems of equations, with dynamics.

2.1 Recall: Properties

- Dividend stream y_t
- Discount factor β
- Present discounted value = price :
 $P = \sum_{t=0}^{\infty} \beta^t y_t$, and if $y_t = \bar{y}$, $P = \bar{y}(1 - \beta)^{-1}$
- How to model the evolution of y_t ?
 - Will use systems of linear difference equations in an underlying state x_t
- Example: dividends are a linear function of evolving aggregate and idiosyncratic variables

Recall: Recursive Formulation $P_t = y_t + \beta P_{t+1}$

2.2 Applying to Dynamics

- Let x_t be a n dimensional vector of states.
- Let A, G be matrices.
- Stack first order difference equations, giving another *canonical form*:

$$x_{t+1} = A \cdot x_t, \quad (A \text{ is } n \times n \text{ matrix, } x \text{ is } n \times 1 \text{ vector}) \quad (14)$$

$$y_t = G \cdot x_t, \quad (G \text{ is } 1 \times n \text{ vector, } y_t \text{ is a scalar, i.e. } 1 \times 1) \quad (15)$$

- ‘ A ’ gives evolution of the state, given x_0
- ‘ G ’ gives observation of the state
“Finding the state is an art”

Example:

- Asset payoff follows difference equation (not first order!):

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1} \quad (16)$$

- What is the value of this asset at time t ?

State

Guess: $x_t \equiv \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}$, 2×1 vector.

What is the difference equation for x_t ?

$$\underbrace{\begin{bmatrix} y_{t+1} \\ y_t \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t} \quad (17)$$

and observation:

$$y_t = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_G \underbrace{\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}}_{x_t} \quad (18)$$

Therefore, the set of difference equations in our *canonical form* are:

$$x_{t+1} = Ax_t \quad (19)$$

$$y_t = Gx_t \quad (20)$$

Price is:

$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} \quad (21)$$

$$= \sum_{j=0}^{\infty} \beta^j G \cdot x_{t+j} \quad (22)$$

If $x_{t+1} = A \cdot x_t$, then $x_{t+2} = A \cdot (Ax_t) = A^2 x_t$, and $x_{t+j} = A^j x_t$

$$\Rightarrow P_t = \sum_{j=0}^{\infty} \beta^j G \cdot A^j \cdot x_t \quad (23)$$

$$= G \cdot \left[\sum_{j=0}^{\infty} (\beta A)^j \right] x_t \quad (24)$$

Remember that if λ is scalar: $\sum_{j=0}^{\infty} (\beta\lambda)^j = (1 - \beta\lambda)^{-1} = \frac{1}{1-\beta\lambda}$.

With matrices and inverses, this is similar: $\sum_{j=0}^{\infty} \beta^j A^j = (I - \beta A)^{-1}$,

where the matrices' dimensions are: $A : n \times n$, $I = n \times n$ identity, $(I - \beta A)^{-1} : n \times n$

$$\boxed{P_t = G(I - \beta A)^{-1} x_t} \quad (* : \text{very important memorize}) \quad (25)$$

- Asset pricing formula for first-order linear difference equations.

- Summary of sizes:

- $P_t : 1 \times 1$ scalar
- $G : 1 \times n$ vector
- $A : n \times n$ matrix
- $I : n \times n$ identity matrix
- $\beta : 1 \times 1$ scalar
- $x_t : n \times 1$ state vector

2.3 Stability

- Recall in the example with $x_t = \lambda^t$ that $|\beta\lambda| < 1$ for the series to converge.

- For matrix equations, need a similar condition where eigenvalues of βA are all < 1 , or $\max |\text{eig}(A)| < \frac{1}{\beta}$

- Can use software to check the eigenvalues.

A Connection to Differential Equations

Difference equations are just differential equations in discrete time.

- Let $y(t)$ be the flow dividends, a function of t .
- Let r be the instantaneous interest rate.
- Let the length of a period be Δ , and take the limit as it goes to 0.
- Dividends over Δ period $\approx \Delta y(t) \equiv y_t(\Delta)$
- Discounting over Δ period $\approx 1 - \Delta r \equiv \beta(\Delta)$

The difference equation is: $P_t = y_t + \beta P_{t+1}$.

Using the above \Rightarrow Let function $p(t)$ be the price of asset:

$$p(t) = \Delta \cdot y(t) + (1 - \Delta r) \cdot p(t + \Delta) \quad (\text{A.1})$$

Rearrange:

$$\Delta r \cdot p(t + \Delta) = \Delta \cdot y(t) + p(t + \Delta) - p(t) \quad (\text{A.2})$$

$$\Rightarrow rp(t + \Delta) = y(t) + \frac{p(t + \Delta) - p(t)}{\Delta} \quad (\text{A.3})$$

Take limit as $\Delta \rightarrow 0$, i.e. discrete \rightarrow continuous t

$$\partial p(t) = \frac{p(t + \Delta) - p(t)}{\Delta}, \text{ definition of a derivative} \quad (\text{A.4})$$

where $\partial p(t) = \frac{d}{dt}p(t)$

$$\Rightarrow \underbrace{rp(t)}_{\substack{\text{opportunity cost} \\ \text{of buying a unit} \\ \text{of the asset}}} = \underbrace{y(t)}_{\substack{\text{flow} \\ \text{dividends}}} + \underbrace{\partial p(t)}_{\substack{\text{capital} \\ \text{gains}}} \quad (\text{A.5})$$

- Consider this pricing equation and arbitrage:

What if $rp(t) < y(t) + \partial p(t)$ instead of being an equation?

B Popping Bubbles

- In our discrete time model, keep $y_t = \bar{y}$ deterministic for simplicity:

- Let the bubble term have a chance of popping each period.
- Therefore, prices are a random variable.
- Linear asset pricing if random:

$$P_t = y_t + \beta \mathbb{E}_t [P_{t+1}] \quad (\text{Expected value of } P_{t+1} \text{ given information at } t) \quad (\text{B.1})$$

B.1 Bubble Evolution

$$\text{Let } C_{t+1} = \begin{cases} \frac{1}{\lambda} C_t & \text{with prob. } \lambda \in (0, 1) \\ 0 & \text{with prob. } 1 - \lambda \end{cases} \quad (\text{B.2})$$

i.e., C_t multiplied by $\frac{1}{\lambda}$ each time until bubble breaks. Then $C_t = 0 \forall t$

Note:

$$\mathbb{E}_t [C_{t+1}] = \lambda \left(\frac{1}{\lambda} C_t \right) + (1 - \lambda) \cdot 0 = C_t \quad (\text{B.3})$$

If $\mathbb{E}_t [y_{t+1}] = y_t$, then this term is called a *martingale*.

B.2 Price Level

We can check that for any C_0 :

$$P_t = \begin{cases} \frac{\bar{y}}{1-\beta} + (\beta\lambda)^{-t} \cdot C_0 & \text{if bubble hasn't popped} \\ \frac{\bar{y}}{1-\beta} & \text{after bubble pops} \end{cases} \quad (\text{B.4})$$

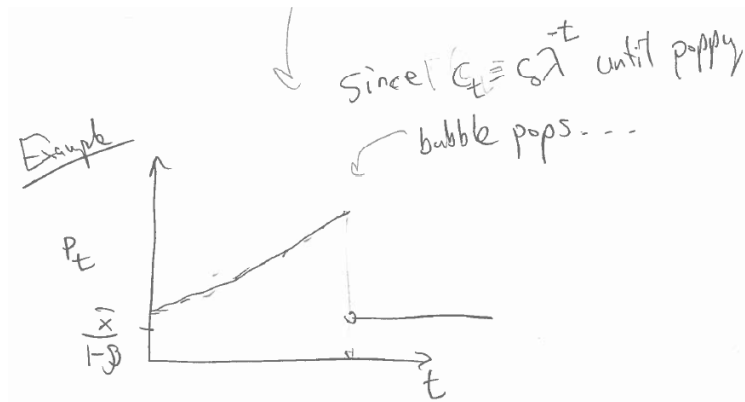


Figure 1: Graphical representation of the price level when the bubble pops