Doubling Down on Debt: Limited Liability as a Financial Friction
Online Appendix

Jesse Perla* Carolin Pflueger† Michal Szkup*

Draft Date: December 20, 2021

This appendix includes material omitted in the main paper and its appendix. In Appendix A we provide proofs of Main Propositions 4 to 6 together with a number of preliminary claims needed to establish these results. In addition, we discuss why equity holders' choices lead to a decrease the leverage when the initial leverage is already high. Finally, we also discuss briefly what happens if equity holders could separately sell coupons and bankruptcy claims. In Appendix B we provide more detailed data description, include additional empirical results, and explain our calibration strategy.

Appendix A Proofs for Main Section 3

In this section, we provide proofs of propositions stated in Main Section 3 (Main Propositions 4 to 6). We begin by establishing a number of useful preliminary results and by discussing feasibility of equity holders' choice of $\{g, \hat{\ell}, m, \psi\}$ (Appendix A.1). In Appendices A.2 to A.4 we then provide proofs of Main Propositions 4 to 6, respectively.

A.1 Preliminary Results

We establish first a number of preliminary results that we will use to prove Propositions 4 and 5. Throughout this section we assume that $\theta = 0$. Readers primarily interested in main results may choose to skip this section.

*University of British Columbia, Vancouver School of Economics; and †University of Chicago, Harris School of Public Policy, NBER, and CEPR

Lemma 1. *Define* $\bar{\ell}$ *as the unique solution to*

$$1 = \chi \bar{\ell}^{\eta}, \tag{A.1}$$

Then $p(\bar{\ell})\bar{\ell} = \frac{1}{r-\mu}$.

Proof. Plugging $\bar{\ell}$ into the expression for $p(\ell)$ (see Main (15)) we obtain $p(\bar{\ell}) = \frac{\eta}{1+\eta}$. From the definitions of $\bar{\ell}$ and χ ((A.1) and Main (9), respectively) we obtain

$$\bar{\ell} = \frac{1+\eta}{\eta}\frac{1}{r-\mu} \tag{A.2}$$

Combining these observations we obtain $p(\bar{\ell})\bar{\ell} = \frac{1}{r-\mu}$. □

In the proof of Lemma 1 we derived expressions for $\bar{\ell}$ and $p(\bar{\ell})$. Since we make use of these expressions repeatedly, we report them in the following Corollary.

Corollary 2. *We have* $\bar{\ell} = \frac{1}{r-\mu}\frac{1+\eta}{\eta}$ *and* $p(\bar{\ell}) = \frac{\eta}{1+\eta}$

Next, we show the constraint $p(\ell) \geq p^B(\ell)$ is satisfied if and only if $\ell \in [0, \bar{\ell}]$.

Corollary 3. *We have* $p(\ell) \geq p^B(\ell)$ *if and only if* $\ell \in [0, \bar{\ell}]$.

Proof. From the definitions of $p(\ell)$ and $p^B(\ell)$ (see Main Proposition 2) we have $p(\ell) - p^B(\ell) = 1 - \chi\ell^{\eta}$. Since $\chi > 0$ and $\eta > 0$ it follows that $p(\ell) - p^B(\ell)$ is strictly decreasing in ℓ. Moreover, from the definition of $\bar{\ell}$ we see that $p(\bar{\ell}) - p^B(\bar{\ell}) = 0$. This establishes the claim. □

Having characterized the highest feasible leverage, $\bar{\ell}$, we now establish two useful results regarding the value of outstanding debt.

Lemma 4. *For all* $\hat{\ell} < \bar{\ell}$ *we have*

$$\frac{\partial}{\partial \hat{\ell}}\left[p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right)\right] = p'(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) + p(\hat{\ell})(1+g) > 0 \tag{A.3}$$

Proof. We have

$$\frac{\partial}{\partial \hat{\ell}}\left[p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right)\right] = \left[1 - \chi\hat{\ell}^{\eta}\right](1+g) + \frac{\eta\chi}{1+\eta}\hat{\ell}^{\eta-1}\ell, \tag{A.4}$$

where we used the definition of $p(\ell)$ (see Main (15) with $\theta = 0$). The claim follows from the observation that, since $\hat{\ell} < \bar{\ell}$, we have $\left[1 - \chi\hat{\ell}^{\eta}\right] > 0$. □

Lemma 5. *The value of outstanding debt,* $p(\ell)\ell$*, is strictly increasing in* ℓ *for all* $\ell \in [0, \bar{\ell})$.

Proof. We have

$$\frac{\partial}{\partial \ell} p(\ell)\ell = p'(\ell)\ell + p(\ell) = 1 - \chi \ell^{\eta} > 0, \tag{A.5}$$

where the last inequality follows since $\ell < \bar{\ell}$. □

Finally, we discuss which equity holders' choices of $g, \psi, m,$ and $\hat{\ell}$ are feasible (i.e., satisfy equity holders' budget constraint). Note that the equity holders' choices of $g, \psi, m,$ and $\hat{\ell}$ have to jointly satisfy the equity holders' budget constraint

$$p(\hat{\ell})((1+g)\hat{\ell} - \ell) = \psi q(g) + m \tag{A.6}$$

The budget constraint implies that once the equity holders make choices of $g, \psi,$ and m the post-investment leverage $\hat{\ell}$ is determined implicitly by (A.6). Thus, we can treat the post-investment leverage as an implicit function of $g, \psi,$ and m, and denote it by $\hat{\ell}(g, \psi, m)$. This leads to the following definition of feasibility of equity holders choices.

Definition 1. *The equity holders choices of g, ψ, m are feasible if $\hat{\ell}(g, \psi, m) \leq \bar{\ell}$*

We now derive a feasibility constraint on issuance of m.

Lemma 6. *Fix g and ψ such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$. Then m is feasible if $m \in [0, \bar{m}(g, \psi)]$, where*

$$\bar{m}(g, \psi) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g) \tag{A.7}$$

Moreover, at $m = \bar{m}(g, \psi)$ we have $\hat{\ell}(g, \psi, m) = \bar{\ell}$.

Proof. Since ψ and g are such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$ then, given choices of g and ψ there exists $m > 0$ that satisfies

$$p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) + m \tag{A.8}$$

By applying the implicit function theorem to the above equation we see that

$$\frac{\partial \hat{\ell}}{\partial m} = \frac{1}{p'(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) + p(\hat{\ell})(1+g)} > 0, \tag{A.9}$$

where the inequality follows from Lemma 4. It follows that at the highest feasible m, which we denote by $\bar{m}(g,\psi)$, we have $\hat{\ell} = \bar{\ell}$. Setting $\hat{\ell} = \bar{\ell}$ in (A.8) and rearranging, we obtain

$$\bar{m}(g,\psi) = \frac{1+g}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi q(g)$$

$\square$

With the above results in hand, we now prove Main Propositions 4 and 5.

A.2 Proof of Main Proposition 4 ($\kappa = 0$)

Proof of Main Proposition 4. **(Part 1)** Equity financing implies that $\psi = 0$. Therefore, the post-investment leverage $\hat{\ell}$ is given by $\hat{\ell} = \frac{\ell}{1+g}$ and equity holders' problem simplifies to

$$\max_{g\geq 0} \frac{1+g}{r-\mu} - p\left(\frac{\ell}{1+g}\right)\ell - q(g)$$

The first-order condition associated with the above problem is given by

$$\frac{1}{r-\mu} + p'\left(\frac{\ell}{1+g}\right)\left(\frac{\ell}{1+g}\right)^2 - q'(g) = 0 \tag{A.10}$$

Recall that g^u denotes the first-best investment (see Main Definition 1) and let g_e^* denote equity holders' optimal investment when they finance their investment only with equity. Then, Main (27) and (A.10) imply that

$$\frac{1}{r-\mu} - q'(g^u) = 0 = \frac{1}{r-\mu} + p'\left(\frac{\ell}{1+g_e^*}\right)\left(\frac{\ell}{1+g_e^*}\right)^2 - q'(g_e^*) < \frac{1}{r-\mu} - q'(g_e^*), \tag{A.11}$$

where the inequality follows from the observation that $p'(\ell) < 0$ for all ℓ. Since the cost function q is strictly increasing in g, (A.11) implies that $g_e^* < g^u$.

(Part 2) We now allow the equity holders to choose their financing of investment optimally so that $\psi \in [0,1]$. In this case, the equity holders' problem is given by

$$\max_{\substack{g,\hat{\ell}\geq 0\\ \psi\in[0,1]}} \quad \frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) \tag{A.12}$$

$$\text{s.t. } p(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) \tag{A.13}$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \tag{A.14}$$

where $m = 0$ since $\kappa = 0$. Recall that $\hat{\ell}(g, \psi, m)$ denotes the level of post-investment leverage implied by equity holders' choices via the budget constraint. Since $m = 0$ in what follows we slightly abuse notation and write $\hat{\ell}(g, \psi)$ instead to $\hat{\ell}(g, \psi, 0)$.

It is easy to see that, as long as $\ell < \bar{\ell}$ the equity holders will never choose g, ψ such that $\hat{\ell}(g, \psi) = \bar{\ell}$. This is because when $\hat{\ell}(g, \psi) = \bar{\ell}$ then equity holders' post-investment value of equity is 0 (and the equity holders immediately default) while $\ell < \bar{\ell}$ implies that the pre-investment value of equity is strictly positive. It follows that the constraint (A.14) (which, as shown in Corollary 3 is equivalent to the constraint $\hat{\ell} \leq \bar{\ell}$) is not binding. Hence, the equity holders' problem can be written as

$$\max_{\substack{g \geq 0 \\ \psi \in [0,1]}} \quad \frac{1+g}{r-\mu} - p\left(\hat{\ell}(g, \psi)\right)\ell - q(g), \tag{A.15}$$

subject to $\psi \in [0, 1]$, where $\hat{\ell}(g, \psi)$ is implicitly defined by (A.13). Note that the first-order derivative of equity holders' objective function (A.15) w.r.t. ψ is given by

$$-p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial \psi} > 0 \tag{A.16}$$

since $p'(\hat{\ell}) < 0$ and $\partial\hat{\ell}/\partial\psi$ (obtained by applying the implicit function theorem to (A.13)) is given by

$$\frac{\partial \hat{\ell}}{\partial \psi} = \frac{q(g)}{p'(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) + p(\hat{\ell})(1+g)} > 0 \tag{A.17}$$

It follows that equity holders find it optimal to finance all of their investment with debt, that is, $\psi^* = 1$.

Consider next the optimal choice of g. The optimal choice of g, which we denote by g^*, satisfies the following first-order condition

$$\frac{1}{r-\mu} - p'\left(\hat{\ell}\right)\ell \left.\frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=1 \\ g=g^*}} - q'(g^*) = 0, \tag{A.18}$$

where

$$\frac{\partial \hat{\ell}}{\partial g} = -\frac{p(\hat{\ell})\hat{\ell} - \psi q'(g)}{p'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p(\hat{\ell})(1+g)} \tag{A.19}$$

We now argue that $\partial\hat{\ell}/\partial g$ evaluated at $g = g^*, \psi = \psi^* = 1$ is strictly positive. To see this, note that (A.18) implies that

$$-q'(g^*) = -\frac{1}{r-\mu} + p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}}$$

Using the above expression in (A.19) evaluated at $g = g^*, \psi = 1$ yields

$$\frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}} = -\frac{p(\hat{\ell})\hat{\ell} - \frac{1}{r-\mu} + p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}}}{p'(\hat{\ell})(\hat{\ell}(1+g^*) - \ell) + p(\hat{\ell})(1+g^*)}$$

Rearranging the above equation, we obtain

$$\frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}} \left[\frac{(1+g^*)\left(p(\hat{\ell}) + p'(\hat{\ell})\hat{\ell}\right)}{p'(\hat{\ell})(\hat{\ell}(1+g^*) - \ell) + p(\hat{\ell})(1+g^*)}\right] = \frac{-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu}}{p'(\hat{\ell})(\hat{\ell}(1+g^*) - \ell) + p(\hat{\ell})(1+g^*)} \tag{A.20}$$

From Lemma 1 and Lemma 5 we know that $-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu} \geq 0$ with a strict inequality if $\hat{\ell} < \bar{\ell}$. However, as we argued above, choosing $\hat{\ell} = \bar{\ell}$ is not optimal. Thus, at optimal choices of investment and financing we have $-p(\hat{\ell})\hat{\ell} + \frac{1}{r-\mu} > 0$. Next, note that by Lemma 4 the denominator on the RHS of (A.20) is strictly positive. Thus, it follows that the RHS of (A.20) is strictly positive. Furthermore, (A.20). Lemmas 4 and 5 imply that the expression in square brackets on the LHS of (A.20) is strictly positive. Therefore, we conclude that

$$\frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}} > 0 \tag{A.21}$$

Having established that $\partial \hat{\ell}/\partial g|_{\{g=g^*,\psi=1\}} > 0$ we consider again (A.18). Since, $p'(\hat{\ell}) < 0$ we have

$$0 = \frac{1}{r-\mu} - q'(g^*) - p'\left(\hat{\ell}\right)\ell \frac{\partial \hat{\ell}}{\partial g}\Big|_{\substack{\psi=1\\ g=g^*}} > \frac{1}{r-\mu} - q'(g^*) \tag{A.22}$$

Since the cost function q is strictly increasing, (A.22) implies that $g^* > g^u$.

□

A.3 Proof of Main Proposition 5 ($\kappa > 0$)

Before we prove Main Proposition 5, we establish an important intermediate result.

Lemma 7. *Equity holders choose to issue as much dividend as they can. That is, given g and ψ such that $\hat{\ell}(g, \psi, 0) < \bar{\ell}$, the equity holders choose*

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\}, \tag{A.23}$$

where $\bar{m}(g, \psi)$ is defined in (A.7).

Proof. Recall that, when $\theta = 0$, the equity holders' objective function is given by (see Main Proposition 3)

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g), \tag{A.24}$$

where $\hat{\ell} = \hat{\ell}(g, \psi, m)$.[1] Thus, the first-derivative of equity holders' objective function w.r.t. m is given by

$$-p'(\hat{\ell})\ell \frac{\partial \hat{\ell}}{\partial m} > 0$$

since $p'(\hat{\ell}) < 0$ and $\partial \hat{\ell}/\partial m > 0$ (see the proof of Lemma 6). Thus, it follows that $m^* = \min\{\kappa, \bar{m}(g, \psi)\}$ □

Lemma 7 tells us that equity holders' problem can be simplified to

$$\max_{\substack{g \geq 0 \\ \psi \in [0,1]}} \frac{1+g}{r-\mu} - p(\hat{\ell})\ell - q(g) \tag{A.25}$$

$$\text{s.t. } p(\hat{\ell}) \left(\hat{\ell}(1+g) - \ell\right) = \psi q(g) + m^* \tag{A.26}$$

$$p(\hat{\ell}) \geq p^B(\hat{\ell}) \tag{A.27}$$

$$m^* = \min\{\kappa, \bar{m}(g, \psi)\} \tag{A.28}$$

In other words, we can think of equity holders' problem as choosing first g and ψ and then setting m to the highest value that is feasible.

Proof of Main Proposition 5. The proof consists of four parts. First, we characterize the solution when $\kappa = \infty$. We refer to this solution as the "unconstrained" solution. Next, we determine $\bar{\kappa}$ such that if $\kappa \geq \bar{\kappa}$ then the unconstrained solution is attainable. Thus, for all $\kappa \geq \bar{\kappa}$ the equity payout constraint, $m \leq \kappa$, is not binding. We then show that there exists $\underline{\kappa}$

[1] As explained in Appendix A.1 we can treat post-investment leverage as a function of $\{g, \psi, m\}$ defined implicitly by equity holders' budget constraint.

with $\underline{\kappa} < \bar{\kappa}$ such that if $\kappa \in [\underline{\kappa}, \bar{\kappa})$ then the equity holders can still attain the same payoff as in the case of $\kappa = \infty$ but with an additional restriction on their financing choices. Finally, we determine the equity holders' choices when $\kappa < \underline{\kappa}$.

When $\kappa = \infty$ then $m^* = \bar{m}(g, \psi)$ (see Lemma 7). Then the first-order derivative of equity holders' objective function w.r.t. ψ is given by

$$-p'(\hat{\ell}) \left[\frac{\partial \hat{\ell}}{\partial \psi} + \frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^*}{\partial \psi} \right] \tag{A.29}$$

Since $m^* = \bar{m}(g, \psi)$, we have that

$$\frac{\partial \bar{m}(\psi, g)}{\partial \psi} = -q(g) \tag{A.30}$$

Moreover,

$$\frac{\partial \hat{\ell}}{\partial \psi} = \frac{q(g)}{p(\hat{\ell})(1+g) + p'(\hat{\ell})(\hat{\ell}(1+g) - \ell)} \tag{A.31}$$

$$\frac{\partial \hat{\ell}}{\partial m} = \frac{1}{p(\hat{\ell})(1+g) + p'(\hat{\ell})(\hat{\ell}(1+g) - \ell)} \tag{A.32}$$

Therefore, it follows that

$$p'(\hat{\ell}) \left[\frac{\partial \hat{\ell}}{\partial \psi} + \frac{\partial \hat{\ell}}{\partial m} \frac{\partial m^*}{\partial \psi} \right] = 0 \tag{A.33}$$

That is, when equity holders' equity payout choices are unconstrained they are indifferent between any $\psi \in [0, 1]$.

Fix $\psi^* \in [0, 1]$. Consider now the first-order condition that determines the optimal investment, g^*, and which is given by

$$\frac{1}{r-\mu} + p'\left(\hat{\ell}\right) \ell \left[\left.\frac{\partial \hat{\ell}}{\partial g}\right|_{\substack{\psi=\psi^* \\ g=g^*}} + \left.\frac{\partial \hat{\ell}}{\partial m}\right|_{\substack{\psi=\psi^* \\ g=g^*}} \left.\frac{\partial m^*}{\partial g}\right|_{\substack{\psi=\psi^* \\ g=g^*}} \right] - q'(g^*) = 0 \tag{A.34}$$

Since $m^* = \bar{m}(g, \psi)$, it follows that

$$\frac{\partial m^*}{\partial g} = \frac{1}{r-\mu} - \psi q'(g) \tag{A.35}$$

Moreover, since $m^* = \bar{m}(g,\psi)$ we know that $\hat{\ell} = \bar{\ell}$ (see Lemma 6). Therefore,

$$\frac{\partial \hat{\ell}}{\partial g}\bigg|_{\substack{\psi=\psi^*\\ g=g^*}} + \frac{\partial \hat{\ell}}{\partial m}\bigg|_{\substack{\psi=\psi^*\\ g=g^*}} \frac{\partial m^*}{\partial g}\bigg|_{\substack{\psi=\psi^*\\ g=g^*}} = \frac{-p(\bar{\ell})\bar{\ell} + \psi^* q'(g^*) + \frac{1}{r-\mu} - \psi^* q'(g^*)}{p(\bar{\ell})(1+g^*) + p'(\bar{\ell})(\bar{\ell}(1+g^*) - \ell)} = 0, \tag{A.36}$$

where the last equality follows from the fact that $p(\bar{\ell})\bar{\ell} = 1/(r-\mu)$ (see Lemma 5). Thus, the first-order condition determining g^* simplifies to

$$\frac{1}{r-\mu} - q'(g^*) = 0 \tag{A.37}$$

implying that $g^* = g^u$. Thus, we conclude that if $\kappa = \infty$ then the solution to the equity holders' problem is given by $g^* = g^u$, $\psi^* \in [0,1]$, and $m^* = \bar{m}(g^u, \psi^*)$.

Next, we investigate for which κ the above unconstrained solution is feasible. This is the case if $\kappa \geq \bar{m}(g^u, \psi)$ for all $\psi \in [0,1]$. Note that $\bar{m}(g^u,\psi)$ is decreasing in ψ. Therefore, if we define $\bar{\kappa} \equiv \bar{m}(g^u, 0)$ then for all $\kappa \geq \bar{\kappa}$ the "unconstrained solution" is attainable.

Before proceeding further, note for all $\kappa \geq \bar{\kappa}$ equity holders' payoff is given by

$$v^*(\ell) = \frac{1+g^u}{r-\mu} - \frac{\eta}{1+\eta}\ell - q(g^u) \tag{A.38}$$

(A.38) shows us that the equity holders capture all the value of new investment and extract as much as possible from the old debt holders given the constraint that $\hat{\ell} \leq \bar{\ell}$. Moreover, note that the post-investment value of equity is independent of m, ψ, and $\hat{\ell}$.

Next, consider a situation where $\kappa < \bar{m}(g^u, 0)$ but $\kappa \geq \bar{m}(g^u, 1)$. In this case, the unconstrained solution characterized above is not feasible for some choices of ψ. However, the equity holders can still attain the payoff defined in (A.38) by choosing $g^* = g^u$, $\psi^* \in [\underline{\psi}_\kappa, 1]$, $m^* = \bar{m}(g^*, \psi^*)$, where $\underline{\psi}_\kappa$ is the unique solution to

$$\kappa = \bar{m}(g^u, \underline{\psi}_\kappa) \tag{A.39}$$

Therefore, if we define $\underline{\kappa} \equiv \bar{m}(g^u, 1)$ then the above discussion implies that for all $\kappa \geq \underline{\kappa}$ the equity holders invest the first-best amount. Finally, note that from the definition of $\bar{m}(g^u, 1)$ we have

$$\frac{\partial \underline{\kappa}}{\partial \ell} < 0, \quad \frac{\partial \underline{\kappa}}{\partial \sigma^2} > 0, \quad \frac{\partial \underline{\kappa}}{\partial r} < 0 \tag{A.40}$$

It remains to determine the equity holders' choices when $0 < \kappa < \underline{\kappa}$ (the case of $\kappa = 0$ is covered by Main Proposition 4). We first argue, by contradiction, that in this case the equity holders' optimal choices $\{g^*, \psi^*, m^*\}$ are such that $m^* = \kappa < \bar{m}(g^*, \psi^*)$. To see this assume, to the contrary, that $\{g^*, \psi^*, m^*\}$ are such that $m^* = \bar{m}(g^*, \psi^*) < \kappa$. Then, from Lemma 6 we know that $\hat{\ell}(g^*, \psi^*, m^*) = \bar{\ell}$, which implies that the equity holders' payoff is given by

$$\frac{1+g^*}{r-\mu} - \frac{\eta}{1+\eta}\ell - q(g^*) \tag{A.41}$$

Note that $\frac{1+g}{r-\mu} - q(g)$ is strictly increasing in g for all $g < g^u$ and strictly decreasing in g for all $g > g^u$ and recall that since $\kappa < \underline{\kappa}$ it must be the case that $g^* \neq g^u$. Furthermore, note that if $g^* < g^u$ then the equity holders would have incentives to increase their investment from g^* to $g^* + \varepsilon$ for small $\varepsilon > 0$. This is feasible by setting

$$m' = \frac{1+(g^*+\varepsilon)}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi^* q(g^*+\varepsilon) \tag{A.42}$$

as long as ε is small enough so that $m' \leq \kappa$. Hence $g < g^u$ cannot be optimal. By a similar argument, if $g^* > g^u$ then the equity holders would find it optimal and feasible to decrease their investment. Thus, we conclude that a choice of $\{g^*, \psi^*, m^*\}$ such that $m^* = \bar{m}(g^*, \psi^*) < \kappa$ is not optimal.

Next, suppose that $\{g^*, \psi^*, m^*\}$ are such that $m^* = \bar{m}(g^*, \psi^*) = \kappa$. In this case, the budget constraint implies that

$$\kappa = \frac{1+g^*}{r-\mu} - \frac{\eta}{1+\eta}\ell - \psi^* q(g^*) \tag{A.43}$$

while the equity holders' payoff is given by

$$\frac{1+g^*}{r-\mu} - \frac{\eta}{1+\eta}\ell - q(g^*) \tag{A.44}$$

Using (A.43) in (A.44) we see that the equity holders' payoff can be expressed as

$$\kappa - (1-\psi^*)\, q(g^*) \tag{A.45}$$

We now argue that the equity holders can attain a strictly higher payoff than the payoff in (A.45) by choosing $\psi' = 1$, $m' = \kappa$, and an investment g' such that g' solves the budget constraint

$$p(\hat{\ell})\left(\hat{\ell}(1+g') - \ell\right) = \kappa + q(g') \tag{A.46}$$

with $\hat{\ell}(g', \psi', \kappa) < \bar{\ell}$. If equity holders make such choices then their payoff would be given by

$$\frac{1+g'}{r-\mu} - p(\hat{\ell})\ell - q(g') = \frac{1+g'}{r-\mu} - p(\hat{\ell})\hat{\ell}(1+g') + \kappa > \kappa, \tag{A.47}$$

where the first equality follows from (A.46), while the final inequality follows from the observation that $p(\hat{\ell})\hat{\ell} < \frac{1+g}{r-\mu}$ for all $\hat{\ell} < \bar{\ell}$ (see Lemma 1 and Lemma 5). Therefore, we conclude that choice of $\{g^*, \psi^*, m^*\}$ such that $m^* = \bar{m}(g^*, \psi^*) = \kappa$ cannot be optimal. It follows that we must have $m^* = \kappa < \bar{m}(g^*, \psi^*)$.

We argued above that if $\kappa \in (0, \underline{\kappa})$ then $m^* = \kappa < \bar{m}(g^*, \psi^*)$. Therefore, we have

$$\left.\frac{\partial m^*}{\partial \psi}\right|_{\substack{g=g^* \\ \psi=\psi^*}} = 0 \quad \text{and} \quad \left.\frac{\partial m^*}{\partial g}\right|_{\substack{g=g^* \\ \psi=\psi^*}} = 0 \tag{A.48}$$

This implies that, when $\kappa < \underline{\kappa}$, the first-order conditions that determine equity holders' choices of g and ψ are identical to those when $\kappa = 0$. Thus, using the same argument as in the proof of Proposition 4 we conclude that $g^* > g^u$ and $\psi^* = 1$. This concludes the proof.

□

A.4 Proof of Main Proposition 6 (Bankruptcy Costs)

In this section, we prove Main Proposition 6. We start with a number of preliminary results mostly of technical nature. A reader may wish to skip these intermediate claims and go straight to the proof of Main Proposition 6.

Lemma 8. *For all $\theta \in [0,1]$ $p(\hat{\ell})\left((1+g)\hat{\ell} - \ell\right)$ is a strictly concave function of $\hat{\ell}$ and achieves its maximum for some $\hat{\ell} > \frac{\ell}{1+g}$.*

Proof. Differentiating twice $p(\hat{\ell})((1+g)\hat{\ell} - \ell)$ w.r.t. $\hat{\ell}$ we obtain

$$p''(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) + 2p'(\hat{\ell})(1+g) = -\eta(1+\theta\eta)\chi\hat{\ell}^{\eta-2}\left[\hat{\ell}(1+g) - \frac{\eta-1}{\eta+1}\ell\right] < 0$$

since $\eta, \theta, \chi > 0$ and $\left(\hat{\ell}(1+g) - \ell\right) \geq 0$ establishing strict concavity. Next, note that the first derivative of $p(\hat{\ell})\left((1+g)\hat{\ell} - \ell\right)$ w.r.t. $\hat{\ell}$ evaluated at $\hat{\ell}(1+g) = \ell$ is positive, which implies that the maximum must occur for some $\hat{\ell} > \frac{\ell}{1+g}$. □

Next, we note that if $\theta > 0$ it is possible that investment g is consistent with two different levels of leverage $\hat{\ell}$. This is because when $\theta > 0$ then the LHS of the budget constraint is a single-peaked concave function. The next results, shows that if $\hat{\ell}_1$ and $\hat{\ell}_2$ with $\hat{\ell}_1 < \hat{\ell}_2$ both satisfy the budget constraint then the equity holders strictly prefer the smaller leverage.

Lemma 9. *Fix $g > 0$ and suppose that both $\hat{\ell}_1$ and $\hat{\ell}_2$ both satisfy the budget constraint and that $\hat{\ell}_1 < \hat{\ell}_2$. Then equity holders strictly prefer $\hat{\ell}_1$ to $\hat{\ell}_2$.*

Proof. This claim follows immediately from the fact the payoff to equity holders (as shown in Main Proposition 3) is given by

$$\frac{1+g}{r-\mu} - p(\hat{\ell})\hat{\ell}(1+g) - H(\hat{\ell})\hat{\ell}(1+g),$$

which is strictly decreasing in $\hat{\ell}$ holding g constant. □

Corollary 10. *The post-investment $\hat{\ell}$ satisfies*

$$p'(\hat{\ell})((1+g)\hat{\ell} - \ell) + p(\hat{\ell})(1+g) \geq 0 \tag{A.49}$$

Proof. The equity holders would never choose $\hat{\ell}$ such that $p'(\hat{\ell})((1+g)\hat{\ell}-\ell)+p'(\hat{\ell})(1+g) < 0$ since then there exists $\hat{\ell}' < \hat{\ell}$, which also satisfies the equity holders' budget constraint and which is preferred by the equity holders (Lemma 9). Such $\hat{\ell}'$ exists since the LHS of the budget constraint is concave and increasing for small values of $\hat{\ell}$ (see Lemma 8 and the discussion that follows). □

Lemma 11. *Suppose that investment is financed fully with debt. Then $\partial\hat{\ell}/\partial g \geq (>)0$ if and only if $g \geq (>)g_0$, where g_0 is the unique solution to*

$$p(\hat{\ell})\hat{\ell} - q'(g) = 0 \tag{A.50}$$

Proof. Recall that

$$\frac{\partial\hat{\ell}}{\partial g} = -\frac{p(\hat{\ell})\hat{\ell} - q'(g)}{p'(\hat{\ell})\left(\hat{\ell}(1+g) - \ell\right) + p(\hat{\ell})(1+g)} \tag{A.51}$$

From Corollary 10 we know that the denominator in (A.51) is positive. Therefore, $\partial\hat{\ell}/\partial g \geq 0$ if and only if

$$p(\hat{\ell})\hat{\ell} - q'(g) \leq 0 \tag{A.52}$$

Note that at $g = 0$ the LHS of (A.52) is positive implying that for small values of g we have $\partial\hat{\ell}/\partial g < 0$. Similarly, for sufficiently high g the LHS of (A.52) is negative since

$\lim_{g\to\infty} q'(g) = \infty$. Finally, let g_0 be a solution to (A.50). Then at $g = g_0$ we have

$$\frac{\partial}{\partial g}\left[p(\hat{\ell})\hat{\ell} - q'(g)\right] = \left[p'(\hat{\ell})\hat{\ell} + p(\hat{\ell})\right]\frac{\partial \hat{\ell}}{\partial g} - q''(g) = -q''(g) < 0 \tag{A.53}$$

implying that g_0 is unique. □

With these preliminary results, we can now prove Main Proposition 6.[2]

Proof of Main Proposition 6. **(Part 1):** From Main Proposition 3 it is immediate that the first-order condition that determines optimal investment is given by

$$\frac{1}{r-\mu} - q'(g) - \partial_\ell p(\hat{\ell})\frac{\partial \hat{\ell}}{\partial g}\ell - \left[(1+g)H'(\hat{\ell})\hat{\ell}\frac{\partial \hat{\ell}}{\partial g} + (1+g)H(\hat{\ell})\frac{\partial \hat{\ell}}{\partial g} + H(\hat{\ell})\hat{\ell}\right] = 0 \tag{A.54}$$

If investment is financed with equity (that is, when $\psi = 0$) then $\hat{\ell} = \ell/(1+g)$ and so $\partial\hat{\ell}/\partial g = -\ell/(1+g)^2$. Using this observation in (A.54), we conclude that optimal investment under equity financing, which we denote by g_e^*, has to satisfy

$$\frac{1}{r-\mu} - q'(g_e^*) - H(\hat{\ell})\hat{\ell} + \left[\partial_\ell p(\hat{\ell})\ell + (1+g_e^*)H'(\hat{\ell})\hat{\ell} + (1+g_e^*)H(\hat{\ell})\right]\frac{\ell}{(1+g_e^*)^2} = 0, \tag{A.55}$$

where $\hat{\ell}$ in the above equation is the leverage implied by the equity holders' choices, and is given by $\hat{\ell} = \ell/(1+g_e^*)$. Using this observation, we obtain

$$\partial_\ell p(\hat{\ell})\ell + (1+g_e^*)H'(\hat{\ell})\hat{\ell} + (1+g_e^*)H(\hat{\ell}) = -\eta\chi\frac{1}{1+\eta}\hat{\ell}^\eta(1+g_e^*)\,(1-\theta) < 0$$

Therefore, we conclude that at $g = g_e^*$ we have

$$-H(\hat{\ell})\hat{\ell} + \left[\partial_\ell p(\hat{\ell})\ell + (1+g_e^*)H'(\hat{\ell})\hat{\ell} + (1+g_e^*)H(\hat{\ell})\right]\frac{\ell}{(1+g_e^*)^2} < 0 \tag{A.56}$$

It follows that

$$\frac{1}{r-\mu} - q'(g_e^*) > 0 = \frac{1}{r-\mu} - q'(g^u), \tag{A.57}$$

[2] Below, we implicitly assume that assume g^u is feasible under debt financing. It is easy to show that this is indeed the case for small enough bankruptcy cost, θ.

where we used the characterization of g^u established in Main (27). Since q is strictly increasing it follows that $g_e^* < g^u$.

(Part 2): We establish this results in three steps. (1) We show that for a sufficiently small θ equity holders find it optimal to finance their investment fully with debt. (2) We next show that under debt financing at the optimal investment we have $\partial\hat{\ell}/\partial g > 0$ and provide a lower bound for this derivative. In light of Lemma 11 this implies that $g^* > g_0$. (3) Finally, we show that for any $\ell > 0$ there exists small enough θ such that for all $g \in [g_0, g^u]$, under debt financing, equity holders' marginal benefit from investing always exceed their marginal benefit when $\ell = 0$, which implies that $g^* > g^u$ for sufficiently low θ.[3]

The derivative of equity holders' objective function (derived in Main Proposition 3) w.r.t. ψ is given by

$$\left[-\partial_\ell p(\hat{\ell})\ell - (1+g)H'(\hat{\ell})\hat{\ell} - (1+g)H(\hat{\ell})\right]\frac{\partial\hat{\ell}}{\partial\psi}, \tag{A.58}$$

where $\partial\hat{\ell}/\partial\psi > 0$ by the analogous argument to the one used in the proof of Main Proposition 4. Moreover,

$$-\partial_\ell p(\hat{\ell})\ell - (1+g)H'(\hat{\ell})\hat{\ell} - (1+g)H(\hat{\ell}) = \eta\frac{1+\theta\eta}{1+\eta}\chi\hat{\ell}^{\eta-1}\ell - \eta\theta\chi\hat{\ell}^{\eta-1}\hat{\ell}(1+g) \tag{A.59}$$

First, we note that $\psi = 0$ is never optimal. This is because $\psi = 0$ implies that $\hat{\ell}(1+g) = \ell$ and so

$$\eta\frac{1+\theta\eta}{1+\eta}\chi\hat{\ell}^{\eta-1}\ell - \eta\theta\chi\hat{\ell}^{\eta-1}\hat{\ell}(1+g) = \frac{\eta}{1+\eta}\chi\hat{\ell}^{\eta} > 0 \tag{A.60}$$

Therefore, equity holders always finance their investment at least partially with debt (and this also means that g_e^* is not optimal). Next, note that for all $g \in (0, g^u)$ we have

$$\frac{1+\theta\eta}{1+\eta}\chi\hat{\ell}^{\eta-1}\ell - \eta\theta\chi\hat{\ell}^{\eta-1}\hat{\ell}(1+g) > \chi\hat{\ell}^{\eta-1}\left[\frac{1}{1+\eta}\ell - \theta\eta\bar{\ell}(1+g^u)\right], \tag{A.61}$$

where $\bar{\ell}$ is the maximum leverage the firm can have (see Corollary 3). The expression in square brackets is strictly decreasing in θ and positive at $\theta = 0$. It follows that there exists

[3] Recall that equity holders invest first-best amount when $\ell = 0$.

$\underline{\theta}_1(\ell) > 0$ such that for all $\theta \in [0, \underline{\theta}_1(\ell)]$ we have

$$\frac{1+\theta\eta}{1+\eta}\chi\hat{\ell}^{\eta-1}\ell - \eta\theta\chi\hat{\ell}^{\eta-1}\hat{\ell}(1+g) > 0 \tag{A.62}$$

Therefore, we conclude that for all $\theta \in [0, \underline{\theta}_1(\ell)]$ the equity holders would finance any $g \in [0, g^u]$ fully with debt. Next, suppose that $\theta \in [0, \underline{\theta}_1(\ell)]$ and that the optimal investment g^* (which can be larger than g^u) is financed partially with equity, that is, $\psi^* \in (0,1)$. Then,

$$\left[\partial_\ell p(\hat{\ell})\ell + (1+g)H'(\hat{\ell})\hat{\ell} + (1+g)H(\hat{\ell})\right] = 0 \tag{A.63}$$

When (A.63) holds, the F.O.C. (A.54) simplifies to

$$\frac{1}{r-\mu} - q'(g) - H(\hat{\ell})\hat{\ell} = 0, \tag{A.64}$$

which implies that $g^* < g^u$ as $H(\hat{\ell})\hat{\ell} > 0$. But this is a contradiction since we showed that if $\theta \in [0, \underline{\theta}_1(\ell)]$ then for all $g \in [0, g^u]$ optimal financing is $\psi = 1$. Thus, we conclude that optimal investment has to be financed fully with debt. This concludes the first part of our argument.

Above we established that $\theta \in [0, \underline{\theta}_1(\ell)]$ the optimal investment has to be fully financed with debt. Therefore, we can follow the same steps as in the proof of Main Proposition 4 to show that

$$\frac{\partial\hat{\ell}}{\partial g} = \frac{\frac{1}{r-\mu} - p(\hat{\ell})\hat{\ell} - H(\hat{\ell})\hat{\ell}}{p^C(\hat{\ell})(1+g)} > 0, \tag{A.65}$$

which implies that $g^* > g_0$ (where g_0 is defined in Lemma 11). If $g^u < g_0$ then we are done (i.e., we conclude that equity holders overinvest). Thus, in what follows we assume that $g^u > g_0$.

Denote by $\hat{\ell}(g,\theta)$ the post-investment leverage when equity holders invest g (and finance their investment with debt) and bankruptcy costs are θ. Then, under debt financing, for all $g \in [g_0, g^u]$ we have

$$\hat{\ell}(g_0, 0) < \hat{\ell}(g,\theta) < \hat{\ell}(g^u,\theta) \tag{A.66}$$

Therefore,

$$\frac{\partial\hat{\ell}}{\partial g} \geq \frac{\frac{1}{r-\mu} - p\left(\hat{\ell}(g,\theta)\right)\hat{\ell}(g,\theta) - H\left(\hat{\ell}(g,\theta)\right)\hat{\ell}(g,\theta)}{p^C(\hat{\ell}(g_0,0))(1+g^u)} \equiv \Delta_{\hat{\ell}}(\theta) > 0, \tag{A.67}$$

where the last inequality follows from the observation that for all $\hat{\ell} < \bar{\ell}$ we have

$$p(\hat{\ell})\hat{\ell} + H(\hat{\ell})\hat{\ell} < \frac{1}{r-\mu}$$

Observe that $\Delta_{\hat{\ell}}(\theta)$ is decreasing in θ. Next, consider (A.54), the F.O.C. for optimal investment g. We want to show that for small enough θ if $g \in [g_0, g^u]$ then

$$-\partial_\ell p(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g}\ell - \left[(1+g)H'(\hat{\ell})\hat{\ell}\frac{\partial\hat{\ell}}{\partial g} + (1+g)H(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g} + H(\hat{\ell})\hat{\ell}\right] > 0 \tag{A.68}$$

We note that

$$H(\hat{\ell})\ell \leq \frac{\theta\eta}{1+\eta}\bar{\ell} \tag{A.69}$$

Therefore, for all $\theta \in [0, \underline{\theta}_1(\ell)]$ and all $g \in [0, g^u]$ we have

$$\begin{aligned}
&-\partial_\ell p(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g}\ell - \left[(1+g)H'(\hat{\ell})\hat{\ell}\frac{\partial\hat{\ell}}{\partial g} + (1+g)H(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g} + H(\hat{\ell})\hat{\ell}\right] \\
&> \frac{\partial\hat{\ell}}{\partial g}\left[-\partial_\ell p(\hat{\ell}) - (1+g)H'(\hat{\ell})\hat{\ell} - (1+g)H(\hat{\ell})\right] - \frac{\theta\eta}{1+\eta}\bar{\ell} \\
&> \Delta_{\hat{\ell}}(\theta)\left[\frac{\eta}{1+\eta}\chi\ell^\eta - \theta\eta\hat{\ell}(g^u,\theta)\left((1+g^u)\hat{\ell}(g^u,\theta) - \frac{\eta}{1+\eta}\ell\right)\right] - \frac{\theta\eta}{1+\eta}\bar{\ell},
\end{aligned} \tag{A.70}$$

We note that both $\Delta_{\hat{\ell}}(\theta)$ and

$$\left[\frac{\eta}{1+\eta}\chi\ell^\eta - \theta\eta\hat{\ell}(g,\theta)\left((1+g^u)\hat{\ell}(g^u,\theta) - \frac{\eta}{1+\eta}\ell\right)\right]$$

are strictly decreasing in θ. Moreover,

$$\lim_{\theta\to 0}\left\{\Delta_{\hat{\ell}}(\theta)\left[\frac{\eta}{1+\eta}\chi\ell^\eta - \theta\eta\hat{\ell}(g,\theta)\left((1+g^u)\hat{\ell}(g^u,\theta) - \frac{\eta}{1+\eta}\ell\right)\right] - \frac{\theta\eta}{1+\eta}\bar{\ell}\right\} = \Delta_{\hat{\ell}}(0)\frac{\eta}{1+\eta}\chi\ell^\eta > 0 \tag{A.71}$$

It follows that there exists $\underline{\theta}_2(\ell) > 0$ such that for all $\theta \in [0, \underline{\theta}_2(\ell)]$ and all $g \in [g_0, g^u]$ inequality (A.68) holds. As a consequence, for all $\theta \in [0, \underline{\theta}_2(\ell)]$ and all $g \in [g_0, g^u]$ we have

$$\frac{1}{r-\mu} - q'(g) - \partial_\ell p(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g}\ell - \left[(1+g)H'(\hat{\ell})\hat{\ell}\frac{\partial\hat{\ell}}{\partial g} + (1+g)H(\hat{\ell})\frac{\partial\hat{\ell}}{\partial g} + H(\hat{\ell})\hat{\ell}\right] > \frac{1}{r-\mu} - q'(g) \tag{A.72}$$

Since g^u satisfies $\frac{1}{r-\mu} - q(g^u) = 0$ and q is strictly increasing, we conclude that equity holders optimal investment exceeds g^u if $\theta \leq \min\{\underline{\theta}_1, \underline{\theta}_2\}$, which establishes the claim. $\square$

A.5 Optimal Choices of Leverage

In this section, we describe in more detail equity holders' optimal choices of investment, g, and leverage, $\hat{\ell}$. Recall that the equity holders face the following trade-off. On the one hand, overinvesting and financing investment with debt allows the equity holders either dilute existing creditors (as depicted in Figure 3) or (if the leverage is already relatively high) limit the amount of cash flows from new investment captured by the existing debt holders. On the other hand, the cost of inefficient investment is fully borne by equity holders. Thus, equity holders' choices balance the benefits and costs of overinvestment.

For sufficiently high levels of ℓ the costs of increasing leverage above its initial level becomes too high and equity holders' optimally choose to deleverage. However, the rate of overinvestment initially continues to increase. Rather by incentives to dilute, this is driven now by equity holders' desire to limit the value of cash flows generated by new investment that is captured by existing debt holders.

The next lemma shows that increasing leverage requires a larger investment as ℓ increases. Since the cost of investment is a strictly convex function and is borne by equity holders (see Main Proposition 3), this result implies that the cost of increasing leverage increases with ℓ.

Lemma 12. *Let $\bar{g}(\ell)$ be the investment such that at $g = \bar{g}(\ell)$ we have*

$$\hat{\ell}(\bar{g}, \ell) = \ell \tag{A.73}$$

Then, $\bar{g}$ is unique and $\partial\bar{g}/\partial\ell > 0$.

Proof. Form Lemma 11 we know that there exists a unique g_0 such that for all $g < g_0$ we have $\partial\hat{\ell}/\partial g < 0$. Therefore, it has to be the case that $\bar{g} > g_0$ and so $\partial\hat{\ell}/\partial g|_{g=\bar{g}} > 0$.

Now, applying Implicit Function Theorem to (A.73), we obtain

$$\frac{\partial\bar{g}}{\partial\ell} = -\frac{\frac{\partial\hat{\ell}}{\partial\ell}|_{g=\bar{g}} - 1}{\frac{\partial\hat{\ell}}{\partial g}|_{g=\bar{g}}} \tag{A.74}$$

Since, $\frac{\partial\hat{\ell}}{\partial g}|_{g=\bar{g}} > 0$, it is sufficient to determine the sign of the numerator in (A.74). Here, we note that

$$\frac{\partial\hat{\ell}}{\partial\ell} = \frac{p(\ell)}{p'(\hat{\ell})(\hat{\ell}(1+g) - \ell) + p(\hat{\ell})(1+g)} > 0 \tag{A.75}$$

Therefore,

$$\left.\frac{\partial\hat{\ell}}{\partial\ell}\right|_{g=\bar{g}} - 1 = \frac{p(\hat{\ell}) - p'(\hat{\ell})(\hat{\ell}(1+g)-\ell) - p(\hat{\ell})(1+g)}{p'(\hat{\ell})(\hat{\ell}(1+g)-\ell) + p(\hat{\ell})(1+g)} \tag{A.76}$$

$$= \frac{-p'(\hat{\ell})\hat{\ell}g - p(\hat{\ell})g}{p'(\hat{\ell})(\hat{\ell}(1+g)-\ell) + p(\hat{\ell})(1+g)} \tag{A.77}$$

$$< 0, \tag{A.78}$$

where the first equality follows by the fact that at $\bar{g}$ we have $\hat{\ell} = \ell$ and the final inequality follows by Lemma 4 and (A.5). This establishes that $\partial\bar{g}/\partial\ell > 0$. □

A.6 Finite-maturity Debt

In this section, we show that our results naturally extend to the case when debt has finite maturity. In particular, we establish that Main Proposition 4 and Main Proposition 5 continue to hold in this setup.

To model finite maturity debt, we follow Leland (1998) and consider debt that has no stated maturity but is continuously retired at par at a constant fractional rate ξ. That is, at each instance of time fraction ξ of existing debt matures. It follows that $1/\xi$ is the average maturity of debt and higher ξ is associated with shorter average maturity. Each unit of debt pays a constant coupon rate of 1 and has the face value F. Finally, as in Leland (1998), we assume that equity holders are committed to always rollover their debt (i.e., keep leverage fixed), except possibly at the time of investment.[4,5]

Proposition 1. *Main Proposition 4 and Main Proposition 5 hold in an unchanged manner in the setup with finite maturity debt.*

Proof. Let T be a random default time and assume there are K units of debt outstanding. First, we compute equity holders' liability (i.e., the PDV of equity holders' promises).

$$L = \left[\int_0^\infty e^{(r+\xi)t}(1+\xi F)dt\right] K = \frac{1+\xi F}{r+\xi} K = \varrho \frac{K}{r}, \tag{A.79}$$

[4]Modeling finite maturity debt in this way leads to analytically tractable problem and has been popular both in corporate finance (see, for example, Leland (1998), Dangl and Zechner (2021), DeMarzo and He (2021), He and Xiong (2012)) and in sovereign debt literature (see, for example, Chatterjee and Eyigungor (2012)).

[5]The commitment to rolling over debt is a common assumption in this literature. See the discussion in Dangl and Zechner (2021) who also relax this assumption.

where

$$\varrho = \frac{r(1+\xi F)}{r+\xi} \tag{A.80}$$

and $\varrho \to 1$ as $\xi \to 0$. Therefore,

$$K = \frac{rL}{\varrho} \tag{A.81}$$

In what follows, we use L as the state variable to make analysis easily comparable to the analysis in the main paper.

The price of finite-maturity debt is given by

$$P(Z,L;\xi) = \mathbb{E}\left[\int_0^T e^{-(r+\xi)t}(1+\xi F)dt + e^{-(r+\xi)T}\frac{\varrho V^D}{rL}\right], \tag{A.82}$$

where V^D is the value of the firm at default. Suppose that $\underline{Z}$ is the value of Z at which firm default so that $V^D = \frac{\underline{Z}}{r-\mu}$. Then,

$$P(Z,L;\xi) = \frac{1+\xi F}{r+\xi}\left[1-\left(\frac{\underline{Z}}{Z_0}\right)^{\eta}\right] + \frac{\varrho \underline{Z}}{(r-\mu)rL}\left(\frac{\underline{Z}}{Z_0}\right)^{\eta}, \tag{A.83}$$

where

$$\eta \equiv \frac{\mu - \frac{1}{2}\sigma^2\sqrt{(\mu-\frac{1}{2}\sigma^2)^2+2(r+\xi)\sigma^2}}{\sigma^2} \tag{A.84}$$

As $\xi \to 0$ the above price converges to the price in the baseline model.

Next, we consider the total value of the firm (total enterprise value). We have

$$TEV = \mathbb{E}\left[\int_0^\infty e^{-rt}Z_t dt\right] = \frac{Z}{r-\mu} \tag{A.85}$$

Since equity holders are residual owners, it follows that the value of equity is given by

$$V(Z,L) = TEV - P(Z,L)\frac{rL}{\varrho} = \frac{Z_0}{r-\mu} - \left[1-\left(\frac{\underline{Z}}{Z_0}\right)^{\eta}\right]L - \frac{\underline{Z}}{r-\mu}\left(\frac{\underline{Z}}{Z_0}\right)^{\eta} \tag{A.86}$$

It remains to determine the optimal default threshold. We know that $V(Z,L)$ has to satisfy the smooth-pasting condition and, hence,

$$0 = V_Z(\underline{Z}, L) = \frac{1}{r-\mu} - \left[L - \frac{\underline{Z}}{r-\mu}\right]\eta\frac{1}{\underline{Z}} \tag{A.87}$$

Solving the above equation for $\underline{Z}$, we obtain

$$\underline{Z} = \frac{\eta}{1+\eta}(r-\mu)L \tag{A.88}$$

Note that the expression for $\underline{Z}$ is the same as in the baseline model. The only difference between these two version of the model is that the mapping between the quantity of debt issues and equity holders' liability.

Using the expression for $\underline{Z}$ found above in the expression for the value of equity (A.86) and simplifying, we obtain

$$V(Z, L) = \frac{Z}{r-\mu} - \left(1 - \chi\frac{1}{1+\eta}\left(\frac{L}{Z}\right)^{\eta}\right)L, \tag{A.89}$$

where χ is defined in Main (8). Thus, as in the baseline model, we have $V(Z, L) = v(\ell)Z$, where

$$v(\ell) = \frac{1}{r-\mu} + \left(1 - \chi\frac{1}{1+\eta}\ell^{\eta}\right)\ell \tag{A.90}$$

Note that the expression for $v(\ell)$ in (A.90) is the same as in Main (11). Therefore, conditional on the PDV of equity holders' liabilities, the equity holders' default decision is unchanged.

Similarly, using the expression for $\underline{Z}$ we can simplify the expression for $P(Z, L; \xi)$. In particular, we have

$$P(Z, L; \xi) = \left(1 - \chi\frac{1}{1+\eta}\ell^{\eta}\right)\frac{\varrho}{r} \tag{A.91}$$

It remains to consider the budget constraint that equity holders face at the time of investment. As in the baseline model, let $\hat{L}$ denote post-investment liabilities. From (A.81), we know that if equity holders issue K' units of debt to finance their investment g then the associated increase in liabilities is given by

$$\frac{r}{\varrho}(\hat{L} - L), \tag{A.92}$$

where $\hat{L} = \frac{\varrho}{r}(K' + K)$, and the funds the equity holders obtain are equal to

$$\left(1 - \chi \frac{1}{1+\eta} \ell^{\eta}\right) (\hat{L} - L) \tag{A.93}$$

Therefore, the equity holders' budget constraint (divided by Z) is given by

$$\left(1 - \chi \frac{1}{1+\eta} \ell^{\eta}\right) (\hat{\ell}(1+g) - \ell) = \psi q(g) + m, \tag{A.94}$$

Thus, comparing the model with finite maturity debt to the baseline model, we see that the only difference is that η now depends on ξ. It follows that we can use exactly the same argument as used in the proofs of Main Proposition 4 and Main Proposition 5 to show that these results also hold in the model with finite maturity debt. □

Appendix B Empirical Appendix

B.1 Data Definitions

Quarterly firm accounting data is from Compustat 2000.Q1 to 2018.Q1. We exclude financial firms (SIC codes 6000-6999), utilities (SIC codes 4900-4949), and quasi-government companies (SIC codes 9000-9999). We also drop firms with total assets of less than \$100m or missing, as well as firms with negative or missing sales, negative or missing cash holdings, or firms with less than 20 quarterly observations for the investment rate. If there are multiple observations per quarter for one firm due to a change in the reporting period, we use the latest one. All items are winsorized at the 1% level.

The investment rate, leverage, and interest coverage are computed from Compustat. We compute quarterly CAPX taking into account that capital expenditures are reported cumulated over the year. [Investment rate=CAPX/Lagged ATQ; Leverage=Book value of debt/(Book value of debt+Market value of equity); Book value of debt=DLTTQ+DLCQ; Market value of equities=CSHOQ·PRCCQ]. Tobin's Q is defined as market value of equity plus total assets minus book value of equity and deferred taxes and investment tax credit, divided by total assets. Return on assets follows Irvine and Pontiff (2009) and is defined as earnings plus interest expenses over the past four quarters divided by lagged assets.

To compare high- versus low leverage firms, we sort firms by their prior-quarter leverage each quarter. We re-do the sort each quarter to ensure that portfolios reflect firms with constant characteristics and to control for time fixed effects that might drive all firms' leverage. We sort by prior quarter leverage to ensure that the sorting variable is not mechanically affected by the current quarter investment decision. Main Figure 8 shows the equal-weighted investment rate for firms in the bottom leverage quartile in solid blue and the equal-weighted investment rate for firms in the top leverage quarter in dashed red.

B.2 Impulse Response Regressions

Table A2 shows the regression results of the Jordà (2005) impulse responses, and verifies that they are robust across various forecasting horizons and to including different fixed effects. By including quarter and time fixed-effects, we flexibly control for time-invariant firm characteristics and aggregate macro trends that might affect all firms at the same time.

B.3 Calibration

To discipline parameters, we calibrate to moments from the firm dynamics (Sterk et al. (2021)) and investment spikes (Gourio and Kashyap (2007)) literature.

Discount Rate We base our discount rate of cash flows on long-term real risk-free rate, measured as the 10-year nominal Treasury rate (from FRED) minus 1-year Survey of Professional Forecasters inflation expectations for the GDP deflator. This proxy for the long-term real rate averages 3.55% for the period 1979-2012, matching the sample period in Sterk et al. (2021). We choose a long-term real rate because bonds in our model are perpetuities and hence extremely long-term and closer to the opportunity cost of equity holders. In addition, since we do not have exit in our model (but empirical investment rates would reflect the exit probability) we add the estimated 4.1% exogenous exit rate estimated in in Sterk et al. (2021). Combining the exit rate and real risk-free rate leads to a discount rate of $r = 0.0765$.

First-Best Dynamics Assuming that the first-best, g^u, calculated from Main (34) exists, the dynamics of the unconstrained $Z(t)$ follow Main (30). Since g^u is independent of $Z(t)$, the process is a simple jump diffusion process with constant drift, volatility, and jump sizes. Following standard techniques and using Ito's Lemma, the solution to this SDE given $Z(0)$ is $\log\left(\frac{Z_t}{Z_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + Y_t$ where W_t is a Wiener process and Y_t is a compound Poisson process with rate λ and jumps size given by g^u, i.e., $Y_t = \sum_{k=1}^{N_t} \log(1+g^u)$. Use Wald's identity and differentiate to find moments of the stochastic process,

$$\mathbb{E}\left[d\log Z_t\right] = \frac{\partial}{\partial t} E\left[\log\left(\frac{Z_t}{Z_0}\right)\right] = \mu - \frac{1}{2}\sigma^2 + \lambda \log(1+g^u) \tag{B.1}$$

$$\mathbb{V}\left[d\log Z_t\right] \equiv \mathbb{E}\left[(d\log Z_t)^2\right] - \mathbb{E}\left[d\log Z_t\right]^2 = \sigma^2 + \lambda\log(1+g^u)^2 \tag{B.2}$$

Note that in the case of the one-shot model, with $\lambda = 0$, $\mathbb{E}\left[d \log Z_t\right] = \mu - \frac{1}{2}\sigma^2$ and $\mathbb{V}\left[d \log Z_t\right] = \sigma^2$, the familiar formulas for the GBM process in Main (1).

Firm Dynamics For disciplining parameters, we will compare moments in the data to the dynamics implied by given by the first-best solutions given a calibration. We expect them to be fairly close to the version with financial distortions after averaging out across firms (i.e. distortions in the $g \neq g^u$ would partially cancel out for different leverage levels). Our results are not very sensitive to the g^u target.

To estimate parameters of the cash flow process, we take moments from Sterk et al. (2021). The authors estimate firm dynamics using covariances from the Longitudinal Business Database (LBD), an administrative panel covering nearly all private employers in the United States from 1976 to 2012. They find that—controlling for ex-ante heterogeneity—the volatility of a random walk using the 1-year growth rate for a balanced panel of LBD firms provides a target of $\sqrt{\mathbb{V}\left[d \log Z_t\right]} = 0.1846$.

While the covariance matrices in Sterk et al. (2021) provide the ideal moments to target for volatility, due to the structure of their data, they cannot provide guidance on the drift. Since we do not have a model with exit selection, we instead target a backwards drift equal to 0.7% from the average United States TFP growth rate from 1976 to 2012 (ie. $\mathbb{E}\left[d \log Z_t\right] = -0.007$).

Investment Arrival Rates and Size For the size and frequency of investments, we connect our large investment jumps to the investment spikes literature and in particular the empirical evidence of Gourio and Kashyap (2007) (Table 1 of the NBER version). They find that in close to 30% of US plant-years have an investment spike of 12% or more of total assets. In particular, as a proportion of investment relative to assets, 11.6% invest between 0.12 and 0.2, 8% invest between 0.2 and 0.35, and 8.3% invest more than 0.35). For the size of the investment, we take the weighted average of these over the minimum of each bin. Hence, fixing the arrival rate as $\lambda = 0.3$, the target jump-size for calibration is $g^u = 0.21$.

Joint Calibration The parameters ζ, σ, and μ are jointly matched to the three target moments for g^u, $\mathbb{E}\left[d \log Z_t\right]$, and $\mathbb{V}\left[d \log Z_t\right]$ using the system of equations (B.1) and (B.2) and Main (34) along with the targets $g^u = 0.21, \mathbb{E}\left[d \log Z_t\right] = -0.007$ and, $\mathbb{V}\left[d \log Z_t\right] = 0.1846^2$. The solution provides: $\sigma = 0.1534, \mu = -0.0514$, and $\zeta = 50.036$.

Guidance on Equity Payout Constraints Grullon and Michaely (2002) estimate an equal-weighted ratio of total of equity payouts (including dividends, repurchases, etc.) to earnings of around 0.5 in 2000. We take this as an upper bound for equity payouts in our model because in practice not all dividends/equity buybacks are financed through new debt issuance. Given our calibrated $\lambda = 0.3$, we therefore obtain an upper bound $\kappa \leq 0.5/\lambda = 1.7$.

Guidance on Interest Coverage Ratios A firm with state (Z, L) in our model has flow profits of Z and a present value of liabilities of L. At interest rate r, they pay a flow of rL in interest. Putting this together, the interest rate coverage ratio is $Z/(rL) = 1/(r\ell)$.

Palomino et al. (2019) report an average interest coverage ratio of around 4 for the period 1970-2017. They also find that an interest coverage ratio of 1.5 is associated with a default probability of 3%, that 30% of creditors had an interest coverage ratio of 2 or less, and about 10% of borrowers had an interest coverage ratio of 1 or less. Further, Blume et al. (1998) report that the average interest coverage ratio for BBB-rated (i.e. low investment grade) companies is 4. We therefore consider $\ell(0) \in \{3, 9\}$ corresponding to interest coverage ratios of 4 and 1.5 to capture a highly levered and an average firm.

Table A1: Summary Statistics

	Capex/Assets	*Leverage*	*log(Assets)*	*ROA*	TQ	*Delta debt*	*Gross equity payout*
mean	5.40	0.23	7.09	4.57	1.79	0.62	1.01
sd	6.63	0.24	1.55	10.20	1.14	5.53	1.91
p25	1.61	0.03	5.87	2.31	1.07	-0.64	0.00
p50	3.22	0.17	6.88	5.99	1.44	0.00	0.17
p75	6.35	0.35	8.05	9.37	2.09	0.47	1.13
min	0.00	0.00	4.61	-40.82	0.52	-15.78	0.00
max	40.97	0.96	13.01	26.75	7.03	34.73	10.46
No. Obs.	105365	105365	105365	91022	92440	105365	105365

Summary statistics for the Compustat sample 2004.Q1-2018.Q1. *Capex/Assets* are quarterly capital expenditures divided by previous quarter total assets expressed in annualized percent. *Leverage* is book value of debt divided by the sum of book value of debt and market value of equity. Market value of equity is computed using CRSP share price times CRSP shares outstanding where available, aggregated by GVKEY, and Compustat share price times Compustat shares outstanding otherwise. *log(Assets)* is the natural logarithm of total assets. *Delta debt* is the one quarter change in the book value of debt divided by the 8-quarter moving average of the firm's total assets in percent. *Gross equity payout* equals dividends plus equity repurchases (DVY+PRSTKCY) divided by the 8-quarter moving average of the firm's total assets in annualized percent.

Table A2: Firm Profitability and Debt Issuance Responses

$$y_{i,t+h} = \alpha_i + \alpha_t + \beta_0 CAPX/Assets_{i,t} \times HighLev_{i,t-1} + \beta_1 CAPX/Assets_{i,t} + \beta_2 HighLev_{i,t-1} + \gamma y_{i,t-1} + \varepsilon_{i,t}$$

	Profitability (Return on Assets)				Debt Issuance			
	$h=4$		$h=8$		$h=4$		$h=8$	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
CAPX/Assets$_{i,t}$ x HighLev$_{i,t}$	-1.289***	-0.924***	-1.593***	-1.237***	2.445***	4.015***	2.840***	4.144***
	(-4.88)	(-3.03)	(-4.58)	(-3.83)	(-8.50)	(12.13)	(-5.27)	(8.74)
CAPX/Assets$_{i,t}$	1.142***	1.255***	0.718***	0.673***	0.155*	0.063	0.031	-0.156
	(-6.04)	(5.15)	(-2.99)	(2.44)	(-1.88)	(0.38)	(-0.33)	(-1.09)
HighLev$_{i,t}$	-0.662*	-3.956***	-0.134	-0.783	-1.373***	-12.296***	0.110	-18.564***
	(-1.67)	(-3.38)	(-0.27)	(-0.62)	(-3.27)	(-8.93)	(-0.14)	(-5.13)
y$_{i,t-1}$	0.477***	0.006	0.382***	-0.081***	0.173***	-0.088***	0.234***	-0.172***
	(-14.88)	(0.230)	(-11.51)	(-3.68)	(-4.36)	(-3.28)	(-4.26)	(-3.75)
N	25154	25154	19257	19257	30968	30968	21892	21892
Time FE	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Firm FE	No	Yes	No	Yes	No	Yes	No	Yes

Regression results of the Jordà (2005) local projections for firm profitability and debt issuance onto investment. We run regressions of the following form: $y_{i,t+h} = \alpha_i+\alpha_t+\beta_0 CAPX/Assets_{i,t}\times HighLev_{i,t-1}+\beta_1 CAPX/Assets_{i,t}+\beta_2 HighLev_{i,t-1}+\gamma y_{i,t-1}+\varepsilon_{i,t}$. The variable HighLev$_{i,t}$ takes a value of one if $Lev_{i,t}$ is in the top quartile, and zero otherwise. Our sample consists of a quarterly panel 2004.Q1-2018.Q1 and includes the top and bottom quartiles of firms by $t-1$ leverage. When forecasting ROA, y_{t+h} is the 4-quarter return on assets (in %) ending in quarter $t+h$. When forecasting debt issuance, y_{t+h} is the cumulative debt issuance from time t to $t+h$ (in %). T-statistics based on double-clustered standard errors by firm and quarter in parentheses. Significance levels are indicated by * p<0.10, ** p<0.05, ***<0.01. For summary statistics and detailed variable definitions see Appendix B.

References

Blume, M. E., F. Lim, and A. C. MacKinlay (1998): "The declining credit quality of US corporate debt: Myth or reality?" *Journal of Finance*, 53, 1389–1413.

Chatterjee, S. and B. Eyigungor (2012): "Maturity, indebtedness, and default risk," *American Economic Review*, 102, 2674–99.

Dangl, T. and J. Zechner (2021): "Debt maturity and the dynamics of leverage," *The Review of Financial Studies*, 34, 5796–5840.

DeMarzo, P. M. and Z. He (2021): "Leverage dynamics without commitment," *The Journal of Finance*, 76, 1195–1250.

Gourio, F. and A. K. Kashyap (2007): "Investment spikes: New facts and a general equilibrium exploration," *Journal of Monetary Economics*, 54, 1 – 22.

Grullon, G. and R. Michaely (2002): "Dividends, share repurchases, and the substitution hypothesis," *Journal of Finance*, 57, 1649–1684.

He, Z. and W. Xiong (2012): "Rollover risk and credit risk," *The Journal of Finance*, 67, 391–430.

Irvine, P. J. and J. Pontiff (2009): "Idiosyncratic return volatility, cash flows, and product market competition," *Review of Financial Studies*, 22, 1149–1177.

Jordà, Ò. (2005): "Estimation and inference of impulse responses by local projections," *American Economic Review*, 95, 161–182.

Leland, H. E. (1998): "Agency Costs, Risk Management, and Capital Structure," *The Journal of Finance*, 53, 1213–1243.

Palomino, F., S. Paolillo, A. Perez, and G. Sanz-Maldonado (2019): "The Information in Interest Coverage Ratios of the US Nonfinancial Corporate Sector," *FED Notes*, 10.

Sterk, V., P. Sedláček, and B. Pugsley (2021): "The Nature of Firm Growth," *American Economic Review*, 111, 547–79.