0.1 Einführung

In the following we mainly follow Bartholdi's hands on approach from [1] to define the forested graph complex. The more general definition from Conant and Vogtmann from [2] has been postponed as it is too complex and unintuitive for the beginning. Very useful in the general understanding of what a graph complex is and how the boundary map acts was Bar-Natan's draft []. Inspired by this similar examples for the forested graph complex are presented.

Definition. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$. A loop is an edge having the same start and endpoint. We call a graph *connected* if the CW complex is connected in the topological sense. A graph is n-connected if it remains connected after removing $n-1$ arbitrary edges. A graph is said to be n-*regular* if every vertex has valency n i.e. for every vertex the number of incident edges is n.

For a collection of edges Φ of G we denote by G/Φ the graph quotient, which is the quotient space of the CW complex G over its topological subspace Φ.

Lastly a *tree* is a connected graph which contains no cycles and a *forest* is a collection of disjoint trees.

Definition. As the fundamental group of a connected Graph is isomorphic to a free group, it makes sense to define the rank of a graph as the rank of its fundamental group $\pi_1(G)$.

For a proof of this see for example [4, p. 43f]

Proposition. *The rank of a graph G is equal to:*

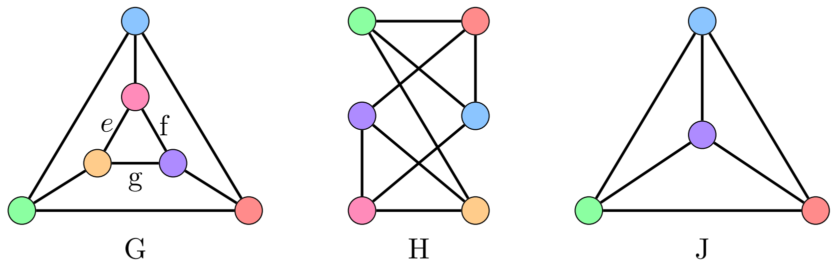
1. *The number of independent cycles in G*
2. *The first Betti number i.e. the rank of $H_1(G)$*
3. *the Euler Characteristic of G given by $\chi(G) = |C(G)| - |V(G)| + |E(G)|$, where $|C(G)|$ is the number of connected components of G.*

Proof Sketch. That the rank is equal to the first Betti number follows from Hurewicz Theorem and the fact that the abelianisation of the free group of rank n is the free abelian group of rank n.

The equality of 2) and 3) follows from the argument for the fundamental group in [4, p. 43f]. Lastly a proof of the equality of 1) and 3) can be found in [3, p. 37-40]. $\square$

Example. Consider the following graph G.

Then G is 3-connected and 3-regular. Moreover G is isomorphic to H. An isomorphism between them is given by mapping the same colored nodes to each other. The graph quotient $G/\{e, f, g\}$ is given by J.



Definition. Let G, H be two graphs. A map $f : V(G) \to V(H)$ is said to be a graph isomorphism if f is a bijection such that

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H).$$

Let F_n denote the free group of rank n.

Definition. An *admissible graph of rank n* is a 2-connected loop less graph G with fundamental group isomorphic to F_n and with vertex-valency ≥ 2.

We often just write admissible graph for an admissible graph of rank n.

Definition. Let G be an admissible graph. Its *degree* is given by

$$\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).$$

In particular G has $2n - 2 - \deg(G)$ vertices and $3n - 3 - \deg(G)$ edges and is 3-regular iff $\deg(G) = 0$.

Definition. An orientation σ of a graph G is an ordering of the edges i.e. σ is an injective function from $E(G)$ to $\{1, \ldots, |E(G)|\}$. We call the tuple (G, σ) an oriented graph and note that Sym acts on (G, σ) by $\pi(G, \sigma) = (G, \pi\sigma)$ for $\pi \in$ Sym.

If it is clear that (G, σ) is an oriented graph we often just write G.

Definition. A *forested graph* is a pair (G, Φ) where G is an admissible graph and Φ is an oriented forest which contains all vertices of G.

All vertices cant be either because then boundary doesnt work?

A map f between two forested graphs $(G, \Phi), (H, \Psi)$ is said to be a forested graph isomorphism if f is a graph isomorphism and $\sigma \circ f\mid_\Phi$ is the identity, where σ is the orientation on Φ.

Is this correct?

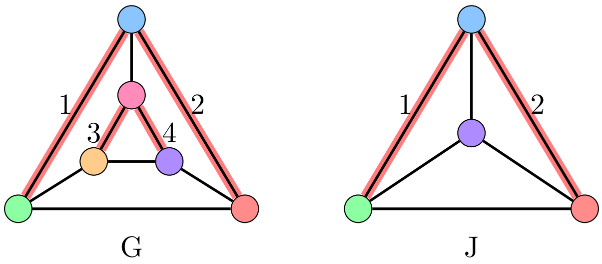
For $k \in \mathbb{N}$ let C_k denote the $\mathbb{Q}$-vector space spanned by isomorphism classes of forested graphs of rank n with a forest of size k, subject to the relation

$$(G, \pi\Phi) = \operatorname{sgn}\pi \cdot (G, \Phi) \quad \text{for all} \quad \pi \in \mathrm{Sym}(k).$$

Observe that if $(G, \Phi) \simeq (G, \pi\Phi)$ for an odd permutation π then $(G, \Phi) \simeq (G, \pi\Phi) = -(G, \Phi)$ and thus $(G, \Phi) = 0$ in C_k.

If C_k is only of rank n how can the boundary map be well defined when it reduces edge number by 1 thus reducing rank by 1

Example. Consider the graphs G, J as in Example 0.1. Then G and J are admissible graphs of rank 3 and 2. Thus if we equip them with oriented forests Φ, Ψ as below (where the red edges represent the forest and the numbers the orientation) we get forested graphs.



Observe, that $(J, \Psi) = 0$ in C_k, since (12) is an odd permutation and $(12)(J, \Psi)$ is isomorphic to (J, Ψ) via the isomorphism mirroring along the vertical passing through the blue and purple vertex.

(G, Φ) however is not trivial which can be seen as follows :

show that

To turn these spaces into a chain complex we define a differential as follows:

Definition. Let $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ be a forested graph. Then let

$$\partial_C(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}),$$
$$\partial_R(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G, \Phi \setminus \{e_i\})$$

where if $\sigma : E(\Phi) \to \{1, \dots, p\}$ is the orientation on Φ the orientation $\tau : E(\Phi \setminus \{e_i\}) \to \{1, \dots, p-1\}$ on $\Phi \setminus \{e_i\}$ is given by

$$\tau(e) = \begin{cases} \sigma(e) & \text{if } \sigma(e) < i \\ \sigma(e) - 1 & \text{if } \sigma(e) > i \end{cases}.$$

Notice that the case $\sigma(e) = i$ can't happen as e_i is not contained in $\Phi \setminus \{e_i\}$. Finally define the boundary map $\partial = \partial_C - \partial_R$.

Is this renumbering correct or do we have to move e_i first to the end of the orientation and then remove it inducing a sign of -1^{i-1}

Proposition. *∂ is well-defined and $\partial^2 = 0$.*

Proof. Let $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ be a forested graph and denote the edges in $\Phi \setminus \{e_i\}$ by $\{e'_1, \dots, e'_{p-1}\}$. Firstly we observe that

$$(G/e_i)/e'_j = \begin{cases} (G/e_j)/e'_{i-1} & \text{if } i > j \\ (G/e_{j+1})/e'_i & \text{if } i \leq j \end{cases} \quad \text{as well as} \quad (\Phi \setminus \{e_i\}) \setminus \{e'_j\} = \begin{cases} (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\} & \text{if } i > j \\ (\Phi \setminus \{e_{j+i}\}) \setminus \{e'_i\} & \text{if } i \leq j \end{cases}$$

Now we compute:

$$\begin{aligned} \partial_C^2 &= \partial_C \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}) = \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \end{aligned}$$

We claim that the right and left sum cancel. For this first apply the observations above to the left sum and then change variables by setting $l = j$ and $m = i - 1$ to obtain:

$$\begin{aligned} \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) &= \sum_{j<i} (-1)^{i+j} ((G/e_j)/e'_{i-1}, (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\}) \\ &= \sum_{l \leq m} (-1)^{l+m+1} ((G/e_l)/e'_m, (\Phi \setminus \{e_l\}) \setminus \{e'_m\}) \end{aligned}$$

This last expression is the same as the left sum above but with opposite sign. Thus they cancel and we have shown $\partial_C^2 = 0$. The same argument shows that $\partial_R^2 = 0$.

For the mixed terms we compute as follows

$$\begin{aligned} \partial_R \partial_C &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \qquad (*) \end{aligned}$$

and

$$
\begin{aligned}
\partial_C \partial_R &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} (G/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\
&= \sum_{j<i} (-1)^{i+j} (G/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \le j} (-1)^{i+j} (G/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\
&\overset{(\heartsuit)}{=} \sum_{j<i} (-1)^{i+j} (G/e_j, (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\}) + \sum_{i \le j} (-1)^{i+j} (G/e_{j+1}, (\Phi \setminus \{e_{j+1}\}) \setminus \{e'_i\}) \\
&\overset{(\dagger)}{=} \sum_{l \le m} (-1)^{m+l+1} (G/e_l, (\Phi \setminus \{e_l\}) \setminus \{e'_m\}) + \sum_{k<n} (-1)^{n+k-1} (G/e_k, (\Phi \setminus \{e_k\}) \setminus \{e'_n\}) \quad (**)
\end{aligned}
$$

Where in $(\heartsuit)$ we used that if $j < i$ then $e_j = e'_j$ and if $i \le j$ then $e_{j+1} = e'_j$, as well as the observation above. In $(\dagger)$ we used the substitution $m = i - 1$, $l = j$ on the left and $m = i$, $k = j + 1$ on the right sum. Comparing the sums in $(*)$ and $(**)$ we see that they differ by a sign and thus cancel. Hence $\partial_C \partial_R - \partial_R \partial_C = 0$.

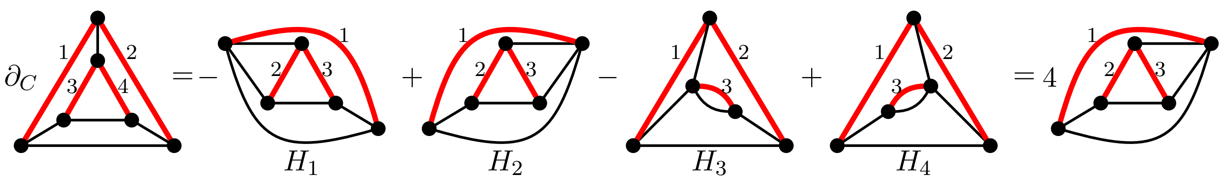
Combining the above we get:

$$
\partial^2 = (\partial_C - \partial_R)^2 = \partial_C^2 - (\partial_C \partial_R + \partial_R \partial_C) + \partial_R^2 = 0
$$

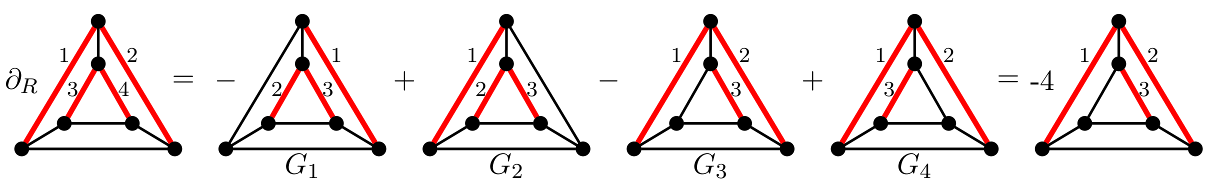
□

Thus the spaces $(C_\bullet)$ with the differential $\partial_\bullet$ form a chain complex.

Example. Once again we consider the graph (G, Φ) from above and calculate its boundary operator:



Where we used that $-H_1$ is equal to H_2 by mirroring along the vertical and applying (23), $-H_3$ is equal to H_2 by exchanging inner and outer vertices, mirroring along the vertical and applying (13) and H_4 is equal to H_2 by exchanging inner and outer vertices and applying (13)(23). Where we have used the permutation (23) on the first, (13) on the third, (13)(23) on the fourth graph to get the result.



Where we have used that $-G_1$ is equal to $-G_3$ by exchanging inner and outer vertices and applying (13)(12), G_2 is equal to $-G_3$ by exchanging, mirroring along the vertical and applying (13) and G_4 is equal to $-G_3$ by mirroring along the vertical and applying (12).

Thus we can conclude that $\partial_\bullet G = 4H_2 - 4G_3$. Moreover we have that $H_2 - G_3 \in \operatorname{Im} \partial_\bullet$ and as $\partial_\bullet^2 = 0$ also $H_2 - G_3 \in \operatorname{Ker} \partial_\bullet$

Bibliography

[1] Laurent Bartholdi. *The rational homology of the outer automorphism group of F_7*. 2016. arXiv: `1512.03075 [math.GR]`.

[2] James Conant and Karen Vogtmann. “On a theorem of Kontsevich”. In: *Algebraic & Geometric Topology* 3.2 (Dec. 2003), pp. 1167–1224.

[3] Frank Harary. *Graph Theory*. Reading Mass., Menlo Park Calif.: Addison-Wesley, 1969.

[4] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000.

0.2 Draftparts

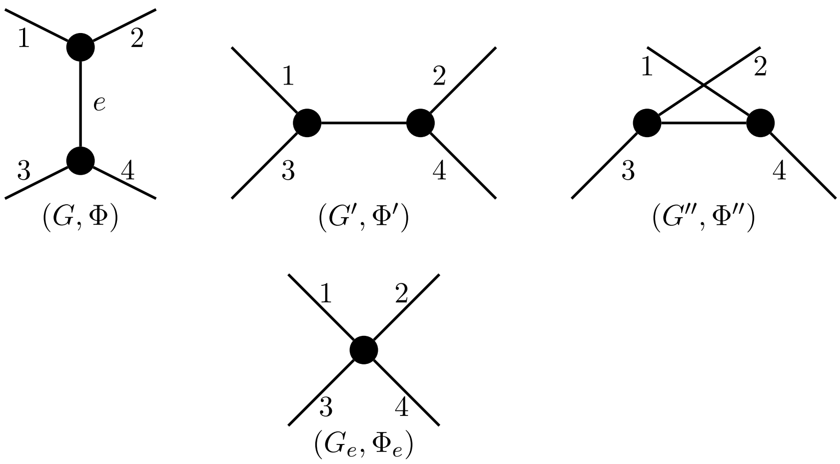
Definition. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$ and the set of half edges by $H(G)$. We call a graph *connected* if the CW complex is connected in the topological sense. A graph is said to be *n-valent* if every vertex has valency n i.e. for every vertex the number of edges incident is n.

Lastly a *tree* is a graph which contains no loops and a *forest* is a collection of disjoint trees.

On trivalent connected graphs we call an *orientation* a choice of cyclic orders of all vertices up to an even number of changes.

Definition. A *forested graph* is a pair (G, Φ), where G is a finite connected trivalent graph and Φ is an oriented forest which contains all vertices of G.

Definition. Let (G, Φ) be a forested graph and let $e \in \Phi$. Moreover let (G_e, Φ_e) be the graph where e has been collapsed. Then there exist exactly two other graphs whose edge collapse results in (G_e, Φ_e). This is visualised in the figure below. Where $1, 2, 3, 4$ represent the rest of the graph.



Now the vector

$$(G, \Phi) + (G', \Phi') + (G'', \Phi'')$$

is called the *basic IHX relator* associated to (G, Φ, e).

Denote by $\widehat{f\mathcal{G}_k}$ the vector space spanned by all forested graphs containing k trees modulo the relations $(G, \Phi) = -(G, -\Phi)$. Moreover let $f\mathcal{G}_k$ be the quotient of $\widehat{f\mathcal{G}_k}$ modulo the subspace spanned by all basic IHX relators.

Definition. Let $\widehat{\partial}_E(G, \Phi) : \widehat{f\mathcal{G}_k} \to \widehat{f\mathcal{G}_{k-1}}$ be given by

$$\widehat{\partial}_E(G, \Phi) = \sum (G, \Phi \cup e).$$

where the sum is over all edges e of $G \setminus \Phi$ such that $\Phi \cup e$ is still a forest. Notice that this only happens if the two vertices of e lie in different trees of Φ. Thus $\Phi \cup e$ has $k - 1$ components. The orientation of $\Phi \cup e$ is determined by ordering the edges of Φ with labels $1, \ldots, k$ consistent with its orientation and then labeling the new edge e with $k + 1$.

Now let the boundary map $\partial_E : f\mathcal{G}_k \to f\mathcal{G}_{k-1}$ be the map induced by $p \circ \widehat{\partial}_E$ where p is the quotient map $\widehat{f\mathcal{G}_k} \to f\mathcal{G}_k$.

Proposition. *∂_E is well-defined and $\partial_E^2 = 0$.*

Proof. ∎

The *forested graph complex* is thus defined as the sequence $f\mathcal{G}_k$ with boundary map ∂_E and is well-defined by the above proposition.