Thesis Title

Thesis Subtitle

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1 Introduction

As groups are a fundamental concept of mathematics understanding them has been a central aspect ever since their discovery. Free groups are then the class of groups where no relation exists between their generators: The elements are words of generators and their inverses. The group relation is given by concatenating words and reducing them i.e. if two adjacent elements are inverse to each other they will be omitted. The free group on n generators will be denoted by F_n. The importance of free groups now arises from the fact that every group is isomorphic to a quotient group of a free group an in particular any finitely generated group is isomorphic to a quotient group of F_n for some n.

Another way to approach a group G is to understand its automorphism group $\mathrm{Aut}(G)$ as it describes G's symmetries. Now $\mathrm{Aut}(G)$ can be separated into the inner automorphism group $\mathrm{Inn}(G)$, the group of automorphisms that arise from conjugation, and the outer automorphism group $\mathrm{Out}(G)$, the quotient of $\mathrm{Aut}(G)$ by $\mathrm{Inn}(G)$.

Combining those two aspects it is only natural that mathematicians study the automorphism group of F_n since the beginning of the subject. The inner automorphism group is well understood and isomorphic to F_n itself. The outer automorphism group $\mathrm{Out}(F_n)$ however still leaves many open questions. As $F_1 \cong \mathbb{Z}$,

$$\mathrm{Out}(F_1) = \mathrm{Out}(\mathbb{Z}) = \mathrm{GL}_1(\mathbb{Z}).$$

For F_2 Nielsen proved in [6] that $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$. For higher groups only the existence of a surjection $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ can be guaranteed.

Historically $\mathrm{GL}_n(\mathbb{Z})$ has been studied by by its action on the symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. Due to the relation between $\mathrm{Out}(F_n)$ and $\mathrm{GL}_n(\mathbb{Z})$ early attempts of the study of the outer automorphism group examined its action on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ induced by the above surjection. However this action is not proper meaning that inverse of compact sets under the action are not necessarily compact. Hence it behaves quite badly and a different approach was needed.

Therefore Culler and Vogtmann introduced a new space $\mathcal{X}_n$ known as "Outer space" in [7]. To understand this space however we first have to introduce basic concepts of graph theory.

1.1 Graphs

Definition 1.1. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$. We call an edge having the same start and end vertex a loop.

We call a graph *connected* if the CW complex is connected in the topological sense. A graph is n-edge-connected if it remains connected after removing $n-1$ arbitrary edges.

A graph is said to be *n-regular* if every vertex has valency n i.e. for every vertex the number of incident edges is n. The valency of a vertex $v \in G$ is often also called degree of v and denoted by $\deg(v)$.

For a subset of edges Φ of G we denote by G/Φ the graph quotient, which is the quotient space of the CW complex G over its topological subspace Φ.

Remark 1.2. Note that sometimes these types of graphs are called multigraphs, as they are allowed to have multiple edges between vertices as well as loops. The word graph there normally refers to simple graphs which do not allow multi-edges and loops.

In the context of algebraic topology however multigraphs are needed and thus the word graph here denotes multigraphs.

Definition 1.3. A subgraph G' of a graph G is a subcomplex of the CW-complex G. As a subcomplex is itself a CW-complex of dimension smaller or equal to the original complex, G' is itself a graph.

A cycle in a graph G is a subgraph that is homeomorphic to S^1. A tree is a connected graph containing no cycles. A forest is a collection of disjoint trees.

Theorem 1.4. *Let G be a graph. Then its fundamental group $\pi_1(G)$ is isomorphic to a free group.*

By the theorem it makes sense to define the *rank* of a graph $\mathrm{rank}(G)$ as the rank of its fundamental group.

A proof of this theorem will be given later in ...

Finally a *metric graph* is a finite connected graph with

graph metric, path metric

1.2 Outer space

We can now come back to the study of $\mathrm{Out}(F_n)$ and the aforementioned Outer space.

To define Outer space we first consider the graph R_n given by one vertex and n edges each forming a loop. By Theorem 3.1 we can identify $\pi_1(R_n)$ with the free group F_n by identifying the generators of F_n $x_1, \dots, x_n$ with oriented edges of R_n. Then every reduced word in F_n corresponds to a reduced edge-path loop starting and ending at the base-point of R_n. An automorphism ϕ on F_n is now a homotopy equivalence sending the loop corresponding to x_i to the one identified with $\phi(x_i)$.

Outer space can now be defined as the set of marked metric graphs (g, G) under an equivalence relation. The graphs (g, G) have to satisfy

- G is a finite graph with vertex valency at least 3,
- $g : R_n \to G$ is a homotopy equivalence called the marking of G, and
- each edge has a positive length such that the sum of all edge lengths is one. This turns G into a metric space via the path metric.

The equivalence relation is given by $(g, G) \equiv (g', G')$ if and only if there exists an isometry $h : G \to G'$ such that $g \circ h$ is homotopic to g'.

2 Introduction[1]

Free groups are one of the most basic examples of infinite finitely-generated groups. The automorphism groups of free groups are of particular interest since they are tied to many other areas of mathematics. The automorphism group $\mathrm{Aut}(G)$ of a group G can be separated into the inner automorphism group $\mathrm{Inn}(G)$, the group of automorphisms that arise from conjugation, and the outer automorphism group $\mathrm{Out}(G)$, the quotient of $\mathrm{Aut}(G)$ by $\mathrm{Inn}(G)$.

Most often the inner automorphism groups are well understood. To understand the outer automorphism groups the standard approach has been to study it via its action on some topological space. For the outer automorphism group of the free group on n generators $\mathrm{Out}(F_n)$ the space $\mathcal{X}_n$ known as "Outer space" has been introduced by Culler and Vogtmann in [7]. As the points of the Outer space correspond to finite graphs with fundamental group F_n, the structure of Outer space is connected to such diverse areas as the study of Lie algebras of derivations, degenerations of algebraic varieties, the computation of Feynman integrals, and the statistics of phylogenetic trees.

A central concept of the Outer space is its spine K_n which is an equivariant deformation retract of $\mathcal{X}_n$. By being able to compute the rational homology of $K_n/\mathrm{Out}(F_n)$ one can also compute $\mathrm{Out}(F_n)$. By identifying the spine K_n with a cube complex one arrives at the forested graph complex which has been introduced by Conant and Vogtmann in [3].

The forested graph complex is also the main focus of this paper. We will begin by defining basic concepts of graphs, the Outer space and the forested graph complex as well as show the connection via the cube complex.

Afterwards we will focus on Morita's classes which are an infinite sequence of cocycles representing potentially nontrivial cohomology classes $\mu_k \in H^{4k}(\mathrm{Out}(F_{2k+2})$.

3 Basic Definitions

3.1 Further results on graphs

Theorem 3.1. *Let G be a graph. Then its fundamental group $\pi_1(G)$ is isomorphic to a free group.*

By the theorem it makes sense to define the *rank* of a graph $\mathrm{rank}(G)$ as the rank of its fundamental group.

Proof. The following proof is from [5, p. 43f] Let G be a graph. W.l.o.g. G is connected as else we consider each connected component separately. Then let T be a spanning tree on G i.e. T is a tree containing every vertex of G. Then T is contractible. Now choose for every $e_\alpha \in E \setminus T$ an open neighborhood A_α of $T \cup e_\alpha$ that deformation retracts onto $T \cup e_\alpha$. The intersection of such A_α is T and thus contractible. Moreover as G is connected as a graph A_α and T are path connected. Now the A_α form an open cover of G and as T is simply connected by Van Kampen's Theorem we get that $\pi_1(G) = *_\alpha \pi_1(A_\alpha)$. Finally A_α deformation retracts onto S^1 and thus $\pi_1(A_\alpha) = \mathbb{Z}$. Now there are exactly $|E| - |T|$ many A_α, which as T is a spanning tree results in $\pi_1(G)$ being free on $|E| - |V| + 1$ generators. □

To understand the rank better we will need the following definitions

[1] This section follows closely Vogtmann's survey article [8]

Definition 3.2. For a finite CW-complex X the *Euler Characteristic* is defined as the alternating sum

$$\chi(X) = k_0 - k_1 + k_2 - \dots$$

where k_i denotes the number of cells of dimension i in the complex X.

For graphs we thus get $\chi(G) = k_0 - k_1$, as they are 1-dimensional which is equal to $\chi(G) = |V| - |E|$.

Definition 3.3. Let G be a graph. Then its cycle space is the set of even-degree subgraphs of G. This space forms a vector space over $\mathbb{F}_2$ where the vector addition is given by the symmetric difference of two or more subgraphs. A basis of this space is called cycle basis and two cycles are independent if they are linearly independent in the vector space.

Remark 3.4. The cycle space is equal to the one homology group of G with coefficients in $\mathbb{F}_2$ i.e. $H_1(G, \mathbb{F}_2)$.

The following proposition

Proposition 3.5. *Let G be a connected graph. Then the following are equal:*

1. *The rank of G.*
2. *The number of independent cycles in G i.e. the size of the cycle basis of G.*
3. *The first Betti number i.e. the rank of $H_1(G)$.*
4. $1 - \chi(G) = |E| - |V| + 1$.

For the proof we will need the following Lemma:

Lemma 3.6. *Let A be a set. Then the abelianization of the free group on A is isomorphic the free abelian group of A.*

Proof. Consider the space $X = \bigvee_{a \in A} S^1$. Then by Van Kampen's Theorem $\pi_1(X) = *_{a \in A}\mathbb{Z}$. On the other hand we have $H_1(X) = \bigoplus_{a \in A} H_1(S_1) = \bigoplus_{a \in A} \mathbb{Z}$ which follows from the relative Homeomorphism Theorem. Now using Hurewicz Theorem we get that the abelianization of $\pi_1(X)$ is isomorphic to $H_1(X)$ and thus the desired statement $\square$

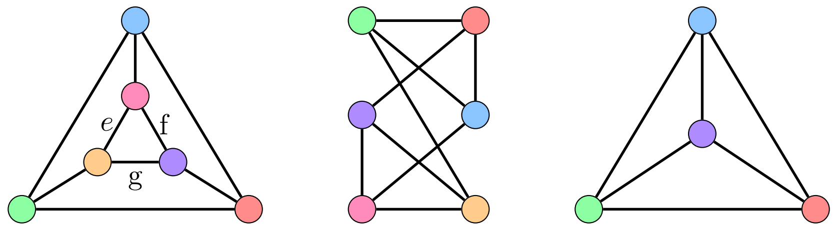
Proof of the Theorem. (1) = (4): This was shown in the proof of Theorem 3.1.
(1) = (3): From Hurewicz Theorem we get that the abelianization of $\pi_1(G)$ is equal to $H_1(G)$ and thus by the previous Lemma that the rank of $\pi_1(G)$ is equal to the rank of $H_1(G)$ which is the first Betti number.
(4) = (2)[2]: Consider again the sets A_α from the proof of Theorem 3.1. Then each of them deformation retracts onto a cycle in G. Let $Z(T)$ be the set of cycles obtained in this way. Then $Z(T)$ is independent as each cycle contains an edge not contained in any other cycle. Moreover every cycle Z in G can be written as the symmetric difference over the cycles corresponding to the edges in $(E \setminus T) \cap Z$. Thus $Z(T)$ spans the cycle space and consequently is a cycle basis. Now the size of $Z(T)$ is given by $|E| - |V| + 1$ and thus we conclude. $\square$

Definition 3.7. Let G, H be two graphs. A map $f : V(G) \to V(H)$ is said to be a graph isomorphism if f is a bijection such that

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H).$$

Example 3.8. Consider the following graphs.

[2]This proof is based on Harary's proof in [4, p. 37-40].



G H J

Then G is 3-connected and 3-regular. Moreover G is isomorphic to H. An isomorphism between them is given by mapping the same colored nodes to each other. The graph quotient $G/\{e, f, g\}$ is given by J.

Finally we introduce the notion of degree of a graph also sometimes called excess.

Definition 3.9. Let G be a connected graph of rank n with vertex-valency ≥ 3. Its *degree* is defined by

$$\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).$$

Proposition 3.10. *Let $G = (V, E)$ be a graph of rank n. Then we have the following identities:*

1. $\deg(G) = 2|E| - 3|V|$
2. $|V| = 2n - 2 - \deg(G)$
3. $|E| = 3n - 3 - \deg(G)$
4. *G is 3-regular* $\Leftrightarrow \deg(G) = 0$.

Proof. By counting half edges we get $2|E| = \sum_{v \in V} \deg(V)$. Combining this with the definition of degree we directly get the first identity. Using Proposition 3.5 and the first identity we have

$$2n - 2 - \deg(G) = 2|E| - 2|V| + 2 - 2 - 2|E| + 3|V| = |V|$$
$$3n - 3 - \deg(G) = 3|E| - 3|V| + 3 - 3 - 2|E| + 3|V| = |E|$$

which proves the second and third. The last statement follows as every element in the sum of the degree is non-negative. Thus $\deg(G) = 0$ if and only if every term is 0 and thus if and only if every vertex has valency 3. □

3.2 The Outer space

We closely follow Vogtmann's definition from [8, p. 2 ff.]

Definition 3.11. By a *metric graph* we mean a finite connected graph with positive real edge lengths, equipped with the path metric. We fix a model rose R_n (a graph with one vertex and n petals), and identify the petals of R_n with the generators of the free group F_n. A point in $\mathcal{X}_n$ is then a metric graph G together with a homotopy equivalence $g : R_n \to G$ called a *marking*; the marking serves to identify the fundamental group of G with F_n. Marked graphs (g, G) and (g', G') are considered the same if there is an isometry $f : G \to G'$ with $f \circ g$ homotopic to g'.

To get a finite dimensional space we assume G has no uni- and bivalent vertices (see Theorem 3.17). Moreover we normalize our objects i.e. we assume the sum of edge lengths to be 1 and assume that G is 2-connected.

To make $\mathcal{X}_n$ a space we need to define a topology. We proceed as follows: For every marked graph (g, G) we define the open simplex $\sigma(g, G)$ as the set obtained by varying the edge lengths of G, keeping their sum equal to 1. The simplex $\sigma(g', G')$ is then a *face* of $\sigma(g, G)$ if (g', G') can be obtained from (g, G) by collapsing some edges to points.

Finally $\mathcal{X}_n$ is the quotient space obtained from the disjoint union of the open simplices $\sigma(g, G)$ by face identification.

However not all faces of these simplices are in $\mathcal{X}_n$. To rectify this we replace each open simplex $\sigma(g, G)$ by a closed simplex $\overline{\sigma}(g, G)$ and take the quotient as before. This new space, denoted by $\mathcal{X}_n^*$, is a simplical complex and called the *simplical closure* of Outer space. The points in $\mathcal{X}_n^*$ which are not in $\mathcal{X}_n$ are said to be at infinity.

Now the group $\mathrm{Out}(F_n)$ acts on $\mathcal{X}_n$ by changing the marking in particular any $\varphi \in \mathrm{Out}(F_n)$ can be realized by a homotopy equivalence $f : R_n \to R_n$ by mapping petals to each other according to their identification with generators of F_n. The group action by φ on (g, G) is then defined by

$$(g, G)\varphi = (g \circ f, G).$$

Finally $\mathcal{X}_n$ contains an equivariant deformation retract K_n, the spine of Outer space. It is a subcomplex of the barycentric subdivision of the simplical closure $\mathcal{X}_n^*$, consisting of simplices spanned by vertices which are not at infinity.

In other language, K_n is the geometric realisation of the partially ordered set of open simplices $\sigma(g, G)$ in $\mathcal{X}_n$, where the partial order is given by the face relation.

We have the following vital result providing the relationship between $\mathcal{X}_n$ and $\mathrm{Out}(F_n)$. This was proved by Culler and Vogtmann in [7].

Theorem 3.12. *$\mathcal{X}_n$ is contractible and the action of $\mathrm{Out}(F_n)$ is proper. The spine K_n is an equivariant deformation retract of dimension $2n - 3$ with compact quotient*

3.3 Forested graph Complex

The forested graph complex has first been introduced by Conant and Vogtmann in [3]. We however follow the simplified construction and definition of the forested graph complex given by Bartholdi in [2]. Very useful in the general understanding of what a graph complex is and how the boundary map acts was Bar-Natan's and McKay's draft [1]. Inspired by this similar examples for the forested graph complex are presented.

Let F_n denote the free group of rank n. We also denote by $\mathbb{S}_n$ the symmetric group of degree n.

Definition 3.13. An *admissible graph of rank n* is a 2-edge-connected graph G with fundamental group isomorphic to F_n and with vertex-valency ≥ 3.

We often just write admissible graph for an admissible graph of rank n.

Definition 3.14. Let $G = (V, E)$ be a graph. An ordering on its edges is a bijective function σ from E to $\{1, \ldots, |E(G)|\}$. Notice that $\mathbb{S}_{|E|}$ acts on σ by $\pi \circ \sigma$ for $\pi \in \mathbb{S}_n$. We call the tuple (G, σ) an ordered graph and note that $\mathbb{S}_{|E|}$ acts on (G, σ) by $\pi(G, \sigma) = (G, \pi\sigma)$ for $\pi \in \mathbb{S}_{|E|}$.

A *forested graph* is a triple (G, Φ, σ) where G is an admissible graph Φ is a subset of edges that spans a forest on G and σ is an ordering on Φ i.e. $\sigma : \Phi \to \{1, \ldots, |\Phi|\}$.

A map f between two forested graphs $(G, \Phi, \sigma) \to (H, \Psi, \tau)$ is said to be a forested graph isomorphism if f is a graph isomorphism on G, $f(\Phi) = \Psi$ and $\sigma = \tau \circ f$

We know want to construct our graph complex. For this we remember the notion of a graded vector space:

Definition 3.15. A graded vector space, is a vector space V and with a decomposition $(V_n)_{n=0}^{\infty}$ such that

$$V = \bigoplus_{k=0}^{\infty} V_k.$$

We know consider the $\mathbb{Q}$-vector space C spanned by isomorphism classes of forested graphs, subject to the relation

$$(G, \Phi, \pi \circ \sigma) = \operatorname{sgn} \pi \cdot (G, \Phi, \sigma) \quad \text{for all} \quad \pi \in \mathbb{S}_n(|\Phi|).$$

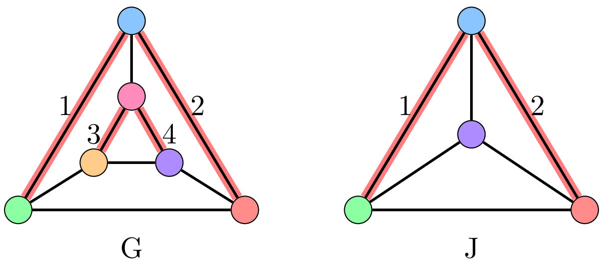
Under this relation we call σ an orientation. Observe that if $(G, \Phi, \sigma) \simeq (G, \Phi, \pi \circ \sigma)$ for an odd permutation π then $(G, \Phi, \sigma) \simeq (G, \Phi, \pi \circ \sigma) = -(G, \Phi, \sigma)$ and thus $(G, \Phi, \sigma) = 0$ in C_k.

We can define the following three gradings on C:

- Let $C^n \subseteq C$ be the subspace spanned by forested graphs of rank n. Then clearly $C^n \cap C^m = \varnothing$ for $n \neq m$ and as every graph has a rank, we get that the C^n form a grading on C.
- Let $C_k \subseteq C$ be the subspace spanned by forested graphs (G, Φ, σ) with $k = |\Phi|$. Clearly this also yields a decomposition of C into a direct sum and thus yields another grading on C.
- Let $C_d \subseteq C$ be the subspace spanned by forested graphs of degree d. Once again this yields a grading on C.

In the following we will mostly be concerned with the first two gradings. In particular we will consider the double-grading C_k^n, where k denotes the forest size and n the rank.

Example 3.16. Consider the graphs G, J as in Example 3.8. Then G and J are admissible graphs of rank 4 and 3. Thus if we equip them with ordered forests $(\Phi, \sigma), (\Psi, \tau)$ as below (where the red edges represent the forest and the numbers the orientation) we get forested graphs in C_4^4 and C_2^3 respectively.



Observe, that $(J, \Psi) = 0$ in C_2^3, since (12) is an odd permutation and $(12)(J, \Psi)$ is isomorphic to (J, Ψ) via the isomorphism mirroring along the vertical passing through the blue and purple vertex.

(G, Φ) however is not trivial as the automorphism group is given by the identity, mirroring along the vertical, exchanging inner and outer vertices and their composition. None of these automorphisms induce an odd permutation and hence (G, Φ) does not vanish.

Before we construct the chain complex we show that the C^n are finitely generated and thus so are the C_k^n.

Theorem 3.17. *For all n, C^n is finitely generated and $C_k^n = 0$ for all $k > 2n - 3$.*

To prove this theorem we first need the following lemma:

Lemma 3.18. *For $n, m \in \mathbb{N}$ There are only finitely many admissible graphs $G = (V, E)$ with $|V| \leq n$ vertices and $|E| \leq m$ edges.*

Proof. Every graph on n vertices can be written as incidence matrix and each entry is $\leq m$ if the graph has maximally m edges. Thus there are maximally m^{n^2} many different incidence matrices for graphs with n vertices and maximally m edges. As every graph corresponds to an incidence matrix this also gives an upper bound on the number of different graphs with n vertices.

Thus the maximal possible number of admissible graphs with $\leq n$ vertices and $\leq m$ edges is bounded by

$$\sum_{k=1}^{n} m^{k^2}$$

which is finite. $\square$

Proof of Theorem 3.17. By counting half edges we have

$$2|E| = \sum_{v \in V} \deg v.$$

Using that admissible graphs have vertex-valency ≥ 3 and rearranging yields $|E| \geq \frac{3}{2}|V|$. From Proposition 3.5 we get that $|E| = |V| + n - 1$. Combining yields

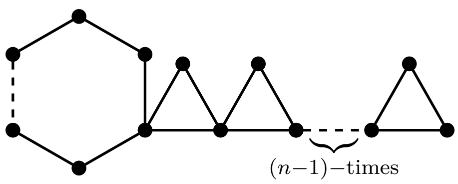
$$|V| + n - 1 \geq \frac{3}{2}|V| \Leftrightarrow 2(n-1) \geq |V|$$

and plugging in the above in the identity from the Proposition yields $|E| \leq 3(n-1)$.

Thus by the above lemma we have that there are only finitely many graphs with $|V| \leq 2(n-1)$ and $|E| \leq 3(n-1)$. As each graph also only has a finite number of forests and each of them has a finite number of orientations we get that C^n is finitely generated.

That $C^n_k = 0 \; \forall k > 2n - 3$ follows from the bound on the number of vertices and the fact that a forest in a graph has maximally $|V| - 1$ edges. $\square$

Remark 3.19. Notice that the constraint of vertex-valency ≥ 3 in the definition of admissible graphs is necessary for C^n being finitely generated. As else we can consider the following family of graphs:



They all have rank n (which can be checked via the Euler characteristic), are 2-edge-connected and not isomorphic as they have different number of vertices/edges.

Remark 3.20. The bound on the C^n_k can not be improved as the graph J from the example above with the tree extended by the edge between purple and green has rank 3 and tree size 3 which equals $2 \cdot 3 - 3$.

To construct our chain complex we fix the rank n and define a differential as follows:

Definition 3.21. Let $(G, \Phi, \sigma) = (G, \{e_1, \ldots, e_k\}, \sigma)$ be a forested graph. Then let $\partial_C, \partial_R : C^n_k \to C^n_{k-1}$ be given by

$$\partial_C(G, \Phi, \sigma) = \sum_{i=1}^{k} (-1)^i (G/e_i, \Phi \setminus \{e_i\}, \sigma_{e_i}),$$

$$\partial_R(G, \Phi, \sigma) = \sum_{i=1}^{k} (-1)^i (G, \Phi \setminus \{e_i\}, \sigma_{e_i})$$

where $\sigma_{e_i} : \Phi \setminus \{e_i\} \to \{1, \ldots, k-1\}$ is given by

$$\sigma_{e_i}(e) = \begin{cases} \sigma(e) & \text{if } \sigma(e) < i \\ \sigma(e) - 1 & \text{if } \sigma(e) > i \end{cases}.$$

Notice that the case $\sigma(e) = i$ cannot happen as e_i is not contained in $\Phi \setminus \{e_i\}$. Finally define the boundary map $\partial = \partial_C - \partial_R$.

Proposition 3.22. *∂ is well-defined and $\partial^2 = 0$.*

For better readability we will omit the orientation σ in the proof.

Proof. We proof the result in three steps:
Step 1: Contracting an edge of a graph does not change the Euler characteristic as both the vertex number and the edge number decreases by one. Thus ∂_C preserves the rank of the graph. Moreover the vertex-valency stays ≥ 3 and the graph continues to be 2-edge-connected. Hence it is admissible. Moreover ∂_C as well as ∂_R remove one edge from each forest. Thus decreasing k by 1. Hence both maps are well-defined from C_k^n to C_{k-1}^n and thus so is ∂.

Let $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ be a forested graph and denote the edges in $\Phi \setminus \{e_i\}$ by $\{e'_1, \dots, e'_{p-1}\}$. For the consecutive steps we need the following observations:

$$(G/e_i)/e'_j = \begin{cases} (G/e_j)/e'_{i-1} & \text{if } i > j \\ (G/e_{j+1})/e'_i & \text{if } i \leq j \end{cases} \quad \text{and} \quad (\Phi \setminus \{e_i\}) \setminus \{e'_j\} = \begin{cases} (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\} & \text{if } i > j \\ (\Phi \setminus \{e_{j+i}\}) \setminus \{e'_i\} & \text{if } i \leq j \end{cases}$$

Step 2: *Claim:* $\partial_C^2 = 0$ and $\partial_R^2 = 0$
We compute:

$$\begin{aligned} \partial_C^2 = \partial_C \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}) &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \end{aligned}$$

We claim that the right and left sum cancel. For this first apply the observations above to the left sum and then change variables by setting $l = j$ and $m = i - 1$ to obtain:

$$\begin{aligned} \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) &= \sum_{j<i} (-1)^{i+j} ((G/e_j)/e'_{i-1}, (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\}) \\ &= \sum_{l \leq m} (-1)^{l+m+1} ((G/e_l)/e'_m, (\Phi \setminus \{e_l\}) \setminus \{e'_m\}) \end{aligned}$$

This last expression is the same as the left sum above but with opposite sign. Thus they cancel and we have shown $\partial_C^2 = 0$. The same argument shows that $\partial_R^2 = 0$.
Step 3: *Claim:* $\partial_C \partial_R - \partial_R \partial_C = 0$
For the mixed terms we compute as follows

$$\begin{aligned} \partial_R \partial_C &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \qquad (*) \end{aligned}$$

and

$$\begin{aligned}
\partial_C\partial_R &= \sum_{i=1}^{p}\sum_{j=1}^{p-1}(-1)^{i+j}(G/e'_j,(\Phi\setminus\{e_i\})\setminus\{e'_j\}) \\
&= \sum_{j<i}(-1)^{i+j}(G/e'_j,(\Phi\setminus\{e_i\})\setminus\{e'_j\}) + \sum_{i\le j}(-1)^{i+j}(G/e'_j,(\Phi\setminus\{e_i\})\setminus\{e'_j\}) \\
&\overset{(\heartsuit)}{=} \sum_{j<i}(-1)^{i+j}(G/e_j,(\Phi\setminus\{e_j\})\setminus\{e'_{i-1}\}) + \sum_{i\le j}(-1)^{i+j}(G/e_{j+1},(\Phi\setminus\{e_{j+1}\})\setminus\{e'_i\}) \\
&\overset{(\dagger)}{=} \sum_{l\le m}(-1)^{m+l+1}(G/e_l,(\Phi\setminus\{e_l\})\setminus\{e'_m\}) + \sum_{k<n}(-1)^{n+k-1}(G/e_k,(\Phi\setminus\{e_k\})\setminus\{e'_n\}) \quad (**)
\end{aligned}$$

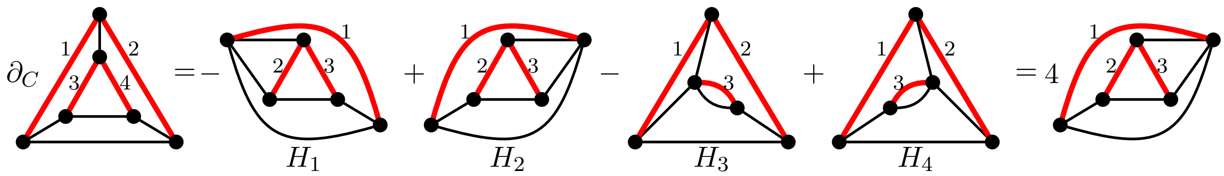
Where in $(\heartsuit)$ we used that if $j<i$ then $e_j=e'_j$ and if $i\le j$ then $e_{j+1}=e'_j$, as well as the observation above. In $(\dagger)$ we used the substitution $m=i-1$, $l=j$ on the left and $m=i$, $k=j+1$ on the right sum. Comparing the sums in $(*)$ and $(**)$ we see that they differ by a sign and thus cancel. Hence $\partial_C\partial_R-\partial_R\partial_C=0$.

Combining Step 2 and 3 we get:

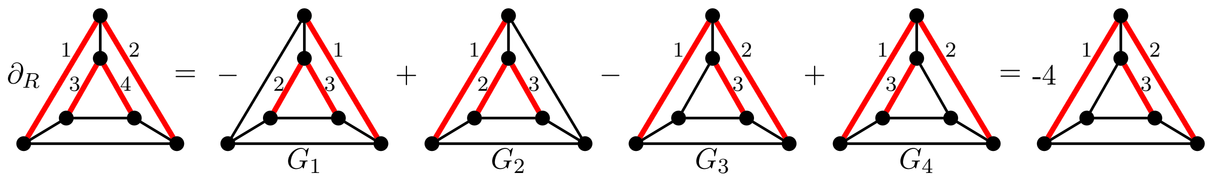
$$\partial^2=(\partial_C-\partial_R)^2=\partial_C^2-(\partial_C\partial_R+\partial_R\partial_C)+\partial_R^2=0 \qquad \square$$

Thus the spaces $(C^n_\bullet)$ with the differential $\partial_\bullet$ form a chain complex.

Example 3.23. Once again we consider the graph (G,Φ) from above and calculate its boundary operator:



Where we used that $-H_1$ is equal to H_2 by mirroring along the vertical and applying (23), $-H_3$ is equal to H_2 by exchanging inner and outer vertices, mirroring along the vertical and applying (13) and H_4 is equal to H_2 by exchanging inner and outer vertices and applying (13)(23). Where we have used the permutation (23) on the first, (13) on the third, (13)(23) on the fourth graph to get the result.



Where we have used that $-G_1$ is equal to $-G_3$ by exchanging inner and outer vertices and applying (13)(12), G_2 is equal to $-G_3$ by exchanging, mirroring along the vertical and applying (13) and G_4 is equal to $-G_3$ by mirroring along the vertical and applying (12).

Thus we can conclude that $\partial_\bullet G=4H_2-4G_3$. Moreover we have that $H_2-G_3\in\operatorname{Im}\partial_\bullet$ and as $\partial_\bullet^2=0$ also $H_2-G_3\in\operatorname{Ker}\partial_\bullet$

3.4 Cubical Chain Complex

The above constructed complex $C_\bullet$ can also be viewed as a cubical chain complex. Here we think of a graph $(G,\Phi)=(G,\{e_1,\ldots,e_k\})\in C_k$ as the k-dimensional $[0,1]$-cube embedded in R^k. The

Graph G_Φ where all edges in Φ have been collapsed sits at the origin and the graph G where all edges have been removed from Φ but not collapsed sits diagonally opposite at $(1, \dots, 1)$. We can assign a graph to every face in the following way: Consider a face F of dimension $n < k$, let bF be its barycenter. Then $bF \in \{0, 1/2, 1\}^k$ and denote by bF_i its i-th coordinate. Moreover let

$$C := \{e_i \in \Phi \mid bF_i = 0\} \quad \text{and} \quad R := \{e_i \in \Phi \mid bF_i = 1\}.$$

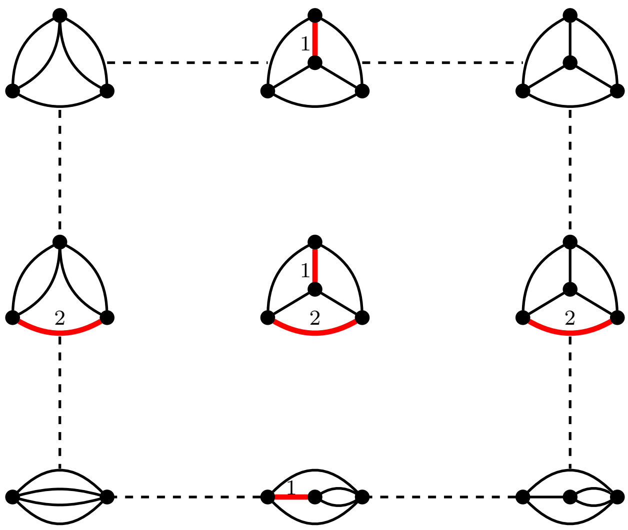
Then the graph associated to F is given by $(G/C, \Phi \setminus (C \cup D)$. Thus an edge gets contracted if $bF_i = 0$ and an edge gets removed from Φ but not contracted if $bF_i = 1$. Now if a face is of dimension n then n coordinates of bF equal $1/2$ and thus the resulting graph has a forest of size n and is hence in C_n.

The above description gives us a bijection between the reduced graphs of G and the half integral points. Denote the reduced graph of G associated to bF by G_{bF}. Then we can define the boundary operator via this bijection as follows:

Formulate this

To visualize this construction we consider the following example:

Example 3.24. Consider again the graph J from example 3.8 with the forest Φ given by an edge between the top and middle vertex and between the left and write vertex. Then its 2-dimensional cube is given as below:



4 Morita-Cycles

Formulate this whole section better and in an easier way

The goal of this section is to show that there exist a cycle in every C_n. For this we define the Morita-Cycle graphs and show that there exists a chain of those graphs that vanishes under the boundary ∂.

Definition 4.1. A general *Morita-Cycle* M_n, for $n \geq 3$ odd, is a forested graph (G, Φ, τ) defined as follows: G has $2 \cdot n$ vertices which form two separate cycles each of order n. More over each vertex of one cycle is connected via an edge to exactly one vertex of the other cycle. The forest Φ are $n - 1$ edges in each of the two cycles. The orientation τ is some numbering of the edges in Φ.

We call one of the cycles the "left" and the other one the "right" cycle. Moreover the missing edge in the cycle from the left/right tree in Φ is called missing left/right edge.

As the cycles have n edges each. The forest Φ has exactly two trees of size $n-1$ of the form of a line. For every even n it can easily be seen that M_n has an odd automorphism (mirroring between the two cycles) and thus vanishes.

Definition 4.2. We say a Morita-Cycle M_n is (drawn) in standard form if the edges in the forest in the left cycle are numbered in ascending order starting with 1 on one edge adjacent to the missing left edge and ending with $n-1$ on the other edge adjacent to the missing left. For the right cycle the edges in Φ are numbered analogous just from n to $2n-1$. Moreover when we draw the graph we always draw the missing left/right to the outside of the cycles.

To make this definition more intuitive it is best to view the illustration below of a Morita cycle M_5 in standard form.

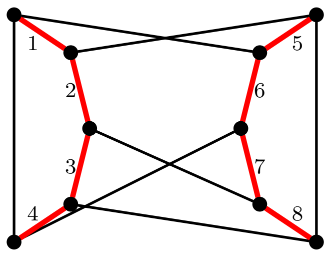
Proposition 4.3. *Every Morita-Cycle M_n in standard form is fully determined by n and a permutation $\sigma \in S_n$. Thus we denote such a Graph by $M_n(\sigma)$. Moreover every general Morita-Cycle M_n can be written in standard form.*

Proof. We show the second part of the statement first. By renumbering the edges in the forest we can always get to the desired ordering. Notice that this might introduce a factor of -1 if the renumbering is an odd permutation. Now by rotating the left/right cycle in our drawing we can also get the missing left/right in its desired place.

Now to the first part of the proposition. By the properties of the standard form the left and right cycle are fully fixed and equal for all Morita-Cycles of size $2n$. Thus the only difference are the edges between the two cycles. For these we number the nodes in the left cycle from 1 to n from top to bottom and on the right from n to $2n$ also from top to bottom. Then each edge can be identified with a pair (u, v). We now define the permutation $\sigma : 1, \ldots, n \to 1, n$ via $\sigma(u) = v - n$.

On the other side if we have a permutation $\sigma \in S_n$. We can define the edges between the left and right cycle by $(i, \sigma(i) + n)$ for $i \in 1, \ldots, n$ and get a Morita-Cycle in standard form $\square$

Example 4.4. Below is a Morita-Cycle of order 5 in standard form defined by the permutation (12)(345).



Theorem 4.5. *For all $n \in \mathbb{N}_{\geq 3}$ odd it holds that*

$$\partial \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_n(\sigma) \right) = 0.$$

We prove this statement in two parts first for ∂_C and then for ∂_R, from which the final result follows. Moreover when we talk about a Morita-Cycle we will always talk about it drawn in its standard form.

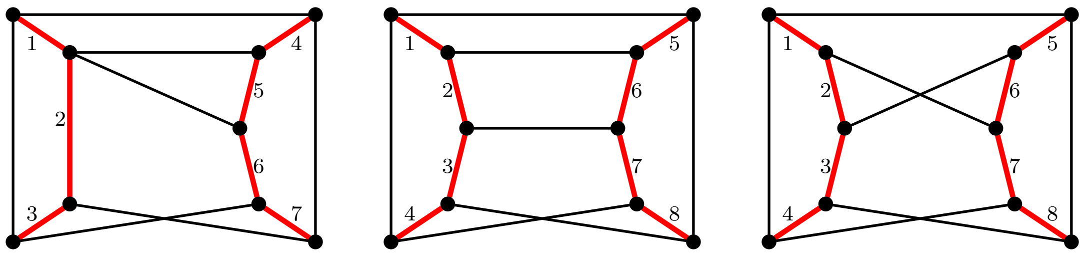
Proof for ∂_C. Let (H, Ψ, η) be in the sum $\partial_C \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_n(\sigma)\right)$. Then as it is an element of the boundary of some Morita-Cycle it has to have precisely one vertex of degree 4, in either the left or right cycle and the same cycle has order $n-1$ and only $n-2$ edges in the tree. W.l.o.g.

we assume that the edge is missing in the left tree as else one can mirror along the horizontal and apply the permutation $(1 \dots 2n-1)^n$ to the orientation.

Now assume that the vertex of degree 4 is the k-th one from the top. An example is shown in the figure below on the left for $k = 2$. Then there exist exactly two Morita-Cycles in standard form in whose boundary H lies. One is obtained by splitting the vertex k into two vertices where each new vertex contains one edge from the forest and one connecting to the right cycle. Moreover the two new nodes get connected by an edge which is also added to the forest and given the number k in the ordering. All other edges with ordering number $> k$ get increased by one. Lets denote this Morita-Cycle by $M_n(\sigma)$. The other one is obtained in the same way however the two edges connecting to the right cycle are permuted i.e. if the new node is numbered by $k+1$ then this Morita-Cycle equals $M_n((k\ k+1)\sigma)$. Examples for both are also shown in the figure below in the middle and right respectively.

Thus we see that the two permutations σ and $(k\ k+1)\sigma$ have opposite parity and thus $M_n(\sigma)$ and $M_n((k\ k+1)\sigma)$ have opposite sign in the sum from the theorem. Thus the elements in their boundary corresponding to H also have opposite sign and hence cancel.

As H was an arbitrary element of the sum we get that every summand has coefficient 0 and the sum vanishes.



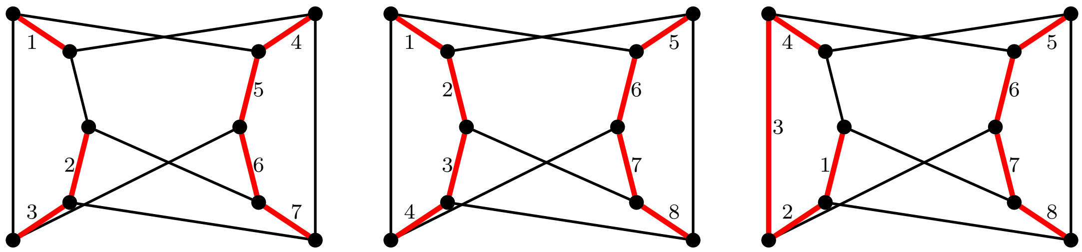
$\square$

Proof for ∂_R. Again let (H, Ψ, η) be in the sum $\partial_R \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_n(\sigma)\right)$. Let σ be the edge assignment between the left and right cycle as described in the standard form. Then H is a Morita-Cycle in standard form missing one edge in either the left or right tree of Ψ. W.l.o.g. we assume that the edge is missing in the left tree as else one can mirror along the horizontal and apply the permutation $(1 \dots 2n-1)^n$ to the orientation.

Now assume the k-th edge from the top is missing from Ψ; an example is shown in the figure below on the left for $k = 2$. Then there are exactly two Morita-Cycles in standard form which have H in their boundary. One where the k-th edge has been added to the forest with number k in the ordering i.e. $M_n(\sigma)$. And the other where the left most edge has been added, the numbering in the left cycle been changed such that 1 is the first edge after the missing one, the left most edge has order $n-k$ and so on. Notice that the second graph is not drawn in standard form. To bring it in standard form one has to "rotate" the left cycle by k nodes counter clockwise. Each of this rotation induces the permutation $(1 \dots n)$ on the edges connecting the left and right cycle i.e. the resulting Morita cycle is given by $M_n(\tau) := M_n(\sigma(1 \dots n)^k)$. Examples for both are also shown in the figure below in the middle and right respectively.

As $(1 \dots n)$ for n odd has even parity τ and σ have the same parity and thus $M_n(\tau)$ and $M_n(\sigma)$ have the same sign in the sum from the theorem. Finally taking the boundary the elements corresponding to H have different orientations. The element in $\partial_R(M_n(\sigma))$ is exactly H with sign $(-1)^k$. The element in $\partial_R(M_n(\tau))$ however differs from H by the permutation $(1 \dots n-2)^{k-1}$ with even parity as $n-2$ is odd and has sign $(-1)^{n-k}$. Now for n odd if k is even $n-k$ is odd and vice versa. Thus the elements corresponding to H have opposite sign and cancel.

As H was an arbitrary element in the sum we get that every summand has coefficient 0 and the sum vanishes.



Thus we have shown the result for both ∂_C and ∂_R and as $\partial = \partial_C - \partial_R$ it also follows for ∂. $\square$

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