1 Introduction

Out Fn. Reason for interest in forested graph complexes

In the following we mainly follow Bartholdi's hands on approach from [1] to define the forested graph complex. The more general definition by Conant and Vogtmann from [2] has been postponed as it is too complex and unintuitive for the beginning. Very useful in the general understanding of what a graph complex is and how the boundary map acts was Bar-Natan's draft []. Inspired by this similar examples for the forested graph complex are presented.

2 Basic Definitions

2.1 Graphs

Definition. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$. We call an edge having the same start and end vertex a loop.

We call a graph *connected* if the CW complex is connected in the topological sense. A graph is n-edge-connected if it remains connected after removing $n-1$ arbitrary edges.

A graph is said to be *n-regular* if every vertex has valency n i.e. for every vertex the number of incident edges is n. The valency of a vertex $v \in G$ is often also called degree of v and denoted by $\deg(v)$.

For a subset of edges Φ of G we denote by G/Φ the graph quotient, which is the quotient space of the CW complex G over its topological subspace Φ.

Remark. Note that in classical graph theory these types of graphs are called multigraphs, as they are allowed to have multiple edges between vertices as well as loops. The word graph there normally refers to simple graphs which don't allow multi-edges and loops.

In the context of algebraic topology however multigraphs are needed and thus the word graph here denotes multigraphs.

Definition. A subgraph G' of a graph G is a subcomplex of the CW-complex G. As a subcomplex is itself a CW-complex of dimension smaller or equal to the original complex, G' is itself a graph.

A cycle in a graph G is a subgraph that is homeomorphic to S^1. A tree is a connected graph containing no cycles. A forest is a collection of disjoint trees.

Theorem. *Let G be a graph. Then its fundamental group $\pi_1(G)$ is isomorphic to a free group.*

By the theorem it makes sense to define the *rank* of a graph $\text{rank}(G)$ as the rank of its fundamental group.

The following proof is from [4, p. 43f]

Proof. Let G be a graph. W.l.o.g. G is connected as else we consider each connected component separately. Then let T be a spanning tree on G i.e. T is a tree containing every vertex of G. Then T is contractible. Now choose for every $e_\alpha \in E \setminus T$ an open neighborhood A_α of $T \cup e_\alpha$ that deformation retracts onto $T \cup e_\alpha$. The intersection of such A_α is T and thus contractible. Moreover as G is connected as a graph A_α and T are path connected. Now the A_α form an open cover of G and as T is simply connected by Van Kampen's Theorem we get that $\pi_1(G) = *_\alpha \pi_1(A_\alpha)$. Finally A_α deformation retracts onto S^1 and thus $\pi_1(A_\alpha) = \mathbb{Z}$. Now there are exactly $|E| - |T|$ many A_α, which as T is a spanning tree results in $\pi_1(G)$ being free on $|E| - |V| + 1$ generators. $\square$

To understand the rank better we will need the following definitions

Definition. For a finite CW-complex X the *Euler Characteristic* is defined as the alternating sum

$$\chi(X) = k_0 - k_1 + k_2 - \dots$$

where k_i denotes the number of cells of dimension i in the complex X.

For graphs we thus get $\chi(G) = k_0 - k_1$, as they are 1-dimensional which is equal to $\chi(G) = |V| - |E|$.

Definition. Let G be a graph. Then its cycle space is the set of even-degree subgraphs of G. This space forms a vector space over $\mathbb{F}_2$ where the vector addition is given by thesymmetric difference of two or more subgraphs. A basis of this space is called cycle basis and two cycles are independent if they are linearly independent in the vector space.

The following proposition

Proposition. *Let G be a connected graph. Then the following are equal:*

1. *The rank of G*
2. *The number of independent cycles in G i.e. the size of the cycle basis of G.*
3. *The first Betti number i.e. the rank of $H_1(G)$*
4. $1 - \chi(G) = |E| - |V| + 1$

For the proof we will need the following Lemma:

Lemma. *Let A be a set. Then the abelianization of the free group on A is isomorphic the free abelian group of A.*

Proof. Consider the space $X = \bigvee_{a \in A} S^1$. Then by Van Kampen's Theorem $\pi_1(X) = *_{a \in A} \mathbb{Z}$. On the other hand we have $H_1(X) = \bigoplus_{a \in A} H_1(S_1) = \bigoplus_{a \in A} \mathbb{Z}$ which follows from the relative Homeomorphism Theorem. Now using Hurewicz Theorem we get that the abelianization of$\pi_1(X)$ is isomorphic to $H_1(X)$ and thus the desired statement $\square$

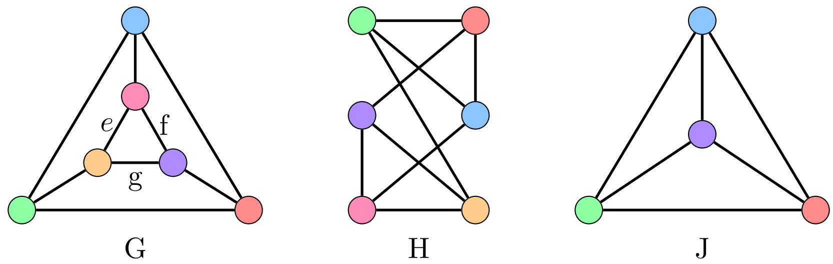
Proof of the Theorem. (1) = (4): This was shown in the proof of Theorem 2.1.
(1) = (3): From Hurewicz Theorem we get that the abelianization of $(\pi_1(G)$ is equal to $H_1(G)$ and thus by the previous Lemma that the rank of $\pi_1(G)$ is equal to the rank of $H_1(G)$ which is the first Betti number.
(4)= (2)[1]: Consider again the sets A_α from the proof of Theorem 2.1. Then each of them deformation retracts onto a cycle in G. Let $Z(T)$ be the set of cycles obtained in this way. Then $Z(T)$ is independent as each cycle contains an edge not contained in any other cycle. Moreover every cycle Z in G can be written as the symmetric difference over the cycles corresponding to the edges in $(E \setminus T) \cap Z$. Thus $Z(T)$ spans the cycle space and consequently is a cycle basis. Now the size of $Z(T)$ is given by $|E| - |V| + 1$ and thus we conclude. $\square$

Definition. Let G, H be two graphs. A map $f : V(G) \to V(H)$ is said to be a graph isomorphism if f is a bijection such that

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H).$$

[1]This proof is based on Harary's proof in [3, p. 37-40].

Example. Consider the following graphs.



Then G is 3-connected and 3-regular. Moreover G is isomorphic to H. An isomorphism between them is given by mapping the same colored nodes to each other. The graph quotient $G/\{e, f, g\}$ is given by J.

Finally we introduce the notion of degree of a graph also sometimes called excess.

Definition. Let G be a connected graph of rank n with vertex-valency ≥ 3. Its *degree* is given by

$$\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).$$

Proposition. *We have the following identities*

1. $\deg(G) = 2|E| - 3|V|$
2. $|V| = 2n - 2 - \deg(G)$
3. $|E| = 3n - 3 - \deg(G)$
4. G *is* 3*-regular* $\Leftrightarrow \deg(G) = 0$.

Proof. A general fact in graph theory is that $2|E| = \sum_{v \in V} \deg(V)$. Combining this with the definition of degree we directly get the first identity. Using Proposition 2.1 and the first identity we have

$$\begin{aligned} 2n - 2 - \deg(G) &= 2|E| - 2|V| + 2 - 2 - 2|E| + 3|V| = |V| \\ 3n - 3 - \deg(G) &= 3|E| - 3|V| + 3 - 3 - 2|E| + 3|V| = |E| \end{aligned}$$

which proves the second and third. The last statement follows as every element in the sum of the degree is positive. Thus $\deg(G) = 0$ if and only if every term is 0 and thus iff $\deg(v) = 0 \; \forall v \in V$. $\square$

2.2 Forested graph Complex

Let F_n denote the free group of rank n.

Definition. An *admissible graph of rank n* is a 2-edge-connected graph G with fundamental group isomorphic to F_n and with vertex-valency ≥ 3.

We often just write admissible graph for an admissible graph of rank n.

Definition. An orientation σ of a graph G is an ordering of the edges i.e. σ is an injective function from $E(G)$ to $\{1, \ldots, |E(G)|\}$. We call the tuple (G, σ) an oriented graph and note that Sym acts on (G, σ) by $\pi(G, \sigma) = (G, \pi\sigma)$ for $\pi \in \mathrm{Sym}$.

Definition. A *forested graph* is a triple (G, Φ, σ) where G is an admissible graph Φ is a subset of edges that spans a forest on G and σ is an orientation on Φ.

A map f between two forested graphs $(G, \Phi, \sigma) \to (H, \Psi, \tau)$ is said to be a forested graph isomorphism if f is a graph isomorphism on G, $f(\Phi) = \Psi$ and $\sigma = \tau \circ f$

We know want to construct our graph complex. For this we remember the notion of a graded vector space:

Definition. A graded vector space, is a vector space V and with a decomposition $(V_n)_{n=0}^{\infty}$ such that

$$V = \bigoplus_{k=0}^{\infty} V_n$$

We know consider the $\mathbb{Q}$-vector space C spanned by isomorphism classes of forested graphs, subject to the relation

$$(G, \pi\Phi) = \operatorname{sgn}\pi \cdot (G, \Phi) \quad \text{for all} \quad \pi \in \operatorname{Sym}(k).$$

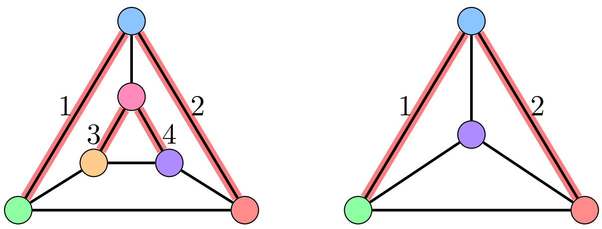
Observe that if $(G, \Phi) \simeq (G, \pi\Phi)$ for an odd permutation π then $(G, \Phi) \simeq (G, \pi\Phi) = -(G, \Phi)$ and thus $(G, \Phi) = 0$ in C_k.

We can define the following three gradings on C:

- Let $C^n \subseteq C$ be the subspace spanned by forested graphs of rank n. Then clearly $C^n \cap C^m = \varnothing$ for $n \neq m$ and as every graph has a rank, we get that the C^n form a grading of C.
- Let $C_k \subseteq C$ be the subspace spanned by forested graphs with a forest of size k. Clearly this also yields a decomposition of C into a direct sum and thus yields another grading of C
- Let $C_d \subseteq C$ be the subspace spanned by forested graphs of degree d. Once again this yields a grading on C

In the following we will mostly be concerned with the first two gradings. In particular we will consider the double-grading C_k^n, where k denotes the forest size and n the rank.

Example. Consider the graphs G, J as in Example 2.1. Then G and J are admissible graphs of rank 3 and 2. Thus if we equip them with oriented forests Φ, Ψ as below (where the red edges represent the forest and the numbers the orientation) we get forested graphs.



G J

Observe, that $(J, \Psi) = 0$ in C_2^3, since (12) is an odd permutation and $(12)(J, \Psi)$ is isomorphic to (J, Ψ) via the isomorphism mirroring along the vertical passing through the blue and purple vertex.

(G, Φ) however is not trivial which can be seen as follows :

show that

Before we construct the chain complex we show that the C^n are finitely generated and thus so are the C_k^n.

Theorem. *For all n, C^n is finitely generated and for $C_k^n = 0\ \forall k > 2n-3$.*

Proof. We have the following general fact

$$2|E| = \sum_{v \in V} \deg v.$$

Using that admissible graphs have vertex-valency ≥ 3 and rearranging yields $|E| \geq \frac{3}{2}|V|$. From Proposition 2.1 we get that $|E| = |V| + n - 1$. Combining yields

$$|V| + n - 1 \geq \frac{3}{2}|V| \Leftrightarrow 2(n-1) \geq |V|$$

and plugging in the above in the identity from the Proposition yields $|E| \leq 3(n-1)$.

With this we can get a loose upper bound on the amount of admissible graphs. As every graph can be written as incidence matrix and each entry is $\leq |E|$ we get that there are maximally $|E|^{|V|^2}$ many different incidence matrices for graphs with $|V|$ vertices. As every graph corresponds to an incidence matrix this also gives an upper bound on the number of different graphs with $|V|$ vertices.

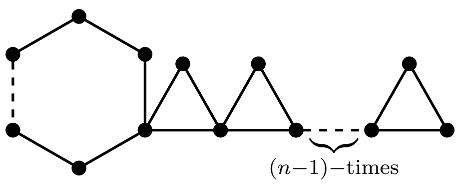
Thus the maximal possible number of admissible graphs of rank n is bounded by

$$\sum_{k=1}^{2(n-1)} (3(n-1))^{k^2}$$

As on each graph there also only exists a finite number of forests and each of them has a finite number of orientations we get that C^n is finitely generated.

That $C^n_k = 0\ \forall k > 2n-3$ follows from the bound on the number of vertices and the fact that a forest in a graph has maximally $|V| - 1$ edges. $\square$

Remark. Notice that the constraint of vertex-valency ≥ 3 in the definition of admissible graphs is necessary for C^n being finitely generated. As else we can consider the following family of graphs:



They all have rank n (which can be checked via the Euler characteristic), are 2-edge-connected and not isomorphic as they have different number of vertices/edges.

Remark. The bound on the C^n_k can not be improved as the graph J from the example above with the tree extended by the edge between purple and green has rank 3 and tree size 3 which equals $2 \cdot 3 - 3$.

To construct our chain complex we fix the rank n and define a differential as follows:

Definition. Let $(G, \Phi, \sigma) = (G, \{e_1, \ldots, e_k\}, \sigma)$ be a forested graph. Then let $\partial_C, \partial_R : C^n_k \to C^n_{k-1}$ be given by

$$\partial_C(G, \Phi) = \sum_{i=1}^{k} (-1)^k (G/e_i, \Phi \setminus \{e_i\}, \sigma_{e_i}),$$

$$\partial_R(G, \Phi) = \sum_{k=1}^{k} (-1)^k (G, \Phi \setminus \{e_i\}, \sigma_{e_i})$$

where $\sigma_{e_i} : \Phi\ \{e\} \to \{1, \ldots, k-1\}$ is given by

$$\sigma_{e_i}(e) = \begin{cases} \sigma(e) & \text{if } \sigma(e) < i \\ \sigma(e) - 1 & \text{if } \sigma(e) > i \end{cases}.$$

Notice that the case $\sigma(e) = i$ can't happen as e_i is not contained in $\Phi \setminus \{e_i\}$. Finally define the boundary map $\partial = \partial_C - \partial_R$.

Proposition. *∂ is well-defined and $\partial^2 = 0$.*

For better readability we will omit the orientation σ in the proof.

Proof. We proof the result in three steps:
Step 1: Contracting an edge of a graph doesn't change the Euler characteristic as both the vertex number and the edge number decreases by one. Thus ∂_C preserves the rank of the graph. Moreover the vertex-valency stays ≥ 3 and the graph continues to be 2-edge-connected. Hence is admissible. Moreover ∂_C as well as ∂_R remove one edge from each graph. Thus decreasing k by 1. Hence both maps are well-defined from C_k^n to C_{k-1}^n and thus so is ∂.

Let $(G, \Phi) = (G, \{e_1, \ldots, e_p\})$ be a forested graph and denote the edges in $\Phi \setminus \{e_i\}$ by $\{e'_1, \ldots, e'_{p-1}\}$. For the consecutive steps we need the following observations:

$$(G/e_i)/e'_j = \begin{cases} (G/e_j)/e'_{i-1} & \text{if } i > j \\ (G/e_{j+1})/e'_i & \text{if } i \leq j \end{cases} \quad \text{as well as} \quad (\Phi \setminus \{e_i\}) \setminus \{e'_j\} = \begin{cases} (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\} & \text{if } i > j \\ (\Phi \setminus \{e_{j+i}\}) \setminus \{e'_i\} & \text{if } i \leq j \end{cases}$$

Step 2: *Claim:* $\partial_C^2 = 0$ and $\partial_R^2 = 0$

We compute:

$$\begin{aligned} \partial_C^2 &= \partial_C \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}) = \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \end{aligned}$$

We claim that the right and left sum cancel. For this first apply the observations above to the left sum and then change variables by setting $l = j$ and $m = i - 1$ to obtain:

$$\begin{aligned} \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) &= \sum_{j<i} (-1)^{i+j} ((G/e_j)/e'_{i-1}, (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\}) \\ &= \sum_{l \leq m} (-1)^{l+m+1} ((G/e_l)/e'_m, (\Phi \setminus \{e_l\}) \setminus \{e'_m\}) \end{aligned}$$

This last expression is the same as the left sum above but with opposite sign. Thus they cancel and we have shown $\partial_C^2 = 0$. The same argument shows that $\partial_R^2 = 0$.

Step 3: *Claim:* $\partial_C \partial_R - \partial_R \partial_C = 0$

For the mixed terms we compute as follows

$$\begin{aligned} \partial_R \partial_C &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \qquad (*) \end{aligned}$$

and

$$
\begin{aligned}
\partial_C\partial_R &= \sum_{i=1}^{p}\sum_{j=1}^{p-1}(-1)^{i+j}(G/e'_j, (\Phi\setminus\{e_i\})\setminus\{e'_j\}) \\
&= \sum_{j<i}(-1)^{i+j}(G/e'_j, (\Phi\setminus\{e_i\})\setminus\{e'_j\}) + \sum_{i\leq j}(-1)^{i+j}(G/e'_j, (\Phi\setminus\{e_i\})\setminus\{e'_j\}) \\
&\overset{(\heartsuit)}{=} \sum_{j<i}(-1)^{i+j}(G/e_j, (\Phi\setminus\{e_j\})\setminus\{e'_{i-1}\}) + \sum_{i\leq j}(-1)^{i+j}(G/e_{j+1}, (\Phi\setminus\{e_{j+1}\})\setminus\{e'_i\}) \\
&\overset{(\dagger)}{=} \sum_{l\leq m}(-1)^{m+l+1}(G/e_l, (\Phi\setminus\{e_l\})\setminus\{e'_m\}) + \sum_{k<n}(-1)^{n+k-1}(G/e_k, (\Phi\setminus\{e_k\})\setminus\{e'_n\}) \quad (**)
\end{aligned}
$$

Where in $(\heartsuit)$ we used that if $j<i$ then $e_j = e'_j$ and if $i\leq j$ then $e_{j+1} = e'_j$, as well as the observation above. In $(\dagger)$ we used the substitution $m = i-1$, $l = j$ on the left and $m = i$, $k = j+1$ on the right sum. Comparing the sums in $(*)$ and $(**)$ we see that they differ by a sign and thus cancel. Hence $\partial_C\partial_R - \partial_R\partial_C = 0$.

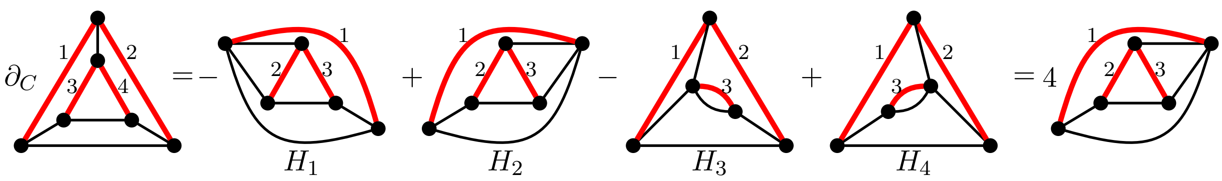
Combining Step 2 and 3 we get:

$$\partial^2 = (\partial_C - \partial_R)^2 = \partial_C^2 - (\partial_C\partial_R + \partial_R\partial_C) + \partial_R^2 = 0$$

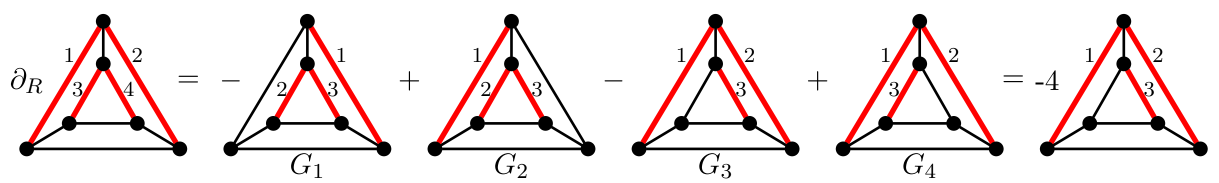
$\square$

Thus the spaces $(C_\bullet)$ with the differential $\partial_\bullet$ form a chain complex.

Example. Once again we consider the graph (G,Φ) from above and calculate its boundary operator:



Where we used that $-H_1$ is equal to H_2 by mirroring along the vertical and applying (23), $-H_3$ is equal to H_2 by exchanging inner and outer vertices, mirroring along the vertical and applying (13) and H_4 is equal to H_2 by exchanging inner and outer vertices and applying (13)(23). Where we have used the permutation (23) on the first, (13) on the third, (13)(23) on the fourth graph to get the result.



Where we have used that $-G_1$ is equal to $-G_3$ by exchanging inner and outer vertices and applying (13)(12), G_2 is equal to $-G_3$ by exchanging, mirroring along the vertical and applying (13) and G_4 is equal to $-G_3$ by mirroring along the vertical and applying (12).

Thus we can conclude that $\partial_\bullet G = 4H_2 - 4G_3$. Moreover we have that $H_2 - G_3 \in \operatorname{Im}\partial_\bullet$ and as $\partial_\bullet^2 = 0$ also $H_2 - G_3 \in \operatorname{Ker}\partial_\bullet$

2.3 Cubical Chain Complex

The above constructed complex $C_\bullet$ can also be viewed as a cubical chain complex. Here we think of a graph $(G, \Phi) = (G, \{e_1, \dots, e_k\}) \in C_k$ as the k-dimensional $[0,1]$-cube embedded in R^k. The Graph G_Φ where all edges in Φ have been collapsed sits at the origin and the graph G where all edges have been removed from Φ but not collapsed sits diagonally opposite at $(1, \dots, 1)$. We can assign a graph to every face in the following way: Consider a face F of dimension $n < k$, let bF be its barycenter. Then $bF \in \{0, 1/2, 1\}^k$ and denote by bF_i its i-th coordinate. Moreover let

$$C := \{e_i \in \Phi \mid bF_i = 0\} \quad \text{and} \quad R := \{e_i \in \Phi \mid bF_i = 1\}.$$

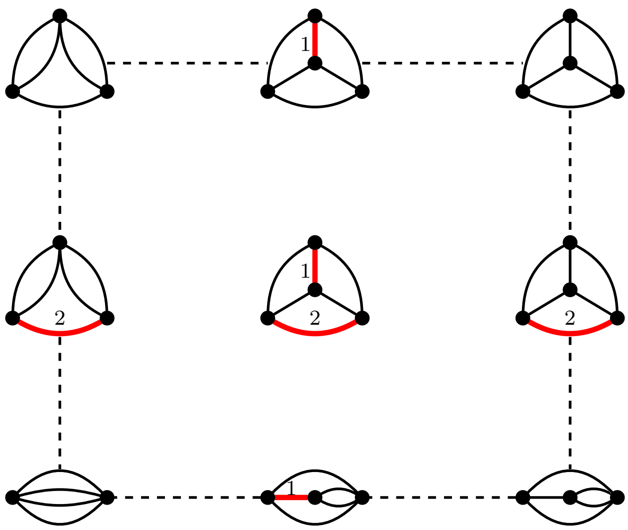
Then the graph associated to F is given by $(G/C, \Phi \setminus (C \cup D)$. Thus an edge gets contracted if $bF_i = 0$ and an edge gets removed from Φ but not contracted if $bF_i = 1$. Now if a face is of dimension n then n coordinates of bF equal $1/2$ and thus the resulting graph has a forest of size n and is hence in C_n.

The above description gives us a bijection between the reduced graphs of G and the half integral points. Denote the reduced graph of G associated to bF by G_{bF}. Then we can define the boundary operator via this bijection as follows:

Formulate this

To visualize this construction we consider the following example:

Example. Consider again the graph J from example 2.1 with the forest Φ given by an edge between the top and middle vertex and between the left and write vertex. Then its 2-dimensional cube is given as below:



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[2] James Conant and Karen Vogtmann. "On a theorem of Kontsevich". In: *Algebraic & Geometric Topology* 3.2 (Dec. 2003), pp. 1167–1224.

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3 Draftparts

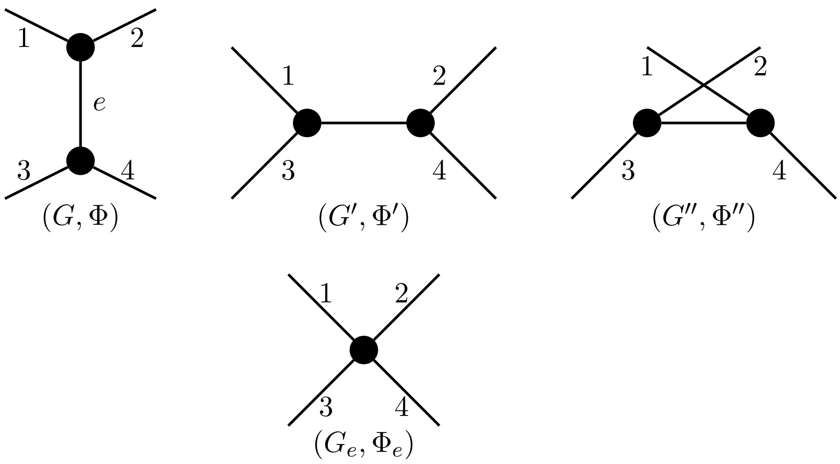
Definition. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$ and the set of half edges by $H(G)$. We call a graph *connected* if the CW complex is connected in the topological sense. A graph is said to be *n-valent* if every vertex has valency n i.e. for every vertex the number of edges incident is n.

Lastly a *tree* is a graph which contains no loops and a *forest* is a collection of disjoint trees.

On trivalent connected graphs we call an *orientation* a choice of cyclic orders of all vertices up to an even number of changes.

Definition. A *forested graph* is a pair (G, Φ), where G is a finite connected trivalent graph and Φ is an oriented forest which contains all vertices of G.

Definition. Let (G, Φ) be a forested graph and let $e \in \Phi$. Moreover let (G_e, Φ_e) be the graph where e has been collapsed. Then there exist exactly two other graphs whose edge collapse results in (G_e, Φ_e). This is visualised in the figure below. Where $1, 2, 3, 4$ represent the rest of the graph.



Now the vector

$$(G, \Phi) + (G', \Phi') + (G'', \Phi'')$$

is called the *basic IHX relator* associated to (G, Φ, e).

Denote by $\widehat{f\mathcal{G}_k}$ the vector space spanned by all forested graphs containing k trees modulo the relations $(G, \Phi) = -(G, -\Phi)$. Moreover let $f\mathcal{G}_k$ be the quotient of $\widehat{f\mathcal{G}_k}$ modulo the subspace spanned by all basic IHX relators.

Definition. Let $\widehat{\partial}_E(G, \Phi) : \widehat{f\mathcal{G}_k} \to \widehat{f\mathcal{G}}_{k-1}$ be given by

$$\widehat{\partial}_E(G, \Phi) = \sum (G, \Phi \cup e).$$

where the sum is over all edges e of $G \setminus \Phi$ such that $\Phi \cup e$ is still a forest. Notice that this only happens if the two vertices of e lie in different trees of Φ. Thus $\Phi \cup e$ has $k-1$ components. The orientation of $\Phi \cup e$ is determined by ordering the edges of Φ with labels $1, \ldots, k$ consistent with its orientation and then labeling the new edge e with $k+1$.

Now let the boundary map $\partial_E : f\mathcal{G}_k \to f\mathcal{G}_{k-1}$ be the map induced by $p \circ \widehat{\partial}_E$ where p is the quotient map $\widehat{f\mathcal{G}_k} \to f\mathcal{G}_k$.

Proposition. *∂_E is well-defined and $\partial_E^2 = 0$.*

Proof. □

The *forested graph complex* is thus defined as the sequence $f\mathcal{G}_k$ with boundary map ∂_E and is well-defined by the above proposition.