The forested graph complex

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Bachelor Thesis

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1 Introduction

Groups one of the fundamental concepts of mathematics has been a central aspect of mathematical studies since the 19th century. Free groups are then the class of groups where no relation between the generators exists: The elements are words of generators and their inverses. The group relation is given by concatenating words and reducing them i.e. if two adjacent elements are inverse to each other they will be omitted. The free group on n generators will be denoted by F_n. The importance of free groups arises from the fact that every group is isomorphic to a quotient group of a free group and in particular any finitely generated group is isomorphic to a quotient group of F_n for some n.

Another way to approach a group G is to understand its automorphism group $\mathrm{Aut}(G)$ as it describes G's symmetries. $\mathrm{Aut}(G)$ can be separated into the inner automorphism group $\mathrm{Inn}(G)$, the group of automorphisms that arise from conjugation, and the outer automorphism group $\mathrm{Out}(G)$, the quotient of $\mathrm{Aut}(G)$ by $\mathrm{Inn}(G)$.

Combining those two aspects it is only natural that mathematicians intensely study the automorphism group of F_n. The inner automorphism group is well understood and isomorphic to F_n itself. The outer automorphism group $\mathrm{Out}(F_n)$ however still leaves many open questions. As $F_1 \cong \mathbb{Z}$,

$$\mathrm{Out}(F_1) = \mathrm{Out}(\mathbb{Z}) = \mathrm{GL}_1(\mathbb{Z}).$$

For F_2 Nielsen proved in [15] that $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$. For higher groups only the existence of a surjection $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ can be guaranteed.

Historically $\mathrm{GL}_n(\mathbb{Z})$ has been studied by by its action on the symmetric space $\mathrm{SL}_n(\mathbb{R})/\,\mathrm{SO}_n(\mathbb{R})$. Due to the relation between $\mathrm{Out}(F_n)$ and $\mathrm{GL}_n(\mathbb{Z})$ early attempts of the study of the outer automorphism group of F_n examined its action on $\mathrm{SL}_n(\mathbb{R})/\,\mathrm{SO}_n(\mathbb{R})$ induced by the above surjection. However this action is not proper that is the inverse of compact sets under the group action are not necessarily compact. Hence it behaves quite badly and a different approach was needed.

Therefore Culler and Vogtmann introduced a new space $\mathcal{X}_n$ known as "Outer space" in [17]. To understand this space we first have to introduce basic concepts of graph theory.

1.1 Graphs

Definition 1.1. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$. We call an edge having the same start and end vertex a *loop*.

We call a graph *connected* if the CW complex is connected in the topological sense. A graph is *n-edge-connected* if it remains connected after removing $n-1$ arbitrary edges.

A graph is said to be *n-regular* if every vertex has valency n i.e. for every vertex the number of incident edges is n. The valency of a vertex $v \in G$ is often also called *degree* of v and denoted by $\deg(v)$.

For a subset of edges Φ of G we denote by G/Φ the *graph quotient*, which is the quotient space of the CW complex G over its topological subspace Φ.

Remark 1.2. Note that sometimes these types of graphs are called multigraphs, as they are allowed to have multiple edges between vertices as well as loops. The word graph then normally refers to simple graphs which do not allow multi-edges and loops.

In the context of algebraic topology however multigraphs are needed and thus the word graph here denotes multigraphs.

Definition 1.3. A *subgraph* G' of a graph G is a subcomplex of the CW-complex G. Since a subcomplexes are themself CW-complexes of dimension smaller or equal to the original complex, G' itself is a graph.

A *cycle* in a graph G is a subgraph that is homeomorphic to S^1. A *tree* is a connected graph containing no cycles. A *forest* is a collection of disjoint trees.

The following result gives the fundamental connection between graphs and free groups:

Theorem 1.4. *Let G be a graph. Then its fundamental group $\pi_1(G)$ is isomorphic to a free group.*

By the theorem it makes sense to define the *rank* of a graph $\text{rank}(G)$ as the rank of its fundamental group. A proof of this theorem will be given later at the beginning of section 2.

Finally a *metric graph* is a finite connected graph where each edge is assigned a positive real value, its length. Moreover on these lengths we define the path metric i.e. the distance between two points is the length of the shortest path between them. Here the length of the path is the sum of the lengths of the edges that are (partially) traversed.

1.2 Outer space[1]

We can now continue with the study of $\text{Out}(F_n)$ and the aforementioned Outer space. For this we follow Vogtmann's survey article [18].

To define Outer space we first consider the graph R_n given by one vertex and n edges each forming a loop. By Theorem 1.4 we can identify $\pi_1(R_n)$ with the free group F_n by identifying the generators of F_n $x_1, \dots, x_n$ with oriented edges of R_n. Then every reduced word in F_n corresponds to a reduced edge-path loop starting and ending at the base-point of R_n. An automorphism ϕ on F_n is now a homotopy equivalence sending the loop corresponding to x_i to the loop identified with $\phi(x_i)$.

Outer space is now the set of marked metric graphs (g, G) under an equivalence relation. The graphs (g, G) have to satisfy the following:

- G is a finite graph with vertex valency at least 3,
- $g : R_n \to G$ is a homotopy equivalence called the marking of G, and
- each edge has a positive length such that the sum of all edge lengths is one. This turns G into a metric space via the path metric.

The equivalence relation is given by $(g, G) \sim (g', G')$ if and only if there exists an isometry $h : G \to G'$ such that $g \circ h$ is homotopic to g'.

Intuitively each point (g, G) can be represented by drawing the graph G, finding a maximal forest T on G and labelling all edges not in T with an orientation and an element of F_n. The labels determine a map $f : G \to R_n$ by sending T to the vertex of R_n and sending each edge in $G \setminus T$ to the corresponding loop in R_n given by the labelling. The labelling is chose in such a way that f is a homotopy inverse to g. An example is given in figure **??**.

To define a topology on $\mathcal{X}_n$ we look at the set $\mathcal{C}$ of conjugacy classes of F_n, the cyclic reduced words, that is words for which every cyclic permutation is reduced. Now we can define a map from Outer space $\mathcal{X}_n$ to the infinite projective space $\mathbb{RP}^{\mathcal{C}}$. For a fixed marked graph (g, G) the map assigns to each cyclically reduced word w the length of the unique cyclically reduced path loop in G homotopic to $g(w)$. As this map is injective, we can view $\mathcal{X}_n$ as s subspace of $\mathbb{RP}^{\mathcal{C}}$ and thus give $\mathcal{X}_n$ the subspace topology.

[1] The figures in this section are taken from [18]

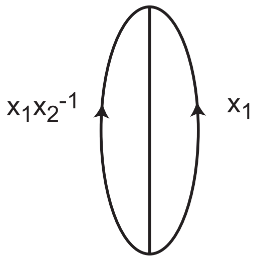
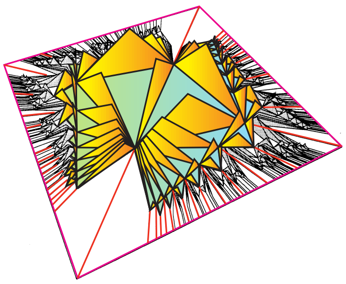
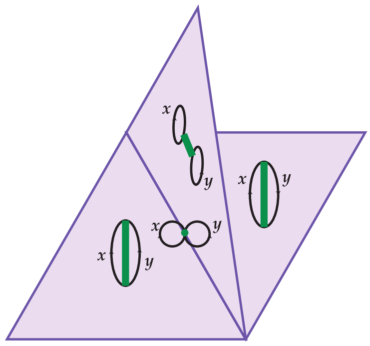


Figure 1: A point of Outer Space

This definition might seem quite ad hoc, however under it $\mathcal{X}_n$ decomposes nicely into a disjoint union of open simplices: Every marked graph (g, G) belongs to the open simplex containing all graphs that can be reached by varying the non-zero edge lengths such that the total edge length of G remains 1. The faces of the simplex are then the marked graphs where one edge of (g, G) has been fully contracted. Of course a contracted edge can be extended again in multiple ways thus connecting the different simplices. An example is given in the figure below on the right. On the left we see a depiction of $\mathcal{X}_2$.



(a) Outer space $\mathcal{X}_2$



(b) Simplices in $\mathcal{X}_2$

For a marked graph with $k+1$ edges the corresponding simplex is k dimensional. Moreover the identification works the other way round i.e. every open simplex in X_n is a face of a maximal simplex which corresponds to a trivalent marked graph. Taking the argument from the proof of Theorem 3.5 we see that the dimension of $\mathcal{X}_n$ is equal to $3n-4$.

Now with the Outer space defined we can look at the right group action of $\mathrm{Out}(F_n)$ on $\mathcal{X}_n$: Every $\phi \in \mathrm{Out}(F_n)$ induces a map $f : R_n \to R_n$ by mapping the edge labelled by x to the edge labelled by $\phi(x)$. Then the right group action is defined by $(g, G)\phi = (g \circ f, G)$.

An inconvenience that arises is that the quotient of $\mathcal{X}_n$ by $\mathrm{Out}(F_n)$ is not compact. Resolving this leads us to the next construction.

1.3 Spine of Outer Space

The beginning of this section follows Culler and Vogtmanns original definition from [17]. In the latter half we adhere to [19].

In a first step we define a more convenient somewhat simpler version of $\mathcal{X}_n$ known as reduced Outer space and denote it by Y_n. The points in the subspace Y_n are the marked graphs (g, G) that do not contain any separating edges, i.e. no edges e such that $G \setminus e$ is disconnected. An equivariant deformation retraction from $\mathcal{X}_n$ to Y_n is given by shrinking the lengths of the separating edges to zero while uniformly extending the lengths of the other edges in order to preserve the total edge length of 1.

We can now define the *spine of Outer space* K_n as follows: K_n is the maximal full subcomplex of the barycentric subdivision of Y_n which is disjoint from the boundaries of the open simplices in Y_n. The vertices of K_n are the barycenters of simplices in Y_n i.e. the subspace of Y_n of marked graphs whose edges are of equal length. The deformation retraction from Y_n to K_n is given by collapsing every simplex τ of the barycentric subdivision of Y_n to the face of τ contained in K_n. This can be done equivariantly. Thus K_n can be thought of as ignoring the metric structure on Y_n and only focusing on its combinatorial structure.

Going the other way if we have to vertices (g, G) and (g', G') in K_n. Then the open simplex in $\mathcal{X}_n$ determined by (g, G) is a face of the one determined by (g', G') exactly when G is obtained from G' by collapsing a forest of edges in G' and g is homotopic to the composition of g' with the collapsing map. This collapsing is also called a *forest collapse*. Thus K_n has the structure of a simplical complex where a k-simplex is a chain of k forest collapses.

With this in mind, as collapsing an edge decreases the number of vertices by 1 we can determine the dimension of K_n. Thus with a similar argument as in the proof of Theorem 3.5 we see that $\dim(K) = 2n - 3$. An example of a part of the spine of $\mathcal{X}_2$ is given below[2].

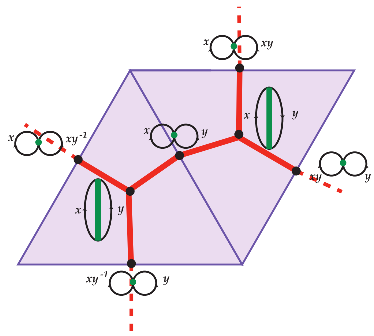


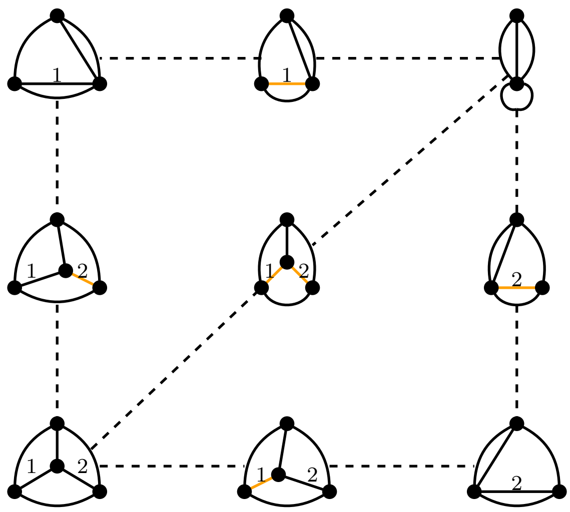
Figure 3: A section of the spine of Outer space $\mathcal{X}_2$

Coming back to $\mathrm{Out}(F_n)$ acting on spaces; its action on $\mathcal{X}_n$ extends to a simplical action on K_n. Culler and Vogtmann proved in [17] that this action has finite stabilizers. Thus the rational homology of $\mathrm{Out}(F_n)$ can be computed as the quotient of K_n by $\mathrm{Out}(F_n)$

$$H_\bullet(\mathrm{Out}(F_n), \mathbb{Q}) \cong H_\bullet(K_n / \mathrm{Out}(F_n), \mathbb{Q}).$$

In order to calculate this homology we turn K_n into a cube complex i.e. a CW-complex where the cells are homeomorphic to Euclidean cubes and the attaching maps identify faces with lower dimensional cubes via homeomorphisms. For a forest Φ in G with k edges we can now define the k-cube: From Φ we get a chain of k-forest collapses by collapsing each edge in Φ at a time. This yields a k-simplex. Collapsing the edges in another order yields another k-simplex. All these different k-simplices can now be fit together to triangulate a k-dimensional cube. Thus every k-cube is given by a graph G and a forest Φ of size k. The faces of dimension $k - 1$ are the graphs obtained from G where one edge in Φ has been collapsed. An example is given in the figure below where the orange edges represent the edges that are being contracted along that face.

[2] This figure is taken from [18]



Viewing K_n as a cube complex with one cube for every tuple (G, Φ) we can define an orientation up to even permutation via ordering the edges of Φ. Then the rational homology of $K_n/\operatorname{Out}(F_n)$ can be computed from a chain complex with one generator for each pair (G, Φ) that has no orientation reversing-automorphism.

1.4 The Forested graph complex

This chain complex of pairs (G, Φ) is generally known as the forested graph complex and has been introduced by Conant and Vogtmann in [5]. In this introductory section we will describe the original construction. In the later chapter we are going to introduce a more practical hands-on but equivalent definition from [3].

Definition 1.5. A *forested graph* is a pair (G, Φ) of a finite connected trivalent graph G and an oriented forest Φ containing all vertices of G. The orientation on the forest is given by an ordering of its edges where interchanging any two edges reverses the orientation.

We now denote by $\widehat{f\mathcal{G}_k}$ the vector space spanned by forested graphs with forest size k modulo the relations $(G, \Phi) = -(G, -\Phi)$.

If we consider a forested graph (G, Φ) and we collapse an edge e in Φ then the graph obtained (G_e, Φ_e) has exactly one 4 valent vertex. As the image below shows there are exactly two other graphs whose edge collapse leads to the graph (G_e, Φ_e).

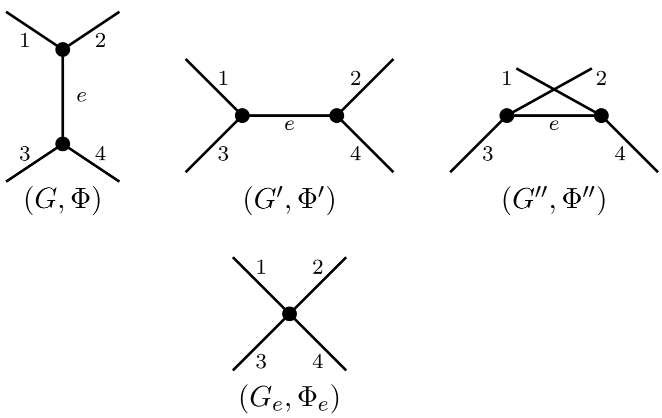


Figure 4: The numbers represent the four parts of the graph that get connected at e. The graphs G, G', G'' then represent the three different ways how this can happen so that collapsing e leads to G_e.

If we denote those two by (G', Φ') and (G'', Φ'') then we call the vector

$$(G, \Phi) + (G', \Phi') + (G'', \Phi'').$$

the basic IHX relator associated to (G, Φ, e). We denote by IHX_k the subspace of $\widehat{f\mathcal{G}}_k$ spanned by all basic IHX relators and define $f\mathcal{G}_k$ as the quotient space $\widehat{f\mathcal{G}}_k / IHX_k$.

Finally we can define a boundary map $\partial : f\mathcal{G}_k \to f\mathcal{G}_{k+1}$ induced by the map on $\widehat{f\mathcal{G}}_k$ given by

$$\partial_E(G, \Phi) = \sum (G, \Phi \cup e)$$

where we sum over all edges in $G \setminus \Phi$ such that $\Phi \cup e$ is still a forest and e gets the label $k+1$ in the orientation.

One can check that ∂_E is really a boundary map that is $\partial_E^2 = 0$. With this shown we get a chain complex $f\mathcal{G}_\bullet$ with boundary map ∂_E. The rational homology of this complex now computes the rational homology of $\mathrm{Out}(F_n)$ as explained at the end of the previous section.

1.5 Group homology and the known homology of Outer space

To conclude this introduction we try to summarize the known rational homology of $\mathrm{Out}(F_n)$. We will only consider the rational homology here and thus omit the field $\mathbb{Q}$.

For this we need to first introduce two concepts:

Definition 1.6. If a group G contains a torsion free subgroup of finite index, then the *virtual cohomological dimension* $\mathrm{vcd}(G)$ is the cohomological dimension of this subgroup. It is independent of the choice of subgroup.

Moreover a series of groups $G_1 \subseteq G_2 \subseteq \ldots$ is called *homological stable* if for every k there exists a N such that

$$H_k(G_n) \cong H_k(G_{n+1}) \quad \text{for all} \quad n \geq N$$

that is for n large enough the homology is independent of n.

Culler and Vogtmann showed in [17] that $H_k(\mathrm{Out}(F_n))$ is finitely generated and vanishes for k greater than the vcd of $\mathrm{Out}(F_n)$ which is $2n-3$. This gives us an upper bound on homology. Moreover they showed that the spine K_n is path connected from which we conclude that $H_0(\mathrm{Out}(F_n)) = \mathbb{Q}$.

On the other side homological stability was proven for $\mathrm{Out}(F_n)$ for $n \geq 5(k+1)/4$ in [12, 11]. Moreover Galatius proved in [7] that these groups are in fact zero, giving us a lower bound.

Only very few of the non-trivial homology groups are explicitly known. For the groups $H_4(\mathrm{Out}(F_4))$, $H_8(\mathrm{Out}(F_6))$ and $H_{12}(\mathrm{Out}(F_8))$ it is known that the Morita Classes do not vanish and in fact as they have dimension 1 the Morita Classes even generate these groups. We will expand on this in the last chapter of this work. Ohashi showed in [16] that all other groups for $n \leq 6$ are trivial. The only other known groups were determined by Bartholdi in [2] and are $H_8(\mathrm{Out}(F_7))$ and $H_{11}(\mathrm{Out}(F_7))$. He also showed that all other homologies for $n = 7$ are trivial. The graphic below has been adapted from Figure 1 in [6] and summarizes these results.

Euler Characteristic computations for $n \leq 11$ by Morita, Sakasai and Suzuki in [13, 14] are shown below. If this negative trend continues it would mean that there are lots of odd-dimensional

n	3	4	5	6	7	8	9	19	11	12
$\chi(\mathrm{Out}(F_n))$	1	2	1	2	1	1	-21	-124	-1202	?

homology classes of $\mathrm{Out}(F_n)$ of which the only one known is Bartholdi's $H_{11}(\mathrm{Out}(F_7))$. Thus the homology of $\mathrm{Out}(F_n)$ still remains mostly unknown.

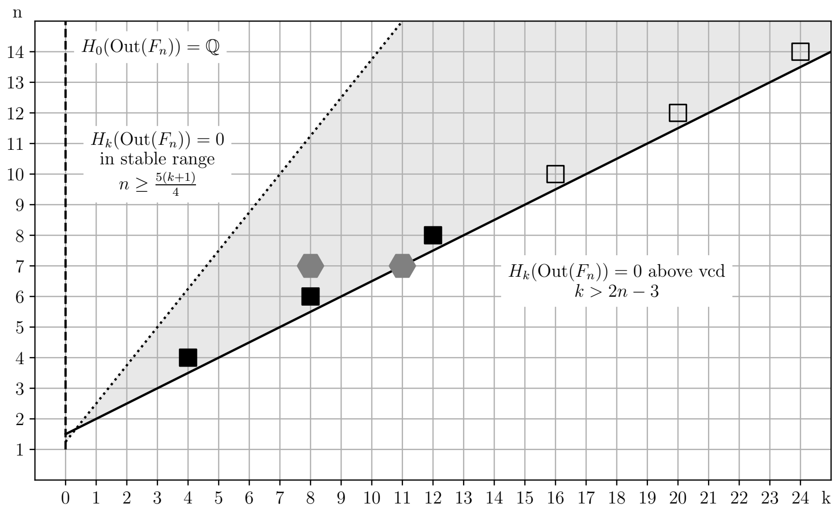


Figure 5: Homology Classes of $\mathrm{Out}(F_n)$. The squares represent the Morita classes and are filled in if they are known to be non-trivial. The hexagons are Bartholdis non-trivial classes.

2 The rank of a graph

We start this section of by proving Theorem 1.4 which we state again below. Afterwards we expand on the concept of rank and prove several identities and relations.

Theorem 2.1. *Let G be a graph. Then its fundamental group $\pi_1(G)$ is isomorphic to a free group.*

Proof. The following proof is from [10, p. 43f]. Let G be a graph. W.l.o.g. G is connected as else we consider each connected component separately. Then let T be a spanning tree on G i.e. T is a tree containing every vertex of G.

Now choose for every $e_\alpha \in E(G) \setminus T$ an open neighborhood A_α of $T \cup e_\alpha$ that deformation retracts onto $T \cup e_\alpha$. The intersection of such A_α deformation retracts onto T and is thus contractible. Moreover as G is connected as a graph A_α and T are path connected. Now the A_α form an open cover of G and as T is simply connected by Van Kampen's theorem we get that $\pi_1(G) = *_\alpha \pi_1(A_\alpha)$. Finally A_α deformation retracts onto S^1 and thus $\pi_1(A_\alpha) = \mathbb{Z}$. Now there are exactly $|E(G)| - |E(T)|$ many A_α, which as T is a spanning tree and hence $|E(T)| = |V(G)| - 1$ results in $\pi_1(G)$ being free on $|E(G)| - |V(G)| + 1$ generators. □

To understand the rank better we will need the following definitions

Definition 2.2. For a finite CW-complex X the *euler characteristic* is defined as the alternating sum

$$\chi(X) = k_0 - k_1 + k_2 - \dots$$

where k_i denotes the number of cells of dimension i in the CW-complex X.

For graphs we thus get $\chi(G) = k_0 - k_1$, as they are 1-dimensional which is equal to $\chi(G) = |V| - |E|$.

Definition 2.3. Let G be a graph. Then its *cycle space* is the set of even-degree subgraphs of G that is the subgraphs of G all of whose vertices have even degree. This space forms a vector space over $\mathbb{F}_2$ where the vector addition is given by the symmetric difference of two or more subgraphs. A basis of this space is called *cycle basis* and two cycles are *independent* if they are linearly independent in the vector space.

Remark 2.4. The cycle space is equal to the one homology group of G with coefficients in $\mathbb{F}_2$ i.e. $H_1(G, \mathbb{F}_2)$.

The following proposition relates the rank to different invariants and gives an easy way to compute it:

Proposition 2.5. *Let G be a connected graph. Then the following are equal:*

1. *The rank of G.*
2. *The number of independent cycles in G i.e. the size of the cycle basis of G.*
3. *The first Betti number i.e. the rank of $H_1(G)$.*
4. $1 - \chi(G) = |E| - |V| + 1$.

For the proof we will need the following lemma:

Lemma 2.6. *Let A be a set. Then the abelianization of the free group on A is isomorphic the free abelian group of A.*

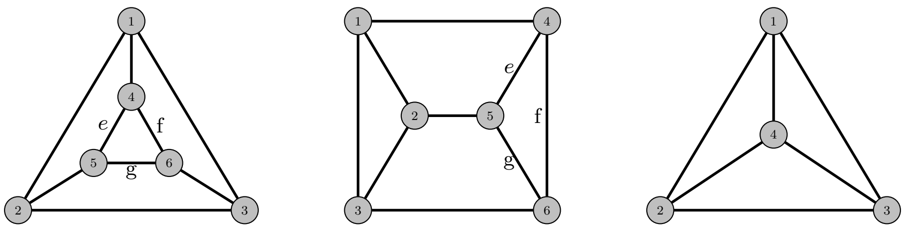
Proof. Consider the space $X = \bigvee_{a \in A} S^1$. Then by Van Kampen's theorem $\pi_1(X) = *_{a \in A}\mathbb{Z}$ that is $\pi_1(X)$ is isomorphic to the free group on A. On the other hand we have $H_1(X) = \bigoplus_{a \in A} H_1(S_1) = \bigoplus_{a \in A} \mathbb{Z}$ that is $H_1(X)$ is isomorphic to the free abelian group on A, which follows from the relative homeomorphism theorem. Now using Hurewicz theorem we get that the abelianization of $\pi_1(X)$ is isomorphic to $H_1(X)$ and thus the desired statement. $\square$

Proof of the Theorem. (1) = (4): This was shown in the proof of Theorem 1.4.
(1) = (3): From Hurewicz Theorem we get that the abelianization of $\pi_1(G)$ is equal to $H_1(G)$ and thus by the previous Lemma that the rank of $\pi_1(G)$ is equal to the rank of $H_1(G)$ which is the first Betti number.
(4) = (2)[3]: Consider again the sets A_α from the proof of Theorem 1.4. Then each of them deformation retracts onto a cycle in G. Let $Z(T)$ be the set of cycles obtained in this way. Then $Z(T)$ is independent as each cycle contains an edge not contained in any other cycle. Moreover every cycle Z in G can be written as the symmetric difference over the cycles corresponding to the edges in $(E \setminus T) \cap Z$. Thus $Z(T)$ spans the cycle space and consequently is a cycle basis. Now the size of $Z(T)$ is given by $|E| - |V| + 1$ and thus we conclude. $\square$

Definition 2.7. Let G, H be two graphs. A map $f : V(G) \to V(H)$ is said to be a *graph isomorphism* if f is a bijection such that

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H).$$

Example 2.8. Consider the following graphs.



G H J

[3] This proof is based on Harary's proof in [9, p. 37-40].

Then G is 3-connected and 3-regular. Moreover G is isomorphic to H. An isomorphism between them is given by mapping the equally labelled nodes to each other. The graph quotient $G/\{e, f, g\}$ is given by J.

Finally we introduce the notion of degree of a graph also sometimes called excess.

Definition 2.9. Let G be a connected graph of rank n with vertex-valency ≥ 3. Its *degree* is defined by

$$\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).$$

Proposition 2.10. *Let $G = (V, E)$ be a graph of rank n. Then we have the following identities:*

1. $\deg(G) = 2|E| - 3|V|$
2. $|V| = 2n - 2 - \deg(G)$
3. $|E| = 3n - 3 - \deg(G)$
4. *G is 3-regular* $\Leftrightarrow \deg(G) = 0$.

Proof. By counting half edges we get $2|E| = \sum_{v \in V} \deg(V)$. Combining this with the definition of degree we directly get the first identity. Using Proposition 2.5 and the first identity we have

$$\begin{aligned} 2n - 2 - \deg(G) &= 2|E| - 2|V| + 2 - 2 - 2|E| + 3|V| = |V| \\ 3n - 3 - \deg(G) &= 3|E| - 3|V| + 3 - 3 - 2|E| + 3|V| = |E| \end{aligned}$$

which proves the second and third identities. The last statement follows as every element in the sum of the degree is non-negative. Thus $\deg(G) = 0$ if and only if every term is 0 and thus if and only if every vertex has valency 3. $\square$

3 The forested graph complex

As mentioned before the forested graph complex has first been introduced by Conant and Vogtmann in [5]. Here however we will introduce the simplified construction and definition of the forested graph complex given by Conant and Vogtmann in [3]. Very useful in the general understanding of what a graph complex is and how the boundary map acts was Bar-Natan's and McKay's draft [1]. Inspired by this similar examples for the forested graph complex are presented.

Let F_n denote the free group of rank n. We also denote by $\mathbb{S}_n$ the symmetric group of degree n.

Definition 3.1. An *admissible graph of rank n* is a 2-edge-connected graph G with fundamental group isomorphic to F_n and with vertex-valency ≥ 3.

We often just write graph for an admissible graph of rank n.

Definition 3.2. Let $G = (V, E)$ be a graph. An *ordering* on its edges is a bijective function σ from E to $\{1, \ldots, |E(G)|\}$. Notice that $\mathbb{S}_{|E|}$ acts on σ by $\pi \circ \sigma$ for $\pi \in \mathbb{S}_n$. We call the tuple (G, σ) an ordered graph and note that $\mathbb{S}_{|E|}$ acts on (G, σ) by $\pi(G, \sigma) = (G, \pi\sigma)$ for $\pi \in \mathbb{S}_{|E|}$.

A *forested graph* is a triple (G, Φ, σ) where G is an admissible graph Φ is a subset of edges that spans a forest on G and σ is an ordering on Φ i.e. $\sigma : \Phi \to \{1, \ldots, |\Phi|\}$.

A map f between two forested graphs $(G, \Phi, \sigma) \to (H, \Psi, \tau)$ is said to be a *forested graph* isomorphism if f is a graph isomorphism on G, $f(\Phi) = \Psi$ and $\sigma = \tau \circ f$

We now want to construct our graph complex. For this we remember the notion of a graded vector space:

Definition 3.3. A graded vector space, is a vector space V and with a decomposition $(V_n)_{n=0}^{\infty}$ such that

$$V = \bigoplus_{k=0}^{\infty} V_k.$$

We now consider the $\mathbb{Q}$-vector space C spanned by isomorphism classes of forested graphs, subject to the relation

$$(G, \Phi, \pi \circ \sigma) = \operatorname{sgn} \pi \cdot (G, \Phi, \sigma) \quad \text{for all} \quad \pi \in \mathbb{S}_n(|\Phi|).$$

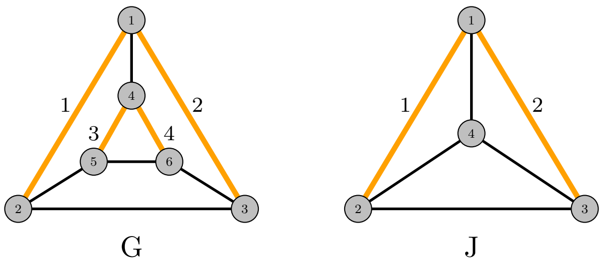
Under this relation we call σ an *orientation*. Observe that if $(G, \Phi, \sigma) \simeq (G, \Phi, \pi \circ \sigma)$ for an odd permutation π then $(G, \Phi, \sigma) \simeq (G, \Phi, \pi \circ \sigma) = -(G, \Phi, \sigma)$ and thus $(G, \Phi, \sigma) = 0$ in C_k.

We can define the following three gradings on C:

- Let $C^n \subseteq C$ be the subspace spanned by forested graphs of rank n. Then clearly $C^n \cap C^m = \varnothing$ for $n \neq m$ and as every graph has a rank, we get that the C^n form a grading on C.
- Let $C_k \subseteq C$ be the subspace spanned by forested graphs (G, Φ, σ) with $k = |\Phi|$. Clearly this also yields a decomposition of C into a direct sum and thus yields another grading on C.
- Let $C_d \subseteq C$ be the subspace spanned by forested graphs of degree d. Once again this yields a grading on C.

In the following we will mostly be concerned with the first two gradings. In particular we will consider the double-grading C_k^n, where k denotes the forest size and n the rank.

Example 3.4. Consider the graphs G, J from example 2.8. Then G and J are admissible graphs of rank 4 and 3. Thus if we equip them with ordered forests $(\Phi, \sigma), (\Psi, \tau)$ as below (where the orange edges represent the forest and the numbers the orientation) we get forested graphs in C_4^4 and C_2^3 respectively.



Observe, that $(J, \Psi) = 0$ in C_2^3, since (12) is an odd permutation and $(12)(J, \Psi)$ is isomorphic to (J, Ψ) via the isomorphism mirroring that mirrors vertices along the vertical passing through the vertices labelled 1 and 4.

(G, Φ) however is not trivial as the automorphism group is given by the identity, mirroring along the vertical, exchanging inner and outer vertices and their composition. None of these automorphisms induce an odd permutation and hence (G, Φ) does not vanish.

Before we construct the chain complex we show that the C^n are finitely generated and thus so are the C_k^n.

Theorem 3.5. *For all n, C^n is finitely generated and $C_k^n = 0$ for all $k > 2n - 3$.*

To prove this theorem we first need the following lemma:

Lemma 3.6. *For $n, m \in \mathbb{N}$ There are only finitely many admissible graphs $G = (V, E)$ with $|V| \leq n$ vertices and $|E| \leq m$ edges.*

Proof. Every graph on n vertices can be written as incidence matrix and each entry is $\leq m$ if the graph has maximally m edges. Thus there are maximally m^{n^2} many different incidence matrices for graphs with n vertices and maximally m edges. As every graph corresponds to an incidence matrix this also gives an upper bound on the number of different graphs with n vertices and maximally m edges.

Thus the maximal possible number of admissible graphs with $\leq n$ vertices and $\leq m$ edges is bounded by

$$\sum_{k=1}^{n} m^{k^2}$$

which is finite. $\square$

Proof of Theorem 3.5. By counting half edges we have

$$2|E| = \sum_{v \in V} \deg v.$$

Using that admissible graphs have vertex-valency ≥ 3 and rearranging yields $|E| \geq \frac{3}{2}|V|$. From Proposition 2.5 we get that $|E| = |V| + n - 1$. Combining the two yields

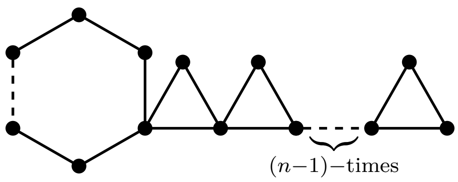
$$|V| + n - 1 \geq \frac{3}{2}|V| \Leftrightarrow 2(n-1) \geq |V|$$

and plugging in the result in the identity from Proposition 2.5 yields $|E| \leq 3(n-1)$.

Thus by the above lemma we have that there are only finitely many graphs with $|V| \leq 2(n-1)$ and $|E| \leq 3(n-1)$. As each graph only has a finite number of forests and each of them has a finite number of orientations we get that C^n is finitely generated.

That $C_k^n = 0 \ \forall k > 2n-3$ follows from the bound on the number of vertices and the fact that a forest in a graph has maximally $|V| - 1$ edges. $\square$

Remark 3.7. Notice that the constraint of vertex-valency ≥ 3 in the definition of admissible graphs is necessary for C^n being finitely generated. As else we can consider the following family of graphs where the most left polygon is of arbitrary size:



They all have rank n (which can be checked via the Euler characteristic), are 2-edge-connected and not isomorphic as they have different number of vertices/edges.

Remark 3.8. The bound on the C_k^n can not be improved as the graph J from the example above with the tree extended by the edge between the vertex labelled 1 and 4 has rank 3 and tree size 3 which equals $2 \cdot 3 - 3$.

To construct forested graph complex we fix the rank n and define a differential as follows:

Definition 3.9. Let $(G, \Phi, \sigma) = (G, \{e_1, \ldots, e_k\}, \sigma)$ be a forested graph. Then let ∂_C, ∂_R :

$C_k^n \to C_{k-1}^n$ be given by

$$\partial_C(G, \Phi, \sigma) = \sum_{i=1}^{k} (-1)^i (G/e_i, \Phi \setminus \{e_i\}, \sigma_{e_i}),$$
$$\partial_R(G, \Phi, \sigma) = \sum_{i=1}^{k} (-1)^i (G, \Phi \setminus \{e_i\}, \sigma_{e_i})$$

where $\sigma_{e_i} : \Phi \setminus \{e_i\} \to \{1, \dots, k-1\}$ is given by

$$\sigma_{e_i}(e) = \begin{cases} \sigma(e) & \text{if } \sigma(e) < i \\ \sigma(e) - 1 & \text{if } \sigma(e) > i \end{cases}.$$

Notice that the case $\sigma(e) = i$ cannot happen as e_i is not contained in $\Phi \setminus \{e_i\}$. Finally define the boundary map $\partial := \partial_C - \partial_R$.

Proposition 3.10. *∂ is well-defined and $\partial^2 = 0$.*

For better readability we will omit the orientation σ in the proof.

Proof. We proof the result in three steps:
Step 1: Contracting an edge of a graph does not change the Euler characteristic as both the vertex number and the edge number decreases by one. Thus ∂_C preserves the rank of the graph. Moreover the vertex-valency stays ≥ 3 and the graph continues to be 2-edge-connected. Hence it is admissible. Moreover ∂_C as well as ∂_R remove one edge from each forest. Thus decreasing k by 1. Hence both maps are well-defined from C_k^n to C_{k-1}^n and thus so is ∂.
Let $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ be a forested graph and denote the edges in $\Phi \setminus \{e_i\}$ by $\{e'_1, \dots, e'_{p-1}\}$, where $e'_j = e_j$ for $j < i$ and $e'_j = e_{j+1}$ for $j > i$. For the consecutive steps we need the following observations:

$$(G/e_i)/e'_j = \begin{cases} (G/e_j)/e'_{i-1} & \text{if } i > j \\ (G/e_{j+1})/e'_i & \text{if } i \leq j \end{cases} \quad \text{and} \quad (\Phi \setminus \{e_i\}) \setminus \{e'_j\} = \begin{cases} (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\} & \text{if } i > j \\ (\Phi \setminus \{e_{j+i}\}) \setminus \{e'_i\} & \text{if } i \leq j \end{cases}$$

Step 2: *Claim:* $\partial_C^2 = 0$ and $\partial_R^2 = 0$
We compute:

$$\begin{aligned} \partial_C^2 &= \partial_C \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}) = \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\ &= \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \leq j} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \qquad (\star) \end{aligned}$$

We claim that the two sums cancel. For this first apply the observations above to the first sum and then change variables by setting $l = j$ and $m = i - 1$ to obtain:

$$\begin{aligned} \sum_{j<i} (-1)^{i+j} ((G/e_i)/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) &= \sum_{j<i} (-1)^{i+j} ((G/e_j)/e'_{i-1}, (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\}) \\ &= \sum_{l \leq m} (-1)^{l+m+1} ((G/e_l)/e'_m, (\Phi \setminus \{e_l\}) \setminus \{e'_m\}) \end{aligned}$$

This last expression is the same as the second sum in $(\star)$ but with opposite sign. Thus they cancel and we have shown $\partial_C^2 = 0$. The same argument shows that $\partial_R^2 = 0$.
Step 3: *Claim:* $\partial_C \partial_R - \partial_R \partial_C = 0$

For the mixed terms we compute as follows

$$
\begin{aligned}
\partial_R \partial_C &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\
&= \sum_{j<i} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \le j} (-1)^{i+j} (G/e_i, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \qquad (*)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_C \partial_R &= \sum_{i=1}^{p} \sum_{j=1}^{p-1} (-1)^{i+j} (G/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\
&= \sum_{j<i} (-1)^{i+j} (G/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) + \sum_{i \le j} (-1)^{i+j} (G/e'_j, (\Phi \setminus \{e_i\}) \setminus \{e'_j\}) \\
&\overset{(\heartsuit)}{=} \sum_{j<i} (-1)^{i+j} (G/e_j, (\Phi \setminus \{e_j\}) \setminus \{e'_{i-1}\}) + \sum_{i \le j} (-1)^{i+j} (G/e_{j+1}, (\Phi \setminus \{e_{j+1}\}) \setminus \{e'_i\}) \\
&\overset{(\dagger)}{=} \sum_{l \le m} (-1)^{m+l+1} (G/e_l, (\Phi \setminus \{e_l\}) \setminus \{e'_m\}) + \sum_{k<n} (-1)^{n+k-1} (G/e_k, (\Phi \setminus \{e_k\}) \setminus \{e'_n\}) \quad (**)
\end{aligned}
$$

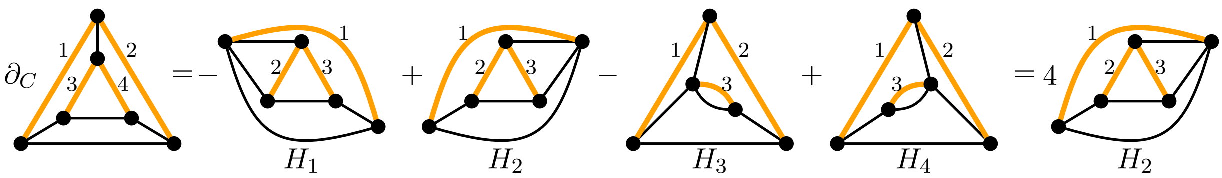
Where in $(\heartsuit)$ we used that if $j < i$ then $e_j = e'_j$ and if $i \le j$ then $e_{j+1} = e'_j$, as well as the observation above. In $(\dagger)$ we used the substitution $m = i - 1$, $l = j$ on the left and $m = i$, $k = j + 1$ on the right sum. Comparing the sums in $(*)$ and $(**)$ we see that they differ by a sign and thus cancel. Hence $\partial_C \partial_R - \partial_R \partial_C = 0$.

Combining Step 2 and 3 we get:

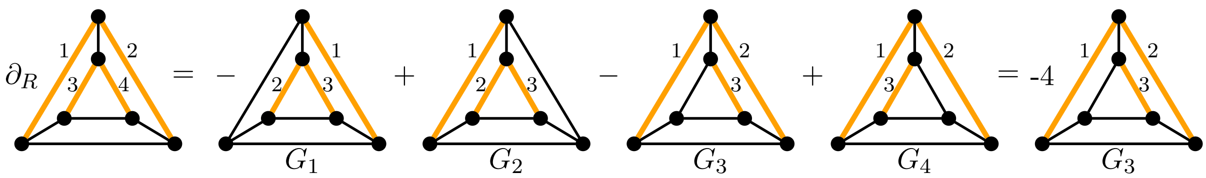
$$\partial^2 = (\partial_C - \partial_R)^2 = \partial_C^2 - (\partial_C \partial_R + \partial_R \partial_C) + \partial_R^2 = 0 \qquad \square$$

Thus the spaces $(C_\bullet^n)$ with the differential $\partial_\bullet$ form a chain complex.

Example 3.11. Once again we consider the graph (G, Φ) from example 3.4 and calculate its boundary operator:



Where we used that $-H_1$ is equal to H_2 by mirroring along the vertical and applying (23), $-H_3$ is equal to H_2 by exchanging inner and outer vertices, mirroring along the vertical and applying (13) and H_4 is equal to H_2 by exchanging inner and outer vertices and applying (13)(23). Where we have used the permutation (23) on the first, (13) on the third, (13)(23) on the fourth graph to get the result.



Where we have used that $-G_1$ is equal to $-G_3$ by exchanging inner and outer vertices and applying (13)(12), G_2 is equal to $-G_3$ by exchanging, mirroring along the vertical and applying (13) and G_4 is equal to $-G_3$ by mirroring along the vertical and applying (12).

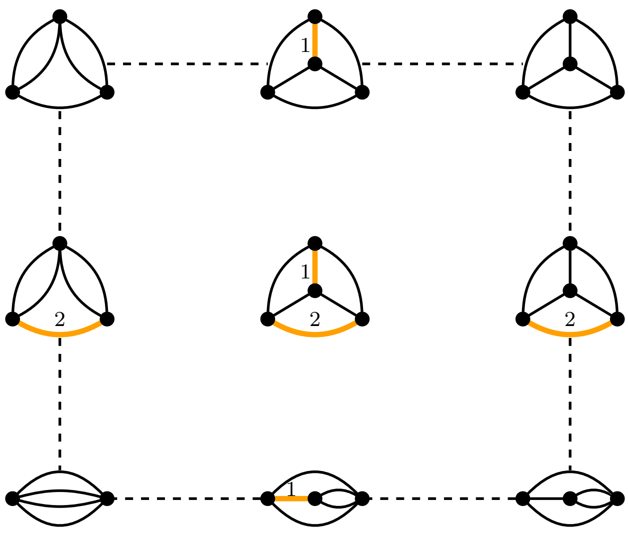
Thus we can conclude that $\partial_\bullet G = 4H_2 - 4G_3$. Moreover we have that $H_2 - G_3 \in \operatorname{Im} \partial_\bullet$ and as $\partial_\bullet^2 = 0$ also $H_2 - G_3 \in \operatorname{Ker} \partial_\bullet$

Remark 3.12. As the example above shows these calculations, especially identifying isomorphic graphs and finding odd automorphisms become rather tedious rather quickly. To this end these calculations can be left to the computer and an implementation of this in python can be found in the Appendix.

Coming back to the forested graph complex C_n it can be viewed as a cubical complex in a similar way as we did with the spine of Outer space in the introduction: The k-cubes are given by graphs (G, Φ, σ) with forest size k and the faces are $k-1$-cubes obtained from collapsing an edge in Φ or removing an edge in Φ from the forest. Collapsing the edge $e \in \Phi$ is on the opposite side of removing e from Φ on the k-cube. An orientation on the cube is induced by the signs from the boundary maps ∂_C and ∂_R.

To visualize this construction we consider the following example:

Example 3.13. Consider again the graph J from example 2.8 with the forest Φ given by an edge between the top and middle vertex and between the left and write vertex. Then its 2-dimensional cube is given as below:



4 Morita-Cycles

The goal of this section is to show that there exist a cycle in every C_n. For this we define the Morita-Cycle graphs and show that there exists a chain of those graphs that vanishes under the boundary ∂.

Definition 4.1. A Morita-Cycle $M_n(\sigma)$ for $n \in \mathbb{N}_{\geq 3}$, $\sigma \in \mathbb{S}_n$ is a forested graph (G, Φ, τ) defined as follows: G consists of two polygons with n vertices, each of which is connected to precisely one vertex on the other polygon. The vertices are labelled from 1 to n on one polygon and $n+1$ to $2n$ on the other then the connecting edges between them are given by $(i, \sigma(i)+n)$ i.e. sigma describes how the edges connecting the two polygons are permuted.

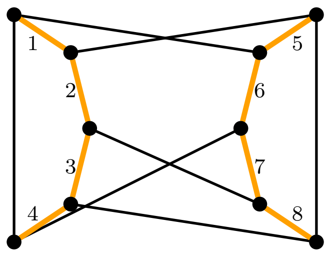
The forest Φ consists of the edges $(i, i+1)$ for $i \in \{1, \dots, n-1, n+1, \dots, 2n-1\}$ i.e. the forest consists of two linear trees of size $n-1$ each of which is obtained by removing one edge from the corresponding polygon. Finally the ordering τ is given by

$$\tau((i, i+1)) = \begin{cases} i & \text{for } 1 \leq i \leq n-1 \\ i-1 & \text{for } n+1 \leq i \leq 2n-1 \end{cases}.$$

Notice that for all even n the graph $M_n(\sigma)$ has an odd automorphism which is given by exchanging the two polygons and applying the permutation $(1\ n)\dots(n-1\ 2n-1)$. Thus those Morita-Cycles vanish. Moreover as $M_n(\sigma)$ has $2n$ vertices and $3n$ edges by Proposition 2.5 it has rank $n+1$. Furthermore $M_n(\sigma)$ is clearly an admissible graph and thus $M_n(\sigma) \in C_n$.

To give a better understanding of this definition we give the following example:

Example 4.2. Below is a Morita-Cycle of order 5 given by the permutation $(12)(345)$.



The following result proves that there exists a cycle in every C_n for $n \geq 4$.

Theorem 4.3. *For all $n \in \mathbb{N}_{\geq 3}$ odd it holds that*

$$\partial \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_n(\sigma) \right) = 0.$$

Let this sum be denoted by Z_n.

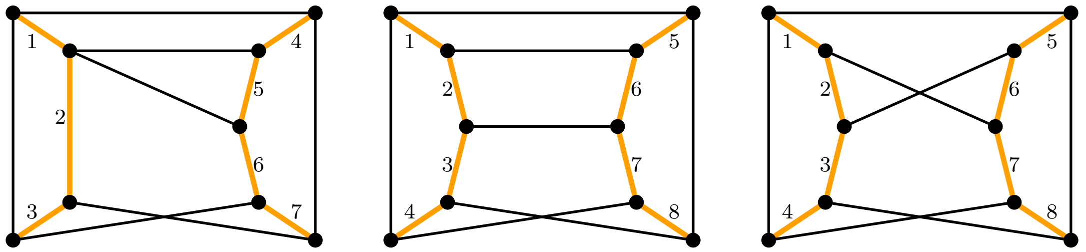
We prove this statement in two parts first for ∂_C and then for ∂_R, from which the final result follows.

Proof for ∂_C. Let (H, Ψ, η) be an element of the chain $\partial_C Z_n$. Then as it is an element of the boundary of some Morita-Cycle it has to have precisely one vertex of degree 4. W.l.o.g. we can assume that it is in the polygon containing the edge with the label 1 as else we can apply the even permutation $(1 \dots 2n-1)^n$ to the orientation.

Let k be the label of the degree 4 vertex an example is shown in the figure below on the left for $k = 2$. Then there exist exactly two Morita-Cycles in whose boundary H lies and whose induced node labeling corresponds to the one of H. One is obtained by splitting the vertex k into two vertices labeled k and $k+1$ where each new vertex contains one edge from the forest and one connecting to the other polygon. Moreover the two new nodes get connected by an edge which is also added to the forest and given the number k in the ordering. All other edges with ordering number $> k$ get increased by one. Lets denote this Morita-Cycle by $M_n(\sigma)$. The other one is obtained in the same way however the two edges connecting to the other polygon are permuted i.e. this Morita-Cycle equals $M_n((k\ k+1)\sigma)$. Examples for both are also shown in the figure below in the middle and right respectively.

Finally we see that the two permutations σ and $(k\ k+1)\sigma$ have opposite parity and thus $M_n(\sigma)$ and $M_n((k\ k+1)\sigma)$ have opposite sign in Z_n. Thus the elements in their boundary corresponding to H also have opposite sign and hence cancel.

As H was an arbitrary element of Z_n we get that every summand has coefficient 0 and that Z_n vanishes.



□

Proof for ∂_R. Again let (H, Ψ, η) be an element of the chain $\partial_R Z_n$. Then H is a Morita-Cycle whose forest Ψ is missing one edge in one of its trees. We denote by σ the edge permutation of H between its two polygons as given in the definition of Morita-Cycles. W.l.o.g. we can assume that the tree with the missing edge contains the edge with the label 1 as else we can apply the even permutation $(1 \dots 2n-1)^n$ to the orientation.

Let $(j, j+1)$ denote the extra edge missing from Φ. An example is shown in the figure below on the left for $j = 2$. Then there are exactly two Morita-Cycles in the sum which have H in their boundary and whose induced node labelling corresponds to the one of H.

One where the edge $(j, j+1)$ has been added to the forest with number j in the ordering and later edges have gotten their numbering increased by 1 i.e. $M_n(\sigma)$. And the other where the edge $(1, n-1)$ has been added and the ordering is given by

$$\tau((i, i+1)) = \begin{cases} i-j & \text{for } j+1 \leq i \leq n-1 \\ n-j+i & \text{for } 1 \leq i \leq j-1 \\ i-1 & \text{for } n+1 \leq i \leq 2n-1 \end{cases} \quad \text{and} \quad \tau(1, n-1) = n-j$$

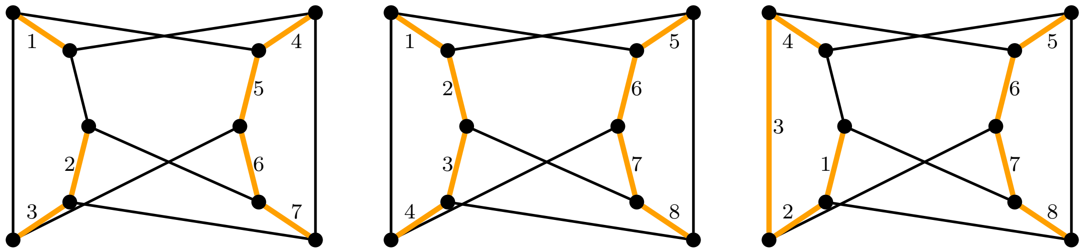
i.e. the numbering starts with 1 at the edge $(j+1, j)$ and increases until it reaches $n-1$ on the edge $(j-1, j)$. The ordering on the other polygon has not been altered.

Notice that the second graph is not part of Z_n as the node labelling and the forest ordering do not correspond anymore. To resolve this we have to change the node labelling which induces the permutation $(1 \dots n)^k$ on σ. Thus the element in the sum corresponding to this graph is given by $M_n(\tau) := M_n(\sigma(1 \dots n)^k)$. Examples for both are also shown in the figure below in the middle and right respectively. As $(1 \dots n)$ for n odd has even parity τ and σ have the same parity and thus $M_n(\tau)$ and $M_n(\sigma)$ have the same sign in the sum.

Finally taking the boundary the elements corresponding to H have different orientations. The element in $\partial_R(M_n(\sigma))$ is exactly H with sign $(-1)^k$. The element in $\partial_R(M_n(\tau))$ has sign $(-1)^{n-k}$

however it differs from H by the permutation $(1 \dots n-2)^{k-1}$ with even parity as $n-2$ is odd. Now for n odd if k is even $n-k$ is odd and vice versa. Thus the elements corresponding to H have opposite signs and cancel.

As H was an arbitrary element of Z_n we get that every summand has coefficient 0 and that Z_n vanishes.



Thus we have shown the result for both ∂_C and ∂_R and as $\partial = \partial_C - \partial_R$ it also follows that $\partial Z_n = 0$. □

Remark 4.4. The Morita-Cycles are also included in the python implementation in the Appendix and the above formula can be checked for small n. For $n = 3$ and $n = 5$ explicit isomorphism reduced representations of Z_n are also shown in the appendix. For larger n this becomes difficult very quickly as the number of Morita-Cycles is proportional to $n!$.

A more general result showing that a similar sum vanishes for n polygons instead of just two has been shown by Conant and Vogtmann in [3].

As we have shown that these sums of Morita-Cycles vanish under the boundary the question arises if they are non trivial in homology i.e. if they lie in the image of ∂ or not. Morita showed himself that the first cycle ($n = 3$) is non-trivial and conjectured that all of his classes are non-trivial. Conant and Vogtmann showed in [4] that the second cycle ($n = 5$) is also non-trivial, Gray extended this to show in [8] that the third cycle ($n = 7$) is non-trivial. Both of these calculations relied partly on computer calculations which increase extremely quickly with n. Thus Morita's conjecture remains a challenging open problem to this day.

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A Python implementation

To run the ensuing code graphviz, a version of python 3 as well as the following packages are needed:

- numpy
- sympy
- pydot
- networkx

The code consists of four files. In the `boundaries.py` file ∂_C,∂_R, ∂ as well as the reduction up to isomorphisms and the vanishing of graphs with odd automorphisms is implemented. The file `plotting.py` includes functions for plotting a single forested graph as well as chain. In `MoritaCycles.py` the generation of Morita-Cycles with regard to n and permutation as well as generating Z_n and producing positions for plotting is implemented. Finally in `main.py` a small working example reducing and plotting Z_n as well as calculating its boundary and printing it has been implemented.

The following two figures show the reduced versions of Z_3 and Z_5.

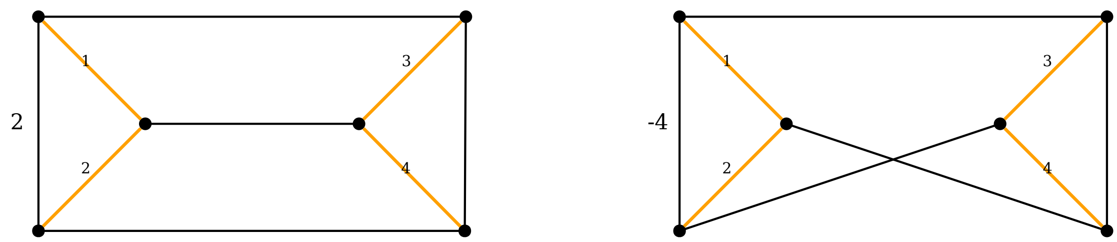

Figure 6: Isomorphism reduced version of Z_3

A.1 main.py

```
from boundaries import *
from MoritaCycles import *
from plotting import *

n=5
path = "MCCycle" + str(n)
C = createAllMCOf(n)
C = resolveIsos(C)
C = resolveVanishing(C)
pltChain(C,lambda x: createMCPos(n, center=x),path,lineWidth=3)

dC = delta(C)
print(dC)
```

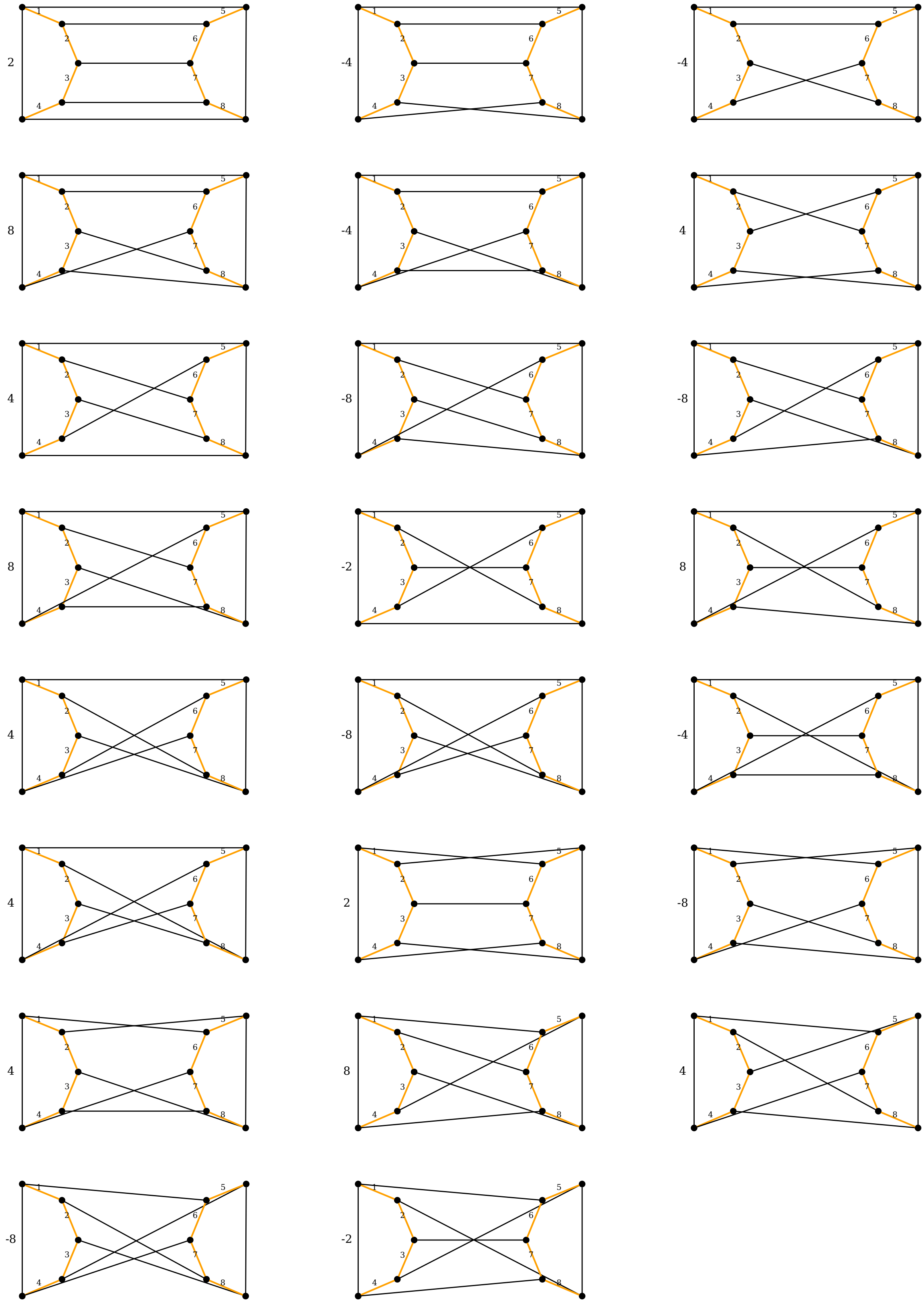

Figure 7: Isomorphism reduced version of Z_5

A.2 boundaries.py

```
import numpy as np
import networkx as nx
from networkx.algorithms import isomorphism as nxiso
from sympy.combinatorics import Permutation

def getForest(G):
    o = nx.get_edge_attributes(G, 'order')
    return {key: val for key, val in o.items() if val != -1}


def contractEdge(G, e):
    H = G.copy()
    H.remove_edges_from([e])
    return nx.contracted_nodes(H, e[0], e[1])


edgeEquality = nx.isomorphism.categorical_multiedge_match("forest",
    False)


def deltaC(C):
    dC = []
    for k,G in C:
        F = getForest(G)
        dG = []
        for e, i in F.items():
            H = contractEdge(G, e)
            FH = getForest(H)
            for a, j in FH.items():
                if j > i:
                    H.edges[a]['order'] = j - 1
            dG.append([(-1) ** i * k, H])
        dC += dG
    return dC


def deltaR(C):
    dC = []
    for k,G in C:
        F = getForest(G)
        dG = []
        for e, i in F.items():
            H = G.copy()
            H.edges[e]['forest'] = False
            H.edges[e]['order'] = -1
            FH = getForest(H)
            for a, j in FH.items():
                if j > i:
```

```
                    H.edges[a]['order'] = j - 1
            dG.append([(-1) ** i * k, H])
        dC += dG
    return dC


def getForestPerm(F1, F2, g):
    F1 = {(k[0], k[1]): v for k, v in F1.items()}
    F2 = {(k[0], k[1]): v for k, v in F2.items()}
    sig = np.zeros(len(F1))
    for e, i in F1.items():
        se = (g[e[0]], g[e[1]])
        if se in F2:
            sig[i - 1] = F2[se]
        else:
            sig[i - 1] = F2[(se[1], se[0])]
    perm = Permutation(sig - 1.0)
    return (-1) ** perm.parity()


def resolveIsos(dG):
    i = 0
    n = len(dG)
    while i < n - 1:
        H1 = dG[i]
        j = i + 1
        while j < n:
            H2 = dG[j]
            GM = nxiso.GraphMatcher(H1[1], H2[1],
   edge_match=edgeEquality)
            if GM.is_isomorphic():
                sgn = getForestPerm(getForest(H1[1]),
   getForest(H2[1]), GM.mapping)
                H1[0] += sgn * H2[0]
                del dG[j]
                n += -1
            else:
                j += 1
        i += 1

    # remove 0s
    result = [G for G in dG if G[0] != 0]
    return result

def resolveVanishing(dG):
    def hasOddAuto(H):
        GM = nxiso.GraphMatcher(H, H, edge_match=edgeEquality)
        for g in GM.isomorphisms_iter():
            F = getForest(H)
            if getForestPerm(F, F, g) == -1:
```

```python
                return 1
        return 0

    result = [G for G in dG if not hasOddAuto(G[1])]
    return result


def delta(G):
    dGC = deltaC(G)
    dGR = deltaR(G)
    dGC = resolveIsos(dGC)
    dGR = resolveIsos(dGR)

    dGR = np.array(dGR, dtype=object)
    dGR[:, 0] *= -1
    dG = dGC + dGR.tolist()
    return resolveVanishing(dG)
```

A.3 plotting.py

```python
import os
import networkx as nx
from boundaries import *

def pltFG(H, pos,path = "out"):
    G = H.copy()
    forest = getForest(G)
    nx.set_edge_attributes(G,{key: {"color": "red", "label": val}
   for key, val in forest.items()})
    for n, npos in pos.items():
        if(G.has_node(n)):
            G.nodes[n]['pos'] = '"%d,%d"'%(npos[0],npos[1])
            G.nodes[n]['shape'] = "point"
    p = nx.drawing.nx_pydot.to_pydot(G)
    p.write(path + ".dot")
    os.system("neato -n2 -Tpng " + path + ".dot -o " + path + ".png")

def pltChain(C,posFunction,path="out",lineWidth=5):
    masterG = nx.MultiGraph()
    hPos = 0
    for j, (count,H) in enumerate(C):
        if j!= 0 and j % lineWidth == 0:
            hPos -= 3.0
        vPos = 6 * (j % lineWidth)
        pos = posFunction((vPos, hPos))
        G = H.copy()
        nx.set_edge_attributes(G,2.0,name="penwidth")
        forest = getForest(G)
        nx.set_edge_attributes(G, {key: {"color": "#ffa000",
   "penwidth": 3, "label": val} for key, val in forest.items()})
```

```
        for node, nodePos in pos.items():
            if(G.has_node(node)):
                G.nodes[node]['pos'] = '"%d,%d"' % (nodePos[0],
nodePos[1])
                G.nodes[node]['shape'] = "point"
                G.nodes[node]['width'] = "0.15pt"
        masterG = nx.disjoint_union(masterG,G)
        cof = str(j) + "coef"
        masterG.add_node(cof)
        masterG.nodes[cof]['label'] = count
        masterG.nodes[cof]['shape'] = "plaintext"
        masterG.nodes[cof]['fontsize'] = "20pt"
        masterG.nodes[cof]['pos'] = '"%d,%d"' %(100*(-2.2 + vPos),
hPos*100)


    p = nx.drawing.nx_pydot.to_pydot(masterG)

    p.write(path + ".dot")
    os.system("neato -n2 -Tpdf " + path + ".dot -o " + path + ".pdf")
```

A.4 MoritaCycles.py

```
import numpy as np
import networkx as nx
from itertools import permutations
from sympy.combinatorics import Permutation

def createMC(n, i=0,j=0, perm = 0):
    H1 = nx.cycle_graph(np.arange(n))
    H2 = nx.cycle_graph(np.arange(n, 2 * n))
    G = nx.MultiGraph(nx.compose(H1, H2))
    if perm == 0:
        G.add_edges_from(np.arange(2 * n).reshape((2, -1)).T)
    else:
        G.add_edges_from(np.vstack((np.arange(n),perm)).T)
    nx.set_edge_attributes(G, False, 'forest')
    nx.set_edge_attributes(G, -1, 'order')

    forest = np.hstack((np.vstack((np.vstack((np.arange(n - 1),
np.arange(1, n))).T,
                        np.vstack((np.arange(n, 2 * n - 1),
np.arange(n + 1, 2 * n))).T)),np.zeros((2*n-2,1))))
    if i != n-1:
        forest[i] = [0, n - 1,0]
    if j != n-1:
        forest[n-1 + j] = [n,2*n-1,0]

    for i, e in enumerate(forest):
        G.edges[e]['forest'] = True
        G.edges[e]['order'] = i + 1
```

```
    return G

def createAllMCOf(n):
    dG = []
    for perm in permutations(np.arange(n,2*n)):
        G = createMC(n,n-1,n-1,perm)
        perm = Permutation(np.array(perm)-n)
        dG += [[(-1) ** perm.parity(),G]]
    return dG

def createMCPos(n, center=(0,0)):
    scale = 100.0
    lAng = np.linspace(np.pi / 2, -np.pi / 2, n)
    lPoints = scale*np.vstack((np.cos(lAng) - 2.0 + center[0],
   np.sin(lAng) + center[1])).T
    lPos = dict(enumerate(lPoints.tolist(), 0))

    rAng = np.linspace(np.pi / 2, 3 * np.pi / 2, n)
    rPoints = scale*np.vstack((np.cos(rAng) + 2.0 + center[0],
   np.sin(rAng) + center[1])).T
    rPos = dict(enumerate(rPoints.tolist(), n))
    return lPos | rPos

def rotateMCPos(pos,r):
    n = int(len(pos)/2)
    pos = np.array(list(pos.values()))
    lPos = np.roll(pos[:n],r,axis=0)
    rPos = np.roll(pos[n:], r, axis=0)
    return dict(enumerate(lPos.tolist(), 0)) |
   dict(enumerate(rPos.tolist(), n))
```