0.1 Einführung

In the following we mainly follow Bartholdi's hands on approach from [1] to define the forested graph complex. The more general definition from Conant and Vogtmann from [2] has been postponed as it is too complex and unintuitive for the beginning. Very useful in the general understanding of what a graph complex is and how the boundary map acts was Bar-Natan's draft []. Inspired by this similar examples for the forested graph complex are presented.

Definition. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$. A loop is an edge having the same start and endpoint. We call a graph *connected* if the CW complex is connected in the topological sense. A graph is n-connected if it remains connected after removing $n-1$ arbitrary edges. A graph is said to be *n-regular* if every vertex has valency n i.e. for every vertex the number of incident edges is n.

For a collection of edges Φ of G we denote by G/Φ the graph quotient, which is the quotient space of the CW complex G over its topological subspace Φ.

Lastly a *tree* is a connected graph which contains no cycles and a *forest* is a collection of disjoint trees.

Definition. As the fundamental group of a connected Graph is isomorphic to a free group, it makes sense to define the rank of a graph as the rank of its fundamental group $\pi_1(G)$.

For a proof of this see for example [4, p. 43f]

Proposition. *The rank of a graph G is equal to:*

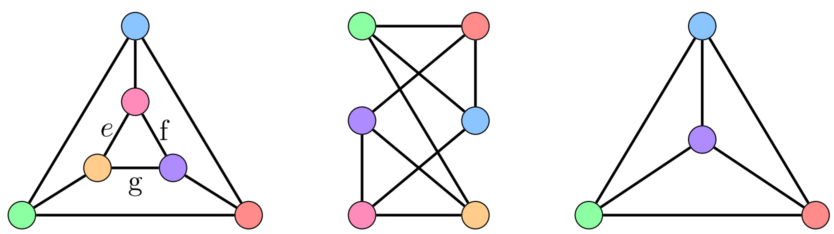
1. *The number of independent cycles in G*
2. *The first Betti number i.e. the rank of $H_1(G)$*
3. *the Euler Characteristic of G given by $\chi(G) = |C(G)| - |V(G)| + |E(G)|$, where $|C(G)|$ is the number of connected components of G.*

Proof Sketch. That the rank is equal to the first Betti number follows from Hurewicz Theorem and the fact that the abelianisation of the free group of rank n is the free abelian group of rank n.

The equality of 2) and 3) follows from the argument for the fundamental group in [4, p. 43f]. Lastly a proof of the equality of 1) and 3) can be found in [3, p. 37-40]. □

Beispiel. Consider the following graph G.

Then G is 3-connected and 3-regular. Moreover G is isomorphic to H. An isomorphism between them is given by mapping the same colored nodes to each other. The graph quotient $G/\{e, f, g\}$ is given by J.



G H J

Definition. Let G, H be two graphs. A map $f : V(G) \to V(H)$ is said to be a graph isomorphism if f is a bijection such that

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H).$$

Let F_n denote the free group of rank n.

Definition. An *admissible graph of rank* n is a 2-connected loop less graph G with fundamental group isomorphic to F_n and with vertex-valency ≥ 2.

We often just write admissible graph for an admissible graph of rank n.

Definition. Let G be an admissible graph. Its *degree* is given by

$$\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).$$

In particular G has $2n - 2 - \deg(G)$ vertices and $3n - 3 - \deg(G)$ edges and is 3-regular iff $\deg(G) = 0$.

Definition. An orientation σ of a graph G is an ordering of the edges i.e. σ is an injective function from $E(G)$ to $\{1, \dots, |E(G)|\}$. We call the tuple (G, σ) an oriented graph and note that Sym acts on (G, σ) by $\pi(G, \sigma) = (G, \pi\sigma)$ for $\pi \in \mathrm{Sym}$.

If it is clear that (G, σ) is an oriented graph we often just write G.

Definition. A *forested graph* is a pair (G, Φ) where G is an admissible graph and Φ is an oriented forest which contains all vertices of G.

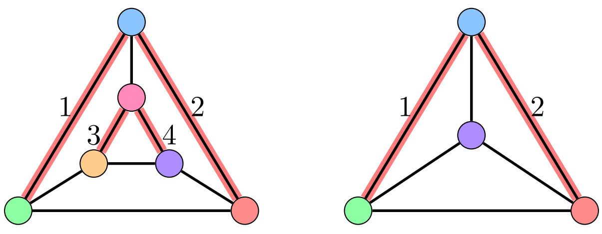
A map f between two forested graphs $(G, \Phi), (H, \Psi)$ is said to be a forested graph isomorphism if f is a graph isomorphism and $\sigma \circ f\mid_\Phi$ is the identity, where σ is the orientation on Φ. Is this correct?

For $k \in \mathbb{N}$ let C_k denote the $\mathbb{Q}$-vector space spanned by isomorphism classes of forested graphs of rank n with a forest of size k, subject to the relation

$$(G, \pi\Phi) = \operatorname{sgn} \pi \cdot (G, \Phi) \quad \text{for all} \quad \pi \in \mathrm{Sym}(k).$$

Observe that if $(G, \Phi) \simeq (G, \pi\Phi)$ for an odd permutation π then $(G, \Phi) \simeq (G, \pi\Phi) = -(G, \Phi)$ and thus $(G, \Phi) = 0$ in C_k.

Beispiel. Consider the graphs G, J as in Example 0.1. Then G and J are admissible graphs of rank 3 and 2. Thus if we equip them with oriented forests Φ, Ψ as below (where the red edges represent the forest and the numbers the orientation) we get forested graphs.



Observe, that $(J, \Psi) = 0$ in C_k, since (12) is an odd permutation and $(12)(J, \Psi)$ is isomorphic to (J, Ψ) via the isomorphism mirroring along the vertical passing through the blue and purple vertex.

(G, Φ) however is not trivial which can be seen as follows :

show that

To turn these spaces into a chain complex we define a differential as follows:

Definition. Let $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ be a forested graph. Then let

$$\partial_C(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}),$$
$$\partial_R(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G, \Phi \setminus \{e_i\})$$

and define $\partial = \partial_C - \partial_R$.

define how to renumber edges

Proposition. *∂ is well-defined and $\partial^2 = 0$.*

Beweis. □

Thus the spaces $(C_\bullet)$ with the differential $\partial_\bullet$ form a chain complex.

Beispiel.

Literatur

[1] Laurent Bartholdi. *The rational homology of the outer automorphism group of F_7*. 2016. arXiv: `1512.03075 [math.GR]`.

[2] James Conant und Karen Vogtmann. „On a theorem of Kontsevich“. In: *Algebraic & Geometric Topology* 3.2 (Dez. 2003), S. 1167–1224.

[3] Frank Harary. *Graph Theory*. Reading Mass., Menlo Park Calif.: Addison-Wesley, 1969.

[4] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000.

0.2 Draftparts

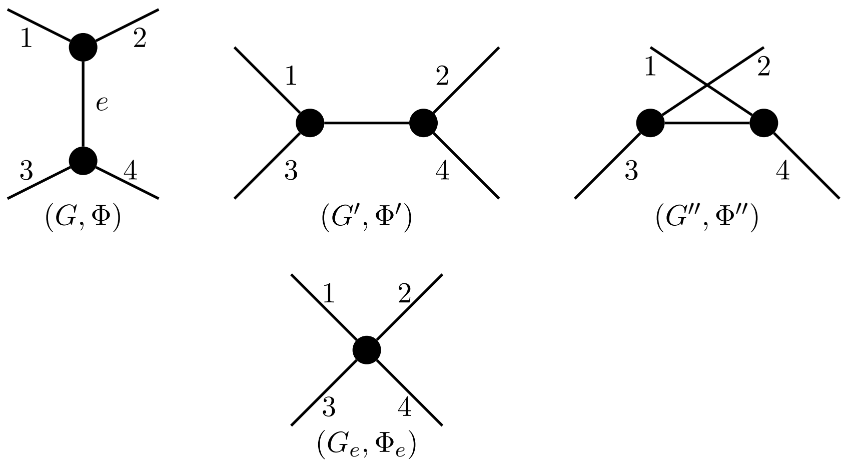
Definition. A *graph* G is a finite 1-dimensional CW complex. The set of edges is denoted by $E(G)$, the set of vertices by $V(G)$ and the set of half edges by $H(G)$. We call a graph *connected* if the CW complex is connected in the topological sense. A graph is said to be *n-valent* if every vertex has valency n i.e. for every vertex the number of edges incident is n.

Lastly a *tree* is a graph which contains no loops and a *forest* is a collection of disjoint trees.

On trivalent connected graphs we call an *orientation* a choice of cyclic orders of all vertices up to an even number of changes.

Definition. A *forested graph* is a pair (G, Φ), where G is a finite connected trivalent graph and Φ is an oriented forest which contains all vertices of G.

Definition. Let (G, Φ) be a forested graph and let $e \in \Phi$. Moreover let (G_e, Φ_e) be the graph where e has been collapsed. Then there exist exactly two other graphs whose edge collapse results in (G_e, Φ_e). This is visualised in the figure below. Where $1, 2, 3, 4$ represent the rest of the graph.



Now the vector

$$(G, \Phi) + (G', \Phi') + (G'', \Phi'')$$

is called the *basic IHX relator* associated to (G, Φ, e).

Denote by $\widehat{f\mathcal{G}_k}$ the vector space spanned by all forested graphs containing k trees modulo the relations $(G, \Phi) = -(G, -\Phi)$. Moreover let $f\mathcal{G}_k$ be the quotient of $\widehat{f\mathcal{G}_k}$ modulo the subspace spanned by all basic IHX relators.

Definition. Let $\widehat{\partial}_E(G, \Phi) : \widehat{f\mathcal{G}_k} \to \widehat{f\mathcal{G}_{k-1}}$ be given by

$$\widehat{\partial}_E(G, \Phi) = \sum (G, \Phi \cup e).$$

where the sum is over all edges e of $G \setminus \Phi$ such that $\Phi \cup e$ is still a forest. Notice that this only happens if the two vertices of e lie in different trees of Φ. Thus $\Phi \cup e$ has $k-1$ components. The orientation of $\Phi \cup e$ is determined by ordering the edges of Φ with labels $1, \ldots, k$ consistent with its orientation and then labeling the new edge e with $k+1$.

Now let the boundary map $\partial_E : f\mathcal{G}_k \to f\mathcal{G}_{k-1}$ be the map induced by $p \circ \widehat{\partial}_E$ where p is the quotient map $\widehat{f\mathcal{G}_k} \to f\mathcal{G}_k$.

Proposition. *∂_E is well-defined and $\partial_E^2 = 0$.*

Beweis. □

The *forested graph complex* is thus defined as the sequence $f\mathcal{G}_k$ with boundary map ∂_E and is well-defined by the above proposition.