Tichu strategies depending on course of play

Matthias Schmickler   Paul Sander   Jean-Luc Portner   Tom Haidinger
Andrey Bryutkin

Swiss Federal Institute of Technology Zurich (ETH)

June 03, 2020

1 Motivation and goals

A prevalent characteristic of both mathematics and physics is their elaborate structure of interdependent concepts. It is only natural, then, for us students to thrive off of complexity and intricateness both within our courses as well as throughout our daily life. One such endeavour we satisfy our free time with is the game of Tichu. Tichu is a team card game taught to us by older students of our courses during the, albeit not very wide-ranging, cultural melting-pot between years – that unproductive period when our courses' study hall slowly turns into the Mensa.

In addition to luck and strategy, this game became popular within our friend-group due to its fairly unique aspect of interaction. Allowing for multiple possible card combinations, special cards and a process of card switching, Tichu derives its complexity not through complicated rules, but rather through diverseness of strategy. Indeed, given one must adapt to one's opponents' and teammate's strategies, one may say its strategies are interactive and reactive in nature. Therefore – although we note that card games are hard to analyse in general – we believe Tichu is indeed approachable through game-theoretic methods.

Fascinated by this type of game and equipped with tools from our game theory course, we pose the general question: "which strategy should we pursue as players?". We shall further differentiate this admittedly vague question into: "what strategy should we pursue given": a) "we know what player-type all other players are" and b) "we do not know anything about other players".

2 Introduction on Tichu

Tichu is played by four players, where two players form a team. Tichu is subdivided into subgames, which are played repeatedly until one team reaches 1000 points and in this way wins the game. At the beginning of a game, every player gets 14 cards. He exchanges one card with every other player. This will be called the "Exchange Stage", which we will discuss later on. After the exchange of the cards, the regular game takes place. There are two major ways to achieve points: 1st Claiming Cards with a value, 2nd Announcing a Tichu. The Game ends, when one team finished of their cards. If one team finished of their cards before one of the opposing team finished, then the winner team receives the cards from both opponent players. If the team finished of their cards and someone from the opposing team finished before, only the last player passes away the cards to the winner team. Then you count the values of the cards and this value to your score.

A Tichu-set consists of the cards 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A in four different colors: red, blue, green and black. There are four special cards Dragon, Phoenix, Dogs and Mah Jong. In total we have 56 cards, every player receives 14 cards. However, only a few cards have a value as listed in the table:

5	$\sim$	**5 P**
10	$\sim$	**10 P**
K	$\sim$	**10 P**
Dragon	$\sim$	**25 P**
Phoenix	$\sim$	**-25 P**

In total, there are 100 Points every round. It is also possible to win a game with 125 or also -25 Points. Announcing a Tichu can earn your team additional points. A game starts, when a player plays a valid combination. The other players can now try to play a higher version of the same combination (as many cards as the combination). If no player can or wants to play a higher combination, the player with the highest combination takes the cards and can play a new combination. There are several valid combinations:

High Card	$\sim$	**2** $<$ **3** $<$ ... $<$ **A**
Pair	$\sim$	**(2,2)** $<$ ... $<$ **(A,A)**
Triple	$\sim$	**(2,2,2)** $<$ ... $<$ **(A,A,A)**
Following Pairs	$\sim$	**(2,2,3,3, ...)** $<$ ... $<$ **(..., K,K,A,A)**
Full House	$\sim$	**(2,2,2,A,A)** $<$ ... $<$ **(A,A,A,K,K)**
Street(more than 5 cards)	$\sim$	**(2,3,4,5,6, ...)** $<$ ... $<$**(...,10,J,Q,K,A)**

This combinations can only be played on the same combination when it is the players turn. Following Pairs and Streets can be continued up to 14 cards. There are also bombs, which can be played at any time and on every combination (except higher or equal bombs). We differ between two kinds of bombs:

Bomb	~	**(2,2,2,2), < ... < (A,A,A,A)**
Street Bomb	~	**(A,A,A,A) < (2,3,4,5,6) < (2,3,4,5,6, ...) < ... < (...,10,J,Q,K,A)**

In a street Bomb all colors have the same color. A Street Bomb is always higher than a shorter Street Bomb or a regular Bomb.

The four special cards can be played under certain conditions,but special cars can not be player in a Bomb or Streetbomb:

Mah Jong The player with this card starts the game. Mah Jong can be played as a **1** as a High Card or in a Street. The player is also allowed to request a not special card. Whoever has this card and can play it, has to play it.

Dog The dogs can not be played in any combination. When a player uses the dog, his partner is allowed to play the next combination. If his partner is already done, the next active player is allowed to play.

Phoenix The phoenix can be used in two different ways. It is a valid substitute for every not special card and can be played in combinations. It can also be played on a card and count as Card + 0.5. It can also be used on an A but not on the Dragon.

Dragon The dragon is the highest High Card and can be played on every high card including the phoenix. After winning the combination with the dragon, the player has to give the cards to qqone of his enemies.

Announcing a Tichu

A Player can announce a Tichu as long as he has not played any card yet. If he announces a Tichu, he has to be the first one who finishes the game. If he succeeds, he will earn his team 100 additional points. If he fails, his team loses 100 points. A stronger version of Tichu is the Great Tichu, which is worth 200 points if won. A Great Tichu has to be announced after the player takes the first eight cards before the regular game starts. It also has an impact on the Exchange Stage. If the two players from one team finish the game first, they get 200 points and the cards will not be counted. This will be called a "Double Victory".

3 Our Model and approximations

3.1 Basic assumptions

Our basic approximation consists of two distinctions regarding player behavior and transparency of information.

In our consideration we would like to define the following two types of players. The first player is the "aggressive player" - he is characterized by the fact that he takes a higher risk and pays more attention to the value of his cards and less to his playing partner. In case of doubt he will keep the good cards and give his partner something worse than the other way round. The willingness to take risks is characterized by the fact that even with below average cards (relative to cards with which the average player would announce a tichu) he already announces the tichu.

The second player is the "defensive player". This stands in contrast to the "aggressive player type". He is rather risk-averse, which means he needs above average (relative to cards with which the average player would announce a tichu) good cards to announce a tichu. When exchanging cards he decides to give the good card to his partner instead of keeping it himself.

To make it easier to identify the players in the following, the following notation is introduced: We will store information on the players behaviour in a vector $\alpha \in \{0,1\}^4$.

- A: "aggressive player" will be assigned to $\alpha_i = 1$
- D: "defensive player" will be assigned to $\alpha_i = 0$

In our approximation of Tichu, we consider two possible situations. The first situation is characterized by transparency. Every player knows exactly which type the other players are and acts up to this information. In the second situation, the players are not aware of the tactics. They only know how they act by themself.

Like the different types of players, we will identify the situations with the following notation. The information is stored in a vector $\beta \in \{0,1\}^4$.

- T: "transparent" will be assigned to $\beta_i = 1$
- $\overline{T}$: "not transparent" will be assigned to $\beta_i = 0$

We will assume $\beta_i = \beta_j \forall i, j \in \{1,2,3,4\}$ in the beginning.

We also introduce the possibility to take a higher risk depending on the current score. If a player is risky, he will definitely announce a Tichu. If he is shy, he will never announce a Tichu. We will store information on the player's possibility to announce a tichu in a vector $\gamma \in \{0,1\}^4$.

- R: "risky" will be assigned to $\gamma_i = 1$
- $\overline{R}$: "not risky" will be assigned to $\gamma_i = 0$

While α is invariant, γ is dynamically changing throughout the game.

Our last variables will be the actual score of each team and the rounds played, which will be stored in $\delta \in \mathbb{Z}^4$. $\delta_{1,2} \widehat{=}$ total score, $\delta_{3,4} \widehat{=}$ last round score

We have also adopted the following approximations about the game. The Great Tichu, i.e. the announcement of a Tichu before the exchange of cards, is very rare and is therefore negligible for our basic model. Also the possibility to announce a Tichu during the game, i.e. when other players have already played cards, is neglected. Furthermore, all rules are known to the players ("common knowledge") and they act rational to win the game and not harm their own team.

3.2 Introducing the game model

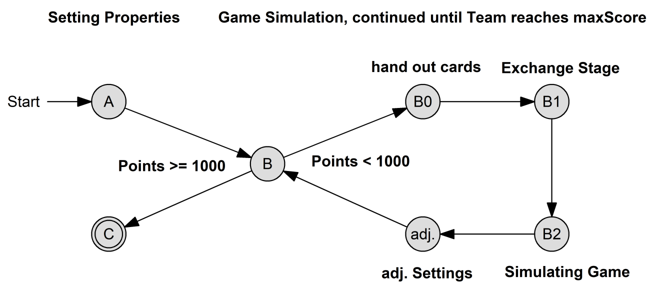

Figure 1: Game model

The game up to 1000 points is represented as a game tree. At the beginning 4 players, divided into 2 teams enter the game (A). We will set α and β. The next part of the game (B) will be repeated until the score reaches at least 1000 points. We divided it into three subgames (B.0, B.1, B.2). First, we will give every player a set of cards (B.0). This set of cards will be represented by a value X_i in [0,1]. The sum of all sets can be higher than 1 in total. The value of the set is not connected to single cards but to a probability to play these cards in combinations. We will discuss this in detail later on.

In the basic model, B.0 is only connected to statistics and no game in meanings of game theory. Later on, we can introduce game-theoretical elements to this subgame.

The second part (B.1) represents the "Exchange Stage". Depending on his Cards and his strategy, every player will give away three Δx and will receive three Δx which can be positive or negative. The payoff will be represented by a Util$\in [0, 1]$ with "new" card values.

The third subgame (B.2) takes the card values X from subgame B.1 and the score. They are used to determine the outcome of the actual game, reflecting on the players strategies and possibilities to take a higher risk because of the score. This will result in a Util function which allocates the 100 points (and additional Tichu-points) to the teams.

To every game B, the parameters $\alpha, \beta, \gamma, \delta$ are considered and can be adjusted. The repetition of subgame B will end when one team reaches at least 1000 points.

In first approximations we will also consider β and γ as invariant.

Subgame A

To start a round of Tichu, we will assign a player α_i and β_i. As assumed before, all β_i are equal. Therefore, we will have two options on β and 16 options on α. Some of those options are only permutations, because of the team-aspect. We will reduce the number of possible α to 6 as shown in the table. We will call this combination $\tilde{\alpha}_i$:

Player 1 \ Player 2	A	B
A	1 (AA)	2 (AD = DA)
B	2 (DA = AD)	3 (DD)

Player 1 \ Player 2	1	2	3
1	α_1 (AAAA)	α_2 (AAAD)	α_3 (AADD)
2	α_2 (ADAA)	α_4 (ADAD)	α_5 (DDAD)
3	α_3 (DDAA)	α_5 (ADDD)	α_6 (DDDD)

3.3 Subgame B.0

In our basic model, B.0 is invariant under all α, β, γ and δ. It is only based on statistics. To start off with a simple approximation, we assign X_i to be an element of the normal distribution with parameters μ, σ. Our first assumptions is, that we set μ to 0.5 and will define X_i higher or equal to 1 as 1, X_i lower or equal to 0 as 0. According to this definition, X_i is always in $[0, 1]$. We will set σ later according to realistic data and surveys. We will now assume that we will always get an $X = \{X_1, \ldots, X_4\}$ from subgame B.0. This will be the card value used in later games.

3.4 Subgame B.1

To understand the "Exchange Stage", we have to give some basic information on Tichu which leads to our approximation of this subgame. First of all we definie some terms:

Definition 3.1 "GOOD" CARDS

"Good" cards are for example bombs, high cards like ace or long, specific combinations that other players do not have.

Definition 3.2 "BAD" CARDS

Bad cards can rarely be played, for example deep cards or combinations that are never the highest combinations, so you rarely win a trick

1. A player can give “good” cards to his partner if and only if he has “good” cards.

2. A player can give “bad” cards to his opponents if and only if he has “bad” cards.

3. The value of a card is subjective to a player.

4. There is no absolute solution for every situation, the best solution is subjective to a player

Definition 3.3 PLAYER, TEAM, PARTNER AND OPPONENT

Given α, β and X, a player P_i is called the triple (α_i, β_i, X_i). On the set of players, we define an equivalence relation, where each class forms a Team T_j. Following conditions are true for players A (α_A, β_A, X_A), B(α_B, β_B, X_B), C(α_C, β_C, X_C):

(P1) A $\sim$ B (Reflexiv)

(P2) A $\sim$ B $\Leftrightarrow$ B $\sim$ A (Symmetric)

(P3) A $\sim$ B, B $\sim$ C $\Rightarrow$ A $\sim$ C (Transitivity)

Players in the same Team will be called partners while players in another Team will be called opponents. We will write PT_i for the Partner Team of P_i and OT_i for the Opponent Team of P_i.

We define $T_1 = [P_1, P_2]$ and $T_2 = [P_3, P_4]$. Therefore, $PT_1 = T_1$, $OT_1 = T_2$, $PT_2 = T_2$, $OT_2 = T_1$.

Definition 3.4 SYMMETRIC GAME PERSPECTIVE

We will say, two players P_i, P_j with $i \neq j$ share a symmetric game perspective if $P_i = P_j$, $PT_i = PT_j$, $OT_i = OT_j$, in words: They have the same cards and the teams are identically from their point of view.

Definition 3.5 TOTAL EXCHANGE-FUNCTION

We assume we have given α, β, X. A function $\pi : \{0,1\}^8 \times [0,1]^4 \to [0,1]^4, (\alpha, \beta, X) \mapsto (X')$ will be called an Exchange-function to B.1 if it applies to the following rules:

Randomness and Average α, β, X are fixed. Then there is $\Delta X \subset [0,1]^4$ with $X' \in \Delta X$. We will write $\pi^*(\alpha, \beta, X) = X^* \in \Delta X$ for the average of $\pi(\alpha, \beta, X)$. π^* is a well-defined function, while π allows random values around π^*.

Continuous and monotony of X α, β and X are fixed. If we change X in only one value that X_i becomes $wide\hat{X}_i$, it applies to the following rules:

(C1) $\forall \epsilon > 0, \exists \delta > 0$ with $|X_i - \widehat{X_i}| < \delta$
$\Rightarrow |\pi^*(X_i) - \pi^*(\widehat{X_i})| < \epsilon$

(C2) $\widehat{X_i} \leq X_i \Rightarrow \pi^*(\widehat{X_i}) \leq \pi(X_i)$

Symmetric outcome If two players P_i, P_j have a symmetric game perspective, $\pi^*(X_i) = \pi^*(X_j)$

Based on our approximations of the subgame, we now try to construct an Exchange Function. First of all, we try to create an easier function, which describes how a single Player selects cards he wants to give away. If we only look at this function, we will see some facts:

(F1) For Player P_i, only α, β_i and X_i are relevant.

(F2) Player P_i does not differ between Opponents.

(F3) The Player $P_i's$ strategy does not change his behaviour on Opponents.

(F4) In fact, he only needs to know his strategy, who he selects cards for and if known, his partner's strategy.

We put these facts together in a Diagram (the numbers represent diffrent exchange functions and P_P - is a Player passing; P_R - is a Player receiving):

Table 1: Table 1.1

	$\beta_i = 0$			$\beta_i = 1$		
OT	$P_R \backslash P_P$	A	D		A	D
	A	1	1	A	2	2
	D	1	1	D	2	2
PT	$P_R \backslash P_P$	A	D		A	D
	A	3	4	A	5	6
	D	3	4	D	7	8

1. The Player is exchanging with an unknown Opponent. Therefore he will select a "bad" card, independent of his own strategy

2. The Player knows his opponent. He will also give him a "bad" card. The only difference is, that he knows better how the player will react on the card. This will change how an Opponent receives the card, but is equivalent to 1.

3./4. The Player gives cards to his Partner depending on his strategy. He does not know the strategy of his Partner.

5.-8. The Player knows his own and his Partners strategy. Therefore his results will be better and the card difference will change.

Definition 3.6 PASS-FUNCTION

A function $\xi_{ji} : \{0,1\}^5 \to [-1,1], (\alpha, \beta_i) \to \hat{\xi}_{ji}$ will be called a pass function, if it describes what P_i gives to P_j according to certain rules:

Randomness and Average α, β_i are fixed. Then there is $\Delta\hat{\xi_{ji}} \subset [-1,1]$ with $\hat{\xi_{ji}} \in \Delta\hat{\xi_{ji}}$. We will write $\{xi_{ji}\}^*(\alpha, \beta_i) = \hat{\xi_{ji}} \in \Delta\hat{\xi_{ji}}$ for the average of $\xi(\alpha, \beta_i)$. ξ^*_{ji} is a well-defined function, while ξ allows random values around ξ^*.

Symmetric outcome If two players P_i, P_j have a symmetric game perspective towards each other, $\hat{\xi_{ji}}^* = \hat{\xi_{ji}}^*$

Normative aspects $\sum_{j=1}^4 \hat{\xi_{ji}} = 1$

Realistic passing $\hat{\xi_{ii}} >> \hat{\xi_{ji}}$ with $i \neq j$

Definition 3.7 RECEIVE-FUNCTION

A function $\eta_{ji} : \{0,1\} \times [-1,1] \to [-1,1], (\beta_i, \hat{\xi_{ji}}) \mapsto \hat{\eta_{ji}}$ will be called a "Receive-Function", if it describes how P_j receives cards from P_i according to certain rules:

Randomness and Average β_i, $\hat{\xi_{ji}}$ are fixed. Then there is $\Delta\hat{\eta_{ji}} \subset [-1,1]$ with $\hat{\eta_{ji}} \in \Delta\hat{\eta_{ji}}$. We will write $\{\eta_{ji}\}^*(\beta_i, \hat{\xi_{ji}}) = \hat{\eta_{ji}} \in \Delta\hat{\eta_{ji}}$ for the average of $\eta(\beta_i, \hat{\xi_{ji}})$. η^*_{ji} is a well-defined function, while η allows random values around η^*.

Symmetric outcome If two players P_i, P_j have a symmetric game perspective towards each other: $\hat{\xi_{ji}}^* = \hat{\xi_{ij}}^*$

Self passing $\eta_{ii}(\hat{\xi_{ii}}) = \hat{\xi_{ii}}$

Definition 3.8 EXCHANGE-FUNCTION

A function $\lambda_{ji} : \{0,1\}^5 \to [-1,1], \lambda_{ji} = \eta_{ji} \circ \xi_{ji}$ with η_{ji} a Receive-Function and ξ_{ji} a Pass-Function with existing α, β_i.

Remark (Construction). We will now construct a Total Exchange Function based on an Exchange Function λ_{ji}. We will define $\Lambda = (\lambda_{ji})$ as a quadratic matrix. The Total Exchange-Function will be $\Lambda : X \mapsto \Lambda X = X'$ We now have to prove that this construction fits the definition.

(1) Randomness and Average: This is induced from the combination of ξ_{ji} and η_{ji} in every argument. Therefore we can find an $\Delta\Lambda$ and Λ^* which lead to ΔX and X^*

(2) Λ^* is a linear function, therefore continuous and monotone

(3) Symmetric outcome is induced from the combination of ξ_{ji} and η_{ji} in every argument as in 1

We have now created a matrix Λ which describes the Exchange Stage. To complete the definitions, we will introduce $\Pi : \{0,1\}^8 \to M(4 \times 4, \mathbb{R}), (\alpha, \beta) \mapsto \Lambda$

3.5 Subgame B.2

We shall now explore the active game, i.e. the laying and trumping of cards. We will first define a few new terms and three basic functions, including a double win probability function as well as a tichu announcement and tichu win probability function. We will then simulate the theory of an entire game consisting of multiple rounds and establish the different possible cases that can occur throughout a round.

We further assume players are completely rational and do not harm their own chances. Concretely, we presuppose that no counter-tichu will be called on a teammate's tichu announcement. At this stage we can therefore treat two teammates as simply one element.

As a reminder, teams are defined by combining the two player triples P_1, P_2 and P_3, P_4into one element $T_1 = \{P1, P2\}$ and $T_2 = \{P3, P4\}$.

We shall start by defining the relevant terms and functions needed for this discussion. Let S define the game score, given by $S = \{S_1, S_2\} \in \mathbb{Z}^2$ where S_1 and S_2 are the scores of T_1 and T_2 respectively.

Definition 3.9 Double Win Function

For each team we define a double win function $D_i(T_1, T_2)$.

$$D_i : (0, 1^5 \times [0, 1])^4 \to [0, 1](T1, T2) \mapsto d_i$$

This function returns the probability of a double win of each team for a round. As teams are always aiming for this bonus through a double win, we describe this function as independent of the current score and probably more dependent on the team strategies and the value of the cards. Whether or not player strategies have any influence on this probability will be heuristically/ empirically determined in the realisation section of this essay.

Next, as scoring a Tichu requires both calling and winning a Tichu, we separate the Tichu scoring opportunity into a Tichu announcement and Tichu winning function. For each team we define a Tichu announcement function $C_i(T_i, S)$, where T_i are the respective teams and S is the current score.

Definition 3.10 Tichu Announcment Function

For each team we define a Tichu announcement function $C_i(T_i, S)$, where T_i are the respective teams and S is the current score:

$$C_i : (0, 1^5 \times [0, 1])^2 \times \mathbb{Z}^2 \to [0, 1](T_i, S) \mapsto c_i$$

This function returns the Tichu announcement probability of each team for a round. It depends on a host of different variables including the player and team strategies as well as the current score and score difference. Such a function allows us to, for example, represent a greater probability of calling a Tichu when the opposing team has a large score margin or is very close to 1000 points.

Definition 3.11 Tichu Win Function

Finally, we define the Tichu winning function $W_i(X_1, X_2, X_3, X_4)$ for each team as follows.

$$W_i : [0, 1]^4 \to [0, 1]$$
$$(X_1, X_2, X_3, X_4) \mapsto t_i$$

This function indicates the probability that a team wins their Tichu bet in a given round. This too, is independent of score and game strategies as both teams will always want to win the Tichu bet. Rather, it depends completely on the teams' card values.

With the above functions we can now simulate the theory of an entire game consisting of multiple rounds and establish the different possible cases that can occur throughout a round. We shall first explain how we simulate the probability of one or multiple Tichu calls and the scoring of a double win. Next, we shall explain how we calculate points for a team and then explore the following possible scenarios case by case.

Definition 3.12 binary random varibale

We initially want to define a helper function, namely a **binary random variable** $Z(x) : [0, 1] \to [0, 1]$ as a random variable that takes the value 1 with probability x and 0 with probability $1 - x$.

Course of the game

At the beginning we determine, through two binary random variables $Z(c_1)$ and $Z(c_2)$, whether team 1 or 2 announces a Tichu ($Z(c_1) = 1$ or $Z(c_2) = 1$). Similarly, a binary random variable $Z(d_1)$ determines whether team 1 makes a double victory ($Z(d_1) = 1$) or not ($Z(d_1) = 0$). If $Z(d_1) = 0$, another binary random variable $Z(d_2')$ determines if team 2 makes a double victory ($Z(d_2') = 1$) or not ($Z(d_2') = 0$). d_2' is given by $d_2' = d_2 \cdot \frac{1}{1-d_1}$ because this case is only determined in $1 - d_1 \cdot 100$ percent of the cases. If $Z(d_1) = 1$, then $Z(d_2)$ is automatically 0, since a double victory of one team strictly excludes a double victory of the other.

Here we make the first differentiation between cases depending on whether a double win is scored or not. This is because a double win immediately awards $+200$ points to the scoring team regardless of the points distribution during the round. If a double win is not achieved by either team the points distribution must be calculated.

In the case $Z(d_1) = 1 \lor Z(d_2) = 1$, i.e. a double victory has been achieved, the score of team T_i for which $Z(d_i) = 1$ applies will be increased by 200 points ($S_i+ = 200$). Furthermore, if $Z(c_i) = 1$, Team T_i has won its announced Tichu and therefore gets another 100 points ($S_i+ = 100$). However, if the opposing team $T_j(i! = j)$ has announced Tichu ($Z(c_j) = 1$), this team loses 100 points ($S_j- = 100$) as the calling player was unable to exit the round first.

The alternative case, $Z(d_1) = 0 \wedge Z(d_2) = 0$, requires more theory as we must determine how many points each team scores during a round. We assume that the announcement of Tichus has no effect on the distribution of points, although the opposing team may, for example, prioritise exiting the round first over scoring maximum points through won card values. We argue any such shift by a Tichu announcement in any specific round would generally be balanced out over all rounds of the game. Thus, we must only determine the points scored through the distribution of won cards.

For this calculation, we introduce two normally distributed random variables n_1 and n_2 whose mean value and standard deviation are to be determined in the realisation section of this essay. The purpose of these random variables is to simulate the fluctuations in scores obtained every round by both teams throughout the game. This will be explained further in the next section. The scored points for a round are to be calculated as follows: Let $X_{tot} = \sum_{i=1}^{4} X_i$ be the total sum of card values in this round, then

$$\Delta S_1 = (X_1 \cdot n_1 + X_2 \cdot n_2)/X_{tot} \cdot 125 - 25$$
$$\Delta S_2 = (X_3 \cdot n_1 + X_4 \cdot n_2)/X_{tot} * 125 - 25$$

represent the change in score for each team for a round, where ΔS_1 and ΔS_2 are rounded to the nearest multiple of 5. Thus the change in score is calculated as a percentage of the card values for a specific round. The correctional factors 125 and -25 as well as the rounding serve to adjust the point value to the frame of a Tichu round.

Compounded onto the above theory are the following three cases as they relate to possible Tichu scenarios, namely: none, one or both teams announcing a Tichu:

(C1) If no team announces a Tichu ($Z(c_1) = 0 \wedge Z(c_2) = 0$), then the point changes are simply added to the score.

$$S_1+ = \Delta S_1 \qquad S_2+ = \Delta S_2$$

(C2) If one of the teams announces a Tichu ($Z(c_1) = 1$ XOR $Z(c_2) = 1$) then a binary random variable $Z(w_i)$ determines whether this team makes the announced tichu. Here w_i is, as defined earlier, the team's win probability. If $Z(w_i) = 1$ then the change of points is

$$S_i+ = \Delta S_i + 100 \qquad S_j+ = \Delta S_j$$

Where S_j represents the score of the opposing team. If $Z(w_i) = 0$ for the announced Tichu, then 100 points are subtracted from S_i. Other than that, the scores are added up as in case 1.

(C3) Both teams have announced a Tichu ($Z(c_1) = 1 \wedge Z(c_2) = 1$). Since we assume every player to be rational, we assume that one of the two Tichus is definitely made. Now the probability that team 1 will make the Tichu is $w_1' = \frac{w_1}{w_1+w_2}$, while $w_2' = \frac{w_2}{w_1+w_2} = 1 - w_1'$ represents the probability for team 2. Again, with the help of a binary random variable $Z(w_1')$, we determine whether team 1 ($Z(w_1') = 1$) or team 2 ($Z(w_1') = 0$) has made the Tichu. If T_i makes the Tichu and T_j is the opponent team, the score is:

$$S_i+ = \Delta S_i + 100 \qquad S_j+ = \Delta S_j - 100$$

This completes a round of the game. Now the subgames $B.0, B.1, B.2$ are repeated until $S_1 > 1000$ or $S_2 > 1000$ applies. In the following flowchart the process of $B.2$ is visualized once again:

(Schaubild)

4 Realisation

We will now adjust our basic model to realistic data which helps to achieve an approximation of Tichu. We will not change α and β during the game. Therefore we will start with modeling a function for B.0.

4.1 Subgame B.0

In our basic approach, we assume an average hand to have a value $X_i = 0.5$, this value is based on the data from tichuonline.ch, the winning rate is almost normal distributed by 0.5 with (skewness) to 0, in our idealistic scenario we assume that the distribution is symmetric. $X_i = 0$ has no chance in winning the game and $X_i = 1$ is equal to an guaranteed win. The only question is, to which percentage do we get really good cards? Assuming, the card value is symmetric around 0.5, we can agree, that instant loose and guaranteed win are very rare. We now have to make an arbitrary choice, which value corresponds to which card combinations. Given, that a Great Tichu is only won in 2.21% of all games, We will set $P(X_i \geq 0.9) = 0.221$, the share of Winning a great Tichu in reference to all games. This gives us a σ of 0.199. A Tichu is only won in 8.53% of all games, which is equal to $X \geq 0.78$ We will receive this plot which shows the possibility to get cards with a certain value.

We will always use this approximation and will not discuss it furthermore.

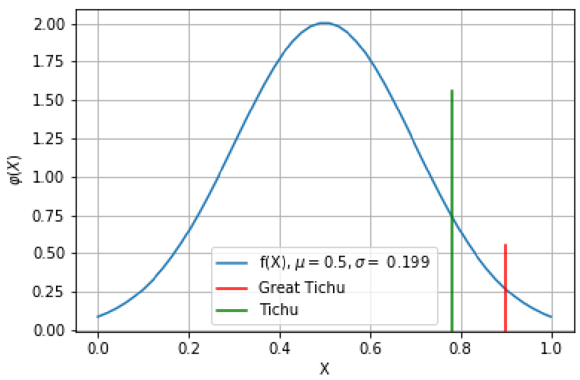

Figure 2: cards distribution

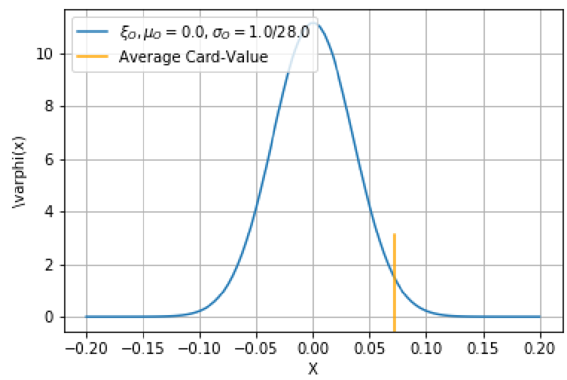

Figure 4: pass function for oppoenent player

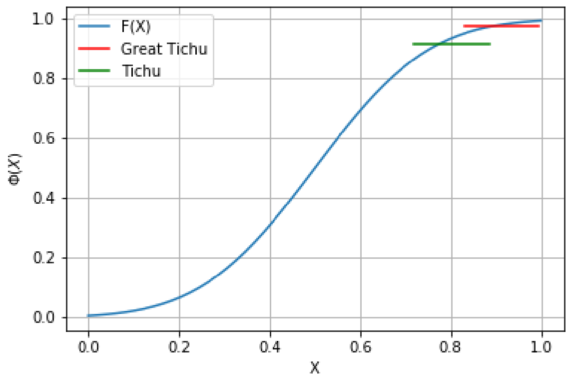

Figure 3: cards distribution cumultative

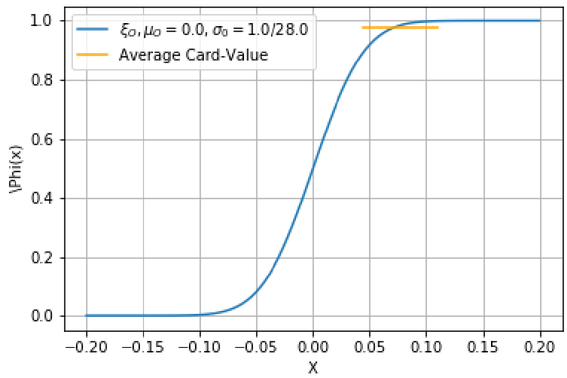

Figure 5: pass function for the opponent player cumultative

4.2 Subgame B.1

To approximate the game, we will start with the most simple approach possible. A player can only pass one card, he has 14 cards on his hand. Therefore, if he passes all cards away, the average value will be $\frac{X}{14}$. This is not a perfect approximation, but it will work for the beginning. Later on, we can redefine this value if necessary.

Keeping it easy, we will start looking at the exchange with an enemy. We only discovered one difference, whether the Player knows his Opponents strategy or not. But this difference is marginal and we will do not distinguish between these cases in this approach. Therefore, there is only one possible function towards an Opponent. We will call this function ξ_O and assume it as normal distributed. We will set $\mu = 0$, because normally giving away one card does not change the value of your hand. According to our "stupid approach", we will set $2\sigma = \frac{1}{14}$, because it is the value of an average card in most of the cases (further explanation in work).

The Pass-function towards your partner is more difficult to define: To realize it, let us imagine you are a new player to this game. You have no strategy and no knowledge. So every card is equal to you and as described before, the average card value is $\frac{X}{14}$. The cards are probably normal distributed within a wide range. Therefore we define ξ_P as the Pass Function towards a Partner with $\mu_P = \frac{1}{14}$ and $\sigma_P = \frac{1}{14}$.

If a player is aggressive, he will pass cards with less value and if he is defensive, he will pass cards with higher value. If he knows, his partner is defensive, he will pass lower cards and if he knows, that his partner is aggressive, he passes higher cards. This leads us to the following table:

We can construct a formula using this table:

Table 2: Table 2

$\beta = 0$			$\beta = 1$		
$P_R \backslash P_P$	A	D	$P_R \backslash P_P$	A	D
A	-1	+1	A	0	+2
D	-1	+1	D	-2	0

$$z_{ji} = (1 - 2 \cdot \alpha_i) + \beta_i \cdot (2 \cdot \alpha_j - 1)$$

We will set $\mu = \mu_P + \frac{\sigma_P}{2} \cdot z_{ji}(\alpha, \beta)$. This will give us one function for ξ_P:

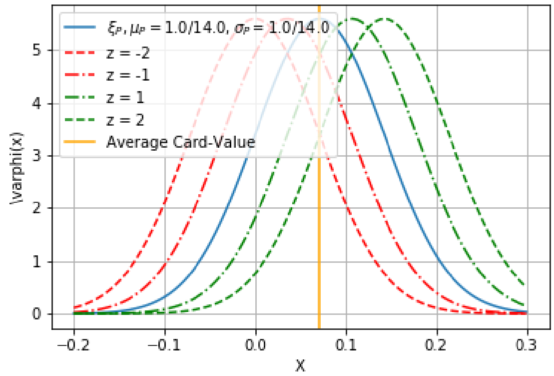

Figure 6: pass function for your team partner

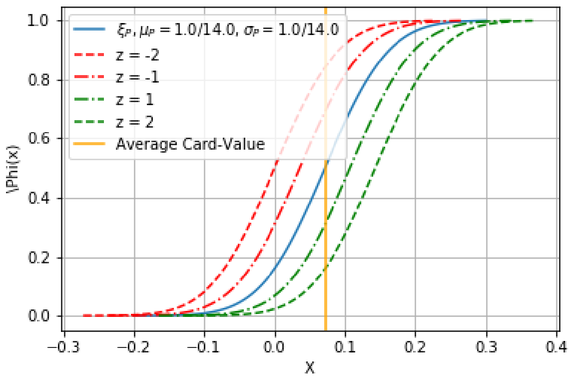

Figure 7: pass function for your teampartner cumultative

A card which has a negative value to player P_i can have a positive value to P_j (can complete a bomb). Therefore the value can change during the exchange from $\hat{\xi}_{ij}$ to $\hat{\eta}_{ij}$. In this approach, we will set $\eta_{ij} = id$, because the effect is very small in a large number of games.

We now have created Pass-functions ξ_{ji} and a Receive-Function $\eta_{ji} = 1$. We can use this functions to analyse the outcome of the Exchange-Stage $\lambda_{ji} = \xi_{ji}$. We can write Λ using the average:

$$\Lambda = \begin{pmatrix} 1-\xi_{21} & \xi_{12} & 0 & 0 \\ \xi_{21} & 1-\xi_{12} & 0 & 0 \\ 0 & 0 & 1-\xi_{43} & \xi_{34} \\ 0 & 0 & \xi_{43} & 1-\xi_{34} \end{pmatrix}$$

4.3 Subgame B.2

With the actual Tichus play there are several combinations and possibilities how this can run off. Especially decisive can be for example, who starts. Because of the combination of the cards it could be that all 4 players have the possibility to announce the Tichu. To approximate this problem as best as possible, we have considered that we simply look at the probability of winning Tichu, rather than the individual hand cards.

So first we will calculate the probability that a certain team will announce a Tichu. However, this will be done independently of the cards, because we assume that we simulate so many games that the average probabilities of winning the tichu fit better than imaginary probabilities of each card. We assume that the team announces a tichu and not the single player. There are in theory game situations in which both team partners announce a tichu, but since we assume common knowledge, these situations are extremely rare. For example, player 1 from team 1 and player 2 from team 2 has announced a tichu. Player 3 from team 1 notices that player 1 probably won't make it and says tichu himself and wins it. This case occurs very rarely, because it requires, among other things, that player 3 has not yet played a card and is already sure that he will make the tichu and his partner will not. In addition, we say that if both teams announce, one of the two teams will surely win the Tichu. Of course, in theory a third player could win the game (even without announcing Tichu), but this player would also have to be sure that his partner will not make it, because otherwise the common knowledge would be violated. But since this situation of absolute certainty about the cards of the partner is very rare, we neglect this case here.

So to calculate the probabilities for the announcement of the Tichu, we need 2 basic factors:

1. Basic aggressiveness of the players (team_{α_1},team_{α_2})
2. Additional risk tolerance depending on the score

The basic aggressiveness of the players is converted to a basic aggressiveness of the team. This basic aggressiveness is derived from the player model. The willingness to take risks increases. However, this cannot be directly dependent on the player model, because due to the common knowledge, both the defensive and the aggressive player must be prepared to take extreme risks at some point. To determine this, we have conducted a survey. It asks for the willingness to take risks in 3 categories:

(1) diffrence in scores

(2) edge of wedge

(3) own distance to victory

The scale here was the willingness to take risks from 1 to 10, whereby in category (1) and (2) 10 meant a lot of risk and 1 normal. For (3), 1 was normal risk and

10 was particularly low risk. As the survey revealed, Category (1) > Category (2) > Category (3) is in the ranking. The fluctuations in category (1) are 5 points. For category (2) 2.5 points and in (3) 0.5 points. In this weighting, these categories are also considered in the function. The survey revealed the following data points (based on this point we generated with numpy a regression):

Cat.1 $X = [0, 100, 200, 300, 400, 500, 600, 700, 800, 900]$

$Y = [\frac{86}{21}, \frac{83}{21}, \frac{96}{21}, \frac{114}{21}, \frac{122}{21}, \frac{142}{21}, \frac{150}{21}, \frac{170}{21}, \frac{180}{21}, \frac{183}{21}]$

Cat.2 $X = [100, 200, 300, 400, 500, 600, 700, 800]$

$Y = [\frac{140}{21}, \frac{133}{21}, \frac{111}{21}, \frac{105}{21}, \frac{100}{21}, \frac{89}{21}, \frac{88}{21}, \frac{93}{21}]$

Cat.3 $X = [100, 200, 300, 400, 500, 600, 700, 800]$

$Y = [\frac{94}{21}, \frac{104}{21}, \frac{105}{21}, \frac{92}{21}, \frac{105}{21}, \frac{104}{21}, \frac{92}{21}, \frac{94}{21}]$ This increase

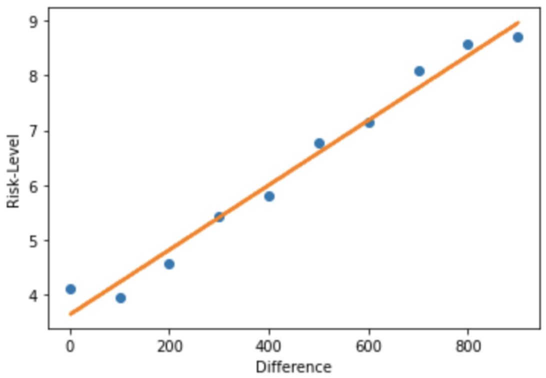

Figure 8: $f_1(x) = 0.00591631 \cdot x + 3.65194805$

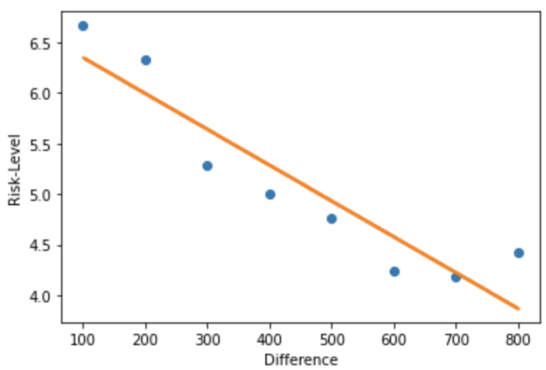

Figure 9: $f_2(x) = -0.00354308390 \cdot x + 6.70748299$

should now be offset against the basic aggressiveness.

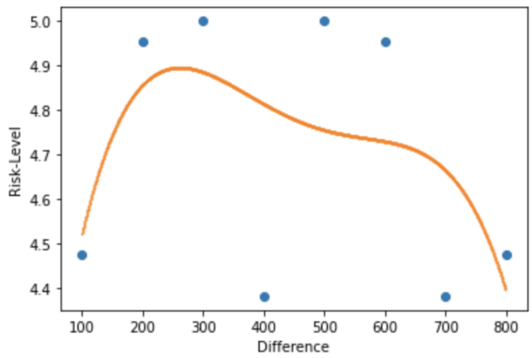

Figure 10: $f_3(x) = -3.87806638 \cdot 10^{-11} \cdot x^4 + 7.31721982 \cdot 10^{-8} \cdot x^3 + -4.95400433 \cdot 10^{-5} \cdot x^2 + 1.36745860 \cdot 10^{-2} \cdot x + 3.57993197$

The following additional conditions are set: If in category 1 the willingness to take risks is 8.5, the team should announce a 100% Tichu. With 4 it should have no effect. In between it runs linear, as the function shows. While category 1 can make a difference of up to 100%, category 2 should have a maximum of 50%. Again, linear. Category 3 should bring in a maximum of 10%. This function has level 4.

```
f(play_1,play_2,t1,t2) :
    a=(play_1+play_2)/2
    if t2>=t1:
        d=(f1(t2-t1)-f1(0))/f1(900)
    else:
        d=0
    if d>=1:
        return 1
    if t2<=200:
        b=0
    else:
        b=0.5*(f2(1000-t2)-f2(800))/
        (f2(100)-f2(800))
    if t1>=100 and t1<=800:
        c=0.1*(f3(1000-t1)-f3(800))/
        (f3(800)-f3(250))
    else:
        c=0
    if a+b-c+d>=1:
        return 1
    elif a+b-c+d>=0:
        return a+b-c+d
    else: return 0
```

In addition, there are other probabilities that must be known for the simulation. For example, the probability that a team will win the double or not. Also the probability that a player will win a Tichu must of course depend on the announcement frequency. Since this data should not come from somewhere, we built a web scraper and downloaded data of 13000 players from the site onlinetichu.com. From this data we have set up a function S, which assigns a probability to a player depending on his aggressiveness, with which he will win the Tichu. So a graph of the tichu rate in relation to tichu announcements / rounds plots. The

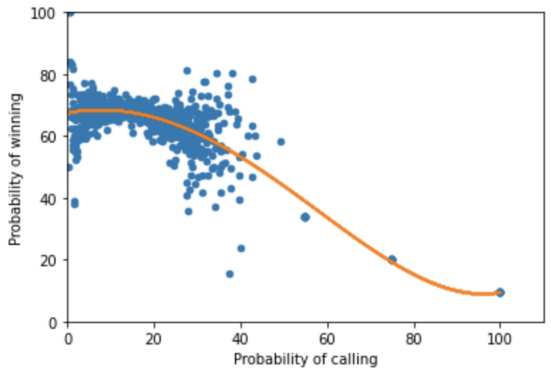

Figure 11: $f4(x) = 5.21140286 \cdot 10^{-10} \cdot x^5 + 7.51385614 \cdot 10^{-07} \cdot x^4 + 1.69471455 \cdot 10^{-06} \cdot x^3 + -1.64932045 \cdot 10^{-02} \cdot x^2 + 2.48230296 \cdot 10^{-01} \cdot x + 67.7185027$

average probability for the double victory is calculated from the average of all players. So total number of doubles / total number of rounds.

5 Simulation

5.1 B.1

First of all, we will have a look on the outcome of the Exchange Stage under different circumstances. But which outcome is the best for a team? We try to get a maximal value in three different ways:

Maximum We are only looking at the card value of the higher cards, because this can lead to a win of Tichu

Square Addition We sum the squares of both card values, because it doesn't look only at the higher card value and consider both card values of the team.

Square Diffrence We take the difference of the squares of both card values and try to minimize it, because it can be good to have balanced players.

We try to find the nash equilibrium in all three cases and set $\beta = 0$. The players will not know which strategies the other players follow. We will only look at case $\beta = 0$ for case 1, because the information on his partners strategy will only create a bigger difference but will result in the same tactics.

To determine the nash equilibrium, we will use the average values, including $\lambda_j i = 0$ for opponents. So we only look at the partner exchange with X = 0.5, $\mu = \frac{1}{14}$, $\sigma = \frac{1}{28}$.

We will show the average card values with $\beta = 0$. The actual payoffs can differ:

Receiving \ Passing	A	D
A	0.5, 0.5	$0.5 + \frac{1}{14}$, $0.5 - \frac{1}{14}$
D	$0.5 - \frac{1}{14}$, $0.5 + \frac{1}{14}$	0.5, 0.5

1. Maximum:

Receiving \ Passing	A	D
A	0.5, 0.5	$0.5 + \frac{1}{14}$, $0.5 + \frac{1}{14}$
D	$0.5 + \frac{1}{14}$, $0.5 + \frac{1}{14}$	0.5, 0.5

2. Square Addition:

Receiving \ Passing	A	D
A	0.5, 0.5	$\frac{25}{49}$, $\frac{25}{49}$
D	$\frac{25}{49}$, $\frac{25}{49}$	0.5, 0.5

Assuming, Player 1 is playing strategy A with a probability of p, we will find the mixed nash-equilibrium p = 0.5, q = 0.5 if Player 2 is playing with a probability of q:

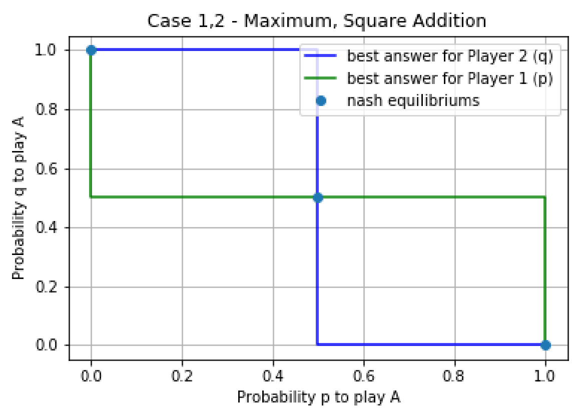

Figure 12: Case 1,2 - Maximum, Square Addition

Anologously we can find the nash equilibrium for the third case.

3. Square Diffrence:

Receiving \ Passing	A	D
A	0.0, 0.0	$-\frac{1}{7}$, $\frac{1}{7}$
D	$-\frac{1}{7}$, $\frac{1}{7}$	0.0, 0.0

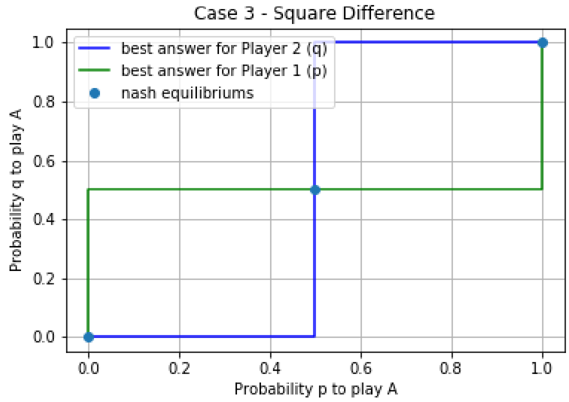

Figure 13: Case 3 - Square Diffrence

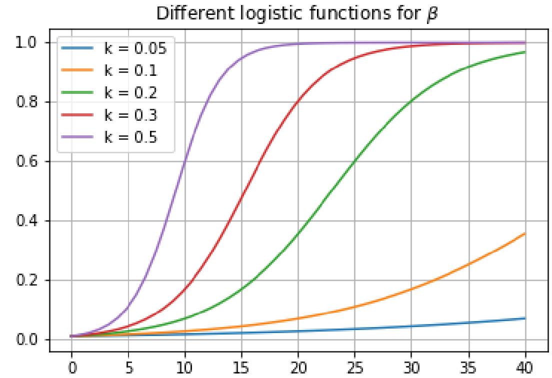

Figure 14: Adjusting k for $\beta_P^{(r)}$

Adjusting β

If we set $\beta = 0$, we will only redefine the values, but the nash equilibrium will stay the same. Just from the cards, it is the best to play a mixed strategy for all cases. To enable this strategy in our simulation, we will now allow α to be in [0,1]. This will give us a well defined Exchange function as described before.

We always used β as a fixed value in $\{0, 1\}$. But in a real game, players can recognize the strategy of another player if they play enough games. Taking this into account, we will adjust β during the game after each round. We decided to simulate the growth of knowledge with a logistic function, this is, in our point of view, the most realistix approximation of β as at the beginning the growth of your information on other players is slowly but after some amount of games it speeded up. At the beginning, a Player has some information on the other players strategies (or does not). This value will be $\beta_P^{(0)} \in \{0, 1\}$ for each Player $P \in \{1, 2, 3, 4\}$. Note that now the knowledge of diffrent players $P_i \neq P_j$ can differ and develop throughout the game in diffrent ways based on the information given before the game. The player can get more information during the game. If he has much information, he will need more rounds to get new information on the other players. We will adjust this value to $\beta_P^{(1)}$ for the first round. This notation will lead us to the following adjusting, which will be described later on:

$$\beta_P^{(r+1)} = \frac{\beta_P^{(r)}}{\beta_P^{(r)} + (1 - \beta_P^{(r)})} \cdot e^{-k}$$

Where k is in $\mathbb{R}_{>0}$ and r is the number of rounds played. If k is low, Players will learn slowly about other players strategies. If k is high, this process will be faster. We will now use $k = 0.2$, because we believe it is the most realistic function for β. An average round has <20 games and a knowledge of 80% is way to high, while 10% is low (see Chart):

6 Conclusion