

Problem 21.96 Two charges are placed as shown in Fig. P21.96. The magnitude of q_1 is $3.00\text{ }\mu\text{C}$, but its sign and the value of the charge q_2 are not known. The direction of the net electric field \mathbf{E} at point P is entirely in the negative y direction.

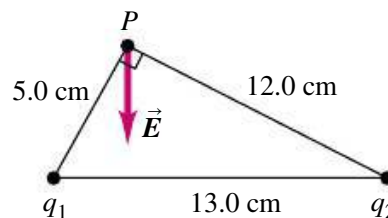
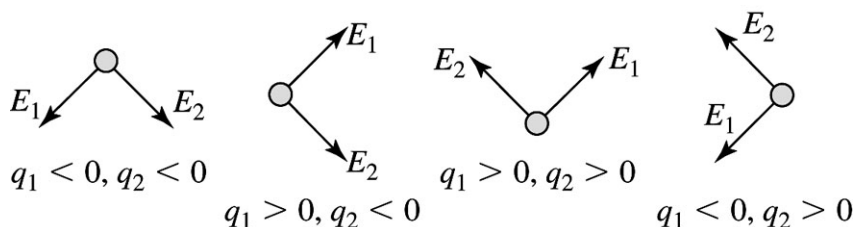


Figure P21.96

- Considering the different possible signs of q_1 and q_2 , four possible diagrams could represent the electric fields \mathbf{E}_1 and \mathbf{E}_2 produced by q_1 and q_2 . Sketch the four possible electric field configurations.
- Using the sketches from part (a) and the direction of \mathbf{E} , deduce the signs of q_1 and q_2 .
- Determine the magnitude of \mathbf{E} .

Solution

- The four possible diagrams are:



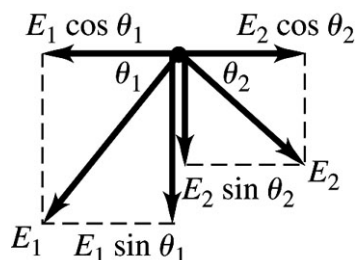
Here E_1 and E_2 indicate the electric fields of q_1 and q_2 , respectively.

- The first diagram is the only one that has the net electric field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ pointing in the $-y$ direction like we want. This means **both q_1 and q_2 are negative**. (We also could have deduced this by placing a positive test charge at P , and realizing we need both q_1 and q_2 to be negative in order to produce an electric field pointing down.)
- Using the expression for the electric field due to a point charge, we can find the magnitudes of E_1 and E_2 :

$$E_1 = \frac{1}{4\pi\epsilon_0} \frac{|q_1|}{(5.0\text{ cm})^2} = -\frac{1}{4\pi\epsilon_0} \frac{3.00\text{ }\mu\text{C}}{(5.0\text{ cm})^2}, \quad E_2 = \frac{1}{4\pi\epsilon_0} \frac{|q_2|}{(12.0\text{ cm})^2}.$$

Notice that we need to solve for q_2 , which we can do by force balance.

To find the magnitude of \mathbf{E} , we need to find and add the x and y components of \mathbf{E}_1 and \mathbf{E}_2 , which we can do by finding their angles from the horizontal. Call these angles θ_1 and θ_2 . Building off of our diagram for (a), we can make something similar to a force diagram, but for the electric field:



From Fig. P21.96, we can write down

$$\cos \theta_1 = \frac{5}{13}, \quad \sin \theta_1 = \frac{12}{13}, \quad \cos \theta_2 = \frac{12}{13}, \quad \sin \theta_2 = \frac{5}{12}.$$

First, let's balance forces in the x direction, in which we know there's 0 net electric field. This will allow us to find the magnitude of q_2 :

$$0 = E_x = E_{1x} + E_{2x} = E_1 \cos \theta_1 + E_2 \cos \theta_2 = -\frac{5}{13} \frac{1}{4\pi\epsilon_0} \frac{3.00 \mu\text{C}}{(5.0 \text{ cm})^2} + \frac{12}{13} \frac{1}{4\pi\epsilon_0} \frac{|q_2|}{(12.0 \text{ cm})^2}$$

$$\Rightarrow \frac{3.00 \mu\text{C}}{5.0 \text{ cm}^2} = \frac{|q_2|}{12.0 \text{ cm}^2} \Rightarrow |q_2| = \frac{36}{5} \mu\text{C} = 7.2 \mu\text{C}.$$

Now we can add the forces in the y direction to find the magnitude of \mathbf{E} :

$$|\mathbf{E}| = E_y = E_{1y} + E_{2y} = E_1 \sin \theta_1 + E_2 \sin \theta_2 = \frac{12}{13} \frac{1}{4\pi\epsilon_0} \frac{3.00 \mu\text{C}}{(5.0 \text{ cm})^2} + \frac{5}{12} \frac{1}{4\pi\epsilon_0} \frac{7.2 \mu\text{C}}{(12.0 \text{ cm})^2}$$

$$= \frac{1}{4\pi\epsilon_0} \left(\frac{12}{13} \frac{3.00 \times 10^{-6} \text{ C}}{25 \times 10^{-4} \text{ m}^2} + \frac{5}{12} \frac{7.2 \times 10^{-6} \text{ C}}{144 \times 10^{-4} \text{ m}^2} \right) = \frac{1}{4\pi\epsilon_0} \left(\frac{36}{325} + \frac{36}{1728} \right) \times 10^{-2} \text{ C m}^{-2}$$

$$= (8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2})(0.131 \times 10^{-2} \text{ C m}^{-2})$$

$$= 1.18 \times 10^7 \text{ N C}^{-1}.$$

Problem 23.82 A hollow, thin-walled insulating cylinder of radius R and length L (like the cardboard tube in a roll of toilet paper) has charge Q uniformly distributed over its surface.

- Calculate the electric potential at all points along the axis of the tube. Take the origin to be at the center of the tube, and take the potential to be zero at infinity.
- Use the result of part (a) to find the electric field at all points along the axis of the tube.

Solution

- We can think of the tube as being made of a stack of charged rings, and integrate over the length of the stack. Example 23.11 in the textbook derives the expression for the electric potential due to a ring, which is

$$V_{\text{ring}} = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{ring}}}{\sqrt{x^2 + a^2}},$$

where Q_{ring} is the total charge of the ring, x is the distance from the center of the ring along its axis, and a is the ring's radius (Fig. 23.20).

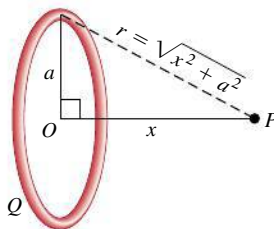


Figure 23.20

Let the ring have charge dQ and be at position z along the x axis. We will need to integrate z from $-L/2$ to $L/2$. The infinitesimal charge dQ of the ring is related to the entire charge of the tube through their respective lengths:

$$\frac{dQ}{dz} = \frac{Q}{L} \Rightarrow dQ = \frac{Q}{L} dz.$$

Our integral is then

$$V = \int \frac{1}{4\pi\epsilon_0} \frac{dQ}{\sqrt{(x-z)^2 + R^2}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{-L/2}^{L/2} \frac{dz}{\sqrt{(x-z)^2 + R^2}}.$$

The denominator of the integrand looks kind of gross, so we probably want to do a u substitution. Let $u = x - z$, so $du = -dz$. Then our integral becomes

$$V = -\frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{x+L/2}^{x-L/2} \frac{du}{\sqrt{u^2 + R^2}} = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{x-L/2}^{x+L/2} \frac{du}{\sqrt{u^2 + R^2}},$$

whose form suggests a trig substitution. Let $u = R \tan \theta$, so $du = R \sec^2 \theta d\theta$. The the bounds of integration are

$$\theta_1 \equiv \tan^{-1}\left(\frac{x-L/2}{R}\right), \quad \theta_2 \equiv \tan^{-1}\left(\frac{x+L/2}{R}\right),$$

and our integral becomes

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{\theta_1}^{\theta_2} \frac{R \sec^2 \theta}{\sqrt{R^2 \tan^2 \theta + R^2}} d\theta = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{\theta_1}^{\theta_2} \frac{\sec^2 \theta}{\sqrt{\tan^2 \theta + 1}} d\theta = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{\theta_1}^{\theta_2} \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}} d\theta \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \int_{\theta_1}^{\theta_2} \sec \theta d\theta. \end{aligned}$$

This looks simple, but it's actually quite tricky to integrate.¹ It's not obvious what substitution we should use, since differentiating the cosine will give us a sine factor that won't cancel. So it's best to consult a table of integrals for this one, which tells us

$$\int \sec \theta = \ln(\sec \theta + \tan \theta).$$

Now we can plug in our bounds of integration:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \left[\ln(\sec \theta + \tan \theta) \right]_{\theta_1}^{\theta_2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} [\ln(\sec \theta_2 + \tan \theta_2) - \ln(\sec \theta_1 + \tan \theta_1)] \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \ln\left(\frac{\sec \theta_2 + \tan \theta_2}{\sec \theta_1 + \tan \theta_1}\right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \ln\left(\frac{\sec\left(\tan^{-1}\left(\frac{x+L/2}{R}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x+L/2}{R}\right)\right)}{\sec\left(\tan^{-1}\left(\frac{x-L/2}{R}\right)\right) + \tan\left(\tan^{-1}\left(\frac{x-L/2}{R}\right)\right)}\right) \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \ln\left(\frac{\sqrt{1 + \left(\frac{x+L/2}{R}\right)^2} + \frac{x+L/2}{R}}{\sqrt{1 + \left(\frac{x-L/2}{R}\right)^2} + \frac{x-L/2}{R}}\right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{L} \ln\left(\frac{\sqrt{R^2 + (x+L/2)^2} + x+L/2}{\sqrt{R^2 + (x-L/2)^2} + x-L/2}\right). \end{aligned}$$

This is a really difficult problem. No wonder we couldn't figure it out during the discussion period!

¹In fact, it's so famously difficult there's even a Wikipedia article about it!

(c) Since the expression we found for V depends only on x , we have

$$\mathbf{E} = -\frac{dV}{dx} \hat{\mathbf{i}},$$

so we “just” have to differentiate with respect to x . Judging by the expression we found for V , this will involve a lot of nasty algebra. It’s probably easier to integrate the expression for the electric field due to a ring over the length of the tube, like we did for the potential in (a). We already did most of the substitution work. Plus, the expression for the field has a different power of $x^2 + a^2$ in the denominator than the expression for the potential, so we won’t even have to use a table of integrals.

The electric field due to a ring of charge was found in Ex. 21.9 (and also in class):

$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{ring}} x}{(x^2 + a^2)^{3/2}}.$$

Setting up the integral over the length of the tube as we did before, we get

$$E = \int \frac{1}{4\pi\epsilon_0} \frac{x}{[(x-z)^2 + R^2]^{3/2}} dQ = \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \int_{-L/2}^{L/2} \frac{dz}{[(x-z)^2 + R^2]^{3/2}}.$$

Making the first substitution $u = x - z$,

$$E = \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \int_{x-L/2}^{x+L/2} \frac{du}{(u^2 + R^2)^{3/2}},$$

and then the second substitution $u = R \tan \theta$,

$$\begin{aligned} E &= \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \int_{\theta_1}^{\theta_2} \frac{R \sec^2 \theta}{(R \tan^2 \theta + R^2)^{3/2}} d\theta = \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \int_{\theta_1}^{\theta_2} \frac{\sec^2 \theta}{(\sec^2 \theta)^{3/2}} d\theta = \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sec \theta} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \int_{\theta_1}^{\theta_2} \cos \theta d\theta = \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \left[\sin \theta \right]_{\theta_1}^{\theta_2} = \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} (\sin \theta_2 - \sin \theta_1). \end{aligned}$$

Much easier, right? Now we just need to substitute back to our original variables:

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \left\{ \sin \left[\tan^{-1} \left(\frac{x+L/2}{R} \right) \right] - \sin \left[\tan^{-1} \left(\frac{x-L/2}{R} \right) \right] \right\} \hat{\mathbf{i}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \left(\frac{(x+L/2)/R}{\sqrt{(x+L/2)^2/R^2 + 1}} - \frac{(x-L/2)/R}{\sqrt{(x-L/2)^2/R^2 + 1}} \right) \hat{\mathbf{i}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Qx}{L} \left(\frac{x+L/2}{\sqrt{(x+L/2)^2 + R^2}} - \frac{x-L/2}{\sqrt{(x-L/2)^2 + R^2}} \right) \hat{\mathbf{i}}. \end{aligned}$$

I promise Prof. Gazes will never assign you a problem with calculus this difficult!

Problem 22.32 A cube has sides of length $L = 0.300$ m. One corner is at the origin (Fig. E22.6). The nonuniform electric field is given by $\mathbf{E} = (-5.00 \text{ N C}^{-1} \text{ m}^{-1})x \hat{\mathbf{i}} + (3.00 \text{ N C}^{-1} \text{ m}^{-1})z \hat{\mathbf{k}}$.

- Find the electric flux through each of the six cube faces S_1 , S_2 , S_3 , S_4 , S_5 , and S_6 .
- Find the total electric charge inside the cube.

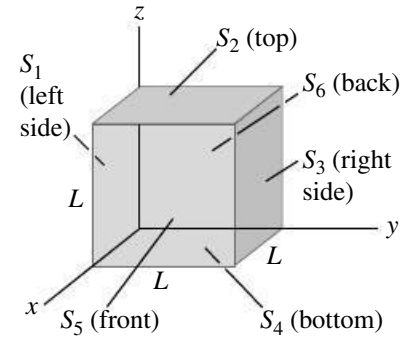


Figure E22.6

Solution

- This is a problem about flux, so it's a problem about Gauss's law:

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

For this first part, we'll be using the first equality with

$$\oint \mathbf{E} \cdot d\mathbf{A} = \mathbf{E} \cdot \mathbf{A}.$$

For each face of the cube, we define the magnitude of the vector \mathbf{A} by the area of its face. We define the direction by the unit normal (perpendicular) to the face, pointing outward. Looking at the picture, this gives us

$$\mathbf{A}_1 = -L^2 \hat{\mathbf{j}}, \quad \mathbf{A}_2 = L^2 \hat{\mathbf{k}}, \quad \mathbf{A}_3 = L^2 \hat{\mathbf{j}}, \quad \mathbf{A}_4 = -L^2 \hat{\mathbf{k}}, \quad \mathbf{A}_5 = L^2 \hat{\mathbf{i}}, \quad \mathbf{A}_6 = -L^2 \hat{\mathbf{i}}.$$

Now we can find the flux through each of the faces:

$$\Phi_1 = \mathbf{E} \cdot \mathbf{A}_1 = 0, \quad \Phi_3 = \mathbf{E} \cdot \mathbf{A}_3 = 0, \quad \Phi_4 = \mathbf{E} \cdot \mathbf{A}_4 = 0, \quad \Phi_6 = \mathbf{E} \cdot \mathbf{A}_6 = 0,$$

and

$$\begin{aligned} \Phi_2 &= \mathbf{E} \cdot \mathbf{A}_2 = E_x L^3 = (3.00 \text{ N C}^{-1} \text{ m}^{-1})(3.00 \times 10^{-1} \text{ m})^3 = 3^4 \times 10^{-3} \text{ N m}^2 \text{ C}^{-1} \\ &= 81.0 \times 10^{-3} \text{ N m}^2 \text{ C}^{-1}, \end{aligned}$$

$$\begin{aligned} \Phi_5 &= \mathbf{E} \cdot \mathbf{A}_5 = E_z L^3 = (-5.00 \text{ N C}^{-1} \text{ m}^{-1})(3.00 \times 10^{-1} \text{ m})^3 = (5)3^3 \times 10^{-3} \text{ N m}^2 \text{ C}^{-1} \\ &= -135.0 \times 10^{-3} \text{ N m}^2 \text{ C}^{-1}. \end{aligned}$$

- Now we'll use the second equality of Gauss's law to find the charge. The total flux is simply the sum of the (nonzero) results from (a):

$$\Phi_E = \Phi_2 + \Phi_5 = -54.0 \times 10^{-3} \text{ N m}^2 \text{ C}^{-1}.$$

Then

$$\begin{aligned} Q_{\text{enc}} &= \epsilon_0 \Phi_E = (8.86 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2})(-5.400 \times 10^{-2} \text{ N m}^2 \text{ C}^{-1}) = (8.86)(5.4) \times 10^{-14} \text{ C} \\ &= -4.78 \times 10^{-13} \text{ C}. \end{aligned}$$

Problem 24.11 A spherical capacitor contains a charge of 3.30 nC when connected to a potential difference of 220 V. If its plates are separated by vacuum and the inner radius of the outer shell is 4.00 cm, calculate

- (a) the capacitance,
- (b) the radius of the inner sphere, and
- (c) the electric field just outside the surface of the inner sphere.

Solution

- (a) We just need to use the definition of capacitance:

$$C = \frac{Q}{V} = \frac{3.30 \times 10^{-9} \text{ C}}{2.20 \times 10^2 \text{ V}} = \frac{3.3}{2.2} \times 10^{-11} \text{ F} = 1.50 \times 10^{-11} \text{ F}.$$

- (b) We can relate the capacitor's capacitance to its geometry using the equation for a spherical capacitor, which is derived in Ex. 24.3. The expression is

$$C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a},$$

where r_a is the inner radius and r_b is the outer radius. First we need to solve for the inner radius:

$$4\pi\epsilon_0 = C \frac{r_b - r_a}{r_a r_b} = \frac{C}{r_a} - \frac{C}{r_b} \implies \frac{C}{r_a} = \frac{C}{r_b} + 4\pi\epsilon_0 \implies r_a = \frac{C r_b}{C + 4\pi\epsilon_0 r_b}.$$

Plugging in the given r_b and our result from (a) gives us

$$\begin{aligned} r_a &= \frac{(1.50 \times 10^{-11} \text{ F})(4.00 \times 10^{-2} \text{ m})}{(1.50 \times 10^{-11} \text{ F}) + 4\pi(8.86 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2})(4.00 \times 10^{-2} \text{ m})} \\ &= \frac{(1.5)(4) \times 10^{-13}}{[1500 + 4(3.142)(8.86)(4)] \times 10^{-14}} \text{ m} = \frac{6}{1945} \times 10^1 \text{ m} = 3.08 \text{ cm}. \end{aligned}$$

- (c) We can use Gauss's law. If we enclose the inner sphere in a spherical Gaussian surface, our $Q_{\text{enc}} = 3.30 \text{ nC}$. We know that the electric field will be the same as for a point charge located at the sphere's center. Just outside the surface of the inner sphere, we are a distance r_a away from the center. Then

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{enc}}}{r_a^2} \hat{\mathbf{r}} = (8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}) \frac{3.00 \times 10^{-9} \text{ C}}{(3.08 \times 10^{-2} \text{ m})^2} \hat{\mathbf{r}} = \frac{(8.99)(3)}{3.08} \times 10^4 \text{ N C}^{-1} \hat{\mathbf{r}} \\ &= (3.12 \times 10^4 \text{ N C}^{-1}) \hat{\mathbf{r}}. \end{aligned}$$