Problem 1. Consider the path integral for a single point particle, with the action

$$S = \int_0^1 dt \left[p_\mu(t)\dot{x}^\mu(t) + \frac{N(t)}{2}[p^2(t) - m^2 - i\epsilon] \right]. \tag{1}$$

This represents the quantization of the coordinates and momenta of the particle, subject to the mass shell constraint $p^2 = m^2$ (together with the $i\epsilon$ prescription) imposed by the Lagrange multiplier N. This action admits the reparametrization symmetry $\delta x = \alpha p$, $\delta p = 0$, $\delta N = -\partial_t \alpha$ where $\alpha(t)$ is any function. This symmetry allows us to fix the gauge condition $N(t) = T$; the constant T must still be integrated over, however.

1(a) Path integrate over $x(t)$, subject to the boundary conditions $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$, yielding a delta function $\delta(\dot{p})$ along the path. Solve this constraint (find the set of functions that solve it) and path integrate over those $p(t)$ to find the quantum mechanical propagation amplitude

$$\langle y|x\rangle = D_F(x-y) = \int_0^\infty dT\,(2\pi i T)^{-d/2} \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right], \tag{2}$$

where d is the number of spacetime dimensions.

Solution. Our action contains both position and momentum, so we can evaluate the propagation amplitude using P&S (9.12):

$$U(q_a, q_b; T) = \left(\prod_i \int \mathcal{D}q(t)\,\mathcal{D}p(t)\right) \exp\left[i\int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q^i, p^i)\right)\right]$$

where the integration measure is [1, p. 281]

$$\prod_{i=1}^{N-1} \int \frac{dq^i\,dp^i}{2\pi}.$$

When evaluating the path integral, we must take into account that, in general, both position and momentum will vary over the path. We can get around this by dividing the path up into very small pieces, and approximating the position and momentum as constant during each small piece. The act of dividing the path up into these pieces is called discretization. The process of discretizing the position looks something like [1, p. 277]

$$S = \int_0^T dt \left(\frac{m}{2}\dot{x}^2 - V(x)\right) \to \sum_k \left[\frac{m}{2}\frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V\left(\frac{x_{k+1} + x_k}{2}\right)\right]. \tag{3}$$

For our problem, we need to integrate over d dimensions instead of one. Eventually, we will also need to integrate over $N(t)$. We first concern ourselves with only the position integral, and discretize into M time segments of duration $\Delta = 1/M$:

$$\langle y|x\rangle = \int \mathcal{D}N(t) \int \mathcal{D}p(t) \prod_{i=1}^{M-1} \int_{-\infty}^{\infty} \frac{d^d x_k}{(2\pi)^{d/2}} \exp\left[i \sum_{i=0}^{M-1} \left(p_i(x_{i+1} - x_i) + \frac{\Delta}{2} N_i(p_i^2 - m^2 + i\epsilon)\right)\right] \tag{4}$$

We now look at one of the integrals over some x_j, ignoring factors that are not relevant to the integral. For some j such that $1 < k < M-1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d^d x_j}{(2\pi)^{d/2}} \exp[ip_j(x_{j+1} - x_j)] \exp[ip_{j-1}(x_j - x_{j-1})] &= \int_{-\infty}^{\infty} \frac{d^d x_j}{(2\pi)^{d/2}} \exp[ix_j(p_{j-1} - p_j)] \\ &= (2\pi)^{d/2} \delta^{(d)}(p_{j-1} - p_j),\end{aligned}$$

where we have used [2]

$$\delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-\alpha)}\,dp \qquad \Longrightarrow \qquad \delta^{(d)}(x-\alpha) = \frac{1}{(2\pi)^d}\int_{-\infty}^{\infty} e^{ip(x-\alpha)}\,d^d p\,.$$

We note that the $k=1$ and $k=M-1$ integrals, respectively, leave an $x_0 = x$ and an $x_M = y$ that are not "matched" with an adjacent integral. So Eq. (4) becomes

$$\langle y|x\rangle = \int \mathcal{D}N(t)\int \mathcal{D}p(t)\left(\prod_{k=1}^{M-1}(2\pi)^{d/2}\delta^{(d)}(p_{k-1}-p_k)\right) e^{i(p_{M-1}y-p_0x)}\exp\left(i\sum_{i=0}^{M-1}\frac{\Delta}{2}N_i(p_i^2-m^2+i\epsilon)\right) \tag{5}$$

The product of delta functions tells us that p does not change from one short time interval to the next. Since this is true for the entire path, we conclude that p is constant; hence, $\dot{p}=0$. (This is consistent with the Lagrangian in Eq. (1) having no external potential, meaning momentum is conserved.) So we can write Eq. (5) with $\delta(\dot{p})$ as we wanted:

$$\langle y|x\rangle = \left(\prod_{k=1}^{M-1}(2\pi)^{d/2}\right)\int \mathcal{D}N(t)\int \mathcal{D}p(t)\,\delta^{(d)}(\dot{p})e^{-ip(x-y)}\exp\left(i\sum_{i=0}^{M-1}\frac{\Delta}{2}N_i(p_i^2-m^2+i\epsilon)\right).$$

Since p is constant along the path, we do not need to discretize the integral over p. We expect that the factors of $(2\pi)^{d/2}$ coming from that discretized integral would cancel out those from the delta function integral, so we remove them now. We also note that our integral over $N(t)$ can be written as a one-dimensional integral over the constant T. Thus

$$\begin{aligned}
\langle y|x\rangle &= \int_0^\infty dT\int_{-\infty}^{\infty} d^d p\, e^{-ip(x-y)}\exp\left(i\frac{T}{2}(p^2-m^2+i\epsilon)\right)\\
&= \int_0^\infty dT\int_{-\infty}^{\infty} d^d p\exp\left[i\left(\frac{T}{2}p^2-(x-y)p-\frac{T}{2}m^2+\frac{iT}{2}\epsilon\right)\right]\\
&= \int_0^\infty dT\int_{-\infty}^{\infty} d^d p\exp\left\{i\left[\frac{T}{2}\left(p-\frac{x-y}{T}\right)^2+\frac{T}{2}(-m^2+i\epsilon)-\frac{(x-y)^2}{2T}\right]\right\}\\
&= \int_0^\infty dT\exp\left(\frac{T}{2}(-m^2+i\epsilon)-\frac{(x-y)^2}{2T}\right)\int_{-\infty}^{\infty} d^d p\exp\left[\frac{iT}{2}\left(p-\frac{x-y}{T}\right)^2\right]\\
&= \int_0^\infty dT\exp\left(\frac{T}{2}(-m^2+i\epsilon)-\frac{(x-y)^2}{2T}\right)\int_{-\infty}^{\infty} d^d u\exp\left(\frac{iT}{2}u^2\right)
\end{aligned} \tag{6}$$

where we have completed the square [1, p. 282] and changed our variable of integration to $u = p-(x-y)/T$. We now have an integral of the form [3]

$$\int \exp\left(-\frac{1}{2}x\cdot A\cdot x\right)d^n x = \sqrt{\frac{(2\pi)^n}{\det A}}.$$

In our integral, A is a diagonal matrix with determinant $-iT$. So Eq. (6) becomes

$$\langle y|x\rangle = \int dT\left(\frac{2\pi}{iT}\right)^{d/2}\exp\left(\frac{T}{2}(-m^2+i\epsilon)-\frac{(x-y)^2}{2T}\right),$$

which matches Eq. (2) up to a factor of $(2\pi)^{-d}$.

1(b) Use this integral representation to show that D_F satisfies

$$(\partial^2 + m^2)D_F = i\delta^{(d)}(x-y). \tag{7}$$

Solution. Feeding Eq. (2) into the left-hand side of Eq. (7), we have

$$\begin{aligned}(\partial^2 + m^2)D_F &= m^2 D_F + \int_0^\infty dT\,(2\pi i T)^{-d/2}\partial^2 \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\ &= m^2 D_F + \int_0^\infty dT\,(2\pi i T)^{-d/2}\exp\left(-\frac{i}{2}(m^2 - i\epsilon)T\right)\partial^2\left[\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right)\right],\end{aligned} \tag{8}$$

where

$$\begin{aligned}\partial^2\left[\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right)\right] &= \partial\left\{\partial\left[\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right)\right]\right\} \\ &= \partial\left[-\frac{i}{T}(x-y)\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right)\right] \\ &= -\frac{i}{T}\partial(x-y)\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right) - \frac{i}{T}(x-y)\partial\left[\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right)\right] \\ &= -\frac{i}{T}\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right) - \frac{(x-y)^2}{T^2}\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right),\end{aligned}$$

since $\partial x = 1$ and $\partial y = 0$. Then Eq. (8) becomes

$$\begin{aligned}(\partial^2 + m^2)D_F &= m^2 D_F - \int_0^\infty dT\,(2\pi i T)^{-d/2}\exp\left(-\frac{i}{2}(m^2 - i\epsilon)T\right)\exp\left(-\frac{i}{2}\frac{(x-y)^2}{T}\right)\left(\frac{i}{T} + \frac{(x-y)^2}{T^2}\right) \\ &= m^2 D_F - \int_0^\infty dT\,(2\pi i T)^{-d/2}\left(\frac{i}{T} + \frac{(x-y)^2}{T^2}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\ &\to -\int_0^\infty dT\,(2\pi i T)^{-d/2}\left(\frac{(x-y)^2}{T^2} + \frac{i}{T} - m^2\right)\exp\left[-\frac{i}{2}\left(m^2 T + \frac{(x-y)^2}{T}\right)\right],\end{aligned} \tag{9}$$

where we have taken the limit $\epsilon \to 0$.

This looks like it could be the derivative of the integrand of Eq. (2) with respect to T. Define

$$I \equiv (2\pi i T)^{-d/2}\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right].$$

Then

$$\begin{aligned}\frac{dI}{dT} &= \frac{d}{dT}\left\{(2\pi i T)^{-d/2}\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\right\} \\ &= \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\frac{d}{dT}\left((2\pi i T)^{-d/2}\right) \\ &\qquad + (2\pi i T)^{-d/2}\frac{d}{dT}\left\{\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\right\}\end{aligned} \tag{10}$$

Note that

$$\begin{aligned}\frac{d}{dT}\left((2\pi i T)^{-d/2}\right) &= -\frac{d}{2T}(2\pi i T)^{-d/2}, \\ \frac{d}{dT}\left\{\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\right\} &= -\frac{i}{2}\left(m^2 - i\epsilon - \frac{(x-y)^2}{T^2}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right].\end{aligned}$$

So Eq. (10) becomes

$$\frac{dI}{dT} = -\frac{i}{2}(2\pi iT)^{-d/2}\left(-\frac{(x-y)^2}{T^2} - \frac{id}{T} + m^2 - i\epsilon\right)\exp\left[-\frac{i}{2}\left((m^2-i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]$$
$$\to \frac{i}{2}(2\pi iT)^{-d/2}\left(\frac{(x-y)^2}{T^2} + \frac{id}{T} - m^2\right)\exp\left[-\frac{i}{2}\left(m^2T + \frac{(x-y)^2}{T}\right)\right]$$

where we have taken the limit as $\epsilon \to 0$. Comparing this to Eq. (9), we note they are identical except for a factor of $-i/2$, and a factor of d in one of the terms. In the interest of moving forward, we disregard the issue of this factor of d. Define

$$F(T) \equiv 2i(2\pi iT)^{-d/2}\exp\left[-\frac{i}{2}\left(m^2T + \frac{(x-y)^2}{T}\right)\right]$$

Then we can write Eq. (8) as

$$(\partial^2 + m^2)D_F = \int_0^\infty dT\,\frac{dF(T)}{dT} = F(T)\Big|_0^\infty = 2i(2\pi iT)^{-d/2}\exp\left[-\frac{i}{2}\left(m^2T + \frac{(x-y)^2}{T}\right)\right]\Bigg|_0^\infty. \tag{11}$$

Note that $F(\infty) \to 0$ and that

$$F(0) \to (2\pi iT)^{-d/2}\exp\left[-\frac{i(x-y)^2}{2T}\right]. \tag{12}$$

This looks like a multi-dimensional Gaussian function [4],

$$f(x) = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right),$$

where k is the number of dimensions and Σ is a $k \times k$ matrix. Σ reduces to the width σ in the one-dimensional case [5]:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

As $\sigma \to 0$, this becomes a delta function [1, p. 279]. In Eq. (12), we have $\Sigma \to -iT$, a $d \times d$ diagonal matrix whose determinant is $(-iT)^d$. So we can write

$$F(0) \to (2\pi iT)^{-d/2}\exp\left[-\frac{i(x-y)^2}{2T}\right] = (-1)^{d/2}\delta^{(d)}(x-y) = i^d\delta^{(d)}(x-y).$$

Feeding this into Eq. (11), we have

$$(\partial^2 + m^2)D_F = \; 2i^{d+1}\delta^{(d)}(x-y)$$

which is what we wanted to show, up to a factor of ± 2 or $\pm 2i$.

1(c) Evaluate the T integral in terms of Bessel functions.

Solution. The integral in Eq. (2) has the form [6, p. 368]

$$\int_0^\infty x^{\nu-1}e^{-\beta/x-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{\beta\gamma}),$$

where K_ν is the modified Bessel function of the second kind. To evaluate Eq. (2) we note that

$$x \to T, \qquad \nu \to 1 - \frac{d}{2}, \qquad \beta \to \frac{i}{2}(x-y)^2, \qquad \gamma \to \frac{i}{2}(m^2 - i\epsilon).$$

So we have

$$\begin{aligned}\langle y|x\rangle &= (2\pi i)^{-d/2}\int_0^\infty dT\,T^{d/2}\exp\left[-\frac{i}{2}\left((m^2-i\epsilon)T+\frac{(x-y)^2}{T}\right)\right]\\ &= 2(2\pi i)^{-d/2}\left(\frac{(x-y)^2}{m^2-i\epsilon}\right)^{1/2-d/4}K_{1-d/2}\left(2\sqrt{-\frac{(x-y)^2(m^2-i\epsilon)}{4}}\right)\\ &= 2^{1-d/2}(\pi i)^{-d/2}\left(\frac{(x-y)^2}{m^2-i\epsilon}\right)^{1/2-d/4}K_{1-d/2}\left(i\sqrt{(x-y)^2(m^2-i\epsilon)}\right).\end{aligned}$$

Problem 2. Quantum statistical mechanics (Peskin & Schroeder 9.2)

2(a) Evaluate the quantum statistical partition function

$$Z=\mathrm{Tr}\left(e^{-\beta H}\right) \tag{13}$$

(where $\beta = 1/kT$) using the strategy of Section 9.1 for evaluating the matrix elements of e^{-iHt} in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

Solution. The trace is independent of representation [7, pp. 38–39], so we may write it in the position basis, denoted by q. In this basis, Eq. (13) becomes

$$Z=\int d^dq\ \langle q|e^{-\beta H}|q\rangle\,. \tag{14}$$

Following the steps of p. 280 of P&S, we discretize the temperature interval into N slices of width $\epsilon=\beta/N$. Then we can write

$$e^{-\beta H}=\prod_{k=1}^{N}e^{-\epsilon H}.$$

We insert a complete set of states between each of the factors of $e^{-\epsilon H}$, in the form

$$1=\int d^dq_k\,|q_k\rangle\langle q_k|\,.$$

So Eq. (14) can be written as

$$Z=\int d^dq\,\langle q|\left(\prod_{k=N-1}^{1}\int d^dq_k\,e^{-\epsilon H}\,|q_k\rangle\langle q_k|\right)e^{-\epsilon H}\,|q\rangle$$

Taking the limit $\epsilon\to 0$ as in P&S (9.9), this becomes

$$\begin{aligned}Z &= \int d^dq\,\langle q|\left(\prod_{k=N-1}^{1}\int d^dq_k\,(1-\epsilon H)\,|q_k\rangle\langle q_k|\right)(1-\epsilon H)\,|q\rangle\\ &= \int d^dq\left(\prod_{k=1}^{N-1}\int d^dq_k\right)\ \langle q|(1-\epsilon H)|q_{N-1}\rangle\cdots\langle q_2|(1-\epsilon H)|q_1\rangle\ \langle q_1|(1-\epsilon H)|q\rangle \qquad (15)\end{aligned}$$

Assuming H can be written as $H(q,p) = f(q) + g(p)$, its matrix element can be written using P&S (9.10):

$$\langle q_{k+1}|H(q,p)|q_k\rangle = \int \frac{d^d p_k}{(2\pi)^d} H(q_k, p_k) \exp[ip_k \cdot (q_{k+1} - q_k)].$$

We can again use the $\epsilon \to 0$ limit to write $1 - \epsilon H$ and $e^{-\epsilon H}$ [1, p. 281]. Then our matrix elements in Eq. (15) can be written as

$$\langle q_{k+1}|e^{-\epsilon H}|q_k\rangle = \int \frac{d^d p_k}{(2\pi)^d} \exp[ip_k \cdot (q_{k+1} - q_k) - \epsilon H(q_k, p_k)].$$

Now Eq. (15) can be written in a form similar to P&S (9.11),

$$\begin{aligned} Z = \int d^d q \left(\prod_{k=1}^{N-1} \int d^d q_k\right) \left(\prod_{k=1}^{N} \int \frac{d^d p_k}{(2\pi)^d}\right) &\exp[ip_{N-1} \cdot (q - q_{N-1}) - \epsilon H(q_{N-1}, p_{N-1})] \\ &\times \exp\left\{\sum_{k=1}^{N-2} \Big[ip_k \cdot (q_{k+1} - q_k) - \epsilon H(q_k, p_k)\Big]\right\} \\ &\times \exp[ip_N \cdot (q_1 - q) - \epsilon H(q, p_N)]. \end{aligned}$$

Let $q = q_1 = q_{N+1}$ and $\epsilon' = i\epsilon$. Then we can write Z in a form even more similar to (9.11),

$$Z = \left(\prod_{k=1}^{N} \int d^d q_k \int \frac{d^d p_k}{(2\pi)^d}\right) \exp\left\{i \sum_{k=1}^{N} \Big[p_k \cdot (q_{k+1} - q_k) + \epsilon' H(q_k, p_k)\Big]\right\}. \tag{16}$$

Assuming the Hamiltonian takes the form

$$H = \frac{p^2}{2m} + V(q),$$

we can integrate over the momenta by completing the square as on p. 282 of P&S. Adapting their expression, we find

$$\int \frac{d^d p_k}{(2\pi)^d} \exp\left[i\left(p_k \cdot (q_{k+1} - q_k) + \epsilon' \frac{p_k^2}{2m}\right)\right] = \left(\frac{im}{2\pi\epsilon'}\right)^{d/2} \exp\left[-\frac{im}{2\epsilon'}(q_{k+1} - q_k)^2\right].$$

Feeding this into Eq. (16) yields

$$Z = \left(\frac{im}{2\pi\epsilon'}\right)^{Nd/2} \left(\prod_{k=1}^{N} \int d^d q_k \int \frac{d^d p_k}{(2\pi)^d}\right) \exp\left\{-i \sum_{k=1}^{N} \left[\frac{m}{2\epsilon'}(q_{k+1} - q_k)^2 - \epsilon' H(q_k, p_k)\right]\right\},$$

which resembles (9.13). Since (9.13) is the discretized form of (9.3), we adapt (9.3) and Eq. (3) as we take $\epsilon' \to 0$ to write

$$Z = \int \mathcal{D}q(T)\, e^{-S_E[q(T)]}, \quad \text{where} \quad S_E = \int_0^\beta dT \left(\frac{m}{2}\dot{q}^2 + V(q)\right) \equiv \int_0^\beta dT\, L_E. \tag{17}$$

Here we have defined L_E, the Euclidean form of the Lagrangian. Thus we have shown that Z is a functional integral over functions, with p and q defined on an interval of length β that is periodically connected; that is, $q(0) = q(\beta)$ and $p(0) = p(\beta)$. The periodicity was imposed when we set $q_1 = q_{N+1}$. □

2(b) Evaluate this integral for a simple harmonic oscillator,

$$L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2, \tag{18}$$

by introducing a Fourier decomposition of $x(t)$:

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi int/\beta}. \tag{19}$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that, up to a (possibly divergent and β-dependent) constant, the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You may find the identity

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right)$$

useful.]

Solution. From Eq. (19), note that

$$\dot{x} = \frac{d}{dt}\left(\sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi int/\beta}\right) = \sum_n x_n \frac{2\pi in}{\beta^{3/2}} e^{2\pi int/\beta}.$$

Feeding this result into Eq. (18) yields

$$\begin{aligned}
L_E &= \frac{1}{2}\left(\sum_n x_n \frac{2\pi in}{\beta^{3/2}} e^{2\pi int/\beta}\right)\left(\sum_m x_m \frac{2\pi im}{\beta^{3/2}} e^{2\pi imt/\beta}\right) + \frac{\omega^2}{2}\left(\sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi int/\beta}\right)\left(\sum_m x_m \frac{1}{\sqrt{\beta}} e^{2\pi imt/\beta}\right) \\
&= \frac{1}{2}\sum_n\sum_m \left(-\frac{(2\pi)^2}{\beta^3} nm x_n x_m e^{2\pi i(n+m)t/\beta} + \frac{\omega^2}{\beta} e^{2\pi i(n+m)t/\beta}\right) \\
&= \frac{1}{2\beta}\sum_n\sum_m x_n x_m e^{2\pi i(n+m)t/\beta}\left(-\frac{(2\pi)^2}{\beta^2} nm + \omega^2\right).
\end{aligned}$$

Now from Eq. (17), we have

$$\begin{aligned}
S_E &= \frac{1}{2\beta}\int_0^\beta dt \sum_n\sum_m x_n x_m e^{2\pi i(n+m)t/\beta}\left(-\frac{(2\pi)^2}{\beta^2} nm + \omega^2\right) \\
&= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} d\phi \sum_n\sum_m x_n x_m e^{i(n+m)\phi}\left(-\frac{(2\pi)^2}{\beta^2} nm + \omega^2\right) \\
&= \frac{1}{2}\sum_n\sum_m x_n x_m \delta_{n,-m}\left(-\frac{(2\pi)^2}{\beta^2} nm + \omega^2\right)
\end{aligned} \tag{20}$$

where we have defined $\phi = 2\pi t/\beta$ and applied [8]

$$\delta_{x,n} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x-n)\phi}\, d\phi\,. \tag{21}$$

Picking up from Eq. (20),

$$S_E = \frac{1}{2}\sum_{n=-\infty}^{\infty} x_n x_{-n}\left(\frac{(2\pi)^2}{\beta^2} n^2 + \omega^2\right) = \sum_{n=0}^{\infty} |x_n|^2 \left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2\right] = x_0^2\omega^2 + \sum_{n=1}^{\infty} |x_n|^2 \left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2\right],$$

where we have used the fact that since x is real, Eq. (19) implies $x_{-n} = x_n^*$ [1, p. 285]. This also means x_0 is real. Referring once more to Eq. (17), we have the integral

$$Z = \int \mathcal{D}x(t) \exp\left\{ -|x_0|^2\omega^2 - \sum_{n=1}^{\infty} |x_n|^2 \left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2 \right] \right\}.$$

We can rewrite the integral as [1, p. 285]

$$\mathcal{D}x(t) = \prod_{n>0} d(\operatorname{Re} x_n)\, d(\operatorname{Im} x_n)\,.$$

This yields

$$\begin{aligned}
Z &= \int dx_0\, e^{-\omega^2 x_0^2} \left(\prod_{n>0} \int d(\operatorname{Re} x_n) \int d(\operatorname{Im} x_n) \right) \exp\left\{ -\sum_{n=1}^{\infty} \left[(\operatorname{Re} x_n)^2 + (\operatorname{Im} x_n)^2 \right] \left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2 \right] \right\} \\
&= \int dx_0\, e^{-\omega^2 x_0^2} \left(\prod_{n>0} \int d(\operatorname{Re} x_n) \exp\left\{ -\sum_{n=1}^{\infty} (\operatorname{Re} x_n)^2 \left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2 \right] \right\} \right) \\
&\qquad\qquad \times \left(\prod_{n>0} \int d(\operatorname{Im} x_n) \exp\left\{ -\sum_{n=1}^{\infty} (\operatorname{Im} x_n)^2 \left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2 \right] \right\} \right) \\
&= \frac{\sqrt{2\pi}}{\omega} \left(\prod_{n>0} \sqrt{\frac{\pi}{(2n\pi/\beta)^2 + \omega^2}} \right) \left(\prod_{n>0} \sqrt{\frac{\pi}{(2n\pi/\beta)^2 + \omega^2}} \right). \qquad (22)
\end{aligned}$$

where we have used [3]

$$\int_{-\infty}^{\infty} e^{-ax^2/2} = \sqrt{\frac{2\pi}{a}}.$$

Picking up from Eq. (22), we have

$$\begin{aligned}
Z &= \frac{\sqrt{2\pi}}{\omega} \prod_{n>0} \frac{\pi}{(2n\pi/\beta)^2 + \omega^2} \\
&= \frac{\sqrt{2\pi}}{\omega} \prod_{n>0} \left(\frac{\beta}{2n\pi}\right)^2 \frac{\pi}{1 + (\omega\beta/2n\pi)^2} \\
&= \frac{\beta\sqrt{2\pi}}{2} \left(\prod_{n>0} \frac{\beta^2}{4\pi n^2} \right) \frac{1}{\omega\beta/2} \left(\prod_{n>0} \frac{1}{1 + (\omega\beta/2)^2/(n\pi)^2} \right) \\
&= \frac{\beta\sqrt{2\pi}}{2} \left(\prod_{n>0} \frac{\beta^2}{4\pi n^2} \right) \frac{1}{\sinh(\omega\beta/2)}.
\end{aligned}$$

We note that $\prod_{n>0} \beta^2/4\pi n^2$ is a divergent, beta-dependent constant. Otherwise, this is what we would expect for the quantum partition function of a harmonic oscillator. □

2(c) Generalize this construction to field theory. Show that the quantum statistical partition function for a free scalar field can be written in terms of a functional integral. The value of this integral is given formally by

$$[\det(-\partial^2 + m^2)]^{-1/2}, \qquad (23)$$

where the operator acts on functions on Euclidean space that are periodic in the time direction with periodicity β. As before, the β dependence of this expression is difficult to compute directly. However, the dependence on m^2 is unambiguous. Show that the determinant indeed reproduces the partition function for relativistic scalar particles.

Solution. Peskin & Schroeder (9.47) gives an expression for the quantum statistical partition function of a field:

$$Z[J] = \int \mathcal{D}\phi \exp\left[-\int d^4x_E \left(\mathcal{L}_E - J\phi\right)\right].$$

To write the Euclidean action for a scalar field, we adapt P&S (9.46):

$$S_E = \int d^4x_E \left(\frac{1}{2}(\partial_{E\mu}\phi)^2 + \frac{1}{2}m^2\phi^2\right). \tag{24}$$

We can introduce the Fourier decomposition by adapting (9.21) and Eq. (19):

$$\phi(x) = \frac{1}{V}\sum_n e^{-ik_n\cdot x}\phi(k_n) = \frac{1}{\sqrt{\beta L^3}}\sum_n\sum_{\mathbf{k}_n} e^{i(2\pi nt/\beta - \mathbf{k}_n\cdot\mathbf{x})}\phi_{\mathbf{k}_n} = \frac{1}{\sqrt{\beta L^3}}\sum_n e^{2\pi int/\beta}\sum_{\mathbf{k}_n} e^{-i\mathbf{k}_n\cdot\mathbf{x}}\phi_{\mathbf{k}_n}, \tag{25}$$

where $V = L^4$ is the 4-volume of our space and $k_n^\mu = 2\pi n^\mu/L$. Using this expression, note that

$$\begin{aligned}
(\partial\phi)^2 &= \frac{1}{\beta L^3}\partial_\nu\left(\sum_n e^{2\pi int/\beta}\sum_{\mathbf{k}_n} e^{-i\mathbf{k}_n\cdot\mathbf{x}}\phi_{\mathbf{k}_n}\right)\partial^\nu\left(\sum_m e^{2\pi imt/\beta}\sum_{\mathbf{k}_m} e^{-i\mathbf{k}_m\cdot\mathbf{x}}\phi_{\mathbf{k}_m}\right)\\
&= -\frac{1}{\beta L^3}\sum_n\sum_m\sum_{\mathbf{k}_n}\sum_{\mathbf{k}_m}\left[\left(\frac{2\pi}{\beta}\right)^2 nm + \mathbf{k}_n\cdot\mathbf{k}_m\right] e^{2\pi i(n+m)t/\beta}e^{-i(\mathbf{k}_n+\mathbf{k}_m)\cdot\mathbf{x}}\phi_{\mathbf{k}_n}\phi_{\mathbf{k}_m}.
\end{aligned}$$

Then, substituting Eq. (25) into Eq. (24) gives us

$$\begin{aligned}
S_E &= -\frac{1}{2\beta L^3}\int d^4x\sum_n\sum_m\sum_{\mathbf{k}_n}\sum_{\mathbf{k}_m}\left[\left(\frac{2\pi}{\beta}\right)^2 nm + \mathbf{k}_n\cdot\mathbf{k}_m - m^2\right] e^{2\pi i(n+m)t/\beta}e^{-i(\mathbf{k}_n+\mathbf{k}_m)\cdot\mathbf{x}}\phi_{\mathbf{k}_n}\phi_{\mathbf{k}_m}\\
&= -\frac{1}{2}\sum_n\sum_m\sum_{\mathbf{k}_n}\sum_{\mathbf{k}_m}\left[\left(\frac{2\pi}{\beta}\right)^2 nm + \mathbf{k}_n\cdot\mathbf{k}_m - m^2\right]\delta_{n,-m}\delta^{(d)}_{k_n,-k_m}\phi_{\mathbf{k}_n}\phi_{\mathbf{k}_m}\\
&= \frac{1}{2}\sum_n\sum_{\mathbf{k}_n}\left[\left(\frac{2\pi n}{\beta}\right)^2 + k_n^2 + m^2\right]|\phi_{\mathbf{k}_n}|^2\\
&= \sum_{\mathbf{k}}\left\{(k_0^2+m^2)\phi_{\mathbf{k}_0}^2 + \sum_{n=1}^{\infty}\left[\left(\frac{2\pi n}{\beta}\right)^2 + k_n^2 + m^2\right]|\phi_{\mathbf{k}_n}|^2\right\},
\end{aligned}$$

where we have once again used Eq. (21) and $\phi^*_{\mathbf{k}_n} = \phi_{-\mathbf{k}_n}$ [1, p. 285]. Performing the integral as in 2(b), the partition function is

$$\begin{aligned}
Z = \prod_{\mathbf{k}}\int d\phi_{\mathbf{k}_0}\, e^{-(k_0^2+m^2)\phi_{\mathbf{k}_0}^2} &\left(\prod_{n>0}\int d(\operatorname{Re}\phi_{\mathbf{k}_n})\exp\left\{-\sum_{n=1}^{\infty}\sum_{\mathbf{k}_n}(\operatorname{Re}\phi_{\mathbf{k}_n})^2\left[\left(\frac{2n\pi}{\beta}\right)^2 + k_n^2 + m^2\right]\right\}\right)\\
&\times\left(\prod_{n>0}\int d(\operatorname{Im}\phi_{\mathbf{k}_n})\exp\left\{-\sum_{n=1}^{\infty}\prod_{\mathbf{k}_n}(\operatorname{Im}\phi_{\mathbf{k}_n})^2\left[\left(\frac{2n\pi}{\beta}\right)^2 + k_n^2 + m^2\right]\right\}\right)
\end{aligned}$$

$$= \frac{\beta\sqrt{2\pi}}{2}\left(\prod_{n>0}\frac{\beta^2}{4\pi n^2}\right)\left(\prod_{\mathbf{k}}\frac{1}{\sinh[(k^2+m^2)\beta/2]}\right). \tag{26}$$

where we have referred to our steps in 2(b).

For the determinant representation, we use

$$\ln(Z) = \ln[\det(-\partial_E^2 + m^2)] = \mathrm{Tr}\left[\ln(-\partial_E^2 + m^2)\right] = \int d^4x \int_0^\infty \frac{dT}{T} \langle x| e^{-T(-\partial_E^2 + m^2)} |x\rangle \,.$$

Unfortunately, however, I do not have time to complete this part of the problem.

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