Problem 1. The CP^N model (P&S 13.3) The nonlinear sigma model discussed in the text can be thought of as a quantum theory of fields that are coordinates on the unit sphere. A slightly more complicated space of high symmetry is complex projective space, CP^N. This space can be defined as the space of $(N+1)$-dimensional complex vectors $(z_1, \ldots, z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1,$$

with points related by an overall phase rotation identified, that is,

$$(e^{i\alpha} z_1, \ldots, e^{i\alpha} z_{N+1},) \text{ identified with } (z_1, \ldots, z_{N+1}).$$

In this problem, we study that two-dimensional quantum field theory whose fields are coordinates on this space.

1(a) One way to represent a theory of coordinates on CP^N is to write a Lagrangian depending on fields $z_j(x)$, subject to the constraint, which also has the total symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x),$$

independently at each point x. Show that the following Lagrangian has this symmetry:

$$\mathcal{L} = \frac{1}{g^2}\left[|\partial_\mu z_j|^2 + \left|z_j^* \partial_\mu z_j\right|^2 \right].$$

To prove the invariance, you will need to use the constraint on the z_j, and its consequence

$$z_j^* \partial_\mu z_j = -(\partial_\mu z_j^*) z_j.$$

Show that the nonlinear sigma model for the case $N=3$ can be converted to the CP^N model for the case $N=1$ by the substitution

$$n^i = z^* \sigma^i z,$$

where σ^i are the Pauli sigma matrices.

1(b) To write the Lagrangian in a simpler form, introduce a scalar Lagrange multiplier λ which implements the constraint and also a vector Lagrange multiplier A_μ to express the local symmetry. More specifically, show that the Lagrangian of the CP^N model is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\left[|D_\mu z_j|^2 - \lambda(|z_j| - 1) \right],$$

where $D_\mu = (\partial_\mu + iA_\mu)$, by functionally integrating over the fields λ and A_μ.

1(c) We can solve the CP^N model in the limit $N \to \infty$ by integrating over the fields z_j. Show that this integral leads to the expression

$$Z = \int \mathcal{D}A\, \mathcal{D}\lambda \exp\left(-N \operatorname{tr} \ln\left(-D^2 - \lambda\right) + \frac{i}{g^2} \int d^2x\, \lambda \right),$$

where we have kept only the leading terms for $N \to \infty$, $g^2 N$ fixed. Using methods similar to those we used for the nonlinear sigma model, examine the conditions for minimizing the exponent with respect to λ and A_μ. Show that these conditions have a solution at $A_\mu = 0$ and $\lambda = m^2 > 0$. Show that, if g^2 is renormalized at the scale M, m can be written as

$$m = M \exp\left(-\frac{2\pi}{g^2 N} \right).$$

1(d) Now expand the exponent about $A_\mu = 0$. Show that the first nontrivial term in this expansion is proportional to the vacuum polarization of massive scalar fields. Evaluate this expression using dimensional regularization, and show that it yields a standard kinetic energy term for A_μ. Thus the strange nonlinear field theory that we started with is finally transformed into a theory of $N + 1$ massive scalar fields interacting with a massless photon.