Problem 1. (Jackson 14.1) Verify by explicit calculation that the Liénard-Wiechert expressions for *all* components of $\mathbf{E}$ and $\mathbf{B}$ for a particle moving with constant velocity agree with the ones obtained in the text by means of a Lorentz transformation. Follow the general method at the end of Section 14.1.

Solution. The Liénard-Wiechert expressions for the fields are given by Jackson (14.13–14):

$$\mathbf{B} = [\hat{\mathbf{n}} \times \mathbf{E}]_{\mathrm{ret}}, \qquad \mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\mathrm{ret}} + \frac{e}{c}\left[\frac{\hat{\mathbf{n}} \times \{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R}\right]_{\mathrm{ret}}, \tag{1}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ with $\mathbf{v}$ being the particle's velocity, R is the distance from the observation point to the particle's position, and $\hat{\mathbf{n}}$ is a unit vector defined by $\mathbf{x} - \mathbf{r}(\tau) = R\,\hat{\mathbf{n}}$. Here, $\mathbf{r}(\tau)$ is the particle's present position and τ the proper time.

The expressions for the components of $\mathbf{E}$ and $\mathbf{B}$ obtained by a Lorentz transformation are given by Jackson (11.152):

$$E_1 = -\frac{e\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2 = \frac{e\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_3 = B_1 = B_2 = 0, \qquad B_3 = \beta E_2, \tag{2}$$

where the particle is moving in the x_1 direction at impact parameter b on the x_2 axis, as shown in Fig. (1).

For a particle moving with constant velocity in the x_1 direction with velocity v as shown in Fig. (1), $\boldsymbol{\beta} = \beta\,\hat{\mathbf{x}}_\mathbf{1}$ and $\dot{\boldsymbol{\beta}} = 0$. From Jackson (14.16), note that

$$(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2R^2 = b^2 + v^2t^2 - \beta^2b^2 = \frac{b^2+\gamma^2v^2t^2}{\gamma^2} \qquad \Longrightarrow \qquad (1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2 = \frac{(b^2+\gamma^2v^2t^2)^{3/2}}{R\gamma^3}.$$

This calculation comes from Fig. (2), where O is the observation point, P is the present position of the particle, and P' its retarded position. Also from Fig. 2,

$$\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}}_\mathbf{1} + \sin\theta\,\hat{\mathbf{x}}_\mathbf{2} = \frac{\beta R - vt}{R}\,\hat{\mathbf{x}}_\mathbf{1} + \frac{b}{R}\,\hat{\mathbf{x}}_\mathbf{2}.$$

Making these substitutions in the expression for $\mathbf{E}(\mathbf{x}, t)$ in Eq. (1),

$$\mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\mathrm{ret}} = e\left[\frac{(\beta - vt/R - \beta)\,\hat{\mathbf{x}}_\mathbf{1} + (b/R)\,\hat{\mathbf{x}}_\mathbf{2}}{\gamma^2(b^2+\gamma^2v^2t^2)^{3/2}}R\gamma^3\right]_{\mathrm{ret}} = e\gamma\frac{-vt\,\hat{\mathbf{x}}_\mathbf{1} + b\,\hat{\mathbf{x}}_\mathbf{2}}{(b^2+\gamma^2v^2t^2)^{3/2}}. \tag{3}$$

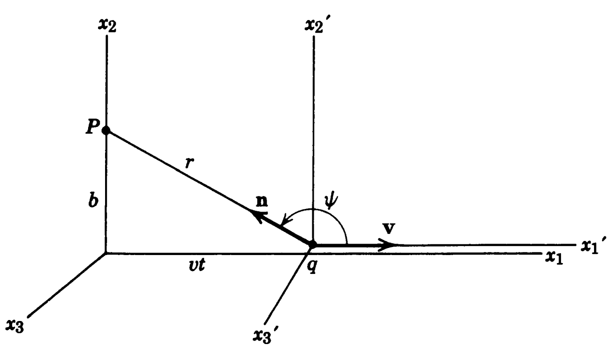

Figure 1: (Jackson Fig. 11.8) Particle of charge q moving at constant velocity $\mathbf{v}$ passes an observation point P at impact parameter b.

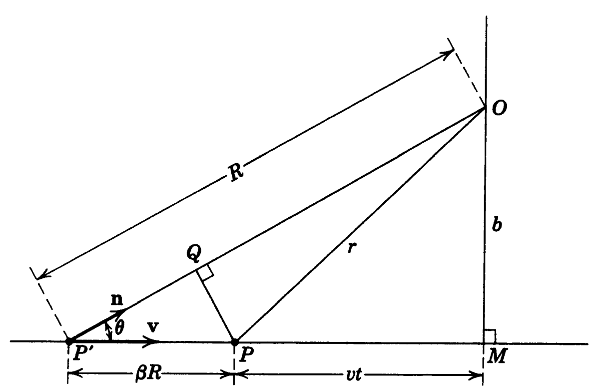

Figure 2: (Jackson Fig. 14.2) Present and retarded positions of a charge in uniform motion.

For $\mathbf{B}(\mathbf{x}, t)$, note that

$$\hat{\mathbf{n}} \times \mathbf{E} \propto \left(\frac{\beta R - vt}{R}\,\hat{\mathbf{x}}_{\mathbf{1}} + \frac{b}{R}\,\hat{\mathbf{x}}_{\mathbf{2}}\right) \times (-vt\,\hat{\mathbf{x}}_{\mathbf{1}} + b\,\hat{\mathbf{x}}_{\mathbf{2}}) = \left(b\frac{\beta R - vt}{R} + \frac{bvt}{R}\right)\hat{\mathbf{x}}_{\mathbf{3}} = \beta b,$$

so

$$\mathbf{B}(\mathbf{x}, t) = e\gamma\frac{\beta b\,\hat{\mathbf{x}}_{\mathbf{3}}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \tag{4}$$

Writing Eqs. (3) and (4) in component notation, we find

$$E_1 = -\frac{e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_2 = \frac{e\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_3 = 0,$$

$$B_1 = 0, \qquad B_2 = 0, \qquad B_3 = \frac{e\gamma\beta b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \beta E_2,$$

which are identical to Eq. (2) as desired. □

Problem 2. **(Jackson 14.3)** The Heaviside-Feynman expression for the electric field of a particle of charge e in arbitrary motion, an alternative to the Liénard-Wiechert expression (14.14) is

$$\mathbf{E} = e\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + e\left[\frac{R}{c}\right]_{\text{ret}}\frac{d}{dt}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + \frac{e^2}{c^2}\frac{d^2}{dt^2}[\hat{\mathbf{n}}]_{\text{ret}},$$

where the time derivatives are with respect to the time at the observation point. The magnetic fields are given by (14.3).

Using the fact that the retarded time is $t' = t - R(t')/c$ and that, as a result,

$$\frac{dt}{dt'} = 1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t'),$$

show that the form above yields (14.14) when the time differentiations are performed.

Problem 3. **(Jackson 14.4)** Using the Liénard-Wiechart fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a particle with charge e. Sketch the angular distribution of the radiation and determine the total power radiated in each case.

3(a) The particle is moving along the z axis with instantaneous position $z(t) = \alpha \cos \omega_0 t$.

3(b) The particle is moving in a circle of radius R in the xy plane with constant angular frequency ω_0.