1 Problem 1

The motion of a particle in a cubic potential is governed by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{k^2}{2}q^2 - \frac{A}{3}q^3. \tag{1}$$

Here m is the particle mass, k is the spring constant, and A is a positive dimensional constant.

1.a Sketch the potential and the contours of H. Identify any fixed points (mechanical equilibrium states) that exist. Classify them as stable (elliptic) or unstable (hyperbolic).

Solution. Define the potential of (1) as

$$V(q) \equiv \frac{k^2}{2}q^2 - \frac{A}{3}q^3 \equiv g(q) + g(q), \tag{2}$$

where we have defined $f(q) = k^2q^2/2$ and $g(q) = -Aq^3/3$. Figures 1 and 2 and show sketches of $f(q)$ and $g(q)$, respectively. Their sum $V(q)$ may be obtained by summing them graphically, and is shown in figure 3.

Fixed points are located where $dV/dq\,|_{q^*} = 0$. They are stable where $V(q)$ has a local minimum ($d^2V\big/dq^2\,|_{q^*} > 0$) and unstable where $V(q)$ has a local maximum ($d^2V\big/dq^2\,|_{q^*} < 0$). There are two fixed points, indicated by circles in figure 3. The stable (unstable) fixed point is indicated by a closed (open) circle.

Hamilton's equations for (1) are given by

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \implies p = m\dot{q}, \\ \dot{p} &= -\frac{\partial H}{\partial q} = k^2q - Aq^2.\end{aligned} \tag{3}$$

Fixed points occur where $\dot{q} = \dot{p} = 0$; that is, the solutions of the equation

$$p^* = k^2q^* - Aq^{*2}.$$

From (3), $\dot{q} = 0 \implies \dot{p} = 0$. Thus, the stable fixed point is located at $(q^*, p^*) = 0$, and the unstable fixed point is located at $(q^*, p^*) = (k^2/A, 0)$.

Contours are curves in the phase plane for which H is constant. Several contours are shown in figure 4.

1.b Sketch qualitatively both representative and interesting trajectories in the phase space. If there is a separatrix, a trajectory separating qualitatively different types of motion, specify the equation governing its shape.

Solution. Trajectories lie along contours of H. The directions of the trajectories may be deduced by (3), which indicates that time evolution flows in the $+q$ ($-q$) direction when $p > 0$ (< 0). This corresponds to the top (bottom) half of the phase plane. Representative trajectories corresponding to some of the contours in figure 4 are shown in figure 5.

There is a separatrix in figure 5, shown in red. The separatrix passes through the unstable fixed point at $(q^*, p^*) = (k^2/A, 0)$. Feeding these values into (1), we obtain

$$E \equiv \frac{k^2}{2}\left(\frac{k^2}{A}\right)^2 - \frac{A}{3}\left(\frac{k^2}{A}\right)^3 = \frac{1}{6}\frac{k^6}{A^2}$$

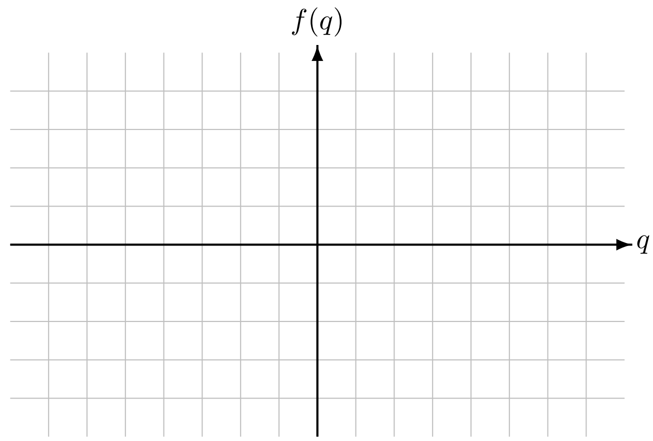

Figure 1: Sketch of $f(q)$ as defined in (2).

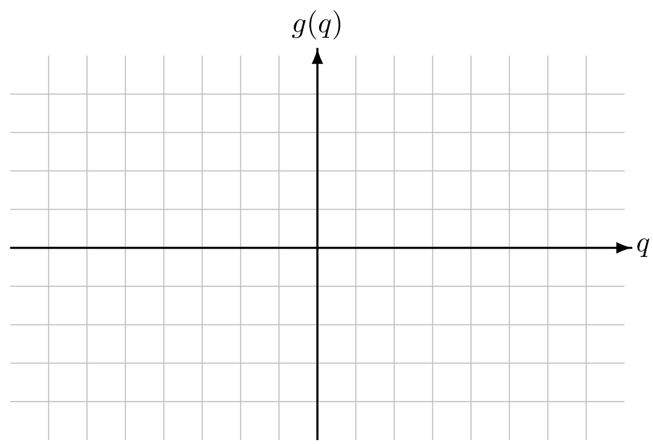

Figure 2: Sketch of $g(q)$ as defined in (2).

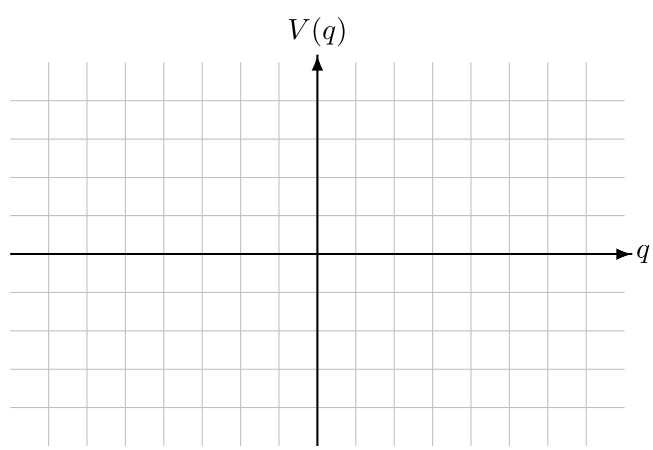

Figure 3: Sketch of $V(q)$ obtained by summing $f(q)$ and $g(q)$. The stable (unstable) fixed point is represented by a closed (open) circle.

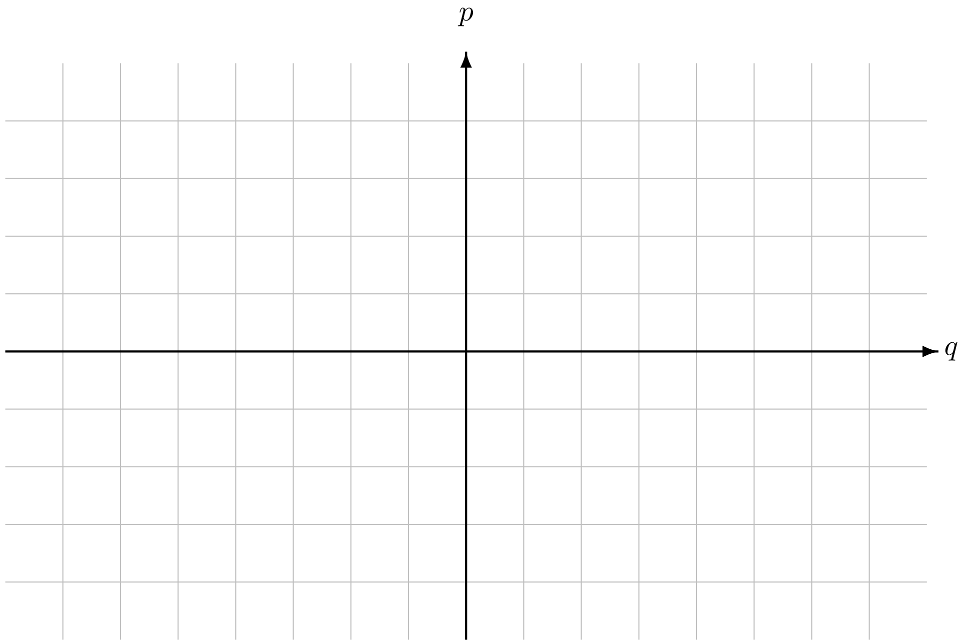

Figure 4: Contours of H. The stable (unstable) fixed point is represented by a closed (open) circle.

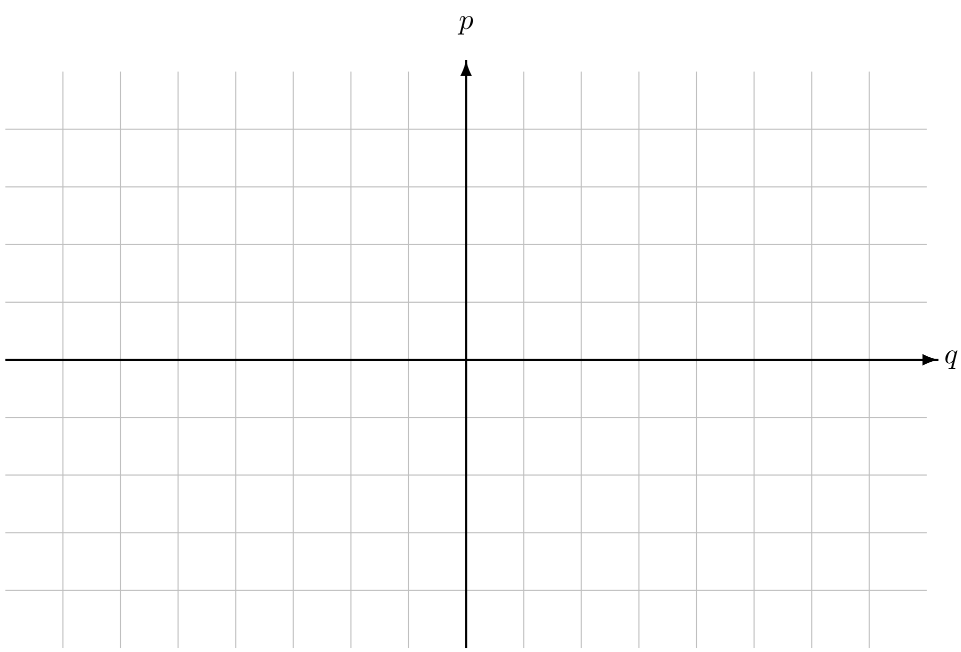

Figure 5: Trajectories of H, with the direction of time evolution indicated by arrows. The stable (unstable) fixed point is represented by a closed (open) circle. The separatrix is drawn in red.

as the constant energy of the separatrix. Substituting once more into (1) yields

$$\frac{1}{6}\frac{k^6}{A^2} = \frac{p^2}{2m} + \frac{k^2}{2}q^2 - \frac{A}{3}q^3 \iff p^3 = m\left(\frac{1}{3}\frac{k^6}{A^2} - k^2q^2 + \frac{2}{3}Aq^3\right)$$

as the equation governing the shape of the separatrix.

2 Problem 2

A particle in three spatial dimensions moves in a force field give by the Yukawa potential

$$U(r) = -\frac{k}{r}e^{-r/a},$$

where k and a are positive, and r is the radial distance between the particle and the origin.

2.a Show that this central force problem can be reduced to an equivalent one-dimensional problem with an effective potential. Specify the effective potential.

Solution. $U(r)$ is a central potential, so it has a corresponding central force

$$\mathbf{F} = -\nabla U(r) = -\frac{ke^{-r/a}}{a}\left(\frac{a}{r^2} + \frac{1}{r}\right)\hat{\mathbf{r}}, \tag{4}$$

which is radially symmetric. This means that the particle's torque $\boldsymbol{\tau}$ is zero, and therefore

$$0 = \boldsymbol{\tau} = \frac{d\mathbf{J}}{dt},$$

where $\mathbf{J}$ is the particle's angular momentum. This shows that $\mathbf{J}$ is constant over time; that is, it is a conserved quantity. Notably, the *direction* of $\mathbf{J}$ does not change over time. $\mathbf{J}$ is defined by

$$\mathbf{J} = \mathbf{r} \times \mathbf{p}.$$

Because $\mathbf{r}$ is perpendicular to $\mathbf{J}$ by definition, $\mathbf{J}$'s not changing direction implies that $\mathbf{r}$ is confined to a plane perpendicular to $\mathbf{J}$ for all time.

Confining ourselves to such a plane, we may write the Lagrangian for the system in the polar coordinates (r, θ). We note that r retains its definition as the particle's distance from the origin. The Lagrangian is given by

$$L(r, \theta, \dot{r}, \dot{\theta}) = T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}e^{-r/a},$$

which has no explicit θ dependence. From Noether's theorem, this implies a conserved quantity, given by

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \equiv J.$$

Here we have defined J, which is the magnitude of the angular momentum $\mathbf{J}$.

The Euler-Lagrange equation for θ is redundant. The Euler-Lagrange equation for r is

$$0 = \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = mr\dot{\theta}^2 - \frac{ke^{-r/a}}{a}\left(\frac{a}{r^2} + \frac{1}{r}\right) - m\ddot{r},$$

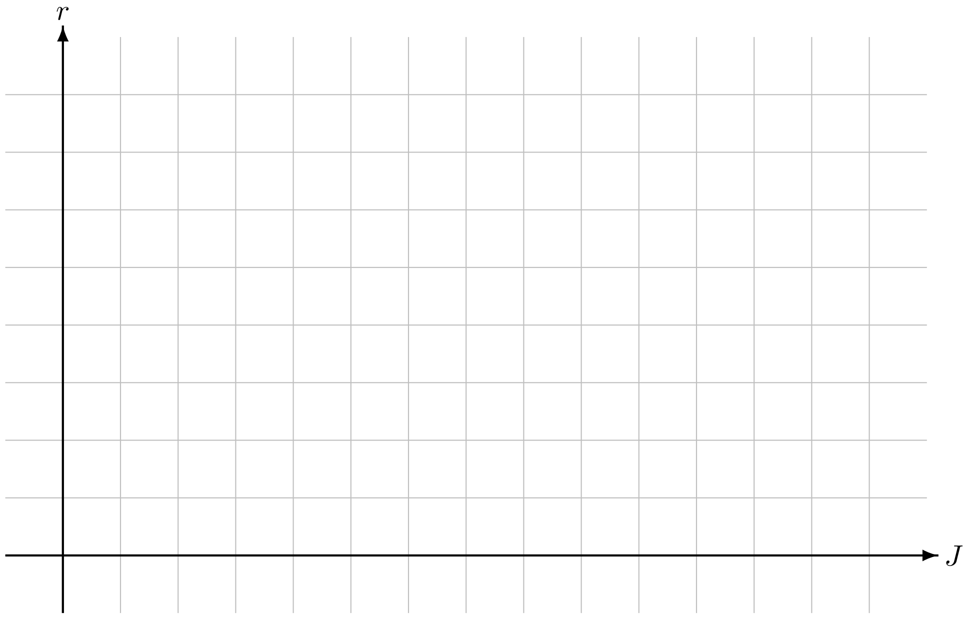

Figure 6: Bifurcation diagram for the Yukawa potential, indicating the number and stability of the fixed points of the system as J^2 is varied. The unstable fixed point is indicated by a dashed line and the stable fixed point by a solid line.

which can be rewritten in terms of J:

$$m\ddot{r} = \frac{J^2}{mr^3} - \frac{ke^{-r/a}}{a}\left(\frac{a}{r^2} + \frac{1}{r}\right) \equiv -\frac{\partial U_{\text{eff}}}{\partial r}.$$

This equation describes the complete motion of the system and depends on only r and its time derivatives, so this is a problem in only one dimension. Here, we have defined the effective potential $U_{\text{eff}}(r)$ by

$$U_{\text{eff}}(r) = \frac{1}{2}\frac{J^2}{mr^2} - \frac{k}{r}e^{-r/a}.$$

2.b Describe qualitatively the different types of motion possible as the system parameters are varied. If you think a sketch clarifies your answer, include it.

Solution. Since r is the particle's distance from the origin, it is positive definite. The system will have a fixed point at $r = r^*$ when

$$0 = \left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r^*} = -\frac{J^2}{mr^{*3}} + \frac{ke^{-r^*/a}}{a}\left(\frac{a}{r^{*2}} + \frac{1}{r^*}\right) \implies J^2 = \frac{mke^{-r^*/a}}{a}(ar^* + r^{*2}). \tag{5}$$

The roots of the right-hand side of (5) are determined by the polynomial $ar + r^2$. So there are at most two fixed points, and only for a certain range of J^2 values. The system cannot have a fixed point if $J = 0$, because this would require $r^* = 0$ and U_{eff} has a singularity there. If J^2 is too large, the right-hand side of (5) decays too quickly to ever reach equality.

Denote the maximal value of J^2 by J^{*2}. In mathematical terms, J^{*2} is a bifurcation point (corresponding to a saddle-node bifurcation). If $J^2 > J^{*2}$, there are no fixed points, and the particle will always have a hyperbolic orbit. A bifurcation diagram is shown in figure 6, indicating the existence and stability of the fixed points as J^2 is varied.

There are two fixed points in the regime $J^2 \in (0, J^{*2})$. The stable fixed point is closer to the origin because $U_{\text{eff}} \to \infty$ as $r \to 0$. Call the stable and unstable fixed points r_s^* and r_u^*, respectively. Then $r_s^* < r^* < r_u^*$. The particle will have a closed (elliptic) orbit if $r_0 < r_u^*$ and its energy is smaller than $U_{\text{eff}}(r_u^*)$. A circular orbit is stable for some specific energy. However, if the particle's energy is larger than $U_{\text{eff}}(r_u^*)$, or it has $r_0 > r_u^*$, it will have a hyperbolic orbit.

If the system has exactly one fixed point, it is an inflection point and not a local maximum or minimum of U_{eff}. Thus it is only accessible at precisely J^{*2}, and is located at $r = r^*$. Essentially, the two fixed points in the above case are overlapping. The particle will have a closed orbit if $r_0 < r^*$ and its energy is smaller $U_{\text{eff}}(r^*)$, and a hyperbolic orbit otherwise.

3 Problem 3

A physical process described by a multivariable function $\phi(x, y)$ satisfies a variational principle:

$$S[\phi(x,y)] = \frac{1}{2}\int_U \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] dx\, dy\,.$$

The solution $\phi^0(x, y)$ that gives an extremum value of $S[\phi]$ obtains in the units disk $U : x^2 + y^2 < 1$ bounded by the curve $\partial U : x^2 + y^2 = 1$ and satisfies the boundary condition $\phi(x, y)|_{\partial U} = \phi_0$, where ϕ_0 is a constant.

Derive the corresponding Euler-Lagrange partial differential equation. Indentify one (or more) physical process that is described by this variational principle.

Solution. The Lagrangian density $\mathcal{L}$ is defined by $S[\phi] = \int \mathcal{L}\, dx\, dy$, so

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right].$$

In general, the Euler-Lagrange equation is given by

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\phi_t} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\phi_x} - \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial\phi_y}.$$

Note that

$$\frac{\partial\mathcal{L}}{\partial\phi} = 0, \qquad \frac{\partial\mathcal{L}}{\partial\phi_t} = 0, \qquad \frac{\partial\mathcal{L}}{\partial\phi_x} = \phi_x, \qquad \frac{\partial\mathcal{L}}{\partial\phi_y} = \phi_y,$$

and that

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\phi_x} = \frac{\partial^2\phi}{\partial x^2}, \qquad \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial\phi_y} = \frac{\partial^2\phi}{\partial y^2}.$$

So the Euler-Lagrange equation is

$$0 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \nabla^2\phi.$$

This is Laplace's equation in two dimensions. Therefore, this variational principle describes a two-dimensional electric field $\phi(x, y)$ in the absence of external charge. It also describes the flow of an incompressible, irrotational (that is, curl free) fluid in two dimensions.