Problem 1. (Jackson 12.3) A particle with mass m and charge e moves in a uniform, static, electric field $\mathbf{E}_0$.

1(a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\mathbf{v}_0$ was perpendicular to the electric field.

Solution. Jackson (12.1) gives the force exerted on a charged particle in an external electromagnetic field:

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right),$$

where $\mathbf{u}$ is the velocity of the particle. In this problem $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{B} = \mathbf{0}$, so

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E}_0.$$

Since $\mathbf{E}_0$ is constant, we can easily find $\mathbf{p}$ as a function of time by solving this expression as a differential equation. This gives us

$$\int_0^\infty d\mathbf{p} = e\mathbf{E}_0 \int_0^\infty dt \qquad \Longrightarrow \qquad \mathbf{p}(t) = e\mathbf{E}_0 t + \mathbf{p}_0.$$

In order to use this result to find the velocity of the particle, we need to write the particle's velocity in terms of its momentum. According to Jackson (11.46), (11.51), and (11.55), $\mathbf{p} = \gamma m v$, $\mathcal{E} = m\gamma c^2$, and $\mathcal{E} = \sqrt{c^2p^2 + m^2c^4}$, where $\mathcal{E}$ is the total energy of the particle. Combining these gives us

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{c^2\mathbf{p}}{\mathcal{E}} = \frac{c\mathbf{p}}{\sqrt{m^2c^2 + \mathbf{p}^2}}. \tag{1}$$

Then, substituting into Eq. (1), we have

$$\mathbf{v}(t) = c\frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + (e\mathbf{E}_0 t + \mathbf{p}_0)^2}} = c\frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + 2e\mathbf{p}_0 \cdot \mathbf{E}_0 t + \mathbf{p}_0^2}} = c\frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}},$$

where in going to the final equality we have used the fact that $\mathbf{p}_0 = \gamma m \mathbf{v}_0$ is perpendicular to $\mathbf{E}_0$.

Finally, we can solve this as a differential equation to find the position of the particle as a function of time. Let $\mathbf{v}(t) = d\mathbf{r}/dt$, where $\mathbf{r}(t)$ is the position of the particle. Then

$$\begin{aligned}\int_0^\infty d\mathbf{r} &= c\int_0^\infty \frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}}\, dt \\ &= ce\mathbf{E}_0 \int_0^\infty \frac{t}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}}\, dt + c\mathbf{p}_0 \int_0^\infty \frac{dt}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}}.\end{aligned} \tag{2}$$

For the first integral on the right side,

$$\int_0^\infty \frac{t}{\sqrt{e^2\mathbf{E}_0^2 t^2 + m^2c^2 + \mathbf{p}_0^2}}\, dt = \frac{1}{2e^2\mathbf{E}_0^2}\int_{u_0}^\infty \frac{1}{\sqrt{u}}\, du = \frac{\sqrt{u} - \sqrt{u_0}}{e^2\mathbf{E}_0^2},$$

where we have used the substitution $u = e^2\mathbf{E}_0^2 t^2 + m^2c^2 + \mathbf{p}_0^2$, with $u_0 = m^2c^2 + \mathbf{p}_0^2$.

For the second integral on the right side of Eq. (2),

$$\int_0^\infty \frac{dt}{\sqrt{e^2\mathbf{E}_0^2 t^2 + m^2c^2 + \mathbf{p}_0^2}} = \frac{1}{\sqrt{m^2c^2 + \mathbf{p}_0^2}}\int_0^\infty \frac{dt}{\sqrt{e^2\mathbf{E}_0^2 t^2/(m^2c^2 + \mathbf{p}_0^2) + 1}} = \frac{1}{e|\mathbf{E}_0|}\int_0^\infty \frac{du}{\sqrt{u^2+1}} = \frac{\sinh^{-1} u}{e|\mathbf{E}_0|},$$

where we have used the substitution $u = e|\mathbf{E}_0|t/\sqrt{m^2c^2 + \mathbf{p}_0^2}$, the fact that $d\sinh^{-1} z/dz = 1/\sqrt{1+z^2}$, and the fact that $\sinh u_0 = \sinh(0) = 0$ [**?**].

With these solutions, Eq. (2) becomes

$$\mathbf{r}(t) = \frac{c}{e|\mathbf{E}_0|}\left[\frac{\mathbf{E}_0}{|\mathbf{E}_0|}\left(\sqrt{e^2\mathbf{E}_0^2t^2 + m^2c^2 + \mathbf{p}_0^2} - \sqrt{m^2c^2 + \mathbf{p}_0^2}\right) + \mathbf{p}_0 \sinh^{-1}\left(\frac{e|\mathbf{E}_0|t}{\sqrt{m^2c^2 + \mathbf{p}_0^2}}\right)\right] + \mathbf{r}_0,$$

where $\mathbf{r}_0$ is the initial position of the particle.

To make the equation a little neater, we can write it in terms of the particle's initial energy, which, by another application of Jackson (11.55), is $\mathcal{E}_0 = \sqrt{m^2c^4 + c^2\mathbf{p}_0^2}$. This gives us

$$\mathbf{r}(t) = \frac{1}{e|\mathbf{E}_0|}\left[\frac{\mathbf{E}_0}{|\mathbf{E}_0|}\left(\sqrt{c^2e^2\mathbf{E}_0^2t^2 + \mathcal{E}_0^2} - \mathcal{E}_0\right) + c\mathbf{p}_0 \sinh^{-1}\left(\frac{ce|\mathbf{E}_0|t}{\mathcal{E}_0}\right)\right] + \mathbf{r}_0.$$

For the velocity, we have

$$\mathbf{v}(t) = \frac{ce\mathbf{E}_0t + c\mathbf{p}_0}{\sqrt{e^2\mathbf{E}_0^2t^2 + m^2c^2 + \mathbf{p}_0^2}} = \frac{c^2e\mathbf{E}_0t + c^2\mathbf{p}_0}{\sqrt{c^2e^2\mathbf{E}_0^2t^2 + \mathcal{E}_0^2}}.$$

1(b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

Solution. Let $\mathbf{r}_0 = \mathbf{0}$, and let $r_\perp(t)$ and $r_\parallel(t)$ denote the components of the particle's position that are, respectively, parallel to and perpendicular to its original velocity. Then

$$r_\perp(t) = \frac{\sqrt{c^2e^2\mathbf{E}_0^2t^2 + \mathcal{E}_0^2} - \mathcal{E}_0}{e|\mathbf{E}_0|}, \qquad r_\parallel(t) = \frac{cp_0}{e|\mathbf{E}_0|}\sinh^{-1}\left(\frac{ce|\mathbf{E}_0|t}{\mathcal{E}_0}\right). \tag{3}$$

It is easiest to solve $r_\parallel(t)$ for t, which gives us

$$\frac{e|\mathbf{E}_0|r_\parallel}{cp_0} = \sinh^{-1}\left(\frac{ce|\mathbf{E}_0|t}{\mathcal{E}_0}\right) \quad\Longrightarrow\quad t = \frac{\mathcal{E}_0}{ce|\mathbf{E}_0|}\sinh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right).$$

Substituting into the expression for $r_\perp$, we find

$$\begin{aligned} r_\perp &= \frac{1}{e|\mathbf{E}_0|}\left(\sqrt{c^2e^2\mathbf{E}_0^2\left[\frac{\mathcal{E}_0}{ce|\mathbf{E}_0|}\sinh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right)\right]^2 + \mathcal{E}_0^2} - \mathcal{E}_0\right) = \frac{\mathcal{E}_0}{e|\mathbf{E}_0|}\left[\sqrt{\sinh^2\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) + 1} - 1\right] \\ &= \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\left[\cosh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) - 1\right], \end{aligned}$$

where we have used $\cosh^2 x - \sinh^2 x = 1$ [**?**]. Then the trajectory of the particle is given by

$$r_\perp = \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\left[\cosh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) - 1\right] = \frac{\sqrt{mc^2 + \mathbf{p}_0^2}}{|\mathbf{E}_0|}\left[\cosh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) - 1\right].$$

For short times, the argument of $\sinh^{-1}$ in Eq. (3) must be small. Note also that $r_\perp(t)$ can be written as

$$r_\perp(t) = \mathcal{E}_0\frac{\sqrt{c^2e^2\mathbf{E}_0^2t^2/\mathcal{E}_0^2 + 1} - 1}{e|\mathbf{E}_0|}, \tag{4}$$

so we can conclude that $t \ll \mathcal{E}_0/ce|\mathbf{E}_0|$ for short times. Likewise, $t \gg \mathcal{E}_0/ce|\mathbf{E}_0|$ for long times.

To obtain the trajectory for short times, we note that $u = \mathcal{E}_0/ce|\mathbf{E}_0| \ll 1$ implies that $r_\parallel \ll 1$. Thus, we can Taylor expand Eq. (4) around $r_\parallel = 0$. The Taylor series for $\cosh z$ is [**?**]

$$\cosh z = 1 + \frac{z^2}{2} + \cdots ,$$

so we find

$$\lim_{u\to 0} r_\perp = \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\frac{r_\parallel^2}{2} = \frac{\sqrt{m^2c^2 + \mathbf{p}_0^2}}{|\mathbf{E}_0|}\frac{r_\parallel^2}{2},$$

indicating that the trajectory of the particle is parabolic for short times. As soon as the field is turned on, it will start pulling the particle in a direction that is perpendicular to its original velocity. This is just like projectile motion under the influence of gravity.

To obtain the trajectory for long times, we note that $\lim_{u\to\infty} \sinh^{-1} u = \infty$ [**?**], so taking u to be large is the same as taking $r_\parallel$ to be large. Note that

$$\lim_{z\to\infty} \cosh z = \lim_{z\to\infty} \frac{e^z + e^{-z}}{2} = \frac{e^z}{2},$$

so we find

$$\lim_{u\to\infty} r_\perp = \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\frac{e^{r_\parallel}}{2} = \frac{\sqrt{m^2c^2 + \mathbf{p}_0^2}}{|\mathbf{E}_0|}\frac{e^{r_\parallel}}{2},$$

indicating that the trajectory of the particle is exponential for long times. When the field has been turned on for a long time, the particle has been accelerating parallel to the field for a long time. At infinite time, the particle's original direction of velocity has been completely washed out by the force of the electric field.

Problem 2. (Jackson 12.5) A particle of mass m and charge e moves in the laboratory in crossed, static, uniform, electric and magnetic fields. $\mathbf{E}$ is parallel to the x axis; $\mathbf{B}$ is parallel to the y axis.

2(a) For $|\mathbf{E}_0| < |\mathbf{B}|$ make the necessary Lorentz transformation described in Section 12.3 to obtain explicitly parametric equations for the particle's trajectory.

Solution. The boost described in Section 12.3 of Jackson for $|\mathbf{E}_0| < |\mathbf{B}|$ is into a frame K', which moves with velocity $\mathbf{u}$ with respect to the laboratory frame. This $\mathbf{u}$ is the particle's drift velocity. According to Jackson (12.43), it is given by

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}^2}.$$

In K', the fields are given by Jackson (12.44):

$$\mathbf{E}'_\parallel = \mathbf{E}'_\perp = \mathbf{B}'_\parallel = \mathbf{0}, \qquad \mathbf{B}'_\perp = \frac{\mathbf{B}}{\gamma} = \sqrt{\frac{\mathbf{B}^2 - \mathbf{E}^2}{\mathbf{B}^2}}\mathbf{B},$$

where $\mathbf{E}'_\parallel$ and $\mathbf{B}'_\parallel$ are parallel to $\mathbf{u}$, and $\mathbf{E}'_\perp$ and $\mathbf{B}'_\perp$ are perpendicular to $\mathbf{u}$. This means that the particle's motion in this frame is the same as motion in a uniform, static magnetic field. The particle's trajectory in a uniform magnetic field $\mathbf{B}$ that points in the y direction is found by modifying Jackson (12.41):

$$\mathbf{r}(t) = \mathbf{r}_0 + v_\parallel t\,\hat{\mathbf{y}} + ia(\hat{\mathbf{z}} - i\,\hat{\mathbf{x}})e^{-i\omega_B t}, \tag{5}$$

where $v_\parallel$ is the component of the particle's velocity along the field, $\boldsymbol{\omega}_B$ its gyration frequency, and a its gyration radius. These quantities are given by Jackson (12.39) and the formula immediately following (12.41), respectively:

$$\boldsymbol{\omega}_B = \frac{e\mathbf{B}}{\gamma mc} = \frac{ec\mathbf{B}}{\mathcal{E}}, \qquad\qquad cp_\perp = eBa,$$

where $p_\perp$ is the particle's transverse momentum.

In K', we have

$$\boldsymbol{\omega}_B = \frac{ec\mathbf{B}'_\perp}{\mathcal{E}'_0}, \qquad\qquad a = \frac{cp'_\perp}{e\big|\mathbf{B}'_\perp\big|} = \frac{\mathcal{E}_0 v'_\perp}{ec\big|\mathbf{B}'_\perp\big|}, \tag{6}$$

where $v'_\perp$ is the component of the particle's velocity perpendicular to the field in K' and $\mathcal{E}'_0$ its initial energy in K', with $\mathcal{E}' = \mathcal{E}'_0$ since $d\mathcal{E}'/dt = 0$ in this frame by Jackson (12.37).

Then Eq. (5) gives us

$$\mathbf{r}'(t') = \mathbf{r}'_0 + v'_\parallel t'\,\hat{\mathbf{y}} + ia(\hat{\mathbf{z}} - i\,\hat{\mathbf{x}})e^{-i\omega_B t'}.$$

Taking the real part and letting $\mathbf{r}'_0 = \mathbf{0}$, we find the equations

$$z'(t') = a\sin\big(\omega_B t'\big), \qquad x'(t') = a\cos\big(\omega_B t'\big), \qquad y'(t') = v'_\parallel t', \tag{7}$$

where we have used $e^{-ix} = \cos x - i\sin x$.

Now we will return to the lab frame, where $\mathbf{u}$ points in the z direction. Note that $|\mathbf{u}| = c|\mathbf{E}|/|\mathbf{B}|$, so $\beta = |\mathbf{E}|/|\mathbf{B}|$. The inverse Lorentz transformation for a boost in the z direction is found by modifying Jackson (11.18), which yields

$$ct = \gamma(ct' + \beta z'), \qquad x = x', \qquad y = y', \qquad z = \gamma(z' + \beta ct'). \tag{8}$$

Applying these to Eq. (7) and substituting for β and γ, we find the parametric equations

$$t(t') = \sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{B}^2}}^{\,-1}\left[t' + \frac{a|\mathbf{E}|}{c|\mathbf{B}|}\sin\big(\omega_B t'\big)\right], \qquad x(t') = a\cos\big(\omega_B t'\big),$$

$$y(t') = v'_\parallel t', \qquad z(t') = \sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{B}^2}}^{\,-1}\left[a\sin\big(\omega_B t'\big) + \frac{c|\mathbf{E}|}{|\mathbf{B}|}t'\right],$$

where ω_B and a are given by Eq. (6).

2(b) Repeat the calculation of part (a) for $|\mathbf{E}_0| > |\mathbf{B}|$.

Solution. For $|\mathbf{E}_0| > |\mathbf{B}|$, the boost described in Sec. (12.3) of Jackson is into a frame K'' which moves with velocity $\mathbf{u}'$ with respect to the laboratory frame, where

$$\mathbf{u}' = c\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{E}^2},$$

according to Jackson (12.46). The electric and magnetic fields in this frame are given by Jackson (12.46):

$$\mathbf{E}''_\perp = \frac{\mathbf{E}}{\gamma'} = \sqrt{\frac{\mathbf{E}^2 - \mathbf{B}^2}{\mathbf{E}^2}}\mathbf{E}, \qquad\qquad \mathbf{E}''_\parallel = \mathbf{B}''_\perp = \mathbf{B}''_\parallel = \mathbf{0}, \tag{9}$$

where $\mathbf{E}''_\parallel$ and $\mathbf{B}''_\parallel$ are parallel to $\mathbf{u}'$, and $\mathbf{E}''_\perp$ and $\mathbf{B}''_\perp$ are perpendicular to $\mathbf{u}'$. Then the particle's trajectory in this frame is described by Eq. (3). Since $\mathbf{E}''_\perp$ points in the x direction, we have

$$x''(t'') = \frac{\sqrt{c^2e^2{\mathbf{E}''_\perp}^2\,t''^2 + \mathcal{E}_0''^2} - \mathcal{E}_0''}{e|\mathbf{E}''_\perp|} + {v_0}''_x t'',$$

$$y''(t'') = \frac{c{p_0}''_y}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right) = \frac{\mathcal{E}_0''{v_0}''_y}{ce|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right),$$

$$z''(t'') = \frac{c{p_0}''_z}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right) = \frac{\mathcal{E}_0''{v_0}''_z}{ce|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right),$$

where ${v_0}''_x$, ${v_0}''_y$, and ${v_0}''_z$ are the x'', y'', and z'' components, respectively, of the particle's initial velocity in K'', and $\mathcal{E}_0''$ is the particle's initial energy in K''.

We can transform back to the lab frame in a similar way as in Eq. (8), except now we boost by $\beta' = |\mathbf{B}|/|\mathbf{E}|$:

$$ct = \gamma'(ct'' + \beta' z''), \qquad x = x'', \qquad y = y'', \qquad z = \gamma'(z'' + \beta' ct'').$$

Substituting, we find

$$t(t'') = \sqrt{1 - \frac{\mathbf{B}^2}{\mathbf{E}^2}}^{\,-1}\left[t'' + \frac{\mathcal{E}_0''{v_0}''_z}{c^2e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right)\right],$$

$$x(t'') = \frac{\sqrt{c^2e^2{\mathbf{E}''_\perp}^2\,t''^2 + \mathcal{E}_0''^2} - \mathcal{E}_0''}{e|\mathbf{E}''_\perp|} + {v_0}''_x t'',$$

$$y(t'') = \frac{\mathcal{E}_0''{v_0}''_y}{ce|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right),$$

$$z(t'') = \sqrt{1 - \frac{\mathbf{B}^2}{\mathbf{E}^2}}^{\,-1}\left[\frac{\mathcal{E}_0''{v_0}''_z}{ce|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}_0''}\right) + \frac{|\mathbf{B}|}{|\mathbf{E}|}ct''\right],$$

where $\mathbf{E}''_\perp$ is given by Eq. (9).

Problem 3. (Jackson 12.19) Source-free electromagnetic fields exist in a localized region of space. Consider the various conservation laws that are contained in the integral of $\partial_\alpha M^{\alpha\beta\gamma} = 0$ over all space, where

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta. \tag{10}$$

3(a) Show that when β and γ are both space indices conservation of the total field angular momentum follows.

Solution. Conservation of angular momentum means that $d\mathbf{L}/dt = 0$, where $\mathbf{L}$ is defined by the equation just before Jackson (12.109):

$$\mathbf{L} = \frac{1}{4\pi c}\int \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3x\,. \tag{11}$$

Let $\beta = i$ and $\gamma = j$, which are both spatial indices. Note that

$$0 = \partial_\alpha M^{\alpha ij} = \frac{\partial M^{0ij}}{\partial(ct)} + \frac{\partial M^{1ij}}{\partial x} + \frac{\partial M^{2ij}}{\partial y} + \frac{\partial M^{3ij}}{\partial z}, \tag{12}$$

and from Jackson (12.114),

$$\Theta^{00} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}, \qquad \Theta^{0i} = \frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi}, \qquad \Theta^{ij} = -\frac{E_iE_j + B_iB_j - \delta_{ij}(\mathbf{E}^2 + \mathbf{B}^2)/2}{4\pi}. \tag{13}$$

We will examine each term of Eq. (12) separately. For the first term,

$$M^{0ij} = \Theta^{0i}x^j - \Theta^{0j}x^i = \frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi}x^j - \frac{(\mathbf{E}\times\mathbf{B})_j}{4\pi}x^i$$

and

$$\frac{\partial M^{0ij}}{\partial(ct)} = \frac{1}{4\pi}\begin{cases} \dfrac{\partial}{\partial t}[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_k & i \neq j, \\ 0 & i = j. \end{cases}$$

For the remaining terms,

$$M^{kij} = \Theta^{ki}x^j - \Theta^{kj}x^i = -\frac{E_kE_i + B_kB_i - \delta_{ki}(\mathbf{E}^2 + \mathbf{B}^2)/2}{4\pi}x^j + \frac{E_kE_j + B_kB_j - \delta_{kj}(\mathbf{E}^2 + \mathbf{B}^2)/2}{4\pi}x^i,$$

and

$$\frac{\partial M^{kij}}{\partial x^k} = \frac{1}{4\pi}\begin{cases} \dfrac{\partial}{\partial x^k}\left[\left(E_kE_j + B_kB_j - \delta_{kj}\dfrac{\mathbf{E}^2 + \mathbf{B}^2}{2}\right)x^i - \left(E_kE_i + B_kB_i - \delta_{ki}\dfrac{\mathbf{E}^2 + \mathbf{B}^2}{2}\right)x^j\right] & i \neq j, \\ 0 & i = j. \end{cases}$$

Note that this is also 0 if $k \neq i$ and $k \neq j$. Summing the only nonzero terms, we have

$$\frac{\partial M^{iij}}{\partial x^i} + \frac{\partial M^{jij}}{\partial x^j} = \frac{(E_iE_j + B_iB_j) - (E_jE_i + B_jB_i)}{4\pi} = 0.$$

Combining these results, Eq. (12) becomes

$$0 = \frac{1}{4\pi}\frac{\partial}{\partial(ct)}[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_k,$$

where we have stipulated that $i \neq j$ (otherwise, the equation is trivial). Integrating over all of space, we find

$$0 = \frac{1}{c}\int \frac{\partial M^{0ij}}{\partial t}\,d^3x = \frac{1}{4\pi c}\int \frac{\partial}{\partial t}[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_k\,d^3x = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c}\int [\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_k\,d^3x\right),$$

where we have moved $\partial/\partial t$ outside the integral since $[\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]_k$ has no explicit time dependence.

Then, applying Eq. (11), we have

$$\frac{\partial L_k}{\partial t} = 0 \quad \Longrightarrow \quad \frac{\partial \mathbf{L}}{\partial t} = \mathbf{0}$$

as desired. □

3(b) Show that when $\beta = 0$ the conservation law is

$$\frac{d\mathbf{X}}{dt} = \frac{c^2\mathbf{P}_{\text{em}}}{E_{\text{em}}}, \tag{14}$$

where $\mathbf{X}$ is the coordinate of the center of mass of the electromagnetic fields, defined by

$$\mathbf{X}\int u\, d^3x = \int \mathbf{x}u\, d^3x\,, \tag{15}$$

where u is the electromagnetic energy density and E_{em} and $\mathbf{P}_{\text{em}}$ are the total energy and momentum of the fields.

Solution. From Wald (5.9–10), the energy density and momentum density of the electromagnetic field are, respectively,

$$u = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}, \qquad \boldsymbol{\mathcal{P}} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi}. \tag{16}$$

Then the total energy and momentum of the fields are, respectively,

$$E_{\text{em}} = \int u\, d^3x = \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{B}^2)\, d^3x\,, \qquad \mathbf{P}_{\text{em}} = \int \boldsymbol{\mathcal{P}}\, d^3x = \frac{1}{4\pi}\int \mathbf{E}\times\mathbf{B}\, d^3x\,. \tag{17}$$

Note that

$$\partial_\alpha M^{\alpha 0\gamma} = \frac{\partial M^{00\gamma}}{\partial(ct)} + \frac{\partial M^{10\gamma}}{\partial x} + \frac{\partial M^{20\gamma}}{\partial y} + \frac{\partial M^{30\gamma}}{\partial z}. \tag{18}$$

Again, we will proceed one term at a time. Applying Eq. (10) to the first,

$$M^{00\gamma} = \Theta^{00}x^\gamma - \Theta^{0\gamma}x^0.$$

Clearly $M^{000} = 0$, so we need only concern ourselves with the case in which $\gamma = i$ is a space index. Making these substitutions in Eq. (13) and applying Eq. (16),

$$M^{00i} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}x^i - \frac{(\mathbf{E}\times\mathbf{B})_i}{4\pi}ct = ux^i - \mathcal{P}_i ct \quad \Longrightarrow \quad \frac{\partial M^{00i}}{\partial(ct)} = \frac{\partial}{\partial(ct)}\big(ux^i\big) - \mathcal{P}_i.$$

For the remaining terms of Eq. (18), the $\gamma = 0$ case is also trivial. Taking advantage of the symmetry of $\Theta^{\alpha\beta}$, we have

$$M^{k0i} = \Theta^{k0}x^i - \Theta^{ki}x^0 = \Theta^{0k}x^i - \Theta^{ki}x^0 = \frac{(\mathbf{E}\times\mathbf{B})_k}{4\pi}x^i + \frac{E_kE_i + B_kB_i - \delta_{ki}(\mathbf{E}^2 + \mathbf{B}^2)/2}{4\pi}ct$$

and

$$\frac{\partial M^{k0i}}{\partial x^k} = \begin{cases} \mathcal{P}_i & k = i, \\ 0 & k \neq i. \end{cases}$$

Again taking the derivative and integrating, we find

$$\int \partial_\alpha M^{\alpha 0i}\, d^3x = 0 \qquad \Longrightarrow \qquad 0 = \frac{1}{c}\int \frac{\partial M^{00i}}{\partial t}\, d^3x = \frac{1}{c}\int \frac{\partial}{\partial t}\left(ux^i - ct\mathcal{P}_i\right) d^3x$$

In vector notation,

$$\begin{aligned} 0 &= \frac{1}{c}\int \frac{\partial}{\partial t}(u\mathbf{x})\, d^3x - c\int \frac{\partial}{\partial t}(t\mathbf{P})\, d^3x = \frac{1}{c}\frac{\partial}{\partial t}\left(\int u\mathbf{x}\, d^3x\right) - c\int \mathbf{P}\, d^3x = \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{X}\int u\, d^3x\right) - c\int \mathbf{P}\, d^3x \\ &= \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{X}E_{\text{em}}) - c\mathbf{P}_{\text{em}} = \frac{1}{c}\left(E_{\text{em}}\frac{\partial \mathbf{X}}{\partial t} + \mathbf{X}\frac{\partial E_{\text{em}}}{\partial t}\right) - c\mathbf{P}_{\text{em}} = \frac{E_{\text{em}}}{c}\frac{d\mathbf{X}}{dt} - c\mathbf{P}_{\text{em}} \end{aligned}$$

where we have applied Eqs. (15) and (17), and the fact that $\partial E_{\text{em}}/\partial t = 0$. Rearranging, we have

$$\frac{d\mathbf{X}}{dt} = \frac{c^2\mathbf{P}_{\text{em}}}{E_{\text{em}}}$$

as desired. $\square$

Problem 4. We discussed in class the construction of linearly polarized electromagnetic waves.

4(a) Generalize the discussion to circularly polarized waves (see also Wald Sec. 5.5). Discuss both right-handed and left-handed polarizations.

Solution. The plane waves are given on p. 149 in the lecture notes:

$$\mathbf{E}(\mathbf{r},t) = C\exp(ik_\mu x^\mu)\boldsymbol{\xi}_1, \qquad \mathbf{B}(\mathbf{r},t) = C\exp(ik_\mu x^\mu)\boldsymbol{\xi}_2,$$

where $k^\mu = (\omega/c, \mathbf{k})$ with ω being the wave frequency and $\mathbf{k}$ the wave vector, C is the field strength amplitude, and $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are polarization vectors which are related to the polarization 4-vectors ${\xi_1}^\mu$ and ${\xi_2}^\mu$.

From p. 146 in the lecture notes, the polarization 4-vectors must both satisfy the constraint $k_\mu \xi^\mu = 0$ and the identification $\xi_\mu \sim \xi_\mu + \alpha k_\mu$, where α is an arbitrary constant. We found in class that this means $\xi^\mu = (0, \boldsymbol{\xi})$. Then we must have [**?** , p. 299]

$$\boldsymbol{\xi}_1 \cdot \hat{\mathbf{k}} = \boldsymbol{\xi}_2 \cdot \hat{\mathbf{k}} = \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 = 0, \qquad \boldsymbol{\xi}_2 = \hat{\mathbf{k}} \times \boldsymbol{\xi}_1,$$

where $\hat{\mathbf{k}}$ is a unit vector in the direction of $\mathbf{k}$, and the second equality comes from $\mathbf{k} \propto \mathbf{E} \times \mathbf{B}$.

Circularly polarized waves are the linear combination of two linearly-polarized waves that have the same amplitude and are out of phase by $\pi/2$. The circularly-polarized fields may then be written as [**?** , p. 299]

$$\mathbf{E}(\mathbf{r},t) = C(\boldsymbol{\xi}_1 \pm i\boldsymbol{\xi}_2)\exp(ik_\mu x^\mu), \qquad \mathbf{B}(\mathbf{r},t) = C(\boldsymbol{\xi}_2 \mp i\boldsymbol{\xi}_1)\exp(ik_\mu x^\mu), \tag{19}$$

where the upper signs correspond to left-handed polarization, and the lower signs correspond to right-handed polarization. Incoming waves with left-handed polarization appear to be rotating counter-clockwise to an observer, while incoming right-handed waves appear to be rotating clockwise [**?** , p. 300].

Taking the real part of Eq. (19), and choosing $\boldsymbol{\xi}_1 = \hat{\mathbf{x}}$, $\boldsymbol{\xi}_2 = \hat{\mathbf{y}}$, and $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, we have [**?** , p. 299]

$$E_x(z,t) = C\cos(kz - \omega t), \qquad E_y(z,t) = \mp C\sin(kz - \omega t), \tag{20}$$

$$B_x(z,t) = \pm C\sin(kz - \omega t), \qquad B_y(z,t) = C\cos(kz - \omega t). \tag{21}$$

We will show that these fields satisfy Maxwell's equations for a source-free region. From Wald (5.4–7), the equations are

$$\boldsymbol{\nabla}\cdot\mathbf{E}=0,\qquad \boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\mathbf{0},\qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0,\qquad \boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0}.$$

For the first equation,

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0.$$

For the second,

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}&=-\frac{\partial B_y}{\partial z}\,\hat{\mathbf{x}}+\frac{\partial B_x}{\partial z}\,\hat{\mathbf{y}}+\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\hat{\mathbf{z}}-\frac{1}{c}\left(\frac{\partial E_x}{\partial t}\,\hat{\mathbf{x}}+\frac{\partial E_y}{\partial t}\,\hat{\mathbf{y}}\right)\\&=Ck\sin(kz-\omega t)\,\hat{\mathbf{x}}\pm Ck\cos(kz-\omega t)\,\hat{\mathbf{y}}-\frac{C\omega\sin(kx-\omega t)\,\hat{\mathbf{x}}\pm C\omega\cos(kx-\omega t)\,\hat{\mathbf{y}}}{c}=\mathbf{0},\end{aligned}$$

since $k=\omega/c$ from p. 138.

For the third,

$$\boldsymbol{\nabla}\cdot\mathbf{B}=\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}=0.$$

For the fourth,

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}&=-\frac{\partial E_y}{\partial z}\,\hat{\mathbf{x}}+\frac{\partial E_x}{\partial z}\,\hat{\mathbf{y}}+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\hat{\mathbf{z}}+\frac{1}{c}\left(\frac{\partial B_x}{\partial t}\,\hat{\mathbf{x}}+\frac{\partial B_y}{\partial t}\,\hat{\mathbf{y}}\right)\\&=\pm Ck\cos(kz-\omega t)\,\hat{\mathbf{x}}-Ck\sin(kz-\omega t)\,\hat{\mathbf{y}}-\frac{\pm C\omega\cos(kz-\omega t)\,\hat{\mathbf{x}}-C\omega\sin(kx-\omega t)\,\hat{\mathbf{y}}}{c}=\mathbf{0}.\end{aligned}$$

Thus, we have shown that the circularly-polarized waves given by Eqs. (19–21) are valid solutions to the Maxwell equations for a source-free field.

4(b) Compute the angular momentum of the circularly polarized waves of part (a) using the formula for angular momentum derived in class.

Solution. The angular momentum of an electromagnetic field is given by Eq. (11). Define the angular momentum density by [**?** , p. 358]

$$\boldsymbol{\mathcal{L}}=\frac{\mathbf{x}\times(\mathbf{E}\times\mathbf{B})}{4\pi c},$$

which implies

$$\mathbf{L}=\int\boldsymbol{\mathcal{L}}\,d^3x\,.$$

From the inside cover of Jackson, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. So, using the fields in Eqs. (20) and (21),

$$\begin{aligned}\mathbf{x}\times(\mathbf{E}\times\mathbf{B})&=(\mathbf{x}\cdot\mathbf{B})\mathbf{E}-(\mathbf{x}\cdot\mathbf{E})\mathbf{B}=(xB_x+yB_y)(E_x\,\hat{\mathbf{x}}+E_y\,\hat{\mathbf{y}})-(xE_x+yE_y)(B_x\,\hat{\mathbf{x}}+B_y\,\hat{\mathbf{y}})\\&=[E_x(xB_x+yB_y)-B_x(xE_x+yE_y)]\,\hat{\mathbf{x}}+[E_y(xB_x+yB_y)-B_y(xE_x+yE_y)]\,\hat{\mathbf{y}}\\&=(E_xB_y-E_yB_x)y\,\hat{\mathbf{x}}+(E_yB_x-E_xB_y)x\,\hat{\mathbf{y}}\\&=C^2[\cos^2(kz-\omega t)+\sin^2(kz-\omega t)]y\,\hat{\mathbf{x}}-C^2[\sin^2(kz-\omega t)+\cos^2(kz-\omega t)]x\,\hat{\mathbf{y}}\\&=C^2(y\,\hat{\mathbf{x}}-x\,\hat{\mathbf{y}}),\end{aligned}$$

which is true for both left- and right-handed polarizations. Then the angular momentum density is

$$\boldsymbol{\mathcal{L}} = C^2 \frac{y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}}}{4\pi c}.$$

For the total angular momentum, we integrate over all of space:

$$\begin{aligned}
\mathbf{L} = \frac{C^2}{4\pi c}\int (y\,\hat{\mathbf{x}} - x\,\hat{\mathbf{y}})\,d^3x &= \frac{C^2}{4\pi c}\left(\hat{\mathbf{x}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,dy\,dx\,dz - \hat{\mathbf{y}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,dx\,dy\,dz\right)\\
&= \frac{C^2}{4\pi c}\left(\hat{\mathbf{x}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{y^2}{2}\right]_{-\infty}^{\infty} dx\,dz - \hat{\mathbf{y}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{x^2}{2}\right]_{-\infty}^{\infty} dy\,dz\right)\\
&= \frac{C^2}{4\pi c}\left(\hat{\mathbf{x}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 0\,dx\,dz - \hat{\mathbf{y}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 0\,dy\,dz\right) = \mathbf{0}.
\end{aligned}$$

So we see that the total angular momentum (for all space) is zero. However, the angular momentum density $\mathcal{L}$ is nonzero. If we were to calculate the angular momentum for an arbitrary region of space, it would not necessarily be zero. Thus, the statement on p. 152 in the lecture notes that circularly polarized waves carry angular momentum is still true.

Problem 5. We wrote in class the Lagrangian of a charged particle coupled to the electromagnetic field (see pp. 159–160) in the lecture notes).

5(a) Show that the Euler-Lagrange equations that follow from this Lagrangian give rise to the Lorentz force law

$$\frac{dp_i}{dt} = q\left[E^i + \frac{1}{c}(\mathbf{v}\times\mathbf{B})^i\right].$$

Solution. The action of a charged particle in an electromagnetic field is given on p. 159 of the lecture notes as

$$S[\mathbf{x}] = \int\left(-mc^2\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}} - q\phi + \frac{q}{c}\dot{\mathbf{x}}\cdot\mathbf{A}\right)dt \equiv \int \mathcal{L}(\mathbf{x},\dot{\mathbf{x}})\,dt\,, \tag{22}$$

where we have fixed $\lambda = t$, and we have defined $\mathcal{L}$. The Euler-Lagrange equations are given on p. 94 of the lecture notes:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q_i}.$$

For the Lagrangian defined in Eq. (22), we have firstly

$$\frac{\partial\mathcal{L}}{\partial\dot{x}_i} = -mc^2\left(\frac{1}{2}\frac{1}{\sqrt{1-\dot{\mathbf{x}}^2/c^2}}\right)\left(-\frac{2\dot{x}_i}{c^2}\right) + \frac{q}{c}A_i = \frac{m\dot{x}_i}{\sqrt{1-\dot{\mathbf{x}}^2/c^2}} + \frac{q}{c}A_i = p_i + \frac{q}{c}A_i,$$

since $\mathbf{p} = m\gamma\dot{\mathbf{x}}$ and $\beta = \dot{\mathbf{x}}/c$.

Secondly, we have [**?** , p. 50]

$$\frac{\partial\mathcal{L}}{\partial\mathbf{x}} = \boldsymbol{\nabla}\mathcal{L} = \frac{q}{c}\boldsymbol{\nabla}(\dot{\mathbf{x}}\cdot\mathbf{A}) - q\boldsymbol{\nabla}\phi.$$

One of the vector identities on the inside cover of Jackson is

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}),$$

so [**?** , p. 50]

$$\nabla(\dot{\mathbf{x}}\cdot\mathbf{A}) = (\dot{\mathbf{x}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{x}}\times(\nabla\times\mathbf{A}) \quad\implies\quad \frac{\partial\mathcal{L}}{\partial\mathbf{x}} = \frac{q}{c}[(\dot{\mathbf{x}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{x}}\times(\nabla\times\mathbf{A})] - q\nabla\phi.$$

Then the Euler-Lagrange equations become

$$\frac{d\mathbf{p}}{dt} + \frac{q}{c}\frac{d\mathbf{A}}{dt} = \frac{q}{c}[(\dot{\mathbf{x}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{x}}\times(\nabla\times\mathbf{A})] - q\nabla\phi. \tag{23}$$

The total derivative of $\mathbf{A}$ is given by [**?** , p. 50],

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \dot{\mathbf{x}}\cdot\frac{\partial\mathbf{A}}{\partial\dot{\mathbf{x}}} = \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{x}}\cdot\nabla)\mathbf{A}.$$

Then Eq. (23) becomes

$$\frac{d\mathbf{p}}{dt} = \frac{q}{c}\dot{\mathbf{x}}\times(\nabla\times\mathbf{A}) - q\nabla\phi - \frac{q}{c}\frac{d\mathbf{A}}{dt}. \tag{24}$$

According to Wald (5.2–3),

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \qquad\qquad \mathbf{B} = \nabla\times\mathbf{A}.$$

Making these substitutions and $\dot{\mathbf{x}}\to\mathbf{v}$ in Eq. (24), we have

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) \tag{25}$$

as desired. □

5(b) Show that the Lorentz force law can be written covariantly in the form

$$\frac{dU^\mu}{d\tau} = \frac{q}{mc}F^{\mu\nu}U_\nu. \tag{26}$$

Solution. The 4-velocity U^μ is defined by Jackson (11.36) as $U^\mu = \gamma(c,\mathbf{v}) = (U^0,\mathbf{U})$. The 4-momentum is defined by the equation immediately preceding (11.125) as $P^\mu = (\mathcal{E}/c,\mathbf{p})$, where $\mathcal{E}$ is the total energy of the particle. Also from this equation is the relation $P^\mu = mU^\mu$.

According to Jackson (11.26), $d\tau = dt/\gamma$. Making this substitution in Eq. (25) and dividing by m, we find [**?** , p. 553]

$$\frac{1}{\gamma}\frac{d\mathbf{p}}{d\tau} = q\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) \quad\implies\quad \frac{d\gamma\mathbf{v}}{d\tau} = \frac{d\mathbf{U}}{d\tau} = \frac{q\gamma}{mc}\left(c\mathbf{E} + \mathbf{v}\times\mathbf{B}\right). \tag{27}$$

From Wald (5.17), $d\mathcal{E}/dt = \mathbf{J}\cdot\mathbf{E}$. For a point charge, $\mathbf{J} = q\mathbf{v}$. Making this substitution, dividing by mc, and changing to a derivative of τ, we find [**?** , p. 553]

$$\frac{d\mathcal{E}}{dt} = q\mathbf{v}\cdot\mathbf{E} \quad\implies\quad \frac{1}{mc}\frac{d\mathcal{E}}{dt} = \frac{dU^0}{d\tau} = \frac{q\gamma}{mc}\mathbf{v}\cdot\mathbf{E}. \tag{28}$$

This corresponds to the derivative of the temporal part of U^μ.

Now we will work directly from Eq. (26) and write $F^{\mu\nu}U_\nu$ in terms of the fields. From Jackson (11.137),

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}.$$

Then we have

$$F^{\mu\nu}U_\nu = \gamma \begin{bmatrix} v_x E_x + v_y E_y + v_z E_z \\ cE_x + v_y B_z - v_z B_y \\ cE_y - v_x B_z + v_z B_x \\ cE_z + v_x B_y - v_y B_x \end{bmatrix} = \gamma(\mathbf{v} \cdot \mathbf{E}, c\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Using this result, we may combine Eqs. (27) and (28) to write

$$\frac{dU^\mu}{dd\tau} = \frac{q}{mc} F^{\mu\nu} U_\nu$$

as desired. $\square$