Problem 1. The Bertotti-Robsinson solution of the Einstein field equation (MCP 26.2)
Bruno Bertotti and Ivor Rovinson independently solved the Einstein field equation to obtain the following metric for a universe endowed with a uniform magnetic field:

$$ds^2 = Q^2(-\,dt^2 + \sin^2 t\, dz^2 + d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

Here

$$Q = \text{const}, \qquad 0 \leqslant t \leqslant \pi, \qquad -\infty < z < +\infty, \qquad 0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

Discuss the geometry of this universe and the nature of the coordinates $\{t, z, \theta, \phi\}$.

1(a) Which coordinate increases in a timelike direction and which coordinates in spacelike directions?

Solution. The coordinate t increases in a timelike direction because dt^2 appears in the metric with a minus sign. The coordinates z, θ, and ϕ all increase in a spacelike direction because dz^2, $d\theta^2$, and $d\phi^2$ appear in the metric with plus signs.

1(b) Is this universe spherically symmetric?

Solution. Yes. We can think of $\{\theta, \phi\}$ as a coordinate system on the 2-surface of constant t and z. The metric for this 2-surface is

$$^{(2)}ds^2 = Q^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which is the line element of a 2-dimensional sphere in spherical coordinates. This means the spacetime in Eq. (1) is spherically symmetric [1, p. 1244].

1(c) Is this universe cylindrically symmetric?

Solution. No. In order for the universe to be cylindrically symmetric, we would need a constant-r 2-surface parametrized by some coordinates $\{\hat{\phi}, \hat{z}\}$ with the line element

$$^{(2)}ds^2 = r^2\, d\hat{\phi}^2 + d\hat{z}^2\,.$$

This does not appear in Eq. (1).

1(d) Is this universe asymptotically flat?

Solution. No. In order for the spacetime to be asymptotically flat, we would need to find some limit in which Eq. (1) took the form

$$ds^2 = -\,dt^2 + dx^2 + dy^2 + dz^2\,.$$

No such limit exists.

1(e) How does the geometry of this universe change as t ranges from 0 to π? [Hint: show that the curves $\{z, \theta, \phi = \text{const},\ t = \tau/Q\}$ are timelike geodesics—the world lines of the static observers referred to above. Then argue from symmetry, or use the result of Ex. 25.4a.]

Solution. A timelike geodesic has a tangent vector $\vec{u}$ which can be normalized such that $\vec{u} = d/d\tau$ [1, p. 1202]. The geodesic equation is $\nabla_{\vec{u}}\vec{u} = 0$ by MCP (25.11d).

1(f) Give as complete a characterization as you can of the coordinates $\{t, z, \theta, \phi\}$.

Problem 2. Gravitational redshift of light from a star's surface (MCP 26.4) Consider a photon emitted by an atom at rest on the surface of a static star with mass M and radius R. Analyze the photon's motion in the Schwarzschild coordinate system of the star's exterior, $r \geqslant R > 2M$. In particular, compute the "gravitational redshift" of the photon by the following steps.

2(a) Since the emitting atom is nearly an ideal clock, it gives the emitted photon nearly the same frequency ν_{em}, as measured in the emitting atom's proper reference frame (as it would give were it in an Earth laboratory or floating in free space). Thus the proper reference frame of the emitting atom is central to a discussion of the photon's properties and behavior. Show that the orthonormal basis vectors of that proper reference frame are

$$\vec{e}_{\hat{0}} = \frac{1}{\sqrt{1-2M/r}}\frac{\partial}{\partial t}, \qquad \vec{e}_{\hat{r}} = \sqrt{1-2\frac{M}{r}}\frac{\partial}{\partial r}, \qquad \vec{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}, \qquad \vec{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

with $r = R$ (the star's radius).

Solution. We know that the sun's exterior has a Schwarzschild spacetime geometry [1, p. 1250]. According to MCP (26.1), the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

We also know that the wordline of an atom at rest on its surface is timelike ($r = R$, $\theta = \text{const}$, $\phi = \text{const}$) with the squared proper time

$$d\tau^2 = -\,ds^2 = \left(1 - \frac{2M}{R}\right) dt^2\,.$$

This can be read off the Schwarzschild metric, since for $r > 2M$ t is a time coordinate and r is a space coordinate [1, pp. 1248, 1250]. This means that the proper rest frame of the atom has a Schwarzschild coordinate system. From (4) of Box 26.2, the orthonormal basis associated with the Schwarzschild solution of Einstein's equation are [1, p. 1243]

$$\vec{e}_{\hat{0}} = \frac{1}{\sqrt{1-2M/r}}\frac{\partial}{\partial t}, \qquad \vec{e}_{\hat{r}} = \sqrt{1-2\frac{M}{r}}\frac{\partial}{\partial r}, \qquad \vec{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}, \qquad \vec{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

We have already asserted that $r = R$. □

2(b) Explain why the photon's energy as measured in the emitter's proper reference frame is $\mathcal{E} = h\nu_{\text{em}} = -p_{\hat{0}} = -\vec{p}\cdot\vec{e}_{\hat{0}}$. (Here and below h is Planck's constant, and $\vec{p}$ is the photon's 4-momentum.)

Solution. A photon's energy is, by definition, $\mathcal{E} = h\nu$ [1, p. 106]. We know from 2(a) that ν_{em} is the photon's frequency in the emitter's proper reference frame; thus, $\mathcal{E} = h\nu_{\text{em}}$. A photon's momentum always has magnitude $|p| = \mathcal{E}c$, so in natural units $\mathcal{E} = -\vec{p}$ [1, p. 35]. Why the minus sign? The zeroth component of any particle's four-momentum is its energy, so it follows that $\mathcal{E} = -\vec{p}\cdot\vec{e}_{\hat{0}}$. □

2(c) Show that the quantity $\mathcal{E}_\infty = -p_t = -\vec{p}\cdot\partial/\partial t$ is conserved as the photon travels outward from the emitting atom to an observer at very large radius, which we idealize as $r \to \infty$. [Hint: Recall the result of Ex. 25.4a.] Show, further, that $\mathcal{E}_\infty$ is the photon's energy, as measured by the observer at $r = \infty$—which is why it is called the photon's "energy-at-infinity" and denoted $\mathcal{E}_\infty$. The photon's frequency, as measured by that observer, is given, of course, by $h\nu_\infty = \mathcal{E}_\infty$.

Solution. As $r \to \infty$,

$$\mathcal{E} = -\vec{p} \cdot \vec{e}_{\hat{0}} = -\vec{p} \cdot \frac{1}{\sqrt{1 - 2M/r}} \frac{\partial}{\partial t} \to -\vec{p} \cdot \frac{\partial}{\partial t} = \mathcal{E}_\infty$$

as we wanted to show. □

In this limit, we know the Schwarzschild spacetime is flat and takes the form of MCP (26.4),

$$ds^2 = -\,dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2}$$

Then the metric coefficients are easily found by adapting (2) of Box 26.2:

$$g_{tt} = -1, \qquad g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2 \sin^2\theta.$$

This metric is independent of t; that is, all $g_{\alpha\beta,t} = 0$ for all α, β. As we showed in Ex. 25.4a (Problem 4 of Homework 4), $p_A = \vec{p} \cdot \partial/\partial x^A$ is a constant of motion for a freely moving particle in a coordinate system with metric coefficients independent of x^A. Applying this result to the metric Eq. (2), we conclude that $\mathcal{E}_\infty = -\vec{p}\cdot \partial/\partial t$ is a constant of the photon's motion; that is, it is conserved. □

2(d) Show that $\mathcal{E}_\infty = \mathcal{E}\sqrt{1 - 2M/R}$ and thence that $\nu_\infty = \nu_{\text{em}}\sqrt{1 - 2M/R}$, and that therefore the photon is redshifted by an amount

$$\frac{\lambda_{\text{rec}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{1}{\sqrt{1 - 2M/R}} - 1.$$

Here λ_{rec} is the wavelength that the photon's spectral line exhibits at the receiver, and λ_{em} is the wavelength that the emitting kind of atom would produce in an Earth laboratory.

Solution. We know from 2(c) that

$$\mathcal{E} = -\vec{p} \cdot \frac{1}{\sqrt{1 - 2M/r}} \frac{\partial}{\partial t}, \qquad \mathcal{E}_\infty = -\vec{p} \cdot \frac{\partial}{\partial t}.$$

It is then obvious that

$$\mathcal{E} = \frac{1}{\sqrt{1 - 2M/r}} \mathcal{E}_\infty \quad \Longrightarrow \quad \mathcal{E}_\infty = \mathcal{E}\sqrt{1 - \frac{2M}{r}}$$

and

$$h\nu_\infty = h\nu_{\text{em}}\sqrt{1 - \frac{2M}{r}} \quad \Longrightarrow \quad \nu_\infty = \nu_{\text{em}}\sqrt{1 - \frac{2M}{r}}$$

as we wanted to show. □

We know from 2(c) that the observer is located at infinity, so $\lambda_{\text{rec}} = \lambda_\infty$. In natural units, $\lambda = 1/\nu$. Then

$$\frac{\lambda_{\text{rec}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{1/\nu_\infty - 1/\nu_{\text{em}}}{1/\nu_{\text{em}}} = \frac{\nu_{\text{em}}}{\nu_\infty} - 1 = \frac{1}{\sqrt{1 - 2M/R}} - 1$$

as we wanted to show. □

2(e) Evaluate this redshift for Earth, for the Sun, and for a 1.4-solar-mass, 10-km-radius neutron star.

Solution. In SI units [2],

$$z = \frac{\lambda_{\text{rec}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{1}{\sqrt{1 - 2GM/Rc^2}} - 1$$

where $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ and $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$ [3, p. A-7].

For Earth, $R_\oplus \approx 6.37 \times 10^6\,\mathrm{m}$ and $M_\oplus \approx 5.97 \times 10^{24}\,\mathrm{kg}$ [3, p. A-8]. Then

$$z_\oplus = \frac{1}{\sqrt{1 - 2GM_\oplus/R_\oplus c^2}} - 1 \approx \sqrt{1 - \frac{2(6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}})(5.97 \times 10^{24}\,\mathrm{kg})}{(6.37 \times 10^6\,\mathrm{m})(3.00 \times 10^8\,\mathrm{m\,s^{-1}})^2}}^{\,-1} - 1 \approx 6.95 \times 10^{-10}.$$

For the Sun, $R_\odot \approx 6.96 \times 10^8\,\mathrm{m}$ and $M_\odot \approx 1.99 \times 10^{30}\,\mathrm{kg}$ [3, p. A-8]. Then

$$z_\odot = \frac{1}{\sqrt{1 - 2GM_\odot/R_\odot c^2}} - 1 \approx \sqrt{1 - \frac{2(6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}})(1.99 \times 10^{30}\,\mathrm{kg})}{(6.96 \times 10^8\,\mathrm{m})(3.00 \times 10^8\,\mathrm{m\,s^{-1}})^2}}^{\,-1} - 1 \approx 2.12 \times 10^{-6}.$$

For the neutron star, $R_{\text{ns}} \approx 10 \times 10^3\,\mathrm{m}$ and $M_{\text{ns}} = 1.4M_\odot \approx 2.79 \times 10^{30}\,\mathrm{kg}$. Then

$$z_{\text{ns}} = \frac{1}{\sqrt{1 - 2GM_{\text{ns}}/R_{\text{ns}} c^2}} - 1 \approx \sqrt{1 - \frac{2(6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}})(2.79 \times 10^{30}\,\mathrm{kg})}{(10 \times 10^3\,\mathrm{m})(3.00 \times 10^8\,\mathrm{m\,s^{-1}})^2}}^{\,-1} - 1 \approx 0.306.$$

Problem 3. Mass-radius relation for neutron stars (MCP 26.7) We can illustrate the approach using a simple functional form, which, around nuclear density ($\rho_{\text{nuc}} \simeq 2.3 \times 10^{17}\,\mathrm{kg\,m^{-3}}$), is a fair approximation to some of the models:

$$P = 3 \times 10^{32} \left(\frac{\rho}{\rho_{\text{nuc}}}\right)^3 \mathrm{N\,m^{-2}}.$$

For this equation of state, use the equations of stellar structure (26.38a) and (26.38c) to find the masses and radii of stars with a range of central pressures, and hence deduce a mass-radius relation, $M(R)$. You should discover that, as the central pressure is increased, the mass passes through a maximum, while the radius continues to decrease.

Solution. MCP (26.38a) and (26.38c) are, respectively,

$$\frac{dm}{dr} = 4\pi r^2 \rho, \qquad\qquad \frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)}.$$

what are we even supposed to do here?

Problem 4. An astronaut on a rocket ship has just crossed the event horizon of a Schwarzschild black hole. Show that, no matter how the rocket engines are fired, they will reach $r = 0$ in a proper time $\tau \leqslant \pi M$.

Problem 5. Fermat's principle for a photon's path in static spacetime (MCP 27.4) Show that the Euler-Lagrange equation for the action principle,

$$\Delta t = \int_0^1 \sqrt{\gamma_{jk} \frac{dx^j}{d\eta} \frac{dx^k}{d\eta}}\, d\eta\,, \qquad \text{where } \gamma_{jk} = \frac{g_{jk}}{-g_{00}}, \tag{3}$$

is equivalent to the geodesic equation for a photon in the static spacetime metric $g_{00}(x^k)$, $g_{ij}(x^k)$. Specifically, do the following.

5(a) The action Eq. (3) is the same as that for a geodesic in a 3-dimensional space with the metric γ_{jk} and with t playing the role of proper distance traveled [Eq. (25.19) converted to a positive-definite, 3-dimensional metric]. Therefore, the Euler-Lagrange equation for Eq. (3) is the geodesic equation in that (fictitious) space [Eq. (25.14) with t the affine parameter]. Using Eq. (24.38c) for the connection coefficients, show that the geodesic equation can be written in the form

$$\gamma_{jk} \frac{d^2 x^k}{dt^2} + \frac{1}{2}(\gamma_{jk,l} + \gamma_{jl,k} - \gamma_{kl,j}) \frac{dx^k}{dt} \frac{dx^l}{dt} = 0. \tag{4}$$

Solution. MCP (25.14) is

$$\frac{d^2 x^\alpha}{d\zeta^2} + \Gamma^\alpha{}_{\mu\nu} \frac{dx^\mu}{d\zeta} \frac{dx^\nu}{d\zeta} = 0 \tag{5}$$

with ζ the affine parameter. With t as the affine parameter, this can be written

$$0 = \frac{d^2 x_j}{dt^2} + \Gamma_{jkl} \frac{dx^k}{dt} \frac{dx^l}{dt} = g_{jk} \frac{d^2 x^k}{dt^2} + \Gamma_{jkl} \frac{dx^k}{dt} \frac{dx^l}{dt},$$

where we have also lowered the first index. Using the definition $\gamma_{jk} = g_{jk}/-g_{00}$, we can write

$$\gamma_{jk} \frac{d^2 x^k}{dt^2} + \frac{1}{-g_{00}} \Gamma_{jkl} \frac{dx^k}{dt} \frac{dx^l}{dt}. \tag{6}$$

MCP (24.28c) is

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\beta\gamma} - c_{\beta\gamma\alpha}).$$

Assuming a coordinate basis, the commutation coefficients $c_{\alpha\beta\gamma}$ vanish [1, p. 1171]. Then, again using the definition of γ_{jk}, we have

$$\frac{1}{-g_{00}} \Gamma_{jkl} = \frac{1}{2}(\gamma_{jk,l} + \gamma_{jl,k} - \gamma_{kl,j}).$$

Making this substitution in Eq. (6), we have

$$\gamma_{jk} \frac{d^2 x^k}{dt^2} + \frac{1}{2}(\gamma_{jk,l} + \gamma_{jl,k} - \gamma_{kl,j}) \frac{dx^k}{dt} \frac{dx^l}{dt} = 0$$

as desired. □

5(b) Take the geodesic equation Eq. (5) for the light ray in the real spacetime, with spacetime affine parameter ζ, and change parameters to coordinate time t. Thereby obtain

$$g_{jk} \frac{d^2 x^k}{dt^2} + \Gamma_{jkl} \frac{dx^k}{dt} \frac{dx^l}{dt} - \Gamma_{j00} \frac{g_{kl}}{g_{00}} \frac{dx^k}{dt} \frac{dx^l}{dt} + \frac{d^2 t/d\zeta^2}{(dt/d\zeta)^2} g_{jk} \frac{dx^k}{dt} = 0, \qquad \frac{d^2 t/d\zeta^2}{(dt/d\zeta)^2} + 2\Gamma_{0k0} \frac{dx^k/dt}{g_{00}} = 0. \tag{7}$$

5(c) Insert the second of Eqs. (7) into the first, and write the connection coefficients in terms of derivatives of the spacetime metric. With a little algebra, bring your result into the form Eq. (4) of the Fermat-principle Euler-Lagrange equation.

Problem 6. Consider a collapsing spherical shell of dust with radius $R(\tau)$ where τ is the proper time of the dust. Exterior to the shell the metric is Schwarzschild with some mass parameter M, while interior the geometry is that of flat empty space.

6(a) Show that the rest mass of the shell, $\mu = 4\pi R^2(\tau)\sigma$ is constant, where σ is the surface mass density.

6(b) Derive a differential equation of motion for $R(\tau)$:

$$M = \mu\sqrt{1 + \left(\frac{dR}{d\tau}\right)^2} - \frac{\mu^2}{2R}.$$

6(c) Solve the equation (implicitly) in the special case where the dust begins collapse at rest from infinite radius.

References

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