

1 Problem 1

Consider operators J and K acting in a three-dimensional space as

$$J|e_1\rangle = i|e_2\rangle, \quad J|e_2\rangle = -i|e_1\rangle, \quad J|e_3\rangle = 0, \quad (1)$$

$$K|e_1\rangle = 0, \quad K|e_2\rangle = i|e_3\rangle, \quad K|e_3\rangle = -i|e_2\rangle, \quad (2)$$

where $|e_1\rangle, |e_2\rangle, |e_3\rangle$ form a complete orthonormal basis.

1.1 Compute the matrix elements of J and K .

Solution. The matrix elements of J are

$$J_{11} = \langle e_1|J|e_1\rangle = i\langle e_1|e_2\rangle = 0, \quad J_{12} = \langle e_1|J|e_2\rangle = -i\langle e_1|e_1\rangle = -i, \quad J_{13} = \langle e_1|J|e_3\rangle = 0, \quad (3)$$

$$J_{21} = \langle e_2|J|e_1\rangle = i\langle e_2|e_2\rangle = i, \quad J_{22} = \langle e_2|J|e_2\rangle = -i\langle e_2|e_1\rangle = 0, \quad J_{23} = \langle e_2|J|e_3\rangle = 0, \quad (4)$$

$$J_{31} = \langle e_3|J|e_1\rangle = i\langle e_3|e_2\rangle = 0, \quad J_{32} = \langle e_3|J|e_2\rangle = -i\langle e_3|e_1\rangle = 0, \quad J_{33} = \langle e_3|J|e_3\rangle = 0. \quad (5)$$

The matrix elements of K are

$$K_{11} = \langle e_1|K|e_1\rangle = 0, \quad K_{12} = \langle e_1|K|e_2\rangle = i\langle e_1|e_3\rangle = 0, \quad K_{13} = \langle e_1|K|e_3\rangle = -i\langle e_1|e_2\rangle = 0, \quad (6)$$

$$K_{21} = \langle e_2|K|e_1\rangle = 0, \quad K_{22} = \langle e_2|K|e_2\rangle = i\langle e_2|e_3\rangle = 0, \quad K_{23} = \langle e_2|K|e_3\rangle = -i\langle e_2|e_2\rangle = -i, \quad (7)$$

$$K_{31} = \langle e_3|K|e_1\rangle = 0, \quad K_{32} = \langle e_3|K|e_2\rangle = i\langle e_3|e_3\rangle = i, \quad K_{33} = \langle e_3|K|e_3\rangle = -i\langle e_3|e_2\rangle = 0. \quad (8)$$

1.2 Consider $O = AJ + BK$ where A, B are real numbers. Show that O is Hermitian.

Solution. Using (3)–(8), the matrix elements of O are

$$O_{11} = O_{13} = O_{22} = O_{31} = O_{33} = 0, \quad (9)$$

$$O_{12} = -iA, \quad (10)$$

$$O_{21} = iA, \quad (11)$$

$$O_{23} = -iB, \quad (12)$$

$$O_{32} = iB. \quad (13)$$

O is Hermitian if and only if $O_{ij} = O_{ji}^*$ for all O_{ij} . Recall that $(z^*)^* = z$ for any $z \in \mathbb{C}$. From (9)–(13), note that

$$O_{11} = 0 = O_{11}^*, \quad (14)$$

$$O_{12} = -iA = (iA)^* = O_{21}^*, \quad (15)$$

$$O_{13} = 0 = O_{31}^*, \quad (16)$$

$$O_{22} = 0 = O_{22}^*, \quad (17)$$

$$O_{23} = -iB = (iB)^* = O_{32}^*, \quad (18)$$

$$O_{33} = 0 = O_{33}^*, \quad (19)$$

so O is indeed Hermitian. □

1.3 If $|p_\lambda\rangle$ is an eigenvector of O , we have $O|p_\lambda\rangle = \lambda|p_\lambda\rangle$ where λ is the corresponding eigenvalue. $|p_\lambda\rangle$ can be expanded as $|p_\lambda\rangle = \sum_{i=1}^3 u_{\lambda,i} |e_i\rangle$. Denote the three eigenvalues and the corresponding normalized eigenvectors of O as $\lambda_+, \lambda_0, \lambda_-$ and $|p_+\rangle, |p_0\rangle, |p_-\rangle$ where λ_+ (λ_-) is the largest (smallest) eigenvalue. Find $\lambda_+, \lambda_0, \lambda_-$ and $|p_+\rangle, |p_0\rangle, |p_-\rangle$.

Solution. Using a matrix representation in the $|e_1\rangle, |e_2\rangle, |e_3\rangle$ basis, we can write

$$O = \begin{bmatrix} 0 & iA & 0 \\ -iA & 0 & iB \\ 0 & -iB & 0 \end{bmatrix}. \quad (20)$$

λ is an eigenvalue of O if $\det(O - \lambda I) = 0$, where I is the identity matrix. That is,

$$0 = \begin{vmatrix} -\lambda & iA & 0 \\ -iA & -\lambda & iB \\ 0 & -iB & -\lambda \end{vmatrix} \quad (21)$$

$$= (-\lambda)^3 - (-\lambda)(-iB)(iB) - (-iA)(iA)(-\lambda) \quad (22)$$

$$= \lambda(\lambda^2 - A^2 - B^2) \quad (23)$$

$$= \lambda^2 - A^2 - B^2. \quad (24)$$

From (23) we obtain $\lambda_0 = 0$, and from (24) we obtain $\lambda_\pm = \pm\sqrt{A^2 + B^2}$.

Let $|\lambda_0\rangle, |\lambda_\pm\rangle$ be the not-necessarily-normalized eigenvectors corresponding to λ_0, λ_\pm . Beginning with λ_0 , we will find the corresponding eigenvector $|\lambda_0\rangle = \lambda_{0,1}|e_1\rangle + \lambda_{0,2}|e_2\rangle + \lambda_{0,3}|e_3\rangle$. We seek $\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3}$ such that

$$\begin{bmatrix} 0 & iA & 0 \\ -iA & 0 & iB \\ 0 & -iB & 0 \end{bmatrix} \begin{bmatrix} \lambda_{0,1} \\ \lambda_{0,2} \\ \lambda_{0,3} \end{bmatrix} = 0 \begin{bmatrix} \lambda_{0,1} \\ \lambda_{0,2} \\ \lambda_{0,3} \end{bmatrix}. \quad (25)$$

The algebraic equations corresponding to (25) are

$$iA \lambda_{0,2} = 0, \quad (26)$$

$$iB \lambda_{0,3} - iA \lambda_{0,1} = 0, \quad (27)$$

$$-iB \lambda_{0,2} = 0. \quad (28)$$

(26) and (28) imply that $\lambda_{0,2} = 0$. We may fix $\lambda_{0,3} = A$ without loss of generality. Then (27) implies $\lambda_{0,1} = B$. Thus, $|\lambda_0\rangle = B|e_1\rangle + A|e_3\rangle$.

For $|\lambda_\pm\rangle = \lambda_{\pm,1}|e_1\rangle + \lambda_{\pm,2}|e_2\rangle + \lambda_{\pm,3}|e_3\rangle$, we seek $\lambda_{\pm,1}, \lambda_{\pm,2}, \lambda_{\pm,3}$ such that

$$\begin{bmatrix} 0 & iA & 0 \\ -iA & 0 & iB \\ 0 & -iB & 0 \end{bmatrix} \begin{bmatrix} \lambda_{\pm,1} \\ \lambda_{\pm,2} \\ \lambda_{\pm,3} \end{bmatrix} = \pm\sqrt{A^2 + B^2} \begin{bmatrix} \lambda_{\pm,1} \\ \lambda_{\pm,2} \\ \lambda_{\pm,3} \end{bmatrix}. \quad (29)$$

The algebraic equations corresponding to (29) are

$$iA \lambda_{\pm,2} = \pm\sqrt{A^2 + B^2} \lambda_{\pm,1}, \quad (30)$$

$$iB \lambda_{\pm,3} - iA \lambda_{\pm,1} = \pm\sqrt{A^2 + B^2} \lambda_{\pm,2}, \quad (31)$$

$$-iB \lambda_{\pm,2} = \pm\sqrt{A^2 + B^2} \lambda_{\pm,3}. \quad (32)$$

Summing (30), (31), and (32), we have

$$\pm\sqrt{A^2 + B^2}(\lambda_{\pm,1} + \lambda_{\pm,2} + \lambda_{\pm,3}) = iA(\lambda_{\pm,2} - \lambda_{\pm,1}) + iB(\lambda_{\pm,3} - \lambda_{\pm,2}) \quad (33)$$

$$\pm i\sqrt{A^2 + B^2}(\lambda_{\pm,1} + \lambda_{\pm,2} + \lambda_{\pm,3}) = B(\lambda_{\pm,2} - \lambda_{\pm,3}) - A(\lambda_{\pm,2} - \lambda_{\pm,1}). \quad (34)$$

From the form of (34), we make the ansatz $\lambda_{\pm,1} = -A$, $\lambda_{\pm,3} = B$. Making the relevant substitutions in (30) and (32), we have

$$iA\lambda_{\pm,2} = \mp A\sqrt{A^2 + B^2}, \quad (35)$$

$$-iB\lambda_{\pm,2} = \pm B\sqrt{A^2 + B^2} \quad (36)$$

which both imply $\lambda_{\pm,2} = \mp i\sqrt{A^2 + B^2}$. Therefore, $|\lambda_{\pm}\rangle = -A|e_1\rangle \mp i\sqrt{A^2 + B^2}|e_2\rangle + B|e_3\rangle$.

Now we will compute $|p_+\rangle, |p_0\rangle, |p_-\rangle$ by normalizing $|\lambda_0\rangle, |\lambda_{\pm}\rangle$. Note that

$$\|\lambda_0\|^2 = \langle\lambda_0|\lambda_0\rangle = A^2 + B^2, \quad (37)$$

$$\|\lambda_{\pm}\|^2 = \langle\lambda_{\pm}|\lambda_{\pm}\rangle = A^2 + (A^2 + B^2) + B^2 = 2A^2 + 2B^2, \quad (38)$$

so

$$|p_+\rangle = \frac{|\lambda_+\rangle}{\|\lambda_+\|} = \frac{-A|e_1\rangle - i\sqrt{A^2 + B^2}|e_2\rangle + B|e_3\rangle}{\sqrt{2}\sqrt{A^2 + B^2}}, \quad (39)$$

$$|p_0\rangle = \frac{|\lambda_0\rangle}{\|\lambda_0\|} = \frac{B|e_1\rangle + A|e_3\rangle}{\sqrt{A^2 + B^2}}, \quad (40)$$

$$|p_-\rangle = \frac{|\lambda_-\rangle}{\|\lambda_-\|} = \frac{-A|e_1\rangle + i\sqrt{A^2 + B^2}|e_2\rangle + B|e_3\rangle}{\sqrt{2}\sqrt{A^2 + B^2}}. \quad (41)$$

1.4 Define a new state $|e'_1\rangle$ by $|e'_1\rangle = |h_1\rangle/\|h_1\|$ where $\|h_1\| = \sqrt{\langle h_1|h_1\rangle}$ and $|h_1\rangle = (1 - |p_0\rangle\langle p_0|)|e_1\rangle$. Find the probability that the state $|e'_1\rangle$ is found to have the eigenvalue $\lambda_+, \lambda_0, \lambda_-$.

Solution. First, we can find an $|e'_1\rangle$ using the result (40) for $|p_0\rangle$. Beginning with $|h_1\rangle$, we have

$$|h_1\rangle = |e_1\rangle - \langle p_0|e_1\rangle|p_0\rangle = |e_1\rangle - \frac{B}{\sqrt{A^2 + B^2}}|p_0\rangle = |e_1\rangle - \frac{B}{\sqrt{A^2 + B^2}}\left(\frac{B|e_1\rangle + A|e_3\rangle}{\sqrt{A^2 + B^2}}\right) \quad (42)$$

$$= \left(1 - \frac{B^2}{A^2 + B^2}\right)|e_1\rangle - \frac{AB}{A^2 + B^2}|e_3\rangle. \quad (43)$$

Then

$$\|h_1\|^2 = \left(1 - \frac{B^2}{A^2 + B^2}\right)^2 - \left(\frac{AB}{A^2 + B^2}\right)^2 = 1 - \frac{2B^2}{A^2 + B^2} + \frac{B^4}{(A^2 + B^2)^2} - \frac{A^2B^2}{(A^2 + B^2)^2} \quad (44)$$

$$= \frac{(A^2 + B^2)^2 - 2B^2(A^2 + B^2) + B^4 - A^2B^2}{(A^2 + B^2)^2} = \frac{A^2(A^2 + B^2)}{(A^2 + B^2)^2} \quad (45)$$

$$= \frac{A^2}{A^2 + B^2} \quad (46)$$

so

$$|e'_1\rangle = \frac{|h_1\rangle}{\|h_1\|} = \frac{\sqrt{A^2 + B^2}}{A} \left[\left(1 - \frac{B^2}{A^2 + B^2}\right) |e_1\rangle - \frac{AB}{A^2 + B^2} |e_3\rangle \right] \quad (47)$$

$$= \frac{A}{\sqrt{A^2 + B^2}} |e_1\rangle - \frac{B}{\sqrt{A^2 + B^2}} |e_3\rangle \quad (48)$$

The probability that $|e'_1\rangle$ has the eigenvalue λ is $|\langle p_\lambda | e'_1 \rangle|^2$. Thus,

$$|\langle p_\pm | e'_1 \rangle|^2 = \left| -\frac{A}{\sqrt{A^2 + B^2}} \frac{A}{\sqrt{2}\sqrt{A^2 + B^2}} - \frac{B}{\sqrt{A^2 + B^2}} \frac{B}{\sqrt{2}\sqrt{A^2 + B^2}} \right|^2 = \left| -\frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}, \quad (49)$$

$$|\langle p_0 | e'_1 \rangle|^2 = \left| -\frac{A}{\sqrt{A^2 + B^2}} \frac{B}{\sqrt{A^2 + B^2}} + \frac{B}{\sqrt{A^2 + B^2}} \frac{A}{\sqrt{A^2 + B^2}} \right|^2 = 0. \quad (50)$$