1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1}$$

1.1 Verify the following:

a. $i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle$,

b. $i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle \langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle$,

c. $i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = -\dfrac{\hbar^2}{2m} \left[\langle\Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right]$,

d. $\langle\Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle = \dfrac{\hbar}{i} \left[\langle\Phi(t)|x\rangle \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right]$

e. $\dfrac{\hbar}{i}\partial_x \left[\langle\Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - mel\Phi(t)p^2x \langle x|\Psi(t)\rangle$

Solution.

a. Beginning with Schrödinger's equation, note that

$$i\hbar\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \tag{2}$$
$$i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle x|H|\Psi(t)\rangle \tag{3}$$
$$(i\hbar\partial_t \langle x|\Psi(t)\rangle)^* = (\langle x|H|\Psi(t)\rangle)^* \tag{4}$$
$$-i\hbar\partial_t \langle\Psi(t)|x\rangle = \langle\Psi(t)|H|x\rangle \tag{5}$$
$$i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle, \tag{6}$$

where in going to (5) we have used the fact that H is Hermitian, and (6) is what we sought to prove. □

b. Beginning with what was proven in (a),

$$i\hbar\partial_t \langle\Phi(t)|x\rangle = -\langle\Phi(t)|H|x\rangle \tag{7}$$
$$i\hbar(\partial_t \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle = -\langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle. \tag{8}$$

From (3), we can write
$$\langle\Phi(t)|x\rangle \, i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle \langle x|H|\Psi(t)\rangle. \tag{9}$$

Adding (15) and (17) yields

$$\langle\Phi(t)|x\rangle \, i\hbar\partial_t \langle x|\Psi(t)\rangle + i\hbar(\partial_t \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle \langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle \tag{10}$$
$$i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle \langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle, \tag{11}$$

where in going to (11) we have used the product rule of differentiation. (11) is what we sought to prove. □

c. Using (1), note that:

$$\langle x|H|\Psi(t)\rangle = \langle x| \left[\frac{p^2}{2m} + V(x) \right] |\Psi(t)\rangle \tag{12}$$

$$= \frac{1}{2m} \langle x|p^2|\Psi(t)\rangle + \langle x|V(x)|\Psi(t)\rangle \tag{13}$$

$$= \frac{(-i\hbar\partial_x)^2}{2m} \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle \tag{14}$$

$$= -\frac{\hbar^2}{2m}\partial_x^2 \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle, \tag{15}$$

where in going to (14) we have used the fact that

$$\langle x|p|\Psi(x)\rangle = -i\hbar\partial_x \langle x|\Psi(t)\rangle. \tag{16}$$

Similarly, note that

$$\langle \Phi(t)|H|x\rangle = -\frac{\hbar^2}{2m}\partial_x^2 \langle \Phi(t)|x\rangle + V(x) \langle \Phi(t)|x\rangle \tag{17}$$

where we have used

$$\langle \Phi(t)|p|x\rangle = i\hbar\partial_x \langle \Phi(t)|x\rangle. \tag{18}$$

Making the substitutions (15) and (17) into what was proven in (b),

$$i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle \left[-\frac{\hbar^2}{2m}\partial_x^2 \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle \right]$$

$$- \left[-\frac{\hbar^2}{2m}\partial_x^2 \langle \Phi(t)|x\rangle + V(x) \langle \Phi(t)|x\rangle \right] \langle x|\Psi(t)\rangle \tag{19}$$

$$= -\frac{\hbar^2}{2m} \left[\langle \Phi(t)|x\rangle \partial_x^2 \langle \Phi(t)|x\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle \right]$$

$$+ V(x) \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle - V(x) \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle \tag{20}$$

$$= -\frac{\hbar^2}{2m} \left[\langle \Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle \right], \tag{21}$$

as we sought to prove. □

d. Applying (16) and (18) to the left-hand side of (d),

$$\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle (-i\hbar\partial_x \langle x|\Psi(t)\rangle) + (i\hbar\partial_x \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle \tag{22}$$

$$= \frac{\hbar}{i} \left[\langle \Phi(t)|x\rangle \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle \right] \tag{23}$$

as we sought to prove. □

e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle) = (\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \partial_x \langle x|p|\Psi(t)\rangle \tag{24}$$

Multiplying through by $\hbar/i$,

$$\frac{\hbar}{i}\partial_x(\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle) = (-i\hbar\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle - \langle \Phi(t)|x\rangle i\hbar\partial_x \langle x|p|\Psi(t)\rangle \tag{25}$$

$$= -\langle \Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle, \tag{26}$$

where in going to (26) we have used (16) and (18). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i} \partial_x \left(\langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle \right) = \left(-i\hbar \partial_x \langle \Phi(t)|p|x \rangle \right) \langle x|\Psi(t) \rangle - \langle \Phi(t)|p|x \rangle \, i\hbar \partial_x \langle x|\Psi(t) \rangle \tag{27}$$

$$= - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|p|\Psi(t) \rangle . \tag{28}$$

Adding the results of (26) and (28),

$$\frac{\hbar}{i} \partial_x \left[\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle \right] = \langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p|x \rangle \langle x|p|\Psi(t) \rangle$$

$$+ \langle \Phi(t)|p|x \rangle \langle x|p|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \tag{29}$$

$$= \langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \tag{30}$$

as we sought to prove. $\qquad \square$

1.2 Define

$$\rho(x,t) = \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle , \tag{31}$$

$$J_x(x,t) = \frac{1}{2m} \left[\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle \right] . \tag{32}$$

Show that $\rho(x,t) + \partial_x J_x(x,t) = 0$.

Solution. From (31),

$$\partial_t \rho(x,t) = \partial_t (\langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle), \tag{33}$$

and from what was proven in 1(c),

$$\partial_t (\langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle) = -\frac{1}{i\hbar} \left[\langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle \right] \tag{34}$$

$$= -\frac{1}{2m} \frac{i}{\hbar} \left[\langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \right] , \tag{35}$$

where we have applied (16) and (18) in going to (35). Equating (33) and (35),

$$\partial_t \rho(x,t) = -\frac{1}{2m} \frac{i}{\hbar} \left[\langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \right] . \tag{36}$$

Beginning from (32),

$$\partial_x J_x(x,t) = \frac{1}{2m} \partial_x \left[\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle \right] \tag{37}$$

$$= \frac{1}{2m} \frac{i}{\hbar} \left[\langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \right] , \tag{38}$$

where in going to (38) we have used what was proven in 1(e). Summing (36) and (38), we have

$$\partial_t \rho(x,t) + \partial_x J_x(x,t) = -\frac{1}{2m} \frac{i}{\hbar} \left[\langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \right]$$

$$+ \frac{1}{2m} \frac{i}{\hbar} \left[\langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle \right] \tag{39}$$

$$= 0 \tag{40}$$

as we sought to prove. $\qquad \square$