

1 Problem 1

Let's consider coherent states of a one-dimensional quantum particle with mass m confined in a one-dimensional harmonic potential $V(X) = m\omega^2 X^2/2$:

$$a|\lambda\rangle = \lambda|\lambda\rangle, \quad |\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right) \exp(\lambda a^\dagger) |0\rangle.$$

Here, λ is a complex parameter.

1.1 Compute $\langle x|\lambda\rangle$.

Solution. Since $a|0\rangle = 0|0\rangle$, $\exp(\lambda a)|0\rangle = |0\rangle$ and therefore we can write

$$\langle x|\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right) \langle x|\exp(\lambda a^\dagger)\exp(\lambda a)|0\rangle. \quad (1)$$

For two operators A and B , $e^{A+B} = e^{-[A,B]/2}e^Ae^B$ if $[A, B]$ commutes with each A and B . Here, we have

$$\exp[\lambda(a^\dagger + a)] = \exp\left(\frac{\lambda^2}{2}\right) \exp(\lambda a^\dagger) \exp(\lambda a) \implies \exp(\lambda a^\dagger) \exp(\lambda a) = \exp\left(-\frac{\lambda^2}{2}\right) \exp[\lambda(a^\dagger + a)],$$

where we have used $[a, a^\dagger] = 1$. From (2.3.24) in Sakurai,

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a), \quad (2)$$

so

$$\exp[\lambda(a^\dagger + a)] = \exp\left(\lambda X \sqrt{\frac{2m\omega}{\hbar}}\right).$$

Making these substitutions into (1) yields

$$\begin{aligned} \langle x|\lambda\rangle &= \exp\left(-\frac{1}{2}|\lambda|^2\right) \exp\left(-\frac{\lambda^2}{2}\right) \langle x|\exp\left(\lambda X \sqrt{\frac{2m\omega}{\hbar}}\right)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\lambda|^2\right) \exp\left(-\frac{\lambda^2}{2}\right) \exp\left(\lambda x \sqrt{\frac{2m\omega}{\hbar}}\right) \langle x|0\rangle. \end{aligned} \quad (3)$$

From (2.3.30) in Sakurai,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

so (3) becomes

$$\langle x|\lambda\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{1}{2}|\lambda|^2 - \frac{\lambda^2}{2} + \lambda x \sqrt{\frac{2m\omega}{\hbar}} - \frac{m\omega}{2\hbar}x^2\right).$$

1.2 Compute $\langle \lambda | X | \lambda \rangle$, $\langle \lambda | P | \lambda \rangle$, $\langle \lambda | X^2 | \lambda \rangle$, and $\langle \lambda | P^2 | \lambda \rangle$. Also, compute $\langle \lambda | (\Delta X)^2 | \lambda \rangle$ $\langle \lambda | (\Delta P)^2 | \lambda \rangle$ where $\Delta A = A - \langle A \rangle$.

Solution. For $\langle \lambda | X | \lambda \rangle$,

$$\begin{aligned} \langle \lambda | X | \lambda \rangle &= \frac{1}{|\lambda|^2} \langle \lambda | a^\dagger X a | \lambda \rangle \\ &= \frac{1}{|\lambda|^2} \sqrt{\frac{\hbar}{2m\omega}} \langle \lambda | a^\dagger (a + a^\dagger) a | \lambda \rangle = \frac{1}{|\lambda|^2} \sqrt{\frac{\hbar}{2m\omega}} \langle \lambda | (a^\dagger a^2 + a^{\dagger 2} a) | \lambda \rangle = \frac{|\lambda|^2 (\lambda^* + \lambda)}{|\lambda|^2} \sqrt{\frac{\hbar}{2m\omega}} \\ &= 2 \operatorname{Re}(\lambda) \sqrt{\frac{\hbar}{2m\omega}}, \end{aligned} \quad (4)$$

where we have again used (2). For $\langle \lambda | P | \lambda \rangle$,

$$\begin{aligned} \langle \lambda | P | \lambda \rangle &= \frac{1}{\lambda^2} \langle \lambda | a^\dagger P a | \lambda \rangle \\ &= \frac{i}{|\lambda|^2} \sqrt{\frac{\hbar m \omega}{2}} \langle \lambda | a^\dagger (a^\dagger - a) a | \lambda \rangle = \frac{i}{|\lambda|^2} \sqrt{\frac{\hbar m \omega}{2}} \langle \lambda | (a^{\dagger 2} a - a^\dagger a^2) | \lambda \rangle = \frac{i |\lambda|^2 (\lambda^* - \lambda)}{|\lambda|^2} \sqrt{\frac{\hbar m \omega}{2}} \\ &= 2 \operatorname{Im}(\lambda) \sqrt{\frac{\hbar m \omega}{2}}. \end{aligned} \quad (5)$$

From (2), note that

$$X^2 = \frac{\hbar}{2m\omega} (a^2 + a a^\dagger + a^\dagger a + a^{\dagger 2}), \quad P^2 = -\frac{\hbar m \omega}{2} (a^{\dagger 2} - a^\dagger a - a a^\dagger + a^2).$$

Then for $\langle \lambda | X^2 | \lambda \rangle$,

$$\begin{aligned} \langle \lambda | X^2 | \lambda \rangle &= \frac{1}{|\lambda|^2} \langle \lambda | a^\dagger X^2 a | \lambda \rangle = \frac{1}{|\lambda|^2} \frac{\hbar}{2m\omega} \langle \lambda | a^\dagger (a^2 + a a^\dagger + a^\dagger a + a^{\dagger 2}) a | \lambda \rangle \\ &= \frac{1}{|\lambda|^2} \frac{\hbar}{2m\omega} \langle \lambda | (a^\dagger a^3 + a^\dagger a a^\dagger a + a^{\dagger 2} a^2 + a^{\dagger 3} a) | \lambda \rangle = \frac{1}{|\lambda|^2} \frac{\hbar}{2m\omega} \langle \lambda | (a^\dagger a^3 + a^\dagger a + 2a^{\dagger 2} a^2 + a^{\dagger 3} a) | \lambda \rangle \\ &= (\lambda^2 + 1 + 2|\lambda|^2 + \lambda^{*2}) \frac{\hbar}{2m\omega} = (1 + 2[\operatorname{Re}(\lambda)^2 + \operatorname{Im}(\lambda)^2] + 2[\operatorname{Re}(\lambda)^2 - \operatorname{Im}(\lambda)^2]) \frac{\hbar}{2m\omega} \\ &= [1 + 4 \operatorname{Re}(\lambda)^2] \frac{\hbar}{2m\omega}, \end{aligned} \quad (6)$$

where we have again used $[a, a^\dagger] = 1$. For $\langle \lambda | P^2 | \lambda \rangle$,

$$\begin{aligned} \langle \lambda | P^2 | \lambda \rangle &= \frac{1}{|\lambda|^2} \langle \lambda | a^\dagger P^2 a | \lambda \rangle = -\frac{1}{|\lambda|^2} \frac{\hbar m \omega}{2} \langle \lambda | a^\dagger (a^{\dagger 2} - a^\dagger a - a a^\dagger + a^2) a | \lambda \rangle \\ &= -\frac{1}{|\lambda|^2} \frac{\hbar m \omega}{2} \langle \lambda | (a^{\dagger 2} a - a^{\dagger 2} a^2 - a^\dagger a a^\dagger a + a^\dagger a^3) | \lambda \rangle = -\frac{1}{|\lambda|^2} \frac{\hbar m \omega}{2} \langle \lambda | (a^{\dagger 3} a - a^\dagger a - 2a^{\dagger 2} a^2 + a^\dagger a^3) | \lambda \rangle \\ &= -(\lambda^{*2} - 1 - 2|\lambda|^2 + \lambda^2) \frac{\hbar m \omega}{2} = (1 + 2[\operatorname{Re}(\lambda)^2 + \operatorname{Im}(\lambda)^2] - 2[\operatorname{Re}(\lambda)^2 - \operatorname{Im}(\lambda)^2]) \frac{\hbar m \omega}{2} \\ &= [1 + 4 \operatorname{Im}(\lambda)^2] \frac{\hbar m \omega}{2}. \end{aligned} \quad (7)$$

From (1.4.51) in Sakurai, $\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2$. Then

$$\langle \lambda | (\Delta X)^2 | \lambda \rangle = \langle \lambda | X^2 | \lambda \rangle - \langle \lambda | X | \lambda \rangle^2 = [1 + 4 \operatorname{Re}(\lambda)^2] \frac{\hbar}{2m\omega} - 4 \operatorname{Re}(\lambda)^2 \frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega},$$

where we have used (4) and (6), and

$$\langle \lambda | (\Delta P)^2 | \lambda \rangle = \langle \lambda | P^2 | \lambda \rangle - \langle \lambda | P | \lambda \rangle^2 = [1 + 4 \operatorname{Im}(\lambda)^2] \frac{\hbar m \omega}{2} - 4 \operatorname{Im}(\lambda)^2 \frac{\hbar m \omega}{2} = \frac{\hbar m \omega}{2},$$

where we have used (5) and (7). Finally,

$$\langle \lambda | (\Delta X)^2 | \lambda \rangle \langle \lambda | (\Delta P)^2 | \lambda \rangle = \frac{\hbar^2}{4},$$

which shows that the coherent state $|\lambda\rangle$ satisfies the minimum uncertainty relation.

1.3 Starting from $|\psi(0)\rangle = |\lambda\rangle$ at $t = 0$, we let $|\psi(t)\rangle$ evolve in time. What is the state $|\psi(t)\rangle$ for $t > 0$?

Solution. The Hamiltonian for the harmonic oscillator,

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}, \quad (8)$$

is time independent, so the time evolution operator $U(t)$ for the coherent state in general is given by

$$U(t) = \exp\left(-\frac{iHt}{\hbar}\right), \quad (9)$$

which is (2.1.28) in Sakurai. Rewriting $|\lambda\rangle$ in the power series representation,

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\lambda^n a^{\dagger n}}{n!} |0\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |n\rangle. \quad (10)$$

The time evolution operator $U(t)$ for an energy eigenket $|n\rangle$ of the harmonic oscillator is given by

$$U(t) |n\rangle = \exp\left(-\frac{iE_n t}{\hbar}\right) |n\rangle = \exp\left[-i\left(n + \frac{1}{2}\right)\omega t\right] |n\rangle = e^{-in\omega t} e^{-i\omega t/2},$$

where E_n are given by (2.3.9) in Sakurai. Then, using (10), we have

$$\begin{aligned} |\psi(t)\rangle &= U(t) |\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-in\omega t} e^{-i\omega t/2} |n\rangle = e^{-i\omega t/2} \exp\left(-\frac{|\lambda|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\lambda e^{-i\omega t})^n}{n!} |n\rangle \\ &= e^{-i\omega t/2} |\lambda e^{-i\omega t}\rangle, \end{aligned} \quad (11)$$

where $\lambda e^{-i\omega t}$ is a complex number (albeit one that is changing in time). Thus, $|\lambda e^{-i\omega t}\rangle$ is another coherent state.

1.4 Compute $\langle \psi(t) | X | \psi(t) \rangle$ and $\langle \psi(t) | P | \psi(t) \rangle$, and their time derivatives $d\langle X \rangle/dt$ and $d\langle P \rangle/dt$.

Solution. Proceeding similarly to 1.2 and using (11), we have

$$\begin{aligned} \langle \psi(t) | X | \psi(t) \rangle &= \langle \lambda e^{-i\omega t} | e^{i\omega t/2} X e^{-i\omega t/2} | \lambda e^{-i\omega t} \rangle = \langle \lambda e^{-i\omega t} | X | \lambda e^{-i\omega t} \rangle \\ &= \frac{1}{|\lambda e^{-i\omega t}|^2} \sqrt{\frac{\hbar}{2m\omega}} \langle \lambda e^{-i\omega t} | (a^\dagger a^2 + a^{\dagger 2} a) | \lambda e^{-i\omega t} \rangle = \frac{|\lambda|^2 (\lambda^* e^{i\omega t} + \lambda e^{-i\omega t} e^{-i\omega t})}{|\lambda|^2} \sqrt{\frac{\hbar}{2m\omega}} \end{aligned}$$

$$\begin{aligned}
 &= \{ [\operatorname{Re}(\lambda) - i \operatorname{Im}(\lambda)] e^{i\omega t} + [\operatorname{Re}(\lambda) + i \operatorname{Im}(\lambda)] e^{-i\omega t} \} \sqrt{\frac{\hbar}{2m\omega}} \\
 &= \left[\operatorname{Re}(\lambda) (e^{i\omega t} + e^{-i\omega t}) - \frac{\operatorname{Im}(\lambda)}{i} (e^{i\omega t} - e^{-i\omega t}) \right] \sqrt{\frac{\hbar}{2m\omega}} \\
 &= 2[\operatorname{Re}(\lambda) \cos(\omega t) + \operatorname{Im}(\lambda) \sin(\omega t)] \sqrt{\frac{\hbar}{2m\omega}}.
 \end{aligned} \tag{12}$$

Likewise,

$$\begin{aligned}
 \langle \psi(t) | P | \psi(t) \rangle &= \langle \lambda e^{-i\omega t} | e^{i\omega t/2} P e^{-i\omega t/2} | \lambda e^{-i\omega t} \rangle = \langle \lambda e^{-i\omega t} | P | \lambda e^{-i\omega t} \rangle \\
 &= \frac{i}{|\lambda e^{-i\omega t}|^2} \sqrt{\frac{\hbar m \omega}{2}} \langle \lambda e^{-i\omega t} | (a^\dagger{}^2 a - a^\dagger a^2) | \lambda e^{-i\omega t} \rangle = i \frac{|\lambda|^2 (\lambda^* e^{i\omega t} - \lambda e^{-i\omega t} e^{-i\omega t})}{|\lambda|^2} \sqrt{\frac{\hbar m \omega}{2}} \\
 &= i \{ [\operatorname{Re}(\lambda) - i \operatorname{Im}(\lambda)] e^{i\omega t} - [\operatorname{Re}(\lambda) + i \operatorname{Im}(\lambda)] e^{-i\omega t} \} \sqrt{\frac{\hbar m \omega}{2}} \\
 &= \left[\operatorname{Im}(\lambda) (e^{i\omega t} + e^{-i\omega t}) - \frac{\operatorname{Re}(\lambda)}{i} (e^{i\omega t} - e^{-i\omega t}) \right] \sqrt{\frac{\hbar m \omega}{2}} \\
 &= 2[\operatorname{Im}(\lambda) \cos(\omega t) - \operatorname{Re}(\lambda) \sin(\omega t)] \sqrt{\frac{\hbar m \omega}{2}}.
 \end{aligned} \tag{13}$$

For the time derivatives, the harmonic oscillator Hamiltonian is given by (8). Note that

$$[X, H] = i\hbar \frac{P}{m}, \quad [X, P] = -i\hbar m \omega^2 X,$$

where we have used the results of problem 2.1 on the previous homework. Then, using the Ehrenfest theorem,

$$\begin{aligned}
 \frac{d\langle X \rangle}{dt} &= -\frac{i}{\hbar} \langle \psi(t) | [X, H] | \psi(t) \rangle = \frac{1}{m} \langle \psi(t) | P | \psi(t) \rangle = \frac{2}{m} [\operatorname{Im}(\lambda) \cos(\omega t) - \operatorname{Re}(\lambda) \sin(\omega t)] \sqrt{\frac{\hbar \omega}{2}}, \\
 &= 2\omega [\operatorname{Im}(\lambda) \cos(\omega t) - \operatorname{Re}(\lambda) \sin(\omega t)] \sqrt{\frac{\hbar}{2m\omega}}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{d\langle P \rangle}{dt} &= -\frac{i}{\hbar} \langle \psi(t) | [P, H] | \psi(t) \rangle = -m\omega^2 \langle \psi(t) | X | \psi(t) \rangle = -2m\omega^2 [\operatorname{Re}(\lambda) \cos(\omega t) + \operatorname{Im}(\lambda) \sin(\omega t)] \sqrt{\frac{\hbar}{2m\omega}} \\
 &= -2\omega [\operatorname{Re}(\lambda) \cos(\omega t) + \operatorname{Im}(\lambda) \sin(\omega t)] \sqrt{\frac{\hbar m \omega}{2}},
 \end{aligned}$$

which are what we would get by differentiating (12) and (13), respectively.

1.5 Compute $\langle \lambda'' | \exp(-iHt/\hbar) | \lambda' \rangle$.

Solution. From (9), we are looking for $\langle \lambda'' | U(t) | \lambda' \rangle$. From (11),

$$U(t) | \lambda \rangle = e^{-i\omega t/2} | \lambda e^{-i\omega t} \rangle \implies U(t) | \lambda' \rangle = e^{-i\omega t/2} | \lambda' e^{-i\omega t} \rangle,$$

so

$$\langle \lambda'' | U(t) | \lambda' \rangle = e^{-i\omega t/2} \langle \lambda'' | \lambda' e^{-i\omega t} \rangle.$$

Using the power series representation as in (10),

$$|\lambda' e^{-i\omega t}\rangle = \exp\left(-\frac{|\lambda'|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\lambda' e^{-i\omega t})^n}{n!} |n\rangle,$$

so

$$\langle \lambda'' | \lambda' e^{-i\omega t} \rangle = \exp\left(-\frac{|\lambda''|^2}{2}\right) \exp\left(-\frac{|\lambda'|^2}{2}\right) \sum_{n=0}^{\infty} \frac{(\lambda''^* \lambda' e^{-i\omega t})^n}{n!} \langle n | n \rangle = \exp\left(-\frac{|\lambda''|^2}{2} + \lambda''^* \lambda' e^{-i\omega t} - \frac{|\lambda'|^2}{2}\right).$$

Finally, we have

$$\langle \lambda'' | \exp\left(-\frac{iHt}{\hbar}\right) | \lambda' \rangle = \exp\left(-\frac{i\omega t}{2} - \frac{|\lambda''|^2}{2} + \lambda''^* \lambda' e^{-i\omega t} - \frac{|\lambda'|^2}{2}\right).$$

2 Problem 2

Consider a quantum system which has coordinate X_1 and momentum P_1 , and another system which has coordinate X_2 and momentum P_2 . (An operator from the first system always commutes with an operator of the second system.) We think of the second system as a “probe” which we can use to detect the properties of the first system. For a short time T , the two systems are coupled by a coupling Hamiltonian H_c , given by

$$H_c = \frac{X_1 P_2}{T}.$$

The coupling between the two systems disturbs the momentum of the first system. The disturbance operator is defined to be

$$D \equiv P_1(T) - P_1(0). \quad (14)$$

The probe introduces measurement error or “noise” into the system. The noise operator is defined by

$$N \equiv X_2(T) - X_1(0).$$

The state of the system at $t = 0$ is $|\Psi(0)\rangle = |\phi_1(0) \phi_2(0)\rangle$, and all expectation values are taken in this state.

2.1 With H_c as the Hamiltonian, find the Heisenberg operators $X_1(t)$, $P_1(t)$, $X_2(t)$, and $P_2(t)$ in terms of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$. Time is restricted to the range $t \in [0, T]$.

Solution. In general, a Heisenberg operator $O(t)$ is defined by

$$O(t) = U^\dagger(t) O(0) U(t),$$

where $U(t)$ is the time evolution operator. For H_c , it is given by

$$U(t) = \exp\left(-\frac{iH_c t}{\hbar}\right) = \exp\left(-\frac{it}{\hbar T} X_1(0) P_2(0)\right).$$

(2.2.23) in Sakurai gives the commutation relations

$$[X_i, F(\mathbf{P})] = i\hbar \frac{\partial F}{\partial P_i} \quad [P_i, G(\mathbf{X})] = -i\hbar \frac{\partial G}{\partial X_i}.$$

Using these, we have

$$\begin{aligned}
 [X_1(0), U(t)] &= 0, \\
 [X_2(0), U(t)] &= i\hbar \left(-\frac{it}{\hbar T} X_1(0) \right) U(t) = \frac{t}{T} X_1(0) U(t) = \frac{t}{T} U(t) X_1(0), \\
 [P_1(0), U(t)] &= -i\hbar \left(-\frac{it}{\hbar T} P_2(0) \right) U(t) = -\frac{t}{T} P_2(0) U(t) = -\frac{t}{T} U(t) P_2(0), \\
 [P_2(0), U(t)] &= 0.
 \end{aligned}$$

Then

$$X_1(t) = U^\dagger(t) X_1(0) U(t) = X_1(0), \quad (15)$$

$$P_1(t) = U^\dagger(t) P_1(0) U(t) = U^\dagger(t) \left(U(t) P_1(0) - \frac{t}{T} U(t) P_2(0) \right) = P_1(0) - \frac{t}{T} P_2(0), \quad (16)$$

$$X_2(t) = U^\dagger(t) X_2(0) U(t) = U^\dagger(t) \left(U(t) X_2(0) + \frac{t}{T} U(t) X_1(0) \right) = X_2(0) + \frac{t}{T} X_1(0), \quad (17)$$

$$P_2(t) = U^\dagger(t) P_2(0) U(t) = P_2(0). \quad (18)$$

2.2 Derive an expression for $\sigma(D)$ which involves only the standard deviations of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$. Here, we denote the standard deviation of an operator O as $\sigma(O) = \sqrt{\langle (O - \langle O \rangle)^2 \rangle}$.

Solution. Substituting (18) into (14),

$$D = P_1(0) - \frac{T}{T} P_2(0) - P_1(0) = -P_2(0).$$

Note that for an operator O ,

$$\sigma(-O) = \sqrt{\langle (-O - \langle -O \rangle)^2 \rangle} = \sqrt{\langle (\langle O \rangle - O)^2 \rangle} = \sigma(O),$$

so

$$\sigma(D) = \sigma(P_2(0)). \quad (19)$$

2.3 Derive an expression for $\sigma(N)$ which involves only the standard deviations of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$.

Solution. Substituting (17) into (14),

$$N = X_2(0) + \frac{T}{T} X_1(0) - X_1(0) = X_2(0).$$

which implies

$$\sigma(N) = \sigma(X_2(0)). \quad (20)$$

2.4 Now consider the product $\sigma(N)\sigma(D)$. Assume

$$\sigma(X_1(0))\sigma(P_1(0)) \geq \frac{\hbar}{2}, \quad \sigma(X_2(0))\sigma(P_2(0)) \geq \frac{\hbar}{2}$$

both hold. Is $\sigma(N)\sigma(D) \geq \hbar/2$ satisfied? What conditions are required for equality?

Solution. From (19) and (20),

$$\sigma(N)\sigma(D) = \sigma(P_2(0))\sigma(X_2(0)) \geq \frac{\hbar}{2},$$

where the final inequality is satisfied by assumption. For equality, we would need

$$\sigma(X_2(0))\sigma(P_2(0)) = \frac{\hbar}{2},$$

which is satisfied if the “probe” system is a Gaussian wave packet.

3 Problem 3

Answer the following questions about the angular momentum operator L_i .

3.1 Calculate $[L_i, \mathbf{r}]$ where $i = x, y, z$.

Solution. Firstly, note that

$$L_x = YP_z - ZP_y, \quad L_y = ZP_x - XP_z, \quad L_z = XP_y - YP_x,$$

where the expression for L_z was given in problem 2 of Homework 1, and L_x and L_y are cyclic permutations. Then

$$\begin{aligned} [L_x, X] &= (YP_z - ZP_y)X - X(YP_z - ZP_y) = 0, \\ [L_x, Y] &= (YP_z - ZP_y)Y - Y(YP_z - ZP_y) = YP_zY - ZP_yY - YYP_z + YZP_y = [Y, P_y]Z = i\hbar Z, \\ [L_x, Z] &= (YP_z - ZP_y)Z - Z(YP_z - ZP_y) = YP_zZ - ZP_yZ - ZYP_z + ZZP_y = -[Z, P_z]Y = -i\hbar Y. \end{aligned}$$

Generalizing these results to L_y and L_z ,

$$[L_x, \mathbf{r}] = i\hbar \begin{bmatrix} 0 \\ Z \\ -Y \end{bmatrix}, \quad [L_y, \mathbf{r}] = i\hbar \begin{bmatrix} -Z \\ 0 \\ X \end{bmatrix}, \quad [L_z, \mathbf{r}] = i\hbar \begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix},$$

where $\mathbf{r} = [X \ Y \ Z]^T$.

3.2 Let us now compare the above results with classical mechanics. Rotations around the x , y , and z axes by an angle θ in three-dimensional Cartesian space are represented by the following matrices:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}, \quad R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate $R_i(\theta)\mathbf{r}$. Then expand $R_i(\theta)\mathbf{r}$ for a small angle θ and consider $\mathbf{r} - R_i(\theta)\mathbf{r}$ to first order in θ ,

$$\mathbf{r} - R_i(\theta)\mathbf{r} = \theta M_i \mathbf{r} + \mathcal{O}(\theta^2).$$

Calculate the matrices M_i .

Solution. For $R_i(\theta) \mathbf{r}$, we have

$$R_x(\theta) \mathbf{r} = \begin{bmatrix} X \\ \cos \theta Y - \sin \theta Z \\ \sin \theta Y + \cos \theta Z \end{bmatrix}, \quad R_y(\theta) \mathbf{r} = \begin{bmatrix} \cos \theta X + \sin \theta Z \\ Y \\ \cos \theta Z - \sin \theta X \end{bmatrix}, \quad R_z(\theta) \mathbf{r} = \begin{bmatrix} \cos \theta X - \sin \theta Y \\ \sin \theta X + \cos \theta Y \\ Z \end{bmatrix},$$

In the small angle approximation, to first order $\cos \theta \approx 1$ and $\sin \theta \approx \theta$. In this approximation,

$$R_x(\theta) \mathbf{r} \approx \begin{bmatrix} X \\ Y - \theta Z \\ \theta Y + Z \end{bmatrix}, \quad R_y(\theta) \mathbf{r} \approx \begin{bmatrix} X + \theta Z \\ Y \\ Z - \theta X \end{bmatrix}, \quad R_z(\theta) \mathbf{r} \approx \begin{bmatrix} X - \theta Y \\ \theta X + Y \\ Z \end{bmatrix},$$

and so

$$\mathbf{r} - R_x(\theta) \mathbf{r} \approx \begin{bmatrix} 0 \\ \theta Z \\ -\theta Y \end{bmatrix}, \quad \mathbf{r} - R_y(\theta) \mathbf{r} \approx \begin{bmatrix} -\theta Z \\ 0 \\ \theta X \end{bmatrix}, \quad \mathbf{r} - R_z(\theta) \mathbf{r} \approx \begin{bmatrix} \theta Y \\ -\theta X \\ 0 \end{bmatrix}.$$

These results suggest the matrices

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad M_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.3 Calculate the matrix elements of the angular momentum operator L_i in the basis ket $|l, m\rangle$ when $l = 1$ and $l = 2$. Here, $|l, m\rangle$ is the simultaneous eigenket of L^2 and L_z with the eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$, respectively.

Solution. The ladder operators are defined by (3.5.5) in Sakurai:

$$J_{\pm} = L_x \pm iL_y.$$

Clearly,

$$L_x = \frac{J_+ + J_-}{2}, \quad L_y = \frac{J_+ - J_-}{2i}$$

From (3.5.39) and (3.5.40),

$$J_+ |l, m\rangle = \sqrt{(l-m)(l+m+1)} \hbar |l, m+1\rangle, \quad J_- |l, m\rangle = \sqrt{(l+m)(l-m+1)} \hbar |l, m-1\rangle.$$

Then the matrix elements of L_x are given by

$$\begin{aligned} \langle 1, m' | L_x | 1, m \rangle &= \langle 1, m' | \frac{J_+ + J_-}{2} | 1, m \rangle = \frac{\hbar}{2} \left(\delta_{m+1, m'} \sqrt{2-m-m^2} + \delta_{m-1, m'} \sqrt{2+m-m^2} \right), \\ \langle 1, m' | L_x | 2, m \rangle &= 0, \\ \langle 2, m' | L_x | 1, m \rangle &= 0, \\ \langle 2, m' | L_x | 2, m \rangle &= \langle 1, m' | \frac{J_+ + J_-}{2} | 1, m \rangle = \frac{\hbar}{2} \left(\delta_{m+1, m'} \sqrt{6-m-m^2} + \delta_{m-1, m'} \sqrt{6+m-m^2} \right), \end{aligned}$$

where the integers $m, m' \in [-l, l]$.

The matrix elements of L_y are given by

$$\begin{aligned}\langle 1, m' | L_y | 1, m \rangle &= \langle 1, m' | \frac{J_+ - J_-}{2i} | 1, m \rangle = -\frac{i\hbar}{2} \left(\delta_{m+1, m'} \sqrt{2 - m - m^2} - \delta_{m-1, m'} \sqrt{2 + m - m^2} \right), \\ \langle 1, m' | L_y | 2, m \rangle &= 0, \\ \langle 2, m' | L_y | 1, m \rangle &= 0, \\ \langle 2, m' | L_y | 2, m \rangle &= \langle 1, m' | \frac{J_+ - J_-}{2i} | 1, m \rangle = -\frac{i\hbar}{2} \left(\delta_{m+1, m'} \sqrt{6 - m - m^2} - \delta_{m-1, m'} \sqrt{6 + m - m^2} \right),\end{aligned}$$

where again $m, m' \in [-l, l]$.

Since $|l, m\rangle$ are eigenkets of L_z , its matrix elements are given by

$$\langle l', m' | L_y | l, m \rangle = \hbar m \delta_{m, m'} \delta_{l, l'},$$

where $l, l' \in \{1, 2\}$ and $m, m' \in [-l, l]$.

Explicitly, let

$$R = \begin{matrix} & \begin{matrix} (1,-1) & (1,0) & (1,1) & (2,-2) & (2,-1) & (2,0) & (2,1) & (2,2) \end{matrix} \\ \begin{matrix} (1,-1) \\ (1,0) \\ (1,1) \\ (2,-2) \\ (2,-1) \\ (2,0) \\ (2,1) \\ (2,2) \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix} \end{matrix},$$

where the row labels represent (l', m') and the column labels represent (l, m) . Then

$$L_x = \frac{\hbar}{\sqrt{2}} R, \quad L_y = -\frac{i\hbar}{\sqrt{2}} R,$$

and

$$L_z = \hbar \begin{matrix} & \begin{matrix} (1,-1) & (1,0) & (1,1) & (2,-2) & (2,-1) & (2,0) & (2,1) & (2,2) \end{matrix} \\ \begin{matrix} (1,-1) \\ (1,0) \\ (1,1) \\ (2,-2) \\ (2,-1) \\ (2,0) \\ (2,1) \\ (2,2) \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \end{matrix}.$$