Problem 3. A transverse sinusoidal wave with wavelength 15 cm and wave speed $20\,\mathrm{m\,s^{-1}}$ is traveling on a 5 m-long string of mass 2 g. The average power of the wave is 35 W. What is the amplitude of the wave? What is the average power if the wave speed is tripled?

Solution. The average power $\langle P\rangle$ of a wave is given by

$$\langle P\rangle = \frac{1}{2}\mu\omega^2 z_0^2 v, \tag{1}$$

where $\mu = m/L$ is the mass density of the string, ω is the wave's angular frequency, z_0 is its amplitude, and v is the wave speed. Solving for the amplitude, we find

$$z_0 = \sqrt{\frac{2\langle P\rangle}{\mu\omega^2 v}}. \tag{2}$$

We need to find ω in terms of given quantities. We know $\omega = kv$ and $k = 2\pi/\lambda$, where k is the wave number and λ the wavelength. Thus,

$$\omega = \frac{2\pi v}{\lambda}.$$

Substituting this and $\mu = m/L$ into Eq. (2) gives us

$$z_0 = \sqrt{\frac{2L\langle P\rangle}{mv}\frac{\lambda^2}{4\pi^2 v^2}} = \frac{1}{\pi}\sqrt{\frac{L\lambda^2\langle P\rangle}{2mv^3}}.$$

Substituting in the given quantities, and recalling that $1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}} = 1\,\mathrm{kg\,m^2\,s^{-3}}$, we have

$$z_0 = \frac{1}{\pi}\sqrt{\frac{(5\,\mathrm{m})(15\times10^{-2}\,\mathrm{m})^2(35\,\mathrm{kg\,m^2\,s^{-3}})}{2(2\times10^{-3}\,\mathrm{kg})(20\,\mathrm{m\,s^{-1}})^3}} = \frac{1}{\pi}\sqrt{\frac{(5)(15)^2(35)\times10^{-4}}{2(2)(20)^3\times10^{-3}}\mathrm{m}^2} = \frac{1}{\pi}\sqrt{\frac{39375}{32000}\times10^{-1}\,\mathrm{m}^2} = \frac{\sqrt{0.123}}{\pi}\mathrm{m}$$
$$= 0.11\,\mathrm{m} = 11\,\mathrm{cm}.$$

When we change the amplitude, we will hold all quantities fixed other than the wave speed. Referring back to Eq. (1), we can write

$$\langle P\rangle \propto v \quad\implies\quad \frac{\langle P\rangle_f}{\langle P\rangle_i} = \frac{v_f}{v_i},$$

where v_f and v_i are the wave speeds before and after tripling, respectively, and $\langle P\rangle_i$ and $\langle P\rangle_f$ are the corresponding average powers. We know $v_f/v_i = 3$. Plugging in the given average power for the original amplitude, we find

$$\langle P\rangle_f = 3\langle P\rangle_i = 3(35\,\mathrm{W}) = 105\,\mathrm{W}.$$

If we instead allow the frequency vary as well, $\omega = kv$ tells us that $\omega_f/\omega_i = 3$ as well. Then we will get

$$\frac{\langle P\rangle_f}{\langle P\rangle_i} = \left(\frac{\omega_f}{\omega_i}\right)^2\frac{v_f}{v_i} = (3^2)(3) = 27,$$

and so

$$\langle P\rangle_f = 27\langle P\rangle_i = 27(35\,\mathrm{W}) = 945\,\mathrm{W}.$$