Problem 1. Consider a particle which, as viewed by an observer in an inertial lab, is in a circular orbit in the (x, y) plane with angular velocity ω and radius r. Suppose that this particle carries a spin angular momentum $\vec{S}$ (treated classically in this problem) which is Fermi-Walker transported. Compute the time dependence of this angular momentum $\vec{S}(t)$ where t is the inertial time in the laboratory frame. Show that in the non-relativistic limit, the complex vector $S_x + iS_y$ precesses about the z axis with frequency $\omega_T = r^2\omega^3/2$.

Solution. Since the particle is in a circular orbit in the xy plane, we can write its position in the lab frame as

$$\vec{x} = (t, r\cos(\omega t), r\sin(\omega t), 0).$$

Then from MCP (2.7), its velocity is

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \gamma(1, -r\omega\sin(\omega t), r\omega\cos(\omega t), 0), \tag{1}$$

since $d\tau = dt\,/\gamma$ [1, p. 201]. The particle's acceleration is

$$\vec{a} = \frac{d\vec{U}}{d\tau} = -r\omega^2\gamma^2(0, \cos(\omega t), \sin(\omega t), 0).$$

The Fermi-Walker transport is, as given by MCP (24.62),

$$\nabla_{\vec{U}}\vec{S} = \vec{U}(\vec{a}\cdot\vec{S}), \tag{2}$$

and we know that the spin vector is always orthogonal to the particle's 4-velocity [2, p. 1184]. This means that the spin vector can be written

$$\vec{S} = (0, \mathbf{S})$$

since $\vec{U}\cdot\vec{S} = 0$ is Lorentz invariant, and the spatial components of the particle's velocity are zero in its rest frame. Moreover, this means

$$\nabla_{\vec{U}}\vec{S} = \frac{d\vec{S}}{d\tau} = \gamma\frac{d\vec{S}}{dt},$$

so Eq. (2) can be written

$$\frac{d\vec{S}}{dt} = \frac{1}{\gamma}\vec{U}(\vec{a}\cdot\vec{S}) = -\gamma r\omega^2[S_x\cos(\omega t) + S_y\sin(\omega t)]\vec{U}.$$

Feeding in the relevant components of Eq. (1), we have the system of coupled differential equations

$$\begin{aligned}\frac{dS_x}{dt} &= \gamma^2 r^2\omega^3\sin(\omega t)[S_x\cos(\omega t) + S_y\sin(\omega t)],\\ \frac{dS_y}{dt} &= -\gamma^2 r^2\omega^3\cos(\omega t)[S_x\cos(\omega t) + S_y\sin(\omega t)],\end{aligned}$$

or, in matrix form,

$$\frac{d}{dt}\begin{bmatrix}S_x\\ S_y\end{bmatrix} = \gamma^2 r^2\omega^3\begin{bmatrix}\cos(\omega t)\sin(\omega t) & \sin^2(\omega t)\\ -\cos^2(\omega t) & -\cos(\omega t)\sin(\omega t)\end{bmatrix}\begin{bmatrix}S_x\\ S_y\end{bmatrix}. \tag{3}$$

To solve the system, we define the polar components of $\vec{S}$ by [3, inside cover]

$$S_r = S_x\cos(\omega t) + S_y\sin(\omega t), \qquad S_\theta = -S_x\sin(\omega t) + S_y\cos(\omega t).$$

Then we can transform into these polar coordinates using the matrix [3, inside cover]

$$\begin{bmatrix}S_x\\ S_y\end{bmatrix} = \begin{bmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{bmatrix}\begin{bmatrix}S_r\\ S_\theta\end{bmatrix}, \tag{4}$$

which implies

$$\frac{d}{dt}\begin{bmatrix} S_x \\ S_y \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix} + \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}$$
$$= -\omega\begin{bmatrix} \sin(\omega t) & \cos(\omega t) \\ -\cos(\omega t) & \sin(\omega t) \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix} + \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}. \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (3) yields

$$-\omega\begin{bmatrix} \sin(\omega t) & \cos(\omega t) \\ -\cos(\omega t) & \sin(\omega t) \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix} + \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}$$
$$= \gamma^2 r^2 \omega^3\begin{bmatrix} \cos(\omega t)\sin(\omega t) & \sin^2(\omega t) \\ -\cos^2(\omega t) & -\cos(\omega t)\sin(\omega t) \end{bmatrix}\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}$$
$$= \gamma^2 r^2 \omega^3\begin{bmatrix} \sin(\omega t) & 0 \\ -\cos(\omega t) & 0 \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix},$$

where the matrix multiplication has been carried out with Mathematica. This implies

$$\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\frac{d}{dt}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix} = \left(\omega\begin{bmatrix} \sin(\omega t) & \cos(\omega t) \\ -\cos(\omega t) & \sin(\omega t) \end{bmatrix} + \gamma^2 r^2 \omega^3\begin{bmatrix} \sin(\omega t) & 0 \\ -\cos(\omega t) & 0 \end{bmatrix}\right)\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}$$
$$= \omega\begin{bmatrix} (1+\gamma^2 r^2\omega^2)\sin(\omega t) & \cos(\omega t) \\ -(1+\gamma^2 r^2\omega^2)\cos(\omega t) & \sin(\omega t) \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}.$$

Note that

$$\gamma^2 = \frac{1}{1 - r^2\omega^2} \qquad \Longrightarrow \qquad \gamma^2 = \gamma^2 r^2 \omega^2 + 1.$$

Making this substiution and multiplying both sides by the inverse of the first matrix, we find

$$\frac{d}{dt}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix} = \omega\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}\begin{bmatrix} \gamma^2\sin(\omega t) & \cos(\omega t) \\ -\gamma^2\cos(\omega t) & \sin(\omega t) \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix} = \omega\begin{bmatrix} 0 & 1 \\ -\gamma^2 & 0 \end{bmatrix}\begin{bmatrix} S_r \\ S_\theta \end{bmatrix}.$$

In other words, we have a system of coupled first-order ODEs:

$$\frac{dS_r}{dt} = \omega S_\theta, \qquad\qquad \frac{dS_\theta}{dt} = -\omega\gamma^2 S_r. \tag{6}$$

Differentiating each equation by t once more and substituting, we get a system of uncoupled second-order ODEs with well-known solutions:

$$\frac{d^2 S_r}{dt^2} = \omega\frac{dS_\theta}{dt} = -\omega^2\gamma^2 S_r, \qquad\qquad \frac{d^2 S_\theta}{dt^2} = -\omega\gamma^2\frac{dS_r}{dt} = -\omega^2\gamma^2 S_\theta.$$

The solutions are [4, p. 207]

$$S_r(t) = C_1\cos(\omega\gamma t) + C_2\sin(\omega\gamma t), \qquad\qquad S_\theta(t) = D_1\cos(\omega\gamma t) + D_2\sin(\omega\gamma t).$$

We choose the initial conditions $S_r(0) = S$ and $S_\theta(0) = 0$. This gives us

$$S = S_r(0) = C_1, \qquad\qquad 0 = S_\theta(0) = D_1.$$

Then by Eq. (6),

$$\omega D_2\sin(\omega\gamma t) = \frac{dS_r}{dt} = \omega\gamma[-S\sin(\omega\gamma t) + C_2\cos(\omega\gamma t)],$$
$$-\omega\gamma^2[S\cos(\omega\gamma t) + C_2\sin(\omega\gamma t)] = \frac{dS_\theta}{dt} = \omega\gamma D_2\cos(\omega\gamma t).$$

Solving this system of equations with Mathematica yields $C_2 = 0$ and $D_2 = -\gamma S$. So our equations are

$$S_r(t) = S\cos(\omega\gamma t), \qquad S_\theta(t) = -\gamma S\sin(\omega\gamma t).$$

Transforming back into Cartesian components using Eq. (4), we have

$$\begin{bmatrix} S_x \\ S_y \end{bmatrix} = S\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}\begin{bmatrix} \cos(\omega\gamma t) \\ -\gamma\sin(\omega\gamma t) \end{bmatrix} = S\begin{bmatrix} \cos(\omega t)\cos(\omega\gamma t) + \gamma\sin(\omega t)\sin(\omega\gamma t) \\ \sin(\omega t)\cos(\omega\gamma t) - \gamma\cos(\omega t)\sin(\omega\gamma t) \end{bmatrix}.$$

We can make these expressions look a little nicer by using the following product-to-sum identities [5]:

$$\begin{aligned} 2\cos\theta\cos\phi &= \cos(\theta-\phi) + \cos(\theta+\phi), & 2\sin\theta\sin\phi &= \cos(\theta-\phi) - \cos(\theta+\phi), \\ 2\sin\theta\cos\phi &= \sin(\theta+\phi) + \sin(\theta-\phi), & 2\cos\theta\sin\phi &= \sin(\theta+\phi) - \sin(\theta-\phi). \end{aligned}$$

Then we have

$$\begin{aligned} S_x(t) &\propto \frac{\cos(\omega t - \omega\gamma t) + \cos(\omega t + \omega\gamma t)}{2} + \gamma\frac{\cos(\omega t - \omega\gamma t) - \cos(\omega t + \omega\gamma t)}{2}, \\ S_y(t) &\propto \frac{\sin(\omega t + \omega\gamma t) + \sin(\omega t - \omega\gamma t)}{2} - \gamma\frac{\sin(\omega t + \omega\gamma t) - \sin(\omega t - \omega\gamma t)}{2}, \end{aligned}$$

and finally

$$\begin{aligned} S_x(t) &= S\frac{(1+\gamma)\cos[(1-\gamma)\omega t] + (1-\gamma)\cos[(1+\gamma)\omega t]}{2}, \\ S_y(t) &= S\frac{(1+\gamma)\sin[(1-\gamma)\omega t] + (1-\gamma)\sin[(1+\gamma)\omega t]}{2}. \end{aligned}$$

is the time dependence of $\vec{S}$.

Then

$$\begin{aligned} S_x + iS_y &= S\left[(1+\gamma)\frac{\cos[(1-\gamma)\omega t] + i\sin[(1-\gamma)\omega t]}{2} + (1-\gamma)\frac{\cos[(1+\gamma)\omega t] + i\sin[(1+\gamma)\omega t]}{2}\right] \\ &= \frac{S}{2}\left[(1+\gamma)e^{i(1-\gamma)\omega t} + (1-\gamma)e^{i(1-\gamma)\omega t}\right]. \end{aligned} \tag{7}$$

In the non-relativistic limit, $r\omega \ll 1$ so

$$\gamma = \frac{1}{\sqrt{1 - r^2\omega^2}} \approx 1 + \frac{r^2\omega^2}{2},$$

where we have evaluated the series expansion with Mathematica. Using also $\gamma \to 1$ in this limit, Eq. (7) becomes

$$S_x + iS_y \to Se^{i(1-\gamma)\omega t} \approx S\exp\left[i\left(1 - 1 - \frac{r^2\omega^2}{2}\right)\omega t\right] = Se^{-ir^2\omega^3 t/2}.$$

So we have shown that $S_x + iS_y$ precesses about the z axis with frequency $\omega_T = r^2\omega^3/2$. □

Problem 2. Gravitational redshift (MCP 24.16) Inside a laboratory on Earth's surface the effects of spacetime curvature are so small that current technology cannot measure them. Therefore, experiments performed in the laboratory can be analyzed using special relativity.

2(a) Explain why the spacetime metric in the proper reference frame of the laboratory's floor has the form

$$ds^2 = (1 + 2gz)(dx^{\hat{0}})^2 + dx^2 + dy^2 + dz^2 \,, \tag{8}$$

plus terms due to the slow rotation of the laboratory walls, which we neglect in this exercise. Here g is the acceleration of gravity measured on the floor.

Solution. We can transform coordinates from the proper reference frame of the laboratory floor to another inertial frame. We choose this other frame such that it is only a very small "distance" away at $\vec{x} = 0$ in the proper frame. That is, the frames are identical in the immediate vicinity of event (small $\vec{x}$). Then the coordinate transformation from the proper reference frame to the other inertial frame is given by MCP (24.60a),

$$x^i = x^{\hat{i}} + \frac{1}{2} a^{\hat{i}} (x^{\hat{0}})^2 + \epsilon^{\hat{i}}_{\hat{j}\hat{k}} \Omega^{\hat{j}} x^{\hat{k}} x^{\hat{0}}, \qquad\qquad x^0 = x^{\hat{0}}(1 + a_{\hat{j}} x^{\hat{j}}),$$

where terms to quadratic order in $x^{\hat{\alpha}}$ are included, and $\Omega^{\hat{j}}$ is the rotational angular velocity of the laboratory. Since the metric in the inertial frame is $ds^2 = -(dx^0)^2 + \delta_{ij}\, dx^i\, dx^j$ [2, p. 1183], the metric in the proper reference frame is given by MCP (24.60b),

$$ds^2 = -(1 + 2\mathbf{a} \cdot \mathbf{x})(dx^{\hat{0}})^2 + 2(\boldsymbol{\Omega} \times \mathbf{x}) \cdot d\mathbf{x}\, dx^{\hat{0}} + \delta_{jk}\, dx^{\hat{j}}\, dx^{\hat{k}} \,,$$

which is accurate to linear order in $x^{\hat{\alpha}}$. For this problem, we ignore rotations so $\boldsymbol{\Omega} \to \mathbf{0}$, eliminating the second term. Also, $\mathbf{a} = g\hat{\mathbf{z}}$ so $\mathbf{a} \cdot \mathbf{x} = -gz$. Finally, we note that $x^{\hat{i}} \in \{x, y, z\}$ since these coordinates coincide with the inertial frame [2, p. 1186]. Then the spacetime metric is

$$ds^2 = -(1 + 2gz)(dx^{\hat{0}})^2 + dx^2 + dy^2 + dz^2 \,,$$

which is what we want. Evidently, there is a typo (missing minus sign) in the problem statement. □

2(b) An electromagnetic wave is emitted from the floor, where it is measured to have wavelength λ_o, and is received at the ceiling. Using the metric of Eq. (8), show that, as measured in the proper reference frame of an observer on the ceiling, the received wave has wavelength $\lambda_r = \lambda_o(1 + gh)$, where h is the height of the ceiling above the floor (i.e., the light is *gravitationally redshifted* by $\Delta\lambda/\lambda_o = gh$).

Solution. As measured by an observer on the floor, say that one crest of the wave is emitted at some time $x^{\hat{0}} = t$, and the next crest is emitted at $x^{\hat{0}} = t_2$. Call these events $\mathcal{P}_1$ and $\mathcal{P}_2$. Both events occur on the floor at $z = 0$. The interval between the two events is

$$(ds_{12})^2 = -(t_2 - t_1)^2 \equiv T_o^2,$$

where T_o is the period of the wave as measured at the floor. We know that this interval represents the period because it is a timelike separation [2, p. 45]. At some time later t_3, the first crest hits the ceiling; at t_4, the second crest hits the ceiling. The interval between these two events $\mathcal{P}_3$ and $\mathcal{P}_4$, which both occur at $z = R + h$, is

$$(ds_{34})^2 = -(1 + 2gh)(t_4 - t_3)^2 \equiv T_r^2,$$

where T_r is the period of the wave as measured at the ceiling.

We are concerned with the propagation of an electromagnetic wave, which means each crest travels at the speed of light. Thus, it must be true that $t_2 - t_1 = t_4 - t_3$. Noting that $T_r/T_o = \lambda_r/\lambda_o$, it follows that

$$\frac{\lambda_r^2}{\lambda_o^2} = \frac{(1 + 2gh)(t_4 - t_3)^2}{(t_2 - t_1)^2} = 1 + 2gh \qquad \Longrightarrow \qquad \frac{\lambda_r}{\lambda_o} = \sqrt{1 + 2gh} \approx 1 + gh,$$

in the limit that $gh \ll 1$ (where $c = 1$). Thus we have shown that $\lambda_r = \lambda_o(1 + gh)$. □

Problem 3. Rigidly rotating disk (MCP 24.17) Consider a thin disk with radius R at $z = 0$ in a Lorentz reference frame. The disk rotates rigidly with angular velocity Ω. In the early years of special relativity there was much confusion over the geometry of the disk: In the inertial frame it has physical radius (proper distance from center to edge) R and physical circumference $\mathcal{C} = 2\pi R$. But Lorentz contraction dictates that, as measured on the disk, the circumference should be $\sqrt{1-v^2}\mathcal{C}$ (with $v = \Omega R$), and the physical radius, R, should be unchanged. This seemed weird. How could an obviously flat disk in spacetime have a curved, non-Euclidean geometry, with physical circumference divided by physical radius smaller than 2π? In this exercise you will explore this issue.

3(a) Consider a family of observers who ride on the edge of the disk. Construct a circular curve, orthogonal to their world lines, that travels around the disk (at $\sqrt{x^2+y^2} = R$). This curve can be thought of as lying in a 3-surface of constant time $x^{\hat{0}}$ of the observers' proper reference frames. Show that it spirals upward in a Lorentz-frame spacetime diagram, so it cannot close on itself after traveling around the disk. Thus the 3-planes, orthogonal to the observers' world lines at the edge of the disk, cannot mesh globally to form global 3-planes.

Solution. We can write the position of an observer in an inertial reference frame as

$$\vec{x} = (t, R\cos(\Omega t), R\sin(\Omega t), 0).$$

Then the velocity of the observer, which is tangent to his/her worldline, is

$$\vec{u} = \frac{d\vec{x}}{d\tau} = \gamma(1, -R\Omega\sin(\Omega t), R\Omega\cos(\Omega t), 0).$$

By inspection, a vector orthogonal to this velocity is

$$\vec{v} = \gamma(R^2\Omega^2, -R\Omega\sin(\Omega t), R\Omega\cos(\Omega t), 0) = \frac{d\vec{y}}{d\tau},$$

where $\vec{y}$ traces out the curve orthogonal to the world line. It is given by

$$\vec{y} = (R^2\Omega^2 t, R\cos(\Omega t), R\sin(\Omega t), 0).$$

We note that $\vec{y}$ traces out a helix in a Lorentz-frame spacetime diagram. Thus, it cannot close on itself after traveling around the disk. □

3(b) Next, consider a 2-dimensional family of observers who ride on the surface of the rotating disk. Show that at each radius $\sqrt{x^2+y^2} = \text{const}$, the constant-radius curve that is orthogonal to their world lines spirals upward in spacetime with a different slope. Show that this means that even locally, the 3-planes orthogonal to each of their world lines cannot mesh to form larger 3-planes—thus there does not reside in spacetime any 3-surface orthogonal to these observers' world lines. There is no 3-surface that has the claimed non-Euclidean geometry.

Solution. For a given radius $r = \sqrt{x^2+y^2}$, the constant-radius curve that is orthogonal to the worldline is given by

$$\vec{y} = (r^2\Omega^2 t, -r\Omega\sin(\Omega t), r\Omega\cos(\Omega t), 0).$$

The slope at which this curve spirals upward in the spacetime diagram is $r^2\Omega^2/r = r\Omega^2$ [6], which has a linear dependence on the radius. Thus, the slope is different for different radii. □

Consider the worldline of one observer who is riding at some radius r. A nearby observer at radius $r + dr$ has a slightly different worldline, as we just showed. The slope of her worldline is different by about $r\Omega^2\,dr$. Since these worldlines have slightly different slopes, the 3-planes orthogonal to them are not parallel to each other; they are separated by a small angle. Thus, the planes cannot mesh to form a larger plane. □

Problem 4. Constant of geodesic motion in a spacetime with symmetry (MCP 25.4)

4(a) Suppose that in some coordinate system the metric coefficients are independent of some specific coordinate x^A: $g_{\alpha\beta,A} = 0$ (e.g., in spherical polar coordinates $\{t, r, \theta, \phi\}$ in flat spacetime $g_{\alpha\beta,\phi} = 0$, so we could set $x^A = \phi$). Show that

$$p_A \equiv \vec{p} \cdot \frac{\partial}{\partial x^A}$$

is a constant of the motion for a freely moving particle [p_ϕ = (conserved z component of angular momentum)] in the above, spherically symmetric example.]

Hint: Show that the geodesic equation can be written in the form

$$\frac{dp_\alpha}{d\zeta} - \Gamma_{\mu\alpha\nu} p^\mu p^\nu = 0,$$

where $\Gamma_{\mu\alpha\nu}$ is the covariant connection of Eqs. (24.38c), (24.38d) with $c_{\alpha\beta\gamma} = 0$, because we are using a coordinate basis.

Solution. The general form of the geodesic equation is given by MCP (25.11c),

$$\nabla_{\vec{p}} \vec{p} = 0. \tag{9}$$

Dotting both sides by $\vec{p}$ yields

$$0 = \vec{p} \cdot \nabla_{\vec{p}} \vec{p} \quad \Longrightarrow \quad 0 = p^\mu \nabla_\mu p_\alpha.$$

The covariant components of the gradient are given by (24.36), $A_{\alpha;\beta} = A_{\alpha,\beta} - \Gamma^\mu{}_{\alpha\beta} A_\mu$, where $A_{\alpha,\beta} = \partial_\beta A_\alpha$ and $A_{\alpha;\beta} = \nabla_\beta A_\alpha$. Applying this, we have

$$0 = p^\mu (p_{\alpha,\mu} - \Gamma^\gamma{}_{\alpha\mu} p_\gamma) = p^\mu \partial_\mu p_\alpha - \Gamma^\gamma{}_{\alpha\mu} p^\mu p_\gamma. \tag{10}$$

Using $\vec{p}/m = d\vec{x}/d\tau$ and $\tau = m\zeta$ [2, p. 1202], we can rewrite the first term of Eq. (10):

$$p^\mu \partial_\mu p_\alpha = m \frac{dx^\mu}{d\tau} \partial_\mu p_\alpha = m \frac{dp_\alpha}{d\tau} = \frac{dp_\alpha}{d\zeta}.$$

For the second term of Eq. (10), we multiply by the metric $g_{\beta\alpha}$ as in (24.38d), $\Gamma^\mu{}_{\beta\gamma} = g^{\mu\alpha} \Gamma_{\alpha\beta\gamma}$. Then we have

$$\Gamma^\gamma{}_{\alpha\mu} p^\mu p_\gamma = g^{\gamma\nu} \Gamma_{\nu\alpha\mu} p^\mu p_\gamma = \Gamma_{\nu\alpha\mu} p^\mu p^\nu = \Gamma_{\mu\alpha\nu} p^\mu p^\nu,$$

where in the final step we have relabeled indices. Thus we can write Eq. (10) as

$$\frac{dp_\alpha}{d\zeta} - \Gamma_{\mu\alpha\nu} p^\mu p^\nu = 0, \tag{11}$$

as recommended.

Now we apply (24.38c); in a coordinate basis, it reduces to

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}). \tag{12}$$

Then for the second term of Eq. (11), we can write

$$\begin{aligned}
\Gamma_{\mu\alpha\nu} p^\mu p^\nu &= \frac{1}{2}(g_{\mu\alpha,\nu} + g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu}) p^\mu p^\nu \\
&= \frac{1}{2}(g_{\nu\alpha,\mu} + g_{\nu\mu,\alpha} - g_{\alpha\mu,\nu}) p^\mu p^\nu \\
&= \frac{1}{2}(g_{\alpha\nu,\mu} + g_{\mu\nu,\alpha} - g_{\mu\alpha,\nu}) p^\mu p^\nu \\
&= \frac{1}{2} g_{\mu\nu,\alpha} p^\mu p^\nu,
\end{aligned}$$

where we have again relabeled indices, and also used the symmetry of the metric. Then for Eq. (11), we have

$$\frac{dp_\alpha}{d\zeta} = \frac{1}{2} g_{\mu\nu,\alpha} p^\mu p^\nu .$$

Therefore when $\alpha = A$,

$$\frac{dp_A}{d\zeta} = \frac{1}{2} g_{\mu\nu,A} p^\mu p^\nu = 0 \quad \Longrightarrow \quad \frac{dp_A}{d\tau} = 0 \quad \Longrightarrow \quad p_A = \text{const},$$

as we wanted to show [7, pp. 134–135]. □

4(b) As an example, consider a particle moving freely through a time-independent, Newtonian gravitational field. In Ex. 25.18, we learn that such a gravitational field can be described in the language of general relativity by the spacetime metric

$$ds^2 = -(1+2\Phi)\, dt^2 + (\delta_{jk} + h_{jk})\, dx^j\, dx^k \,, \tag{13}$$

where $\Phi(x, y, z)$ is the time-independent Newtonian potential, and h_{jk} are contributions to the metric that are independent of the time coordinate t and have magnitude of order $|\Phi|$. That the gravitational field is weak means $|\Phi| \ll 1$. The coordinates being used are Lorentz, aside from tiny corrections of order $|\Phi|$, and as this exercise and Ex. 25.18 show, they coincide with the coordinates of the Newtonian theory of gravity. Suppose that the particle has velocity $v^j \equiv dx^j/dt$ through this coordinate system that is $\lesssim |\Phi|^{1/2}$ and thus is small compared to the speed of light. Because the metric is independent of the time coordinate t, the component p_t of the particle's 4-momentum must be conserved along its world line. Since throughout physics, the conserved quantity associated with time-translation invariance is always the energy, we expect that p_t, when evaluated accurate to first order in $|\Phi|$, must be equal to the particle's conserved Newtonian energy, $E = m\Phi + mv^j v^k \delta_{jk}/2$, aside from some multiplicative and additive constants. Show that this, indeed, is true, and evaluate the constants.

Solution. For a timelike interval, $ds^2 = -(d\tau)^2$ [2, p. 45]. Then we can write Eq. (13) as

$$\begin{aligned} -\left(\frac{d\tau}{dt}\right)^2 &= -(1+2\Phi)\left(\frac{dt}{dt}\right)^2 + (\delta_{jk} + h_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &= -(1+2\Phi) + (\delta_{jk} + h_{jk}) v^j v^k \\ &= -1 - 2\Phi + \delta_{jk} v^j v^k, \end{aligned}$$

which means

$$\frac{d\tau}{dt} = \sqrt{1 + 2\Phi - \delta_{jk} v^j v^k} \approx 1 + \Phi - \frac{1}{2} v^j v^k \delta_{jk}.$$

Since $d\tau = dt/\gamma$ [1, p. 201] and $\gamma = u^0 = p^0/m$, $p^t = m\, dt/d\tau$. So [**?**]

$$p^t = m \frac{1}{1 + \Phi - v^j v^k \delta_{jk}/2},$$

and we lower the index by multiplying by the only nonzero metric component. This is g_{tt}, which we read off of Eq. (13). Then

$$p_t = g_{tt} p^t = m \frac{1 + 2\Phi}{1 + \Phi - v^j v^k \delta_{jk}/2} \approx m \left(1 + \Phi + \frac{1}{2} v^j v^k \delta_{jk}\right),$$

where we have performed the Taylor series expansion using Mathematica. Thus we have

$$E = m + m\Phi + \frac{m}{2} v^j v^k \delta_{jk},$$

which is what we expected up to the additive constant m. □

Problem 5. Killing vector field (MCP 25.5) A *Killing vector field* is a coordinate-independent tool for exhibiting symmetries of the metric. It is any vector field $\vec{\xi}$ that satisfies

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{14}$$

(i.e., any vector field whose symmetrized gradient vanishes).

5(a) Let $\vec{\xi}$ be a vector field that might or might not be Killing. Show, by construction, that it is possible to introduce a coordinate system in which $\vec{\xi} = \partial/\partial x^A$ for some coordinate x^A.

Solution. We know that a single vector can be defined as an arrow residing in the tangent space to some curved manifold at a point we call $\mathcal{P}$. We can imagine a projection of $\vec{\xi}$ onto the manifold; such a projection is a curve on the manifold. We may choose the direction of this curve as a coordinate of the manifold and call it x^A. The curve itself we call $\mathcal{P}(x^A)$. Then, by construction, $\vec{\xi}$ is tangent to $\mathcal{P}(x^A)$. This means that $\vec{\xi}$ must be the directional derivative along $\mathcal{P}(x^A)$, which is $\partial/\partial x^A$ [2, pp. 1166–1167].

For a vector field, the situation is slightly more complex. Any given manifold can be described by a set of curves that are nonintersecting and which fill the manifold completely. (Imagine a contour plot of some surface with an infinite number of contours; the contours are the curves in question.) These are called "integral curves" [7, p. 430]. Any two integral curves that are arbitrarily close to one another run in the same direction. So we can use this direction to define a coordinate system with the coordinate x^A along the direction of the integral curves. Any point $\mathcal{P}$ on the manifold has an integral curve running through it, and we call this curve $\mathcal{P}(x^A)$. The vector tangent to the manifold at $\mathcal{P}$ is $\partial/\partial x^A$ [2, pp. 1166–1167]. Since this is true at any point $\mathcal{P}$ on the manifold, the set of vectors tangent to all such integral curves make up a vector field. Running this logic in reverse, then, it is possible to start with an arbitrary vector field $\vec{\xi}$ and construct a coordinate system in which $\vec{\xi} = \partial/\partial x^A$ [7, p. 430]. □

5(b) Show that in the coordinate system of 5(a) the symmetrized gradient of $\vec{\xi}$ is $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = \partial g_{\alpha\beta}/\partial x^A$. From this infer that a vector field $\vec{\xi}$ is Killing if and only if there exists a coordinate system in which (i) $\vec{\xi} = \partial/\partial x^A$ and (ii) the metric is independent of x^A.

Solution. According to MCP (24.35),

$$A^\mu{}_{;\beta} = A^\mu{}_{,\beta} + A^\alpha \Gamma^\mu{}_{\alpha\beta}.$$

Then, multiplying by the metric as in (24.38d),

$$\begin{aligned} \xi_{\alpha;\beta} &= g_{\alpha\mu}\xi^\mu{}_{;\beta} = g_{\alpha\mu}(\xi^\mu{}_{,\beta} + \xi^\nu \Gamma^\mu{}_{\nu\beta}) = \xi_{\alpha,\beta} + \Gamma_{\alpha\nu\beta}\xi^\nu, \\ \xi_{\beta;\alpha} &= g_{\beta\mu}\xi^\mu{}_{;\alpha} = g_{\beta\mu}(\xi^\mu{}_{,\alpha} + \xi^\nu \Gamma^\mu{}_{\nu\alpha}) = \xi_{\beta,\alpha} + \Gamma_{\beta\nu\alpha}\xi^\nu \end{aligned}$$

Applying Eq. (12) to the connection coefficients,

$$\Gamma_{\alpha\nu\beta} = \frac{1}{2}(g_{\alpha\nu,\beta} + g_{\alpha\beta,\nu} - g_{\nu\beta,\alpha}), \qquad \Gamma_{\beta\nu\alpha} = \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\alpha\beta,\nu} - g_{\alpha\nu,\beta}) = \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\alpha\beta,\nu} - g_{\alpha\nu,\beta}),$$

where we have used the symmetry of the metric. Then

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = \xi_{\alpha,\beta} + \frac{1}{2}(g_{\alpha\nu,\beta} + g_{\alpha\beta,\nu} - g_{\nu\beta,\alpha})\xi^\nu + \xi_{\beta,\alpha} + \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\alpha\beta,\nu} - g_{\alpha\nu,\beta})\xi^\nu = \xi_{\alpha,\beta} + \xi_{\beta,\alpha} + g_{\alpha\beta,\nu}\xi^\nu.$$

In the coordinate system of 5(a), $\vec{\xi} = \partial/\partial x^A$ so $\xi^A = 1$ and all other $\xi^\gamma = 0$. Thus $\xi_{\alpha,\beta} = 0$ for all α, β and

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = g_{\alpha\beta,A}\xi^A = \frac{\partial g_{\alpha\beta}}{\partial x^A} \tag{15}$$

as we wanted to show. □

it is only possible to reach Eq. (15) if we have chosen our coordinate system such that $\vec{\xi} = \partial/\partial x^A$. Further, it is only possible for Eq. (14) to hold if $\partial g_{\alpha\beta}/\partial x^A = 0$; that is, if the metric is independent of x^A. Thus both conditions are required for $\vec{\xi}$ to be a Killing field.

5(c) Use Killing's equation (14) to show, without introducing a coordinate system, that, if $\vec{\xi}$ is a Killing vector field and $\vec{p}$ is the 4-momentum of a freely-falling particle, then $\vec{\xi}\cdot\vec{p}$ is conserved along the particle's geodesic world line. This is the same conservation law as we proved in 4(a) using a coordinate-dependent calculation.

Solution. For $\vec{\xi}\cdot\vec{p}$ to be conserved along the particle's geodesic world line, we require that $\nabla_{\vec{p}}(\xi_\nu p^\nu) = 0$. Dotting both sides by $\vec{p}$, this becomes $p^\mu \nabla_\mu(\xi_\nu p^\nu) = 0$. Note that

$$p^\mu \nabla_\mu(\xi_\nu p^\nu) = p^\mu p^\nu \nabla_\mu \xi_\nu + p^\mu \xi_\nu \nabla_\mu(p^\nu) = p^\mu p^\nu \xi_{\nu,\mu}$$

since the geodesic equation (9) implies $p^\mu \nabla_\mu p^\nu = 0$ as in 4(a). We can relabel indices to write

$$p^\mu \nabla_\mu(\xi_\nu p^\nu) = p^\mu p^\nu \xi_{\nu,\mu} = p^\mu p^\nu \xi_{\mu,\nu},$$

but from Eq. (14), $\xi_{\nu,\mu} = -\xi_{\mu,\nu}$. So

$$p^\mu \nabla_\mu(\xi_\nu p^\nu) = -p^\mu \nabla_\mu(\xi_\nu p^\nu) \quad\Longrightarrow\quad p^\mu \nabla_\mu(\xi_\nu p^\nu) = 0 \quad\Longrightarrow\quad \nabla_{\vec{p}}(\xi_\nu p^\nu) = 0$$

as we wanted to show [7, pp. 135–136]. Thus, $\vec{\xi}\cdot\vec{p}$ is conserved along the particle's geodesic world line. □

References

[1] R. Resnick, "Introduction to Special Relativity". John Wiley & Sons, Inc., 1968.

[2] K. S. Thorne and R. D. Blandford, "Modern Classical Physics". Princeton University Press, 2017.

[3] D. J. Griffiths, "Introduction to Electrodynamics". Prentice Hall, 3rd edition, 1999.

[4] C. E. Swartz, "Used Math for the First Two Years of College Science". Prentice-Hall, 1973.

[5] Wikipedia contributors, "List of trigonometric identities." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/List_of_trigonometric_identities`.

[6] Wikipedia contributors, "Helix." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Helix`.

[7] S. M. Carroll, "Spacetime and Geometry: An Introduction to General Relativity". Cambridge University Press, 2019.

[8] Wikipedia contributors, "Four-acceleration." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Four-acceleration`.