1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1}$$

1.1 Verify the following:

a. $i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle$,

b. $i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle$,

c. $i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = -\dfrac{\hbar^2}{2m}\left[\langle\Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right]$,

d. $\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle +\ \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle = \dfrac{\hbar}{i}\left[\langle\Phi(t)|x\rangle \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right]$

e. $\dfrac{\hbar}{i}\partial_x\left[\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle +\ \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - mel\Phi(t)p^2x \langle x|\Psi(t)\rangle$

Solution.

a. Beginning with Schrödinger's equation, note that

$$i\hbar\partial_t |\Psi(t)\rangle = H|\Psi(t)\rangle \tag{2}$$
$$i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle x|H|\Psi(t)\rangle \tag{3}$$
$$(i\hbar\partial_t \langle x|\Psi(t)\rangle)^* = (\langle x|H|\Psi(t)\rangle)^* \tag{4}$$
$$-i\hbar\partial_t \langle\Psi(t)|x\rangle = \langle\Psi(t)|H|x\rangle \tag{5}$$
$$i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle, \tag{6}$$

where in going to (5) we have used the fact that H is Hermitian, and (6) is what we sought to prove. □

b. Beginning with what was proven in (a),

$$i\hbar\partial_t \langle\Phi(t)|x\rangle = -\langle\Phi(t)|H|x\rangle \tag{7}$$
$$i\hbar(\partial_t \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle = -\langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle. \tag{8}$$

From (3), we can write

$$\langle\Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle. \tag{9}$$

Adding (15) and (17) yields

$$\langle\Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle + i\hbar(\partial_t \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle \tag{10}$$
$$i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle, \tag{11}$$

where in going to (11) we have used the product rule of differentiation. (11) is what we sought to prove. □

c. Using (1), note that:

$$\langle x|H|\Psi(t)\rangle = \langle x| \left[\frac{p^2}{2m} + V(x)\right] |\Psi(t)\rangle \tag{12}$$
$$= \frac{1}{2m} \langle x|p^2|\Psi(t)\rangle + \langle x|V(x)|\Psi(t)\rangle \tag{13}$$
$$= \frac{(-i\hbar\partial_x)^2}{2m} \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle \tag{14}$$
$$= -\frac{\hbar^2}{2m}\partial_x^2 \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle , \tag{15}$$

where in going to (14) we have used the fact that

$$\langle x|p|\Psi(x)\rangle = -i\hbar\partial_x \langle x|\Psi(t)\rangle . \tag{16}$$

Similarly, note that

$$\langle \Phi(t)|H|x\rangle = -\frac{\hbar^2}{2m}\partial_x^2 \langle \Phi(t)|x\rangle + V(x) \langle \Phi(t)|x\rangle \tag{17}$$

where we have used

$$\langle \Phi(t)|p|x\rangle = i\hbar\partial_x \langle \Phi(t)|x\rangle , \tag{18}$$

which is the complex conjugate of (16) with $\Psi(t) \mapsto \Phi(t)$. Note that p is Hermitian. Making the substitutions (15) and (17) into what was proven in (b),

$$\begin{aligned} i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle &= \langle \Phi(t)|x\rangle \left[-\frac{\hbar^2}{2m}\partial_x^2 \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle\right] \\ &\quad - \left[-\frac{\hbar^2}{2m}\partial_x^2 \langle \Phi(t)|x\rangle + V(x) \langle \Phi(t)|x\rangle\right] \langle x|\Psi(t)\rangle && (19) \\ &= -\frac{\hbar^2}{2m} \left[\langle \Phi(t)|x\rangle \partial_x^2 \langle \Phi(t)|x\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right] \\ &\quad + V(x) \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle - V(x) \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle && (20) \\ &= -\frac{\hbar^2}{2m} \left[\langle \Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right] , && (21) \end{aligned}$$

as we sought to prove. □

d. Applying (16) and (18) to the left-hand side of (d),

$$\begin{aligned} \langle \Phi(t)|x\rangle \ \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle &= \langle \Phi(t)|x\rangle (-i\hbar\partial_x \langle x|\Psi(t)\rangle) + (i\hbar\partial_x \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle && (22) \\ &= \frac{\hbar}{i} [\langle \Phi(t)|x\rangle \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle] && (23) \end{aligned}$$

as we sought to prove. □

e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle \Phi(t)|x\rangle \ \langle x|p|\Psi(t)\rangle) = (\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \partial_x \langle x|p|\Psi(t)\rangle \tag{24}$$

Multiplying through by $\hbar/i$,

$$\begin{aligned} \frac{\hbar}{i}\partial_x(\langle \Phi(t)|x\rangle \ \langle x|p|\Psi(t)\rangle) &= (-i\hbar\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle - \langle \Phi(t)|x\rangle i\hbar\partial_x \langle x|p|\Psi(t)\rangle && (25) \\ &= -\langle \Phi(t)|p|x\rangle \ \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \ \langle x|p^2|\Psi(t)\rangle , && (26) \end{aligned}$$

where in going to (26) we have used (16) and (18). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i}\partial_x(\langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle) = (-i\hbar\partial_x\langle\Phi(t)|p|x\rangle)\langle x|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle\, i\hbar\partial_x\langle x|\Psi(t)\rangle \tag{27}$$

$$= -\langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle. \tag{28}$$

Adding the results of (26) and (28),

$$\frac{\hbar}{i}\partial_x\left[\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle$$

$$+ \langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle \tag{29}$$

$$= \langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle \tag{30}$$

as we sought to prove. □

1.2 Define

$$\rho(x,t) = \langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle, \tag{31}$$

$$J_x(x,t) = \frac{1}{2m}\left[\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle\right]. \tag{32}$$

Show that $\rho(x,t) + \partial_x J_x(x,t) = 0$.

Solution. From (31),

$$\partial_t\rho(x,t) = \partial_t(\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle), \tag{33}$$

and from what was proven in 1(c),

$$\partial_t(\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle) = -\frac{1}{i\hbar}\left[\langle\Phi(t)|x\rangle\,\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right] \tag{34}$$

$$= -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle\right], \tag{35}$$

where we have applied (16) and (18) in going to (35). Equating (33) and (35),

$$\partial_t\rho(x,t) = -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle\right]. \tag{36}$$

Beginning from (32),

$$\partial_x J_x(x,t) = \frac{1}{2m}\partial_x\left[\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle\right] \tag{37}$$

$$= \frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle\right], \tag{38}$$

where in going to (38) we have used what was proven in 1(e). Summing (36) and (38), we have

$$\partial_t\rho(x,t) + \partial_x J_x(x,t) = -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle\right]$$

$$+ \frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle\right] \tag{39}$$

$$= 0 \tag{40}$$

as we sought to prove. This is is the continuity equation. □

2 Problem 4

Consider a particle moving in three dimensions. Consider an operator

$$U(\phi) = \exp\left(-\frac{i}{\hbar}L_3\phi\right), \qquad L_3 = L_z = XP_y - YP_x, \tag{41}$$

where X, Y and P_x, P_y are position and momentum operators, respectively. Define new operators

$$X(\phi) = U^\dagger(\phi)XU(\phi), \qquad Y(\phi) = U^\dagger(\phi)YU(\phi). \tag{42}$$

Note that $X(0) = Y(0) = 0$.

2.1 Derive the equation

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)[L_3, X]U(\phi) = -Y(\phi), \tag{43}$$

and a similar equation for $\mathrm{d}Y(\phi)/\mathrm{d}\phi$.

Solution. Using the definition of $X = X(\phi)$ in (42) and applying the product rule of differentiation,

$$\frac{\mathrm{d}X}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(U^\dagger XU\right) = \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi}XU + U^\dagger\frac{\mathrm{d}}{\mathrm{d}\phi} = \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi}XU + U^\dagger\frac{\mathrm{d}X}{\mathrm{d}\phi}U + U^\dagger X\frac{\mathrm{d}U}{\mathrm{d}\phi}. \tag{44}$$

Note that

$$\frac{\mathrm{d}X}{\mathrm{d}\phi} = 0, \qquad \frac{\mathrm{d}U}{\mathrm{d}\phi} = -\frac{i}{\hbar}L_3\exp\left(-\frac{i}{\hbar}L_3\phi\right) = -\frac{i}{\hbar}L_3U, \tag{45}$$

$$\tag{46}$$

and

$$U^\dagger = \exp\left(\frac{i}{\hbar}L_3\phi\right) \implies \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi} = \frac{i}{\hbar}L_3\exp\left(\frac{i}{\hbar}L_3\phi\right) = \frac{i}{\hbar}L_3U^\dagger = \frac{i}{\hbar}U^\dagger L_3, \tag{47}$$

where the final equality follows because $[L_3, U] = 0$. Then (44) becomes

$$\frac{\mathrm{d}X}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger L_3XU - \frac{i}{\hbar}U^\dagger XL_3U = \frac{i}{\hbar}U^\dagger(L_3X - XL_3)U = \frac{i}{\hbar}U^\dagger(\phi)[L_3, X]U(\phi), \tag{48}$$

which is the first equality of what we wanted to show in (43).

From the definition of L_3 in (41),

$$[L_3, X] = L_3X - XL_3 = (XP_y - YP_x)X - X(XP_y - YP_x) \tag{49}$$

$$= XP_yX - YP_xX - XXP_x - XYP_x \tag{50}$$