1. **Problem.** Suppose we have a mechanical system with n degrees of freedom. Let $q_1(t), q_2(t), \dots, q_n(t)$ be its generalized coordinates. Now consider a time-dependent coordinate transformation

$$Q_i = Q_i(t, q_1, q_2, \dots, q_n) \qquad i = 1, 2, \dots, n.$$

Show that if $q_i(t)$ solves a system of Euler-Lagrange equations involving a Lagrangian $L(t, q_i, \dot{q}_i)$, then $Q_i(t)$ solves the Euler-Lagrange equations involving $L(t, Q_i, \dot{Q}_i)$ provided the time-dependent coordinate transformation fulfills some minimal standard of good behavior. Specify this "minimal standard of good behavior."

Solution. Suppose that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0; \tag{1}$$

that is, $q_i(t)$ solve a system of Euler-Lagrange equations. We want to show that

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = 0. \tag{2}$$

Beginning with the first term of (1), we can use the chain rule for $L(t, Q_i, \dot{Q}_i)$ to write

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial q_i}, \tag{3}$$

provided there exists an inverse transformation

$$q_i = q_i(t, Q_1, Q_2, \dots, Q_n) \qquad i = 1, 2, \dots, n \tag{4}$$

that allows us to write $L(t, q_i, \dot{q}_i)$ in terms of t, Q_i, and $\dot{Q}_i$. This is only possible if there is a one-to-one correspondence between $q_i(t)$ and $Q_i(t)$, which is the "minimal standard of good behavior" for the transformation. We will assume the transformation is so well behaved.

Again using the chain rule for $Q_j = Q_j(t, q_1, q_2, \dots, q_n)$, note that

$$\dot{Q}_j = \frac{\partial Q_j}{\partial t} + \frac{\partial Q_j}{\partial q_i}\dot{q}_i \tag{5}$$

so (3) becomes

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial q_i\,\partial t} + \frac{\partial^2 Q_j}{\partial q_i\,\partial q_k}\dot{q}_k\right). \tag{6}$$

Applying the chain rule now to the second term of (1), we have

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial Q_j}{\partial q_i} \tag{7}$$

where the right-hand side comes from (5). Then, using the product rule to take the time derivative,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{d}{dt}\frac{\partial Q_j}{\partial q_i}\right). \tag{8}$$

For the second term of (8), the chain rule for $Q_j = Q_j(t, q_1, q_2, \ldots, q_n)$ gives

$$\frac{d}{dt}\frac{\partial Q_j}{\partial q_i} = \frac{\partial^2 Q_j}{\partial t\,\partial q_i} + \frac{\partial^2 Q_j}{\partial q_i\,\partial q_k}\dot{q}_k. \tag{9}$$

Substituting (9) into (8), we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial t\,\partial q_i} + \frac{\partial^2 Q_j}{\partial q_k\,\partial q_i}\dot{q}_k\right) \tag{10}$$

where the terms on the far right appeared in (6). Making this substitution,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i}, \tag{11}$$

and rearranging,

$$\frac{\partial Q_j}{\partial q_i}\left(\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right) = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}, \tag{12}$$

Finally, substituting the original assumption (1), we find

$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j} = 0 \tag{13}$$

which is what we sought to prove. □

2. **Problem.** Look at the Lagrangian

$$L = e^{\sigma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

for one-dimensional motion.

(a) Write down the associated Euler-Lagrange ODE.

(b) Now perform a point transformation

$$Q = e^{\sigma t/2} q$$

where the new position coordinate Q is a function of t and q. What is the equation of motion for $Q(t)$? Are there conserved quantities?

Solution.

(a) Beginning from the general expression for the Euler-Lagrange equations,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -e^{\sigma t}kq - \frac{d}{dt}\left(e^{\sigma t}m\dot{q}\right) = -me^{\sigma t}\left(\ddot{q} + \sigma\dot{q} + \frac{k}{m}q\right) \tag{14}$$

so the ODE is

$$0 = \ddot{q} + \sigma\dot{q} + \frac{k}{m}q. \tag{15}$$

(b) It is possible to invert this transformation and write $q = q(t, Q)$. Explicitly, this is

$$q = Qe^{-\sigma t/2} \tag{16}$$

so

$$\dot{q} = e^{-\sigma t/2}\left(\dot{Q} - \frac{\sigma}{2}Q\right). \tag{17}$$

Rewriting the Lagrangian such that $L = L(t, Q, \dot{Q})$ results in

$$\begin{aligned} L &= e^{\sigma t}\left(\frac{m}{2}\left(e^{-\sigma t/2}\left(\dot{Q} - \frac{\sigma}{2}Q\right)\right)^2 - \frac{k}{2}\left(Qe^{-\sigma t/2}\right)^2\right) && (18)\\ &= \frac{m}{2}\left(\dot{Q} - \frac{\sigma}{2}Q\right)^2 - \frac{k}{2}Q^2 && (19)\\ &= \frac{m}{2}\left(\dot{Q}^2 - \sigma\dot{Q}Q + \left(\frac{\sigma^2}{4} - \frac{k}{m}\right)Q^2\right). && (20) \end{aligned}$$

Then the Euler-Lagrange equations are given by

$$0 = \frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{m}{2}\left(-\sigma\dot{Q} + 2\left(\frac{\sigma^2}{4} - \frac{k}{m}\right)Q - \frac{d}{dt}\left(2\dot{Q} - \sigma Q\right)\right) \tag{21}$$

which simplifies to

$$0 = \ddot{Q} + \left(\frac{k}{m} - \frac{\sigma^2}{4}\right)Q. \tag{22}$$

The solutions to (22) have the form

$$Q(t) = \begin{cases} A_1 \sin\left(\sqrt{\dfrac{k}{m} - \dfrac{\sigma^2}{4}}\,t\right) + A_2\cos\left(\sqrt{\dfrac{k}{m} - \dfrac{\sigma^2}{4}}\,t\right) & \text{if } \dfrac{k}{m} > \dfrac{\sigma^2}{4},\\ B_1 + B_2 t & \text{if } \dfrac{k}{m} = \dfrac{\sigma^2}{4},\\ C_1 \exp\left\{-\sqrt{\dfrac{k}{m} - \dfrac{\sigma^2}{4}}\,t\right\} + C_2\exp\left\{\sqrt{\dfrac{k}{m} - \dfrac{\sigma^2}{4}}\,t\right\} & \text{if } \dfrac{k}{m} < \dfrac{\sigma^2}{4}, \end{cases} \tag{23}$$

where A_i, B_i, C_i are real constants.

The Lagrangian in (20) does not explicitly depend on time (in contrast to the Lagrangian in the problem statement). Thus, the total energy H of the system is conserved. Explicitly,

$$H = \dot{Q}\frac{\partial L}{\partial \dot{Q}} - L = \frac{m}{2}\left(\dot{Q}^2 - \left(\frac{\sigma^2}{4} - \frac{k}{m}\right)Q^2\right) \tag{24}$$

is a conserved quantity.

3. **Problem.** Let $U(\alpha\mathbf{r}_1, \ldots, \alpha\mathbf{r}_N)$ be a potential for N particles that satisfies the relation

$$U(\alpha\mathbf{r}_1, \ldots, \alpha\mathbf{r}_N) = \alpha^k U(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

The factor α can be any nonzero real number. The exponent k is an integer.

(a) Show that the equations of motion associated with such a potential remain unchanged under a dilation of the distance scale if the time scale is also dilated by some other factor β. Find β as a function of α and k.

(b) If $k = 2$, the forces correspond to a system of harmonic oscillators coupled to each other. Show that the result in part (a) implies the frequencies of such a system are independent of the oscillation amplitude.

(c) If $k = -1$, we have an inverse square force law, such as that which arises in mutual gravitational attraction. Show that the result in part (a) implies Kepler's third law: the square of the orbital period of a planet is directly proporitonal to the cube of the semi-major axis of its orbit.

Solution. The Lagrangian $L = L(t, \mathbf{r}_i, \dot{\mathbf{r}}_i)$ for the system of N particles is

$$L = T - U = \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i - U(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{25}$$

where m_i is the mass of the particle located at $\mathbf{r}_i$. The Euler-Lagrange equations for this Lagrangian are

$$\frac{\partial L}{\partial \mathbf{r}_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = 0 \implies \frac{\partial U}{\partial \mathbf{r}_i} + m_i \ddot{\mathbf{r}}_i = 0. \tag{26}$$

Define the time scale transformation

$$T = \beta t, \tag{27}$$

and define the coordinate transformation

$$\mathbf{R}_i = \mathbf{R}_i(T) = \alpha \mathbf{r}_i \tag{28}$$

for all N particles. Using these coordinates, the Lagrangian $L = L(T, \mathbf{R}_i, \dot{\mathbf{R}}_i)$ is

$$L = \frac{1}{2} m_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i - U(\mathbf{R}_1, \ldots, \mathbf{R}_N) \tag{29}$$

and the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \mathbf{R}_i} - \frac{d}{dT}\frac{\partial L}{\partial \dot{\mathbf{R}}_i} = 0 \implies \frac{\partial U}{\partial \mathbf{R}_i} + m_i \ddot{\mathbf{R}}_i = 0. \tag{30}$$

(a) The equations of motion associated to the Lagrangians (25) and (29) are identical if the Euler-Lagrange equations in (26) and (30) are identical. We will now show that this is the case.

The transformation $\mathbf{R}_i = \alpha \mathbf{r}_i$ is invertible, so $\mathbf{r}_i = \mathbf{R}_i/\alpha$. Likewise, $t = T/\beta$. By the chain rule,

$$\frac{d}{dT} = \frac{d}{dt}\frac{dt}{dT} = \frac{1}{\beta}\frac{d}{dt} \tag{31}$$

so

$$\dot{\mathbf{R}} = \alpha \frac{d\mathbf{r}_i}{dT} = \frac{\alpha}{\beta}\dot{\mathbf{r}}_i \tag{32}$$

and, likewise,

$$\ddot{\mathbf{R}} = \frac{\alpha}{\beta^2}\ddot{\mathbf{r}}_i. \tag{33}$$

From the given relationship for U, note that

$$U(\mathbf{R}_1, \ldots, \mathbf{R}_N) = \alpha^k U(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{34}$$

and again using the chain rule,

$$\frac{\partial}{\partial \mathbf{R}_i} = \frac{\partial}{\partial \mathbf{r}_i}\frac{d\mathbf{r}_i}{d\mathbf{R}_i} = \frac{1}{\alpha}\frac{\partial}{\partial \mathbf{r}_i}. \tag{35}$$

Making use of (33), (34), and (35), we can rewrite (30) in terms of the original coordinates:

$$0 = \frac{\alpha^k}{\alpha}\frac{\partial U}{\partial \mathbf{r}_i} + m_i \frac{\alpha}{\beta^2}\ddot{\mathbf{r}}_i \implies \alpha^{k-2}\beta^2 \frac{\partial U}{\partial \mathbf{r}_i} + m_i \ddot{\mathbf{r}}_i = 0 \tag{36}$$

which is equivalent to (26) so long as

$$\alpha^{k-2}\beta^2 = 1 \implies \beta = \alpha^{2-k/2}. \tag{37}$$

□

(b) Fixing $k = 2$, the solution of (a) gives us

$$\beta = \alpha^0 = 1. \tag{38}$$

This result indicates that the time scale is completely independent of the distance scale. That is, if we make the distance scale transformation $\mathbf{r}_i \mapsto \alpha \mathbf{r}_i$, the equations of motion will remain unchanged with respect to time ($t \mapsto \beta t = t$).

For the system of harmonic oscillators, the oscillation amplitudes A_i have units of distance. The frequencies ω_i have units of inverse time. Making the transformation $\mathbf{r}_i \mapsto \alpha \mathbf{r}_i$ will change the distance scale and therefore the amplitudes, but the time scale and hence ω_i will remain unchanged. In other words, the frequencies are independent of the amplitudes. □

(c) Fixing $k = -1$, the solution of (a) gives us

$$\beta = \alpha^{3/2}. \tag{39}$$

Let a be the semi-major axis length of some planet's orbit and T its orbital period. Kepler's third law states that

$$T^2 \propto a^3. \tag{40}$$

Note that a has units of distance and T has units of time. In order to show that (40) holds for any planet, we can consider an arbitrary length αa for the semimajor axis. Thus, we want to show that (40) is unchanged under the transformation $a \mapsto \alpha a$ and the corresponding time dilation $T \mapsto \beta T = \alpha^{3/2} T$. Making these transformations,

$$(\alpha^{3/2}T)^2 \propto (\alpha a)^3 \iff \alpha^3 T^2 \propto \alpha^3 a^3 \iff T^2 \propto a^3 \tag{41}$$

which is indeed equivalent to (40). Thus, Kepler's law holds for any planet. □

4. **Problem.** A particle in three-dimensional space is confined in a central potential

$$U(r) = -U_0 \left(\frac{r_0}{r}\right)^n.$$

Here $r = |\mathbf{r}|$ where $\mathbf{r}(t)$ is the location of the particle at time t, U_0 is a characteristic energy scale and r_0 is a characteristic length scale. The exponent n is an integer that is greater than or equal to 1. Show that the particle motion is confined to a two-dimensional orbital plane. For what values of n are circular orbits stable?

Solution. We want to show that the particle motion is confined to a two-dimensional orbital plane. We will use the spherical coordinates (ρ, θ, ϕ), so r retains its definition as the particle's distance from the origin.

$U(r)$ is a central potential, so it has a corresponding central force

$$\mathbf{F} = -\nabla U(r) = -nU_0 \frac{r_0^n}{r^{n+1}} \hat{\mathbf{r}} \tag{42}$$

which is radially symmetric by inspection. This means that the particle's torque $\boldsymbol{\tau} = 0$. Therefore, the particle's angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{43}$$

is conserved; that is, it is constant over time. Notably, the *direction* of $\mathbf{L}$ does not change over time. Because $\mathbf{r}$ is perpendicular to $\mathbf{L}$ as defined by (43), $\mathbf{L}$'s not changing direction implies that $\mathbf{r}$ is confined to a plane perpendicular to $\mathbf{L}$ for all time.

We will choose $\mathbf{L}$ to point in the $\hat{\boldsymbol{\phi}}$ direction, so $\mathbf{r}$ is confined to the plane (r, θ). Therefore, we can write $\mathbf{r} = \mathbf{r}(r, \theta)$. The particle's potential energy $T = m\mathbf{r}^2/2$ where m is the particle's mass. This gives us the Lagrangian

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + U_0 \frac{r_0^n}{r^n}, \tag{44}$$

which does not depend explicitly on θ or on t.

For r, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = mr\dot{\theta}^2 - nU_0 \frac{r_0^n}{r^{n+1}} - m\ddot{r} = 0. \tag{45}$$

For θ, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 - \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \tag{46}$$

which implies

$$mr^2\dot{\theta} = l \tag{47}$$

where $l = |\mathbf{L}|$ is a constant. Substituting (47) into (46), and rearranging, we obtain

$$m\ddot{r} = \frac{l^2}{mr^3} - nU_0\frac{r_0^n}{r^{n+1}} \equiv -\frac{\partial U_{\text{eff}}}{\partial r} \tag{48}$$

where we have defined the effective potential $U_{\text{eff}} = U_{\text{eff}}(r)$. Explicitly,

$$U_{\text{eff}} = -U_0\frac{r_0^n}{r^n} + \frac{1}{2}\frac{l^2}{mr^2}. \tag{49}$$

If a circular orbit at $r = r_c$ to be stable, small perturbations $r_c \mapsto r_c + \delta r$ will stay close to r_c as time moves forward. In order for this to be the case, $U_{\text{eff}}(r)$ must have a local minimum at r_c. In order to have any kind of extremum at r_c, we require

$$\left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r_c} = 0. \tag{50}$$

Using the definition of (49),

$$\left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r_c} = 0 \implies \frac{l^2}{mr_c^3} = nU_0\frac{r_0^n}{r_c^{n+1}}. \tag{51}$$

In order for the extremum at $r_c = 0$ to be a minimum, we require

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{r_c} > 0. \tag{52}$$

Using the definition of (49),

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{r_c} = -n(n+1)U_0\frac{r_0^n}{r_c^{n+2}} + 3\frac{l^2}{mr_c^4} = -n(n+1)U_0\frac{r_0^n}{r_c^{n+2}} + \frac{3}{r_c}nU_0\frac{r_0^n}{r_c^{n+1}} \tag{53}$$

where in the final equality we are substituting the result of (51). So the condition for a stable circular orbit becomes

$$3nU_0\frac{r_0^n}{r^{n+2}} > n(n+1)U_0\frac{r_0^n}{r^{n+2}} \tag{54}$$

which holds for $n < 2$ by inspection. Thus, for the conditions of this problem, it is only possible to have a stable circular orbit for $n = 1$.

In writing these solutions, I consulted Goldstein's *Classical Mechanics*, Landau and Lifshitz's *Mechanics*, and David Tong's lecture notes.