

1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \quad (1)$$

1.1 Verify the following:

- a. $i\hbar\partial_t \langle \Psi(t)|x \rangle = -\langle \Psi(t)|H|x \rangle,$
- b. $i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle,$
- c. $i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = -\frac{\hbar^2}{2m} [\langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle],$
- d. $\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle = \frac{\hbar}{i} [\langle \Phi(t)|x \rangle \partial_x \langle x|\Psi(t) \rangle - (\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle]$
- e. $\frac{\hbar}{i} \partial_x [\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle] = \langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - m\ell\Phi(t)p^2x \langle x|\Psi(t) \rangle$

Solution.

- a. We will begin with the Schrödinger equation,

$$i\hbar\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle. \quad (2)$$

Since the Hamiltonian given by (1) is time independent, the system evolves in time under the time-evolution operator $U(t) = \exp(-iHt/\hbar)$. Denote the eigenkets of H by $|E_i\rangle$ and the corresponding eigenvalues by E_i . Assuming $V(x)$ is a real-valued function, H is Hermitian, and so $|E_i\rangle$ form a complete orthonormal basis. Then we may rewrite $|\Psi(t)\rangle$ in terms of $U(t)$ and expand it in $|E_i\rangle$:

$$|\Psi(t)\rangle = U(t) |\Psi\rangle = e^{iHt/\hbar} \sum_i |E_i\rangle \langle E_i|\Psi\rangle = \sum_i e^{iE_i t/\hbar} |E_i\rangle \langle E_i|\Psi\rangle. \quad (3)$$

Substituting (3) into (2) and evaluating the time derivative,

$$-\sum_i E_i e^{iE_i t/\hbar} |E_i\rangle \langle E_i|\Psi\rangle = H \sum_i e^{iE_i t/\hbar} |E_i\rangle \langle E_i|\Psi\rangle. \quad (4)$$

Taking the adjoint of (4) yields

$$-\sum_i E_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar} = H \sum_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar}. \quad (5)$$

From the adjoint of (3), note that

$$i\hbar\partial_t \langle \Psi(t)| = i\hbar\partial_t \sum_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar} = \sum_i E_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar}. \quad (6)$$

Making these substitutions into (5), and multiplying by $|x\rangle$ on the right, we have

$$-i\hbar\partial_t \langle \Psi(t)| = H \langle \Psi(t)| \implies i\hbar\partial_t \langle \Psi(t)|x \rangle = -\langle \Psi(t)|H|x \rangle \quad (7)$$

as we sought to prove. \square

- b. Rewriting what was proven in (a) with $\Psi \mapsto \Phi$ and then multiplying by $\Psi(x, t)$ on the right,

$$i\hbar\partial_t \langle \Phi(t)|x \rangle = -\langle \Phi(t)|H|x \rangle \quad (8)$$

$$i\hbar(\partial_t \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle = -\langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle. \quad (9)$$

Multiplying (2) by $\langle \Phi(t)|x \rangle \langle x|$ on the left,

$$\langle \Phi(t)|x \rangle i\hbar\partial_t \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle. \quad (10)$$

Adding (10) and (9) yields

$$\langle \Phi(t)|x \rangle i\hbar\partial_t \langle x|\Psi(t) \rangle + i\hbar(\partial_t \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle \quad (11)$$

$$i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle, \quad (12)$$

where in going to (12) we have used the product rule of differentiation on the left-hand side. (12) is what we sought to prove. \square

- c. Using (1), note that:

$$\langle x|H|\Psi(t) \rangle = \langle x| \left[\frac{p^2}{2m} + V(x) \right] |\Psi(t) \rangle = \frac{1}{2m} \langle x|p^2|\Psi(t) \rangle + \langle x|V(x)|\Psi(t) \rangle \quad (13)$$

$$= -\frac{\hbar^2}{2m} \partial_x^2 \langle x|\Psi(t) \rangle + V(x) \langle x|\Psi(t) \rangle, \quad (14)$$

where in going to (14) we have (twice) used the fact that

$$\langle x|p|\Psi(t) \rangle = -i\hbar\partial_x \langle x|\Psi(t) \rangle. \quad (15)$$

Similarly, note that

$$\langle \Phi(t)|H|x \rangle = -\frac{\hbar^2}{2m} \partial_x^2 \langle \Phi(t)|x \rangle + V(x) \langle \Phi(t)|x \rangle \quad (16)$$

where we have (twice) used the adjoint of (15) with $\Psi \mapsto \Phi$,

$$\langle \Phi(t)|p|x \rangle = i\hbar\partial_x \langle \Phi(t)|x \rangle. \quad (17)$$

This follows because p is Hermitian. Making the substitutions (14) and (16) into what was proven in (b),

$$\begin{aligned} i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle &= \langle \Phi(t)|x \rangle \left[-\frac{\hbar^2}{2m} \partial_x^2 \langle x|\Psi(t) \rangle + V(x) \langle x|\Psi(t) \rangle \right] \\ &\quad - \left[-\frac{\hbar^2}{2m} \partial_x^2 \langle \Phi(t)|x \rangle + V(x) \langle \Phi(t)|x \rangle \right] \langle x|\Psi(t) \rangle \end{aligned} \quad (18)$$

$$\begin{aligned} &= -\frac{\hbar^2}{2m} [\langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle] \\ &\quad + V(x) \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle - V(x) \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle \end{aligned} \quad (19)$$

$$= -\frac{\hbar^2}{2m} [\langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle], \quad (20)$$

as we sought to prove. \square

- d. Applying (15) and (17) to the left-hand side of (d),

$$\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle (-i\hbar\partial_x \langle x|\Psi(t) \rangle) + (i\hbar\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle \quad (21)$$

$$= \frac{\hbar}{i} [\langle \Phi(t)|x \rangle \partial_x \langle x|\Psi(t) \rangle - (\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle] \quad (22)$$

as we sought to prove. \square

- e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle) = (\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \partial_x \langle x|p|\Psi(t)\rangle \quad (23)$$

Multiplying through by \hbar/i ,

$$\frac{\hbar}{i} \partial_x(\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle) = (-i\hbar \partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle - \langle \Phi(t)|x\rangle i\hbar \partial_x \langle x|p|\Psi(t)\rangle \quad (24)$$

$$= -\langle \Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle, \quad (25)$$

where in going to (25) we have used (15) and (17). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i} \partial_x(\langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle) = (-i\hbar \partial_x \langle \Phi(t)|p|x\rangle) \langle x|\Psi(t)\rangle - \langle \Phi(t)|p|x\rangle i\hbar \partial_x \langle x|\Psi(t)\rangle \quad (26)$$

$$= -\langle \Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle. \quad (27)$$

Adding the results of (25) and (27),

$$\begin{aligned} \frac{\hbar}{i} \partial_x [\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle] &= \langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle \Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle \\ &\quad + \langle \Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle - \langle \Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle \end{aligned} \quad (28)$$

$$= \langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle \Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle \quad (29)$$

as we sought to prove. \square

1.2 Define

$$\rho(x, t) = \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle, \quad (30)$$

$$J_x(x, t) = \frac{1}{2m} [\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle]. \quad (31)$$

Show that $\rho(x, t) + \partial_x J_x(x, t) = 0$.

Solution. From (30),

$$\partial_t \rho(x, t) = \partial_t (\langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle), \quad (32)$$

and from what was proven in 1(c),

$$\partial_t (\langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle) = -\frac{1}{i\hbar} [\langle \Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle] \quad (33)$$

$$= -\frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle \Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle], \quad (34)$$

where we have applied (15) and (17) in going to (34). Equating (32) and (34),

$$\partial_t \rho(x, t) = -\frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle \Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle]. \quad (35)$$

Beginning from (31),

$$\partial_x J_x(x, t) = \frac{1}{2m} \partial_x [\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle] \quad (36)$$

$$= \frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle \Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle], \quad (37)$$

where in going to (37) we have used what was proven in 1(e). Summing (35) and (37), we have

$$\begin{aligned} \partial_t \rho(x, t) + \partial_x J_x(x, t) &= -\frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle] \\ &\quad + \frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle] \end{aligned} \quad (38)$$

$$= 0 \quad (39)$$

as we sought to prove. This is the continuity equation for probability. \square

2 Problem 4

Consider a particle moving in three dimensions. Consider an operator

$$U(\phi) = \exp\left(-\frac{i}{\hbar} L_3 \phi\right), \quad L_3 = L_z = X P_y - Y P_x, \quad (40)$$

where X, Y and P_x, P_y are position and momentum operators, respectively. Define new operators

$$X(\phi) = U^\dagger(\phi) X U(\phi), \quad Y(\phi) = U^\dagger(\phi) Y U(\phi). \quad (41)$$

Note that $X(0) = Y(0) = 0$.

2.1 Derive the equation

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi) [L_3, X] U(\phi) = -Y(\phi), \quad (42)$$

and a similar equation for $dY(\phi)/d\phi$.

Solution. Using the definition of $X(\phi)$ in (41) and applying the product rule of differentiation,

$$\frac{dX(\phi)}{d\phi} = \frac{d}{d\phi} (U^\dagger X U) = \frac{dU^\dagger}{d\phi} X U + U^\dagger \frac{d}{d\phi} (X U) \quad (43)$$

$$= \frac{dU^\dagger}{d\phi} X U + U^\dagger \frac{dX}{d\phi} U + U^\dagger X \frac{dU}{d\phi}. \quad (44)$$

We know immediately that $dX/d\phi = 0$ because ϕ is not a parameter of the position operator X . From the definition of $U(\phi)$ in (40), we know that $[L_3, U(\phi)] = 0$. Thus

$$\frac{dU}{d\phi} = -\frac{i}{\hbar} L_3 U = -\frac{i}{\hbar} L_3 \exp\left(-\frac{i}{\hbar} L_3 \phi\right) = -\frac{i}{\hbar} U L_3, \quad (45)$$

and likewise

$$U^\dagger = \exp\left(\frac{i}{\hbar} L_3 \phi\right) \implies \frac{dU^\dagger}{d\phi} = \frac{i}{\hbar} L_3 \exp\left(\frac{i}{\hbar} L_3 \phi\right) = \frac{i}{\hbar} L_3 U^\dagger = \frac{i}{\hbar} U^\dagger L_3 \quad (46)$$

because $[L_3, U^\dagger] = 0$ as well. Then (44) becomes

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger L_3 X U - \frac{i}{\hbar} U^\dagger X L_3 U = \frac{i}{\hbar} U^\dagger (L_3 X - X L_3) U = \frac{i}{\hbar} U^\dagger(\phi) [L_3, X] U(\phi), \quad (47)$$

which is the first equality of what we wanted to show in (42).

From the definition of L_3 in (40),

$$[L_3, X] = L_3X - XL_3 = (XP_y - YP_x)X - X(XP_y - YP_x) \quad (48)$$

$$= XP_yX - YP_xX - XXP_y + XYP_x = YXP_x - YP_xX \quad (49)$$

$$= Y[X, P_x] = i\hbar Y \quad (50)$$

where in (49) we have used $[X, P_y] = [X, Y] = 0$, and in (50) we have used $[X, P_x] = i\hbar$. Making the substitution (50) into (47), we have

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)(i\hbar Y)U(\phi) = -U^\dagger(\phi)YU(\phi) = -Y(\phi), \quad (51)$$

where the last equality is from the definition of $Y(\phi)$ in (41). This is the second equality of what we wanted to show in (42), which completes the proof.

For $dY(\phi)/d\phi$, we can make the substitutions $X(\phi) \mapsto Y(\phi)$, $X \mapsto Y$ in (44) and (47) to obtain

$$\frac{dY(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)[L_3, Y]U(\phi). \quad (52)$$

Then making similar use of commutators $[Y, P_x] = [X, Y] = 0$ and $[Y, P_y] = i\hbar$ as for (49) and (50),

$$[L_3, Y] = L_3Y - YL_3 = (XP_y - YP_x)Y - Y(XP_y - YP_x) \quad (53)$$

$$= XP_yY - YP_xY - YXP_y + YYP_x = XP_yY - XYP_y \quad (54)$$

$$= X[P_y, Y] = -X[Y, P_y] = -i\hbar X. \quad (55)$$

Substituting (55) into (52),

$$\frac{dY(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)(-i\hbar X)U(\phi) = X(\phi), \quad (56)$$

and so we have derived

$$\frac{dY(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)[L_3, Y]U(\phi) = X(\phi). \quad (57)$$

and (42) as desired. \square

2.2 Define $X_\pm(\phi) = X(\phi) \pm iY(\phi)$. From the results of previous parts, show $X_+(\phi) = e^{i\phi}X_+$ where $X_+ = X_+(0)$. Derive the similar expression for $X_-(\phi)$.

Solution. Differentiating $X_\pm(\phi)$ and making use of (42) and (57),

$$\frac{dX_\pm(\phi)}{d\phi} = \frac{dX(\phi)}{d\phi} \pm i \frac{dY(\phi)}{d\phi} = -Y(\phi) \pm iX(\phi) = \pm i[X(\phi) \pm iY(\phi)] \quad (58)$$

$$= \pm iX_\pm(\phi). \quad (59)$$

The differential equation (59) has solutions given by exponential functions of $\pm i\phi$. We will make the ansatz

$$X_\pm(\phi) = e^{\pm i\phi}C_\pm, \quad (60)$$

where C_{\pm} is an operator “constant” in ϕ (that is, independent of it) and is fixed by an initial condition. Inspecting (60), clearly $X_{\pm}(0) = C_{\pm}$ where it is defined $X_{\pm}(0) \equiv X_{\pm}$. All that remains is to show that (60) obeys the relation (59), as follows:

$$\frac{dX_{\pm}(\phi)}{d\phi} = \frac{d}{d\phi} (e^{\pm i\phi}) C_{\pm} = \pm i e^{\pm i\phi} C_{\pm} = \pm i X_{\pm}(\phi). \quad (61)$$

Thus, we have derived

$$X_{+}(\phi) = e^{i\phi} X_{+}, \quad X_{-}(\phi) = e^{-i\phi} X_{-} \quad (62)$$

as desired. \square

2.3 Show that $[L_3, X_{+}] = \hbar X_{+}$. Derive the similar expression for $[L_3, X_{-}]$.

Solution. Firstly, note that

$$X_{\pm} = X_{\pm}(0) = X(0) \pm iY(0) = U^{\dagger}(0)XU(0) \pm iU^{\dagger}(0)YU(0) = X \pm iY \quad (63)$$

because $U(0) = U^{\dagger}(0) = I$. Also applying the definition of L_3 in (40), we have

$$[L_3, X_{\pm}] = [XP_y - YP_x, X \pm iY] = (XP_y - YP_x)(X \pm iY) - (X \pm iY)(XP_y - YP_x) \quad (64)$$

$$= XP_yX \pm iXP_yY - YP_xX \mp iYP_xY - XXP_y + XYP_x \mp iYXP_y \pm iYYP_x \quad (65)$$

$$= \pm iXP_yY - YP_xX + XYP_x \mp iYXP_y = \pm iX[P_y, Y] + Y[X, P_x] \quad (66)$$

$$= \pm \hbar X + i\hbar Y = \pm \hbar[X \pm iY] = \pm \hbar X_{\pm}. \quad (67)$$

Thus, we have shown

$$[L_3, X_{+}] = \hbar X_{+}, \quad [L_3, X_{-}] = -\hbar X_{-} \quad (68)$$

as desired. \square

3 Problem 1

Consider a particle with coordinate $x \in (-\infty, \infty)$, and momentum $p \in (-\infty, \infty)$, along with corresponding operators X and P . We have

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}. \quad (69)$$

3.1 Consider $\langle p|X|\Psi\rangle$. Express it in terms of $\langle p|\Psi\rangle$.

Solution. In the momentum space, the action of X is given by

$$\langle p|X|\Psi\rangle = i\hbar\partial_p \langle p|\Psi\rangle. \quad (70)$$

3.2 Define a state $|\Psi'\rangle$ from $|\Psi\rangle$ by $\langle p - p_0|\Psi\rangle = \langle p|\Psi'\rangle$. Construct the unitary operator $V(p_0)$ such that

$$|\Psi'\rangle = V(p_0) |\Psi\rangle. \quad (71)$$

Solution. For an infinitesimal p_0 ,

$$V^\dagger(p_0) |p\rangle = |p - p_0\rangle = e^{-p_0 \partial_p} |p\rangle \quad (72)$$

and since $\partial_p^\dagger = -\partial_p$ in the momentum basis,

$$V(p_0) = e^{p_0 \partial_p} = e^{ip_0 X/\hbar} \quad (73)$$

because $X = -i\hbar \partial_p$ when acting on the $|p\rangle$ basis, as given by the adjoint of (70). Then

$$\langle p|V(p_0)|\Psi\rangle = \langle p - p_0|\Psi\rangle = \langle p|\Psi'\rangle \quad (74)$$

as desired.

$V(p_0)$ has the following properties that were also required of $U(a)$:

1. In the limit $p_0 \rightarrow 0$, $V(p_0) \rightarrow I$:

$$\lim_{p_0 \rightarrow 0} V(p_0) = \lim_{p_0 \rightarrow 0} e^{ip_0 X/\hbar} = e^0 = I. \quad (75)$$

2. Successive applications are equivalent to a single application:

$$V(p_1)V(p_2) = e^{ip_1 X/\hbar} e^{ip_2 X/\hbar} = e^{i(p_1+p_2)X/\hbar} = V(p_1 + p_2). \quad (76)$$

3. Unitarity:

$$V(p_0)V^\dagger(p_0) = e^{ip_0 X/\hbar} e^{-ip_0 X/\hbar} = I, \quad V^\dagger(p_0)V(p_0) = e^{-ip_0 X/\hbar} e^{ip_0 X/\hbar} = I. \quad (77)$$

3.3 Consider $|\Psi''\rangle = U(a)V(p_0)|\Psi\rangle$ where $U(a)$ is the spatial translation operator. Express $\langle x|\Psi''\rangle$ as

$$\langle x|\Psi''\rangle = \exp(i\Phi(x, a, p_0)) \langle x''|\Psi\rangle \quad (78)$$

where the phase Φ and x'' are to be determined as part of the problem.

Solution. Using the definition of $|\Psi''\rangle$,

$$\langle x|\Psi''\rangle = \langle x|U(a)V(p_0)|\Psi\rangle = \langle x - a|V(p_0)|\Psi\rangle = \langle x - a|e^{ip_0 X/\hbar}|\Psi\rangle = e^{ip_0(x-a)/\hbar} \langle x - a|\Psi\rangle \quad (79)$$

which gives

$$\Phi = \frac{p_0(x - a)}{\hbar}, \quad x'' = x - a. \quad (80)$$

3.4 Defining $\langle X \rangle = \langle \Psi | X | \Psi \rangle$ and $\langle P \rangle = \langle \Psi | P | \Psi \rangle$, define formulas which express $\langle \Psi'' | X | \Psi'' \rangle$ and $\langle \Psi'' | P | \Psi'' \rangle$ in terms of $\langle X \rangle$, $\langle P \rangle$, and constants.

Solution. Beginning with $\langle \Psi'' | V | \Psi'' \rangle$, we may insert the identity operator:

$$\langle \Psi'' | X | \Psi'' \rangle = \iint \langle \Psi'' | x \rangle \langle x | X | x' \rangle \langle x' | \Psi'' \rangle dx dx' \quad (81)$$

$$= \iint \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} x' \delta(x - x') e^{ip_0(x'-a)/\hbar} \langle x' - a | \Psi \rangle dx dx', \quad (82)$$

$$= \int \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} x e^{ip_0(x-a)/\hbar} \langle x - a | \Psi \rangle dx \quad (83)$$

$$= \int \langle \Psi | x - a \rangle x \langle x - a | \Psi \rangle dx, \quad (84)$$

where in going to (82) we have substituted (79) and its adjoint. Now making the change of variable $x - a \mapsto x$, (84) becomes

$$\langle \Psi'' | X | \Psi'' \rangle = \int \langle \Psi | x \rangle (x + a) \langle x | \Psi \rangle dx = a \int \langle \Psi | x \rangle \langle x | \Psi \rangle dx + \int \langle \Psi | x \rangle x \langle x | \Psi \rangle dx = a + \langle X \rangle. \quad (85)$$

Now proceeding similarly for $\langle \Psi'' | P | \Psi'' \rangle$,

$$\langle \Psi'' | P | \Psi'' \rangle = \iint \langle \Psi'' | x \rangle \langle x | P | x' \rangle \langle x' | \Psi'' \rangle dx dx' \quad (86)$$

$$= \iint \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} \left(i\hbar \frac{\partial}{\partial x'} \delta(x - x') e^{ip_0(x'-a)/\hbar} \langle x' - a | \Psi \rangle \right) dx dx', \quad (87)$$

$$= i\hbar \int \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} \left(\frac{\partial}{\partial x} e^{ip_0(x-a)/\hbar} \langle x - a | \Psi \rangle \right) dx, \quad (88)$$

$$= i\hbar \int \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} \left(\frac{\partial}{\partial x} e^{ip_0(x-a)/\hbar} \right) \langle x - a | \Psi \rangle dx \\ + i\hbar \int \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} e^{ip_0(x-a)/\hbar} \left(\frac{\partial}{\partial x} \langle x - a | \Psi \rangle \right) dx, \quad (89)$$

$$= i\hbar \frac{ip_0}{\hbar} \int \langle \Psi | x - a \rangle \langle x - a | \Psi \rangle dx + i\hbar \int \langle \Psi | x - a \rangle \left(\frac{\partial}{\partial x} \langle x - a | \Psi \rangle \right) dx. \quad (90)$$

Again making the change of variable $x - a \mapsto x$, (90) becomes

$$\langle \Psi'' | P | \Psi'' \rangle = i\hbar \int \langle \Psi | x \rangle \left(\frac{\partial}{\partial x} \langle x | \Psi \rangle \right) dx = p_0 \int \langle \Psi | x \rangle \langle x | \Psi \rangle dx = \langle P \rangle - p_0. \quad (91)$$

In summary, we have found $\langle \Psi'' | X | \Psi'' \rangle = \langle X \rangle + a$ and $\langle \Psi'' | P | \Psi'' \rangle = \langle P \rangle - p_0$.

4 Problem 2

Suppose we have a particle moving in one dimension ($-\infty < x < \infty$), with quantum Hamiltonian given by

$$H(t) = H_0 - XF(t) \quad (92)$$

where

$$H_0 = \frac{P^2}{2m} + V(X) \quad (93)$$

where $V(X)$ is the potential and $F(t)$ is a c-number function. Consider