Problem 1. Linear sigma model (Peskin & Schroeder 4.3) The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Psi^i(x)$, $i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\boldsymbol{\nabla}\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \boldsymbol{\Phi} \cdot \boldsymbol{\Phi}$, and

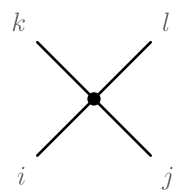
$$V(\Phi^2) = \frac{1}{2}m^2(\Phi^i)^2 + \frac{\lambda}{4}\left((\Phi^i)^2\right)^2 \tag{1}$$

is a function symmetric under rotations of $\boldsymbol{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

1(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$\contraction{}{\Phi}{^i(x)}{\Phi}\Phi^i(x)\Phi^j(y) = \delta^{ij} D_F(x-y),$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}). \tag{2}$$

Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

Solution. The Klein-Gordon Hamiltonian is given by Peskin & Schroeder (2.8),

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \tag{3}$$

For $\lambda = 0$, the linear sigma model has the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\boldsymbol{\nabla}\Phi^i)^2 + \frac{1}{2}m^2(\Phi^i)^2 \right)$$

which is clearly N copies of the Klein-Gordon Hamiltonian, one for each i.

From (4.36) we know that the Feynman propagator is the contraction of two fields:

$$\contraction{}{\phi}{(x)}{\phi}\phi(x)\phi(y) = D_F(x-y).$$

No terms $\Phi^i\Phi^j$ for $i \neq j$ appear in the Hamiltonian, so fields with $i \neq j$ cannot be contracted. Moreover, each field is governed by its own independent Klein-Gordon Hamiltonian to zeroth order. So the propagator must be

$$\contraction{}{\Phi}{^i(x)}{\Phi}\Phi^i(x)\Phi^j(y) = \delta^{ij} D_F(x-y)$$

where $D_F(x-y)$ is the Klein-Gordon propagator. □

In order to determine the Feynman rules, we use Peskin & Schroeder (4.90),

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_n | iT | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p}_1 \cdots \mathbf{p}_n | T \left\{ \exp\left(-i \int_{-T}^{T} dt\, H_I(t) \right) \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

Our interaction Hamiltonian is

$$H_I = \int d^3x\, \frac{\lambda}{4} \left((\Phi^i)^2 \right)^2 = \int d^3x\, \frac{\lambda}{4} (\mathbf{\Phi} \cdot \mathbf{\Phi})^2 = \frac{\lambda}{4} \int d^3x \left(\sum_i (\Phi^i)^4 + 2 \sum_{i\neq j} (\Phi^i)^2 (\Phi^j)^2 \right),$$

W have two final momenta, $\mathbf{p}_1$ and $\mathbf{p}_2$. Now we have

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ \exp\left[-i \int d^4x\, \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i\neq j} (\Phi^i)^2 (\Phi^j)^2 \right) \right] \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

The first term that contributes to leading order is, by analogy to (4.92),

$$\begin{aligned} &{}_0\langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ -i \int d^4x\, \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i\neq j} (\Phi^i)^2 (\Phi^j)^2 \right) \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 \\ &\qquad = {}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i \int d^4x\, \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i\neq j} (\Phi^i)^2 (\Phi^j)^2 \right) + \text{contractions} \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 , \end{aligned}$$

but only the terms in which none of the fields are contracted with each other will contribute [1, p. 111].

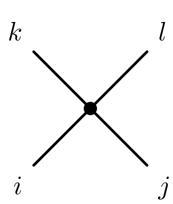
The first term represents the process $\Phi^i \Phi^i \to \Phi^i \Phi^i$:

$${}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i \int d^4x\, \frac{\lambda}{4} \sum_i \Phi^i \Phi^i \Phi^i \Phi^i + \text{contractions} \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

The fields are all the same, so there are 4! ways of contracting the fields with the momenta, and we will obtain a diagram similar to (4.98). Adapting that expression, we find

$$-4!i \int \frac{\lambda}{4}\, d^4x\, e^{-i(p_\mathcal{A}+p_\mathcal{B}-p_1-p_2)\cdot x} = -6i\lambda (4\pi)^4 \delta^4(p_\mathcal{A} + p_\mathcal{B} - p_1 - p_2).$$

The diagram in Eq. (2) is $\Phi^i \Phi^j \to \Phi^k \Phi^l$. Since $i = j = k = l$ for this term, we have



$$= -6i\lambda = -2i\lambda(1+1+1) = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

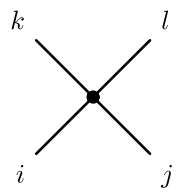
The second term can represent the processes $\Phi^i \Phi^i \to \Phi^j \Phi^j$ or $\Phi^i \Phi^j \to \Phi^i \Phi^j$ (where the indices and the order of the fields on either side is interchangeable):

$${}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i \int d^4x\, \frac{\lambda}{4} 2 \sum_{i\neq j} \Phi^i \Phi^i \Phi^j \Phi^j + \text{contractions} \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

Now there are only $2! \times 2! = 4$ ways to contract the fields with the momenta. We have

$$-4i \int \frac{\lambda}{2} d^4x \, e^{-i(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)\cdot x} = -2i\lambda(4\pi)^4\delta^4(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2).$$

Here, either $i = j$ and $k = l$, $i = l$ and $j = k$, or $i = k$ and $j = l$. We have



$$= -2i\lambda = -2i\lambda(1+0+0) = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

Both of the terms can therefore be represented by Eq. (2) as we wanted to show. □

When all four of the particles in the interaction have the same mass, the differential cross section in the center-of-mass frame is given by Peskin & Schroeder (4.85)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2},$$

where E_{cm} is the center-of-mass energy and $\mathcal{M}$ is the invariant matrix element. We know that the diagrams we calculated before have the form $i\mathcal{M}(2\pi)^4\delta^4(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)$ [1, p. 112]. Then

$$\mathcal{M} = -6\lambda \quad \text{for} \quad \Phi^1\Phi^1 \to \Phi^1\Phi^1, \qquad \mathcal{M} = -2\lambda \quad \text{for} \quad \Phi^1\Phi^2 \to \Phi^1\Phi^2 \text{ and } \Phi^1\Phi^1 \to \Phi^2\Phi^2.$$

So the differential cross sections are, to leading order in λ,

$$(\Phi^1\Phi^2 \to \Phi^1\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-2\lambda|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{\lambda^2}{16\pi^2 E_{\text{cm}}^2},$$

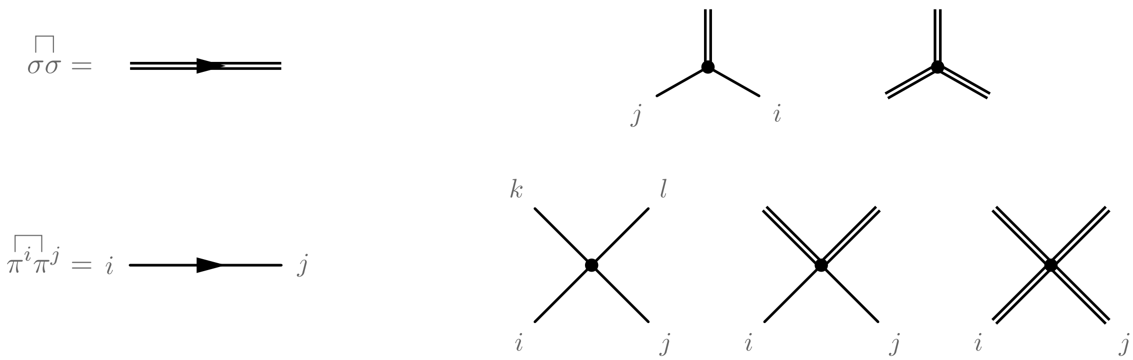
$$(\Phi^1\Phi^2 \to \Phi^1\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-6\lambda|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{9\lambda^2}{16\pi^2 E_{\text{cm}}^2},$$

$$(\Phi^1\Phi^1 \to \Phi^2\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-6\lambda|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{9\lambda^2}{16\pi^2 E_{\text{cm}}^2}.$$

1(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \qquad i = 1, \ldots, N-1, \qquad \Phi^N(x) = v + \sigma(x)$$

where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:



Solution. With the negative mass, Eq. (1) becomes

$$V(\Phi^2) = -\frac{1}{2}\mu^2(\Phi^i)^2 + \frac{\lambda}{4}\left((\Phi^i)^2\right)^2. \tag{4}$$

To find the minimum of V, we differentiate with respect to Φ^N. We stipulate

$$0 = \frac{\partial V}{\partial \Phi^N} = -\mu^2\Phi^N + \lambda(\mathbf{\Phi}\cdot\mathbf{\Phi})\Phi^N = -\mu^2\Phi^N + \lambda(\Phi^N)^3,$$

where we have used the chain rule to evaluate the second term, and the fact that V is minimal for all $\Phi^i = 0$ with $i \neq N$. This implies $\Phi^N = 0$ or

$$(\Phi^N)^2 = \frac{\mu^2}{\lambda} \tag{5}$$

when V is minimal. Thus

$$v = \frac{\mu}{\sqrt{\lambda}}.$$

In order to determine the form of the theory, we need to rewrite $V(\Phi^2)$ in the new coordinates. Note that $\mathbf{\Phi} = (\boldsymbol{\pi}, v+\sigma)$. Then

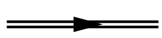
$$\begin{aligned}
V(\Phi^2) &= -\frac{1}{2}\mu^2\left[\boldsymbol{\pi}^2 + (v+\sigma)^2\right] + \frac{\lambda}{4}\left[\boldsymbol{\pi}^2 + (v+\sigma)^2\right]^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2\right) + \frac{\lambda}{4}\left(\boldsymbol{\pi}^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2\right)^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2\right) \\
&\qquad + \frac{\lambda}{4}\left((\boldsymbol{\pi}^2)^2 + 2\frac{\boldsymbol{\pi}^2\mu^2}{\lambda} + \frac{\mu^4}{\lambda^2} + 4\frac{\boldsymbol{\pi}^2\mu\sigma}{\sqrt{\lambda}} + 4\frac{\mu^3\sigma}{\lambda^{3/2}} + 2\boldsymbol{\pi}^2\sigma^2 + 6\frac{\mu^2\sigma^2}{\lambda} + 4\frac{\mu\sigma^3}{\sqrt{\lambda}} + \sigma^4\right) \\
&= -\frac{\boldsymbol{\pi}^2\mu^2}{2} - \frac{\mu^4}{2\lambda} - \frac{\mu^3\sigma}{\sqrt{\lambda}} - \frac{\mu^2\sigma^2}{2} + \frac{(\boldsymbol{\pi}^2)^2\lambda}{4} + \frac{\boldsymbol{\pi}^2\mu^2}{2} + \frac{\mu^4}{4\lambda} \\
&\qquad + \boldsymbol{\pi}^2\mu\sigma\sqrt{\lambda} + \frac{\mu^3\sigma}{\sqrt{\lambda}} + \frac{\boldsymbol{\pi}^2\sigma^2\lambda}{2} + \frac{3\mu^2\sigma^2}{2} + \mu\sigma^3\sqrt{\lambda} + \frac{\sigma^4\lambda}{4} \\
&= \frac{(\boldsymbol{\pi}^2)^2\lambda}{4} + \boldsymbol{\pi}^2\mu\sigma\sqrt{\lambda} + \frac{\boldsymbol{\pi}^2\sigma^2\lambda}{2} + \mu^2\sigma^2 + \mu\sigma^3\sqrt{\lambda} + \frac{\sigma^4\lambda}{4},
\end{aligned}$$

where we have dropped the constant term. This expression includes a $\mu^2\sigma^2$ term, which indicates a massive sigma field. Comparing with Eq. (3), the pion mass is $\sqrt{2}\mu$. However, there is no $\mu^2\boldsymbol{\pi}^2$ term, which indicates that the pion field is massless. The terms of $\mathcal{O}\left(\sqrt{\lambda}\right)$ and $\mathcal{O}(\lambda)$ have factors of $\boldsymbol{\pi}^4$, $\boldsymbol{\pi}^2\sigma$, $\boldsymbol{\pi}^2\sigma^2$, σ^3, and σ^4; these are all cubic and quartic factors. Since they are of $\mathcal{O}\left(\sqrt{\lambda}\right)$ and $\mathcal{O}(\lambda)$, they become small as $\lambda \to 0$. This is what we wanted to show. $\square$

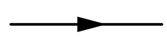
For the propagators, we can use (4.46) of Peskin & Schroeder:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}.$$

Then we can write



$$= \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip\cdot(x-y)}}{p^2 - 2\mu^2 + i\epsilon},$$

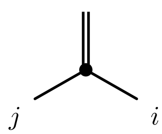
i  j

$$= \delta^{ij} \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip\cdot(x-y)}}{p^2 + i\epsilon}. \tag{6}$$

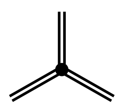
We can associate each of the vertices with a term in $V(\Psi^2)$. The symmetry factors for each of the terms are

$$\boldsymbol{\pi}^2\mu\sigma\sqrt{\lambda} : 2! = 2, \qquad \mu\sigma^3\sqrt{\lambda} : 3! = 6, \qquad \frac{(\boldsymbol{\pi}^2)^2\lambda}{4} : 4! = 24, \qquad \frac{\boldsymbol{\pi}^2\sigma^2\lambda}{2} : 2!2! = 4, \qquad \frac{\sigma^4\lambda}{4} : 4! = 24.$$

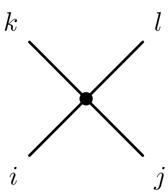
Then the vertices are



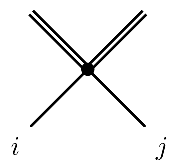
$$= -2i\mu\sqrt{\lambda}\delta^{ij},$$



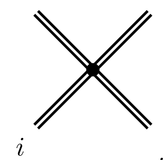
$$= -6i\mu\sqrt{\lambda}, \tag{7}$$



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}),$$



$$= -2i\lambda\delta^{ij},$$

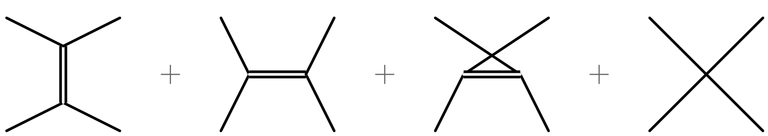


$$= -6i\lambda.$$

1(c) Compute the scattering amplitude for the process

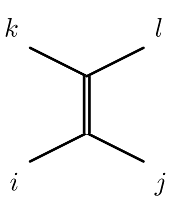
$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:

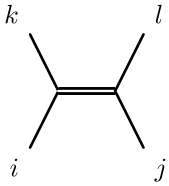


Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. Show that, in the special case $N = 2$ (1 species of pion), the term $\mathcal{O}(p^2)$ also cancels.

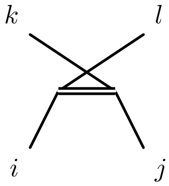
Solution. Using the propagators and vertices of Eqs. (6) and (7), the contributions from each diagram are



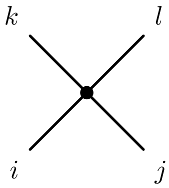
$$= (-2i\mu\sqrt{\lambda}\delta^{ij})\frac{i}{(p_1+p_2)^2 - 2\mu^2}(-2i\mu\sqrt{\lambda}\delta^{kl}) = -\frac{4i\mu^2\lambda}{(p_1+p_2)^2 - 2\mu^2}\delta^{ij}\delta^{kl},$$



$$= (-2i\mu\sqrt{\lambda}\delta^{ik})\frac{i}{(p_1-p_3)^2-2\mu^2}(-2i\mu\sqrt{\lambda}\delta^{jl}) = -\frac{4i\mu^2\lambda}{(p_1-p_3)^2-2\mu^2}\delta^{ik}\delta^{jl},$$



$$= (-2i\mu\sqrt{\lambda}\delta^{il})\frac{i}{(p_1-p_4)^2-2\mu^2}(-2i\mu\sqrt{\lambda}\delta^{jk}) = -\frac{4i\mu^2\lambda}{(p_1-p_4)^2-2\mu^2}\delta^{il}\delta^{jk},$$



$$= -2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}),$$

so the total amplitude is

$$\begin{aligned}
i\mathcal{M} &= -\frac{4i\mu^2\lambda}{(p_1+p_2)^2-2\mu^2}\delta^{ij}\delta^{kl} - \frac{4i\mu^2\lambda}{(p_1-p_3)^2-2\mu^2}\delta^{ik}\delta^{jl} - \frac{4i\mu^2\lambda}{(p_1-p_4)^2-2\mu^2}\delta^{il}\delta^{jk} - 2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) \\
&= -4i\mu^2\lambda\left(\frac{\delta^{ij}\delta^{kl}}{(p_1+p_2)^2-2\mu^2}+\frac{\delta^{ik}\delta^{jl}}{(p_1-p_3)^2-2\mu^2}+\frac{\delta^{il}\delta^{jk}}{(p_1-p_4)^2-2\mu^2}\right) - 2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}),
\end{aligned}$$

where we have referred to (4.119) of Peskin & Schroeder.

When $\mathbf{p}_i = 0$, $p_i = 0$. This is because $p_i = (E_i, \mathbf{p}_i)$ and $m_\pi = 0$, so $E_i = 0$ at zero momentum. Then the amplitude is

$$\begin{aligned}
i\mathcal{M} &= 4i\mu^2\lambda\left(\frac{\delta^{ij}\delta^{kl}}{2\mu^2}+\frac{\delta^{ik}\delta^{jl}}{2\mu^2}+\frac{\delta^{il}\delta^{jk}}{2\mu^2}\right) - 2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) \\
&= 2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) - 2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) \\
&= 0
\end{aligned}$$

as we wanted to show. $\square$

When there is only one species of pion, $i=j=k=l$. So the amplitude is

$$i\mathcal{M} = -4i\mu^2\lambda\left(\frac{1}{(p_1+p_2)^2-2\mu^2}+\frac{1}{(p_1-p_3)^2-2\mu^2}+\frac{1}{(p_1-p_4)^2-2\mu^2}\right) - 6i\lambda.$$

Assuming that the pions have similar momenta, $(p_i \pm p-j)^2 \ll \mu^2$. Let $x = (p_i \pm p-j)^2/\mu^2$. Then the first three terms of $i\mathcal{M}$ each look like

$$\frac{1}{2\mu^2}\frac{1}{x-1} = -\frac{1+x}{2\mu^2} + \mathcal{O}(x^2),$$

where we have performed a Taylor series expansion about 0 to first order in x [2]. Then the amplitude is

$$\begin{aligned}
i\mathcal{M} &= -2i\lambda\left(-1-\frac{(p_1+p_2)^2}{2\mu^2}-1-\frac{(p_1-p_3)^2}{2\mu^2}-1-\frac{(p_1-p_4)^2}{2\mu^2}\right) - 6i\lambda + \mathcal{O}(p^4) \\
&= \frac{i\lambda}{\mu^2}\left[(p_1+p_2)^2+(p_1-p_3)^2+(p_1-p_4)^2\right] + \mathcal{O}(p^4). \qquad (8)
\end{aligned}$$

From Peskin & Schroeder (5.69), we can write the Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \qquad t = (p_3 - p_1)^2 = (p_4 - p_2)^2, \qquad u = (p_4 - p_1)^2 = (p_3 - p_2)^2.$$

For four particles of the same mass m [1, p. 159],

$$s + t + u = 4m^2.$$

The pion has mass $m_\pi = 0$, so $s + t + u = 0$ for the pions. Substituting the Mandelstam variables into Eq. (8), we have

$$i\mathcal{M} = \frac{i\lambda}{\mu^2}\left[s + t + u\right] + \mathcal{O}\big(p^4\big) = \mathcal{O}\big(p^4\big).$$

So we have shown that the $\mathcal{O}\big(p^2\big)$ terms cancel, as desired. □

1(d) Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. Find the new value of v that minimizes V, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to a.

Solution. With this term, Eq. (4) becomes

$$V(\Phi^2) = -\frac{1}{2}\mu^2(\Phi^i)^2 + \frac{\lambda}{4}\left((\Phi^i)^2\right)^2 - a\Phi^N.$$

Differentiating with respect to Φ^N as in 1(b) to find the minimum, we stipulate that

$$0 = \frac{\partial V}{\partial \Phi^N} = -\mu^2\Phi^N + \lambda(\mathbf{\Phi}\cdot\mathbf{\Phi})\Phi^N - a = -\mu^2\Phi^N + \lambda(\Phi^N)^3 - a. \tag{9}$$

We know Φ^N is a function of a. Let $\Phi^N = F(a)$. Since a is small, we can Taylor expand to first order about $a = 0$:

$$\Phi^N \approx F(0) + a\left.\frac{\partial F(a)}{\partial a}\right|_{a=0} = \sqrt{\frac{\mu^2}{\lambda}} + aF'(0),$$

where we have used Eq. (5) as $F(0)$. Then (9) becomes

$$\begin{aligned}
0 &= -\mu^2\left(\sqrt{\frac{\mu^2}{\lambda}} + aF'(0)\right) + \lambda\left(\sqrt{\frac{\mu^2}{\lambda}} + aF'(0)\right)^3 - a \\
&= -\mu^2\sqrt{\frac{\mu^2}{\lambda}} - \mu^2 aF'(0) + \lambda a^3[F'(0)]^2 + 3a\mu^2F'(0) + 2a^2[F'(0)]^2\sqrt{\frac{\mu^2}{\lambda}} + \mu^2\sqrt{\frac{\mu^2}{\lambda}} - a \\
&\approx 2a\mu^2F'(0) - a,
\end{aligned}$$

where we have dropped terms of $\mathcal{O}\big(a^2\big)$. This implies $F'(0) = 0$ or

$$F'(0) = \frac{1}{2\mu^2},$$

when V is mininal, and thus

$$v = \sqrt{\frac{\mu^2}{\lambda}} + \frac{a}{2\mu^2}.$$

In the new coordinates, $V(\Phi^2)$ can be written

$$
\begin{aligned}
V(\Phi^2) &= -\frac{1}{2}\mu^2\left[\boldsymbol{\pi}^2+(v+\sigma)^2\right]+\frac{\lambda}{4}\left[\boldsymbol{\pi}^2+(v+\sigma)^2\right]^2-a(v+\sigma)\\
&= -\frac{1}{2}\mu^2\left[\boldsymbol{\pi}^2+\left(\sqrt{\frac{\mu^2}{\lambda}}+\frac{a}{2\mu^2}+\sigma\right)^2\right]+\frac{\lambda}{4}\left[\boldsymbol{\pi}^2+\left(\sqrt{\frac{\mu^2}{\lambda}}+\frac{a}{2\mu^2}+\sigma\right)^2\right]^2-a\left(\sqrt{\frac{\mu^2}{\lambda}}+\frac{a}{2\mu^2}+\sigma\right)\\
&= \frac{3a^2\lambda\sigma^2}{8\mu^4}+\frac{a^3\lambda\sigma}{8\mu^6}+\frac{3a^2\lambda\sigma}{4\mu^3}\sqrt{\frac{\mu^2}{\lambda}}+\frac{a^4\lambda}{64\mu^8}+\frac{a^3\lambda}{8\mu^6}\sqrt{\frac{\mu^2}{\lambda}}+\frac{a^2\lambda\boldsymbol{\pi}^2}{8\mu^4}-\frac{a^2}{4\mu^2}+\frac{a\lambda\sigma^3}{2\mu^2}+\frac{3a\lambda\sigma^2}{2\mu^2}\sqrt{\frac{\mu^2}{\lambda}}+\frac{a\lambda\boldsymbol{\pi}^2\sigma}{2\mu^2}\\
&\qquad+\frac{a\lambda\boldsymbol{\pi}^2}{2\mu^2}\sqrt{\frac{\mu^2}{\lambda}}-a\sqrt{\frac{\mu^2}{\lambda}}+\lambda\sigma^3\sqrt{\frac{\mu^2}{\lambda}}+\lambda\boldsymbol{\pi}^2\sigma\sqrt{\frac{\mu^2}{\lambda}}-\frac{\mu^4}{4\lambda}+\frac{\lambda\sigma^4}{4}+\frac{\lambda\boldsymbol{\pi}^2\sigma^2}{2}+\frac{\lambda(\boldsymbol{\pi}^2)^2}{4}+\mu^2\sigma^2\\
&\approx \frac{a\lambda\sigma^3}{2\mu^2}+\frac{3a\lambda\sigma^2}{2\mu^2}\sqrt{\frac{\mu^2}{\lambda}}+\frac{a\lambda\boldsymbol{\pi}^2\sigma}{2\mu^2}+\frac{a\lambda\boldsymbol{\pi}^2}{2\mu^2}\sqrt{\frac{\mu^2}{\lambda}}+\lambda\sigma^3\sqrt{\frac{\mu^2}{\lambda}}\\
&\qquad+\lambda\boldsymbol{\pi}^2\sigma\sqrt{\frac{\mu^2}{\lambda}}+\frac{\lambda\sigma^4}{4}+\frac{\lambda\boldsymbol{\pi}^2\sigma^2}{2}+\frac{\lambda(\boldsymbol{\pi}^2)^2}{4}+\mu^2\sigma^2\\
&= \frac{\lambda}{4}(\boldsymbol{\pi}^2)^2+\frac{a}{2}\sqrt{\frac{\lambda}{\mu^2}}\boldsymbol{\pi}^2+\lambda\boldsymbol{\pi}^2\sigma\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)+\frac{\lambda\boldsymbol{\pi}^2\sigma^2}{2}\\
&\qquad+\sigma^2\left(\mu^2+\frac{3a}{2}\sqrt{\frac{\lambda}{\mu^2}}\right)+\lambda\sigma^3\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)+\frac{\lambda\sigma^4}{4},
\end{aligned}
$$

where we have dropped constant terms and terms of $\mathcal{O}(a^2)$. Since the term

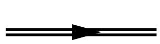
$$\frac{a}{2}\sqrt{\frac{\lambda}{\mu^2}}\boldsymbol{\pi}^2=\frac{m_\pi^2}{2}\boldsymbol{\pi}^2 \tag{10}$$

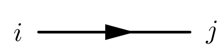
appears, we can say that the pion acquires a mass such that $m_\pi^2\sim a$. $\square$

Note also that the sigma mass has changed:

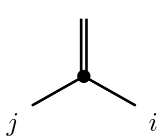
$$\mu^2+\frac{3a}{2}\sqrt{\frac{\lambda}{\mu^2}}=\frac{m_\sigma^2}{2}. \tag{11}$$

The Feynman rules are

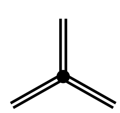

$= \int\frac{d^4p}{(2\pi)^3}\frac{ie^{-ip\cdot(x-y)}}{p^2-m_\sigma^2+i\epsilon}$,

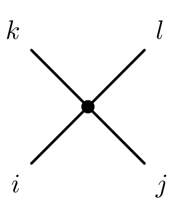


$= \delta^{ij}\int\frac{d^4p}{(2\pi)^3}\frac{ie^{-ip\cdot(x-y)}}{p^2-m_\pi^2+i\epsilon}$,

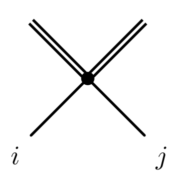


$= -2i\lambda\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{ij}$,

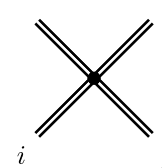

$= -6i\lambda\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)$,



$= -2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl})$,

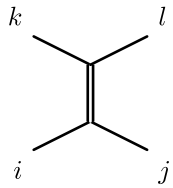


$= -2i\lambda\delta^{ij}$,



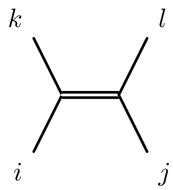
$= -6i\lambda$.

Using the new Feynman rules, the contributions from each diagram are



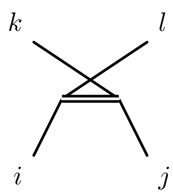
$$= \left[-2i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{ij}\right]\frac{i}{(p_1+p_2)^2-m_\sigma^2}\left[-2i\lambda\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{kl}\right]$$

$$= -\frac{4i\lambda^2}{(p_1+p_2)^2-m_\sigma^2}\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2\delta^{ij}\delta^{kl},$$



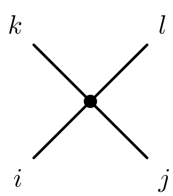
$$= \left[-2i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{ik}\right]\frac{i}{(p_1-p_3)^2-m_\sigma^2}\left[-2i\lambda\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{jl}\right]$$

$$= -\frac{4i\lambda^2}{(p_1-p_3)^2-m_\sigma^2}\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2\delta^{ik}\delta^{jl},$$



$$= \left[-2i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{il}\right]\frac{i}{(p_1-p_4)^2-m_\sigma^2}\left[-2i\lambda\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)\delta^{jk}\right]$$

$$= -\frac{4i\lambda^2}{(p_1-p_4)^2-m_\sigma^2}\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2\delta^{il}\delta^{jk},$$



$$= -2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}),$$

and the total amplitude for their sum is

$$\begin{aligned} i\mathcal{M} = -4i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2&\left(\frac{\delta^{ij}\delta^{kl}}{(p_1+p_2)^2-m_\sigma^2}+\frac{\delta^{ik}\delta^{jl}}{(p_1-p_3)^2-m_\sigma^2}+\frac{\delta^{il}\delta^{jk}}{(p_1-p_4)^2-m_\sigma^2}\right)\\ &-2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}). \end{aligned}$$

When $\mathbf{p}_i = 0$ at the threshold, $p_i = (m_\pi, 0)$. With this substitution,

$$\begin{aligned} i\mathcal{M} &= -4i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2\left(\frac{\delta^{ij}\delta^{kl}}{(m_\pi+m_\pi)^2-m_\sigma^2}+\frac{\delta^{ik}\delta^{jl}}{(m_\pi-m_\pi)^2-m_\sigma^2}+\frac{\delta^{il}\delta^{jk}}{(m_\pi-m_\pi)^2-m_\sigma^2}\right)\\ &\qquad -2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl})\\ &= -4i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2\left(\frac{\delta^{ij}\delta^{kl}}{4m_\pi^2-m_\sigma^2}-\frac{\delta^{ik}\delta^{jl}}{m_\sigma^2}-\frac{\delta^{il}\delta^{jk}}{m_\sigma^2}\right)-2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}). \end{aligned}$$

We can write m_π and m_σ in terms of a using Eqs. (10) and (11), and drop terms of $\mathcal{O}(a^2)$:

$$\begin{aligned}
i\mathcal{M} &= -4i\lambda^2\left(\frac{a}{2\mu^2}+\sqrt{\frac{\mu^2}{\lambda}}\right)^2\left[\delta^{ij}\delta^{kl}\left(4a\sqrt{\frac{\lambda}{\mu^2}}-2\mu^2-3a\sqrt{\frac{\lambda}{\mu^2}}\right)^{-1}-(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})\left(2\mu^2+3a\sqrt{\frac{\lambda}{\mu^2}}\right)^{-1}\right] \\
&\qquad\qquad - 2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) \\
&\approx -4i\lambda^2\left(\frac{\mu^2}{\lambda}+\frac{a}{\mu\sqrt{\lambda}}+\frac{a^2}{4\mu^4}\right)\left(\frac{\delta^{ij}\delta^{kl}}{a\sqrt{\lambda}/\mu-2\mu^2}-\frac{\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}}{3a\sqrt{\lambda}/\mu+2\mu^2}\right)-2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) \\
&\approx 4i\lambda^2\left(\frac{\mu^2}{\lambda}+\frac{a}{\mu\sqrt{\lambda}}\right)\left(\frac{\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}}{2\mu^2}\right)-2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}) \\
&= \frac{2ia\lambda^{3/2}}{\mu^3}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}),
\end{aligned}$$

which is nonzero and proportional to a as we wanted to show. □

Problem 2. Rutherford scattering (Peskin & Schroeder 4.4) The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

2(a) Show that the T-matrix element for electron scattering off a localized classical potential is, to lowest order,

$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\cdot\tilde{A}_\mu(p'-p), \tag{12}$$

where $\tilde{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

Solution. We use Peskin & Schroder (4.94) and the fermion contractions on p. 118 to write

$$\overbrace{\psi\,|\mathbf{p}}, s\rangle = u^s(p)e^{-ip\cdot x}, \qquad\qquad \langle\mathbf{p},s|\,\bar{\psi} = \bar{u}^s(p)e^{ip\cdot x}.$$

We again apply (4.90) in Peskin and Schroeder and the knowledge that only the terms in which none of the fields are contracted with each other will contribute [1, p. 111]. Since A_μ is classical, it cannot be contracted. The matrix element is

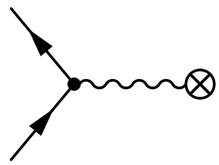
$$\begin{aligned}
\langle p'|iT|p\rangle &= {}_0\langle p'|T\left(-ie\int d^4x\,\bar{\psi}\gamma^\mu\psi A_\mu(x)\right)|p\rangle_0 \\
&= -ie\int d^4x\,{}_0\langle p'|\overbrace{\bar{\psi}\gamma^\mu\psi}|p\rangle_0 A_\mu(x) \\
&= -ie\int d^4x\,\bar{u}(p')e^{ip'\cdot x}\gamma^\mu u(p)e^{-ip\cdot x}A_\mu(x) \\
&= -ie\bar{u}(p')\gamma^\mu u(p)\int d^4x\, e^{i(p'-p)\cdot x}A_\mu(x) \\
&= -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p),
\end{aligned}$$

as desired. □

2(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p'|iT|p\rangle \equiv i\mathcal{M}\cdot(2\pi)\delta(E_f - E_i), \tag{13}$$

where E_i and E_f are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:



$$= -ie\gamma^\mu \tilde{A}_\mu(\mathbf{q}),$$

where $\tilde{A}_\mu(\mathbf{q})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-independent, localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2(2\pi)\delta(E_f - E_i),$$

where v_i is the particle's initial velocity. This formula is a natural modification of (4.79). Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.

Solution. Equation (13) resembles Peskin & Schroeder (4.73). In order to calculate the cross section, we first need to compute the probability for the initial state to scatter into the final state. Peskin & Schroeder (4.74) gives the general probability for scattering into a state of n particles:

$$\mathcal{P}(\mathcal{AB} \to 12\ldots n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|{}_{\text{out}}\langle \mathbf{p}_1 \ldots \mathbf{p}_n|\phi_\mathcal{A}\phi_\mathcal{B}\rangle_{\text{in}}\right|^2, \tag{14}$$

where $|\phi_\mathcal{A}\phi_\mathcal{B}\rangle_{\text{in}}$ is the initial state given by (4.68):

$$|\phi_\mathcal{A}\phi_\mathcal{B}\rangle_{\text{in}} = \int \frac{d^3k_\mathcal{A}}{(2\pi)^3}\int \frac{d^3k_\mathcal{B}}{(2\pi)^3}\frac{\phi_\mathcal{A}(\mathbf{k}_\mathcal{A})\phi_\mathcal{B}(\mathbf{k}_\mathcal{B})e^{-i\mathbf{b}\cdot\mathbf{k}_\mathcal{B}}}{\sqrt{(2E_\mathcal{A})(2E_\mathcal{B})}}\,|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle_{\text{in}}.$$

In this expression, $\mathcal{A}$ represents the target and $\mathcal{B}$ the incoming particle. But for the Rutherford scattering problem, we do not explicitly consider $\mathcal{A}$, only the electron $\mathcal{B}$. Letting $k_\mathcal{B} \to p$, our initial state is

$$|\psi\rangle_{\text{in}} = \int \frac{d^3p}{(2\pi)^3}\frac{\psi(\mathbf{p})e^{-i\mathbf{b}\cdot\mathbf{p}}}{\sqrt{2E_\mathbf{p}}}\,|\mathbf{p}\rangle_{\text{in}}.$$

The out state is also the electron, now with 4-momentum p'. So our adaptation of Eq. (14) is

$$\begin{aligned}
\mathcal{P} &= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\left|{}_{\text{out}}\langle \mathbf{p}'|\psi\rangle_{\text{in}}\right|^2 \\
&= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\left|{}_{\text{out}}\langle \mathbf{p}'|\int \frac{d^3p}{(2\pi)^3}\frac{\psi(\mathbf{p})e^{-i\mathbf{b}\cdot\mathbf{p}}}{\sqrt{2E_\mathbf{p}}}|\mathbf{p}\rangle_{\text{in}}\right|^2 \\
&= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\left|{}_{\text{out}}\langle \mathbf{p}'|\int \frac{d^3p}{(2\pi)^3}\frac{\psi(\mathbf{p})e^{-i\mathbf{b}\cdot\mathbf{p}}}{\sqrt{2E_\mathbf{p}}}|\mathbf{p}\rangle_{\text{in}}\right|^2 \\
&= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\int \frac{d^3p}{(2\pi)^3}\frac{\psi(\mathbf{p})e^{-i\mathbf{b}\cdot\mathbf{p}}}{\sqrt{2E_\mathbf{p}}}\,{}_{\text{out}}\langle \mathbf{p}'|\mathbf{p}\rangle_{\text{in}}\int \frac{d^3\bar{p}}{(2\pi)^3}\frac{\psi^*(\bar{\mathbf{p}})e^{i\mathbf{b}\cdot\bar{\mathbf{p}}}}{\sqrt{2\bar{E}_\mathbf{p}}}\,{}_{\text{out}}\langle \mathbf{p}'|\bar{\mathbf{p}}\rangle_{\text{in}}^* \\
&= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}'}}\int \frac{d^3p\,d^3\bar{p}}{(2\pi)^6}\frac{\psi(\mathbf{p})\psi^*(\bar{\mathbf{p}})e^{i\mathbf{b}\cdot(\bar{\mathbf{p}}-\mathbf{p})}}{2\sqrt{E_\mathbf{p}\bar{E}_\mathbf{p}}}\,\langle \mathbf{p}'|S|\mathbf{p}\rangle\,\langle \mathbf{p}'|S|\bar{\mathbf{p}}\rangle^*,
\end{aligned}$$

where $\bar{p}$ is a real integration variable, and we have used Peskin & Schroeder (4.71):

$$_{\text{out}}\langle \mathbf{p}_1 \dots \mathbf{p}_n | \mathbf{k}_{\mathcal{A}} \mathbf{k}_{\mathcal{B}} \rangle_{\text{in}} \equiv \langle \mathbf{p}_1 \dots \mathbf{p}_n | S | \mathbf{k}_{\mathcal{A}} \mathbf{k}_{\mathcal{B}} \rangle .$$

We note that $S = \mathbf{1} + iT$ by (4.72), the nontrivial part of the matrix element is given by Eq. (13) [1, pp. 104–105]. This gives us

$$\begin{aligned}
\mathcal{P} &= \frac{d^3p'}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}'}} \int \frac{d^3p\, d^3\bar{p}}{(2\pi)^6} \frac{\psi(\mathbf{p})\psi^*(\bar{\mathbf{p}})e^{i\mathbf{b}\cdot(\bar{\mathbf{p}}-\mathbf{p})}}{2\sqrt{E_{\mathbf{p}}\bar{E}_{\mathbf{p}}}} \left[i\mathcal{M}(2\pi)\delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) \right] \left[i\mathcal{M}(2\pi)\delta(E_{\mathbf{p}'} - \bar{E}_{\mathbf{p}}) \right]^* \\
&= \frac{d^3p'}{2\pi} \frac{1}{2E_{\mathbf{p}'}} \int \frac{d^3p\, d^3\bar{p}}{(2\pi)^6} \frac{\psi(\mathbf{p})\psi^*(\bar{\mathbf{p}})e^{i\mathbf{b}\cdot(\bar{\mathbf{p}}-\mathbf{p})}}{2\sqrt{E_{\mathbf{p}}\bar{E}_{\mathbf{p}}}} |\mathcal{M}|^2 \delta(E_{\mathbf{p}'} - E_{\mathbf{p}})\delta(E_{\mathbf{p}'} - \bar{E}_{\mathbf{p}}).
\end{aligned}$$

The cross section is defined by (4.75),

$$\sigma = \int d^2b\, \mathcal{P}(\mathbf{b}).$$

The infinitesimal cross section is then [1, p. 105]

$$\begin{aligned}
d\sigma &= \frac{d^3p'}{2\pi} \frac{1}{2E_{\mathbf{p}'}} \int d^2b \int \frac{d^3p\, d^3\bar{p}}{(2\pi)^6} \frac{\psi(\mathbf{p})\psi^*(\bar{\mathbf{p}})e^{i\mathbf{b}\cdot(\bar{\mathbf{p}}-\mathbf{p})}}{2\sqrt{E_{\mathbf{p}}\bar{E}_{\mathbf{p}}}} |\mathcal{M}|^2 \delta(E_{\mathbf{p}'} - E_{\mathbf{p}})\delta(E_{\mathbf{p}'} - \bar{E}_{\mathbf{p}}) \\
&= \frac{d^3p'}{2\pi} \frac{1}{2E_{\mathbf{p}'}} \int \frac{d^3p\, d^3\bar{p}}{(2\pi)^6} \frac{\psi(\mathbf{p})\psi^*(\bar{\mathbf{p}})}{2\sqrt{E_{\mathbf{p}}\bar{E}_{\mathbf{p}}}} |\mathcal{M}|^2 \delta(E_{\mathbf{p}'} - E_{\mathbf{p}})\delta(E_{\mathbf{p}'} - \bar{E}_{\mathbf{p}})(2\pi)^2\delta^2(p_\perp - \bar{p}_\perp).
\end{aligned}$$

Peskin & Schroeder (4.77) gives

$$\int d\bar{k}^z_{\mathcal{A}}\, d\bar{k}^z_{\mathcal{B}}\, \delta\Big(\bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} - \sum p^z_f\Big)\, \delta\Big(\bar{E}_{\mathcal{A}} + \bar{E}_{\mathcal{B}} + \sum E_f\Big) = \frac{1}{|v_{\mathcal{A}} - v_{\mathcal{B}}|}.$$

Using this, we find

$$d\sigma = \frac{d^3p'}{(2\pi)^3} \frac{2\pi}{v_{\mathbf{p}}} \frac{1}{2E_{\mathbf{p}'}} \int \frac{d^3p_f}{(2\pi)^3} \frac{|\psi(\mathbf{p})|^2}{2E_{\mathbf{p}}} |\mathcal{M}|^2 \delta(E_{\mathbf{p}'} - E_{\mathbf{p}}).$$

Then, pulling everything except the delta function outside the integral, and using the normalization condition [1, pp. 102, 106]

$$\int \frac{d^3k}{(2\pi)^3} |\psi(\mathbf{k})|^2 = 1,$$

we can write

$$d\sigma = \frac{d^3p'}{(2\pi)^3} \frac{2\pi}{v_{\mathbf{p}}} \frac{1}{2E_{\mathbf{p}'}} \frac{1}{2E_{\mathbf{p}}} |\mathcal{M}|^2 \delta(E_{\mathbf{p}'} - E_{\mathbf{p}}).$$

Noting that the subscripts $\mathbf{p} \leftrightarrow i$ and $\mathbf{p}' \leftrightarrow f$, we have

$$d\sigma = \frac{1}{v_i} \frac{1}{2E_i} \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} |\mathcal{M}(p_i \to p_f)|^2 (2\pi)\delta(E_f - E_i)$$

as we wanted to show. □

Integrating over $|p_f|$, we have

$$\begin{aligned} d\sigma &= \frac{1}{v_i}\frac{1}{2E_i}\int d^3p_f\, p_f^2\, d\Omega\, \frac{1}{2E_f}|\mathcal{M}|^2(2\pi)\delta(E_f - E_i) \\ &= \frac{1}{v_i}\frac{1}{2E_i}\int d^3p_f\, p_f^2\, d\Omega\, \frac{1}{2E_f}|\mathcal{M}|^2(2\pi)\frac{E_i}{p_i}\delta(p_f - p_i) \\ &= \frac{1}{v_i}\frac{1}{2E_i}\, d\Omega\, \frac{p_i^2}{2E_i}\frac{|\mathcal{M}|^2}{(2\pi)^2}\frac{E_i}{p_i} \\ &= d\Omega\, \frac{|\mathcal{M}|^2}{4(2\pi)^2}\frac{p_i}{v_iE_i} \\ &= d\Omega\, \frac{|\mathcal{M}|^2}{4(2\pi)^2} \end{aligned}$$

where we have followed the steps in (4.81) and (4.82). Thus,

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2}. \tag{15}$$

2(c) Specialize to the case of electron scattering from a Coulomb potential ($A^0 = Ze/4\pi r$). Working in the nonrelativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4\sin^4(\theta/2)}.$$

Solution. From Eq. (12),

$$\mathcal{M} = -ie\bar{u}(p')\gamma^\mu u(p)\cdot \tilde{A}_\mu(p'-p).$$

The Coulomb potential is similar to the Yukawa potential of Peskin & Schroeder (4.127), whose Fourier transform is given by (4.125):

$$V(r) = -\frac{g^2}{4\pi}\frac{e^{-m_\phi r}}{r}, \qquad \tilde{V}(\mathbf{q}) = -\frac{g^2}{|\mathbf{q}|^2 + m_\phi^2}.$$

When $m_\phi = 0$, this has the same form as the Coulomb potential. Thus

$$\tilde{A}_0(\mathbf{k}) = \frac{Ze}{4\pi}\frac{4\pi}{|\mathbf{k}|^2} = \frac{Ze}{|\mathbf{k}|^2}.$$

So our matrix element is given by

$$\mathcal{M} = -ie\bar{u}(p')\gamma^0 u(p)\tilde{A}_0(\mathbf{p}'-\mathbf{p}) = -\frac{ie^2Z}{(\mathbf{p}'-\mathbf{p})^2}\bar{u}(p')\gamma^0u(p).$$

In the nonrelativistic limit,

$$\bar{u}(p')\gamma^0u(p) \approx 2m\xi'^\dagger\xi = 2m\delta^{ss'}$$

since $\xi^{s'\dagger}\xi^s = \delta^{ss'}$, where s and s' are the initial and final spin states [1, pp. 121, 125]. The matrix element can thus be written as

$$\mathcal{M} = -\frac{2ie^2mZ}{(\mathbf{p}'-\mathbf{p})^2}\delta^{ss'}.$$

Feeding this into Eq. (15),

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\left|-\frac{2ie^2mZ}{(p'-p)^2}\delta^{ss'}\right|^2 = \frac{e^4m^2Z^2}{4\pi^2(\mathbf{p}'-\mathbf{p})^4} = \frac{4\alpha^2m^2Z^2}{(\mathbf{p}'-\mathbf{p})^4},$$

where we require that $s = s'$, so $\delta^{ss'} = 1$, and we have used $\alpha = e^2/4\pi$ [1, p. 126]. Let θ be the angle between $\mathbf{p}$ and $\mathbf{p}'$. Note that

$$(\mathbf{p}'-\mathbf{p})^2 = \mathbf{p}'^2 - 2\mathbf{p}'\cdot\mathbf{p} + \mathbf{p}^2 = \mathbf{p}'^2 - 2|\mathbf{p}'||\mathbf{p}|\cos\theta + \mathbf{p}^2.$$

Since the momenta are very small compared to the energy in the nonrelativistic limit, $\mathbf{p} \approx \mathbf{p}'$. Also in this limit, $p = mv$. This gives us

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha^2m^2Z^2}{[2(\mathbf{p}^2 - \mathbf{p}^2\cos\theta)]^2} = \frac{\alpha^2m^2Z^2}{\mathbf{p}^4(1-\cos\theta)^2} = \frac{\alpha^2m^2Z^2}{4\mathbf{p}^4\sin^4(\theta/2)} = \frac{\alpha^2m^2Z^2}{4m^4v^4\sin^4(\theta/2)} = \frac{\alpha^2Z^2}{4m^2v^4\sin^4(\theta/2)}$$

as desired. □

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] E. W. Weisstein, "Maclaurin Series." From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/MaclaurinSeries.html`.