1 Problem 1

Consider operators J and K acting in a three-dimensional space as

$$J|e_1\rangle = i|e_2\rangle, \qquad J|e_2\rangle = -i|e_1\rangle, \qquad J|e_3\rangle = 0, \tag{1}$$
$$K|e_1\rangle = 0, \qquad K|e_2\rangle = i|e_3\rangle, \qquad K|e_3\rangle = -i|e_2\rangle, \tag{2}$$

where $|e_1\rangle, |e_2\rangle, |e_3\rangle$ form a complete orthonormal basis.

1.1 Compute the matrix elements of J and K.

Solution. The matrix elements of J are

$$J_{11} = \langle e_1|J|e_1\rangle = i\langle e_1|e_2\rangle = 0, \quad J_{12} = \langle e_1|J|e_2\rangle = -i\langle e_1|e_1\rangle = -i, \quad J_{13} = \langle e_1|J|e_3\rangle = 0, \tag{3}$$
$$J_{21} = \langle e_2|J|e_1\rangle = i\langle e_2|e_2\rangle = i, \quad J_{22} = \langle e_2|J|e_2\rangle = -i\langle e_2|e_1\rangle = 0, \quad J_{23} = \langle e_2|J|e_3\rangle = 0, \tag{4}$$
$$J_{31} = \langle e_3|J|e_1\rangle = i\langle e_3|e_2\rangle = 0, \quad J_{32} = \langle e_3|J|e_2\rangle = -i\langle e_3|e_1\rangle = 0, \quad J_{33} = \langle e_3|J|e_3\rangle = 0. \tag{5}$$

The matrix elements of K are

$$K_{11} = \langle e_1|K|e_1\rangle = 0, \quad K_{12} = \langle e_1|K|e_2\rangle = i\langle e_1|e_3\rangle = 0, \quad K_{13} = \langle e_1|K|e_3\rangle = -i\langle e_1|e_2\rangle = 0, \tag{6}$$
$$K_{21} = \langle e_2|K|e_1\rangle = 0, \quad K_{22} = \langle e_2|K|e_2\rangle = i\langle e_2|e_3\rangle = 0, \quad K_{23} = \langle e_2|K|e_3\rangle = -i\langle e_2|e_2\rangle = -i, \tag{7}$$
$$K_{31} = \langle e_3|K|e_1\rangle = 0, \quad K_{32} = \langle e_3|K|e_2\rangle = i\langle e_3|e_3\rangle = i, \quad K_{33} = \langle e_3|K|e_3\rangle = -i\langle e_3|e_2\rangle = 0. \tag{8}$$

1.2 Consider $O = AJ + BK$ where A, B are real numbers. Show that O is Hermitian.

Solution. Using (3)–(8), the matrix elements of O are

$$O_{11} = O_{13} = O_{22} = O_{31} = O_{33} = 0, \tag{9}$$
$$O_{12} = -iA, \tag{10}$$
$$O_{21} = iA, \tag{11}$$
$$O_{23} = -iB, \tag{12}$$
$$O_{32} = iB. \tag{13}$$

O is Hermitian if and only if $O_{ij} = O_{ji}^*$ for all O_{ij}. Recall that $(z^*)^* = z$ for any $z \in \mathbb{C}$. From (9)–(13), note that

$$O_{11} = 0 = O_{11}^*, \tag{14}$$
$$O_{12} = -iA = (iA)^* = O_{21}^*, \tag{15}$$
$$O_{13} = 0 = O_{31}^*, \tag{16}$$
$$O_{22} = 0 = O_{22}^*, \tag{17}$$
$$O_{23} = -iB = (iB)^* = O_{32}^*, \tag{18}$$
$$O_{33} = 0 = O_{33}^*, \tag{19}$$

so O is indeed Hermitian. □

1.3 If $|p_\lambda\rangle$ is an eignevector of O, we have $O|p_\lambda\rangle = \lambda|p_\lambda\rangle$ where λ is the corresponding eigenvalue. $|p_\lambda\rangle$ can be expanded as $|p_\lambda\rangle = \sum i = 1^3 u_{\lambda,i}|e_i\rangle$. Denote the three eigenvalues and the corresponding normalized eigenvectors of O as $\lambda_+, \lambda_0, \lambda_-$ and $|p_+\rangle, |p_0\rangle, |p_-\rangle$ where λ_+ (λ_-) is the largest (smallest) eigenvalue. Find $\lambda_+, \lambda_0, \lambda_-$ and $|p_+\rangle, |p_0\rangle, |p_-\rangle$.

Solution. Using a matrix representation in the $|e_1\rangle, |e_2\rangle, |e_3\rangle$ basis, we can write

$$O = \begin{bmatrix} 0 & -iA & 0 \\ iA & 0 & -iB \\ 0 & iB & 0 \end{bmatrix}. \tag{20}$$

λ is an eigenvalue of O if $\det(O - \lambda I) = 0$, where I is the identity matrix. That is,

$$\begin{aligned}
0 &= \begin{vmatrix} -\lambda & -iA & 0 \\ iA & -\lambda & -iB \\ 0 & iB & -\lambda \end{vmatrix} && (21)\\
&= (-\lambda)^3 - (-\lambda)(-iB)(iB) - (-iA)(iA)(-\lambda) && (22)\\
&= \lambda(\lambda^2 - A^2 - B^2) && (23)\\
&= \lambda^2 - A^2 - B^2. && (24)
\end{aligned}$$

From (23) we obtain $\lambda_0 = 0$, and from (24) we obtain $\lambda_\pm = \pm\sqrt{A^2 + B^2}$.

Let $|\lambda_0\rangle, |\lambda_\pm\rangle$ be the not-necessarily-normalized eigenvectors corresponding to $\lambda_0, \lambda_\pm$. Beginning with λ_0, we will find the corresponding eigenvector $|\lambda_0\rangle = \lambda_{0,1}|e_1\rangle + \lambda_{0,2}|e_2\rangle + \lambda_{0,3}|e_3\rangle$. We seek $\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3}$ such that

$$\begin{bmatrix} 0 & -iA & 0 \\ iA & 0 & -iB \\ 0 & iB & 0 \end{bmatrix} \begin{bmatrix} \lambda_{0,1} \\ \lambda_{0,2} \\ \lambda_{0,3} \end{bmatrix} = 0 \begin{bmatrix} \lambda_{0,1} \\ \lambda_{0,2} \\ \lambda_{0,3} \end{bmatrix}. \tag{25}$$

The algebraic equations corresponding to (25) are

$$\begin{aligned}
-iA\,\lambda_{0,2} &= 0, && (26)\\
iA\,\lambda_{0,1} - iB\,\lambda_{0,3} &= 0, && (27)\\
iB\,\lambda_{0,2} &= 0. && (28)
\end{aligned}$$

(26) and (28) imply that $\lambda_{0,2} = 0$. We may fix $\lambda_{0,3} = A$ without loss of generality. Then (27) implies $\lambda_{0,1} = B$. Thus, $|\lambda_0\rangle = B|e_1\rangle + A|e_3\rangle$.

For $|\lambda_\pm\rangle = \lambda_{\pm,1}|e_1\rangle + \lambda_{\pm,2}|e_2\rangle + \lambda_{\pm,3}|e_3\rangle$, we seek $\lambda_{\pm,1}, \lambda_{\pm,2}, \lambda_{\pm,3}$ such that

$$\begin{bmatrix} 0 & -iA & 0 \\ iA & 0 & -iB \\ 0 & iB & 0 \end{bmatrix} \begin{bmatrix} \lambda_{\pm,1} \\ \lambda_{\pm,2} \\ \lambda_{\pm,3} \end{bmatrix} = \pm\sqrt{A^2 + B^2} \begin{bmatrix} \lambda_{\pm,1} \\ \lambda_{\pm,2} \\ \lambda_{\pm,3} \end{bmatrix}. \tag{29}$$

The algrabraic equations corresponding to (55) are

$$\begin{aligned}
-iA\,\lambda_{\pm,2} &= \pm\sqrt{A^2 + B^2}\,\lambda_{\pm,1}, && (30)\\
iA\,\lambda_{\pm,1} - iB\,\lambda_{\pm,3} &= \pm\sqrt{A^2 + B^2}\,\lambda_{\pm,2}, && (31)\\
iB\,\lambda_{\pm,2} &= \pm\sqrt{A^2 + B^2}\,\lambda_{\pm,3}. && (32)
\end{aligned}$$

Summing (30), (31), and (32), we have

$$\pm\sqrt{A^2+B^2}(\lambda_{\pm,1}+\lambda_{\pm,2}+\lambda_{\pm,3}) = iA(\lambda_{\pm,1}-\lambda_{\pm,2})+iB(\lambda_{\pm,2}-\lambda_{\pm,3}) \tag{33}$$

$$\pm i\sqrt{A^2+B^2}(\lambda_{\pm,1}+\lambda_{\pm,2}+\lambda_{\pm,3}) = A(\lambda_{\pm,2}-\lambda_{\pm,1})-B(\lambda_{\pm,2}-\lambda_{\pm,3}). \tag{34}$$

From the form of (34), we make the ansatz $\lambda_{\pm,1}=-A, \lambda_{\pm,3}=B$. Making the relevant substituions in (30) and (32), we have

$$-iA\,\lambda_{\pm,2} = \pm A\sqrt{A^2+B^2}, \tag{35}$$

$$iB\,\lambda_{\pm,2} = \pm B\sqrt{A^2+B^2} \tag{36}$$

which both imply $\lambda_{\pm,2}=\mp i\sqrt{A^2+B^2}$. Therefore, $|\lambda_\pm\rangle = -A\,|e_1\rangle \mp i\sqrt{A^2+B^2}\,|e_2\rangle + B\,|e_3\rangle$.

Now we will compute $|p_+\rangle, |p_0\rangle, |p_-\rangle$ by normalizing $|\lambda_0\rangle, |\lambda_\pm\rangle$. Note that

$$\|\lambda_0\|^2 = \langle\lambda_0|\lambda_0\rangle = A^2+B^2, \tag{37}$$

$$\|\lambda_\pm\|^2 = \langle\lambda_\pm|\lambda_\pm\rangle = A^2+(A^2+B^2)+B^2 = 2A^2+2B^2, \tag{38}$$

so

$$|p_+\rangle = \frac{|\lambda_+\rangle}{\|\lambda_+\|} = \frac{-A\,|e_1\rangle - i\sqrt{A^2+B^2}\,|e_2\rangle + B\,|e_3\rangle}{\sqrt{2}\sqrt{A^2+B^2}}, \tag{39}$$

$$|p_0\rangle = \frac{|\lambda_0\rangle}{\|\lambda_0\|} = \frac{B\,|e_1\rangle + A\,|e_3\rangle}{\sqrt{A^2+B^2}}, \tag{40}$$

$$|p_-\rangle = \frac{|\lambda_-\rangle}{\|\lambda_-\|} = \frac{-A\,|e_1\rangle + i\sqrt{A^2+B^2}\,|e_2\rangle + B\,|e_3\rangle}{\sqrt{2}\sqrt{A^2+B^2}}. \tag{41}$$

1.4 Define a new state $|e_1'\rangle$ by $|e_1'\rangle = |h_1\rangle/\|h_1\|$ where $\|h_1\| = \sqrt{\langle h_1|h_1\rangle}$ and $|h_1\rangle = (1 - |p_0\rangle\langle p_0|)\,|e_1\rangle$. Find the probability that the state $|e_1'\rangle$ is found to have the eigenvalue $\lambda_+, \lambda_0, \lambda_-$.

Solution. First, we can find an $|e_1'\rangle$ using the result (40) for $|p_0\rangle$. Beginning with $|h_1\rangle$, we have

$$|h_1\rangle = |e_1\rangle - \langle p_0|e_1\rangle\,|p_0\rangle = |e_1\rangle - \frac{B}{\sqrt{A^2+B^2}}\,|p_0\rangle = |e_1\rangle - \frac{B}{\sqrt{A^2+B^2}}\left(\frac{B\,|e_1\rangle + A\,|e_3\rangle}{\sqrt{A^2+B^2}}\right) \tag{42}$$

$$= \left(1-\frac{B^2}{A^2+B^2}\right)|e_1\rangle - \frac{AB}{A^2+B^2}\,|e_3\rangle\,. \tag{43}$$

Then

$$\|h_1\|^2 = \left(1-\frac{B^2}{A^2+B^2}\right)^2 - \left(\frac{AB}{A^2+B^2}\right)^2 = 1 - \frac{2B^2}{A^2+B^2} + \frac{B^4}{(A^2+B^2)^2} - \frac{A^2B^2}{(A^2+B^2)^2} \tag{44}$$

$$= \frac{(A^2+B^2)^2 - 2B^2(A^2+B^2) + B^4 - A^2B^2}{(A^2+B^2)^2} = \frac{A^2(A^2+B^2)}{(A^2+B^2)^2} \tag{45}$$

$$= \frac{A^2}{A^2+B^2} \tag{46}$$

so

$$\left|e_1'\right\rangle = \frac{|h_1\rangle}{\||h_1\|} = \frac{\sqrt{A^2+B^2}}{A}\left[\left(1-\frac{B^2}{A^2+B^2}\right)|e_1\rangle - \frac{AB}{A^2+B^2}|e_3\rangle\right] \tag{47}$$

$$= \frac{A}{\sqrt{A^2+B^2}}|e_1\rangle - \frac{B}{\sqrt{A^2+B^2}}|e_3\rangle \tag{48}$$

The probability that $|e_1'\rangle$ has the eigenvalue λ is $|\langle p_\lambda|e_1'\rangle|^2$. Thus,

$$\left|\langle p_\pm|e_1'\rangle\right|^2 = \left|-\frac{A}{\sqrt{A^2+B^2}}\frac{A}{\sqrt{2}\sqrt{A^2+B^2}} - \frac{B}{\sqrt{A^2+B^2}}\frac{B}{\sqrt{2}\sqrt{A^2+B^2}}\right|^2 = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}, \tag{49}$$

$$\left|\langle p_0|e_1'\rangle\right|^2 = \left|-\frac{A}{\sqrt{A^2+B^2}}\frac{B}{\sqrt{A^2+B^2}} + \frac{B}{\sqrt{A^2+B^2}}\frac{A}{\sqrt{A^2+B^2}}\right|^2 = 0. \tag{50}$$

As expected, $|\langle p_+|e_1'\rangle|^2 + |\langle p_0|e_1'\rangle|^2 + |\langle p_0|e_1'\rangle|^2 = 1$.

2 Problem 2

Consider an operator A acting in a two-dimensional space as

$$A|e_1\rangle = i|e_2\rangle, \quad A|e_2\rangle = -i|e_1\rangle, \tag{51}$$

where $|e_1\rangle, |e_2\rangle$ form a complete orthonormal basis.

2.1 Find the matrix elements A_{ij} $(i,j=1,2)$ of A with respect to $|e_1\rangle, |e_2\rangle$.

Solution. Using (51), the matrix elements of A are

$$A_{11} = \langle e_1|A|e_1\rangle = i\langle e_1|e_2\rangle = 0, \qquad A_{12} = \langle e_1|A|e_2\rangle = -i\langle e_1|e_1\rangle = -i, \tag{52}$$

$$A_{21} = \langle e_2|A|e_1\rangle = i\langle e_2|e_2\rangle = i, \qquad A_{22} = \langle e_2|A|e_2\rangle = -i\langle e_2|e_1\rangle = 0. \tag{53}$$

2.2 The eigenvalues of A are ± 1. Find the corresponding eigenvectors $|e_1'\rangle, |e_2'\rangle$ and represent them in terms of $|e_1\rangle, |e_2\rangle$.

Solution. Using a matrix representation in the $|e_1\rangle, |e_2\rangle$ basis, we can write

$$A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{54}$$

Let $|\lambda_\pm\rangle$ be the not-necessarily-normalized eigenvector corresponding to the eigenvalue ± 1. We seek $\lambda_{\pm 1}, \lambda_{\pm 2}$ such that

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \lambda_{\pm 1} \\ \lambda_{\pm 2} \end{bmatrix} = \pm\begin{bmatrix} \lambda_{\pm 1} \\ \lambda_{\pm 2} \end{bmatrix}. \tag{55}$$

The algebraic equations corresponding to (55) are

$$-i\,\lambda_{\pm 2} = \pm\,\lambda_{\pm 1}, \qquad i\,\lambda_{\pm 1} = \pm\,\lambda_{\pm 2}. \tag{56}$$

By inspection of (56), $\lambda_{\pm 1} = \mp i$ and $\lambda_{\pm 2} = 1$. Thus $|\lambda_{\pm}\rangle = \mp i\,|e_1\rangle + |e_2\rangle$.

Let $|e_1'\rangle$ ($|e_2'\rangle$) be the normalized eigenvector corresponding to eigenvalue 1 (-1). Then

$$\left|e_1'\right\rangle = \frac{|\lambda_+\rangle}{\|\lambda_+\|} = \frac{-i\,|e_1\rangle + |e_2\rangle}{\sqrt{2}}, \qquad \left|e_2'\right\rangle = \frac{|\lambda_-\rangle}{\|\lambda_-\|} = \frac{i\,|e_1\rangle + |e_2\rangle}{\sqrt{2}}. \tag{57}$$

2.3 Let U be the unitary operator such that $|e_i'\rangle = U\,|e_i\rangle$. Find the matrix elements U_{ij} of U with respect to $|e_1\rangle, |e_2\rangle$.

Solution. Using (57), the matrix elements of U are

$$U_{11} = \langle e_1|U|e_1\rangle = \left\langle e_1\middle|e_1'\right\rangle = -\frac{i}{\sqrt{2}}, \qquad U_{12} = \langle e_1|U|e_2\rangle = \left\langle e_1\middle|e_2'\right\rangle = \frac{i}{\sqrt{2}}, \tag{58}$$

$$U_{21} = \langle e_2|U|e_1\rangle = \left\langle e_2\middle|e_1'\right\rangle = \frac{1}{\sqrt{2}}, \qquad U_{22} = \langle e_2|U|e_2\rangle = \left\langle e_2\middle|e_2'\right\rangle = \frac{1}{\sqrt{2}}. \tag{59}$$

U is a unitary operator if and only if $UU^\dagger = U^\dagger U = I$ where I is the identity matrix. Using a matrix representation in the $|e_1\rangle, |e_2\rangle$ basis, we have

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, \qquad U^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \tag{60}$$

so

$$UU^\dagger = \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{61}$$

$$U^\dagger U = \frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{62}$$

so U is indeed unitary.

2.4 Consider the matrix elements of A in the $|e_1'\rangle, |e_2'\rangle$ basis. Represent A_{ij}' using A_{ij} and U_{ij}. (Numerical values of A_{ij}' are not required.)

Solution. Recall that $|e_1\rangle, |e_2\rangle$ form a complete orthonormal basis, so $\sum_{i=1}^{2} |e_i\rangle\langle e_i| = I$. This allows us to write

$$A = \sum_{n=1}^{2}\sum_{m=1}^{2} |e_n\rangle\,\langle e_n|A|e_m\rangle\,\langle e_m| = \sum_{n=1}^{2}\sum_{m=1}^{2} |e_n\rangle\,A_{nm}\,\langle e_m|\,. \tag{63}$$

Then the matrix elements A'_{ij} are

$$A'_{ij} = \langle e'_i | A | e'_j \rangle = \sum_{n=1}^{2} \sum_{m=1}^{2} \left\langle e'_i \middle| e_n \right\rangle A_{nm} \left\langle e_m \middle| e'_j \right\rangle . \tag{64}$$

From (58) and (59) we know that

$$\langle e_m | e'_j \rangle = \langle e_m | U | e_j \rangle = U_{mj}. \tag{65}$$

Similarly,

$$\langle e'_i | e_n \rangle = (\left\langle e_n \middle| e'_i \right\rangle)^* = (\langle e_n | U | e_i \rangle)^* = U^*_{ni}. \tag{66}$$

Making the substitutions (65) and (66), (64) becomes

$$A'_{ij} = \sum_{n=1}^{2} \sum_{m=1}^{2} U^*_{in} A_{nm} U_{mj}. \tag{67}$$

Explicitly in terms of i, j, this is

$$A'_{ij} = U^*_{ii} A_{ii} U_{ij} + U^*_{ii} A_{ij} U_{jj} + U^*_{ij} A_{ji} U_{ij} + U^*_{ij} A_{jj} U_{jj}. \tag{68}$$