

**Problem 1.** Consider the following probabilistic game: There are four doors ( $Q, R, S, T$ ). Behind each door is a device which displays  $\pm 1$  randomly according to the probability  $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$ . Alice and Bob are on the same team. Alice has to choose either  $Q$  and  $R$ , and then Bob has to choose either  $S$  and  $T$ . When the numbers match, they get  $+1$  point; when the numbers do not match, they get  $-1$  point. However, when they open  $Q$  and  $T$ , it's an exception. When the numbers (do not) match, they get  $-1$  ( $+1$ ).

**1.1** Let's assume Alice and Bob open the doors completely randomly. When all numbers are  $+1$  with probability 1, what is the expectation value of the point they get?

**Solution.** Let  $\mathbf{E}$  be the expectation value of the number of points. In this case, the numbers behind the two doors will always match. So

$$\mathbf{E} = \frac{QS + RS + RT - QT}{4} = \frac{1 + 1 + 1 - 1}{4} = \frac{1}{2}.$$

**1.2** As it turns out, irrespective of how hard you fine tune the probability  $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$ , the expectation value of the point Alice and Bob get cannot exceed a certain value Max:

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} \leq \text{Max}.$$

Here,  $\mathbf{E}(QS)$ , etc. is the expectation value of the point when Alice opens  $Q$  and Bob opens  $S$ . This is a Bell inequality. Determine Max.

*Hint:* For a given realization of the numbers  $Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1$ , which occurs with probability  $P(Q, R, S, T)$ , note that  $QS + RS + RT - QT = (Q + R)S + (R - Q)T$ , where one of  $\{(R + Q), (R - Q)\}$  is 2 and the other 0.

**Solution.** In addition to the information provided in the hint, both  $S$  and  $T$  must be  $\pm 1$ . This means the only possibilities for the number of points earned are

$$\frac{(Q + R)S + (R - Q)T}{4} = \begin{cases} \frac{(0)(-1) + (2)(1)}{4} = \frac{1}{2}, \\ \frac{(0)(1) + (2)(-1)}{4} = -\frac{1}{2}. \end{cases}$$

Thus,

$$\text{Max} = \frac{1}{2}.$$

**1.3** Frustrated by the upper bound set by the Bell inequality, Bob decides to cheat. He now changes the value of  $T$  after Alice chooses  $Q$  or  $R$ . Assume  $Q, R, S$  are set to be  $+1$  with probability 1. To make the expectation value of the point they get equal to  $+1$ , what values should Bob set after Alice chooses  $Q$  or  $R$ ?

**Solution.** If Alice chooses  $R$ , Bob should set  $T = 1$ . If Alice chooses  $Q$ , Bob should set  $T = -1$ . This way,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1 + 1 + 1 + 1}{4} = 1.$$

**1.4** Now consider a quantum mechanical version of the game. There are quantum states of two spin-1/2 degrees of freedom shared by Alice and Bob. Alice can measure the  $z$  component or  $x$  components of the first spin  $\mathbf{S}^A$ . (This corresponds to  $Q = \pm 1$  or  $R = \pm 1$ .) Bob can measure the  $-(z + x)$  component or the  $(z - x)$  component of the second spin  $\mathbf{S}^B$ . (This corresponds to  $S = \pm 1$  or  $T = \pm 1$ .)

More specifically, Alice and Bob share the quantum state

$$|\psi\rangle = \frac{|\uparrow_z\rangle \otimes |\downarrow_z\rangle - |\downarrow_z\rangle \otimes |\uparrow_z\rangle}{\sqrt{2}}.$$

The operators to be measured are

$$Q = S_z^A, \quad R = S_x^A, \quad S = -\frac{S_z^B + S_x^B}{\sqrt{2}}, \quad T = \frac{S_z^B - S_x^B}{\sqrt{2}}.$$

Let us consider the case when Alice measures  $Q$  and Bob measures  $T$ . Calculate the probability  $P(Q, T)$  for Alice and Bob getting the measurement outcomes  $(Q, T) = (\pm 1, \pm 1)$ .

**Solution.** From Sakurai (3.9.11), the probability of measuring  $\mathbf{S} \cdot \hat{\mathbf{a}}$  and  $\mathbf{S} \cdot \hat{\mathbf{b}}$  to both be positive is

$$P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{1}{2} \sin^2\left(\frac{\theta_{ab}}{2}\right),$$

where the  $1/2$  comes from the probability of measuring  $\theta_{ab}$  is the angle between the  $\hat{\mathbf{a}}$  and  $\hat{\mathbf{b}}$  directions. For the other combinations, we may generalize this expression using Fig. (3.9) in Sakurai: This gives us

$$P(\hat{\mathbf{a}}-; \hat{\mathbf{b}}-) = \frac{1}{2} \sin^2\left(\frac{\theta_{ab}}{2}\right) = P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+), \quad P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}-) = \frac{1}{2} \sin^2\left(\frac{\theta_{ab} + \pi/2}{2}\right) = \frac{1}{2} \cos^2\left(\frac{\theta_{ab}}{2}\right) = P(\hat{\mathbf{a}}-; \hat{\mathbf{b}}-). \quad (1)$$

For  $Q$  and  $T$ ,  $\theta_{ab} = \pi/4$ . So we have

$$P(Q = \pm 1, T = \pm 1) = \frac{1}{2} \sin^2\left(\frac{\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2} - \sqrt{2}}{2}\right)^2 = \frac{1}{2} \frac{2 - \sqrt{2}}{4} = \frac{2 - \sqrt{2}}{8} \approx 0.073,$$

$$P(Q = \pm 1, T = \mp 1) = \frac{1}{2} \cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2} + \sqrt{2}}{2}\right)^2 = \frac{1}{2} \frac{2 + \sqrt{2}}{4} = \frac{2 + \sqrt{2}}{8} \approx 0.427.$$

**1.5** Similarly, consider the case when Alice measures  $R$  and Bob measures  $T$ . Calculate the probability  $P(R, T)$  for Alice and Bob getting the measurement outcomes  $(R, T) = (\pm 1, \pm 1)$ .

**Solution.** Again applying (??), for  $R$  and  $T$ ,  $\theta_{ab} = 3\pi/4$ . So we have

$$P(R = \pm 1, T = \pm 1) = \frac{1}{2} \sin^2\left(\frac{3\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2} + \sqrt{2}}{2}\right)^2 = \frac{1}{2} \frac{2 + \sqrt{2}}{4} = \frac{2 + \sqrt{2}}{8} \approx 0.427,$$

$$P(R = \pm 1, T = \mp 1) = \frac{1}{2} \cos^2\left(\frac{3\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2} - \sqrt{2}}{2}\right)^2 = \frac{1}{2} \frac{2 - \sqrt{2}}{4} = \frac{2 - \sqrt{2}}{8} \approx 0.073.$$

1.6 Compute the expectation values  $\mathbf{E}(QS)$ ,  $\mathbf{E}(RS)$ ,  $\mathbf{E}(QT)$ , and  $\mathbf{E}(RT)$ . Compute

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4}.$$

**Solution.** We need to find the probabilities of obtaining  $(Q, S) = (\pm 1, \pm 1)$  and  $(R, S) = (\pm 1, \pm 1)$ . For  $Q$  and  $S$ ,  $\theta_{ab} = 3\pi/4$ , so

$$P(Q = \pm 1, S = \pm 1) = P(R = \pm 1, T = \pm 1), \quad P(Q = \pm 1, S = \mp 1) = P(R = \pm 1, T = \mp 1).$$

For  $R$  and  $S$ ,  $\theta_{ab} = 5\pi/4$ , so

$$P(R = \pm 1, S = \pm 1) = \frac{1}{2} \sin^2\left(\frac{5\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2} + \sqrt{2}}{2}\right)^2 = \frac{1}{2} \frac{2 + \sqrt{2}}{4} = \frac{2 + \sqrt{2}}{8} \approx 0.427,$$

$$P(R = \pm 1, S = \mp 1) = \frac{1}{2} \cos^2\left(\frac{5\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2} - \sqrt{2}}{2}\right)^2 = \frac{1}{2} \frac{2 - \sqrt{2}}{4} = \frac{2 - \sqrt{2}}{8} \approx 0.073.$$

The expectation value of a random variable  $X$  is defined

$$E(X) = \sum_i p_i x_i,$$

where  $x_i$  are all of the possible values of  $X$ , and  $p_i$  the probabilities associated with each. Then

$$\begin{aligned} \mathbf{E}(QS) &= 2P(Q = \pm 1, S = \pm 1) - 2P(Q = \pm 1, S = \mp 1) = \frac{2 + \sqrt{2}}{4} - \frac{2 - \sqrt{2}}{4} = \frac{\sqrt{2}}{2}, \\ \mathbf{E}(RS) &= 2P(R = \pm 1, S = \pm 1) - 2P(R = \pm 1, S = \mp 1) = \frac{2 + \sqrt{2}}{4} - \frac{2 - \sqrt{2}}{4} = \frac{\sqrt{2}}{2}, \\ \mathbf{E}(RT) &= 2P(R = \pm 1, T = \pm 1) - 2P(R = \pm 1, T = \mp 1) = \frac{2 + \sqrt{2}}{4} - \frac{2 - \sqrt{2}}{4} = \frac{\sqrt{2}}{2}, \\ \mathbf{E}(QT) &= 2P(Q = \pm 1, T = \pm 1) - 2P(Q = \pm 1, T = \mp 1) = \frac{2 - \sqrt{2}}{4} - \frac{2 + \sqrt{2}}{4} = -\frac{\sqrt{2}}{2}. \end{aligned}$$

Finally,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1}{4} \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2},$$

which is greater than Max, thereby violating Bell's inequality.

**Problem 2.** Consider a quantum particle with mass  $m$  moving in the presence of the square well potential

$$V(r) = \begin{cases} -V_0 & r \leq a, \\ 0 & r > a. \end{cases}$$

**2.1** Writing the wave function in polar coordinates as  $\psi(\mathbf{r}) = R_l(r) Y_{lm}(\theta, \phi)$ , write down the Schrödinger equation obeyed by  $R_l$ .

**Solution.** From (A.5.1) in Sakurai, the full Schrödinger equation is

$$-\frac{\hbar^2}{2m} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi_E}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi_E}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi_E}{\partial \phi^2} \right] + V(r) \psi_E = E \psi_E,$$

where the angular part of  $\psi_E$  satisfies (A.5.4),

$$-\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_{lm} = l(l+1) Y_{lm}.$$

Then the equivalent one-dimensional Schrödinger equation is the equation immediately following (A.5.8),

$$-\frac{\hbar^2}{2m} \frac{d^2 u_E}{dr^2} + \left[ V(r) + \frac{l(l+1)\hbar^2}{2mr^2} \right] u_E = E u_E,$$

where  $u_E(r) = rR_l(r)$ . In terms of  $R_l$ ,

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} (rR_l) + \left[ V(r) + \frac{l(l+1)\hbar^2}{2mr^2} \right] rR_l = E rR_l.$$

or

$$\frac{\hbar^2}{2m} \left[ -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + V(r) + \frac{l(l+1)}{r^2} \right] R_l(r) = E_l R_l(r).$$

From (7.7.1), the effective potential at low energies for the  $l$ th partial wave is

$$V_{\text{eff}} = V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2},$$

so the Schrödinger equation can be rewritten as

$$\left[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + V_{\text{eff}} \right] R_l(r) = E_l R_l(r).$$

**2.2** When  $V_0$  is a certain value, there is one bound state for the  $s$  wave ( $l = 0$ ). The bound state energy  $\varepsilon$  is small ( $0 < |\varepsilon| \ll V_0$ ). Obtain the range of the depth of the well  $V_0$  ( $? \leq V_0 < ?$ ). Also, calculate for the bound state the probability for the particle to exist outside of the well.

**Solution.** Inside the well,  $R_l$  are given by (A.5.16),

$$R_l(r) = \text{constant } j_l(\alpha r),$$

where  $\alpha$  is defined in Eq. (A.5.17),

$$\alpha = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}, \quad r < a,$$

and the spherical Bessel functions  $j_l$  is given by (A.5.12),

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+1/2}(\rho).$$

Outside the well,  $R_l$  are given by (A.5.21),

$$R_l(r) = \text{constant } h_l^{(1)}(i\kappa r) = \text{constant } [j_l(i\kappa r) + in_l(i\kappa r)],$$

where  $\kappa$  is defined in (A.5.20),

$$\kappa = \sqrt{\frac{2m|E|}{\hbar^2}},$$

and  $h_l^{(1)}$  are the spherical Hankel functions.

We need to match the inside and outside solutions at the boundary  $r = a$  and determine the range of  $V_0$  for which exactly one solution exists. For the  $s$  wave, the relevant Bessel function is given by (A.5.12),

$$j_0(\rho) = \frac{\sin \rho}{\rho}.$$

Then we have

$$R_0(r) \propto \sqrt{\frac{\hbar^2}{2m(V_0 - |\varepsilon|)}} \frac{1}{r} \sin\left(\sqrt{\frac{2mV_0}{\hbar^2}} r\right), \quad r < a, \quad (2)$$

for the solution inside the well. For the solution outside, the relevant Hankel function is given by (A.5.22),

$$h_0^{(1)}(i\kappa r) = -\frac{1}{\kappa r} e^{-\kappa r}.$$

Then we have

$$R_0(r) \propto -\sqrt{\frac{\hbar^2}{2m|\varepsilon|}} \frac{1}{r} \exp\left(-\sqrt{\frac{2m|\varepsilon|}{\hbar^2}} r\right), \quad r > a. \quad (3)$$

Matching (??) and (??) at  $r = a$ , we obtain

$$\sqrt{\frac{\hbar^2}{2mV_0}} \frac{1}{a} \sin\left(\sqrt{\frac{2m(V_0 - |\varepsilon|)}{\hbar^2}} a\right) = -\sqrt{\frac{\hbar^2}{2m|\varepsilon|}} \frac{1}{a} \exp\left(-\sqrt{\frac{2m|\varepsilon|}{\hbar^2}} a\right),$$

which simplifies to

$$\sqrt{\frac{|\varepsilon|}{V_0 - |\varepsilon|}} \sin\left(\sqrt{\frac{2mV_0}{\hbar^2}} a\right) = -\exp\left(-\sqrt{\frac{2m|\varepsilon|}{\hbar^2}} a\right).$$

Since  $|\varepsilon| \ll V_0$ , we may Taylor expand both sides about  $|\varepsilon|$ . To first order, we obtain

$$\sqrt{\frac{|\varepsilon|}{V_0}} \sin\left(\sqrt{\frac{2mV_0}{\hbar^2}} a\right) = -1 + \sqrt{\frac{2m|\varepsilon|}{\hbar^2}} a$$

help???

**2.3** Consider the scattering problem by the well. For each  $l$ , for large enough  $r$ , when  $R_l(r)$  is given by  $R_l(r) \sim A_l \sin(kr - l\pi/2 + \delta_l)/r$ ,  $\delta_l$  is called the scattering phase shift. For the value of  $V_0$  within the range you obtained in the above problem, when the energy of the incident wave is  $E = 9V_0/16$ , calculate  $\tan \delta_0$  (where  $\delta_0$  is the scattering phase shift for the  $s$  wave).

**Solution.** From (7.6.35) in Sakurai,

$$\tan \delta_l = \frac{kR j_l'(kR) - \beta_l j_l(kR)}{kR n_l'(kR) - \beta_l n_l(kR)},$$

where (7.6.34) defines  $\beta_l$ ,

$$\beta_l \equiv \left[ \frac{r}{A_l} \frac{dA_l}{dr} \right]_{r=R},$$

and

$$j_l'(kR) = \left. \frac{dj_l}{d(kr)} \right|_{kr=kR}$$

**2.4** Now consider the  $S$  matrix,  $S \equiv \exp(2i\delta_0) = \exp(i\delta_0)/\exp(-i\delta_0)$ . Compare the condition on  $s$  wave bound state energies and the zero of the denominator of  $S$ . Explain their relation.

**Problem 3.** Consider a three dimensional potential

$$V(|r|) = \frac{\hbar^2 \gamma}{2m} \delta(|r| - a).$$

The  $s$  wave Schrödinger equation is given by

$$-\frac{\hbar^2}{2m} \frac{d^2 \chi_0(r)}{dr^2} + \frac{\hbar^2 \gamma}{2m} \delta(r - a) \chi_0(r) = E \chi_0(r).$$

The  $s$  wave function must be regular (zero) at  $r = 0$ . At  $r = a$ , it is continuous, but its derivative can jump.

**3.1** Calculate the  $s$  wave scattering phase shift ( $k$ ), where  $k$  is related to  $E$  as  $E = \hbar^2 k^2 / 2m$ .

**3.2** When  $\gamma \gg k$ ,  $1/a$  and when  $\sin ka$  is not small, discuss the behavior of the scattering phase shift.

**3.3** Obtain the condition to have resonant states and calculate the energy of the resonant states.

**3.4** Calculate the width  $\Gamma$  of the resonance. Discuss its behavior when  $\gamma$  is big.

**3.5** When the velocity of the incident wave is small, obtain the scattering cross section.

I consulted Sakurai's *Modern Quantum Mechanics*, Shankar's *Principles of Quantum Mechanics*, and the Wikipedia article on a particle in a spherically symmetric potential while writing up these solutions.