1 Problem 1

Let's consider coherent states of a one-dimensional quantum particle with mass m confined in a one-dimensional harmonic potential $V(X) = m\omega^2 X^2/2$:

$$a\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle, \qquad \left|\lambda\right\rangle = \exp\!\left(-\frac{1}{2}|\lambda|^2\right)\exp\!\left(\lambda a^\dagger\right)\left|0\right\rangle.$$

Here, λ is a complex parameter.

1.1 Compute $\langle x|\lambda\rangle$.

Solution. Since $a\left|0\right\rangle = 0\left|0\right\rangle$, $\exp(\lambda a)\left|0\right\rangle = \left|0\right\rangle$ and therefore we can write

$$\langle x|\lambda\rangle = \exp\!\left(-\frac{1}{2}|\lambda|^2\right)\left\langle x\right|\exp\!\left(\lambda a^\dagger\right)\exp(\lambda a)\left|0\right\rangle. \tag{1}$$

For two operators A and B, $e^{A+B} = e^{-[A,B]/2}e^A e^B$ if $[A, B]$ commutes with each A and B. Here, we have

$$\exp\!\left[\lambda(a^\dagger + a)\right] = \exp\!\left(\frac{\lambda^2}{2}\right)\exp\!\left(\lambda a^\dagger\right)\exp(\lambda a) \implies \exp\!\left(\lambda a^\dagger\right)\exp(\lambda a) = \exp\!\left(-\frac{\lambda^2}{2}\right)\exp\!\left[\lambda(a^\dagger + a)\right],$$

where we have used $[a, a^\dagger] = 1$. From (2.3.24) in Sakurai,

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a), \tag{2}$$

so

$$\exp\!\left[\lambda(a^\dagger + a)\right] = \exp\!\left(\lambda X\sqrt{\frac{2m\omega}{\hbar}}\right).$$

Making these substitutions into (1) yields

$$\begin{aligned}\langle x|\lambda\rangle &= \exp\!\left(-\frac{1}{2}|\lambda|^2\right)\exp\!\left(-\frac{\lambda^2}{2}\right)\left\langle x\right|\exp\!\left(\lambda X\sqrt{\frac{2m\omega}{\hbar}}\right)\left|0\right\rangle \\ &= \exp\!\left(-\frac{1}{2}|\lambda|^2\right)\exp\!\left(-\frac{\lambda^2}{2}\right)\exp\!\left(\lambda x\sqrt{\frac{2m\omega}{\hbar}}\right)\langle x|0\rangle. \end{aligned} \tag{3}$$

From (2.3.30) in Sakurai,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\!\left(-\frac{m\omega}{2\hbar}x^2\right)$$

so (3) becomes

$$\langle x|\lambda\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\!\left(-\frac{1}{2}|\lambda|^2 - \frac{\lambda^2}{2} + \lambda x\sqrt{\frac{2m\omega}{\hbar}} - \frac{m\omega}{2\hbar}x^2\right).$$

1.2 Compute $\langle\lambda|X|\lambda\rangle$, $\langle\lambda|P|\lambda\rangle$, $\langle\lambda|X^2|\lambda\rangle$, and $\langle\lambda|P^2|\lambda\rangle$. Also, compute $\langle\lambda|(\Delta X)^2|\lambda\rangle\ \langle\lambda|(\Delta P)^2|\lambda\rangle$ where $\Delta A = A - \langle A\rangle$.

Solution. For $\langle\lambda|X|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|X|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger X a|\lambda\rangle \\
&= \frac{1}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|a^\dagger(a+a^\dagger)a|\lambda\rangle = \frac{1}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|(a^\dagger a^2 + {a^\dagger}^2 a)|\lambda\rangle = \frac{|\lambda|^2(\lambda^*+\lambda)}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}} \\
&= 2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar}{2m\omega}}, \qquad (4)
\end{aligned}$$

where we have again used (2). For $\langle\lambda|P|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|P|\lambda\rangle &= \frac{1}{\lambda^2}\langle\lambda|a^\dagger P a|\lambda\rangle \\
&= \frac{i}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}}\langle\lambda|a^\dagger(a^\dagger - a)a|\lambda\rangle = \frac{i}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}}\langle\lambda|({a^\dagger}^2 a - a^\dagger a^2)|\lambda\rangle = \frac{i|\lambda|^2(\lambda^*-\lambda)}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}} \\
&= 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar m\omega}{2}}. \qquad (5)
\end{aligned}$$

From (2), note that

$$X^2 = \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + {a^\dagger}^2), \qquad P^2 = -\frac{\hbar m\omega}{2}({a^\dagger}^2 - a^\dagger a - aa^\dagger + a^2).$$

Then for $\langle\lambda|X^2|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|X^2|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger X^2 a|\lambda\rangle = \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|a^\dagger(a^2 + aa^\dagger + a^\dagger a + {a^\dagger}^2)a|\lambda\rangle \\
&= \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger a^3 + a^\dagger a a^\dagger a + {a^\dagger}^2 a^2 + {a^\dagger}^3 a)|\lambda\rangle = \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger a^3 + a^\dagger a + 2{a^\dagger}^2 a^2 + {a^\dagger}^3 a)|\lambda\rangle \\
&= (\lambda^2 + 1 + 2|\lambda|^2 + \lambda^{*2})\frac{\hbar}{2m\omega} = \left(1 + 2\left[\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2\right] + 2\left[\mathrm{Re}(\lambda)^2 - \mathrm{Im}(\lambda)^2\right]\right)\frac{\hbar}{2m\omega} \\
&= [1 + 4\,\mathrm{Re}(\lambda)^2]\frac{\hbar}{2m\omega}, \qquad (6)
\end{aligned}$$

where we have again used $[a, a^\dagger] = 1$. For $\langle\lambda|P^2|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|P^2|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger P^2 a|\lambda\rangle = -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|a^\dagger({a^\dagger}^2 - a^\dagger a - aa^\dagger + a^2)a|\lambda\rangle \\
&= -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|({a^\dagger}^2 a - {a^\dagger}^2 a^2 - a^\dagger a a^\dagger a + a^\dagger a^3)|\lambda\rangle = -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda)|({a^\dagger}^3 a - a^\dagger a - 2{a^\dagger}^2 a^2 + a^\dagger a^3|\lambda)\rangle \\
&= -(\lambda^{*2} - 1 - 2|\lambda|^2 + \lambda^2)\frac{\hbar m\omega}{2} = \left(1 + 2\left[\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2\right] - 2[\mathrm{Re}(\lambda)^2 - \mathrm{Im}(\lambda)^2]\right)\frac{\hbar m\omega}{2} \\
&= [1 + 4\,\mathrm{Im}(\lambda)^2]\frac{\hbar m\omega}{2}. \qquad (7)
\end{aligned}$$

From (1.4.51) in Sakurai, $\left\langle(\Delta A)^2\right\rangle = \left\langle A^2\right\rangle - \left\langle A\right\rangle^2$. Then

$$\langle\lambda|(\Delta X)^2|\lambda\rangle = \langle\lambda|X^2|\lambda\rangle - \langle\lambda|X|\lambda\rangle^2 = [1 + 4\,\mathrm{Re}(\lambda)^2]\frac{\hbar}{2m\omega} - 4\,\mathrm{Re}(\lambda)^2\frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega},$$

where we have used (4) and (6), and

$$\langle\lambda|(\Delta P)^2|\lambda\rangle = \langle\lambda|P^2|\lambda\rangle - \langle\lambda|P|\lambda\rangle^2 = [1+4\,\mathrm{Im}(\lambda)^2]\frac{\hbar m\omega}{2} - 4\,\mathrm{Im}(\lambda)^2\frac{\hbar m\omega}{2} = \frac{\hbar m\omega}{2},$$

where we have used (5) and (7). Finally,

$$\langle\lambda|(\Delta X)^2|\lambda\rangle\ \langle\lambda|(\Delta P)^2|\lambda\rangle = \frac{\hbar^2}{4},$$

which shows that the coherent state $|\lambda\rangle$ satisfies the minimum uncertainty relation.

1.3 Starting from $|\psi(0)\rangle = |\lambda\rangle$ at $t=0$, we let $|\psi(t)\rangle$ evolve in time. What is the state $|\psi(t)\rangle$ for $t>0$?

Solution. The Hamiltonian for the harmonic oscillator,

$$H = \frac{P^2}{2m} + \frac{m\omega^2X^2}{2}, \tag{8}$$

is time independent, so the time evolution operator $U(t)$ for the coherent state in general is given by

$$U(t) = \exp\left(-\frac{iHt}{\hbar}\right), \tag{9}$$

which is (2.1.28) in Sakurai. Rewriting $|\lambda\rangle$ in the power series representation,

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n a^{\dagger n}}{n!}|0\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}|n\rangle. \tag{10}$$

The time evolution operator $U(t)$ for an energy eigenket $|n\rangle$ of the harmonic oscillator is given by

$$U(t)|n\rangle = \exp\left(-\frac{iE_nt}{\hbar}\right)|n\rangle = \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]|n\rangle = e^{-in\omega t}e^{-i\omega t/2},$$

where E_n are given by (2.3.9) in Sakurai. Then, using (10), we have

$$\begin{aligned}|\psi(t)\rangle = U(t)|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-in\omega t}e^{-i\omega t/2}|n\rangle = e^{-i\omega t/2}\exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\lambda e^{-i\omega t})^n}{n!}|n\rangle \\ &= e^{-i\omega t/2}\left|\lambda e^{-i\omega t}\right\rangle, \end{aligned} \tag{11}$$

where $\lambda e^{-i\omega t}$ is a complex number (albeit one that is changing in time). Thus, $\left|\lambda e^{-i\omega t}\right\rangle$ is another coherent state.

1.4 Compute $\langle\psi(t)|X|\psi(t)\rangle$ and $\langle\psi(t)|P|\psi(t)\rangle$, and their time derivatives $d\langle X\rangle/dt$ and $d\langle P\rangle/dt$.

Solution. Proceeding similarly to 1.2 and using (11), we have

$$\begin{aligned}\langle\psi(t)|X|\psi(t)\rangle &= \left\langle\lambda e^{-i\omega t}\right| e^{i\omega t/2}Xe^{-i\omega t/2}\left|\lambda e^{-i\omega t}\right\rangle = \left\langle\lambda e^{-i\omega t}\middle|X\middle|\lambda e^{-i\omega t}\right\rangle \\ &= \frac{1}{|\lambda e^{-i\omega t}|^2}\sqrt{\frac{\hbar}{2m\omega}}\left\langle\lambda e^{-i\omega t}\middle|(a^\dagger a^2 + a^{\dagger 2}a)\middle|\lambda e^{-i\omega t}\right\rangle = \frac{|\lambda|^2(\lambda^* e^{i\omega t} + \lambda e^{-i\omega t}e^{-i\omega t})}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\end{aligned}$$

$$
\begin{aligned}
&= \left\{[\operatorname{Re}(\lambda) - i\operatorname{Im}(\lambda)]e^{i\omega t} + [\operatorname{Re}(\lambda) + i\operatorname{Im}(\lambda)]e^{-i\omega t}\right\}\sqrt{\frac{\hbar}{2m\omega}} \\
&= \left[\operatorname{Re}(\lambda)(e^{i\omega t} + e^{-i\omega t}) - \frac{\operatorname{Im}(\lambda)}{i}(e^{i\omega t} - e^{-i\omega t})\right]\sqrt{\frac{\hbar}{2m\omega}} \\
&= 2[\operatorname{Re}(\lambda)\cos(\omega t) + \operatorname{Im}(\lambda)\sin(\omega t)]\sqrt{\frac{\hbar}{2m\omega}}. \qquad (12)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
\langle\psi(t)|P|\psi(t)\rangle &= \left\langle\lambda e^{-i\omega t}\right| e^{i\omega t/2} P e^{-i\omega t/2}\left|\lambda e^{-i\omega t}\right\rangle = \left\langle\lambda e^{-i\omega t}\middle|P\middle|\lambda e^{-i\omega t}\right\rangle \\
&= \frac{i}{|\lambda e^{-i\omega t}|^2}\sqrt{\frac{\hbar m\omega}{2}}\left\langle\lambda e^{-i\omega t}\middle|({a^\dagger}^2 a - a^\dagger a^2)\middle|\lambda e^{-i\omega t}\right\rangle = i\frac{|\lambda|^2(\lambda^* e^{i\omega t} - \lambda e^{-i\omega t}e^{-i\omega t})}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}} \\
&= i\left\{[\operatorname{Re}(\lambda) - i\operatorname{Im}(\lambda)]e^{i\omega t} - [\operatorname{Re}(\lambda) + i\operatorname{Im}(\lambda)]e^{-i\omega t}\right\}\sqrt{\frac{\hbar m\omega}{2}} \\
&= \left[\operatorname{Im}(\lambda)(e^{i\omega t} + e^{-i\omega t}) - \frac{\operatorname{Re}(\lambda)}{i}(e^{i\omega t} - e^{-i\omega t})\right]\sqrt{\frac{\hbar m\omega}{2}} \\
&= 2[\operatorname{Im}(\lambda)\cos(\omega t) - \operatorname{Re}(\lambda)\sin(\omega t)]\sqrt{\frac{\hbar m\omega}{2}}. \qquad (13)
\end{aligned}
$$

For the time derivatives, the harmonic oscillator Hamiltonian is given by (8). Note that

$$
[X, H] = i\hbar\frac{P}{m}, \qquad [X, P] = -i\hbar m\omega^2 X,
$$

where we have used the results of problem 2.1 on the previous homework. Then, using the Ehrenfest theorem,

$$
\begin{aligned}
\frac{d\langle X\rangle}{dt} &= -\frac{i}{\hbar}\langle\psi(t)|[X,H]|\psi(t)\rangle = \frac{1}{m}\langle\psi(t)|P|\psi(t)\rangle = \frac{2}{m}[\operatorname{Im}(\lambda)\cos(\omega t) - \operatorname{Re}(\lambda)\sin(\omega t)]\sqrt{\frac{\hbar\omega}{2}}, \\
&= 2\omega[\operatorname{Im}(\lambda)\cos(\omega t) - \operatorname{Re}(\lambda)\sin(\omega t)]\sqrt{\frac{\hbar}{2m\omega}}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\langle P\rangle}{dt} &= -\frac{i}{\hbar}\langle\psi(t)|[P,H]|\psi(t)\rangle = -m\omega^2\langle\psi(t)|X|\psi(t)\rangle = -2m\omega^2[\operatorname{Re}(\lambda)\cos(\omega t) + \operatorname{Im}(\lambda)\sin(\omega t)]\sqrt{\frac{\hbar}{2m\omega}} \\
&= -2\omega[\operatorname{Re}(\lambda)\cos(\omega t) + \operatorname{Im}(\lambda)\sin(\omega t)]\sqrt{\frac{\hbar m\omega}{2}},
\end{aligned}
$$

which are what we would get by differentiating (12) and (13), respectively.

1.5 Compute $\langle\lambda''|\exp(-iHt/\hbar)|\lambda'\rangle$.

Solution. From (9), we are looking for $\langle\lambda''|U(t)|\lambda'\rangle$. From (11),

$$
U(t)|\lambda\rangle = e^{-i\omega t/2}\left|\lambda e^{-i\omega t}\right\rangle \implies U(t)\left|\lambda'\right\rangle = e^{-i\omega t/2}\left|\lambda' e^{-i\omega t}\right\rangle,
$$

so

$$
\langle\lambda''|U(t)|\lambda'\rangle = e^{-i\omega t/2}\left\langle\lambda''\middle|\lambda' e^{-i\omega t}\right\rangle.
$$

Using the power series representation as in (10),

$$\left|\lambda' e^{-i\omega t}\right\rangle = \exp\left(-\frac{|\lambda'|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\lambda' e^{-i\omega t})^n}{n!}\left|n\right\rangle,$$

so

$$\left\langle\lambda''\middle|\lambda' e^{-i\omega t}\right\rangle = \exp\left(-\frac{|\lambda''|^2}{2}\right)\exp\left(-\frac{|\lambda'|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\lambda''^*\lambda' e^{-i\omega t})^n}{n!}\left\langle n|n\right\rangle = \exp\left(-\frac{|\lambda''|^2}{2}+\lambda''^*\lambda' e^{-i\omega t}-\frac{|\lambda'|^2}{2}\right).$$

Finally, we have

$$\langle\lambda''|\exp\left(-\frac{iHt}{\hbar}\right)|\lambda'\rangle = \exp\left(-\frac{i\omega t}{2}-\frac{|\lambda''|^2}{2}+\lambda''^*\lambda' e^{-i\omega t}-\frac{|\lambda'|^2}{2}\right).$$

2 Problem 2

Consider a quantum system which has coordinate X_1 and momentum P_1, and another system which has coordinate X_2 and momentum P_2. (An operator from the first system always commutes with an operator of the second system.) We think of the second system as a "probe" which we can use to detect the properties of the first system. For a short time T, the two systems are coupled by a coupling Hamiltonian H_c, given by

$$H_c = \frac{X_1 P_2}{T}.$$

The coupling between the two systems disturbs the momentum of the first system. The disturbance operator is defined to be

$$D \equiv P_1(T) - P_1(0). \tag{14}$$

The probe introduces measurement error or "noise" into the system. The noise operator is defined by

$$N \equiv X_2(T) - X_1(0).$$

The stste of the system at $t = 0$ is $|\Psi(0)\rangle = |\phi_1(0)\,\phi_2(0)\rangle$, and all expectation values are taken in this state.

2.1 With H_c as the Hamiltonian, find the Heisenberg operators $X_1(t)$, $P_1(t)$, $X_2(t)$, and $P_2(t)$ in terms of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$. Time is restricted to the range $t \in [0, T]$.

Solution. In general, a Heisenberg operator $O(t)$ is defined by

$$O(t) = U^\dagger(t)\,O(0)\,U(t),$$

where $U(t)$ is the time evolution operator. For H_c, it is given by

$$U(t) = \exp\left(-\frac{iH_c t}{\hbar}\right) = \exp\left(-\frac{it}{\hbar T}\,X_1(0)\,P_2(0)\right).$$

(2.2.23) in Sakurai gives the commutation relations

$$[X_i, F(\mathbf{P})] = i\hbar\frac{\partial F}{\partial P_i} \qquad\qquad [P_i, G(\mathbf{X})] = -i\hbar\frac{\partial G}{\partial X_i}.$$

Using these, we have

$$\begin{aligned}
[X_1(0), U(t)] &= 0, \\
[X_2(0), U(t)] &= i\hbar \left(-\frac{it}{\hbar T}\, X_1(0)\right) U(t) = \frac{t}{T}\, X_1(0)\, U(t) = \frac{t}{T}\, U(t)\, X_1(0), \\
[P_1(0), U(t)] &= -i\hbar \left(-\frac{it}{\hbar T}\, P_2(0)\right) U(t) = -\frac{t}{T}\, P_2(0)\, U(t) = -\frac{t}{T}\, U(t)\, P_2(0), \\
[P_2(0), U(t)] &= 0.
\end{aligned}$$

Then

$$X_1(t) = U^\dagger(t)\, X_1(0)\, U(t) = X_1(0), \tag{15}$$

$$P_1(t) = U^\dagger(t)\, P_1(0)\, U(t) = U^\dagger(t) \left(U(t)\, P_1(0) - \frac{t}{T} U(t)\, P_2(0)\right) = P_1(0) - \frac{t}{T}\, P_2(0), \tag{16}$$

$$X_2(t) = U^\dagger(t)\, X_2(0)\, U(t) = U^\dagger(t) \left(U(t)\, X_2(0) + \frac{t}{T}\, U(t)\, X_1(0)\right) = X_2(0) + \frac{t}{T}\, X_1(0), \tag{17}$$

$$P_2(t) = U^\dagger(t)\, P_2(0)\, U(t) = P_2(0). \tag{18}$$

2.2 Derive an expression for $\sigma(D)$ which involves only the standard deviations of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$. Here, we denote the standard deviation of an operator O as $\sigma(O) = \sqrt{\langle (O - \langle O \rangle)^2 \rangle}$.

Solution. Substituting (18) into (14),

$$D = P_1(0) - \frac{T}{T} P_2(0) - P_1(0) = -P_2(0).$$

Note that for an operator O,

$$\sigma(-O) = \sqrt{\langle (-O - \langle -O \rangle)^2 \rangle} = \sqrt{\langle (\langle O \rangle - O)^2 \rangle} = \sigma(O),$$

so

$$\sigma(D) = \sigma\big(P_2(0)\big). \tag{19}$$

2.3 Derive an expression for $\sigma(N)$ which involves only the standard deviations of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$.

Solution. Substituting (17) into (14),

$$N = X_2(0) + \frac{T}{T} X_1(0) - X_1(0) = X_2(0).$$

which implies

$$\sigma(N) = \sigma\big(X_2(0)\big). \tag{20}$$

2.4 Now consider the product $\sigma(N)\,\sigma(D)$. Assume

$$\sigma\big(X_1(0)\big)\,\sigma\big(P_1(0)\big) \geq \frac{\hbar}{2}, \qquad\qquad \sigma\big(X_2(0)\big)\,\sigma\big(P_2(0)\big) \geq \frac{\hbar}{2}$$

both hold. Is $\sigma(N)\,\sigma(D) \geq \hbar/2$ satisfied? What conditions are required for equality?

Solution. From (19) and (20),

$$\sigma(N)\,\sigma(D) = \sigma\big(P_2(0)\big)\,\sigma\big(X_2(0)\big) \geq \frac{\hbar}{2},$$

where the final inequality is satisfied by assumption. For equality, we would need

$$\sigma\big(X_2(0)\big)\,\sigma\big(P_2(0)\big) = \frac{\hbar}{2},$$

which is satisfied if the "probe" system is a Gaussian wave packet.

3 Problem 3

Answer the following questions about the angular momentum operator L_i.

3.1 Calculate $[L_i, \mathbf{r}]$ where $i = x, y, z$.

Solution. Firstly, note that

$$L_x = YP_z - ZP_y, \qquad L_y = ZP_x - XP_z, \qquad L_z = XP_y - YP_x,$$

where the expression for L_z was given in problem 2 of Homework 1, and L_x and L_y are cyclic permutations. Then

$$\begin{aligned}
[L_x, X] &= (YP_z - ZP_y)X - X(YP_z - ZP_y) = 0,\\
[L_x, Y] &= (YP_z - ZP_y)Y - Y(YP_z - ZP_y) = YP_zY - ZP_yY - YYP_z + YZP_y = [Y, P_y]Z = i\hbar Z,\\
[L_x, Z] &= (YP_z - ZP_y)Z - Z(YP_z - ZP_y) = YP_zZ - ZP_yZ - ZYP_z + ZZP_y = -[Z, P_z]Y = -i\hbar Y.
\end{aligned}$$

Generalizing these results to L_y and L_z,

$$[L_x, \mathbf{r}] = i\hbar \begin{bmatrix} 0 \\ Z \\ -Y \end{bmatrix}, \qquad [L_y, \mathbf{r}] = i\hbar \begin{bmatrix} -Z \\ 0 \\ X \end{bmatrix}, \qquad [L_z, \mathbf{r}] = i\hbar \begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix}, \tag{21}$$

where $\mathbf{r} = \begin{bmatrix} X & Y & Z \end{bmatrix}^T$.

3.2 Let us now compare the above results with classical mechanics. Rotations around the x, y, and z axes by an angle θ in three-dimensional Cartesian space are represented by the following matrices:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate $R_i(\theta)\,\mathbf{r}$. Then expand $R_i(\theta)\,\mathbf{r}$ for a small angle θ and consider $\mathbf{r} - R_i(\theta)\,\mathbf{r}$ to first order in θ,

$$\mathbf{r} - R_i(\theta)\,\mathbf{r} = \theta M_i \mathbf{r} + \mathcal{O}\big(\theta^2\big).$$

Calculate the matrices M_i.

Solution. For $R_i(\theta)\,\mathbf{r}$, we have

$$R_x(\theta)\,\mathbf{r} = \begin{bmatrix} X \\ \cos\theta Y - \sin\theta Z \\ \sin\theta Y + \cos\theta Z \end{bmatrix}, \qquad R_y(\theta)\,\mathbf{r} = \begin{bmatrix} \cos\theta X + \sin\theta Z \\ Y \\ \cos\theta Z - \sin\theta X \end{bmatrix}, \qquad R_z(\theta)\,\mathbf{r} = \begin{bmatrix} \cos\theta X - \sin\theta Y \\ \sin\theta X + \cos\theta Y \\ Z \end{bmatrix},$$

In the small angle approximation, to first order $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. In this approximation,

$$R_x(\theta)\,\mathbf{r} \approx \begin{bmatrix} X \\ Y - \theta Z \\ \theta Y + Z \end{bmatrix}, \qquad R_y(\theta)\,\mathbf{r} \approx \begin{bmatrix} X + \theta Z \\ Y \\ Z - \theta X \end{bmatrix}, \qquad R_z(\theta)\,\mathbf{r} \approx \begin{bmatrix} X - \theta Y \\ \theta X + Y \\ Z \end{bmatrix},$$

and so

$$\mathbf{r} - R_x(\theta)\,\mathbf{r} \approx \theta \begin{bmatrix} 0 \\ Z \\ -Y \end{bmatrix}, \qquad \mathbf{r} - R_y(\theta)\,\mathbf{r} \approx \theta \begin{bmatrix} -Z \\ 0 \\ X \end{bmatrix}, \qquad \mathbf{r} - R_z(\theta)\,\mathbf{r} \approx \theta \begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix},$$

which look similar to (21). These results suggest the matrices

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad M_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad M_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.3 Calculate the matrix elements of the angular momentum operator L_i in the basis ket $|l,m\rangle$ when $l = 1$ and $l = 2$. Here, $|l,m\rangle$ is the simultaneous eigenket of L^2 and L_z with the eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$, respectively.

Solution. The ladder operators are defined by (3.5.5) in Sakurai:

$$J_\pm = L_x \pm i L_y.$$

Clearly,

$$L_x = \frac{J_+ + J_-}{2}, \qquad L_y = \frac{J_+ - J_-}{2i}.$$

From (3.5.39) and (3.5.40),

$$J_+ \,|l,m\rangle = \sqrt{(l-m)(l+m+1)}\hbar\,|l,m+1\rangle\,, \qquad J_- \,|l,m\rangle = \sqrt{(l+m)(l-m+1)}\hbar\,|l,m-1\rangle\,.$$

Then the matrix elements of L_x are given by

$$\begin{aligned}
&\langle 1,m'|L_x|1,m\rangle = \langle 1,m'|\frac{J_+ + J_-}{2}|1,m\rangle = \frac{\hbar}{2}\left(\delta_{m+1,m'}\sqrt{2-m-m^2} + \delta_{m-1,m'}\sqrt{2+m-m^2}\right),\\
&\langle 1,m'|L_x|2,m\rangle = 0,\\
&\langle 2,m'|L_x|1,m\rangle = 0,\\
&\langle 2,m'|L_x|2,m\rangle = \langle 1,m'|\frac{J_+ + J_-}{2}|1,m\rangle = \frac{\hbar}{2}\left(\delta_{m+1,m'}\sqrt{6-m-m^2} + \delta_{m-1,m'}\sqrt{6+m-m^2}\right),
\end{aligned}$$

where the integers $m, m' \in [-l, l]$.

The matrix elements of L_y are given by

$$\begin{aligned}
\langle 1,m'|L_y|1,m\rangle &= \langle 1,m'|\frac{J_+ - J_-}{2i}|1,m\rangle = -\frac{i\hbar}{2}\left(\delta_{m+1,m'}\sqrt{2-m-m^2} - \delta_{m-1,m'}\sqrt{2+m-m^2}\right),\\
\langle 1,m'|L_y|2,m\rangle &= 0,\\
\langle 2,m'|L_y|1,m\rangle &= 0,\\
\langle 2,m'|L_y|2,m\rangle &= \langle 1,m'|\frac{J_+ - J_-}{2i}|1,m\rangle = -\frac{i\hbar}{2}\left(\delta_{m+1,m'}\sqrt{6-m-m^2} - \delta_{m-1,m'}\sqrt{6+m-m^2}\right),
\end{aligned}$$

where again $m, m' \in [-l, l]$.

Since $|l,m\rangle$ are eigenkets of L_z, it is diagonal in this basis. Its matrix elements are given by

$$\langle l',m'|L_y|l,m\rangle = \hbar m\,\delta_{m,m'}\,\delta_{l,l'},$$

where $l, l' \in \{1,2\}$ and $m, m' \in [-l, l]$.

Explicitly, let

$$R = \begin{array}{cc}
 & \begin{array}{cccccccc} (1,-1) & (1,0) & (1,1) & (2,-2) & (2,-1) & (2,0) & (2,1) & (2,2) \end{array} \\
\begin{array}{c} (1,-1)\\ (1,0)\\ (1,1)\\ (2,-2)\\ (2,-1)\\ (2,0)\\ (2,1)\\ (2,2) \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0\\
0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0\\
0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0\\
0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{2}\\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0
\end{bmatrix}
\end{array},$$

where the row labels represent (l', m') and the column labels represent (l, m). Then

$$L_x = \frac{\hbar}{\sqrt{2}}R, \qquad L_y = -\frac{i\hbar}{\sqrt{2}}R,$$

and

$$L_z = \hbar \begin{array}{cc}
 & \begin{array}{cccccccc} (1,-1) & (1,0) & (1,1) & (2,-2) & (2,-1) & (2,0) & (2,1) & (2,2) \end{array} \\
\begin{array}{c} (1,-1)\\ (1,0)\\ (1,1)\\ (2,-2)\\ (2,-1)\\ (2,0)\\ (2,1)\\ (2,2) \end{array} &
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\end{array}.$$

In writing up these solutions, I consulted Sakurai's *Modern Quantum Mechanics* and Shankar's *Principles of Quantum Mechanics*.