# 1

Two identical harmonic oscillators with mass $M$ and natural frequency $\omega_0$ are coupled to each other, and to an extra mass $m$, such that the equations of motion have the form

$$\ddot{x}_1 + \frac{m}{M}\ddot{x}_2 + \omega_0^2 x_1 = 0, \tag{1}$$

$$\ddot{x}_2 + \frac{m}{M}\ddot{x}_1 + \omega_0^2 x_2 = 0. \tag{2}$$

What are the normal mode frequencies?

**Solution.** We will begin by rewriting (1) and (2) such that they have no cross terms. Solving (1) for $\ddot{x}_2$ and (2) for $\ddot{x}_1$ gives us

$$\ddot{x}_2 = -\frac{M}{m}(\ddot{x}_1 + \omega_0^2 x_1), \tag{3}$$

$$\ddot{x}_1 = -\frac{M}{m}(\ddot{x}_2 + \omega_0^2 x_2). \tag{4}$$

Now substituting (3) into (2) and (4) into (1) gives us

$$-\frac{M}{m}(\ddot{x}_1 + \omega_0^2 x_1) + \frac{m}{M}\ddot{x}_1 + \omega_0^2 x_2 = 0 \implies \ddot{x}_1 = \frac{\omega_0^2}{m^2 - M^2}(M^2 x_1 - Mm x_2), \tag{5}$$

$$-\frac{M}{m}(\ddot{x}_2 + \omega_0^2 x_2) + \frac{m}{M}\ddot{x}_2 + \omega_0^2 x_1 = 0 \implies \ddot{x}_2 = \frac{\omega_0^2}{m^2 - M^2}(M^2 x_2 - Mm x_1). \tag{6}$$

Then (5) and (6) may be rewritten in a matrix form in the basis $(x_1, x_2)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\omega_0^2}{m^2 - M^2}\begin{bmatrix} M^2 & -Mm \\ -Mm & M^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv \frac{\omega_0^2}{m^2 - M^2} A\mathbf{x}, \tag{7}$$

where we have defined the matrix $A$ and the vector $\mathbf{x}$.

Let $\lambda_\pm$ be the two eigenvalues of $A$. The eigenvalues are given by $\det(A - \lambda I) = 0$, where $I$ is the identity matrix. That is,

$$0 = \begin{vmatrix} M^2 - \lambda & -Mm \\ -Mm & M^2 - \lambda \end{vmatrix} = (M^2 - \lambda)^2 - M^2 m^2 \implies M^2 - \lambda = \pm Mm \implies \lambda_\pm = M^2 \pm Mm. \tag{8}$$

Let $\omega_\pm$ be the normal mode frequencies, which are given by $\omega_\pm^2 = -\omega_0^2 \lambda_\pm / (m^2 - M^2)$. Explicitly,

$$\omega_+ = \omega_0 \sqrt{\frac{M^2 + Mm}{M^2 - m^2}} = \omega_0 \sqrt{\frac{M}{M - m}}, \tag{9}$$

$$\omega_- = \omega_0 \sqrt{\frac{M^2 - Mm}{M^2 - m^2}} = \omega_0 \sqrt{\frac{M}{M + m}}. \tag{10}$$

## 2 Designing a Double Pendulum

Suppose you are asked to design a double pendulum whose lower frequency is half that of the higher frequency by changing the lengths of the strings and/or the masses. What are the possible designs?

**Solution.** Let the upper part of the pendulum have mass $m_1$ and string length $\ell_1$. Let its position be $\mathbf{r}_1 = (x_1, y_1)$ where the pivot is located at the origin, and the $y$ axis points downward. Define $m_2$, $\ell_2$, and $\mathbf{r}_2$ similarly for the lower part. Then the Lagrangian for the system is given by

$$L = T_1 + T_2 - U_1 - U_2 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - mgy_1 - mgy_2. \tag{11}$$

Define the generalized coordinates $\theta_1, \theta_2$ which represent the angle of inclination of each mass with respect to the vertical. Then the Cartesian coordinates representing the position of each mass are

$$x_1 = \ell_1 \sin\theta_1, \qquad y_1 = \ell_1 \cos\theta_1, \tag{12}$$

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2, \qquad y_2 = \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2, \tag{13}$$

which have the time derivatives

$$\dot{x}_1 = \frac{\partial x_1}{\partial \theta_1}\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \ell_1 \cos\theta_1 \dot{\theta}_1, \tag{14}$$

$$\dot{y}_1 = \frac{\partial y_1}{\partial \theta_1}\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = -\ell_1 \sin\theta_1 \dot{\theta}_1, \tag{15}$$

$$\dot{x}_2 = \frac{\partial x_2}{\partial \theta_1}\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\partial x_2}{\partial \theta_2}\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \ell_1 \cos\theta_1 \dot{\theta}_1 + \ell_2 \cos\theta_2 \dot{\theta}_2, \tag{16}$$

$$\dot{y}_2 = \frac{\partial y_2}{\partial \theta_1}\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\partial y_2}{\partial \theta_2}\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = -\ell_1 \sin\theta_1 \dot{\theta}_1 - \ell_2 \sin\theta_2 \dot{\theta}_2. \tag{17}$$

From (14) and (15),

$$\dot{x}_1^2 + \dot{y}_1^2 = \ell_1^2\dot{\theta}_1^2(\cos^2\theta_1 + \sin^2\theta_1) = \ell_1^2\dot{\theta}_1^2. \tag{18}$$

From (16) and (17),

$$\dot{x}_2^2 = \ell_1^2\cos^2\theta_1\dot{\theta}_1^2 + 2\ell_1\ell_2\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2 + \ell_2^2\cos^2\theta_2\dot{\theta}_2^2, \tag{19}$$

$$\dot{y}_2^2 = \ell_1^2\sin^2\theta_1\dot{\theta}_1^2 + 2\ell_1\ell_2\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2 + \ell_2^2\sin^2\theta_2\dot{\theta}_2^2, \tag{20}$$

so

$$\dot{x}_2^2 + \dot{y}_2^2 = \ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \tag{21}$$

$$= \ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2. \tag{22}$$

Writing (11) in terms of the generalized coordinates, we have

$$L = \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right) + m_1 g\ell_1\cos\theta_1$$
$$+ m_2 g\left(\ell_1\cos\theta_1 + \ell_2\cos\theta_2\right) \tag{23}$$

$$= \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + g(m_1 + m_2)\ell_1\cos\theta_1$$
$$+ g\ell_2 m_2\cos\theta_2. \tag{24}$$

The stable equilibrium solution is for the pendulum hanging straight down, which is at the point $(\theta_1, \theta_2) = (0, 0)$. We will approximate the Lagrangian $L = L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ given by (24) using a Taylor series expansion about this stable point in order to find general expressions for the normal modes. Note that

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= -m_2 \ell_1 \ell_2 (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)\dot{\theta}_1 \dot{\theta}_2 - g(m_1 + m_2)\ell_1 \sin\theta_1 && (25)\\
&= -m_2 \ell_1 \ell_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 - g(m_1 + m_2)\ell_1 \sin\theta_1, && (26)\\
\frac{\partial L}{\partial \theta_2} &= m_2 \ell_1 \ell_2 (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)\dot{\theta}_1 \dot{\theta}_2 - g m_2 \ell_2 \sin\theta_2 && (27)\\
&= m_2 \ell_1 \ell_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 - g m_2 \ell_2 \sin\theta_2, && (28)
\end{aligned}
$$

which implies

$$
\left.\frac{\partial L}{\partial \theta_1}\right|_{0,0} \theta_1 = \left.\frac{\partial L}{\partial \theta_2}\right|_{0,0} \theta_2 = 0. \tag{29}
$$

Thus, we must expand to second order. Note that

$$
\begin{aligned}
\frac{\partial^2 L}{\partial \theta_1^2} &= -m_2 \ell_1 \ell_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)\dot{\theta}_1 \dot{\theta}_2 - g(m_1 + m_2)\ell_1 \cos\theta_1 && (30)\\
&= -m_2 \ell_1 \ell_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 - g(m_1 + m_2)\ell_1 \cos\theta_1, && (31)\\
\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} &= m_2 \ell_1 \ell_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)\dot{\theta}_1 \dot{\theta}_2 = m_2 \ell_1 \ell_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 && (32)\\
\frac{\partial^2 L}{\partial \theta_2^2} &= -m_2 \ell_1 \ell_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)\dot{\theta}_1 \dot{\theta}_2 - g m_2 \ell_2 \cos\theta_2 && (33)\\
&= -m_2 \ell_1 \ell_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 - g m_2 \ell_2 \cos\theta_2. && (34)
\end{aligned}
$$

Then, to second order in $\theta_1$ and $\theta_2$,

$$
\begin{aligned}
L &\approx L(0, 0, \dot{\theta}_1, \dot{\theta}_2) + \frac{1}{2} \left.\frac{\partial^2 L}{\partial \theta_1^2}\right|_{0,0} \theta_1^2 + \left.\frac{\partial^2 L}{\partial \theta_1 \partial \theta_2}\right|_{0,0} \theta_1 \theta_2 + \frac{1}{2} \left.\frac{\partial^2 L}{\partial \theta_2^2}\right|_{0,0} \theta_2^2 && (35)\\
&\approx \frac{1}{2}(m_1 + m_2)\ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 + g(m_1 + m_2)\ell_1 + g \ell_2 m_2 \\
&\quad + \frac{1}{2}\Big(-m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 - g(m_1 + m_2)\ell_1\Big)\theta_1^2 + \Big(m_2 \ell_1 \ell_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2\Big)\theta_1 \theta_2 \\
&\quad + \frac{1}{2}\Big(-m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 - g m_2 \ell_2\Big)\theta_2^2 && (36)\\
&\approx \frac{1}{2}(m_1 + m_2)\ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\theta}_2^2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} g(m_1 + m_2)\ell_1 \theta_1^2 - \frac{1}{2} g m_2 \ell_2 \theta_2^2, && (37)
\end{aligned}
$$

where in going to (37) we have omitted constant terms and terms of $\mathcal{O}(\theta^4)$. In doing so, we have made use of the fact that $\dot{\theta} \propto \theta$.

Now we can obtain the equations of motion for the approximated Lagrangian (37), which we will call $\hat{L}$:

$$0 = \frac{\partial \hat{L}}{\partial \theta_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \hat{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2 + g(m_1+m_2)\ell_1\theta_1, \tag{38}$$

$$0 = \frac{\partial \hat{L}}{\partial \theta_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \hat{L}}{\partial \dot{\theta}_2} = m_2\ell_1\ell_2\ddot{\theta}_1 + m_2\ell_2^2\ddot{\theta}_2 + gm_2\ell_2\theta_2. \tag{39}$$

Solving (38) for $\ddot{\theta}_2$ and (39) for $\ddot{\theta}_1$ gives us

$$\ddot{\theta}_2 = -\frac{1}{m_2\ell_1\ell_2}\left(g(m_1+m_2)\ell_1\theta_1 + (m_1+m_2)\ell_1^2\ddot{\theta}_1\right), \tag{40}$$

$$\ddot{\theta}_1 = -\frac{1}{m_2\ell_1\ell_2}\left(m_2\ell_2^2\ddot{\theta}_2 + gm_2\ell_2\theta_2\right). \tag{41}$$

Substituting (40) into (39) gives us

$$0 = m_2\ell_1\ell_2\ddot{\theta}_1 - \frac{\ell_2}{\ell_1}\left(g(m_1+m_2)\ell_1\theta_1 + (m_1+m_2)\ell_1^2\ddot{\theta}_1\right) + gm_2\ell_2\theta_2 \tag{42}$$

$$\implies m_1\ell_1\ell_2\ddot{\theta}_1 = -g(m_1+m_2)\ell_2\theta_1 + gm_2\ell_2\theta_2, \tag{43}$$

and substituting (41) into (38) gives us

$$0 = -(m_1+m_2)\ell_1\left(\ell_2\ddot{\theta}_2 + g\theta_2\right) + m_2\ell_1\ell_2\ddot{\theta}_2 + g(m_1+m_2)\ell_1\theta_1 \tag{44}$$

$$\implies m_1\ell_1\ell_2\ddot{\theta}_2 = g(m_1+m_2)\ell_1\theta_1 - g(m_1+m_2)\ell_1\theta_2. \tag{45}$$

Then (43) and (45) may be rewritten in a matrix form in the basis $\boldsymbol{\theta}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = -\frac{g(m_1+m_2)}{m_1\ell_1\ell_2}\begin{bmatrix}\ell_2 & -m_2\ell_2/(m_1+m_2)\\ -\ell_1 & \ell_1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} \equiv -\frac{g(m_1+m_2)}{m_1\ell_1\ell_2}A\boldsymbol{\theta}, \tag{46}$$

where we have defined the matrix $A$ and the vector $\boldsymbol{\theta}$.

Let $\lambda_\pm$ be the two eigenvalues of $A$. Then

$$0 = \begin{vmatrix}\ell_2 - \lambda & -m_2\ell_2/(m_1+m_2)\\ -\ell_1 & \ell_1 - \lambda\end{vmatrix} = (\ell_1-\lambda)(\ell_2-\lambda) - \frac{m_2}{m_1+m_2}\ell_1\ell_2 \tag{47}$$

$$= \lambda^2 - (\ell_1+\ell_2)\lambda + \frac{m_1}{m_1+m_2}\ell_1\ell_2 \tag{48}$$

which implies

$$\lambda_\pm = \frac{1}{2}\left((\ell_1+\ell_2) \pm \sqrt{(\ell_1+\ell_2)^2 - 4\ell_1\ell_2\frac{m_1}{m_1+m_2}}\right). \tag{49}$$

Let $\omega_\pm$ be the normal mode frequencies, which are given by

$$\omega_\pm^2 = \frac{g(m_1+m_2)}{m_1\ell_1\ell_2}\lambda_\pm. \tag{50}$$

The higher frequency is $\omega_-$. In order for the lower frequency to be half of this, we need $\omega_-^2 = 4\omega_+^2$, or equivalently $\lambda_- = 4\lambda_+$. Using (49), this gives us the condition

$$\left((\ell_1+\ell_2) - \sqrt{(\ell_1+\ell_2)^2 - 4\ell_1\ell_2\frac{m_1}{m_1+m_2}}\right) = 4\left((\ell_1+\ell_2) + \sqrt{(\ell_1+\ell_2)^2 - 4\ell_1\ell_2\frac{m_1}{m_1+m_2}}\right) \tag{51}$$

$$-3(\ell_1+\ell_2) = 5\sqrt{(\ell_1+\ell_2)^2 - 4\ell_1\ell_2\frac{m_1}{m_1+m_2}} \tag{52}$$

$$9(\ell_1+\ell_2)^2 = 25\left((\ell_1+\ell_2)^2 - 4\ell_1\ell_2\frac{m_1}{m_1+m_2}\right) \tag{53}$$

$$100\ell_1\ell_2\frac{m_1}{m_1+m_2} = 16(\ell_1+\ell_2)^2 \tag{54}$$

$$\frac{\ell_1}{\ell_1+\ell_2}\frac{\ell_2}{\ell_1+\ell_2}\frac{m_1}{m_1+m_2} = 0.16. \tag{55}$$

The possible designs are those that satisfy (55).

## 3 Beats and Double Pendulum

Given a double pendulum whose two strings are of equal length, how should the masses be chosen so that the two eigenfrequencies approach each other, i.e. that the system approaches a degeneracy? Show that the resultant motion proceeds in "beats."

**Solution.** The eigenfrequencies $\omega_\pm$ approaching each other is equivalent to $\lambda_\pm$ approaching each other. Substituting $\ell \equiv \ell_1 = \ell_2$ into (49) results in

$$\lambda_\pm = \frac{1}{2}\left(2\ell \pm \sqrt{(2\ell)^2 - 4\ell^2\frac{m_1}{m_1+m_2}}\right) = \ell \pm \ell\sqrt{1 - \frac{m_1}{m_1+m_2}} = \ell \pm \ell\sqrt{\frac{m_2}{m_1+m_2}}, \tag{56}$$

so the system will approach a degeneracy as $m_2/(m_1+m_2) \to 0$. This means the masses should be chosen such that $m_2 \ll m_1$.

In order to show that the resultant motion proceeds in "beats," we will solve for $\theta_1(t)$ and $\theta_2(t)$ of the linearized system given by the Lagrangian (37). Let $\epsilon^2 = m_2/m_1 \ll 1$. From (56),

$$\lambda_\pm = \ell(1 \pm \epsilon). \tag{57}$$

Substituting (57) into (50) to find the oscillation frequencies, we have

$$\omega_+^2 = \frac{g}{\ell}(1+\epsilon)(1+\epsilon) \implies \omega_+ = (1+\epsilon)\sqrt{\frac{g}{\ell}}, \tag{58}$$

$$\omega_-^2 = \frac{g}{\ell}(1+\epsilon)(1-\epsilon) \implies \omega_- = \sqrt{\frac{g}{\ell}(1+\epsilon^2)} \approx \sqrt{\frac{g}{\ell}}, \tag{59}$$

where we are neglecting terms of $\mathcal{O}(\epsilon^2)$. Note that $\omega_+ = (1+\epsilon)\omega_-$. We will now find the corresponding eigenvectors $\mathbf{v}_\pm$, which are the normal modes of the system. For the matrix $A$ defined in (46), we must find $v_1, v_2$ such that

$$A\mathbf{v}_\pm = \lambda_\pm \mathbf{v}_\pm \implies \ell \begin{bmatrix} 1 & -\epsilon^2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \ell(1 \pm \epsilon) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{60}$$

The algebraic equations corresponding to (60) are

$$v_1 - \epsilon^2 v_2 = (1 \pm \epsilon) v_1 \implies -\epsilon^2 = \pm \epsilon v_1, \tag{61}$$
$$-v_1 + v_2 = (1 \pm \epsilon) v_2 \implies -v_1 = \pm \epsilon v_2, \tag{62}$$

which have solutions $v_1 = \mp\epsilon$ and $v_2 = 1$. Thus, the eigenvectors for the normal modes are

$$\mathbf{v}_\pm = \begin{bmatrix} \mp\epsilon \\ 1 \end{bmatrix}. \tag{63}$$

Then we can write down $\theta_1(t)$ and $\theta_2(t)$:

$$\begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = C_+ \begin{bmatrix} \epsilon \\ 1 \end{bmatrix} \cos(\omega_+ t + \phi_+) + C_- \begin{bmatrix} -\epsilon \\ 1 \end{bmatrix} \cos(\omega_- t + \phi_-), \tag{64}$$

where $C_\pm$ and $\phi_\pm$ are the amplitudes and phases, respectively, corresponding to $\omega_\pm$.

We will consider the case where $C_\pm = 1$ and $\phi_\pm = 0$, which is representative of the qualitative behavior of the system. Then (64) simplifies to the two equations

$$\theta_1(t) = \epsilon\cos(\omega_+ t) - \epsilon\cos(\omega_- t) = -2\epsilon \sin[(\omega_+ + \omega_-)t]\,\sin[(\epsilon/2)t], \tag{65}$$
$$\theta_2(t) = \cos(\omega_+ t) + \cos(\omega_- t) = 2\cos[(\omega_+ + \omega_-)t]\,\cos[(\epsilon/2)t], \tag{66}$$

where we have used the identities

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta, \tag{67}$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta. \tag{68}$$

with $\alpha = (\omega_+ + \omega_-)/2$ and $\beta = (\omega_+ - \omega_-)/2 = \epsilon/2$. The forms of (65) and (66) enable us to sketch qualitatively $\theta_1(t)$ and $\theta_2(t)$ as in figure 1, which indicates the motion of the system proceeding in "beats." That is, both masses experience rapid oscillations at the same frequency, with the oscillation amplitude for each mass varying periodically at a lower common frequency. The maximum amplitude for $m_2$ is larger than that of $m_1$ by a factor $1/\epsilon$. The amplitude oscillations are exactly out of phase, so the masses appear to "take turns" oscillating at their respective maximum amplitudes.

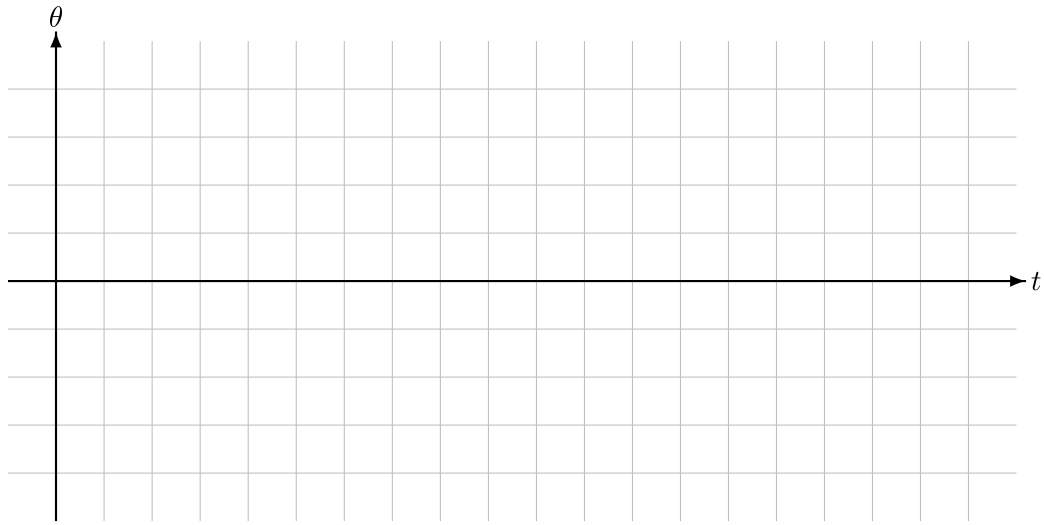


Figure 1: The beat motion of the double pendulum in problem 3. The blue (red) line indicates the motion of $m_1$ ($m_2$).

# 4 Triple Oscillator System

Consider three identical masses connected by identical springs in the shape of an equilateral triangle. Suppose the three springs lie along the arcs of a circle that circumscribes the triangle. Suppose also that the motion of the masses is constrained to move along the circle. Find the normal modes and the eigenfrequencies about the equilibrium state. If there is a zero mode, identify the associated continuous symmetry.

**Solution.** Let $m$ be the mass of each oscillator, and $k$ be the spring constant of each of the springs connecting them. The motion is constrained to a circle, so we will use the generalized coordinates $\theta_1, \theta_2, \theta_3$ to represent the positions of each of the masses. The Lagrangian for this system is given by

$$\begin{aligned} L &= T_1 + T_2 + T_3 - U_{21} - U_{32} - U_{13} && (69) \\ &= \frac{m}{2}(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2) - \frac{k}{2}\left((\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 + (\theta_1 - \theta_3)^2\right) && (70) \\ &= \frac{m}{2}(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2) - k(\theta_1^2 + \theta_2^2 + \theta_3^2 - \theta_1\theta_2 - \theta_2\theta_3 - \theta_1\theta_3). && (71) \end{aligned}$$

The Euler-Lagrange equations for the Lagrangian (71) are

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \theta_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_1} \implies \ddot\theta_1 = -\frac{k}{m}(2\theta_1 - \theta_2 - \theta_3), && (72) \\ 0 &= \frac{\partial L}{\partial \theta_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_2} \implies \ddot\theta_2 = -\frac{k}{m}(2\theta_2 - \theta_1 + \theta_3), && (73) \\ 0 &= \frac{\partial L}{\partial \theta_3} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \theta_3} \implies \ddot\theta_3 = -\frac{k}{m}(2\theta_3 - \theta_1 - \theta_2). && (74) \end{aligned}$$

Then (72)–(74) may be rewritten in a matrix form in the basis $\boldsymbol{\theta}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\end{bmatrix} = -\frac{k}{m}\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}\begin{bmatrix}\theta_1\\\theta_2\\\theta_3\end{bmatrix} \equiv -\frac{k}{m}A\boldsymbol{\theta}, \tag{75}$$

where we have defined the matrix $A$ and the vector $\boldsymbol{\theta}$. Let $\lambda$ be an eigenvalues of $A$. Then

$$0 = \begin{vmatrix}2-\lambda & -1 & -1\\-1 & 2-\lambda & -1\\-1 & -1 & 2-\lambda\end{vmatrix} = (2-\lambda)^3 - (2-\lambda) - 1 - (2-\lambda) - 1 - (2-\lambda) \tag{76}$$

$$= (2-\lambda)^3 - 3(2-\lambda) - 2 = -\lambda^3 + 6\lambda^2 - 9\lambda. \tag{77}$$

By inspection of (77), $\lambda \in \{0, 3, 3\}$. Thus the normal modes of the system are degenerate. The eigenfrequencies of the system are given by $\omega^2 = \lambda k/m$.

In order to find the normal modes, we must find eigenvectors $\mathbf{v}$ corresponding to $\lambda \in \{0, 3, 3\}$. Beginning with $\lambda = 3$, we need to find $v_1, v_2, v_3$ such that

$$A\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = 3\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}. \tag{78}$$

The algebraic equations corresponding to (78) are

$$2v_1 - v_2 - v_3 = 3v_1 \implies v_1 = -(v_2 + v_3), \tag{79}$$

$$2v_2 - v_1 - v_3 = 3v_2 \implies v_2 = -(v_1 + v_3), \tag{80}$$

$$2v_3 - v_1 - v_2 = 3v_3 \implies v_3 = -(v_1 + v_2). \tag{81}$$

Inspecting (79)–(81), we may fix $v_1 = 0$ without loss of generality. Then we are left with $v_2 = -v_3$, so we may fix $v_2 = 1$ which implies $v_3 = -1$. Alternatively, we may instead fix $v_1 = 2$. Then we are left with $v_2 + v_3 = -2$, so we may fix $v_2 = -1$ which implies $v_3 = -1$. For the case $\lambda = 0$, the equations correspsonding to (79)–(81) are trivial and we may fix $v_1 = v_2 = v_3$ without loss of generality.

In summary, the normal modes and corresponding eigenfrequencies are as follows:

a. $m_1$ remains still while $m_2$ and $m_3$ oscillate out of phase with the same amplitude. The oscillation frequency and the eigenvector for this mode are

$$\omega_3 = \sqrt{3\frac{k}{m}}, \qquad \mathbf{v} = \begin{bmatrix}0\\1\\-1\end{bmatrix}. \tag{82}$$

b. $m_1$ oscillates out of phase with $m_2$ and $m_3$, which are in phase with each other, and have half the oscillation amplitude of $m_1$. The oscillation frequency and eigenvector for this mode are

$$\omega_3 = \sqrt{3\frac{k}{m}}, \qquad \mathbf{v} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}. \tag{83}$$

c. All three masses move in phase with the same velocity. This is the "zero mode" in which no actual oscillations occur; the system is simply rotating. This mode is associated with a continuous rotational symmetry. For completeness, the oscillation frequency and the eigenvector for this mode are

$$\omega_0 = 0, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{84}$$

In writing these solutions, I consulted David Tong's lecture notes, Goldstein's *Classical Mechanics*, and Landau and Lifshitz's *Mechanics*.