

**Problem 1.** Consider the charge density  $\rho(\mathbf{x})$  given by

$$\rho(\mathbf{x}) = \begin{cases} (R-r)(1-\cos\theta)^2 & \text{for } |\mathbf{x}| \leq R, \\ 0 & \text{for } |\mathbf{x}| \geq R. \end{cases} \quad (1)$$

Find the electrostatic potential,  $\phi(\mathbf{x})$ , of this charge distribution at all  $\mathbf{x}$  with  $|\mathbf{x}| \geq R$ .

**Solution.** The multipole expansion in spherical harmonics is given by Eq. (2.79) in the course notes,

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi), \quad (2)$$

where the spherical multipole moments  $q_{lm}$  are defined in Eq. (2.80),

$$q_{lm} \equiv \int \rho(\mathbf{x}') r'^l Y_{lm}^*(\theta', \phi') d^3x'.$$

Note that (2) is valid only for  $|\mathbf{x}| \geq R$  when the charge distribution  $\rho(\mathbf{x}')$  is nonzero only within  $|\mathbf{x}'| \leq R$ , which is the regime we are interested in here.

The spherical harmonics  $Y_{lm}$  are given by Eq. (2.58),

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi},$$

and the Lagrange polynomials  $P_l^m$  are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l,$$

although in practice I am taking all spherical harmonics from the table in Jackson.

We can write the angular component of  $\rho(\mathbf{x})$  as an expansion of spherical harmonics. Inspecting (1), we will only have terms of  $l = 0, 1, 2$  and  $m = 0$ . The relevant spherical harmonics are

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta, \quad Y_{20}(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2\theta - \frac{1}{2} \right).$$

Then we have

$$\begin{aligned} \rho(r, \theta, \phi) &= (R-r)(1-2\cos\theta+\cos^2\theta) \\ &= (R-r) \left( \frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) - 2 \sqrt{\frac{4\pi}{3}} Y_{10}(\theta, \phi) + 4 \frac{\sqrt{4\pi}}{3} Y_{00}(\theta, \phi) \right). \end{aligned}$$

The only nonzero  $q_{lm}$  are  $q_{00}$ ,  $q_{10}$ , and  $q_{20}$ :

$$\begin{aligned} q_{00} &= \int_0^{2\pi} \int_{-1}^1 \int_0^R \rho(\mathbf{x}') r'^0 Y_{00}^*(\theta', \phi') r' dr' d(\cos\theta') d\phi' \\ &= 4 \frac{\sqrt{4\pi}}{3} \int_0^{2\pi} \int_{-1}^1 Y_{00}^*(\theta', \phi') Y_{00}(\theta', \phi') d(\cos\theta') d\phi' \int_0^R (R-r') r' dr' \\ &= 4 \frac{\sqrt{4\pi}}{3} \left[ \frac{Rr'^2}{2} - \frac{r'^3}{3} \right]_0^R = 4 \frac{\sqrt{4\pi}}{3} \frac{R^3}{6} = \frac{4\sqrt{\pi}}{9} R^3, \end{aligned}$$

$$\begin{aligned}
 q_{10} &= \int_0^{2\pi} \int_{-1}^1 \int_0^R \rho(\mathbf{x}') r'^1 Y_{10}^*(\theta', \phi') r' dr' d(\cos \theta') d\phi' \\
 &= -2\sqrt{\frac{4\pi}{3}} \int_0^{2\pi} \int_{-1}^1 Y_{10}^*(\theta', \phi') Y_{10}(\theta', \phi') d(\cos \theta') d\phi' \int_0^R (R - r') r'^2 dr' \\
 &= -2\sqrt{\frac{4\pi}{3}} \left[ \frac{Rr'^3}{3} - \frac{r'^4}{4} \right]_0^R = -2\sqrt{\frac{4\pi}{3}} \frac{R^4}{12} = -\frac{1}{3}\sqrt{\frac{\pi}{3}} R^4, \\
 q_{20} &= \int_0^{2\pi} \int_{-1}^1 \int_0^R \rho(\mathbf{x}') r'^2 Y_{20}^*(\theta', \phi') r' dr' d(\cos \theta') d\phi' \\
 &= \frac{2}{3}\sqrt{\frac{4\pi}{5}} \int_0^{2\pi} \int_{-1}^1 Y_{20}^*(\theta', \phi') Y_{20}(\theta', \phi') d(\cos \theta') d\phi' \int_0^R (R - r') r'^3 dr' \\
 &= \frac{2}{3}\sqrt{\frac{4\pi}{5}} \left[ \frac{Rr'^4}{4} - \frac{r'^5}{5} \right]_0^R = \frac{2}{3}\sqrt{\frac{4\pi}{5}} \frac{R^5}{20} = \frac{1}{15}\sqrt{\frac{\pi}{5}} R^5.
 \end{aligned}$$

Then  $\phi$  is given by

$$\begin{aligned}
 \phi(\mathbf{x}) &= \frac{4\pi}{1} \frac{q_{00}}{r^1} Y_{00}(\theta, \phi) + \frac{4\pi}{2+1} \frac{q_{10}}{r^2} Y_{10}(\theta, \phi) + \frac{4\pi}{5} \frac{q_{20}}{r^3} Y_{20}(\theta, \phi) \\
 &= (4\pi) \frac{4\sqrt{\pi}}{9} \frac{R^3}{r} \frac{1}{\sqrt{4\pi}} - \frac{4\pi}{3} \frac{1}{3} \sqrt{\frac{\pi}{3}} \frac{R^4}{r^2} \sqrt{\frac{3}{4\pi}} \cos \theta + \frac{4\pi}{5} \frac{1}{15} \sqrt{\frac{\pi}{5}} \frac{R^5}{r^3} \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \\
 &= \frac{8\pi}{9} \frac{R^3}{r} - \frac{2\pi}{9} \frac{R^4}{r^2} \cos \theta + \frac{\pi}{75} \frac{R^5}{r^3} (2 \cos^2 \theta - 1).
 \end{aligned}$$

**Problem 2.** Let  $\mathcal{V}$  be an arbitrary bounded region of space and suppose that a total charge  $Q$  is to be distributed in  $\mathcal{V}$  in an arbitrary way, with  $\rho = 0$  outside of  $\mathcal{V}$ . Show that the total energy is minimized if the charge is distributed the way that it would be if  $\mathcal{V}$  were a conductor, so that  $\phi = \text{const.}$  within  $\mathcal{V}$  (and thus, in particular, all of the charge lies on the boundary of  $\mathcal{V}$ ).

Hint: Let  $\phi_0(\mathbf{x})$  be the potential one would obtain if  $\mathcal{V}$  were filled by a conducting body. Consider the energy of  $\phi_0 + \phi'$ , where the source  $\rho'$  of  $\phi'$  vanishes outside of  $\mathcal{V}$  and has no net charge within  $\mathcal{V}$ .

**Solution.** Let  $S = \partial\mathcal{V}$  denote the boundary of  $\mathcal{V}$ . We separate space into three mutually exclusive regions:  $\mathcal{V}$ ,  $S$ , and the region outside (in which we are not interested). By the superposition principle, we may write

$$\rho = \rho_0 + \rho', \quad \phi = \phi_0 + \phi',$$

where  $\rho_0$  is the charge of a conducting body filling  $\mathcal{V}$ ,  $\phi_0$  is the electrostatic potential due to  $\rho_0$ ,  $\rho'$  is the charge distribution within  $\mathcal{V}$ , and  $\phi'$  is the electrostatic potential due to  $\rho'$ . In order to eliminate ambiguity on the boundary, we require

$$\rho_0|_{\mathcal{V}} = 0, \quad \rho'|_S = 0. \tag{3}$$

That is,  $\rho_0 = 0$  inside the conductor by definition, and  $\rho'$  vanishes on the boundary where  $\rho_0$  is nonzero. For the entire body to have charge  $Q$ , we need

$$\int \rho_0 d^3x = \int_{\mathcal{V}} \rho' d^3x + \int_S \rho_0 dS = Q.$$

From (3), it follows that

$$\phi_0|_{\mathcal{V}} = \phi_0|_S = \text{const.}$$

The total energy is given by Eq. (2.25) in the course notes,

$$\mathcal{E} = \frac{1}{2} \int \phi \rho \, d^3x.$$

So

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \int (\phi_0 + \phi')(\rho_0 + \rho') \, d^3x = \frac{1}{2} \left( \int \phi_0(\rho_0 + \rho') \, d^3x + \int \phi'(\rho_0 + \rho') \, d^3x \right) \\ &= \frac{1}{2} \left( \phi_0 Q + \int_{\mathcal{V}} \phi' \rho' \, d^3x + \int_S \phi' \rho_0 \, dS \right). \end{aligned}$$

help

**Problem 3.** Charge is distributed on a (nonconducting) sphere of radius  $R$ , i.e., the charge density throughout space is of the form  $\rho(\mathbf{x}) = \sigma(\theta, \varphi) \delta(r - R)$ . The surface charge distribution  $\sigma$  on the sphere is chosen in such a way that the electrostatic potential on the sphere is  $\phi(r = R, \theta, \varphi) = \alpha \cos \theta$ , where  $\alpha$  is a constant.

**3.a** Find the electrostatic potential  $\phi(\mathbf{x})$  at all  $r \leq R$ .

**Solution.** This is a Dirichlet boundary value problem. We are seeking the solution to Poisson's equation  $\nabla^2 \phi = -4\pi\rho$  subject to  $\phi|_S = \psi = \alpha \cos \theta$ . Equation (2.100) in the lecture notes gives the general solution,

$$\phi(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') \, d^3x' - \frac{1}{4\pi} \int_S \psi(\mathbf{x}') \hat{\mathbf{n}}' \cdot \nabla_{\mathbf{x}'} G_D(\mathbf{x}', \mathbf{x}) \, dS_{x'}.$$

The Green's function for a sphere is given by Eq. (2.91),

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + \frac{\beta}{|\mathbf{x} - \mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}' \frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \beta = -\frac{R}{|\mathbf{x}'|}.$$

Spherical harmonic expansion (2.78?)

**3.b** Find the electrostatic potential  $\phi(\mathbf{x})$  at all  $r \geq R$ .

**3.c** Find the surface charge density  $\sigma(\theta, \varphi)$  that was required in order to produce this potential  $\phi$ .

**3.d** Find the total electrostatic energy.

**Problem 4.** A point charge of charge  $q$  is placed at point  $\mathbf{x}'$  inside a conducting spherical shell of radius  $R$ . There is no net charge on the conductor. The potential inside the sphere is thus given by  $q G_D(\mathbf{x}, \mathbf{x}')$ , where the explicit formula for  $G_D(\mathbf{x}, \mathbf{x}')$  for a spherical cavity is given in the lecture notes.

**4.a** Find the surface charge density  $\sigma(\theta, \varphi)$  on the conducting shell.

**4.b** Find the force  $\mathbf{F}$  that must be exerted on the point charge in order to hold it in place.

**Problem 5.** The “mean value theorem” is stated as follows: For any solution  $\phi$  to  $\nabla^2\phi = 0$ , the value of  $\phi$  at  $\mathbf{x}$  is equal to the average value of  $\phi$  on a sphere of radius  $R$  (for any  $R$ ) centered at  $\mathbf{x}$ .

**5.a** Prove the mean value theorem. Hint: Apply Green’s theorem to  $\phi$  and  $1/|\mathbf{x} - \mathbf{x}'|$  for a suitable choice of region and a suitable choice of  $\mathbf{x}'$ .

**5.b** Use this result to show that a point charge can never be in stable equilibrium if placed in an electric field  $\mathbf{E}$  that is source free in a neighborhood of the charge—and, indeed, it can be in neutral equilibrium only if  $\mathbf{E} = 0$  in this neighborhood.