Problem 1. Alternative regulators in QED (Peskin & Schroeder 7.2) In Section. 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Paul-Villars regularization.

1(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

Solution. write the game plan here

In order to find $\delta\Gamma^\mu$ we use Peskin & Schroeder (6.47):

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\frac{2}{D^3} \\ \times \bar{u}(p')\left\{\gamma^\mu\left[-\frac{\ell^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right]\right. \\ \left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m}[2m^2 z(1-z)]\right\} u(p),\end{aligned} \tag{1}$$

where $\Delta \equiv -xyq^2 + (1-z)^2m^2$ by (6.44), $\ell \equiv k + yq - zp$, and $D = \ell^2 - \Delta + i\epsilon$ [**?** , p. 191]. The momenta k and p are assigned in the Feynman diagram on p. 189 of Peskin & Schroeder, and x, y, z are Feynman parameters [**?** , p. 190].

The forms of the two integrals we need to compute are given by Peskin & Schroeder (6.49) and (6.50). Equation (6.49) is

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}.$$

Here $m = 3$ because we have D^{-3} in Eq. (1). We can evaluate the left-hand side using the Euclidian 4-momentum defined in (6.48),

$$\ell^0 \equiv \ell_E^0, \qquad\qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E.$$

Following the steps on p. 193, we have

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} = \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int \frac{d^4\ell_E}{(\ell_E^2+\Delta)^3} = \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\, \frac{\ell_E^3}{(\ell_E^2+\Delta)^3},$$

where we have replaced the upper (infinite) bound of integration by a finite number Λ. Evaluating this integral using Mathematica and using $\int d\Omega_4 = 2\pi^2$ [**?** , p. 193], we find

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} = \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{\Lambda^4}{4\Delta(\Delta+\Lambda^2)^2} = -\frac{i}{8\pi^2}\frac{\Lambda^4}{4\Delta(\Delta+\Lambda^2)^2}. \tag{2}$$

Equation (6.50) in Peskin & Schroeder is

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}.$$

Following similar steps as for Eq. (1), the left-hand side is

$$
\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta)^3} &= -\frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int d^4\ell_E\, \frac{\ell_E^2}{(\ell_E^2+\Delta)^3} \\
&= \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\, \frac{\ell_E^5}{(\ell_E^2+\Delta)^3} \\
&= -\frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{1}{4}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right] \\
&= \frac{i}{32\pi^2}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right] \\
&\approx \frac{i}{32\pi^2}\ln(\Lambda). \qquad (3)
\end{aligned}
$$

Feeding Eqs. (2) and (3) into Eq. (1),