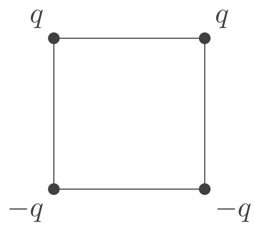
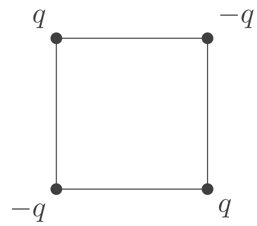
Problem 1. A square of side d lies in the $z = 0$ plane, with the center of the square lying at the origin, $x = y = z = 0$. Charges are placed on the corners of the square in the two different manners, (i) and (ii), shown below. The square is set into rotation with angular velocity Ω about the z axis, where $\Omega d \ll c$. In each case, the radiated power, P, will scale with q, d, and Ω as

$$P \propto q^{n_1} d^{n_2} \Omega^{n_3}$$

for some integers n_1, n_2, n_3. Find n_1, n_2, n_3 for case (i) and for case (ii).

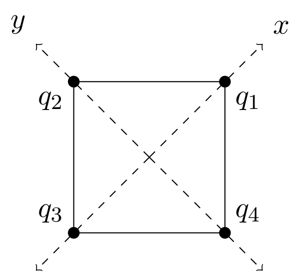


(i)



(ii)

Solution. Inspecting the figures, case (i) is a pure dipole and case (ii) is a pure quadrupole. We will define the x and y axes such that the positive x axis points toward the upper-right corner of each square, and label the charges as shown below.



We can handle case (i) easily using the multipole expansion for the total power given by Eq. (5.75), which goes up to dipole order:

$$P = \frac{2}{3c^3}\left|\frac{d^2\mathbf{p}}{dt^2}\right|^2_{\text{ret}}, \tag{1}$$

where the dipole moment is defined by Eq. (2.36),

$$\mathbf{p} = \int \mathbf{x}\rho\, d^3x\,. \tag{2}$$

Note that each charge is a distance $d/\sqrt{2}$ from the origin. The locations of each of the point charges are

$$\begin{aligned} \mathbf{x}_1(t) &= \frac{d}{\sqrt{2}}(\cos\Omega t\,\hat{\mathbf{x}} + \sin\Omega t\,\hat{\mathbf{y}}), & \mathbf{x}_2(t) &= \frac{d}{\sqrt{2}}(-\sin\Omega t\,\hat{\mathbf{x}} + \cos\Omega t\,\hat{\mathbf{y}}), \\ \mathbf{x}_3(t) &= \frac{d}{\sqrt{2}}(-\cos\Omega t\,\hat{\mathbf{x}} - \sin\Omega t\,\hat{\mathbf{y}}), & \mathbf{x}_4(t) &= \frac{d}{\sqrt{2}}(\sin\Omega t\,\hat{\mathbf{x}} - \cos\Omega t\,\hat{\mathbf{y}}), \end{aligned} \tag{3}$$

The charge densities for the point charges are

$$\rho_1(t,\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}_1), \qquad \rho_2(t,\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}_2), \qquad \rho_3(t,\mathbf{x}) = -q\,\delta(\mathbf{x}-\mathbf{x}_3), \qquad \rho_4(t,\mathbf{x}) = -q\,\delta(\mathbf{x}-\mathbf{x}_4).$$

Then the dipole moment is

$$\mathbf{p} = \int \mathbf{x}[\rho_1(t,\mathbf{x}) + \rho_2(t,\mathbf{x}) + \rho_3(t,\mathbf{x}) + \rho_4(t,\mathbf{x})]\, d^3x = q[\mathbf{x}_1(t) + \mathbf{x}_2(t) - \mathbf{x}_3(t) - \mathbf{x}_4(t)]$$

$$= \sqrt{2} qd[(\cos\Omega t - \sin\Omega t)\,\hat{\mathbf{x}} + (\cos\Omega t + \sin\Omega t)\,\hat{\mathbf{y}}],$$

and its time derivatives are

$$\frac{d\mathbf{p}}{dt} = \sqrt{2} qd\Omega[-(\sin\Omega t + \cos\Omega t)\,\hat{\mathbf{x}} + (\cos\Omega t - \sin\Omega t)\,\hat{\mathbf{y}}],$$
$$\frac{d^2\mathbf{p}}{dt^2} = \sqrt{2} qd\Omega^2[(\cos\Omega t - \sin\Omega t)\,\hat{\mathbf{x}} - (\sin\Omega t + \sin\Omega t)\,\hat{\mathbf{y}}].$$

Replacing t with the retarded time will make no difference to the proportionalities in the above expression, since the time dependence is only in the trigonometric arguments. Thus we have

$$P \propto \left|\frac{d^2\mathbf{p}}{dt^2}\right|^2 \propto q^2 d^2 \Omega^4,$$

so for case (i),

$$n_1 = n_2 = 2, \qquad\qquad n_3 = 4.$$

For case (ii), we need to go to quadrupole order in the multipole expansion. According to the final paragraph of Sec. 5.3.1 in the course notes, the "electric quadrupole radiation" gives a contribution to radiated power proportional to $\left|d^3 Q_{ij}/dt^3\right|^2$. At this order, there is also a contribution to the power from "magnetic dipole radiation" proportional to $\left|d^2\boldsymbol{\mu}/dt^2\right|^2$, where the magnetic dipole moment $\boldsymbol{\mu}$ is defined by (4.32),

$$\boldsymbol{\mu} \equiv \frac{1}{2c}\int \mathbf{x}\times\mathbf{J}(\mathbf{x})\,d^3x\,.$$

However, from (5.68),

$$\int \mathbf{J}(\mathbf{x})\,d^3x = \frac{\partial\mathbf{p}}{\partial t},$$

and $\mathbf{p} = 0$ for case (ii). So the "electric quadrupole radiation" is the only term that contributes for case (ii), and we have

$$P \propto \left|\frac{d^3 Q_{ij}}{dt^3}\right|^2_{\text{ret}},$$

where the quadrupole moment is defined by Eq. (2.47),

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho\,d^3x\,.$$

The locations of the point charges are again given by (3), but now their charge densities are

$$\rho_1(t,\mathbf{x}) = -q\,\delta(\mathbf{x}-\mathbf{x}_1), \qquad \rho_2(t,\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}_2), \qquad \rho_3(t,\mathbf{x}) = -q\,\delta(\mathbf{x}-\mathbf{x}_3), \qquad \rho_4(t,\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}_4).$$

Recall that Q_{ij} is symmetric. Its elements are

$$\begin{aligned}
Q_{11} &= \int (3x^2 - x^2 - y^2 - z^2)[\rho_1(t,\mathbf{x}) + \rho_2(t,\mathbf{x}) + \rho_3(t,\mathbf{x}) + \rho_4(t,\mathbf{x})]\,d^3x \\
&= q\int (2x^2 - y^2 - z^2)[-\delta(\mathbf{x}-\mathbf{x}_1) + \delta(\mathbf{x}-\mathbf{x}_2) - \delta(\mathbf{x}-\mathbf{x}_3) + \delta(\mathbf{x}-\mathbf{x}_4)]\,d^3x \\
&= \frac{qd^2}{2}(-2\cos^2\Omega t - \sin^2\Omega t + 2\sin^2\Omega t + \cos^2\Omega t - 2\cos^2\Omega t - \sin^2\Omega t + 2\sin^2\Omega t + \cos^2\Omega t) \\
&= qd^2(\sin^2\Omega t - \cos^2\Omega t) = -qd^2\cos 2\Omega t,
\end{aligned}$$

$$
\begin{aligned}
Q_{12} &= 3\int xy[\rho_1(t,\mathbf{x}) + \rho_2(t,\mathbf{x}) + \rho_3(t,\mathbf{x}) + \rho_4(t,\mathbf{x})]\, d^3x \\
&= 3q\int [-\delta(\mathbf{x}-\mathbf{x}_1) + \delta(\mathbf{x}-\mathbf{x}_2) - \delta(\mathbf{x}-\mathbf{x}_3) + \delta(\mathbf{x}-\mathbf{x}_4)]\, d^3x = -6qd^2 \sin\Omega t \cos\Omega t = -3qd^2 \sin 2\Omega t \\
&= Q_{21},
\end{aligned}
$$

$$
\begin{aligned}
Q_{22} &= q\int (2y^2 - x^2 - z^2)[-\delta(\mathbf{x}-\mathbf{x}_1) + \delta(\mathbf{x}-\mathbf{x}_2) - \delta(\mathbf{x}-\mathbf{x}_3) + \delta(\mathbf{x}-\mathbf{x}_4)]\, d^3x = qd^2(\cos^2\Omega t - \sin^2\Omega t) \\
&= -Q_{11},
\end{aligned}
$$

$$
Q_{13} = Q_{23} = Q_{33} = 0.
$$

All of the nonzero elements have the same powers of d, q, cos, and sin, as we might have guessed from dimensional analysis. Thus, we need only differentiate one of them to find the general proportionality. Differentiating Q_{11} with respect to time,

$$
\frac{dQ_{11}}{dt} = 2qd^2\Omega \sin 2\Omega t, \qquad \frac{d^2Q_{11}}{dt^2} = 4qd^2\Omega^2 \cos 2\Omega t, \qquad \frac{d^3Q_{11}}{dt^3} = -8qd^2\Omega^3 \sin 2\Omega t.
$$

Again, it makes no difference in the proportionality whether we evaluate the above at the retarded time. Thus,

$$
P \propto \left|\frac{d^3Q_{11}}{dt^3}\right|^2 \propto q^2 d^4 \Omega^6,
$$

so for case (ii),

$$
n_1 = 2, \qquad n_2 = 4, \qquad n_3 = 6.
$$

Problem 2. A point charge of charge q and mass m is placed on the end of a spring of spring constant k. The charge is displaced in the z direction by an amount α away from its equilibrium position and is then released into oscillation. Assume that the resulting motion is nonrelativistic, $v \ll c$.

2.a Assume that the charge oscillates harmonically with amplitude α. To order $1/r$ in distance from the charge and to leading order in v/c, what are the resulting electromagnetic potentials $\phi, \mathbf{A}$?

Solution. The multipole expansions of ϕ and $\mathbf{A}$ are given to order $1/r$ in distance and to leading order in v/c in Eqs. (5.66) and (5.69),

$$
\phi(t,\mathbf{x}) = \frac{q}{|\mathbf{x}|} + \frac{1}{c|\mathbf{x}|}\hat{\mathbf{x}}\cdot\left.\frac{d\mathbf{p}}{dt}\right|_{\text{ret}}, \qquad \mathbf{A}(t,\mathbf{x}) = \frac{1}{c|\mathbf{x}|}\left.\frac{d\mathbf{p}}{dt}\right|_{\text{ret}},
$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and the retarded time $t' = t - |\mathbf{x}|/c$ to this order.

The charge's position is

$$
\mathbf{x}'(t) = \alpha\cos\omega t\,\hat{\mathbf{z}},
$$

where $\omega = \sqrt{k/m}$, and the charge density is

$$
\rho(t,\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}').
$$

Then from (2), the dipole moment is

$$
\mathbf{p} = q\int \mathbf{x}\,\delta(\mathbf{x}-\mathbf{x}')\, d^3x = q\mathbf{x}' = q\alpha\cos\omega t\,\hat{\mathbf{z}},
$$

which has the time derivative

$$\frac{d\mathbf{p}}{dt} = -q\alpha\omega \sin\omega t\,\hat{\mathbf{z}}. \tag{4}$$

The potentials are then

$$\begin{aligned}\phi(t,\mathbf{x}) &= \frac{q}{r} - \frac{q\alpha\omega}{cr}\sin\Big(\omega t - \frac{\omega}{c}r\Big)(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}) = \frac{q}{r}\left[1 - \frac{\alpha\omega\cos\theta}{c}\sin\Big(\omega t - \frac{\omega}{c}r\Big)\right],\\ \mathbf{A}(t,\mathbf{x}) &= -\frac{q}{r}\frac{\alpha\omega}{c}\sin\Big(\omega t - \frac{\omega}{c}r\Big)\,\hat{\mathbf{z}},\end{aligned}$$

where we have used $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$.

2.b What is the radiated power?

Solution. To this order, we may use (1) with the retarded time $t' = t - |\mathbf{x}|/c$. Differentiating (4), we have

$$\frac{d^2\mathbf{p}}{dt^2} = -q\alpha\omega^2\cos\omega t\,\hat{\mathbf{z}},$$

so the radiated power is

$$P = \frac{2}{3c^3}\Big|-q\alpha\omega^2\cos\omega t\,\hat{\mathbf{z}}\Big|^2_{\text{ret}} = \frac{2q^2\alpha^2\omega^4}{3c^3}\left|\cos^2\Big(\omega t - \frac{\omega}{c}r\Big)\right| = \frac{q^2\alpha^2\omega^4}{3c^3}, \tag{5}$$

where we have used the time average $\left|\cos^2 t\right| = 1/2$.

2.c As a result of the radiation of electromagnetic energy, the maximum amplitude of oscillation, α, will slowly decay with time. Find $\alpha(t)$.

Solution. The quality factor Q characterizes energy loss, and is defined in Jackson (8.86) as

$$Q = \omega_0\frac{\text{Stored energy}}{\text{Power loss}},$$

where ω_0 is a resonant frequency. For the oscillating charge,

$$Q = \omega\frac{\mathscr{E}_{\text{osc}}}{P},$$

where $\mathscr{E}_{\text{osc}}$ is the energy stored in the oscillator, and P is the radiated power given by (5). Adapting Eq. (5.75) in the course notes, we can write

$$P = -\frac{d\mathscr{E}_{\text{osc}}}{dt}.$$

Applying the definition of Q gives us a differential equation:

$$\frac{d\mathscr{E}}{dt} = -\frac{\omega}{Q}\mathscr{E}_{\text{osc}} \quad\Longrightarrow\quad \int\frac{d\mathscr{E}_{\text{osc}}}{\mathscr{E}_{\text{osc}}} = -\frac{\omega}{Q}\int dt \quad\Longrightarrow\quad \ln\frac{\mathscr{E}_{\text{osc}}}{\mathscr{E}_{\text{osc}}(0)} = -\frac{\omega}{Q}t,$$

and the final solution is

$$\mathscr{E}_{\text{osc}}(t) = \mathscr{E}_{\text{osc}}(0)\,e^{-\omega t/Q}, \tag{6}$$

where $\mathscr{E}_{\text{osc}}(0)$ is the initial energy stored in the oscillator.

The energy stored in an oscillator at any given time is equivalent to its maximal potential or kinetic energy,

$$\mathscr{E}_{\text{osc}} = \frac{m\omega^2\alpha^2}{2} = \frac{k\alpha^2}{2},$$

and the average power radiated over one oscillation comes directly from (5). We can use these to write Q with no α dependence:

$$Q = \omega\frac{m\omega^2\alpha^2}{2}\frac{3c^3}{q^2\alpha^2\omega^4} = \frac{3mc^3}{2q^2\omega}.$$

Substituting these into (6), we obtain

$$\frac{k}{2}\alpha^2(t) = \frac{k}{2}\alpha^2(0)\,\exp\left(-\omega t\frac{2q^2\omega}{3mc^3}\right) \quad\Longrightarrow\quad \alpha^2(t) = \alpha_0^2\,\exp\left(-\frac{2q^2\omega^2}{3mc^3}t\right)$$

which implies

$$\alpha(t) = \alpha_0\,\exp\left(-\frac{q^2\omega^2}{3mc^3}t\right),$$

where α_0 is the initial displacement from equilibrium.

Problem 3. Consider the wave equation with some arbitrary given "potential" $V(t, \mathbf{x})$,

$$\Box\psi + V\psi = -4\pi f.$$

Although one may not be able to find their form explicitly, it is possible to prove by general arguments that unique retarded and advanced Green's functions exist for this equation, satisfying the "support properties" that

$$G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') = 0 \quad \text{for} \quad t < t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \qquad G_{\text{adv}}(t, \mathbf{x}; t', \mathbf{x}') = 0 \quad \text{for} \quad t > t' + \frac{|\mathbf{x}-\mathbf{x}'|}{c}.$$

3.a Show that a Green's identity holds for any two solutions, ψ_1, ψ_2, of this equation that takes exactly the same form as Eq. (5.100) of the notes.

Solution. Equation (5.100) is

$$\sum_{\mu,\nu}\partial_\mu[\eta^{\mu\nu}(\psi_1\partial_\nu\psi_2 - \psi_2\partial_\nu\psi_1)] = \psi_1\Box\psi_2 - \psi_2\Box\psi_1 = -4\pi(\psi_1 f_2 - \psi_2 f_1), \tag{7}$$

where ψ_1 and ψ_2 are defined in Eqs. (5.98) and (5.99),

$$\Box\psi_1 = -4\pi f_1, \qquad \Box\psi_2 = -4\pi f_2.$$

For this problem, let

$$\Box\psi_1 + V\psi_1 = -4\pi f_1, \qquad \Box\psi_2 + V\psi_2 = -4\pi f_2. \tag{8}$$

Then

$$\begin{aligned}\sum_{\mu,\nu}\partial_\mu[\eta^{\mu\nu}(\psi_1\partial_\nu\psi_2 - \psi_2\partial_\nu\psi_1)] &= \psi_1\Box\psi_2 - \psi_2\Box\psi_1 = \psi_1(-4\pi f_2 - V\psi_2) - \psi_2(-4\pi f_1 - V\psi_1)\\ &= -4\pi f_2\psi_1 + \psi_1 V\psi_2 + 4\pi f_1\psi_2 + \psi_2 V\psi_1 = -4\pi(\psi_1 f_2 - \psi_2 f_1),\end{aligned}$$

which has the same form as (7), as we wanted to show. □

3.b Prove that for the wave equation with a potential, $G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') = G_{\text{adv}}(t', \mathbf{x}'; t, \mathbf{x})$.

Hint: Integrate Green's identity over a suitable spacetime cube for suitable choices of ψ_1 and ψ_2.

Solution. Let x^μ be an arbitrary vector for $\mu = 0, 1, 2, 3$, and let $\mathscr{R}$ be the rectangular region on spacetime defined by $c_0^\mu \le x^\mu \le c_1^\mu$, where c_0^μ, c_1^μ are constants. Let $v^\mu = (v^0, v^1, v^2, v^3)$ be an arbitrary differentiable vector field. Then by (5.101),

$$\int_{\mathscr{R}} \sum_\mu \partial_\mu v^\mu \, d^4x = \left[\int v^0 \, dx^1 \, dx^2 \, dx^3\right]_{c_0^0}^{c_1^0} + \left[\int v^1 \, dx^0 \, dx^2 \, dx^3\right]_{c_0^1}^{c_1^1} + \left[\int v^2 \, dx^0 \, dx^1 \, dx^3\right]_{c_0^2}^{c_1^2} + \left[\int v^3 \, dx^0 \, dx^1 \, dx^2\right]_{c_0^3}^{c_1^3},$$

where $\mathscr{R}$ be the rectangular region on spacetime defined by $c_0^\mu \le x^\mu \le c_1^\mu$.

Following the procedure used in Sec. 2.5 to show that the Dirichlet Green's function is symmetrical, let $\psi_1 = G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}')$ and $\psi_2 = G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'')$ with $x'^0, x''^0 > 0$. As in (5.102), let

$$v^\mu = \sum_\nu \eta^{\mu\nu}(\psi_1 \partial_\nu \psi_2 - \psi_2 \partial_\nu \psi_1).$$

Now we can integrate (7) over $\mathscr{R}$, since we have proven that it holds for (8). Green's theorem is

$$\int_{\mathscr{R}} \sum_\mu \partial_\mu v^\mu \, d^4x = -4\pi \int_{\mathscr{R}} (\psi_1 f_2 - \psi_2 f_1) \, d^4x \,. \tag{9}$$

For $\mathscr{R}$, we choose c_0^μ, c_1^μ such that the bottom face $x^0 = c_0^0 = 0$, the top face $x^0 = c_1^0 > x'^0, x''^0$, and the side faces $c_0^\mu < x'^a - x'^0, x''^a - x''^0$ and $c_1^\mu > x'^a + x'^0, x''^a + x''^0$ for $a = 1, 2, 3$. This way, both the past and future light cones are enclosed in $\mathscr{R}$, and the light cones do not intersect the side faces. Then neither G_{ret} nor G_{adv} contributes to the side faces. So the left side of (9) becomes

$$\begin{aligned}\int_{\mathscr{R}} \sum_\mu \partial_\mu v^\mu \, d^4x &= \left[-\int [G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') \, \partial_0 \, G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'') - G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'') \, \partial_0 \, G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}')] \, dx^1 \, dx^2 \, dx^2\right]_{c_0^0}^{c_1^0} \\ &= 0,\end{aligned} \tag{10}$$

since $G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') = 0$ on the bottom face and $G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'') = 0$ on the top face.

For the right side of (9), applying (7) as permitted by our proof in 3.a,

$$-4\pi \int_{\mathscr{R}} (\psi_1 f_2 - \psi_2 f_1) \, d^4x = \int_{\mathscr{R}} [G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') \, \Box G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'') - G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'') \, \Box G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}')] \, d^4x \,.$$

We know that all Green's functions are solutions to Eq. (5.30),

$$\Box_{(t,\mathbf{x})} G(t, x; t', x') = -4\pi \, \delta(\mathbf{x} - \mathbf{x}') \, \delta(t - t'),$$

so the right side of (9) becomes

$$\begin{aligned}-4\pi \int_{\mathscr{R}} (\psi_1 f_2 - \psi_2 f_1) \, d^4x &= -4\pi \int_{\mathscr{R}} [G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') \, \delta(\mathbf{x} - \mathbf{x}'') \, \delta(t - t'') - G_{\text{adv}}(t, \mathbf{x}; t'', \mathbf{x}'') \, \delta(\mathbf{x} - \mathbf{x}') \, \delta(t - t')] \, d^4x \\ &= -4\pi [G_{\text{ret}}(t'', \mathbf{x}''; t', \mathbf{x}') - G_{\text{adv}}(t', \mathbf{x}'; t'', \mathbf{x}'')].\end{aligned} \tag{11}$$

Equating the left side (10) with the right side (11), and letting $t'' \to t$ and $\mathbf{x}'' \to \mathbf{x}$, we have

$$G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') = G_{\text{adv}}(t', \mathbf{x}'; t, \mathbf{x}),$$

as we sought to prove. $\square$

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics*, Griffiths's *Introduction to Electrodynamics*, and P. R. LeClair's notes on accelerating charges while writing these solutions.