1

Find the Euler-Lagrange equation associated with the functional

$$J[u(x,y,z)] = \int_R \sqrt{1 + u_x^2 + u_y^2 + u_z^2}\, dx\, dy\, dz\,,$$

where R is a region in three-dimensional space.

Solution. We will assume $u(x,y,z)$ has explicit values on the boundary of R, $dx\,dy\,dz$. By the definition of the action,

$$J[u] = \int_R \mathcal{L}\, dx\, dy\, dz \implies \mathcal{L} = \sqrt{1 + u_x^2 + u_y^2 + u_z^2}.$$

In general, the Euler-Lagrange equation is

$$0 = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z}. \tag{1}$$

Here,

$$\frac{\partial \mathcal{L}}{\partial u} = 0, \qquad \frac{\partial \mathcal{L}}{\partial u_x} = \frac{\partial \mathcal{L}}{\partial u_x^2}\frac{\partial u_x^2}{\partial u_x} = \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2 + u_z^2}} = \frac{u_x}{\mathcal{L}} \qquad \frac{\partial \mathcal{L}}{\partial u_y} = \frac{u_y}{\mathcal{L}}, \qquad \frac{\partial \mathcal{L}}{\partial u_z} = \frac{u_z}{\mathcal{L}}.$$

For the $\partial/\partial x$ term of (1),

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} = \frac{\partial}{\partial x}\frac{u_x}{\mathcal{L}} = \frac{\partial u_x}{\partial x}\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} + \frac{\partial u_y}{\partial x}\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} + \frac{\partial u_z}{\partial x}\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}}$$

where

$$\begin{aligned}
\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} &= \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_x}\mathcal{L} - u_x\frac{\partial \mathcal{L}}{\partial u_x}\right) = \frac{1}{\mathcal{L}^2}\left(\mathcal{L} - u_x\frac{u_x}{\mathcal{L}}\right) = \frac{\mathcal{L}^2 - u_x^2}{\mathcal{L}^3}, &(2)\\
\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} &= \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_y}\mathcal{L} - u_x\frac{\partial \mathcal{L}}{\partial u_y}\right) = -\frac{u_x u_y}{\mathcal{L}^3}, &(3)\\
\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}} &= -\frac{u_x u_z}{\mathcal{L}^3}, &(4)
\end{aligned}$$

Generalizing (2)–(4) to the $\partial/\partial y$ and $\partial/\partial z$ terms,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} &= u_{xx}\frac{\mathcal{L}^2 - u_x^2}{\mathcal{L}^3} - u_{yx}\frac{u_x u_y}{\mathcal{L}^3} - u_{zx}\frac{u_x u_z}{\mathcal{L}^3},\\
\frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} &= u_{yy}\frac{\mathcal{L}^2 - u_y^2}{\mathcal{L}^3} - u_{xy}\frac{u_x u_y}{\mathcal{L}^3} - u_{zy}\frac{u_y u_z}{\mathcal{L}^3},\\
\frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z} &= u_{zz}\frac{\mathcal{L}^2 - u_z^2}{\mathcal{L}^3} - u_{xz}\frac{u_x u_z}{\mathcal{L}^3} - u_{yz}\frac{u_y u_z}{\mathcal{L}^3}.
\end{aligned}$$

Then, assuming $u_{xy} = u_{yx}$, $u_{yz} = u_{zy}$, and $u_{xz} = u_{zx}$, (1) becomes

$$\begin{aligned}
0 &= u_{xx}(\mathcal{L}^4 - u_x^2) + u_{yy}(\mathcal{L}^4 - u_y^2) + u_{zz}(\mathcal{L}^4 - u_z^2) - 2u_{xy}u_x u_y - 2u_{yz}u_y u_z - 2u_{xz}u_x u_z\\
&= (u_{xx} + u_{yy} + u_{zz})(1 + u_x^2 + u_y^2 + u_z^2) - u_{xx}u_x^2 - u_{yy}u_y^2 - u_{zz}u_z^2 - 2u_{xy}u_x u_y - 2u_{yz}u_y u_z - 2u_{xz}u_x u_z\\
&= u_{xx}(1 + u_y^2 + u_z^2) + u_{yy}(1 + u_x^2 + u_z^2) + u_{zz}(1 + u_x^2 + u_y^2) - 2u_{xy}u_x u_y - 2u_{yz}u_y u_z - 2u_{xz}u_x u_z.
\end{aligned}$$

2 Plate vibrations (preliminaries)

Start from Green's theorem

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial R} (P\, dx + Q\, dy), \tag{5}$$

where R is the region in the xy plane spanned by the plate, and $dx\, dy\, dz$ its boundary.

2.a Show that

$$\int_R \phi \frac{\partial^2 \psi}{\partial x^2}\, dx\, dy = \int_R \psi \frac{\partial^2 \phi}{\partial x^2}\, dx\, dy + \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right) dy\,. \tag{6}$$

Solution. In (5), let

$$Q = \phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x}, \qquad P = 0.$$

Then

$$\frac{\partial Q}{\partial x} = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \phi \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial x} - \psi \frac{\partial^2 \phi}{\partial x^2} = \phi \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \phi}{\partial x^2}, \qquad \frac{\partial P}{\partial y} = 0.$$

Making these substitutions into (5) gives

$$\begin{aligned} \int_R \left(\phi \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \phi}{\partial x^2} \right) dx\, dy &= \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right) dy \\ \iff \int_R \phi \frac{\partial^2 \psi}{\partial x^2}\, dx\, dy &= \int_R \psi \frac{\partial^2 \phi}{\partial x^2}\, dx\, dy + \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right) dy \end{aligned}$$

as desired. □

2.b Work out analogous expressions for

$$\int_R \phi \frac{\partial^2 \psi}{\partial y^2}\, dx\, dy\,, \tag{7}$$

$$\int_R \phi \frac{\partial^2 \psi}{\partial x \partial y}\, dx\, dy\,. \tag{8}$$

Solution. For (7), let

$$Q = 0, \qquad P = \psi \frac{\partial \phi}{\partial y} - \phi \frac{\partial \psi}{\partial y},$$

in (5). Then, similarly to the proof for (6),

$$\frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial P}{\partial y} = \psi \frac{\partial^2 \phi}{\partial y^2} - \phi \frac{\partial^2 \psi}{\partial y^2}.$$

Substituting into (5),

$$\begin{aligned} \int_R \left(\phi \frac{\partial^2 \psi}{\partial y^2} - \psi \frac{\partial^2 \phi}{\partial y^2} \right) dx\, dy &= \int_{\partial R} \left(\psi \frac{\partial \phi}{\partial y} - \phi \frac{\partial \psi}{\partial y} \right) dx \\ \iff \int_R \phi \frac{\partial^2 \psi}{\partial y^2}\, dx\, dy &= \int_R \psi \frac{\partial^2 \phi}{\partial y^2}\, dx\, dy + \int_{\partial R} \left(\psi \frac{\partial \phi}{\partial y} - \phi \frac{\partial \psi}{\partial y} \right) dy\,. \end{aligned} \tag{9}$$

For (8), let

$$2Q = \phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}, \qquad\qquad 2P = \psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}.$$

Then

$$\begin{aligned}
2\frac{\partial Q}{\partial x} &= \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} + \phi\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y} = \phi\frac{\partial^2\psi}{\partial x\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y},\\
2\frac{\partial P}{\partial y} &= \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} - \phi\frac{\partial^2\psi}{\partial x\partial y} = \psi\frac{\partial^2\phi}{\partial x\partial y} - \phi\frac{\partial^2\psi}{\partial x\partial y}.
\end{aligned}$$

Substituting into (5), we have

$$\begin{aligned}
&\frac{1}{2}\int_R\left(\phi\frac{\partial^2\psi}{\partial x\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y} + \phi\frac{\partial^2\psi}{\partial x\partial y}\right)dx\,dy = \frac{1}{2}\int_{\partial R}\left(\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}\right)dx + \frac{1}{2}\int_{\partial R}\left(\phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}\right)dy\\
&\iff \int_R \phi\frac{\partial^2\psi}{\partial x\partial y}\,dx\,dy = \int_R\psi\frac{\partial^2\phi}{\partial x\partial y}\,dx\,dy + \frac{1}{2}\int_{\partial R}\left(\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}\right)dx + \frac{1}{2}\int_{\partial R}\left(\phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}\right)dy\,. \qquad (10)
\end{aligned}$$

3 Plate vibrations

Start with the action for a vibrating plate whose potential energy is dominated by bending,

$$S[u(x,y,t)] = \frac{1}{2}\int_{t_0}^{t_1}\int_R\left\{\rho u_t^2 - \kappa_1\left[(u_{xx}^2 + u_{yy}^2) - 2(1-\mu)(u_{xx}u_{yy} - u_{xy}^2)\right]\right\}dx\,dy\,dt\,, \qquad (11)$$

where ρ is the mass density per unit area, κ_1 has the dimension of energy and is sometimes called flexural rigidity, and μ is a dimensionless material constant called Poisson's ratio. For isotropic material, $\mu = 1/4$. Notice that there is *no* external bending moment applied to the plate boundary. There is also *no* external forcing.

3.a Using the results of problem 2, show that the variation generated by going from a solution u^0 to $u^0 + \epsilon\psi$ has the form

$$\delta S = \epsilon\int_{t_0}^{t_1}\int_R\left(-\rho u_{tt} - \kappa_1\nabla^4 u\right)\psi\,dx\,dy\,dt + \epsilon\int_{t_0}^{t_1}\int_{\partial R}\left(P(u)\psi + M(u)\frac{\partial\psi}{\partial n}\right)d\ell\,dt\,. \qquad (12)$$

Specify $P(u)$ and $M(u)$.

Solution. Making the substitution $u \mapsto u + \epsilon\psi$ into (11),

$$\begin{aligned}
S[u+\epsilon\psi] &= \int_{t_0}^{t_1}\int_R\left\{\frac{\rho}{2}(u_t + \epsilon\psi_t)^2 - \frac{\kappa_1}{2}\left[(u_{xx}+\epsilon\psi_{xx})^2 + (u_{yy}+\epsilon\psi_{yy})^2\right]\right\}dx\,dy\,dt\\
&\quad + \kappa_1(1-\mu)\int_{t_0}^{t_1}\int_R\left[(u_{xx}+\epsilon\psi_{xx})(u_{yy}+\epsilon\psi_{yy}) - (u_{xy}+\epsilon\psi_{xy})^2\right]dx\,dy\,dt\\
&= \int_{t_0}^{t_1}\int_R\left[\frac{\rho}{2}(u_t^2 + 2\epsilon u_t\psi_t + \epsilon^2\psi_t^2) - \frac{\kappa_1}{2}(u_{xx}^2 + 2\epsilon u_{xx}\psi_{xx} + \epsilon^2\psi_{xx}^2 + u_{yy}^2 + 2\epsilon u_{yy}\psi_{yy} + \epsilon^2\psi_{yy}^2)\right]dx\,dy\,dt\\
&\quad + \kappa_1(1-\mu)\int_{t_0}^{t_1}\int_R(u_{xx}u_{yy} + \epsilon u_{xx}\psi_{yy} + \epsilon u_{yy}\psi_{xx} + \epsilon^2\psi_{xx}\psi_{yy} - u_{xy}^2 - 2\epsilon u_{xy}\psi_{xy} - \epsilon^2\psi_{xy}^2)\,dx\,dy\,dt\,.
\end{aligned}$$

Then

$$\begin{aligned}\Delta S &= S[u+\epsilon\psi] - S[u]\\ &= \int_{t_0}^{t_1}\int_R \left[\frac{\rho}{2}(2\epsilon u_t\psi_t + \epsilon^2\psi_t^2) - \frac{\kappa_1}{2}(2\epsilon u_{xx}\psi_{xx} + \epsilon^2\psi_{xx}^2 + 2\epsilon u_{yy}\psi_{yy} + \epsilon^2\psi_{yy}^2)\right] dx\,dy\,dt\\ &\qquad + \kappa_1(1-\mu)\int_{t_0}^{t_1}\int_R (\epsilon u_{xx}\psi_{yy} + \epsilon u_{yy}\psi_{xx} + \epsilon^2\psi_{xx}\psi_{yy} - 2\epsilon u_{xy}\psi_{xy} - \epsilon^2\psi_{xy}^2)\,dx\,dy\,dt\,,\end{aligned}$$

and so, dropping terms of $\mathcal{O}(\epsilon^2)$,

$$\delta S = \epsilon\int_{t_0}^{t_1}\int_R \{\rho u_t\psi_t - \kappa_1\left[(u_{xx}\psi_{xx} + u_{yy}\psi_{yy}) - (1-\mu)(u_{xx}\psi_{yy} + u_{yy}\psi_{xx} - 2u_{xy}\psi_{xy})\right]\}\,dx\,dy\,dt\,. \tag{13}$$

For the first term in the integrand of (13), using the product rule of differentiation yields

$$u_t\psi_t = \frac{\partial}{\partial t} - u_{tt}\psi.$$

For the second two terms, we may apply what was proven in problem 2. Letting $\phi \mapsto u_{xx}$ and $\psi \mapsto \psi$ in (6) and (9), we have

$$\begin{aligned}\int_{t_0}^{t_1}\int_R u_{xx}\psi_{xx}\,dx\,dy\,dt &= \int_{t_0}^{t_1}\int_R \psi u_{xxxx}\,dx\,dy\,dt + \int_{t_0}^{t_1}\int_{\partial R}(u_{xx}\psi_x - \psi u_{xxx})\,dy\,dt\,,\\ \int_{t_0}^{t_1}\int_R u_{xx}\psi_{yy}\,dx\,dy\,dt &= \int_{t_0}^{t_1}\int_R \psi u_{xxyy}\,dx\,dy\,dt - \int_{t_0}^{t_1}\int_{\partial R}(u_{xx}\psi_y - \psi u_{xxy})\,dx\,dt\,.\end{aligned}$$

Now with $\phi \mapsto u_{yy}$,

$$\begin{aligned}\int_{t_0}^{t_1}\int_R u_{yy}\psi_{xx}\,dx\,dy\,dt &= \int_{t_0}^{t_1}\int_R \psi u_{xxyy}\,dx\,dy\,dt + \int_{t_0}^{t_1}\int_{\partial R}(u_{yy}\psi_x - \psi u_{xyy})\,dy\,dt\,,\\ \int_{t_0}^{t_1}\int_R u_{yy}\psi_{yy}\,dx\,dy\,dt &= \int_{t_0}^{t_1}\int_R \psi u_{yyyy}\,dx\,dy\,dt - \int_{t_0}^{t_1}\int_{\partial R}(u_{yy}\psi_y - \psi u_{yyy})\,dx\,dt\,.\end{aligned}$$

Finally, with $\phi \mapsto u_{xy}$ and $\psi \mapsto \psi$ in (10), we have

$$\begin{aligned}\int_{t_0}^{t_1}\int_R u_{xy}\psi_{xy}\,dx\,dy\,dt = \int_{t_0}^{t_1}\int_R \psi u_{xxyy}\,dx\,dy\,dt &- \frac{1}{2}\int_{t_0}^{t_1}\int_{\partial R}(u_{xy}\psi_x - \psi u_{xxy})\,dx\,dt\\ &+ \frac{1}{2}\int_{t_0}^{t_1}\int_{\partial R}(u_{xy}\psi_y - \psi u_{xyy})\,dy\,dt\,.\end{aligned}$$

Making these substitutions into (13),

$$\begin{aligned}\frac{\delta S}{\epsilon} &= \int_{t_0}^{t_1}\int_R \psi\,\{-\rho u_{tt} - \kappa_1\left[(u_{xxxx} + u_{yyyy}) - (1-\mu)(u_{xxyy} + u_{xxyy} - 2u_{xxyy})\right]\}\,dx\,dy\,dt\\ &\quad + \rho\int_{t_0}^{t_1}\int_R \frac{\partial}{\partial t}\,dx\,dy\,dt - \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[(u_{xx}\psi_x - \psi u_{xxx}) - (1-\mu)(u_{yy}\psi_x - \psi u_{xyy} - u_{xy}\psi_y + \psi u_{xyy})\right] dy\,dt\\ &\quad + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[(u_{yy}\psi_y - \psi u_{yy}) - (1-\mu)(u_{xx}\psi_y - \psi u_{xxy} - u_{xy}\psi_x + \psi u_{xxy})\right] dx\,dt\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\delta S}{\epsilon} &= \int_{t_0}^{t_1}\int_R \psi\left\{-\rho u_{tt} - \kappa_1\left[(u_{xxxx}+u_{yyyy}) - (1-\mu)(u_{xxyy}+u_{xxyy}-2u_{xxyy})\right]\right\} dx\,dy\,dt + \rho\int_R u_t\psi\,dx\,dy \\ &\quad - \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[(u_{xx}\psi_x - \psi u_{xxx}) - (1-\mu)(u_{yy}\psi_x - u_{xy}\psi_y)\right] dy\,dt \\ &\quad + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[(u_{yy}\psi_y - \psi u_{yyy}) - (1-\mu)(u_{xx}\psi_y - u_{xy}\psi_x)\right] dx\,dt\,.\end{aligned}$$

Note that

$$\nabla^4 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}, \tag{15}$$

for a real solution u. Note also that

$$\int_R u_t\psi\,dx\,dy = 0$$

because it is constant in time. Then

$$\begin{aligned}\frac{\delta S}{\epsilon} &= \int_{t_0}^{t_1}\int_R \psi(-\rho u_{tt} - \kappa_1\nabla^4 u)\,dx\,dy\,dt + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\psi(u_{xxx}\,dy - u_{yyy}\,dx) \qquad (16)\\ &\quad + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\psi_x\left\{[u_{xx}+(1-\mu)u_{yy}]\,dy + (1-\mu)u_{xy}\,dx\right\} + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\psi_y\left\{u_{xy}\,dy + [u_{yy}-(1-\mu)u_{xx}]\,dx\right\}.\end{aligned}$$

Thus,

$$\int_{t_0}^{t_1}\int_{\partial R} P(u)\psi\,d\ell\,dt = \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\psi(u_{xxx}\,dy - u_{yyy}\,dx), \tag{17}$$

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\partial R} M(u)\frac{\partial\psi}{\partial n}\,d\ell\,dt &= \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\psi_x\left\{[u_{xx}+(1-\mu)u_{yy}]\,dy + (1-\mu)u_{xy}\,dx\right\} \\ &\quad + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\psi_y\left\{u_{xy}\,dy + [u_{yy}-(1-\mu)u_{xx}]\,dx\right\}. \qquad (18)\end{aligned}$$

Define $\hat{\mathbf{n}}$ as the unit vector normal to the surface and $\hat{\boldsymbol{\ell}}$ as the unit vector tangent to the surface. Then we have the directional derivatives

$$\frac{\partial}{\partial n} = \hat{\mathbf{n}}\cdot\nabla = x_n\frac{\partial}{\partial x} + y_n\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \ell} = \hat{\boldsymbol{\ell}}\cdot\nabla = x_\ell\frac{\partial}{\partial x} + y_\ell\frac{\partial}{\partial y}, \tag{19}$$

and the differentials

$$dn = x_n\,dx + y_n\,dy\,, \qquad d\ell = x_\ell\,dx + y_\ell\,dy\,. \tag{20}$$

In principle, we can use (19) and (20) to rewrite (17) and (18), and obtain $P(u)$ and $M(u)$ explicitly. However, it seems at this point that something has gone wrong. The expression for $P(u)$ needs to include terms with the coefficient $(1-\mu)$, which do not show up on the right-hand side of (17). Perhaps there is a sign error in (14) and the terms proportional to $(1-\mu)\psi$) should not have canceled, or perhaps the reasoning of (15) is incorrect.

From Gelfand and Fomin, the solutions are

$$\delta S = -\epsilon\int_{t_0}^{t_1}\int_R\left(\rho u_{tt} + \kappa_1\nabla^4 u\right)\psi\,dx\,dy\,dt + \epsilon\int_{t_0}^{t_1}\int_{\partial R}\left(P(u)\psi + M(u)\frac{\partial\psi}{\partial n}\right)d\ell\,dt$$

where

$$\begin{aligned}P(u) &= \kappa_1\left\{\frac{\partial}{\partial n}\nabla^2 u + (1-\mu)\frac{\partial}{\partial\ell}\left[u_{xx}x_n x_\ell + u_{xy}(x_n y_\ell + x_\ell y_n) + u_{yy}y_n y_\ell\right]\right\}, \\ M(u) &= -\kappa_1\left[\mu\nabla^2 u + (1-\mu)(u_{xx}x_n^2 + 2u_{xy}x_n y_n + u_{yy}y_n^2)\right].\end{aligned}$$

At any rate, (16) shows that we have correctly derived the volume integral in (12).

3.b Finally, derive the Euler-Lagrange equation and the associated boundary conditions.

Solution. We begin by making the strong assumption that the boundary of the plate remains fixed. Mathematically, we assume that the solution u^0 does not vary on the boundary of the plate, denoted by $\ell \in \partial R$. We further assume that the edges of the plate cannot move; that is, the first derivative of u^0 normal to the plate does not vary either. These assumptions constrain $\psi = \psi(\ell, t)$:

$$u^0(\ell, t) = 0 \implies \psi(\ell, t) = 0, \qquad \frac{\partial u^0(\ell, t)}{\partial n} = 0 \implies \frac{\partial \psi(\ell, t)}{\partial n} = 0.$$

Making these assumptions, the entire surface integral of (12) vanishes, and we are left with

$$\delta S = -\epsilon \int_{t_0}^{t_1} \int_R \left(\rho u_{tt} + \kappa_1 \nabla^4 u \right) \psi \, dx \, dy \, dt \, .$$

By Hamilton's principle, this gives us

$$0 = \rho u_{tt} + \kappa_1 \nabla^4 u$$

as the Euler-Lgrange equation.

Now we use (3) as our assumption and return to (12), which becomes

$$\delta S = \epsilon \int_{t_0}^{t_1} \int_{\partial R} \left(P(u)\psi + M(u) \frac{\partial \psi}{\partial n} \right) d\ell \, dt \, .$$

Once again invoking Hamilton's principle, we find the boundary conditions

$$M(u) = 0, \qquad P(u) = 0. \tag{21}$$

4 Vibrations of a circular disk

The only scenario in which plate vibrations can be described analytically in terms of known functions is a circular disk. Work with polar coordinates (r, θ), the Euler-Lagrange equation

$$u_{tt} + \lambda \nabla^4 u = 0, \tag{22}$$

and the boundary conditions

$$u = 0, \qquad \frac{\partial u}{\partial n} = 0. \tag{23}$$

4.a Show that this problem reduces to an eigenvalue problem if we assume that $u(r, \theta, t)$ is separable:

$$u = v(r, \theta) \, g(t). \tag{24}$$

Write down the general form of $g(t)$.

Solution. Substituting the ansatz (24) into (22), we have

$$v\frac{\partial^2 g}{\partial t^2} + \lambda g\,\nabla^4 v = 0 \implies \frac{1}{g}\frac{\partial^2 g}{\partial t^2} = -\lambda\frac{1}{v}\nabla^4 v \equiv -\lambda^2 \tag{25}$$

where we have fixed λ^2. We may then separate (25) into two differential equations,

$$\nabla^4 v - \lambda v = 0, \tag{26}$$

$$\frac{\partial^2 g}{\partial t^2} + \lambda^2 g = 0. \tag{27}$$

The eigenvalue problem is (26), which we may solve for the eigenvalues λ_n and obtain the eigenfunctions $v_n(r,\theta)$. Then we simply feed λ_n into (27) to obtain $g_n(t)$, which have the general form

$$g(t) = C_1 + C_2 t - \frac{\lambda^2}{6}t^3, \tag{28}$$

where C_1 and C_2 are arbitrary constants. Finally, the solutions to (22) are $u_n(r,\theta,t) = v_n(r,\theta)\,g_n(t)$.

4.b Now consider the eigenvalue problem

$$(\nabla^4 - k^4)v(r,\theta) = 0, \tag{29}$$

with λ set to be k^4. Notice that it factors into

$$(\nabla^2 - k^2)(\nabla^2 + k^2)v(r,\theta) = 0, \tag{30}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Since the disk is circular, we expect the vibration modes to be periodic in θ. This suggests the ansatz

$$v = \sum_{n=-\infty}^{\infty} f_n(r)\,e^{in\theta}. \tag{31}$$

Obtain the ODE governing $f_n(r)$.

Solution. Firstly, note that

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial\theta^4}.$$

Substituting the ansatz of (31) into (29) yields

$$\begin{aligned}
k^4 f_n(r)\,e^{in\theta} &= -\nabla^4 f_n(r)\,e^{in\theta} \\
&= \left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial\theta^4}\right) f_n(r)\,e^{in\theta} \\
&= e^{in\theta}\left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2n^2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2n^2}{r^3}\frac{\partial}{\partial r} + \frac{n^4}{r^4}\right) f_n(r).
\end{aligned}$$

Dividing out $e^{in\theta}$, we have

$$k^4 f_n(r) = \frac{\partial^4 f_n(r)}{\partial r^4} + \frac{2}{r}\frac{\partial^3 f_n(r)}{\partial r^3} + \frac{1-2n^2}{r^2}\frac{\partial^2 f_n(r)}{\partial r^2} - \frac{2n^2}{r^3}\frac{\partial f_n(r)}{\partial r} + \frac{n^4}{r^4}f_n(r)$$

as the ODE governing $f_n(r)$.

4.c What are the appropriate boundary conditions on $f_n(r)$?

Solution. From (24) and (31), the solution u is defined

$$u = v(r,\theta)\, g(t) = g(t) \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}.$$

From (23),

$$u = 0 \implies v = 0 \implies f_n(r) = 0, \tag{32}$$
$$\frac{\partial u}{\partial n} = 0 \implies \frac{\partial v}{\partial n} = 0 \implies \frac{\partial f_n(r)}{\partial r} = 0, \tag{33}$$

for all $n \in (-\infty, \infty)$ on the boundry ∂R of the plate. Note that $\partial/\partial n$ is the normal derivative.

5 Big brother

5.a Let $\mathbf{u}(x, y) = [u_1(x, y), u_2(x, y)]$ be the *unknown* two-dimensional warp map corresponding to a grayscale photograph. Find the Euler-Lagrange equations associated with the elastic energy functional

$$U_b[\mathbf{u}] = \int_R \left[\lambda \operatorname{tr}\big((A + A^T)^2\big) + \mu \operatorname{tr}(A) \operatorname{tr}\big(A^T\big)\right] dx\, dy\,,$$

where λ and μ are elastic constant, the deviation A is given by

$$A = \begin{bmatrix} \partial u_1/\partial x & \partial u_1/\partial y \\ \partial u_2/\partial x & \partial u_2/\partial y \end{bmatrix},$$

and R is the region spanned by a photograph.

Solution. Firstly, note that

$$A^T = \begin{bmatrix} \partial u_1/\partial x & \partial u_2/\partial x \\ \partial u_2/\partial y & \partial u_2/\partial y \end{bmatrix},$$

so

$$A + A^T = \begin{bmatrix} 2\, \partial u_1/\partial x & \partial u_1/\partial y + \partial u_2/\partial x \\ \partial u_2/\partial x + \partial u_1/\partial y & 2\, \partial u_2/\partial y \end{bmatrix},$$
$$(A + A^T)^2 = \begin{bmatrix} 4(\partial u_1/\partial x)^2 + (\partial u_1/\partial y + \partial u_2/\partial x)^2 & \\ & 4(\partial u_2/\partial y)^2 + (\partial u_1/\partial y + \partial u_2/\partial x)^2 \end{bmatrix},$$

where only the diagonal terms of $(A + A^T)^2$ are of interest. Then

$$\begin{aligned}
\operatorname{tr}\big((A + A^T)^2\big) &= 4\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial y}\right)^2 \\
&= 4\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial u_1}{\partial y}\right)^2 + 4\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} + 2\left(\frac{\partial u_2}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial y}\right)^2, \\
\operatorname{tr}(A) \operatorname{tr}\big(A^T\big) &= \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)^2 = \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} + \left(\frac{\partial u_2}{\partial y}\right)^2,
\end{aligned}$$

and

$$U_b[\mathbf{u}] = \int_R \left\{\lambda\left[4\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial y}\right)^2\right] + \mu\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)^2\right\} dx\, dy \equiv \int_R \mathcal{L}\, dx\, dy\,, \tag{34}$$

where we have defined the Lagrangian density $\mathcal{L}$.

We will have two Euler-Lagrange equations, one for each u_1 and u_2. In general, they are given by

$$0 = \frac{\partial \mathcal{L}}{\partial u_1} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{1x}} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{1y}}, \qquad 0 = \frac{\partial \mathcal{L}}{\partial u_2} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{2x}} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{2y}},$$

where $u_{1x} = \partial u_1/\partial x$, and so on. From (34),

$$\frac{\partial \mathcal{L}}{\partial u_1} = 0, \qquad \frac{\partial \mathcal{L}}{\partial u_{1x}} = 2(4\lambda+\mu)u_{1x} + 2\mu u_{2y}, \qquad \frac{\partial \mathcal{L}}{\partial u_{1y}} = 4\lambda u_{1y} + 4\lambda u_{2x},$$
$$\frac{\partial \mathcal{L}}{\partial u_2} = 0, \qquad \frac{\partial \mathcal{L}}{\partial u_{2x}} = 4\lambda u_{2x} + 4\lambda u_{1y}, \qquad \frac{\partial \mathcal{L}}{\partial u_{2y}} = 2(4\lambda+\mu)u_{1y} + 2\mu u_{2x}.$$

Then

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{1x}} = 2(4\lambda+\mu)u_{1xx} + 2\mu u_{2xy}, \qquad \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{1y}} = 4\lambda u_{1yy} + 4\lambda u_{2xy},$$
$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{2x}} = 4\lambda u_{2xx} + 4\lambda u_{1xy}, \qquad \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{2y}} = 2(4\lambda+\mu)u_{2yy} + 2\mu u_{1xy}.$$

So the Euler-Lagrange equations are

$$0 = 2(4\lambda+\mu)\frac{\partial^2 u_1}{\partial x^2} + 4\lambda\frac{\partial^2 u_1}{\partial y^2} + 2(2\lambda+\mu)\frac{\partial^2 u_2}{\partial x \partial y},$$
$$0 = 2(2\lambda+\mu)\frac{\partial^2 u_1}{\partial x \partial y} + 4\lambda\frac{\partial^2 u_2}{\partial x^2} + 2(4\lambda+\mu)\frac{\partial^2 u_2}{\partial y^2},$$

which are coupled.

In writing these solutions, I consulted Gelfand and Fomin's *Calculus of Variations* and Olmstead and Volpert's *Differential Equations in Applied Mathemtics.*