

**Problem 1. (Jackson 9.8)**

**1(a)** Show that a classical oscillating electric dipole  $\mathbf{p}$  with fields given by

$$\mathbf{H} = \frac{ck^2}{4\pi} (\hat{\mathbf{n}} \times \mathbf{p}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right), \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \left\{ k^2 (\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}} \frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\}, \quad (1)$$

radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\mathbf{L}}{dt} = \frac{k^3}{12\pi\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}].$$

**Solution.** According to Jackson (9.20), the time-averaged angular momentum density is

$$\mathbf{l} = \frac{\text{Re}[\mathbf{x} \times (\mathbf{E} \times \mathbf{H}^*)]}{2c^2}.$$

One of the vector identities on the inside cover of Jackson is  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ , so

$$\mathbf{l} = \frac{(\mathbf{x} \cdot \mathbf{H}^*)\mathbf{E} - (\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{2c^2}. \quad (2)$$

From Eq. (1), note that

$$\mathbf{x} \cdot \mathbf{H}^* \propto \mathbf{x} \cdot (\hat{\mathbf{n}} \times \mathbf{p}^*) = \mathbf{p}^* \cdot (\mathbf{x} \times \hat{\mathbf{n}}) = 0,$$

where we have used the identity  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$  and the fact that  $\hat{\mathbf{n}}$  points in the  $\mathbf{x}$  direction. For  $\mathbf{x} \cdot \mathbf{E}$ , note that

$$\begin{aligned} \mathbf{x} \cdot [(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}] &= -\mathbf{x} \cdot [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{p})] = -\mathbf{x} \cdot [(\hat{\mathbf{n}} \cdot \mathbf{p})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{p}] = -(\hat{\mathbf{n}} \cdot \mathbf{p})(\mathbf{x} \cdot \hat{\mathbf{n}}) + \mathbf{x} \cdot \mathbf{p} \\ &= -r(\hat{\mathbf{n}} \cdot \mathbf{p}) + \mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{p} = 0, \end{aligned}$$

$$\mathbf{x} \cdot [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] = 3(\mathbf{x} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3r(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3(\mathbf{x} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 2(\mathbf{x} \cdot \mathbf{p}),$$

since  $|\mathbf{x}| = r$  and  $\mathbf{x} = r\hat{\mathbf{n}}$ . Then

$$\mathbf{x} \cdot \mathbf{E} = \frac{1}{2\pi\epsilon_0} (\mathbf{x} \cdot \mathbf{p}) \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} = \frac{1}{2\pi\epsilon_0} (\hat{\mathbf{n}} \cdot \mathbf{p}) \left( \frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr}.$$

With these substitutions, Eq. (2) becomes

$$\begin{aligned} \mathbf{l} &= -\frac{(\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{c^2} = -\frac{1}{4\pi\epsilon_0 c^2} (\hat{\mathbf{n}} \cdot \mathbf{p}) \left( \frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr} \frac{ck^2}{4\pi} (\hat{\mathbf{n}} \times \mathbf{p}^*) \frac{e^{-ikr}}{r} \left( 1 + \frac{1}{ikr} \right) \\ &= -\frac{k^2}{16\pi^2\epsilon_0 cr} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{r^2} - \frac{ik}{r} \right) \left( 1 - \frac{i}{kr} \right) = -\frac{k^2}{16\pi^2\epsilon_0 c} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{r^2} - \frac{i}{kr^3} - \frac{ik}{r} - \frac{1}{r^2} \right) \\ &= -\frac{ik^2}{16\pi^2\epsilon_0 cr} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{kr^3} + \frac{k}{r^2} \right) = \frac{ik^3}{16\pi^2\epsilon_0 cr^2} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{k^2 r^2} + 1 \right). \end{aligned}$$

Let  $\mathbf{L}$  be the angular momentum radiated to a distance  $R$ . Then

$$\mathbf{L} = \int_R \mathbf{l}(r) d^3x = \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r) r^2 \sin\theta dr d\phi d\theta,$$

and the time derivative is

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt} \left( \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r) r^2 \sin \theta dr d\phi d\theta \right) = \frac{dr}{dt} \frac{d}{dr} \left( \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r) r^2 \sin \theta dr d\phi d\theta \right) \\ &= c \int_0^\pi \int_0^{2\pi} \mathbf{l}(r) r^2 \sin \theta d\phi d\theta = \frac{ik^3}{16\pi^2\epsilon_0} \left( \frac{1}{k^2 r^2} + 1 \right) \int_0^\pi \int_0^{2\pi} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \sin \theta d\phi d\theta.\end{aligned}\quad (3)$$

Note that

$$[(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)]_i = \sum_{j=1}^3 n_j p_j (\hat{\mathbf{n}} \times \mathbf{p}^*)_i = \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \epsilon_{ikl} n_j p_j n_k p_l^*,$$

so

$$\frac{dL_i}{dt} \propto \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \int n_j p_k d\Omega = \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \frac{4\pi}{3} \delta_{jk} = \frac{4\pi}{3} \epsilon_{ikl} p_k p_l^* = \frac{4\pi}{3} (\mathbf{p} \times \mathbf{p}^*)_i,$$

where we have used Jackson (9.47),  $\int n_\beta n_\gamma d\Omega = 4\pi \delta_{\beta\gamma}/3$ . Making this substitution into Eq. (3),

$$\frac{d\mathbf{L}}{dt} = \frac{ik^3}{6\pi\epsilon_0} \left( \frac{1}{k^2 r^2} + 1 \right) (\mathbf{p} \times \mathbf{p}^*).$$

Taking the limit as  $r \rightarrow \infty$ , we find

$$\frac{d\mathbf{L}}{dt} = \text{Re} \left[ \frac{ik^3}{12\pi\epsilon_0} (\mathbf{p} \times \mathbf{p}^*) \right] = \text{Re} \left[ -\frac{ik^3}{12\pi\epsilon_0} (\mathbf{p}^* \times \mathbf{p}) \right] = \frac{k^3}{12\pi\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}], \quad (4)$$

as desired.  $\square$

**1(b)** What is the ratio of angular momentum radiated to energy radiated? Interpret.

**Solution.** According to Jackson (9.24), the total power radiated by an oscillating electric dipole  $\mathbf{p}$  is

$$P = \frac{dE}{dt} = \frac{c^2 Z_0 k^4}{12\pi} |\mathbf{p}|^2.$$

Then the ratio of angular momentum radiated to energy radiated is

$$\frac{d\mathbf{L}/dt}{dE/dt} = \frac{k^3}{12\pi\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}] \frac{12\pi}{c^2 Z_0 k^4 |\mathbf{p}|^2} = \frac{1}{\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}] \frac{1}{c^2 Z_0 k |\mathbf{p}|^2} = \frac{\text{Im}[\mathbf{p}^* \times \mathbf{p}]}{\omega |\mathbf{p}|^2},$$

where we have used  $Z_0 = \sqrt{\mu_0/\epsilon_0} = 1/\sqrt{\epsilon_0^2 c^2} = 1/\epsilon_0 c$ ,  $c^2 = 1/(\epsilon_0 \mu_0)$ , and  $\omega = kc$ .

In the limit of high frequency,  $(d\mathbf{L}/dt)/(dE/dt) \rightarrow 0$ . In this scenario, the energy radiated dominates over the angular momentum radiated. Likewise, in the limit of low frequency,  $(d\mathbf{L}/dt)/(dE/dt) \rightarrow \infty$ , meaning that angular momentum radiation dominates. This is sensible because rotational kinetic energy  $E \propto \omega^2$ , while angular momentum  $L \propto \omega$ .

**1(c)** For a charge  $e$  rotating in the  $xy$  plane at radius  $a$  and angular speed  $\omega$ , show that there is only a  $z$  component of radiated angular momentum with magnitude  $dL_z/dt = e^2 k^3 a^2 / 6\pi\epsilon_0$ . What about a charge oscillating along the  $z$  axis?

**Solution.** We know from Homework 5 that the position of a point charge rotating counterclockwise in the  $xy$  plane is

$$\mathbf{x}(t) = a \cos(\omega t) \hat{\mathbf{x}} + a \sin(\omega t) \hat{\mathbf{y}}.$$

Then the charge distribution is

$$\rho(\mathbf{x}, t) = e\delta[x - a\cos(\omega t)]\delta[y - a\sin(\omega t)]\delta(z).$$

According to Jackson (4.8), the dipole moment is defined

$$\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x'.$$

The components of  $\mathbf{p}$  for the point charge are then

$$\begin{aligned} p_x &= e \iiint x \delta[x - a\cos(\omega t)] \delta[y - a\sin(\omega t)] \delta(z) dx dy dz = ea \cos(\omega t), \\ p_y &= e \iiint y \delta[x - a\cos(\omega t)] \delta[y - a\sin(\omega t)] \delta(z) dx dy dz = ea \sin(\omega t), \\ p_z &= e \iiint z \delta[x - a\cos(\omega t)] \delta[y - a\sin(\omega t)] \delta(z) dx dy dz = 0, \end{aligned}$$

so we can write  $\mathbf{p} = ea e^{-i\omega t}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$ . Substituting into Eq. (4),

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \text{Re} \left[ \frac{ik^3}{12\pi\epsilon_0} e^2 a^2 e^{-i\omega t} e^{i\omega t} [(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \times (\hat{\mathbf{x}} - i\hat{\mathbf{y}})] \right] = \text{Re} \left[ \frac{ie^2 k^3 a^2}{12\pi\epsilon_0} (-2i\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \right] = \text{Re} \left[ \frac{e^2 k^3 a^2}{6\pi\epsilon_0} \hat{\mathbf{z}} \right] \\ &= \frac{e^2 k^3 a^2}{6\pi\epsilon_0} \cos(\omega t) \hat{\mathbf{z}}, \end{aligned}$$

as desired. □

A charge oscillating along the  $z$  axis with amplitude  $a$  has the charge density

$$\rho(\mathbf{x}, t) = ea \delta(x) \delta(y) \delta[z - \cos(\omega t)],$$

which gives the dipole moment

$$\begin{aligned} p_x &= ea \iiint x \delta(x) \delta(y) \delta[z - \cos(\omega t)] dx dy dz = 0, \\ p_y &= ea \iiint y \delta(x) \delta(y) \delta[z - \cos(\omega t)] dx dy dz = 0, \\ p_z &= ea \iiint z \delta(x) \delta(y) \delta[z - \cos(\omega t)] dx dy dz = ea \cos(\omega t). \end{aligned}$$

In complex notation,  $\mathbf{p} = ea e^{-i\omega t} \hat{\mathbf{z}}$ . Substituting into Eq. (4), we find

$$\frac{d\mathbf{L}}{dt} = \text{Re} \left[ \frac{ik^3}{12\pi\epsilon_0} e^2 a^2 e^{-i\omega t} e^{i\omega t} (\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \right] = \mathbf{0}.$$

So we see that a charge undergoing linear motion does not lead to a radiated angular momentum, which is sensible.

**1(d)** What are the results corresponding to Probs. 1(a) and 1(b) for magnetic dipole radiation?

**Solution.** The radiation fields for a magnetic dipole are given by Jackson (19.35–36),

$$\mathbf{H} = \frac{1}{4\pi} \left\{ k^2 (\hat{\mathbf{n}} \times \mathbf{m}) \times \hat{\mathbf{n}} \frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{m}) - \mathbf{m}] \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\}, \quad \mathbf{E} = -\frac{Z_0}{4\pi} k^2 (\hat{\mathbf{n}} \times \mathbf{m}) \frac{e^{ikr}}{r} \left( 1 - \frac{1}{ikr} \right).$$

Comparing with Eq. (1), we see that  $\mathbf{H} \rightarrow -\mathbf{E}/Z_0$ ,  $\mathbf{E} \rightarrow Z_0\mathbf{H}$ , and  $\mathbf{p} \rightarrow \mathbf{m}/c$  as stated in the book [1, p. 413]. Making these substitutions, the results of Probs. 1.1(a) and (b) become

$$\frac{d\mathbf{L}}{dt} = \frac{\mu_0 k^3}{12\pi} \text{Im}[\mathbf{m}^* \times \mathbf{m}], \quad \frac{d\mathbf{L}/dt}{dE/dt} = \frac{\text{Im}[\mathbf{m}^* \times \mathbf{m}]}{\omega |\mathbf{m}|^2}$$

where we have used  $\mu = 1/\epsilon_0 c^2$ .

## Problem 2. (Jackson 10.1)

**2(a)** Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius  $a$ , summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[ \frac{5}{4} - |\epsilon_0 \cdot \hat{\mathbf{n}}|^2 - \frac{1}{4} |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \epsilon_0)|^2 - \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}} \right],$$

where  $\hat{\mathbf{n}}_0$  and  $\hat{\mathbf{n}}$  are the directions of the incident and scattered radiations, respectively, while  $\epsilon_0$  is the (perhaps complex) unit polarization vector of the incident radiation ( $\epsilon_0^* \cdot \epsilon_0 = 1$ ;  $\hat{\mathbf{n}}_0 \cdot \epsilon_0 = 0$ ).

**Solution.** Jackson (10.14) gives the differential cross section for scattering off a small, perfectly conducting sphere with initial polarization  $\epsilon_0$  and outgoing polarization  $\epsilon$ :

$$\frac{d\sigma}{d\Omega} \hat{\mathbf{n}}, \epsilon; \hat{\mathbf{n}}_0, \epsilon_0 = k^4 a^6 \left| (\epsilon^* \cdot \epsilon_0 - \frac{1}{2} (\hat{\mathbf{n}} \times \epsilon^*) \cdot (\hat{\mathbf{n}}_0 \times \epsilon_0)) \right|^2. \quad (5)$$

We will use the polarization vectors  $\epsilon^{(1)}$  and  $\epsilon^{(2)}$ , which are defined in Fig. (1) [1, p. 458]. According to the figure,

$$\begin{aligned} \epsilon^{(2)} &= \frac{\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0}{|\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0|} = \frac{\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}}, \\ \epsilon^{(1)} &= \epsilon^{(2)} \times \hat{\mathbf{n}} = \frac{-\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0)}{\sin \theta} = \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}_0 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}}{\sin \theta} = \frac{\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}}, \end{aligned}$$

which are both real. In the denominator, we have used  $\sin^2 \theta = 1 + \cos^2 \theta = 1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2$ . We also note that  $\hat{\mathbf{n}}_0$ ,  $\hat{\mathbf{n}}$ , and  $\epsilon^{(1)}$  are in the same plane, and that  $\hat{\mathbf{n}} \perp \epsilon^{(1)}$ .

The cross section summed over outgoing polarizations is then found by plugging  $\epsilon = \epsilon^{(1)}$  and  $\epsilon = \epsilon^{(2)}$  into Eq. (5), and taking the sum. For the first term,

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \epsilon^{(1)}; \hat{\mathbf{n}}_0, \epsilon_0) &= k^4 a^6 \left| \epsilon^{(1)*} \cdot \epsilon_0 - \frac{1}{2} (\hat{\mathbf{n}} \times \epsilon^{(1)*}) \cdot (\hat{\mathbf{n}}_0 \times \epsilon_0) \right|^2 \\ &= k^4 a^6 \left| \frac{\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}} \cdot \epsilon_0 - \frac{1}{2} \left( \hat{\mathbf{n}} \times \frac{\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}} \right) \cdot (\hat{\mathbf{n}}_0 \times \epsilon_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| -(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}} \cdot \epsilon_0) - \frac{1}{2} (\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0) \cdot (\hat{\mathbf{n}}_0 \times \epsilon_0) \right|^2. \end{aligned}$$

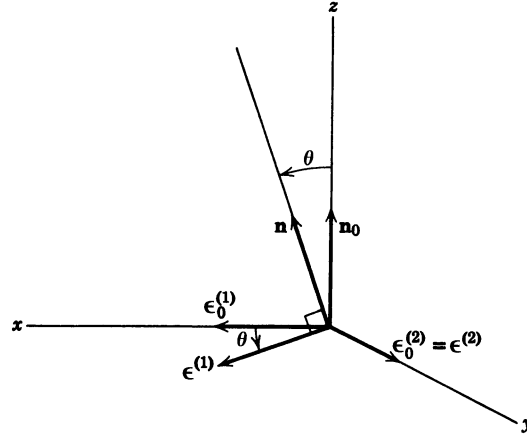


Figure 1: (Jackson 10.1) Polarization and propagation vectors for the incident and scattered radiation.

One of the vector identities on the inside cover of Jackson is  $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ . Applying this, we have

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(1)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) + \frac{1}{2}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}}_0 \cdot \boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0)(\hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}}_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) \right|^2 = \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \left[ (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) - \frac{1}{2} \right]^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \left[ (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{1}{4} \right]. \end{aligned}$$

For the second term,

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(2)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) &= k^4 a^6 \left| \boldsymbol{\epsilon}^{(2)*} \cdot \boldsymbol{\epsilon}_0 - \frac{1}{2}(\hat{\mathbf{n}} \times \boldsymbol{\epsilon}^{(2)*}) \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\ &= k^4 a^6 \left| \frac{\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}} \cdot \boldsymbol{\epsilon}_0 - \frac{1}{2} \left( \hat{\mathbf{n}} \times \frac{\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}} \right) \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) - \frac{1}{2}[(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}_0] \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) - \frac{1}{2}[(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)\hat{\mathbf{n}} - \hat{\mathbf{n}}_0] \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) + \frac{1}{2}\boldsymbol{\epsilon}_0 \cdot (\hat{\mathbf{n}}_0 \times \hat{\mathbf{n}}_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \left( 1 - \frac{1}{2}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \right) \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\ &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left[ 1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{1}{4}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 \right] |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2. \end{aligned}$$

Summing the two terms, we find

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(1)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) + \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(2)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left\{ |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \left[ (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{1}{4} \right] + \left[ 1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{1}{4} (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 \right] |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 \right\} \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left\{ |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 - |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{|\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2}{4} + |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 \right. \\
&\quad \left. - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 + \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2}{4} \right\} \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left\{ \frac{5 |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2}{4} + \frac{5 |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2}{4} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right. \\
&\quad \left. - \frac{|\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2}{4} - \frac{|\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2}{4} + \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2}{4} + (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right\} \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left\{ \left[ \frac{5}{4} - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 \right] \left[ |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right] - [1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2] \left[ \frac{|\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2}{4} + |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right] \right\} \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left[ \frac{5}{4} - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 \right] \left[ |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right] - k^4 a^6 \left[ \frac{|\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2}{4} + |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right]. \quad (6)
\end{aligned}$$

Since  $\hat{\mathbf{n}}_0 \cdot \boldsymbol{\epsilon}_0 = 0$ , we note that

$$\hat{\mathbf{n}} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \hat{\mathbf{n}}_0 + (\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) \boldsymbol{\epsilon}_0 + [\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)] (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \implies 1 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 + |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 + |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2. \quad (7)$$

Substituting into Eq. (6),

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left[ \frac{5}{4} - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 \right] [1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2] - k^4 a^6 \left[ \frac{1}{4} |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \right] \\
&= k^4 a^6 \left[ \frac{5}{4} - |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 - \frac{1}{4} |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 \right], \quad (8)
\end{aligned}$$

as we sought to prove.  $\square$

**2(b)** If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[ \frac{5}{8} (1 + \cos^2 \theta) - \cos \theta - \frac{3}{8} \sin^2 \theta \cos(2\phi) \right],$$

where  $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 = \cos \theta$  and the azimuthal angle  $\phi$  is measured from the direction of linear polarization.

**Solution.** We choose coordinates as in Fig. 1, such that the direction of linear polarization  $\boldsymbol{\epsilon}_0$  points along the  $x$  axis and  $\hat{\mathbf{n}}_0$  points along the  $z$  axis. Then  $\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0$  points along the  $y$  axis. In spherical coordinates, Eq. (7) becomes

$$\hat{\mathbf{n}} = \cos \phi \sin \theta \boldsymbol{\epsilon}_0 + \sin \phi \sin \theta (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) + \cos \theta \hat{\mathbf{n}}_0, \quad (9)$$

which implies

$$\cos \phi \sin \theta = \hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0, \quad \sin \phi \sin \theta = \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0), \quad \cos \theta = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0.$$

Making these substitutions in Eq. (8), we obtain

$$\begin{aligned}
 \frac{d\sigma}{d\Omega}(\theta, \phi) &= k^4 a^6 \left[ \frac{5}{4} - \cos^2 \phi \sin^2 \theta - \frac{1}{4} \sin^2 \phi \sin^2 \theta - \cos \theta \right] \\
 &= k^4 a^6 \left[ \frac{5}{4} - \frac{1}{2} [1 + \cos(2\phi)] \sin^2 \theta - \frac{1}{8} [1 - \cos(2\phi)] \sin^2 \theta - \cos \theta \right] \\
 &= k^4 a^6 \left[ \frac{5}{4} - \frac{1}{2} (1 - \cos^2 \theta) - \frac{1}{2} \cos(2\phi) \sin^2 \theta - \frac{1}{8} (1 - \cos^2 \theta) + \frac{1}{8} \cos(2\phi) \sin^2 \theta - \cos \theta \right] \\
 &= k^4 a^6 \left[ \frac{5}{8} (1 + \cos^2 \theta) - \cos \theta - \frac{3}{8} \sin^2 \theta \cos(2\phi) \right],
 \end{aligned}$$

where we have used the identities  $2 \sin^2 \phi = 1 - \cos(2\phi)$ ,  $2 \cos^2 \phi = 1 + \cos(2\phi)$ , and  $\cos^2 \theta + \sin^2 \theta = 1$ .  $\square$

**2(c)** What is the ratio of scattered intensities at  $\theta = \pi/2$ ,  $\phi = 0$  and  $\theta = \pi/2$ ,  $\phi = \pi/2$ ? Explain physically in terms of the induced multipoles and their radiation patterns.

**Solution.** Firstly, note that

$$\begin{aligned}
 \frac{d\sigma}{d\Omega}(\pi/2, 0) &= k^4 a^6 \left[ \frac{5}{8} - \frac{3}{8} \right] = \frac{k^4 a^6}{4}, \\
 \frac{d\sigma}{d\Omega}(\pi/2, \pi/2) &= k^4 a^6 \left[ \frac{5}{8} + \frac{3}{8} \right] = k^4 a^6,
 \end{aligned}$$

so the ratio is

$$\frac{d\sigma/d\Omega(\pi/2, 0)}{d\sigma/d\Omega(\pi/2, \pi/2)} = \frac{1}{4}.$$

According to Jackson (10.12–13), the electric and magnetic dipole moments of a perfectly conducting sphere are, respectively,

$$\mathbf{p} = 4\pi\epsilon_0 a^3 \mathbf{E}_{\text{inc}}, \quad \mathbf{m} = 2\pi a^3 \mathbf{H}_{\text{inc}},$$

where  $\mathbf{E}_{\text{inc}}$  and  $\mathbf{H}_{\text{inc}}$  are the incident fields. They are given by Jackson (10.1), wherein

$$\mathbf{E}_{\text{inc}} = \epsilon_0 E_0 e^{ik\hat{\mathbf{n}}_0 \cdot \mathbf{x}}, \quad \mathbf{H}_{\text{inc}} = \hat{\mathbf{n}}_0 \times \mathbf{E}_{\text{inc}}/Z_0.$$

The scattered fields are given by Jackson (10.2),

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \left[ (\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}} - \hat{\mathbf{n}} \times \frac{\mathbf{m}}{c} \right], \quad \mathbf{H}_{\text{sc}} = \hat{\mathbf{n}} \times \frac{\mathbf{E}_{\text{sc}}}{Z_0}.$$

When  $\phi = 0$ , Eq. (9) indicates that  $\hat{\mathbf{n}} = \epsilon_0$ . Applying the relations above,  $\hat{\mathbf{n}}$  and  $\mathbf{p}$  therefore point in the same direction. This means  $\hat{\mathbf{n}} \times \mathbf{p} = \mathbf{0}$ , so  $\mathbf{E}_{\text{sc}}$  only has a contribution from the magnetic dipole. However, When  $\phi = \pi/2$ ,  $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0 \times \epsilon_0$  and therefore  $\hat{\mathbf{n}}$  points in the same direction as  $\mathbf{m}$ . This means  $\hat{\mathbf{n}} \times \mathbf{m} = \mathbf{0}$ , so  $\mathbf{E}_{\text{sc}}$  only has a contribution from the electric dipole. The ratio 1/4 indicates that the strength of radiation from a purely electric dipole is four times that from a purely magnetic dipole.

**Problem 3. (Jackson 12.15)** Consider the Proca equation for a localized steady-state distribution of current that has only a static magnetic moment. This model can be used to study the observable effects of a finite photon mass on the earth's magnetic field. Note that if the magnetization is  $\mathcal{M}(\mathbf{x})$  the current density can be written as  $\mathbf{J} = c(\nabla \times \mathcal{M})$ .

**3(a)** Show that if  $\mathcal{M} = \mathbf{m} f(\mathbf{x})$ , where  $\mathbf{m}$  is a fixed vector and  $f(\mathbf{x})$  is a localized scalar function, the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m} \times \nabla \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} d^3x'.$$

**Solution.** The Proca equations of motion in the static limit are given by the equation immediately following Jackson (12.93),

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha,$$

which implies

$$\nabla^2 \mathbf{A} - \mu^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{J}. \quad (10)$$

We will proceed by finding the Green's function for this equation, which satisfies

$$(\nabla^2 - \mu^2) G(\mathbf{x}) = \delta^3(\mathbf{x}). \quad (11)$$

We can Fourier transform  $G(\mathbf{x})$  so long as  $\mathbf{A}$  and its derivatives vanish at infinity [2]. The Fourier transform expression and its inverse in one dimension are, according to Jackson (2.45–46),

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk.$$

Generalizing to three dimensions, the Green's function transforms as [2]

$$G(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int G(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x.$$

Transforming both sides of Eq. (11) in the same way, we have

$$\begin{aligned} \frac{1}{(2\pi)^{3/2}} \int (\nabla^2 - \mu^2) G(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x &= \frac{1}{(2\pi)^{3/2}} \int \nabla^2 G(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x - \frac{\mu^2}{(2\pi)^{3/2}} \int G(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \\ &= \frac{k^2 - \mu^2}{(2\pi)^{3/2}} \int G(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x, \\ \frac{1}{(2\pi)^{3/2}} \int \delta^3(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x &= \frac{1}{(2\pi)^{3/2}}, \end{aligned}$$

so

$$(k^2 - \mu^2) G(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \implies G(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}(k^2 - \mu^2)}.$$

Then we can find  $G(\mathbf{x})$  using the inverse Fourier transform [2]:

$$\begin{aligned} G(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int G(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k = \frac{1}{(2\pi)^3} \int \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2 - \mu^2} d^3k = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{e^{ik|\mathbf{x}|} k^2 \sin \theta}{k^2 - \mu^2} dk d\theta d\phi \\ &= \frac{4\pi}{(2\pi)^3 |\mathbf{x}|} \int_0^\infty \frac{k \sin(k|\mathbf{x}|)}{k^2 - \mu^2} k^2 dk = \frac{e^{-\mu|\mathbf{x}|}}{4\pi |\mathbf{x}|}. \end{aligned}$$



Then the solution to Eq. (10) is

$$\mathbf{A}(\mathbf{x}) = \frac{4\pi}{c} \int \mathbf{J}(\mathbf{x}') G(\mathbf{x} - \mathbf{x}') d^3x' = \frac{1}{c} \int \mathbf{J}(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x'.$$

Note that

$$\frac{\mathbf{J}}{c} = \nabla \times [\mathbf{m} f(\mathbf{x})] = \nabla f(\mathbf{x}) \times \mathbf{m} + f(\mathbf{x}) \nabla \times \mathbf{m} = \nabla f(\mathbf{x}) \times \mathbf{m} = -\mathbf{m} \times \nabla f(\mathbf{x}),$$

where we have used the identity  $\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$ . Making this substitution,

$$\mathbf{A}(\mathbf{x}) = - \int \mathbf{m} \times \nabla' f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x' = -\mathbf{m} \times \int \nabla' f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (12)$$

Integrating by parts,

$$\int \nabla' f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x' = \left[ f(\mathbf{x}') \nabla' \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \right]_{-\infty}^{\infty} - \int f(\mathbf{x}') \nabla' \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x' = - \int f(\mathbf{x}') \nabla' \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x',$$

since  $f(\mathbf{x})$  is a localized scalar function and therefore vanishes at infinity. Since the Green's function depends only on  $|\mathbf{x} - \mathbf{x}'|$ , we can replace  $\nabla'$  by  $-\nabla$ :

$$- \int f(\mathbf{x}') \nabla' \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x' = \int f(\mathbf{x}') \nabla \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x' = \nabla \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x'.$$

Making this substitution in Eq. (12),

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m} \times \nabla \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad (13)$$

as desired. □

**3(b)** If the magnetic dipole is a point dipole at the origin [ $f(\mathbf{x}) = \delta(\mathbf{x})$ ], show that the magnetic field away from the origin is

$$\mathbf{B}(\mathbf{x}) = [3 \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}] \left( 1 + \mu r + \frac{\mu^2 r^2}{3} \right) \frac{e^{-\mu r}}{r^3} - \frac{2}{3} \mu^2 \mathbf{m} \frac{e^{-\mu r}}{r}.$$

**Solution.** Setting  $f(\mathbf{x}) = \delta(\mathbf{x})$  in Eq. (13),

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m} \times \nabla \int \delta(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} d^3x' = -\mathbf{m} \times \nabla \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}.$$

Note that

$$\nabla \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} = \left( -\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2} - \frac{\mu e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} \right) \mathbf{x} = -(1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2} \hat{\mathbf{x}} = -(1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} \mathbf{x},$$

so

$$\mathbf{A}(\mathbf{x}) = (\mathbf{m} \times \mathbf{x})(1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2}.$$

The magnetic field is given by  $\mathbf{B} = \nabla \times \mathbf{A}$ :

$$\mathbf{B}(\mathbf{x}) = \nabla \times \left[ (\mathbf{m} \times \mathbf{x})(1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} \right] = \nabla \left[ (1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} \right] \times (\mathbf{m} \times \mathbf{x}) + (1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} \nabla \times (\mathbf{m} \times \mathbf{x}).$$

For the first term,

$$\nabla \left[ (1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} \right] = \left[ -\mu \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} + (1 + \mu|\mathbf{x}|) \left( -\frac{\mu e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} - \frac{3e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^4} \right) \right] \hat{\mathbf{x}} = -(3 + 3\mu|\mathbf{x}| + \mu^2|\mathbf{x}|^2) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^4} \hat{\mathbf{x}}.$$

Then, using the identity  $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$ ,

$$\nabla \times (\mathbf{m} \times \mathbf{x}) = \mathbf{m}(\nabla \cdot \mathbf{x}) - \mathbf{x}(\nabla \cdot \mathbf{m}) + (\mathbf{x} \cdot \nabla)\mathbf{m} - (\mathbf{m} \cdot \nabla)\mathbf{x} = \mathbf{m}(\nabla \cdot \mathbf{x}) - (\mathbf{m} \cdot \nabla)\mathbf{x}.$$

Making these substitutions,

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= -(3 + 3\mu|\mathbf{x}| + \mu^2|\mathbf{x}|^2) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} \hat{\mathbf{x}} \times (\mathbf{m} \times \hat{\mathbf{x}}) + (1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} [\mathbf{m}(\nabla \cdot \hat{\mathbf{x}}) - (\mathbf{m} \cdot \nabla)\hat{\mathbf{x}}] \\ &= -(3 + 3\mu|\mathbf{x}| + \mu^2|\mathbf{x}|^2) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} [(\hat{\mathbf{x}} \cdot \hat{\mathbf{x}})\mathbf{m} - (\hat{\mathbf{x}} \cdot \mathbf{m})\hat{\mathbf{x}}] + 2(1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2} \mathbf{m} \\ &= -3 \left( 1 + \mu|\mathbf{x}| + \frac{\mu^2|\mathbf{x}|^2}{3} \right) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} [\mathbf{m} - (\hat{\mathbf{x}} \cdot \mathbf{m})\hat{\mathbf{x}}] + 2(1 + \mu|\mathbf{x}|) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2} \mathbf{m} \\ &= [3\hat{\mathbf{x}}(\hat{\mathbf{x}} \cdot \mathbf{m}) - \mathbf{m}] \left( 1 + \mu|\mathbf{x}| + \frac{\mu^2|\mathbf{x}|^2}{3} \right) \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} - \frac{2}{3} \mu^2 \mathbf{m} \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}. \end{aligned}$$

Letting  $\hat{\mathbf{x}} \rightarrow \hat{\mathbf{r}}$  and  $|\mathbf{x}| \rightarrow r$ , we have

$$\mathbf{B}(\mathbf{x}) = [3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}] \left( 1 + \mu r + \frac{\mu^2 r^2}{3} \right) \frac{e^{-\mu r}}{r^3} - \frac{2}{3} \mu^2 \mathbf{m} \frac{e^{-\mu r}}{r} \quad (14)$$

as we sought to show.  $\square$

**3(c)** The result of Prob. 3(b) shows that at fixed  $r = R$  (on the surface of the earth), the earth's magnetic field will appear as a dipole angular distribution, plus an added constant magnetic field (an apparently external field) antiparallel to  $\mathbf{m}$ . Satellite and surface observations lead to the conclusion that the “external” field is less than  $4 \times 10^{-3}$  times the dipole field at the magnetic equator. Estimate a lower limit on  $\mu^{-1}$  in earth radii and an upper limit on the photon mass in grams from this datum.

**Solution.** The earth's magnetic field is parallel to its surface at the magnetic equator [3]. Figure ?? shows the field of a magnetic dipole  $\mathbf{m} \propto \hat{\mathbf{z}}$  [4, pp. 245–246]. Thus, at the magnetic equator,  $\mathbf{m}$  is also parallel to the field, and so the spherical unit vector  $\hat{\mathbf{r}}$  is perpendicular to  $\mathbf{m}$ . So Eq. (14) becomes

$$\mathbf{B}(\mathbf{x}) = -\mathbf{m} \left( 1 + \mu R + \frac{\mu^2 R^2}{3} \right) \frac{e^{-\mu R}}{R^3} - \frac{2}{3} \mu^2 \mathbf{m} \frac{e^{-\mu R}}{R} \equiv \mathbf{B}_{\text{dip}} + \mathbf{B}_{\text{ext}},$$

where we have defined  $\mathbf{B}_{\text{dip}}$  as the dipole field and  $\mathbf{B}_{\text{ext}}$  as the “external” field. The identification is made by comparing with Jackson (5.56), which gives the magnetic field due to a dipole (in SI units):

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{m})}{|\mathbf{x}|^3}.$$

As stated in the problem,  $|\mathbf{B}_{\text{ext}}| < (4 \times 10^{-3})|\mathbf{B}_{\text{dip}}|$ . This gives us

$$\frac{2}{3} \mu^2 \frac{e^{-\mu R}}{R} < (4 \times 10^{-3}) \left( 1 + \mu R + \frac{\mu^2 R^2}{3} \right) \frac{e^{-\mu R}}{R^3} \implies \frac{2}{3} \mu^2 R^2 < (4 \times 10^{-3}) \left( 1 + \mu R + \frac{\mu^2 R^2}{3} \right).$$

Solving the second expression with Mathematica gives the positive  $\mu = 0.0806/R$ . Thus, the lower limit on  $\mu^{-1}$  is  $\mu^{-1} > 12R$ .

The parameter  $\mu$  is defined by  $\mu = m_\gamma c/\hbar$ , where  $m_\gamma$  is the photon mass [1, p. 600]. The earth's radius is  $R = 6.37 \times 10^6$  m [5], so  $\mu < 1.267 \times 10^{-8} \text{ m}^{-1}$ . Since  $c = 2.998 \times 10^8 \text{ m s}^{-1}$  and  $\hbar = h/(2\pi) = (6.63 \times 10^{-34} \text{ J s})/(2\pi) = 1.055 \times 10^{-34} \text{ J s}$  [5], we have

$$m_\gamma = \frac{\hbar\mu}{c} < \frac{(1.055 \times 10^{-34} \text{ J s})(1.267 \times 10^{-8} \text{ m}^{-1})}{2.998 \times 10^8 \text{ m s}^{-1}} \approx 4 \times 10^{-51} \text{ kg} = 4 \times 10^{-48} \text{ g}.$$

So the upper bound on the photon mass is  $m_\gamma < 4 \times 10^{-48} \text{ g}$ .

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