1. **Problem.** Suppose we have a mechanical system with n degrees of freedom. Let

$$q_1(t), q_2(t), \ldots, q_n(t)$$

be its generalized coordinates. Now consider a time-dependent coordinate transformation

$$Q_i = Q_i(t, q_1, q_2, \ldots, q_n) \qquad i = 1, 2, \ldots, n.$$

Show that if $q_i(t)$ solves a system of Euler-Lagrange equations involving a Lagrangian $L(t, q_i, \dot{q}_i)$, then $Q_i(t)$ solves the Euler-Lagrange equations involving $L(t, Q_i, \dot{Q}_i)$ provided the time-dependent coordinate transformation fulfills some minimal standard of good behavior. Specify this "minimal standard of good behavior."

Solution. Suppose that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0; \tag{1}$$

that is, $q_i(t)$ solve a system of Euler-Lagrange equations. We want to show that

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = 0. \tag{2}$$

Beginning with the first term of (1), we can use the chain rule for $L(t, Q_i, \dot{Q}_i)$ to write

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial q_i}, \tag{3}$$

provided there exists an inverse transformation

$$q_i = q_i(t, Q_1, Q_2, \ldots, Q_n) \qquad i = 1, 2, \ldots, n \tag{4}$$

that allows us to write $L(t, q_i, \dot{q}_i)$ in terms of only t, Q_i, and $\dot{Q}_i$. This is only possible if there is a one-to-one correspondence between $q_i(t)$ and $Q_i(t)$, which is the "minimal standard of good behavior" for the transformation. We will assume the transformation is so well behaved.

Again using the chain rule for $Q_j = Q_j(t, q_1, q_2, \ldots, q_n)$, note that

$$\dot{Q}_j = \frac{\partial Q_j}{\partial t} + \frac{\partial Q_j}{\partial q_i}\dot{q}_i \tag{5}$$

so (3) becomes

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial q_i\,\partial t} + \frac{\partial^2 Q_j}{\partial q_i\,\partial q_k}\dot{q}_k\right). \tag{6}$$

Applying the chain rule now to the second term of (1), we have

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial Q_j}{\partial q_i} \tag{7}$$

where the right-hand side comes from (5). Then, using the product rule to take the time derivative,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{d}{dt}\frac{\partial Q_j}{\partial q_i}\right). \tag{8}$$

For the second term of (8), the chain rule for $Q_j = Q_j(t, q_1, q_2, \ldots, q_n)$ gives

$$\frac{d}{dt}\frac{\partial Q_j}{\partial q_i} = \frac{\partial^2 Q_j}{\partial t\,\partial q_i} + \frac{\partial^2 Q_j}{\partial q_i\,\partial q_k}\dot{q}_k. \tag{9}$$

Substituting (9) into (8), we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial t\,\partial q_i} + \frac{\partial^2 Q_j}{\partial q_k\,\partial q_i}\dot{q}_k\right) \tag{10}$$

where the terms on the far right appeared in (6). Making this substitution,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i}, \tag{11}$$

and rearranging,

$$\frac{\partial Q_j}{\partial q_i}\left(\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right) = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}, \tag{12}$$

Finally, substituting the original assumption (1), we find

$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j} = 0 \tag{13}$$

which is what we sought to prove. $\square$

2. **Problem.** Look at the Lagrangian

$$L = e^{\sigma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

for one-dimensional motion.

(a) Write down the associated Euler-Lagrange ODE.

(b) Now perform a point transformation

$$Q = e^{\sigma t/2} q$$

where the new position coordinate Q is a function of t and q. What is the equation of motion for $Q(t)$? Are there conserved quantities?

Solution.

(a) Beginning from the general expression for the Euler-Lagrange equations,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -e^{\sigma t} k q - \frac{d}{dt}\left(e^{\sigma t} m \dot{q}\right) = -m e^{\sigma t}\left(\ddot{q} + \sigma \dot{q} + \frac{k}{m} q\right) \tag{14}$$

so the ODE is

$$0 = \ddot{q} + \sigma \dot{q} + \frac{k}{m} q. \tag{15}$$

(b) It is possible to invert this transformation and write $q = q(t, Q)$. Explicitly, this is

$$q = Q e^{-\sigma t/2} \tag{16}$$

so

$$\dot{q} = e^{-\sigma t/2}\left(\dot{Q} - \frac{\sigma}{2} Q\right). \tag{17}$$

Rewriting the Lagrangian such that $L = L(t, Q, \dot{Q})$ results in

$$\begin{align} L &= e^{\sigma t}\left(\frac{m}{2}\left(e^{-\sigma t/2}\left(\dot{Q} - \frac{\sigma}{2} Q\right)\right)^2 - \frac{k}{2}\left(Q e^{-\sigma t/2}\right)^2\right) \tag{18} \\ &= \frac{m}{2}\left(\dot{Q} - \frac{\sigma}{2} Q\right)^2 - \frac{k}{2} Q^2 \tag{19} \\ &= \frac{m}{2}\left(\dot{Q}^2 - \sigma \dot{Q} Q + \left(\frac{\sigma^2}{4} - \frac{k}{m}\right) Q^2\right). \tag{20} \end{align}$$

Then the Euler-Lagrange equations are given by

$$0 = \frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{m}{2}\left(-\sigma \dot{Q} + 2\left(\frac{\sigma^2}{4} - \frac{k}{m}\right) Q - \frac{d}{dt}\left(2\dot{Q} - \sigma Q\right)\right) \tag{21}$$

which simplifies to

$$0 = \ddot{Q} + \left(\frac{k}{m} - \frac{\sigma^2}{4}\right) Q. \tag{22}$$

The solutions to (22) have the form

$$Q(t) = \begin{cases} A_1 \sin\left(\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\, t\right) + A_2 \cos\left(\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\, t\right) & \frac{k}{m} > \frac{\sigma^2}{4}, \\ B_1 + B_2 t & \frac{k}{m} = \frac{\sigma^2}{4}, \\ C_1 \exp\left\{-\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\, t\right\} + C_2 \exp\left\{\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\, t\right\} & \frac{k}{m} < \frac{\sigma^2}{4}, \end{cases} \tag{23}$$

where A_i, B_i, C_i are real constants.

The Lagrangian in (20) does not explicitly depend on time. (Note that the Lagrangian in the problem statement *does* have an explicit time dependence.) Thus, the total energy H of the system in the new coordinate system is conserved. Explicitly,

$$H = \dot{Q}\frac{\partial L}{\partial \dot{Q}} - L = \frac{m}{2}\left(\dot{Q}^2 - \left(\frac{\sigma^2}{4} - \frac{k}{m}\right)Q^2\right) \tag{24}$$

is a conserved quantity.

3. **Problem.** Let $U(\alpha\mathbf{r}_1, \ldots, \alpha\mathbf{r}_N)$ be a potential for N particles that satisfies the relation

$$U(\alpha\mathbf{r}_1, \ldots, \alpha\mathbf{r}_N) = \alpha^k U(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

The factor α can be any nonzero real number. The exponent k is an integer.

(a) Show that the equations of motion associated with such a potential remain unchanged under a dilation of the distance scale if the time scale is also dilated by some other factor β. Find β as a function of α and k.

(b) If $k = 2$, the forces correspond to a system of harmonic oscillators coupled to each other. Show that the result in part (a) implies the frequencies of such a system are independent of the oscillation amplitude.

(c) If $k = -1$, we have an inverse square force law, such as that which arises in mutual gravitational attraction. Show that the result in part (a) implies Kepler's third law: the square of the orbital period of a planet is directly proporitonal to the cube of the semi-major axis of its orbit.

Solution. The Lagrangian $L = L(t, \mathbf{r}_i, \dot{\mathbf{r}}_i)$ for the system of N particles is

$$L = T - U = \frac{1}{2}m_i\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i - U(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{25}$$

where m_i is the mass of the particle located at $\mathbf{r}_i$. The Euler-Lagrange equations for this Lagrangian are

$$\frac{\partial L}{\partial \mathbf{r}_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = 0 \implies \frac{\partial U}{\partial \mathbf{r}_i} + m_i\ddot{\mathbf{r}}_i = 0. \tag{26}$$

Define the time scale transformation

$$T = \beta t, \tag{27}$$

and define the coordinate transformation

$$\mathbf{R}_i = \mathbf{R}_i(T) = \alpha\mathbf{r}_i \tag{28}$$

for all N particles. Using these coordinates, the Lagrangian $L = L(T, \mathbf{R}_i, \dot{\mathbf{R}}_i)$ is

$$L = \frac{1}{2}m_i\dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i - U(\mathbf{R}_1, \ldots, \mathbf{R}_N) \tag{29}$$

and the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \mathbf{R}_i} - \frac{d}{dT}\frac{\partial L}{\partial \dot{\mathbf{R}}_i} = 0 \implies \frac{\partial U}{\partial \mathbf{R}_i} + m_i\ddot{\mathbf{R}}_i = 0. \tag{30}$$

(a) The equations of motion associated to the Lagrangians (25) and (29) are identical if the Euler-Lagrange equations in (26) and (30) are identical. We will now show that this is the case for a particular value of β.

The transformation $\mathbf{R}_i = \alpha \mathbf{r}_i$ is invertible, so $\mathbf{r}_i = \mathbf{R}_i/\alpha$. Likewise, $t = T/\beta$. By the chain rule,

$$\frac{d}{dT} = \frac{d}{dt}\frac{dt}{dT} = \frac{1}{\beta}\frac{d}{dt} \tag{31}$$

so

$$\dot{\mathbf{R}} = \alpha \frac{d\mathbf{r}_i}{dT} = \frac{\alpha}{\beta}\dot{\mathbf{r}}_i \tag{32}$$

and, likewise,

$$\ddot{\mathbf{R}} = \frac{\alpha}{\beta^2}\ddot{\mathbf{r}}_i. \tag{33}$$

From the given relationship for U, note that

$$U(\mathbf{R}_1, \ldots, \mathbf{R}_N) = \alpha^k U(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{34}$$

and again using the chain rule,

$$\frac{\partial}{\partial \mathbf{R}_i} = \frac{\partial}{\partial \mathbf{r}_i}\frac{d\mathbf{r}_i}{d\mathbf{R}_i} = \frac{1}{\alpha}\frac{\partial}{\partial \mathbf{r}_i}. \tag{35}$$

Making use of (33), (34), and (35), we can rewrite (30) in terms of the original coordinates:

$$0 = \frac{\alpha^k}{\alpha}\frac{\partial U}{\partial \mathbf{r}_i} + m_i \frac{\alpha}{\beta^2}\ddot{\mathbf{r}}_i \implies \alpha^{k-2}\beta^2 \frac{\partial U}{\partial \mathbf{r}_i} + m_i \ddot{\mathbf{r}}_i = 0 \tag{36}$$

which is equivalent to (26) so long as

$$\alpha^{k-2}\beta^2 = 1 \implies \beta = \alpha^{2-k/2}. \tag{37}$$

So we have proven that (26) and (30) are equivalent under the condition (37). □

(b) Fixing $k = 2$, the condition of (a) gives us $\beta = \alpha^0 = 1$. This result indicates that the time scale is completely independent of the distance scale. That is, if we make the distance scale transformation $\mathbf{r}_i \mapsto \alpha \mathbf{r}_i$, the equations of motion will remain unchanged with no change to the time scale ($t \mapsto \beta t = t$).

For the system of harmonic oscillators, the oscillation amplitudes A_i have units of distance. The frequencies ω_i have units of inverse time. Making the transformation $\mathbf{r}_i \mapsto \alpha \mathbf{r}_i$ will change the distance scale and therefore the amplitudes, but the time scale and hence ω_i will remain unchanged. We may thus conclude that the frequencies are independent of the amplitudes. □

(c) Fixing $k = -1$, the solution of (a) gives us $\beta = \alpha^{3/2}$. Consider a planet whose orbit has semi-major axis length a and orbital period T. For this arbitrary planet, there exists some constant j such that

$$T^2 = ja^3. \tag{38}$$

Note that a has units of distance and T has units of time. In order to show that (38) holds for *any* planet, we can consider an arbitrary length αa for the semimajor axis. Thus, we want to show that (38) is unchanged under the transformation $a \mapsto \alpha a$ and the corresponding time dilation $T \mapsto \beta T = \alpha^{3/2} T$. Making these transformations,

$$(\alpha^{3/2} T)^2 = j(\alpha a)^3 \iff \alpha^3 T^2 = j \alpha^3 a^3 \iff T^2 = j a^3 \tag{39}$$

which is indeed equivalent to (38). Thus, we have shown that Kepler's third law holds for any planet. □

4. **Problem.** A particle in three-dimensional space is confined in a central potential

$$U(r) = -U_0 \left(\frac{r_0}{r}\right)^n .$$

Here $r = |\mathbf{r}|$ where $\mathbf{r}(t)$ is the location of the particle at time t, U_0 is a characteristic energy scale and r_0 is a characteristic length scale. The exponent n is an integer that is greater than or equal to 1. Show that the particle motion is confined to a two-dimensional orbital plane. For what values of n are circular orbits stable?

Solution. We want to show that the particle motion is confined to a two-dimensional orbital plane. We will use the spherical coordinates (r, θ, ϕ), so r retains its definition as the particle's distance from the origin.

$U(r)$ is a central potential, so it has a corresponding central force

$$\mathbf{F} = -\nabla U(r) = -nU_0 \frac{r_0^n}{r^{n+1}} \hat{\mathbf{r}} \tag{40}$$

which is radially symmetric. This means that the particle's torque $\boldsymbol{\tau} = 0$. Therefore, the particle's angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{41}$$

is conserved; that is, it is constant over time. Notably, the *direction* of $\mathbf{L}$ does not change over time. Because $\mathbf{r}$ is perpendicular to $\mathbf{L}$ as defined by (41), $\mathbf{L}$'s not changing direction implies that $\mathbf{r}$ is confined to a plane perpendicular to $\mathbf{L}$ for all time. This is what we sought to show. □

Now we will find the values of n for which circular orbits are stable. We will choose $\mathbf{L}$ to point in the $\hat{\boldsymbol{\phi}}$ direction, so $\mathbf{r}$ is confined to the plane (r, θ). Then $\mathbf{r} = \mathbf{r}(r, \theta)$. The particle's potential energy is $T = m\mathbf{r}^2/2$ where m is the particle's mass. In spherical coordinates, this gives us the Lagrangian

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + U_0 \frac{r_0^n}{r^n}, \tag{42}$$

which does not depend explicitly on θ.

For r, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = mr\dot{\theta}^2 - nU_0 \frac{r_0^n}{r^{n+1}} - m\ddot{r} = 0. \tag{43}$$

For θ, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 - \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \tag{44}$$

which implies

$$mr^2\dot{\theta} = l \tag{45}$$

where $l = |\mathbf{L}|$ is a constant. Substituting (45) into (43), and rearranging, we obtain

$$m\ddot{r} = \frac{l^2}{mr^3} - nU_0\frac{r_0^n}{r^{n+1}} \equiv -\frac{\partial U_{\text{eff}}}{\partial r} \tag{46}$$

where we have defined the effective potential $U_{\text{eff}} = U_{\text{eff}}(r)$. Explicitly,

$$U_{\text{eff}} = -U_0\frac{r_0^n}{r^n} + \frac{1}{2}\frac{l^2}{mr^2}. \tag{47}$$

If a circular orbit at $r = r_c$ is stable, small perturbations $r_c \mapsto r_c + \delta r$ will result in orbits that do not "blow up"; that is, they stay close to r_c. In order for this to be the case, $U_{\text{eff}}(r)$ must have a local minimum at r_c. In order to have any kind of extremum at r_c, we require

$$\left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r_c} = 0 \implies \frac{l^2}{mr_c^3} = nU_0\frac{r_0^n}{r_c^{n+1}} \tag{48}$$

where we have applied the definition (47). In order for the extremum at $r_c = 0$ to be a minimum, we require

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{r_c} > 0. \tag{49}$$

Again using (47),

$$\left.\frac{\partial^2 U_{\text{eff}}}{\partial r^2}\right|_{r_c} = -n(n+1)U_0\frac{r_0^n}{r_c^{n+2}} + 3\frac{l^2}{mr_c^4} = -n(n+1)U_0\frac{r_0^n}{r_c^{n+2}} + \frac{3}{r_c}nU_0\frac{r_0^n}{r_c^{n+1}} \tag{50}$$

where in the final equality we are substituting the result of (48). So the condition for a stable circular orbit becomes

$$3nU_0\frac{r_0^n}{r^{n+2}} > n(n+1)U_0\frac{r_0^n}{r^{n+2}}. \tag{51}$$

This holds for $n < 2$. Thus, for the conditions of this problem, it is only possible to have a stable circular orbit for $n = 1$.

In writing these solutions, I consulted David Tong's lecture notes, Goldstein's *Classical Mechanics*, and Landau and Lifshitz's *Mechanics*.