1 Elastically fastened ends

Consider an ideal stretched string in two dimensions with length ℓ, density per unit length ρ, and effective elastic modulus k. Suppose its two ends are fastened *elastically* by two springs with spring constant k_0 so that a nonzero deflection $u(x,t)$ of the end location from either $(0,0)$ or $\ell,0)$ is penalized by a linear restrictive force $-ku$. Adapt the derivation in class for a stretched spring with two fixed ends to this situation. What are the Euler-Lagrange equations and the associated boundary conditions?

Solution. We will begin with the expression for the kinetic energy T of the string. Let $\mathrm{d}x$ denote an infinitesimal length of string. Its mass $\mathrm{d}m = \rho\,\mathrm{d}x$, so its kinetic energy $\mathrm{d}T$ is

$$\mathrm{d}T = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x \implies T = \frac{\rho}{2}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,, \tag{1}$$

where we have integrated over the length of the string to obtain T.

For the potential energy, let U_1 be the work required to stretch the string, and U_2 the work to compress and decompress the springs. (The addition of U_2 is what differs from the derivation in class.). For U_1, consider an infinitesimal length of string $\mathrm{d}x$. If this length is stretched by some amount Δx to a total length

$$\mathrm{d}x + \Delta x = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}u)^2}, \tag{2}$$

it has potential energy $\mathrm{d}U_1 = k\,\Delta x$. Performing a Taylor series expansion for a small Δx and integrating to obtain U_1,

$$\mathrm{d}U_1 = k\,\Delta x = k(\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}u)^2} - \mathrm{d}x) \approx \frac{k}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x \implies U_1 = \frac{k}{2}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,. \tag{3}$$

This approximation is sufficient because we consider only small oscillations. For U_2, the potential energy in the two springs is given by

$$U_2 = \frac{k}{2}u_0^2 + \frac{k}{2}u_\ell^2, \tag{4}$$

where $u_0 = u_0(t) = u(0,t)$ and $u_\ell = u_\ell(t) = u(\ell,t)$. The total potential energy $U = U_1 + U_2$.

Using (1), (3), and (4), we can write an expression for the action of the string:

$$S[u] = \int_{t_0}^{t_1} (T-U)\,\mathrm{d}t = \int_{t_0}^{t_1}\left[\frac{\rho}{2}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x - \frac{k}{2}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x - \frac{k}{2}u_0^2 - \frac{k}{2}u_\ell^2\right]\mathrm{d}t \tag{5}$$

$$= \frac{\rho}{2}\int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{k}{2}\int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{k}{2}\int_{t_0}^{t_1}\left(u_\ell^2 + u_0^2\right)\mathrm{d}t \tag{6}$$

$$= \int_{t_0}^{t_1}\int_0^\ell \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t\,, \tag{7}$$

where $\mathcal{L}$ is the Lagrangian density. Consider some variation of the action $\Delta S = S[u+\epsilon\psi] - S[u]$, where $\psi = \psi(x,t)$ is a function representing a variation and $\epsilon \ll 1$. The principle component of ΔS, δS, is given by

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x}\right)\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{t_0}^{t_1}\int_0^\ell \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\psi\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\psi\right)\right]\mathrm{d}x\,\mathrm{d}t\,, \tag{8}$$

where $u_t = \partial u/\partial t$ and $u_x = \partial u/\partial x$. Note that

$$\frac{\partial \mathcal{L}}{\partial u_t} = \rho\frac{\partial u}{\partial t}, \qquad \frac{\partial \mathcal{L}}{\partial u_x} = -k\frac{\partial u}{\partial x}, \tag{9}$$

so

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1}\int_0^{\ell}\left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2}\right)\psi\,\mathrm{d}x\,\mathrm{d}t - k\int_{t_0}^{t_1}(u_\ell\psi_\ell + u_0\psi_0)\,\mathrm{d}t$$
$$+\rho\int_{t_0}^{t_1}\int_0^{\ell}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\psi\right)\mathrm{d}x\,\mathrm{d}t - k\int_{t_0}^{t_1}\int_0^{\ell}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\psi\right)\mathrm{d}x\,\mathrm{d}t\,, \tag{10}$$

where $\psi_0 = \psi_0(t) = \psi(0,t)$ and $\psi_\ell = \psi_\ell(t) = \psi(\ell,t)$. We stipulate that $\psi(x,t_0) = \psi(x,t_1) = 0$ for $x \in [0,\ell]$ and that $\psi(0,t) = \psi(\ell,t) = 0$ for $t \in [t_0,t_1]$. Then (10) becomes

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1}\int_0^{\ell}\left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2}\right)\psi\,\mathrm{d}x\,\mathrm{d}t - k\int_{t_0}^{t_1}(u_\ell\psi_\ell + u_0\psi_0)\,\mathrm{d}t + k\int_{t_0}^{t_1}\left(\psi_0\left.\frac{\partial u}{\partial x}\right|_{x=0} - \psi_\ell\left.\frac{\partial u}{\partial x}\right|_{x=\ell}\right)\mathrm{d}t \tag{11}$$
$$= \int_{t_0}^{t_1}\int_0^{\ell}\left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2}\right)\psi\,\mathrm{d}x \tag{12}$$

By the principle of least action, $\delta S = 0$ for the actual solution $u(x,t)$:

$$0 = \int_{t_0}^{t_1}\int_0^{\ell}\left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2}\right)\psi\,\mathrm{d}x \implies \frac{\partial^2 u}{\partial t^2} = \frac{k}{\rho}\frac{\partial^2 u}{\partial x^2} \tag{13}$$

for $x \in (0,\ell)$ and for $t \in (-\infty,\infty)$, since the time interval $[t_0,t_1]$ was arbitrary. (1) is the Euler-Lagrange equation for the system. (This is the same as we derived in class.)

In order to evaluate the boundary conditions, we remove the stipulation $\psi(0,t) = \psi(\ell,t) = 0$ for $t \in [t_0,t_1]$. Under the condition that is satisfied, (11) reduces to

$$\frac{\delta S}{\epsilon} = -k\int_{t_0}^{t_1}(u_\ell\psi_\ell + u_0\psi_0)\,\mathrm{d}t + k\int_{t_0}^{t_1}\left(\psi_0\left.\frac{\partial u}{\partial x}\right|_{x=0} - \psi_\ell\left.\frac{\partial u}{\partial x}\right|_{x=\ell}\right)\mathrm{d}t\,, \tag{14}$$

and once again invoking the principle of least action,

$$\delta S = 0 \implies u_\ell\psi_\ell + u_0\psi_0 = \psi_0\left.\frac{\partial u}{\partial x}\right|_{x=0} - \psi_\ell\left.\frac{\partial u}{\partial x}\right|_{x=\ell} = 0. \tag{15}$$

Rearranging the result of (15), we find

$$0 = u(0,t) - \left.\frac{\partial u}{\partial x}\right|_{x=0}, \qquad\qquad 0 = u(\ell,t) + \left.\frac{\partial u}{\partial x}\right|_{x=\ell} \tag{16}$$

as the boundary conditions for (1).