Problem 1. (Jackson 12.3) A particle with mass m and charge e moves in a uniform, static, electric field $\mathbf{E}_0$.

1(a) Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\mathbf{v}_0$ was perpendicular to the electric field.

Solution. Jackson (12.1) gives the force exerted on a charged particle in an external electromagnetic field:

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right),$$

where $\mathbf{u}$ is the velocity of the particle. In this problem $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{B} = \mathbf{0}$, so

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E}_0.$$

Since $\mathbf{E}_0$ is constant, we can easily find $\mathbf{p}$ as a function of time by solving this expression as a differential equation. This gives us

$$\int_0^\infty d\mathbf{p} = e\mathbf{E}_0 \int_0^\infty dt \qquad \Longrightarrow \qquad \mathbf{p}(t) = e\mathbf{E}_0 t + \mathbf{p}_0.$$

In order to use this result to find the velocity of the particle, we need to write the particle's velocity in terms of its momentum. According to Jackson (11.46), (11.51), and (11.55), $\mathbf{p} = \gamma m v$, $\mathcal{E} = m\gamma c^2$, and $\mathcal{E} = \sqrt{c^2p^2 + m^2c^4}$, where $\mathcal{E}$ is the total energy of the particle. Combining these gives us

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{c^2\mathbf{p}}{\mathcal{E}} = \frac{c\mathbf{p}}{\sqrt{m^2c^2 + \mathbf{p}^2}}. \tag{1}$$

Then, substituting into Eq. (1), we have

$$\mathbf{v}(t) = c\frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + (e\mathbf{E}_0 t + \mathbf{p}_0)^2}} = c\frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + 2e\mathbf{p}_0 \cdot \mathbf{E}_0 t + \mathbf{p}_0^2}} = c\frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}},$$

where in going to the final equality we have used the fact that $\mathbf{p}_0 = \gamma m \mathbf{v}_0$ is perpendicular to $\mathbf{E}_0$.

Finally, we can solve this as a differential equation to find the position of the particle as a function of time. Let $\mathbf{v}(t) = d\mathbf{r}/dt$, where $\mathbf{r}(t)$ is the position of the particle. Then

$$\begin{aligned}\int_0^\infty d\mathbf{r} &= c\int_0^\infty \frac{e\mathbf{E}_0 t + \mathbf{p}_0}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}}\, dt \\ &= ce\mathbf{E}_0 \int_0^\infty \frac{t}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}}\, dt + c\mathbf{p}_0 \int_0^\infty \frac{dt}{\sqrt{m^2c^2 + e^2\mathbf{E}_0^2 t^2 + \mathbf{p}_0^2}}.\end{aligned} \tag{2}$$

For the first integral on the right side,

$$\int_0^\infty \frac{t}{\sqrt{e^2\mathbf{E}_0^2 t^2 + m^2c^2 + \mathbf{p}_0^2}}\, dt = \frac{1}{2e^2\mathbf{E}_0^2} \int_{u_0}^\infty \frac{1}{\sqrt{u}}\, du = \frac{\sqrt{u} - \sqrt{u_0}}{e^2\mathbf{E}_0^2},$$

where we have used the substitution $u = e^2\mathbf{E}_0^2 t^2 + m^2c^2 + \mathbf{p}_0^2$, with $u_0 = m^2c^2 + \mathbf{p}_0^2$.

For the second integral on the right side of Eq. (2),

$$\int_0^\infty \frac{dt}{\sqrt{e^2\mathbf{E}_0^2 t^2 + m^2c^2 + \mathbf{p}_0^2}} = \frac{1}{\sqrt{m^2c^2 + \mathbf{p}_0^2}} \int_0^\infty \frac{dt}{\sqrt{e^2\mathbf{E}_0^2 t^2/(m^2c^2 + \mathbf{p}_0^2) + 1}} = \frac{1}{e|\mathbf{E}_0|} \int_0^\infty \frac{du}{\sqrt{u^2 + 1}} = \frac{\sinh^{-1} u}{e|\mathbf{E}_0|},$$

where we have used the substitution $u = e|\mathbf{E}_0|t/\sqrt{m^2c^2 + \mathbf{p}_0^2}$, the fact that $d\sinh^{-1} z/dz = 1/\sqrt{1+z^2}$, and the fact that $\sinh u_0 = \sinh(0) = 0$ [**?**].

With these solutions, Eq. (2) becomes

$$\mathbf{r}(t) = \frac{c}{e|\mathbf{E}_0|}\left[\frac{\mathbf{E}_0}{|\mathbf{E}_0|}\left(\sqrt{e^2\mathbf{E}_0^2t^2 + m^2c^2 + \mathbf{p}_0^2} - \sqrt{m^2c^2 + \mathbf{p}_0^2}\right) + \mathbf{p}_0 \sinh^{-1}\left(\frac{e|\mathbf{E}_0|t}{\sqrt{m^2c^2 + \mathbf{p}_0^2}}\right)\right] + \mathbf{r}_0,$$

where $\mathbf{r}_0$ is the initial position of the particle.

To make the equation a little neater, we can once again apply Jackson (11.55) to define the particle's initial energy as $\mathcal{E}_0 = \sqrt{m^2c^4 + c^2\mathbf{p}_0^2}$. This gives us

$$\mathbf{r}(t) = \frac{1}{e|\mathbf{E}_0|}\left[\frac{\mathbf{E}_0}{|\mathbf{E}_0|}\left(\sqrt{c^2e^2\mathbf{E}_0^2t^2 + \mathcal{E}_0^2} - \mathcal{E}_0\right) + c\mathbf{p}_0 \sinh^{-1}\left(\frac{ce|\mathbf{E}_0|t}{\mathcal{E}_0}\right)\right] + \mathbf{r}_0.$$

For the velocity, we have

$$\mathbf{v}(t) = \frac{ce\mathbf{E}_0t + c\mathbf{p}_0}{\sqrt{e^2\mathbf{E}_0^2t^2 + m^2c^2 + \mathbf{p}_0^2}} = \frac{c^2e\mathbf{E}_0t + c^2\mathbf{p}_0}{\sqrt{c^2e^2\mathbf{E}_0^2t^2 + \mathcal{E}_0^2}}.$$

1(b) Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define "short" and "long" times).

Solution. Let $\mathbf{r}_0 = \mathbf{0}$, and let $r_\perp(t)$ and $r_\parallel(t)$ denote the components of the particle's position that are, respectively, parallel to and perpendicular to its original velocity. Then

$$r_\perp(t) = \frac{\sqrt{c^2e^2\mathbf{E}_0^2t^2 + \mathcal{E}_0^2} - \mathcal{E}_0}{e|\mathbf{E}_0|}, \qquad r_\parallel(t) = \frac{cp_0}{e|\mathbf{E}_0|}\sinh^{-1}\left(\frac{ce|\mathbf{E}_0|t}{\mathcal{E}_0}\right). \tag{3}$$

It is easiest to solve $r_\parallel(t)$ for t, which gives us

$$\frac{e|\mathbf{E}_0|r_\parallel}{cp_0} = \sinh^{-1}\left(\frac{ce|\mathbf{E}_0|t}{\mathcal{E}_0}\right) \quad\Longrightarrow\quad t = \frac{\mathcal{E}_0}{ce|\mathbf{E}_0|}\sinh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right).$$

Substituting into the expression for $r_\perp$, we find

$$\begin{aligned} r_\perp &= \frac{1}{e|\mathbf{E}_0|}\left(\sqrt{c^2e^2\mathbf{E}_0^2\left[\frac{\mathcal{E}_0}{ce|\mathbf{E}_0|}\sinh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right)\right]^2 + \mathcal{E}_0^2} - \mathcal{E}_0\right) = \frac{\mathcal{E}_0}{e|\mathbf{E}_0|}\left[\sqrt{\sinh^2\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) + 1} - 1\right] \\ &= \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\left[\cosh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) - 1\right], \end{aligned}$$

where we have used $\cosh^2 x - \sinh^2 x = 1$ [**?**]. Then the trajectory of the particle is given by

$$r_\perp = \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\left[\cosh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) - 1\right] = \frac{\sqrt{mc^2 + \mathbf{p}_0^2}}{|\mathbf{E}_0|}\left[\cosh\left(\frac{e|\mathbf{E}_0|r_\parallel}{cp_0}\right) - 1\right].$$

For short times, the argument of $\sinh^{-1}$ in Eq. (3) must be small. Note also that $r_\perp(t)$ can be written as

$$r_\perp(t) = \mathcal{E}_0\frac{\sqrt{c^2e^2\mathbf{E}_0^2t^2/\mathcal{E}_0^2 + 1} - 1}{e|\mathbf{E}_0|}, \tag{4}$$

so we can conclude that $t \ll \mathcal{E}_0/ce|\mathbf{E}_0|$ for short times. Likewise, $t \gg \mathcal{E}_0/ce|\mathbf{E}_0|$ for long times.

To obtain the trajectory for short times, we note that $u = \mathcal{E}_0/ce|\mathbf{E}_0| \ll 1$ implies that $r_\parallel \ll 1$. Thus, we can Taylor expand Eq. (4) around $r_\parallel = 0$. The Taylor series for cosh is [**?**]

$$\cosh z = 1 + \frac{z^2}{2} + \cdots,$$

so

$$\lim_{u\to 0} r_\perp = \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\frac{r_\parallel^2}{2} = \frac{\sqrt{m^2c^2+\mathbf{p}_0^2}}{|\mathbf{E}_0|}\frac{r_\parallel^2}{2},$$

indicating that the trajectory of the particle is parabolic for short times. As soon as the field is turned on, it will start pulling the particle in a direction that is perpendicular to its original velocity. This is just like projectile motion under the influence of gravity.

To obtain the trajectory for long times, we note that $\lim_{u\to\infty}\sinh^{-1} u = \infty$ [**?**], so taking u to be large is the same as taking $r_\parallel$ to be large. Note that

$$\lim_{z\to\infty}\cosh z = \lim_{z\to\infty}\frac{e^z+e^{-z}}{2} = \frac{e^z}{2},$$

so

$$\lim_{u\to\infty} r_\perp = \frac{\mathcal{E}_0}{c|\mathbf{E}_0|}\frac{e^{r_\parallel}}{2} = \frac{\sqrt{m^2c^2+\mathbf{p}_0^2}}{|\mathbf{E}_0|}\frac{e^{r_\parallel}}{2},$$

indicating that the trajectory of the particle is exponential for long times. When the field has been turned on for a long time, the particle has been accelerating parallel to the field for a long time. At infinite time, the particle's original direction of velocity has been completely washed out by the force of the electric field.

Problem 2. (Jackson 12.5) A particle of mass m and charge e moves in the laboratory in crossed, static, uniform, electric and magnetic fields. $\mathbf{E}$ is parallel to the x axis; $\mathbf{B}$ is parallel to the y axis.

2(a) For $|\mathbf{E}_0| < |\mathbf{B}|$ make the necessary Lorentz transformation described in Section 12.3 to obtain explicitly parametric equations for the particle's trajectory.

Solution. The boost described in Section 12.3 of Jackson for $|\mathbf{E}_0| < |\mathbf{B}|$ is into a frame K', which moves with velocity $\mathbf{u}$ with respect to the laboratory frame. This $\mathbf{u}$ is the particle's drift velocity. According to Jackson (12.43), it is given by

$$\mathbf{u} = c\frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}^2}.$$

In K', the fields are given by Jackson (12.44):

$$\mathbf{E}'_\parallel = \mathbf{E}'_\perp = \mathbf{B}'_\parallel = \mathbf{0}, \qquad \mathbf{B}'_\perp = \frac{\mathbf{B}}{\gamma} = \sqrt{\frac{\mathbf{B}^2-\mathbf{E}^2}{\mathbf{B}^2}}\mathbf{B},$$

where $\mathbf{E}'_\parallel$ and $\mathbf{B}'_\parallel$ are parallel to $\mathbf{u}$, and $\mathbf{E}'_\perp$ and $\mathbf{B}'_\perp$ are perpendicular to $\mathbf{u}$. This means that the particle's motion in this frame is the same as motion in a uniform, static magnetic field. The particle's trajectory in a uniform magnetic field $\mathbf{B}$ that points in the y direction is given by Jackson (12.41),

$$\mathbf{r}(t) = \mathbf{r}_0 + v_\parallel t\,\hat{\mathbf{y}} + ia(\hat{\mathbf{z}} - i\,\hat{\mathbf{x}})e^{-i\omega_B t}, \tag{5}$$

where $v_\parallel$ is the component of the particle's velocity along the field, $\boldsymbol{\omega}_B$ its gyration frequency, and a its gyration radius. These quantities are given by Jackson (12.39) and the formula immediately following (12.41), respectively:

$$\boldsymbol{\omega}_B = \frac{e\mathbf{B}}{\gamma mc} = \frac{ec\mathbf{B}}{\mathcal{E}}, \qquad cp_\perp = eBa,$$

where $p_\perp$ is the particle's transverse momentum.

In K', we have

$$\boldsymbol{\omega}'_B = \frac{ec\mathbf{B}'_\perp}{\mathcal{E}'} = \frac{e\mathbf{B}'_\perp}{\sqrt{m^2c^2 + \mathbf{p}'^2}}, \qquad a = \frac{cp'_\perp}{e|\mathbf{B}'_\perp|}. \tag{6}$$

Then, since $\mathbf{B}'_\perp$ points in the y' direction, Eq. (5) gives us

$$\mathbf{r}'(t') = \mathbf{r}'_0 + v'_\parallel t'\,\hat{\mathbf{y}} + ia(\hat{\mathbf{z}} - i\,\hat{\mathbf{x}})e^{-i\omega_B t'}.$$

Taking the real part and letting $\mathbf{r}'_0 = \mathbf{0}$, we find the equations

$$z'(t') = a\sin\big(\omega_B t'\big), \qquad x'(t') = a\cos\big(\omega_B t'\big), \qquad y'(t') = v'_\parallel t', \tag{7}$$

where we have used $e^{-ix} = \cos x - i\sin x$.

Now we will return to the lab frame, where $\mathbf{u}$ points in the z direction. Note that $|\mathbf{u}| = c|\mathbf{E}|/|\mathbf{B}|$, so $\beta = |\mathbf{E}|/|\mathbf{B}|$. The inverse Lorentz transformation for a boost in the z direction is found by modifying Jackson (11.18), which yields

$$ct = \gamma(ct' + \beta z'), \qquad x = x', \qquad y = y', \qquad z = \gamma(z' + \beta ct'). \tag{8}$$

Note that $v'_\parallel = v_\parallel$, since $v_\parallel$ is perpendicular to $\mathbf{u}$. Then, applying these to Eq. (7), and substituting for β and γ, we find the parametric equations

$$t(t') = \sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{B}^2}}^{\,-1}\left[t' + \frac{a|\mathbf{E}|}{c|\mathbf{B}|}\sin\big(\omega_B t'\big)\right], \qquad x(t') = a\cos\big(\omega_B t'\big),$$

$$y(t') = v_\parallel t', \qquad z(t') = \sqrt{1 - \frac{\mathbf{E}^2}{\mathbf{B}^2}}^{\,-1}\left[a\sin\big(\omega_B t'\big) + \frac{c|\mathbf{E}|}{|\mathbf{B}|}t'\right],$$

where ω_B and a are given by Eq. (6).

2(b) Repeat the calculation of part (a) for $|\mathbf{E}_0| > |\mathbf{B}|$.

Solution. For $|\mathbf{E}_0| > |\mathbf{B}|$, the boost described in Sec. (12.3) of Jackson is into a frame K'' which moves with velocity $\mathbf{u}'$ with respect to the laboratory frame, where

$$\mathbf{u}' = c\frac{\mathbf{E}\times\mathbf{B}}{E^2},$$

according to Jackson (12.46). The electric and magnetic fields in this frame are given by Jackson (12.46):

$$\mathbf{E}''_\perp = \frac{\mathbf{E}}{\gamma'} = \sqrt{\frac{\mathbf{E}^2 - \mathbf{B}^2}{\mathbf{E}^2}}\mathbf{E}, \qquad \mathbf{E}''_\parallel = \mathbf{B}''_\perp = \mathbf{B}''_\parallel = \mathbf{0}, \tag{9}$$

where $\mathbf{E}''_\parallel$ and $\mathbf{B}''_\parallel$ are parallel to $\mathbf{u}'$, and $\mathbf{E}''_\perp$ and $\mathbf{B}''_\perp$ are perpendicular to $\mathbf{u}'$. Then the particle's trajectory in this frame is described by Eq. (3). Since $\mathbf{E}''_\perp$ points in the x direction, we have

$$x''(t'') = \frac{\sqrt{c^2e^2{\mathbf{E}''_\perp}^2\,t''^2 + \mathcal{E}''^2_0} - \mathcal{E}''_0}{e|\mathbf{E}''_\perp|} + v''_{0x''}t'' = \frac{\sqrt{c^2e^2{\mathbf{E}''_\perp}^2\,t''^2 + \mathcal{E}''^2_0} - \mathcal{E}''_0}{e|\mathbf{E}''_\perp|} + \frac{c^2p''_{0x''}t''}{\mathcal{E}''_0},$$

where we have used Eq. (1), and

$$y''(t'') = \frac{cp''_{0y''}}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}''_0}\right), \qquad z''(t'') = \frac{cp''_{0z''}}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}''_0}\right),$$

where $p''_{0x''}$, $p''_{0y''}$, and $p''_{0z''}$ are the x'', y'', and z'' components, respectively, of the particle's initial momentum in K'', and

$$\mathcal{E}''_0 = c\sqrt{m^2c^2 + \mathbf{p}''^2_0} \tag{10}$$

is the particle's initial energy in K''.

We can transform back to the lab frame similarly to in Eq. (8), except now we boost by $\beta' = |\mathbf{B}|/|\mathbf{E}|$:

$$ct = \gamma'(ct'' + \beta' z''), \qquad x = x'', \qquad y = y'', \qquad z = \gamma'(z'' + \beta' ct'').$$

Substituting, we find

$$t(t'') = \sqrt{1 - \frac{\mathbf{B}^2}{\mathbf{E}^2}}^{\,-1}\left[t'' + \frac{|\mathbf{B}|}{c|\mathbf{E}|}\frac{cp''_{0z''}}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}''_0}\right)\right],$$

$$x(t'') = \frac{\sqrt{c^2e^2{\mathbf{E}''_\perp}^2\,t''^2 + \mathcal{E}''^2_0} - \mathcal{E}''_0}{e|\mathbf{E}''_\perp|} + \frac{c^2p''_{0x''}t''}{\mathcal{E}''_0},$$

$$y(t'') = \frac{cp''_{0y''}}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}''_0}\right),$$

$$z(t'') = \sqrt{1 - \frac{\mathbf{B}^2}{\mathbf{E}^2}}^{\,-1}\left[\frac{cp''_{0z''}}{e|\mathbf{E}''_\perp|}\sinh^{-1}\left(\frac{ce|\mathbf{E}''_\perp|t''}{\mathcal{E}''_0}\right) + \frac{c|\mathbf{B}|}{|\mathbf{E}|}t''\right],$$

where $\mathbf{E}''_\perp$ is given by Eq. (9), $\mathcal{E}''_0$ is given by Eq. (10), and $\mathbf{p}''_0$ is the particle's initial momentum in K''.

Problem 3. (Jackson 12.19) Source-free electromagnetic fields exist in a localized region of space. Consider the various conservation laws that are contained in the integral of $\partial_\alpha M^{\alpha\beta\gamma} = 0$ over all space, where

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta.$$

3(a) Show that when β and γ are both space indices conservation of the total field angular momentum follows.

Solution. Note that

$$\partial_\alpha M^{\alpha\beta\gamma} = \partial_\alpha(\Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta) = x^\gamma\partial_\alpha\Theta^{\alpha\beta} + \Theta^{\alpha\beta}\partial_\alpha x^\gamma - x^\beta\partial_\alpha\Theta^{\alpha\gamma} - \Theta^{\alpha\gamma}\partial_\alpha x^\beta = \Theta^{\alpha\beta}\partial_\alpha x^\gamma - \Theta^{\alpha\gamma}\partial_\alpha x^\beta,$$

where we have used Jackson (12.116), $\partial_\alpha\Theta^{\alpha\beta} = 0$.

3(b) Show that when $\beta = 0$ the conservation law is

$$\frac{d\mathbf{X}}{dt} = \frac{c^2 \mathbf{P}_{\mathrm{em}}}{E_{\mathrm{em}}},$$

where $\mathbf{X}$ is the coordinate of the center of mass of the electromagnetic fields, defined by

$$\mathbf{X} \int u \, d^3x = \int \mathbf{x} u \, d^3x \,,$$

where u is the electromagnetic energy density and E_{em} and $\mathbf{P}_{\mathrm{em}}$ are the total energy and momentum of the fields.

Problem 4. We discussed in class the construction of linearly polarized electromagnetic waves.

4(a) Generalize the discussion to circularly polarized waves (see also Wald Sec. 5.5). Discuss both right-handed and left-handed polarizations.

4(b) Compute the angular momentum of the circularly polarized waves of part (a) using the formula for angular momentum derived in class.

Problem 5. We wrote in class the Lagrangian of a charged particle coupled to the electromagnetic field (see pp. 159–160) in the lecture notes).

5(a) Show that the Euler-Lagrange equations that follow from this Lagrangian give rise to the Lorentz force law

$$\frac{dp_i}{dt} = q\left[E^i + \frac{1}{c}(\mathbf{v}\times\mathbf{B})^i\right].$$

Solution. The action of a charged particle in an electromagnetic field is given on p. 159 of the lecture notes as

$$S[\mathbf{r}] = \int\left(-mc^2\sqrt{1-\frac{\dot{\mathbf{r}}^2}{c^2}} - q\phi + \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}\right)dt \equiv \int \mathcal{L}(\mathbf{r},\dot{\mathbf{r}})\,dt\,, \tag{11}$$

where we have fixed $\lambda = t$, and we have defined $\mathcal{L}$. The Euler-Lagrange equations are given on p. 94 of the lecture notes:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = \frac{\partial\mathcal{L}}{\partial q_i}.$$

For the Lagrangian defined in Eq. (11), we have firstly

$$\frac{\partial\mathcal{L}}{\partial\dot{r}_i} = -mc^2\left(\frac{1}{2}\frac{1}{\sqrt{1-\dot{\mathbf{r}}^2/c^2}}\right)\left(-\frac{2\dot{r}_i}{c^2}\right) + \frac{q}{c}A_i = \frac{m\dot{r}_i}{\sqrt{1-\dot{\mathbf{r}}^2/c^2}} + \frac{q}{c}A_i = p_i + \frac{q}{c}A_i,$$

since $\mathbf{p} = m\gamma\dot{\mathbf{r}}$ and $\beta = \dot{\mathbf{r}}/c$.

Secondly, we have [**?** , p. 50]

$$\frac{\partial\mathcal{L}}{\partial\mathbf{r}} = \boldsymbol{\nabla}\mathcal{L} = \frac{q}{c}\boldsymbol{\nabla}(\dot{\mathbf{r}}\cdot\mathbf{A}) - q\boldsymbol{\nabla}\phi.$$

One of the vector identities on the inside cover of Jackson is

$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} + \mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\boldsymbol{\nabla}\times\mathbf{a}),$$

so [**?** , p. 50]

$$\boldsymbol{\nabla}(\dot{\mathbf{r}}\cdot\mathbf{A}) = (\dot{\mathbf{r}}\cdot\boldsymbol{\nabla})\mathbf{A} + \dot{\mathbf{r}}\times(\boldsymbol{\nabla}\times\mathbf{A}) \qquad\Longrightarrow\qquad \frac{\partial\mathcal{L}}{\partial\mathbf{r}} = \frac{q}{c}[(\dot{\mathbf{r}}\cdot\boldsymbol{\nabla})\mathbf{A} + \dot{\mathbf{r}}\times(\boldsymbol{\nabla}\times\mathbf{A})] - q\boldsymbol{\nabla}\phi.$$

Then the Euler-Lagrange equations become

$$\frac{d\mathbf{p}}{dt} + \frac{q}{c}\frac{d\mathbf{A}}{dt} = \frac{q}{c}[(\dot{\mathbf{r}}\cdot\boldsymbol{\nabla})\mathbf{A} + \dot{\mathbf{r}}\times(\boldsymbol{\nabla}\times\mathbf{A})] - q\boldsymbol{\nabla}\phi. \tag{12}$$

The total derivative of $\mathbf{A}$ is given by [**?** , p. 50],

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + \dot{\mathbf{r}}\cdot\frac{\partial\mathbf{A}}{\partial\dot{\mathbf{r}}} = \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\cdot\boldsymbol{\nabla})\mathbf{A}.$$

Then Eq. (12) becomes

$$\frac{d\mathbf{p}}{dt} = \frac{q}{c}\dot{\mathbf{r}}\times(\boldsymbol{\nabla}\times\mathbf{A}) - q\boldsymbol{\nabla}\phi - \frac{q}{c}\frac{d\mathbf{A}}{dt}. \tag{13}$$

According to Wald (5.2–3),

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \qquad\qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}.$$

Making these substitutions and $\dot{\mathbf{r}}\to\mathbf{v}$ in Eq. (13), we have

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) \tag{14}$$

as desired. □

5(b) Show that the Lorentz force law can be written covariantly in the form

$$\frac{dU^\mu}{d\tau} = \frac{q}{mc} F^{\mu\nu} U_\nu. \tag{15}$$

Solution. The 4-velocity U^μ is defined by Jackson (11.36) as $U^\mu = \gamma(c, \mathbf{v}) = (U^0, \mathbf{U})$. The 4-momentum is defined by the equation immediately preceding (11.125) as $P^\mu = (\mathcal{E}/c, \mathbf{p})$, where $\mathcal{E}$ is the total energy of the particle. Also from this equation is the relation $P^\mu = mU^\mu$.

According to Jackson (11.26), $d\tau = dt/\gamma$. Making this substitution in Eq. (14) and dividing by m, we find [**?** , p. 553]

$$\frac{1}{\gamma}\frac{d\mathbf{p}}{d\tau} = q\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right) \quad\Longrightarrow\quad \frac{d\gamma\mathbf{v}}{d\tau} = \frac{d\mathbf{U}}{d\tau} = \frac{q\gamma}{mc}\left(c\mathbf{E} + \mathbf{v}\times\mathbf{B}\right). \tag{16}$$

From Wald (5.17),

$$\frac{d\mathcal{E}}{dt} = \mathbf{J}\cdot\mathbf{E}.$$

For a point charge, $\mathbf{J} = q\mathbf{v}$. Making this substitution, dividing by mc, and changing to a derivative of τ, we find [**?** , p. 553]

$$\frac{d\mathcal{E}}{dt} = q\mathbf{v}\cdot\mathbf{E} \quad\Longrightarrow\quad \frac{1}{mc}\frac{d\mathcal{E}}{dt} = \frac{dU^0}{d\tau} = \frac{q\gamma}{mc}\mathbf{v}\cdot\mathbf{E}. \tag{17}$$

This corresponds to the derivative of the temporal part of U^μ.

Now we will work directly from Eq. (15) and write $F^{\mu\nu}U_\nu$ in terms of the fields. From Jackson (11.137),

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}.$$

Then we have

$$F^{\mu\nu}U_\nu = \gamma \begin{bmatrix} v_xE_x + v_yE_y + v_zE_z \\ cE_x + v_yB_z - v_zB_y \\ cE_y - v_xB_z + v_zB_x \\ cE_z + v_xB_y - v_yB_x \end{bmatrix} = \gamma(\mathbf{v}\cdot\mathbf{E}, c\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

Using this result, we may combine Eqs. (16) and (17) to write

$$\frac{dU^\mu}{dd\tau} = \frac{q}{mc}F^{\mu\nu}U_\nu$$

as desired. □