Problem 1. Coulomb scattering (Peskin & Schroeder 5.1) Repeat the computation of Problem 4.4, part (c), this time using the full relativistic expression for the matrix element. You should find, for the spin-averaged cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right],$$

where $\mathbf{p}$ is the electron's 3-momentum and β its velocity. This is the *Mott formula* for Coulomb scattering of relativistic electrons. Now derive it in a second way, by working out the cross section for electron-muon scattering, in the muon rest frame, retaining the electron mass but sending $m_\mu \to \infty$.

Solution. From the solution to 2(b) of Homework 4, we know that

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2}. \tag{1}$$

We also know from 2(a) that

$$i\mathcal{M} = -ie\bar{u}(p')\gamma^\mu u(p)\cdot\tilde{A}_\mu(\mathbf{p}'-\mathbf{p}),$$

where the argument of $\tilde{A}$ includes only the spatial components of p and p' because $A_\mu(x)$ is time independent [1, p. 129]. Then

$$-i\mathcal{M}^* = ie\bar{u}(p)\gamma^\mu u(p')\cdot\tilde{A}_\mu(\mathbf{p}'-\mathbf{p}),$$

so

$$|\mathcal{M}|^2 = e^2\bar{u}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu u(p')\tilde{A}_\mu(\mathbf{p}'-\mathbf{p})\tilde{A}_\nu(\mathbf{p}'-\mathbf{p}).$$

Averaging over the spins, we want to compute [1, p. 132]

$$\frac{1}{2}\sum_{s,s'}\left|\mathcal{M}(s,s')\right|^2.$$

By (5.3) of Peskin & Schroeder,

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m.$$

Then, writing in the spinor indices as in the equation above (5.4),

$$\begin{aligned}
\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 &= \frac{e^2}{2}\sum_{s,s'}\bar{u}_a^{s'}(p')\gamma^\mu_{ab}u_b^s(p)\bar{u}_c(p)\gamma^\nu_{cd}u_d^{s'}(p')\tilde{A}_\mu(p'-p)\tilde{A}_\nu(p'-p)\\
&= \frac{e^2}{2}\sum_{s,s'}u_d^{s'}(p')\bar{u}_a^{s'}(p')\gamma^\mu_{ab}u_b^s(p)\bar{u}_c(p)\gamma^\nu_{cd}\tilde{A}_\mu(p'-p)\tilde{A}_\nu(p'-p)\\
&= \frac{e^2}{2}(\not{p}'+m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}\tilde{A}_\mu(\mathbf{p}'-p)\tilde{A}_\nu(\mathbf{p}'-\mathbf{p})\\
&= \frac{e^2}{2}\operatorname{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu\right]\tilde{A}_\mu(\mathbf{p}'-\mathbf{p})\tilde{A}_\nu(\mathbf{p}'-\mathbf{p}).
\end{aligned}$$

Note that

$$\begin{aligned}
\operatorname{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu\right] &= \operatorname{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\nu\right) + \operatorname{Tr}\left(\not{p}'\gamma^\mu m\gamma^\nu\right) + \operatorname{Tr}\left(m\gamma^\mu\not{p}\gamma^\nu\right) + \operatorname{Tr}(m\gamma^\mu m\gamma^\nu)\\
&= \operatorname{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\nu\right) + \operatorname{Tr}(m\gamma^\mu m\gamma^\nu),
\end{aligned} \tag{2}$$

since the other two terms have an odd number of γs [1, p. 133]. Then by Peskin & Schroeder (5.5),

$$\operatorname{Tr}(m\gamma^\mu m\gamma^\nu) = m^2\operatorname{Tr}(\gamma^\mu\gamma^\nu) = 4m^2g^{\mu\nu}. \tag{3}$$

Applying $\not{p} = \gamma^\mu p_\mu$ [1, p. 49],

$$\mathrm{Tr}\big(\not{p}'\gamma^\mu \not{p}\gamma^\nu\big) = p'_\rho p_\sigma \,\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) = 4p'_\rho p_\sigma(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}). \tag{4}$$

Then

$$\begin{aligned}
\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 &= 2e^2\left[p'_\rho p_\sigma(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) + m^2 g^{\mu\nu}\right]\tilde{A}_\mu(\mathbf{p}'-\mathbf{p})\tilde{A}_\nu(\mathbf{p}'-\mathbf{p}) \\
&= 2e^2\left[p'_\rho\tilde{A}^\rho p_\sigma\tilde{A}^\sigma - p'_\rho p^\rho\tilde{A}_\mu\tilde{A}^\nu + p'_\rho\tilde{A}^\nu p_\sigma\tilde{A}^\mu + m^2\tilde{A}_\mu\tilde{A}^\nu\right] \\
&= 2e^2\left\{2(p'\cdot\tilde{A})(p\cdot\tilde{A}) + [m^2 - (p'\cdot p)](\tilde{A}\cdot\tilde{A})\right\},
\end{aligned}$$

where we have omitted the arguments of $\tilde{A}$ for notational simplicity. We know that for Coulomb scattering A_μ has only a zeroth component, $A^0 = Ze/4\pi r$ [1, p. 130]. From the solution to 2(c) of Homework 4, we also know that $\tilde{A}_0(\mathbf{q}) = Ze/|\mathbf{q}|^2$. Then

$$\begin{aligned}
\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 &= 2e^2\left[2EE'\tilde{A}_0^2 + (m^2 - EE' + \mathbf{p}\cdot\mathbf{p}')\tilde{A}_0^2\right] \\
&= 2e^2\tilde{A}_0^2(2EE' + m^2 - E'E + \mathbf{p}\cdot\mathbf{p}') \\
&= 2\frac{Z^2e^4}{|\mathbf{p}'-\mathbf{p}|^4}(EE' + m^2 + \mathbf{p}\cdot\mathbf{p}').
\end{aligned}$$

From 2(c) of Homework 4, we know

$$\big|\mathbf{p}'-\mathbf{p}\big|^2 = \mathbf{p}'^2 - 2\big|\mathbf{p}'\big||\mathbf{p}|\cos\theta + \mathbf{p}^2,$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$. Using this and taking the limit that $p = p'$,

$$\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 = 2Z^2e^4\frac{EE' + m^2 + |\mathbf{p}||\mathbf{p}'|\cos\theta}{(\mathbf{p}'^2 - 2|\mathbf{p}'||\mathbf{p}|\cos\theta + \mathbf{p}^2)^2} = 2Z^2e^4\frac{E^2 + m^2 + \mathbf{p}^2\cos\theta}{4\mathbf{p}^4(1-\cos\theta)^2} = \frac{Z^2e^4}{8}\frac{E^2 + m^2 + \mathbf{p}^2\cos\theta}{\mathbf{p}^4\sin^4(\theta/2)}.$$

Noting that $E^2 = m^2 + p^2$,

$$\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 = \frac{Z^2e^4}{8}\frac{E^2 + E^2 - \mathbf{p}^2 + \mathbf{p}^2\cos\theta}{\mathbf{p}^4\sin^4(\theta/2)} = \frac{Z^2e^4}{8}\frac{2E^2 - \mathbf{p}^2(1-\cos\theta)}{\mathbf{p}^4\sin^4(\theta/2)} = \frac{Z^2e^4}{4}\frac{E^2 - \mathbf{p}^2\sin^2(\theta/2)}{\mathbf{p}^4\sin^4(\theta/2)}.$$

Now using $\mathbf{p}^2 = m^2\beta^2$, $\beta = |\mathbf{p}|/E$, $E/m = 1$, and $\alpha = e^2/4\pi$ [1, p. xxi],

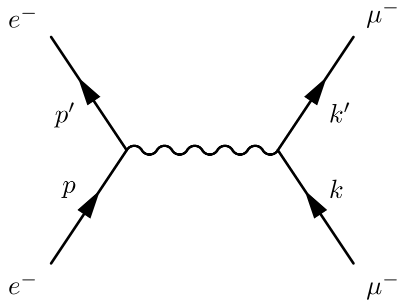
$$\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 = \frac{Z^2e^4E^2}{4}\frac{1-\beta^2\sin^2(\theta/2)}{m^2\beta^2\mathbf{p}^2\sin^4(\theta/2)} = \frac{Z^2e^4E^2}{4}\frac{1-\beta^2\sin^2(\theta/2)}{m^2\beta^2\mathbf{p}^2\sin^4(\theta/2)} = \frac{4\pi^2\alpha^2Z^2}{\beta^2\mathbf{p}^2\sin^4(\theta/2)}\left[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right]$$

Feeding this into Eq. (1) and setting $Z = 1$, we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 = \frac{\alpha^2}{4\beta^2\mathbf{p}^2\sin^4(\theta/2)}\left[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right]$$

as we wanted to show. □

The lowest-order Feynman diagram for electron-muon scattering is [1, p. 153]



$$= \frac{ie^2}{q^2}\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu \bar{u}(k).$$

The corresponding squared amplitude is given the equation above Peskin & Schroeder (5.60):

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\operatorname{Tr}\big[(\not{p}' + m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\big]\operatorname{Tr}\big[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu\big].$$

Referring to Eqs. (2)–(4),

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = 4\frac{e^4}{q^4}\left[p'_\rho p_\sigma(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) + m_e^2 g^{\mu\nu}\right]\left[k'^\alpha k^\beta(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta}) + m_\mu^2 g_{\mu\nu}\right].$$

In the muon rest frame, $\mu = \nu = 0$. Additionally, in sending $m_\mu \to \infty$ we only keep terms f $\mathcal{O}\big(m_\mu^2\big)$. Then

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &\approx 4\frac{e^4}{q^4}\left[p'_\rho p_\sigma(g^{\rho 0}g^{\sigma 0} - g^{\rho\sigma}g^{00} + g^{\rho 0}g^{0\sigma}) + m_e^2 g^{00}\right] m_\mu^2 g_{00} \\
&= 4\frac{e^4 m_\mu^2}{q^4}\left(p'^0 p^0 - p'^\sigma p_\sigma + p'^0 p^0 + m_e^2\right) \\
&= 4\frac{e^4 m_\mu^2}{q^4}\left(E'E - E'E + \mathbf{p}'\cdot\mathbf{p} + E'E + m_e^2\right) \\
&= 4\frac{e^4 m_\mu^2}{q^4}\left(E'E + \mathbf{p}'\cdot\mathbf{p} + m_e^2\right).
\end{aligned}$$

From Peskin & Schroeder (5.69) and (5.72),

$$q = t^2 = -2\mathbf{p}^2(1-\cos\theta).$$

Then, again taking $p' = p$ and proceeding similarly as before,

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= 4\frac{e^4 m_\mu^2}{[-2\mathbf{p}^2(1-\cos\theta)]^2}\left(E'E + \mathbf{p}'\cdot\mathbf{p} + m_e^2\right) \\
&= e^4 m_\mu^2 \frac{E^2 + \mathbf{p}^2\cos\theta + m_e^2}{\mathbf{p}^4(1-\cos\theta)^2} \\
&= e^4 m_\mu^2 \frac{2E^2 - \mathbf{p}^2(1-\cos\theta)}{\mathbf{p}^4(1-\cos\theta)^2} \\
&= \frac{e^4 m_\mu^2}{2}\frac{E^2 - \mathbf{p}^2\sin^2(\theta/2)}{\mathbf{p}^4\sin^4(\theta/2)} \\
&= \frac{e^4 m_\mu^2 E^2}{2}\frac{1 - \beta^2\sin^2(\theta/2)}{m_e^2\beta^2\mathbf{p}^2\sin^4(\theta/2)} \\
&= \frac{8\pi^2\alpha^2 m_\mu^2}{\beta^2\mathbf{p}^2\sin^4(\theta/2)}\left[1 - \beta^2\sin^2\left(\frac{\theta}{2}\right)\right].
\end{aligned}$$

Feeding this into Eq. (1) and sending $m_\mu \to 1$, we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2 = \frac{\alpha^2}{4\beta^2\mathbf{p}^2\sin^4(\theta/2)}\left[1-\beta^2\sin^2\left(\frac{\theta}{2}\right)\right]$$

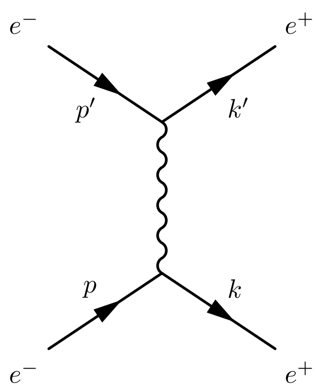
as we wanted to show. □

Problem 2. Bhabha scattering (Peskin & Schroeder 5.2) Compute the differential cross section $d\sigma/d(\cos\theta)$ for Bhabha scattering, $e^+e^- \to e^+e^-$. You may work in the limit $E_{\text{cm}} \gg m_e$, in which it is permissible to ignore the electron mass. There are two Feynman diagrams; these must be added in the invariant matrix element before squaring. Be sure that you have the correct relative sign between these diagrams. The intermediate steps are complicated, but the final result is quite simple. In particular, you may find it useful to introduce the Mandelstam variables s, t, and u. Note that, if we ignore the electron mass, $s+t+u=0$. You should be able to cast the differential cross section into the form

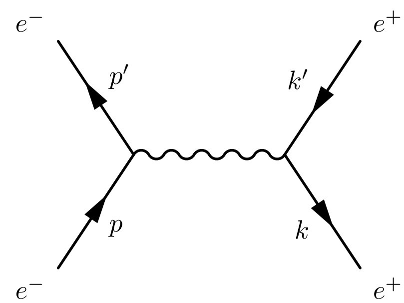
$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right].$$

Rewrite this formula in terms of $\cos\theta$ and graph it. What feature of the diagrams causes the differential cross section to diverge as $\theta \to 0$?

Solution. The two Feynman diagrams are the s- and t-channel diagrams [1, p. 157]



and



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The interaction Hamiltonian for QED is Peskin & Schroeder (4.129),

$$H_{\text{int}} = \int d^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu,$$

so the interaction is $\bar{\psi}A_\mu\gamma^\mu\psi$ [2, p. 226]. Now we can write the matrix element for the interaction:

$$\langle \mathbf{p},\mathbf{k}|(\bar{\psi}A_\mu\gamma^\mu\psi)_x(\bar{\psi}A_\mu\gamma^\mu\psi)_y|\mathbf{p}',\mathbf{k}'\rangle\,.$$

By Peskin & Schroeder (4.118),

$$|\mathbf{p},\mathbf{k}\rangle \sim a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{k}}\,|0\rangle\,, \qquad \langle\mathbf{p}',\mathbf{k}'| \sim \langle 0|\,a_{\mathbf{k}'}a_{\mathbf{p}'}.$$

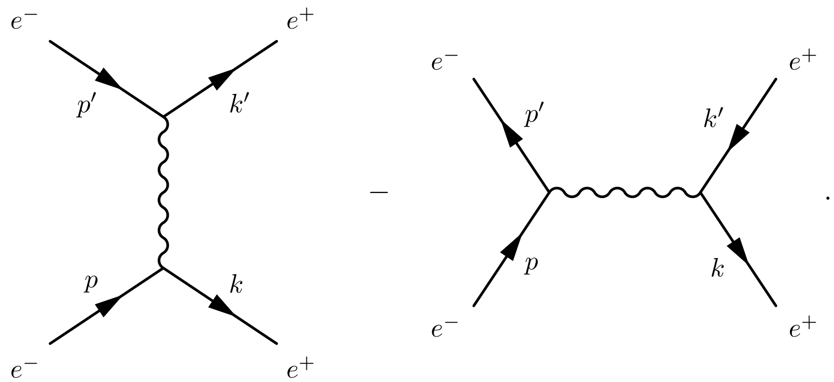
Then the contractions for the s-channel diagram are

$$\langle 0|a_{\mathbf{k}'}a_{\mathbf{p}'}(\bar{\psi}A_\mu\gamma^\mu\psi)_x(\bar{\psi}A_\mu\gamma^\mu\psi)_y a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{k}}|0\rangle\,.$$

For the t channel,

$$
\begin{aligned}
\langle 0| a_{\mathbf{k}'} a_{\mathbf{p}'} (\bar{\psi} A_\mu \gamma^\mu \psi)_x (\bar{\psi} A_\mu \gamma^\mu \psi)_y a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} |0\rangle &= -\langle 0| a_{\mathbf{k}'} a_{\mathbf{p}'} (A_\mu \gamma^\mu \psi \bar{\psi})_x (\bar{\psi} A_\mu \gamma^\mu \psi)_y a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} |0\rangle \\
&= \langle 0| a_{\mathbf{k}'} a_{\mathbf{p}'} (A_\mu \gamma^\mu \psi)_x \bar{\psi}_y \bar{\psi}_x (A_\mu \gamma^\mu \psi)_y a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} |0\rangle \\
&= -\langle 0| a_{\mathbf{k}'} a_{\mathbf{p}'} \bar{\psi}_y (A_\mu \gamma^\mu \psi \bar{\psi})_x (A_\mu \gamma^\mu \psi)_y a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} |0\rangle \,.
\end{aligned}
$$

We needed to swap the order of adjacent fields three times, which gives an overall minus sign [1, p. 118]. This means we subtract the t channel diagram:



The matrix element for the s channel is the same as Peskin & Schroeder (5.1),

$$\frac{ie^2}{s}\bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma_\mu v(k').$$

For the t channel, it is [1, p. 153]

$$\frac{ie^2}{t}\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma_\mu v(k')$$

where we have used the Mandelstam variables [cite]. Their difference is

$$i\mathcal{M} = ie^2\left[\frac{\bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma_\mu v(k')}{s} - \frac{\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma_\mu v(k')}{t}\right],$$

which means [1, p. 132]

$$-i\mathcal{M}^* = -ie^2\left[\frac{\bar{u}(p)\gamma^\mu v(k)\bar{v}(k')\gamma_\mu u(p')}{s} - \frac{\bar{u}(p)\gamma^\mu u(p')\bar{v}(k')\gamma_\mu v(k)}{t}\right].$$

Then

$$
\begin{aligned}
|\mathcal{M}|^2 = e^4 \Bigg[&\frac{\bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma_\mu v(k')\bar{u}(p)\gamma^\nu v(k)\bar{v}(k')\gamma_\nu u(p')}{s^2} - \frac{\bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma_\mu v(k')\bar{u}(p)\gamma^\nu u(p')\bar{v}(k')\gamma_\nu v(k)}{st} \\
&- \frac{\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma_\mu v(k')\bar{u}(p)\gamma^\nu v(k)\bar{v}(k')\gamma_\nu u(p')}{st} + \frac{\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma_\mu v(k')\bar{u}(p)\gamma^\nu u(p')\bar{v}(k')\gamma_\nu v(k)}{t^2}\Bigg].
\end{aligned}
$$

Note that [1, p. 132, 153]

$$
\begin{aligned}
\sum_{\text{spins}} \bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma_\mu v(k')\bar{u}(p)\gamma^\nu v(k)\bar{v}(k')\gamma_\nu u(p') &= \mathrm{Tr}\big(\not{k}\gamma^\mu \not{p}\gamma^\nu\big)\,\mathrm{Tr}\big(\not{p}'\gamma_\mu \not{k}'\gamma_\nu\big), \\
\sum_{\text{spins}} \bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma_\mu v(k')\bar{u}(p)\gamma^\nu u(p')\bar{v}(k')\gamma_\nu v(k) &= \mathrm{Tr}\big(\not{p}'\gamma^\mu \not{p}\gamma^\nu\big)\,\mathrm{Tr}\big(\not{k}\gamma_\mu \not{k}'\gamma_\nu\big),
\end{aligned}
$$

where we have taken $m \to 0$. Writing the other two using components,

$$\begin{aligned}\sum_{\text{spins}} \left[\bar{v}(k)\gamma^\mu u(p)\bar{u}(p')\gamma_\mu v(k')\bar{u}(p)\gamma^\nu u(p')\bar{v}(k')\gamma_\nu v(k) + \text{c.c.}\right] &= \sum_{s,s'}\sum_{r,r'}\left[\bar{v}^r_a\gamma^\mu_{ab}u^s_b\bar{u}^{s'}_c\gamma^{cd}_\mu v^{r'}_d\bar{u}^s_e\gamma^\nu_{ef}u^{s'}_f v^{r'}_g\gamma^{gh}_\nu v^r_h + \text{c.c.}\right] \\ &= \sum_{s,s'}\sum_{r,r'}\left[v^r_h\bar{v}^r_a\gamma^\mu_{ab}u^s_b\bar{u}^s_e\gamma^\nu_{ef}u^{s'}_f\bar{u}^{s'}_c\gamma^{cd}_\mu v^{r'}_d v^{r'}_g\gamma^{gh}_\nu + \text{c.c.}\right] \\ &= \not{k}_{ha}\gamma^\mu_{ab}\not{p}_{be}\gamma^\nu_{ef}\not{p}'_{fc}\gamma^{cd}_\mu\not{k}'_{dg}\gamma^{gh}_\nu + \text{c.c.} \\ &= \operatorname{Tr}\left(\not{k}\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma_\mu\not{k}'\gamma_\nu\right) + \text{c.c.}\end{aligned}$$

where we have used Peskin & Schroeder (5.3),

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m,$$

and $m = 0$. This gives us the matrix element

$$|\mathcal{M}|^2 = e^4\left(\frac{\operatorname{Tr}\left(\not{k}\gamma^\mu\not{p}\gamma^\nu\right)\operatorname{Tr}\left(\not{p}'\gamma_\mu\not{k}'\gamma_\nu\right)}{s^2} - \frac{\operatorname{Tr}\left(\not{k}\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma_\mu\not{k}'\gamma_\nu\right) + \text{c.c.}}{st} + \frac{\operatorname{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\nu\right)\operatorname{Tr}\left(\not{k}\gamma_\mu\not{k}'\gamma_\nu\right)}{t^2}\right).$$

We need to average over spins: $\sum_{\text{spins}}|\mathcal{M}|^2/4$ [1, p. 132]. From (5.70) and (5.71),

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}\frac{e^4}{s^2}\operatorname{Tr}\left(\not{k}\gamma^\mu\not{p}\gamma^\nu\right)\operatorname{Tr}\left(\not{p}'\gamma_\mu\not{k}'\gamma_\nu\right) &= \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right], \\ \frac{1}{4}\sum_{\text{spins}}\frac{e^4}{t^2}\operatorname{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\nu\right)\operatorname{Tr}\left(\not{k}\gamma_\mu\not{k}'\gamma_\nu\right) &= \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right].\end{aligned}$$

For the remaining terms, we adapt Peskin & Schroeder (5.69),

$$s = (p+k)^2 = (p'+k')^2, \qquad t = (p'-p)^2 = (k'-k)^2, \qquad u = (k'-p)^2 = (p'-k)^2.$$

For the s channel in the massless limit [1, p. 156],

$$t = -2p\cdot p' = -2k\cdot k', \qquad u = -2p\cdot k' = -2k\cdot p'.$$

Note that

$$\begin{aligned}\operatorname{Tr}\left(\not{k}\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma_\mu\not{k}'\gamma_\nu\right) &= k_\alpha p_\beta p'_\rho k'_\sigma \operatorname{Tr}\left(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\rho\gamma_\mu\gamma^\sigma\gamma_\nu\right) \\ &= -2k_\alpha p_\beta p'_\rho k'_\sigma \operatorname{Tr}\left(\gamma^\alpha\gamma^\rho\gamma^\nu\gamma^\beta\gamma^\sigma\gamma_\nu\right) \\ &= -8k_\alpha p_\beta p'_\rho k'_\sigma \operatorname{Tr}\left(\gamma^\alpha\gamma^\rho g^{\beta\sigma}\right) \\ &= -8k_\alpha p_\beta p'_\rho k'^\beta \operatorname{Tr}(\gamma^\alpha\gamma^\rho) \\ &= -32k_\alpha p_\beta p'_\rho k'^\beta g^{\alpha\rho} \\ &= -32k_\alpha p'^\alpha p_\beta k'^\beta \\ &= -32(k\cdot p')(p\cdot k') \\ &= -8u^2,\end{aligned}$$

where we have used Peskin & Schroeder (5.9),

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}, \qquad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu,$$

and (5.5), $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$. Then the matrix element is

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}^2| &= \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right] + \frac{4e^4}{st}u^2 + \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right] \\
&= 2e^4\left[\left(\frac{t}{s}\right)^2 + \frac{u^2}{s^2} + 2\frac{u^2}{st} + \left(\frac{s}{t}\right)^2 + \frac{u^2}{t^2}\right] \\
&= 2e^4\left[u^2\left(\frac{1}{s^2} + 2\frac{1}{st} + \frac{1}{t^2}\right) + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right] \\
&= 32\pi^2\alpha^2\left[u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]. \qquad (5)
\end{aligned}$$

By Peskin & Schroeder (4.85), the differential cross section for four particles of the same mass is

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2}.$$

In the center-of-mass frame, $E_{\text{cm}} = (p+k) = \sqrt{s}$. Feeding in Eq. (5),

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\text{cm}}^2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}^2| = \frac{\alpha^2}{2s}\left[u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right].$$

Noting that $d\Omega = d(\cos\theta)\,d\phi$ and $\int d\phi = 2\pi$, we integrate over ϕ to find

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right] \qquad (6)$$

as desired. □

From Peskin & Schroeder (5.72),

$$s = E_{\text{cm}}^2, \qquad t = -2|\mathbf{p}|^2(1-\cos\theta), \qquad u = -2|\mathbf{p}|^2(1+\cos\theta).$$

Feeding these into Eq. (6),

$$\begin{aligned}
\frac{d\sigma}{d(\cos\theta)} &= \frac{\pi\alpha^2}{E_{\text{cm}}^2}\left[4|\mathbf{p}|^4(1+\cos\theta)^2\left(\frac{1}{E_{\text{cm}}^2} - \frac{1}{2|\mathbf{p}|^2(1-\cos\theta)}\right)^2 + \left(-\frac{2|\mathbf{p}|^2(1-\cos\theta)}{E_{\text{cm}}^2}\right)^2 + \left(-\frac{E_{\text{cm}}^2}{2|\mathbf{p}|^2(1-\cos\theta)}\right)^2\right] \\
&= \frac{4\pi\alpha^2|\mathbf{p}|^4}{E_{\text{cm}}^2}\left[\left(\frac{(1+\cos\theta)^4}{E_{\text{cm}}^2} - \frac{(1+\cos\theta)^4}{2|\mathbf{p}|^2(1-\cos\theta)}\right)^2 + \left(\frac{1-\cos\theta}{E_{\text{cm}}^2}\right)^2 + \left(\frac{E_{\text{cm}}^2}{1-\cos\theta}\right)^2\right]. \qquad (7)
\end{aligned}$$

A graph is shown in Fig. 1. The differential cross section diverges as $\theta \to 0$ since the photon in the diagrams is nearly on shell; that is, $q^2 \to 0$ [1, p. 155]. When the mediating photon is massless (on shell), electrons that are separated by an infinite distance can still scatter off each other. Since they are so far apart, however, the angle of scattering approaches 0. Since any electron anywhere in the universe may be scattering off another given electron with $\theta \approx 0$, the scattering cross section must blow up there.

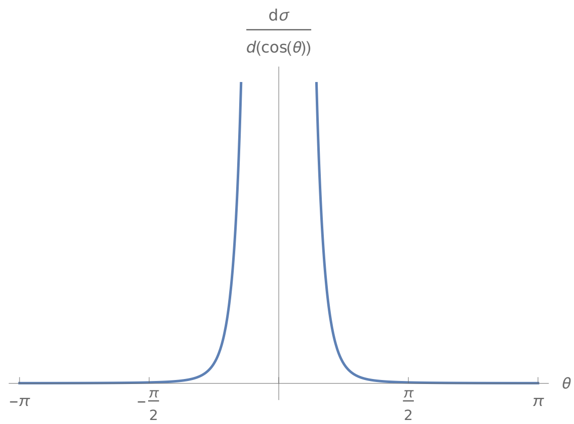


Figure 1: Graph of Eq. (7) showing divergence as $\theta \to 0$.

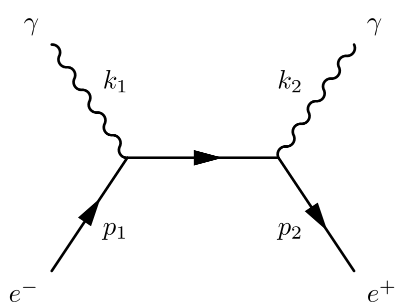
Problem 3. Positronium lifetimes (Peskin & Schroeder 5.4)

3(a) Compute the amplitude $\mathcal{M}$ for e^+e^- annihilation into 2 photons in the extreme nonrelativistic limit (i.e., keep only the term proportional to zero powers of the electron and positron 3-momentum). Use this result, together with our formalism for fermion-antifermion bound states, to compute the rate of annihilation of the $1S$ states of positronium into 2 photons. You should find that the spin-1 states of positronium do not annihilate into 2 photons, confirming the symmetry argument of Problem 3.8. For the spin-0 state of positronium, you should find a result proportional to the square of the $1S$ wavefunction at the origin. Inserting the value of this wavefunction from nonrelativistic quantum mechanics, you should find

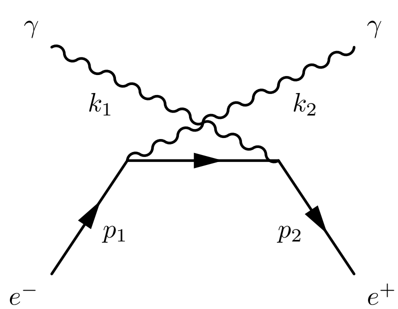
$$\frac{1}{\tau} = \Gamma = \frac{\alpha^5 m_e}{2} \approx 8.03 \times 10^9\,\mathrm{s}^{-1}.$$

A recent measurement gives $\Gamma = 7.994 \pm 0.011\,\mathrm{ns}^{-1}$; the 0.5% discrepancy is accounted for by radiative corrections.

Solution. The amplitude for e^+e^- annihilation into two photons can be obtained from the Feynman diagrams on p. 168 of Peskin & Schroeder:



$$= \bar{v}(p_2)(-ie\gamma^\mu)\epsilon^*_\mu(k_2)\frac{i(\not{p}_1 - \not{k}_1 + m)}{(p_1 - k_1)^2 - m^2}(-ie\gamma^\nu)\epsilon^*_\nu(k_1)u(p_1),$$



$$= \bar{v}(p_2)(-ie\gamma^\nu)\epsilon^*_\nu(k_1)\frac{i(\not{p}_1 - \not{k}_2 + m)}{(p_1 - k_2)^2 - m^2}(-ie\gamma^\mu)\epsilon^*_\mu(k_2)u(p_1).$$

Their sum is

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\bar{v}(p_2)\left(\frac{\gamma^\mu(\not{p}_1 - \not{k}_1 + m)\gamma^\nu}{(p_1-k_1)^2 - m^2} + \frac{\gamma^\nu(\not{p}_1 - \not{k}_2 + m)\gamma^\mu}{(p_1-k_2)^2 - m^2}\right)u(p_1)$$

where we have also referred to the expressions for Compton scattering, as well as the fermion and photon Feynman rules [1, pp. 118, 123, 158–159].

In the extreme nonrelativistic limit and in the center-of-mass frame, let

$$p_1 = (m, \mathbf{0}), \qquad p_2 = (m, \mathbf{0}), \qquad k_1 = (m, m\hat{\mathbf{z}}), \qquad k_2 = (m, -m\hat{\mathbf{z}}). \tag{8}$$

Here we have the two photons being emitted in opposite directions along the z axis. Note that $p_1 - k_1 = -m\hat{\mathbf{z}}$ and $p_1 - k_2 = m\hat{\mathbf{z}}$, so

$$(p_1-k_1)^2 = (p_1-k_1)_\mu(p_1-k_1)^\mu = -m^2 = (p_1-k_2)^2.$$

We also choose the polarization vectors [1, p. 124]

$$\epsilon(k_1) = (0, 1, \pm i, 0), \qquad \epsilon(k_2) = (0, -1, \mp i, 0). \tag{9}$$

Then the amplitude becomes

$$\begin{aligned}
i\mathcal{M} &\approx -ie^2\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\bar{v}(p_2)\left(\frac{\gamma^\mu(m\gamma^0 - m\gamma^0 + m\gamma^3 + m)\gamma^\nu}{-m^2 - m^2} + \frac{\gamma^\nu(m\gamma^0 - m\gamma^0 - m\gamma^3 + m)\gamma^\mu}{-m^2-m^2}\right)u(p_1) \\
&= \frac{ie^2}{2m}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\bar{v}(p_2)\left[\gamma^\mu(1+\gamma^3)\gamma^\nu + \gamma^\nu(1-\gamma^3)\gamma^\mu\right]u(p_1) \\
&= \frac{ie^2}{2m}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\bar{v}(p_2)\left[\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^3\gamma^\nu + \gamma^\nu\gamma^\mu - \gamma^\nu\gamma^3\gamma^\mu\right]u(p_1) \\
&= \frac{ie^2}{2m}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\bar{v}(p_2)\left[\gamma^\mu\gamma^3\gamma^\nu - \gamma^\nu\gamma^3\gamma^\mu\right]u(p_1)
\end{aligned}$$

where we have used $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$ [1, p. 41]. Here, $\delta^{ij} = 0$ since $\epsilon(k_1)$ and $\epsilon(k_2)$ both have zero time component:

$$\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\{\gamma^\mu, \gamma^\nu\} = -2\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\delta^{\mu\nu} = \epsilon^*_\mu(k_2)\epsilon^*_\mu(k_1) = 0$$

since from Eq. (9), $\epsilon(k_1)$ and $\epsilon(k_2)$ are orthogonal. In the nonrelativistic limit, we can use Peskin & Schroeder (5.35),

$$u(k) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}, \qquad v(k') = \sqrt{m}\begin{pmatrix}\xi' \\ -\xi'\end{pmatrix}.$$

So using (3.25), the amplitude can be written

$$\begin{aligned}
i\mathcal{M} &= \frac{ie^2}{2}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\begin{pmatrix}\xi'^\dagger & -\xi'^\dagger\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\left[\begin{pmatrix}0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0\end{pmatrix}\begin{pmatrix}0 & \bar{\sigma}^3 \\ \sigma^3 & 0\end{pmatrix}\begin{pmatrix}0 & \bar{\sigma}^\nu \\ \sigma^\nu & 0\end{pmatrix} - \begin{pmatrix}0 & \bar{\sigma}^\nu \\ \sigma^\nu & 0\end{pmatrix}\begin{pmatrix}0 & \bar{\sigma}^3 \\ \sigma^3 & 0\end{pmatrix}\begin{pmatrix}0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0\end{pmatrix}\right]\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&= \frac{ie^2}{2}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\begin{pmatrix}\xi'^\dagger & -\xi'^\dagger\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\left[\begin{pmatrix}0 & \bar{\sigma}^\mu\sigma^3\bar{\sigma}^\nu \\ \sigma^\mu\bar{\sigma}^3\sigma^\nu & 0\end{pmatrix} - \begin{pmatrix}0 & \bar{\sigma}^\nu\sigma^3\bar{\sigma}^\mu \\ \sigma^\nu\bar{\sigma}^3\sigma^\mu & 0\end{pmatrix}\right]\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&= \frac{ie^2}{2}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\begin{pmatrix}\xi'^\dagger & -\xi'^\dagger\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\left[\begin{pmatrix}0 & \sigma^\mu\sigma^3\sigma^\nu \\ -\sigma^\mu\sigma^3\sigma^\nu & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma^\nu\sigma^3\sigma^\mu \\ -\sigma^\nu\sigma^3\sigma^\mu & 0\end{pmatrix}\right]\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&= \frac{ie^2}{2}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\begin{pmatrix}\xi'^\dagger & -\xi'^\dagger\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu \\ -(\sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu) & 0\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&= \frac{ie^2}{2}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\begin{pmatrix}\xi'^\dagger & -\xi'^\dagger\end{pmatrix}\begin{pmatrix}-(\sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu) & 0 \\ 0 & \sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&= \frac{ie^2}{2}\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\begin{pmatrix}\xi'^\dagger & -\xi'^\dagger\end{pmatrix}\begin{pmatrix}-(\sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu) \\ \sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu\end{pmatrix}\xi \\
&= -ie^2\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\xi'^\dagger(\sigma^\mu\sigma^3\sigma^\nu - \sigma^\nu\sigma^3\sigma^\mu)\xi, \\
&= ie^2\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)\xi'^\dagger(\sigma^\nu\sigma^3\sigma^\mu - \sigma^\mu\sigma^3\sigma^\nu)\xi,
\end{aligned}$$

since $\bar{\sigma}^i = -\sigma^i$.

From Peskin & Schroeder (5.46),

$$i\mathcal{M} = \xi^\dagger \Gamma(\mathbf{k})\xi' \quad \Longrightarrow \quad \Gamma(\mathbf{k}) = ie^2\epsilon^*_\mu(k_2)\epsilon^*_\nu(k_1)(\sigma^\nu\sigma^3\sigma^\mu - \sigma^\mu\sigma^3\sigma^\nu). \tag{10}$$

Adapting their (5.50),

$$i\mathcal{M}(1S \to \gamma\gamma) = \sqrt{\frac{2}{M}}\int \frac{d^3k}{(2\pi)^3}\psi(\mathbf{k})\,\mathrm{Tr}\bigg(\frac{\hat{\mathbf{n}}^*\cdot\boldsymbol{\sigma}}{\sqrt{2}}\Gamma(\mathbf{k})\bigg) \tag{11}$$

which implies

$$-i\mathcal{M}^*(1S \to \gamma\gamma) = \sqrt{\frac{2}{M}}\int \frac{d^3k}{(2\pi)^3}\psi^*(\mathbf{k})\,\mathrm{Tr}\bigg(\frac{\hat{\mathbf{n}}^*\cdot\boldsymbol{\sigma}}{\sqrt{2}}\Gamma(\mathbf{k})\bigg)^*,$$

and so

$$|\mathcal{M}(1S \to \gamma\gamma)|^2 = \frac{1}{m}\int \frac{d^3k}{(2\pi)^3}|\psi(\mathbf{k})|^2\bigg|\mathrm{Tr}\bigg(\frac{\hat{\mathbf{n}}^*\cdot\boldsymbol{\sigma}}{\sqrt{2}}\Gamma(\mathbf{k})\bigg)\bigg|^2, \tag{12}$$

since $M \approx 2m$ [1, p. 149]. Here ψ is the wavefunction of ground state positronium, which is similar to the wavefunction of ground state hydrogen. For positronium, $Z \to m/2$ (the reduced mass) and $a_0 \to 1/\alpha$, where α is the fine-structure constant [1, p. xxi]. Thus [3, pp. 201, 454–455]

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{\pi}}\left(\frac{\alpha m}{2}\right)^{3/2} e^{-\alpha m r/2}. \tag{13}$$

Now we can write an expression for the decay rate using Peskin & Schroeder (5.55) and (4.83):

$$\Gamma = \frac{1}{2M}\int d\Pi_2\,|\mathcal{M}|^2 = \int d(\cos\theta)\,\frac{1}{16\pi}\frac{2|\mathbf{p}_1|}{E_{\mathrm{cm}}}|\mathcal{M}|^2 = \int d(\cos\theta)\,\frac{|\mathcal{M}|^2}{16\pi}, \tag{14}$$

where we have used $|\mathbf{p}_1| = m$ and $E_{\mathrm{cm}} = 2m$ from Eq. (8).

A spin-0 state can be obtained by Peskin & Schroeder (5.49),

$$\xi'\xi^\dagger \to \frac{\mathbf{1}}{\sqrt{2}}.$$

In the case $\epsilon(k_1) = (0,1,i,0)/\sqrt{2}$, $\epsilon(k_2) = (0,-1,i,0)/\sqrt{2}$, $\epsilon^*_\mu(k_1)\sigma^\mu = (\sigma^1 - i\sigma^2)/\sqrt{2}$ and $\epsilon^*_\mu(k_2)\sigma^\mu = -(\sigma^1 + i\sigma^2)/\sqrt{2}$. Then, by Eq. (10),

$$\begin{aligned}
\mathrm{Tr}\bigg(\frac{\mathbf{1}}{\sqrt{2}}\Gamma(\mathbf{k})\bigg) &= \mathrm{Tr}\bigg(-\frac{\mathbf{1}}{2\sqrt{2}}ie^2[(\sigma^1 - i\sigma^2)\sigma^3(\sigma^1 + i\sigma^2) - (\sigma^1 + i\sigma^2)\sigma^3(\sigma^1 - i\sigma^2)]\bigg) \\
&= \mathrm{Tr}\bigg(-\frac{\mathbf{1}}{2\sqrt{2}}ie^2[\sigma^1\sigma^3\sigma^1 + i\sigma^1\sigma^3\sigma^2 - i\sigma^2\sigma^3\sigma^1 + \sigma^2\sigma^3\sigma^2 - \sigma^1\sigma^3\sigma^1 + i\sigma^1\sigma^3\sigma^2 - i\sigma^2\sigma^3\sigma^1 - \sigma^2\sigma^3\sigma^2]\bigg) \\
&= \mathrm{Tr}\bigg(\frac{\mathbf{1}}{\sqrt{2}}e^2[\sigma^1\sigma^3\sigma^2 - \sigma^2\sigma^3\sigma^1]\bigg) \\
&= \mathrm{Tr}\bigg(\frac{\mathbf{1}}{\sqrt{2}}e^2[\sigma^1\sigma^3\sigma^2 + \sigma^1\sigma^3\sigma^2]\bigg) \\
&= \mathrm{Tr}\Big(\sqrt{2}e^2\mathbf{1}\sigma^1\sigma^3\sigma^2\Big) \\
&= \sqrt{2}e^2 2i\epsilon^{132} \\
&= -i2\sqrt{2}e^2,
\end{aligned}$$

where we have used $\mathrm{Tr}\big(\sigma^i\sigma^j\sigma^k\big) = 2i\epsilon^{ijk}$ [4].

In the case $\epsilon(k_1) = (0, 1, -i, 0)$, $\epsilon(k_2) = (0, -1, -i, 0)$, $\epsilon^*_\mu(k_1)\sigma^\mu = (\sigma^1 + i\sigma^2)/\sqrt{2}$ and $\epsilon^*_\mu(k_2)\sigma^\mu = -(\sigma^1 - i\sigma^2)/\sqrt{2}$. Then, once again by Eq. (10),

$$\begin{aligned}
\mathrm{Tr}\left(\frac{\mathbf{1}}{\sqrt{2}}\Gamma(\mathbf{k})\right) &= \mathrm{Tr}\left(-\frac{\mathbf{1}}{2\sqrt{2}}ie^2[(\sigma^1 + i\sigma^2)\sigma^3(\sigma^1 - i\sigma^2) - (\sigma^1 - i\sigma^2)\sigma^3(\sigma^1 + i\sigma^2)]\right) \\
&= \mathrm{Tr}\left(-\frac{\mathbf{1}}{2\sqrt{2}}ie^2[\sigma^1\sigma^3\sigma^1 - i\sigma^1\sigma^3\sigma^2 + i\sigma^2\sigma^3\sigma^1 + \sigma^2\sigma^3\sigma^2 - \sigma^1\sigma^3\sigma^1 - i\sigma^1\sigma^3\sigma^2 + i\sigma^2\sigma^3\sigma^1 - \sigma^2\sigma^3\sigma^2]\right) \\
&= \mathrm{Tr}\left(-\frac{\mathbf{1}}{\sqrt{2}}e^2[\sigma^1\sigma^3\sigma^2 - \sigma^2\sigma^3\sigma^1]\right) \\
&= i2\sqrt{2}e^2.
\end{aligned}$$

In the case $\epsilon(k_1) = (0, 1, i, 0)$, $\epsilon(k_2) = (0, -1, -i, 0)$, $\epsilon^*_\mu(k_1)\sigma^\mu = (\sigma^1 - i\sigma^2)/\sqrt{2}$ and $\epsilon^*_\mu(k_2)\sigma^\mu = -(\sigma^1 - i\sigma^2)/\sqrt{2}$. Then

$$\mathrm{Tr}\left(\frac{\mathbf{1}}{\sqrt{2}}\Gamma(\mathbf{k})\right) = \mathrm{Tr}\left(-\frac{\mathbf{1}}{2\sqrt{2}}ie^2[(\sigma^1 - i\sigma^2)\sigma^3(\sigma^1 - i\sigma^2) - (\sigma^1 - i\sigma^2)\sigma^3(\sigma^1 - i\sigma^2)]\right) = 0.$$

In the case $\epsilon(k_1) = (0, 1, -i, 0)$, $\epsilon(k_2) = (0, -1, i, 0)$, $\epsilon^*_\mu(k_1)\sigma^\mu = (\sigma^1 + i\sigma^2)/\sqrt{2}$ and $\epsilon^*_\mu(k_2)\sigma^\mu = -(\sigma^1 + i\sigma^2)/\sqrt{2}$. Then

$$\mathrm{Tr}\left(\frac{\mathbf{1}}{\sqrt{2}}\Gamma(\mathbf{k})\right) = \mathrm{Tr}\left(-\frac{\mathbf{1}}{2\sqrt{2}}ie^2[(\sigma^1 + i\sigma^2)\sigma^3(\sigma^1 + i\sigma^2) - (\sigma^1 + i\sigma^2)\sigma^3(\sigma^1 + i\sigma^2)]\right) = 0.$$

To summarize, in the spin-0 case we have

$$\mathrm{Tr}\left(\frac{\mathbf{1}}{\sqrt{2}}\Gamma(\mathbf{k})\right) = \begin{cases} -i2\sqrt{2}e^2 & \text{if } \epsilon(k_1) = \dfrac{(0,1,i,0)}{\sqrt{2}},\ \epsilon(k_2) = \dfrac{(0,-1,i,0)}{\sqrt{2}}; \\ i2\sqrt{2}e^2 & \text{if } \epsilon(k_1) = \dfrac{(0,1,-i,0)}{\sqrt{2}},\ \epsilon(k_2) = \dfrac{(0,-1,-i,0)}{\sqrt{2}}; \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

From Peskin & Schroeder (5.48), we can obtain a spin-1 state via

$$\xi'\xi^\dagger \to \frac{\hat{\mathbf{n}}^* \cdot \boldsymbol{\sigma}}{\sqrt{2}},$$

where [1, p. 150]

$$\hat{\mathbf{n}} \in \left\{\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}},\ \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}},\ \hat{\mathbf{z}}\right\}.$$

For $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $\xi'\xi^\dagger \to \sigma^3/\sqrt{2}$. Then for $\epsilon(k_1) = (0, 1, i, 0)/\sqrt{2}$, $\epsilon(k_2) = (0, -1, i, 0)/\sqrt{2}$,

$$\mathrm{Tr}\left(\frac{\sigma^3}{\sqrt{2}}\Gamma(\mathbf{k})\right) = \mathrm{Tr}\left(\sqrt{2}e^2\sigma^3\sigma^1\sigma^3\sigma^2\right) = 0,$$

where we have used $\mathrm{Tr}(\sigma^i\sigma^j\sigma^k\sigma^l) = 2(\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{kl} + \delta^{il}\delta^{jk})$ [4]. The $\epsilon(k_1) = (0, 1, -i, 0)/\sqrt{2}$, $\epsilon(k_2) = (0, -1, -i, 0)/\sqrt{2}$ case is the same with an overall minus sign. For the other two cases, the argument of the trace is identically zero.

For $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})/\sqrt{2}$ and $\hat{\mathbf{n}} = (\hat{\mathbf{x}} - i\hat{\mathbf{y}})/\sqrt{2}$, the argument is identical. We get two trace terms, each being either identically zero or taking the form $\mathrm{Tr}(\sigma^i\sigma^j\sigma^k\sigma^l) = 2(\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{kl} + \delta^{il}\delta^{jk})$ with each term vanishing. Thus we have

$$\mathrm{Tr}\left(\frac{\hat{\mathbf{n}}^* \cdot \boldsymbol{\sigma}}{\sqrt{2}}\Gamma(\mathbf{k})\right) = 0$$

for the spin-1 states. This means, in the spin-1 case, Eq. (11) is $i\mathcal{M}(1S \to \gamma\gamma) = 0$ trivially. Feeding this into Eq. (14) yields

$$\Gamma = \int d(\cos\theta)\, \frac{|0|^2}{16\pi} = 0,$$

meaning that the spin-1 positronium states do not annihilate into two photons, as we wanted to show. □

Returning to the spin-0 states, we average over the polarization states. Note from Eqs. (12) and (15) that

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}(1S \to \gamma\gamma)|^2 &= \frac{1}{2m}\int \frac{d^3k}{(2\pi)^3}|\psi(\mathbf{k})|^2 \sum_{\text{spins}} \left|\mathrm{Tr}\left(\frac{\hat{\mathbf{n}}^* \cdot \boldsymbol{\sigma}}{\sqrt{2}}\Gamma(\mathbf{k})\right)\right|^2 \\
&= \frac{1}{2m}\int \frac{d^3k}{(2\pi)^3}|\psi(\mathbf{k})|^2 \left(\left|-i2\sqrt{2}e^2\right|^2 + \left|i2\sqrt{2}e^2\right|^2\right) \\
&= \frac{2}{2m}\int \frac{d^3k}{(2\pi)^3}|\psi(\mathbf{k})|^2 (2\sqrt{2}e^2)^2 \\
&= \frac{8e^4}{m}|\psi(0)|^2,
\end{aligned}$$

where the integral gives $\psi(0)$ since the amplitude is independent of the momenta [1, p. 149]. Then

$$\sum_{\text{spins}} |\mathcal{M}(1S \to \gamma\gamma)|^2 = 8\frac{16\pi^2\alpha^2}{m}\left[\frac{1}{\sqrt{\pi}}\left(\frac{\alpha m}{2}\right)^{3/2}\right]^2 = \frac{16^2\pi^2\alpha^2}{m}\frac{1}{\pi}\frac{\alpha^3 m^3}{8} = 16\pi\alpha^5 m^2,$$

where we have used Eq. (13). Then, feeding this into Eq. (14) and inserting a factor of 1/2 to account for the two identical photons,

$$\Gamma = \frac{1}{2}\int d(\cos\theta)\, \frac{\sum_{\text{spins}} |\mathcal{M}(1S \to \gamma\gamma)|^2/2}{16\pi} = \frac{1}{2}\int d(\cos\theta)\, \frac{16\pi\alpha^5 m^2}{16\pi} = \frac{\alpha^5 m^2}{2}$$

as we wanted to show. □

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