

1. **Problem.** Suppose we have a mechanical system with  $n$  degrees of freedom. Let  $q_1(t), q_2(t), \dots, q_n(t)$  be its generalized coordinates. Now consider a time-dependent coordinate transformation

$$Q_i = Q_i(t, q_1, q_2, \dots, q_n) \quad i = 1, 2, \dots, n.$$

Show that if  $q_i(t)$  solves a system of Euler-Lagrange equations involving a Lagrangian  $L(t, q_i, \dot{q}_i)$ , then  $Q_i(t)$  solves the Euler-Lagrange equations involving  $L(t, Q_i, \dot{Q}_i)$  provided the time-dependent coordinate transformation fulfills some minimal standard of good behavior. Specify this “minimal standard of good behavior.”

**Solution.** Suppose that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0; \quad (1)$$

that is,  $q_i(t)$  solve a system of Euler-Lagrange equations. We want to show that

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_i} = 0. \quad (2)$$

Beginning with the first term of (1), we can use the chain rule for  $L(t, Q_i, \dot{Q}_i)$  to write

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial q_i}, \quad (3)$$

provided there exists an inverse transformation

$$q_i = q_i(t, Q_1, Q_2, \dots, Q_n) \quad i = 1, 2, \dots, n \quad (4)$$

that allows us to write  $L(t, q_i, \dot{q}_i)$  in terms of  $t, Q_i$ , and  $\dot{Q}_i$ . This is only possible if there is a one-to-one correspondence between  $q_i(t)$  and  $Q_i(t)$ , which is the “minimal standard of good behavior” for the transformation. We will assume the transformation is so well behaved.

Again using the chain rule for  $Q_j = Q_j(t, q_1, q_2, \dots, q_n)$ , note that

$$\dot{Q}_j = \frac{\partial Q_j}{\partial t} + \frac{\partial Q_j}{\partial q_i} \dot{q}_i \quad (5)$$

so (3) becomes

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \left( \frac{\partial^2 Q_j}{\partial q_i \partial t} + \frac{\partial^2 Q_j}{\partial q_i \partial q_k} \dot{q}_k \right). \quad (6)$$

Applying the chain rule now to the second term of (1), we have

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial Q_j}{\partial q_i} \quad (7)$$

where the right-hand side comes from (5). Then, using the product rule to take the time derivative,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_j} \right) \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \left( \frac{d}{dt} \frac{\partial Q_j}{\partial q_i} \right). \quad (8)$$

For the second term of (8), the chain rule for  $Q_j = Q_j(t, q_1, q_2, \dots, q_n)$  gives

$$\frac{d}{dt} \frac{\partial Q_j}{\partial q_i} = \frac{\partial^2 Q_j}{\partial t \partial q_i} + \frac{\partial^2 Q_j}{\partial q_i \partial q_k} \dot{q}_k. \quad (9)$$

Substituting (9) into (8), we have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_j} \right) \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \left( \frac{\partial^2 Q_j}{\partial t \partial q_i} + \frac{\partial^2 Q_j}{\partial q_k \partial q_i} \dot{q}_k \right) \quad (10)$$

where the terms on the far right appeared in (6). Making this substitution,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_j} \right) \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial q_i}, \quad (11)$$

and rearranging,

$$\frac{\partial Q_j}{\partial q_i} \left( \frac{\partial L}{\partial Q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_j} \right) = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}, \quad (12)$$

Finally, substituting the original assumption (1), we find

$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}_j} = 0 \quad (13)$$

which is what we sought to prove.  $\square$

## 2. Problem. Look at the Lagrangian

$$L = e^{\sigma t} \left( \frac{m \dot{q}^2}{2} - \frac{k q^2}{2} \right)$$

for one-dimensional motion.

(a) Write down the associated Euler-Lagrange ODE.

(b) Now perform a point transformation

$$Q = e^{\sigma t/2} q$$

where the new position coordinate  $Q$  is a function of  $t$  and  $q$ . What is the equation of motion for  $Q(t)$ ? Are there conserved quantities?

## Solution.

(a) Beginning from the general expression for the Euler-Lagrange equations,

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = -e^{\sigma t} k q - \frac{d}{dt} (e^{\sigma t} m \dot{q}) = -m e^{\sigma t} \left( \ddot{q} + \sigma \dot{q} + \frac{k}{m} q \right) \quad (14)$$

so the ODE is

$$0 = \ddot{q} + \sigma \dot{q} + \frac{k}{m} q. \quad (15)$$

(b) It is possible to invert this transformation and write  $q = q(t, Q)$ . Explicitly, this is

$$q = Qe^{-\sigma t/2} \quad (16)$$

so

$$\dot{q} = e^{-\sigma t/2} \left( \dot{Q} - \frac{\sigma}{2} Q \right). \quad (17)$$

Rewriting the Lagrangian such that  $L = L(t, Q, \dot{Q})$  results in

$$L = e^{\sigma t} \left( \frac{m}{2} \left( e^{-\sigma t/2} \left( \dot{Q} - \frac{\sigma}{2} Q \right) \right)^2 - \frac{k}{2} \left( Qe^{-\sigma t/2} \right)^2 \right) \quad (18)$$

$$= \frac{m}{2} \left( \dot{Q} - \frac{\sigma}{2} Q \right)^2 - \frac{k}{2} Q^2 \quad (19)$$

$$= \frac{m}{2} \left( \dot{Q}^2 - \sigma \dot{Q} Q + \left( \frac{\sigma^2}{4} - \frac{k}{m} \right) Q^2 \right). \quad (20)$$

Then the Euler-Lagrange equations are given by

$$0 = \frac{\partial L}{\partial Q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{Q}} = \frac{m}{2} \left( -\sigma \dot{Q} + 2 \left( \frac{\sigma^2}{4} - \frac{k}{m} \right) Q - \frac{d}{dt} (2\dot{Q} - \sigma Q) \right) \quad (21)$$

which simplifies to

$$0 = \ddot{Q} + \left( \frac{k}{m} - \frac{\sigma^2}{4} \right) Q. \quad (22)$$

The solutions to (22) have the form

$$Q(t) = \begin{cases} A_1 \sin \sqrt{\frac{k}{m} - \frac{\sigma^2}{4}} t + A_2 \cos \sqrt{\frac{k}{m} - \frac{\sigma^2}{4}} t & \text{if } \frac{k}{m} > \frac{\sigma^2}{4}, \\ B_1 + B_2 t & \text{if } \frac{k}{m} = \frac{\sigma^2}{4}, \\ C_1 \exp \left\{ -\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}} t \right\} + C_2 \exp \left\{ \sqrt{\frac{k}{m} - \frac{\sigma^2}{4}} t \right\} & \text{if } \frac{k}{m} < \frac{\sigma^2}{4}, \end{cases} \quad (23)$$

where  $A_i, B_i, C_i$  are real constants.

The Lagrangian in (20) does not explicitly depend on time. Thus, the total energy  $H$  of the system is conserved. Explicitly,

$$H = \dot{Q} \frac{\partial L}{\partial \dot{Q}} - L = \frac{m}{2} \left( \dot{Q}^2 - \left( \frac{\sigma^2}{4} - \frac{k}{m} \right) Q^2 \right) \quad (24)$$

is a conserved quantity.

3. **Problem.** Let  $U(\alpha \mathbf{r}_1, \dots, \alpha \mathbf{r}_N)$  be a potential for  $N$  particles that satisfies the relation

$$U(\alpha \mathbf{r}_1, \dots, \alpha \mathbf{r}_N) = \alpha^k U(\mathbf{r}_1, \dots, \mathbf{r}_N).$$

The factor  $\alpha$  can be any nonzero real number. The exponent  $k$  is an integer.

- (a) Show that the equations of motion associated with such a potential remain unchanged under a dilation of the distance scale if the time scale is also dilated by some other factor  $\beta$ . Find  $\beta$  as a function of  $\alpha$  and  $k$ .
- (b) If  $k = 2$ , the forces correspond to a system of harmonic oscillators couples to each other. Show that the result in part (a) implies the frequencies of such a system are independent of the oscillation amplitude.
- (c) If  $k = -1$ , we have an inverse square force law, such as that which arises in mutual gravitational attraction. Show that the result in part (a) implies Kepler's third law: the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

**Solution.** The Lagrangian  $L = L(t, \mathbf{r}_i, \dot{\mathbf{r}}_i)$  for the system of  $N$  particles is

$$L = T - U = \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i - U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (25)$$

where  $m_i$  is the mass of the particle located at  $\mathbf{r}_i$ . The Euler-Lagrange equations for this Lagrangian are

$$\frac{\partial L}{\partial \mathbf{r}_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} = 0 \implies \frac{\partial U}{\partial \mathbf{r}_i} + m_i \ddot{\mathbf{r}}_i = 0. \quad (26)$$

Define the time scale transformation

$$T = \beta t, \quad (27)$$

and define the coordinate transformation

$$\mathbf{R}_i = \mathbf{r}_i(T) = \alpha \mathbf{r}_i \quad (28)$$

for all  $N$  particles. Using these coordinates, the Lagrangian  $L = L(T, \mathbf{R}_i, \dot{\mathbf{R}}_i)$  is

$$L = \frac{1}{2} m_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i - U(\mathbf{R}_1, \dots, \mathbf{R}_N) \quad (29)$$

and the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \mathbf{R}_i} - \frac{d}{dT} \frac{\partial L}{\partial \dot{\mathbf{R}}_i} = 0 \implies \frac{\partial U}{\partial \mathbf{R}_i} + m_i \ddot{\mathbf{R}}_i = 0. \quad (30)$$

- (a) The equations of motion associated to the Lagrangians (25) and (29) are identical if the Euler-Lagrange equations in (26) and (30) are identical. We will now show that this is the case.

The transformation  $\mathbf{R}_i = \alpha \mathbf{r}_i$  is invertible, so  $\mathbf{r}_i = \mathbf{R}_i/\alpha$ . Likewise,  $t = T/\beta$ . By the chain rule,

$$\frac{d}{dT} = \frac{d}{dt} \frac{dt}{dT} = \frac{1}{\beta} \frac{d}{dt} \quad (31)$$

so

$$\dot{\mathbf{R}} = \alpha \frac{d\mathbf{r}_i}{dT} = \frac{\alpha}{\beta} \dot{\mathbf{r}}_i \quad (32)$$

and, likewise,

$$\ddot{\mathbf{R}} = \frac{\alpha}{\beta^2} \ddot{\mathbf{r}}_i. \quad (33)$$

From the given relationship for  $U$ , note that

$$U(\mathbf{R}_1, \dots, \mathbf{R}_N) = \alpha^k U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (34)$$

and again using the chain rule,

$$\frac{\partial}{\partial \mathbf{R}_i} = \frac{\partial}{\partial \mathbf{r}_i} \frac{d\mathbf{r}_i}{d\mathbf{R}_i} = \frac{1}{\alpha} \frac{\partial}{\partial \mathbf{r}_i}. \quad (35)$$

Making use of (33), (34), and (35), we can rewrite (30) in terms of the original coordinates:

$$0 = \frac{\alpha^k}{\alpha} \frac{\partial U}{\partial \mathbf{r}_i} + m_i \frac{\alpha}{\beta^2} \ddot{\mathbf{r}}_i \implies \alpha^k \beta^2 \frac{\partial U}{\partial \mathbf{r}_i} + m_i \ddot{\mathbf{r}}_i = 0 \quad (36)$$

which is equivalent to (26) so long as

$$\alpha^k \beta^2 = 1 \implies \beta = \alpha^{-k/2}. \quad (37)$$

(The solution where  $\beta < 1$  is spurious unless we are interested in running time backwards.)

**4. Problem.** A particle in three-dimensional space is confined in a central potential

$$U(r) = -U_0 \left( \frac{r_0}{r} \right)^n.$$

Here  $r = |\mathbf{r}|$  where  $\mathbf{r}(t)$  is the location of the particle at time  $t$ ,  $U_0$  is a characteristic energy scale and  $r_0$  is a characteristic length scale. Show that the particle motion is confined to a two-dimensional orbital plane. For what values of  $n$  are circular orbits stable?