Problem 1. Supersymmetry (Peskin & Schroeder 3.5) It is possible to write field theories with continuous symmetries linking fermions and bosons; such transformations are called *supersymmetries*.

1(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, written in the form

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^* F.$$

Here F is an auxiliary complex scalar field whose field equation is $F = 0$. Show that this Lagrangian is invariant (up to a total divergence) under the infinitesimal transformation

$$\delta\phi = -i\epsilon^T\sigma^2\chi, \qquad \delta\chi = \epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^*, \qquad \delta F = -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi, \tag{1}$$

where the parameter ϵ_a is a 2-component spinor of Grassmann numbers.

Solution. Using the supplied transformations and dropping terms of $\mathcal{O}(\delta^2)$, we have

$$\begin{aligned}
\mathcal{L} &\to \partial_\mu(\phi^* + \delta\phi^*)\partial^\mu(\phi+\delta\phi) + (\chi^\dagger + \delta\chi^\dagger)i\bar{\sigma}\cdot\partial(\chi+\delta\chi) + (F^*\delta F^*)(F+\delta F)\\
&\approx \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \chi^\dagger\bar{\sigma}\cdot\partial\delta\chi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F + F^*\delta F + \delta F^* F\\
&= \mathcal{L} + \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi + \chi^\dagger\bar{\sigma}\cdot\partial\delta\chi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F + \delta F^* F.
\end{aligned} \tag{2}$$

Note that Grassmann numbers satisfy $\alpha\beta = -\beta\alpha$ and $(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*$ for any α, β [1, p. 73]. Then

$$\begin{aligned}
\delta\phi^* &= i(\epsilon^T\sigma^2\chi)^* = i\epsilon^\dagger\sigma^{2*}\chi^* = -i\epsilon^\dagger\sigma^2\chi^* = i\chi^\dagger\sigma^2\epsilon^*,\\
\delta\chi^\dagger &= (\epsilon F)^\dagger + (\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*)^\dagger = F^*\epsilon^\dagger + \epsilon^T\sigma^{2\dagger}\partial_\mu\phi^*\sigma^{\mu\dagger} = F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu,\\
\delta F^* &= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi = i(\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi)^* = -i\epsilon^T\bar{\sigma}^{\mu*}\partial_\mu\chi^* = i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon,
\end{aligned}$$

where we have transposed as needed to obtain $\chi^\dagger$ or χ^*. So the $\mathcal{O}(\delta)$ terms in Eq. (2) are

$$\begin{aligned}
\partial_\mu\phi^*\partial^\mu\delta\phi &= -i\partial_\mu\phi^*\partial^\mu(\epsilon^T\sigma^2\chi), & \partial_\mu\delta\phi^*\partial^\mu\phi &= i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi,\\
\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\epsilon F + \sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*), & \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi &= i(F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)\bar{\sigma}^\nu\partial_\nu\chi,\\
F^*\delta F &= -iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi, & \delta F^* F &= i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon F.
\end{aligned} \tag{3}$$

Adding the fourth and fifth terms above,

$$\delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F = iF^*\epsilon^\dagger\bar{\sigma}^\nu\partial_\nu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi - iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi = i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi.$$

Adding this to the first term of Eq. (3),

$$\partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F = -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi.$$

Note that

$$\sigma^\mu\bar{\sigma}^\nu = \frac{\sigma^\mu\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\mu + \sigma^\mu\bar{\sigma}^\nu - \bar{\sigma}^\nu\sigma^\mu}{2} = \frac{\{\sigma^\mu,\bar{\sigma}^\nu\}}{2} + \frac{[\sigma^\mu,\bar{\sigma}^\nu]}{2} = g^{\mu\nu} + \frac{[\sigma^\mu,\bar{\sigma}^\nu]}{2}$$

where we have used $\{\sigma^\mu,\bar{\sigma}^\nu\} = 2g^{\mu\nu}$ since $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ [2, p. 165]. Then

$$\begin{aligned}
\partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F &= -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^* g^{\mu\nu}\partial_\nu\chi + \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\\
&= -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\partial^\mu\chi + \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\\
&= \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\\
&= \partial_\mu\left(\frac{i}{2}\epsilon^T\sigma^2\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\right).
\end{aligned} \tag{4}$$

Adding the third and sixth terms of Eq. (3),

$$\begin{aligned}\chi^\dagger i\bar\sigma^\mu\partial_\mu\delta\chi + \delta F^* F &= i\chi^\dagger\bar\sigma^\mu\partial_\mu(\epsilon F) + i\chi^\dagger\bar\sigma^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) + i\partial_\mu\chi^\dagger\bar\sigma^{\mu\dagger}\epsilon F\\ &= i\chi^\dagger\bar\sigma^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) + i\bar\sigma^\mu\partial_\mu(\chi^\dagger\epsilon F)\\ &= i\chi^\dagger\bar\sigma^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) + \partial_\mu(i\bar\sigma^\mu\chi^\dagger\epsilon F)\end{aligned}$$

Adding this to the second term of Eq. (3),

$$\chi^\dagger i\bar\sigma^\mu\partial_\mu\delta\chi + \delta F^* F + \partial_\mu\delta\phi^*\partial^\mu\phi = i\chi^\dagger\bar\sigma^\mu\sigma^\nu\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) + i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu(i\bar\sigma^\mu\chi^\dagger\epsilon F).$$

Similar to before,

$$\bar\sigma^\mu\sigma^\nu = \frac{\bar\sigma^\mu\sigma^\nu + \sigma^\nu\bar\sigma^\mu + \bar\sigma^\mu\sigma^\nu - \sigma^\nu\bar\sigma^\mu}{2} = \frac{\{\bar\sigma^\mu,\sigma^\nu\}}{2} + \frac{[\bar\sigma^\mu,\sigma^\nu]}{2} = g^{\mu\nu} + \frac{[\bar\sigma^\mu,\sigma^\nu]}{2},$$

so

$$\chi^\dagger i\bar\sigma^\mu\partial_\mu\delta\chi + \delta F^* F + \partial_\mu\delta\phi^*\partial^\mu\phi = i\chi^\dagger g^{\mu\nu}\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) + \frac{i}{2}\chi^\dagger[\bar\sigma^\mu,\sigma^\nu]\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) + i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu(i\bar\sigma^\mu\chi^\dagger\epsilon F).$$

Note that

$$\chi^\dagger[\bar\sigma^\mu,\sigma^\nu]\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) = \chi^\dagger[\bar\sigma^\nu,\sigma^\mu]\partial_\nu(\partial_\mu\phi\sigma^2\epsilon^*) = -\chi^\dagger[\bar\sigma^\mu,\sigma^\nu]\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) = 0,$$

where we have used $[\bar\sigma^\mu,\sigma^\nu] = -[\bar\sigma^\nu,\sigma^\mu]$, since $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ [2, p. 165]. Then

$$\begin{aligned}\chi^\dagger i\bar\sigma^\mu\partial_\mu\delta\chi + \delta F^* F + \partial_\mu\delta\phi^*\partial^\mu\phi &= i\chi^\dagger\partial_\mu(\partial^\mu\phi\sigma^2\epsilon^*) + i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu(i\bar\sigma^\mu\chi^\dagger\epsilon F)\\ &= \partial_\mu(i\partial^\mu\chi^\dagger\sigma^2\epsilon^*\phi + i\bar\sigma^\mu\chi^\dagger\epsilon F). \qquad (5)\end{aligned}$$

Finally, substituting Eqs. (4) and (5) into Eq. (2),

$$\mathcal{L} \to \mathcal{L} + \partial_\mu\left(\frac{i}{2}\epsilon^T\sigma^2\phi^*\partial_\nu\chi[\sigma^\mu,\bar\sigma^\nu] + i\partial^\mu\chi^\dagger\sigma^2\epsilon^*\phi + i\bar\sigma^\mu\chi^\dagger\epsilon F\right),$$

which is the same up to a total divergence. □

1(b) Show that the term

$$\Delta\mathcal{L} = \left(m\phi F + \frac{i}{2}m\chi^T\sigma^2\chi\right) + (\text{complex conjugate})$$

is also left invariant by the transformation given in 1(a). Eliminate F from the complete Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its field equation, and show that the fermion and boson fields ϕ and χ are given the same mass.

Solution. Taking the complex conjugate,

$$\Delta\mathcal{L} = m\phi F + m\phi^* F^* + \frac{i}{2}m\chi^T\sigma^2\chi - \frac{i}{2}m\chi^\dagger\sigma^2\chi^*.$$

Transforming $\Delta\mathcal{L}$ and dropping terms of $\mathcal{O}(\delta^2)$ yields

$$\begin{aligned}\Delta\mathcal{L} \to\ & m(\phi+\delta\phi)(F+\delta F) + m(\phi^*+\delta\phi^*)(F^*+\delta F^*) + \frac{i}{2}m(\chi^T+\delta\chi^T)\sigma^2(\chi+\delta\chi)\\ & - \frac{i}{2}m(\chi^\dagger+\delta\chi^\dagger)\sigma^2(\chi^*+\delta\chi^*)\\ \approx\ & m\phi F + m\phi\delta F + m\delta\phi F + m\phi^* F^* + m\phi^*\delta F^* + m\delta\phi^* F^* + \frac{i}{2}m\chi^T\sigma^2\chi + \frac{i}{2}m\chi^T\sigma^2\delta\chi + \frac{i}{2}m\delta\chi^T\sigma^2\chi\\ & - \frac{i}{2}m\chi^\dagger\sigma^2\chi^* - \frac{i}{2}m\chi^\dagger\sigma^2\delta\chi^* - \frac{i}{2}m\chi^\dagger\sigma^2\delta\chi^*\\ =\ & \Delta\mathcal{L} + m\phi\delta F + m\delta\phi F + m\phi^*\delta F^* + m\delta\phi^* F^* + \frac{i}{2}m\chi^T\sigma^2\delta\chi + \frac{i}{2}m\delta\chi^T\sigma^2\chi - \frac{i}{2}m\chi^\dagger\sigma^2\delta\chi^* - \frac{i}{2}m\delta\chi^\dagger\sigma^2\chi^*.\end{aligned}$$

Applying Eqs. (1) and (3) to each term, we have

$$\begin{aligned} m\phi\delta F &= -im\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi, & m\delta\phi F &= -im\epsilon^T\sigma^2\chi F, \\ m\phi^*\delta F^* &= im\phi^*\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon, & m\delta\phi^* F^* &= im\chi^\dagger\sigma^2\epsilon^* F^*, \\ \frac{i}{2}m\chi^T\sigma^2\delta\chi &= \frac{i}{2}m\chi^T\sigma^2(\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*), & \frac{i}{2}m\delta\chi^T\sigma^2\chi &= \frac{i}{2}m(F\epsilon^T - \epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T})\sigma^2\chi, \\ -\frac{i}{2}m\delta\chi^\dagger\sigma^2\chi^* &= -\frac{i}{2}m(F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)\sigma^2\chi^*, & -\frac{i}{2}m\chi^\dagger\sigma^2\delta\chi^* &= -\frac{i}{2}m\chi^\dagger\sigma^2(\epsilon^* F^* - \sigma^{\mu *}\partial_\mu\phi^*\sigma^2\epsilon). \end{aligned} \tag{6}$$

where we have used

$$\delta\chi^T = F\epsilon^T - \epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T}, \qquad \delta\chi^* = \epsilon^* F^* - \sigma^{\mu *}\partial_\mu\phi^*\sigma^2\epsilon.$$

Since $\chi^T\sigma^2\epsilon = \epsilon^T\sigma^2\chi$, adding the second, fifth, and sixth terms of Eq. (6) gives us

$$\begin{aligned} m\delta\phi F + \frac{i}{2}m\chi^T\sigma^2\delta\chi + \frac{i}{2}m\delta\chi^T\sigma^2\chi &= -im\epsilon^T\sigma^2\chi F + \frac{i}{2}m\chi^T\sigma^2(\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*) + \frac{i}{2}m(F\epsilon^T - \epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T})\sigma^2\chi \\ &= \frac{i}{2}m\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* - \frac{i}{2}m\epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T}\sigma^2\chi \\ &= im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*. \end{aligned}$$

Similarly $\chi^\dagger\sigma^2\epsilon^* = \epsilon^\dagger\sigma^2\chi^*$, so adding the fourth, seventh, and eighth terms of Eq. (6),

$$\begin{aligned} m\delta\phi^* F^* - \frac{i}{2}m\delta\chi^\dagger\sigma^2\chi^* - \frac{i}{2}m\chi^\dagger\sigma^2\delta\chi^* &= im\chi^\dagger\sigma^2\epsilon^* F^* - \frac{i}{2}m(F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)\sigma^2\chi^* - \frac{i}{2}m\chi^\dagger\sigma^2(\epsilon^* F^* - \sigma^{\mu *}\partial_\mu\phi^*\sigma^2\epsilon) \\ &= -\frac{i}{2}m\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\sigma^2\chi^* + \frac{i}{2}m\chi^\dagger\sigma^2\sigma^{\mu *}\partial_\mu\phi^*\sigma^2\epsilon \\ &= -im\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\sigma^2\chi^*. \end{aligned}$$

Then adding all the remaining terms gives us

$$\begin{aligned} \Delta\mathcal{L} \to \Delta\mathcal{L} &+ im\phi^*\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon - im\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi + im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* - im\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\sigma^2\chi^* \\ = \Delta\mathcal{L} &+ im\phi^*\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon - im\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi + im\chi^T\bar{\sigma}^{\mu *}\partial_\mu\phi\epsilon^* - im\epsilon^T\partial_\mu\phi^*\bar{\sigma}^{\mu *}\chi^* \\ = \Delta\mathcal{L} &+ im\bar{\sigma}^\mu\phi^*\partial_\mu\chi^\dagger\epsilon - im\bar{\sigma}^\mu\phi\epsilon^\dagger\partial_\mu\chi - im\bar{\sigma}^\mu\partial_\mu\phi\epsilon^\dagger\chi + im\bar{\sigma}^\mu\partial_\mu\phi^*\chi^\dagger\epsilon \end{aligned}$$

where we have used $\sigma^2\sigma^\mu\sigma^2 = \bar{\sigma}^{\mu *}$ from Homework 2's 3(a).

fucking complex conjugates

total divergence

$$\mathcal{L} + \Delta\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F + m\phi F + m\phi^* F^* + \frac{i}{2}m\chi^T\sigma^2\chi - \frac{i}{2}m\chi^\dagger\sigma^2\chi^*.$$

The Euler-Lagrange equations are $F = -m\phi^*$ $F^* = -m\phi$

$$\begin{aligned} \mathcal{L} + \Delta\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + m^2|\phi|^2 - m^2|\phi|^2 - m^2|\phi|^2 + \frac{i}{2}m\chi^T\sigma^2\chi - \frac{i}{2}m\chi^\dagger\sigma^2\chi^* \\ &= \left(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\right) + \left[i\chi^\dagger\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]. \end{aligned}$$

The first set of parentheses is the Klein-Gordon Lagrangian describing a particle of mass m (cite), and the second set of brackets is the Majorana Lagrangian for a particle of mass m (cite). CONCLUDE

1(c) It is possible to write supersymmetric nonlinear field equations by adding cubic and higher-order terms to the Lagrangian. Show that the following rather general field theory, containing the field (ϕ_i, χ_i), $i = 1, \dots, n$, is supersymmetric:

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma} \cdot \partial \chi_i + F_i^* F_i + \left(F_i \frac{\partial W[\phi]}{\partial \phi_i} + \frac{i}{2} \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right),$$

where $W[\phi]$ is an arbitrary function of the ϕ_i, called the *superpotential*. For the simple case $n = 1$ and $W = g\phi^3/3$, write out the field equations for ϕ and χ (after elimination of F).

Solution. We already know that the terms outside of the brackets are supersymmetric because that part is equivalent to the Lagrangian from 1(a) (but for the indices; at any rate, it will transform the same way). Then we can say

$$\mathcal{L} \to \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma} \cdot \partial \chi_i + F_i^* F_i + \left((F_i + \delta F_i) \frac{\partial W[\phi + \delta\phi]}{\partial \phi_i} + \frac{i}{2} \frac{\partial^2 W[\phi + \delta\phi]}{\partial \phi_i \partial \phi_j} (\chi_i^T + \delta\chi_i^T) \sigma^2 (\chi_j + \delta\chi_j) + \text{c.c.} \right)$$

$$\begin{aligned}
&\approx \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar{\sigma} \cdot \partial \chi_i + F_i^* F_i + \Bigg[(F_i + \delta F_i) \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j \\
&\qquad + \frac{i}{2} \left(\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} + \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \right) (\chi_i^T \sigma^2 \chi_j + \chi_i^T \sigma^2 \delta\chi_j + \delta\chi_i^T \sigma^2 \chi_j) + \text{c.c.} \Bigg] \\
&= \mathcal{L} + \left[\delta F_i \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j + \frac{i}{2} \left(\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} (\chi_i^T \sigma^2 \delta\chi_j + \delta\chi_i^T \sigma^2 \chi_j) + \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \chi_i^T \sigma^2 \chi_j \right) + \text{c.c.} \right] \\
&= \mathcal{L} + \left(\delta F_i \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \delta\chi_j + \frac{i}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right),
\end{aligned}$$

where we have used i j flip thing. Feeding in Eq. 1,

$$\begin{aligned}
\mathcal{L} \to \mathcal{L} + \Bigg(&- i\epsilon^\dagger \bar{\sigma} \cdot \partial \chi_i \frac{\partial W[\phi]}{\partial \phi_i} - iF_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \epsilon^T \sigma^2 \chi_j + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 (\epsilon F_j + \sigma \cdot \partial \phi_j \sigma^2 \epsilon^*) \\
&+ \frac{1}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j + \text{c.c.} \Bigg) \\
= \mathcal{L} + &\left(-i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \phi_j \epsilon^* + \frac{1}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right) \\
= \mathcal{L} + &\left(-i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \bar{\sigma}^{\mu *} \partial_\mu \phi_j \epsilon^* + \frac{1}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right)
\end{aligned}$$

since

$$-iF_j \frac{\partial^2 W[\phi]}{\partial \phi_j \partial \phi_i} \chi_i^T \sigma^2 \epsilon = -iF_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \epsilon^T \sigma^2 \chi_j$$

Then for the last term, swapping i and j makes it 0

$$\frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j = \frac{\partial^3 W[\phi]}{\partial \phi_j \partial \phi_i \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_j^T \sigma^2 \chi_i = -\frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j = 0.$$

Going back now

$$\mathcal{L} \to \mathcal{L} + \left(-i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \bar{\sigma}^{\mu *} \partial_\mu \phi_j \epsilon^* + \text{c.c.} \right)$$

total divergence

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] J. J. Sakurai, "Modern Quantum Mechanics". Addison-Wesley Publishing Company, revised edition, 1994.