1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1}$$

1.1 Verify the following:

a. $i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle$,

b. $i\hbar\partial_t \langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle\langle x|\Psi(t)\rangle$,

c. $i\hbar\partial_t \langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle = -\frac{\hbar^2}{2m}\left[\langle\Phi(t)|x\rangle\,\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right]$,

d. $\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle = \frac{\hbar}{i}\left[\langle\Phi(t)|x\rangle\,\partial_x\langle x|\Psi(t)\rangle - (\partial_x\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right]$

e. $\frac{\hbar}{i}\partial_x\left[\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle$

Solution.

a. We will begin with the Schrödinger equation,

$$i\hbar\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle. \tag{2}$$

Since the Hamiltonian given by (1) is time independent, the system evolves in time under the time-evolution operator $U(t) = \exp(-iHt/\hbar)$. Denote the eigenkets of H by $|E_i\rangle$ and the corresponding eigenvalues by E_i. Assuming $V(x)$ is a real-valued function, H is Hermitian, and so $|E_i\rangle$ form a complete orthonormal basis. Then we may rewrite $|\Psi(t)\rangle$ in terms of $U(t)$ and expand it in $|E_i\rangle$:

$$|\Psi(t)\rangle = U(t)|\Psi\rangle = e^{iHt/\hbar}\sum_i |E_i\rangle\langle E_i|\Psi\rangle = \sum_i e^{iE_it/\hbar}|E_i\rangle\langle E_i|\Psi\rangle. \tag{3}$$

Substituting (3) into (2) and evaluating the time derivative,

$$-\sum_i E_i e^{iE_it/\hbar}|E_i\rangle\langle E_i|\Psi\rangle = H\sum_i e^{iE_it/\hbar}|E_i\rangle\langle E_i|\Psi\rangle. \tag{4}$$

Taking the adjoint of (4) yields

$$-\sum_i E_i\langle\Psi|E_i\rangle\langle E_i|\,e^{-iE_it/\hbar} = H\sum_i\langle\Psi|E_i\rangle\langle E_i|\,e^{-iE_it/\hbar}. \tag{5}$$

From the adjoint of (3), note that

$$i\hbar\partial_t\langle\Psi(t)| = i\hbar\partial_t\sum_i\langle\Psi|E_i\rangle\langle E_i|\,e^{-iE_it/\hbar} = \sum_i E_i\langle\Psi|E_i\rangle\langle E_i|\,e^{-iE_it/\hbar}. \tag{6}$$

Making these substitutions into (5), and multiplying by $|x\rangle$ on the right, we have

$$-i\hbar\partial_t\langle\Psi(t)| = H\langle\Psi(t)| \implies i\hbar\partial_t\langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle \tag{7}$$

as we sought to prove. □

b. Rewriting what was proven in (a) with $\Psi \mapsto \Phi$ and then multiplying by $\Psi(x,t)$ on the right,

$$i\hbar\partial_t \langle \Phi(t)|x\rangle = -\langle \Phi(t)|H|x\rangle \tag{8}$$
$$i\hbar(\partial_t \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle = -\langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle . \tag{9}$$

Multiplying (2) by $\langle \Phi(t)|x\rangle \langle x|$ on the left,

$$\langle \Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle . \tag{10}$$

Adding (10) and (9) yields

$$\langle \Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle + i\hbar(\partial_t \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle \tag{11}$$
$$i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle , \tag{12}$$

where in going to (12) we have used the product rule of differentiation on the left-hand side. (12) is what we sought to prove. □

c. Using (1), note that:

$$\langle x|H|\Psi(t)\rangle = \langle x| \left[\frac{p^2}{2m} + V(x)\right] |\Psi(t)\rangle = \frac{1}{2m}\langle x|p^2|\Psi(t)\rangle +\ \langle x|V(x)|\Psi(t)\rangle \tag{13}$$
$$= -\frac{\hbar^2}{2m}\partial_x^2 \langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle , \tag{14}$$

where in going to (14) we have (twice) used the fact that

$$\langle x|p|\Psi(x)\rangle = -i\hbar\partial_x \langle x|\Psi(t)\rangle . \tag{15}$$

Similarly, note that

$$\langle \Phi(t)|H|x\rangle = -\frac{\hbar^2}{2m}\partial_x^2 \langle \Phi(t)|x\rangle + V(x)\langle \Phi(t)|x\rangle \tag{16}$$

where we have (twice) used the adjoint of (15) with $\Psi \mapsto \Phi$,

$$\langle \Phi(t)|p|x\rangle = i\hbar\partial_x \langle \Phi(t)|x\rangle . \tag{17}$$

This follows because p is Hermitian. Making the substitutions (14) and (16) into what was proven in (b),

$$\begin{aligned} &i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle \\ &= \langle \Phi(t)|x\rangle \left[-\frac{\hbar^2}{2m}\partial_x^2 \langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle\right] - \left[-\frac{\hbar^2}{2m}\partial_x^2 \langle \Phi(t)|x\rangle + V(x)\langle \Phi(t)|x\rangle\right]\langle x|\Psi(t)\rangle && (18)\\ &= -\frac{\hbar^2}{2m}\left[\langle \Phi(t)|x\rangle\, \partial_x^2 \langle \Phi(t)|x\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right] + [V(x) - V(x)]\langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle && (19)\\ &= -\frac{\hbar^2}{2m}\left[\langle \Phi(t)|x\rangle\, \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right] , && (20) \end{aligned}$$

as we sought to prove. □

d. Applying (15) and (17) to the left-hand side of (d),

$$\langle \Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle +\ \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle (-i\hbar\partial_x \langle x|\Psi(t)\rangle) + (i\hbar\partial_x \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle \tag{21}$$
$$= \frac{\hbar}{i}\left[\langle \Phi(t)|x\rangle\, \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right] \tag{22}$$

as we sought to prove. □

e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle) = (\partial_x\langle\Phi(t)|x\rangle)\,\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\,\partial_x\langle x|p|\Psi(t)\rangle \tag{23}$$

Multiplying through by $\hbar/i$,

$$\begin{aligned}\frac{\hbar}{i}\partial_x(\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle) &= (-i\hbar\partial_x\langle\Phi(t)|x\rangle)\,\langle x|p|\Psi(t)\rangle - \langle\Phi(t)|x\rangle\, i\hbar\partial_x\langle x|p|\Psi(t)\rangle & (24)\\ &= -\langle\Phi(t)|p|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle\,, & (25)\end{aligned}$$

where in going to (25) we have used (15) and (17). Using a similar procedure for the second term of the left-hand side of (e),

$$\begin{aligned}\frac{\hbar}{i}\partial_x(\ \langle\Phi(t)|p|x\rangle\,\langle x|\Psi(t)\rangle) &= (-i\hbar\partial_x\langle\Phi(t)|p|x\rangle)\,\langle x|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle\, i\hbar\partial_x\langle x|\Psi(t)\rangle & (26)\\ &= -\langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\ \langle x|p|\Psi(t)\rangle\,. & (27)\end{aligned}$$

Adding the results of (25) and (27),

$$\begin{aligned}&\frac{\hbar}{i}\partial_x\left[\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\,\langle x|\Psi(t)\rangle\right]\\ &\quad= \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\ \langle x|p|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle & (28)\\ &\quad= \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle & (29)\end{aligned}$$

as we sought to prove. □

1.2 Define

$$\begin{aligned}\rho(x,t) &= \langle\Phi(t)|x\rangle\,\langle x|\Psi(t)\rangle\,, & (30)\\ J_x(x,t) &= \frac{1}{2m}\left[\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\,\langle x|\Psi(t)\rangle\right]. & (31)\end{aligned}$$

Show that $\rho(x,t) + \partial_x J_x(x,t) = 0$.

Solution. From (30),

$$\partial_t\rho(x,t) = \partial_t(\langle\Phi(t)|x\rangle\,\langle x|\Psi(t)\rangle), \tag{32}$$

and from what was proven in 1(c),

$$\begin{aligned}\partial_t(\langle\Phi(t)|x\rangle\,\langle x|\Psi(t)\rangle) &= -\frac{1}{i\hbar}\left[\langle\Phi(t)|x\rangle\,\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\,\langle x|\Psi(t)\rangle\right] & (33)\\ &= -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle\right], & (34)\end{aligned}$$

where we have applied (15) and (17) in going to (34). Equating (32) and (34),

$$\partial_t\rho(x,t) = -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle\right]. \tag{35}$$

Beginning from (31),

$$\begin{aligned}\partial_x J_x(x,t) &= \frac{1}{2m}\partial_x\left[\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\,\langle x|\Psi(t)\rangle\right] & (36)\\ &= \frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle\right], & (37)\end{aligned}$$

where in going to (37) we have used what was proven in 1(e). Summing (35) and (37), we have

$$\partial_t\rho(x,t) + \partial_x J_x(x,t) = \left(-\frac{1}{2m}\frac{i}{\hbar} + \frac{1}{2m}\frac{i}{\hbar}\right)\left[\langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle\right] = 0 \tag{38}$$

as we sought to prove. This is is the continuity equation for probability. □

2 Problem 4

Consider a particle moving in three dimensions. Consider an operator

$$U(\phi) = \exp\left(-\frac{i}{\hbar}L_3\phi\right), \qquad L_3 = L_z = XP_y - YP_x, \tag{39}$$

where X, Y and P_x, P_y are position and momentum operators, respectively. Define new operators

$$X(\phi) = U^\dagger(\phi)XU(\phi), \qquad Y(\phi) = U^\dagger(\phi)YU(\phi). \tag{40}$$

Note that $X(0) = Y(0) = 0$.

2.1 Derive the equation

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)[L_3, X]U(\phi) = -Y(\phi), \tag{41}$$

and a similar equation for $\mathrm{d}Y(\phi)/\mathrm{d}\phi$.

Solution. Using the definition of $X(\phi)$ in (40) and applying the product rule of differentiation,

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(U^\dagger XU\right) = \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi}XU + U^\dagger\frac{\mathrm{d}}{\mathrm{d}\phi} \tag{42}$$

$$= \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi}XU + U^\dagger\frac{\mathrm{d}X}{\mathrm{d}\phi}U + U^\dagger X\frac{\mathrm{d}U}{\mathrm{d}\phi}. \tag{43}$$

We know immediately that $\mathrm{d}X/\mathrm{d}\phi = 0$ because ϕ is not a parameter of the position operator X. From the definition of $U(\phi)$ in (39), we know that $[L_3, U(\phi)] = 0$. Thus

$$\frac{\mathrm{d}U}{\mathrm{d}\phi} = -\frac{i}{\hbar}L_3U = -\frac{i}{\hbar}L_3\exp\left(-\frac{i}{\hbar}L_3\phi\right) = -\frac{i}{\hbar}UL_3, \tag{44}$$

and likewise

$$U^\dagger = \exp\left(\frac{i}{\hbar}L_3\phi\right) \implies \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi} = \frac{i}{\hbar}L_3\exp\left(\frac{i}{\hbar}L_3\phi\right) = \frac{i}{\hbar}L_3U^\dagger = \frac{i}{\hbar}U^\dagger L_3 \tag{45}$$

because $[L_3, U^\dagger] = 0$ as well. Then (43) becomes

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger L_3XU - \frac{i}{\hbar}U^\dagger XL_3U = \frac{i}{\hbar}U^\dagger(L_3X - XL_3)U = \frac{i}{\hbar}U^\dagger(\phi)[L_3, X]U(\phi), \tag{46}$$

which is the first equality of what we wanted to show in (41).

From the definition of L_3 in (39),

$$[L_3, X] = L_3X - XL_3 = (XP_y - YP_x)X - X(XP_y - YP_x) \tag{47}$$

$$= XP_yX - YP_xX - XXP_y + XYP_x = YXP_x - YP_xX \tag{48}$$

$$= Y[X, P_x] = i\hbar Y \tag{49}$$

where in (48) we have used $[X, P_y] = [X, Y] = 0$, and in (49) we have used $[X, P_x] = i\hbar$. Making the substitution (49) into (46), we have

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)(i\hbar Y)U(\phi) = -U^\dagger(\phi)YU(\phi) = -Y(\phi), \tag{50}$$

where the last equality is from the definition of $Y(\phi)$ in (40). This is the second equality of what we wanted to show in (41), which completes the proof.

For $\mathrm{d}Y(\phi)/\mathrm{d}\phi$, we can make the substitutions $X(\phi) \mapsto Y(\phi), X \mapsto Y$ in (43) and (46) to obtain

$$\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)[L_3, Y]U(\phi). \tag{51}$$

Then making similar use of commutators $[Y, P_x] = [X, Y] = 0$ and $[Y, P_y] = i\hbar$ as for (48) and (49),

$$\begin{aligned}
[L_3, Y] &= L_3 Y - Y L_3 = (XP_y - YP_x)Y - Y(XP_y - YP_x) &(52)\\
&= XP_yY - YP_xY - YXP_y + YYP_x = XP_yY - XYP_y &(53)\\
&= X[P_y, Y] = -X[Y, P_y] = -i\hbar X. &(54)
\end{aligned}$$

Substituting (54) into (51),

$$\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)(-i\hbar X)U(\phi) = X(\phi), \tag{55}$$

and so we have derived

$$\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)[L_3, Y]U(\phi) = X(\phi). \tag{56}$$

and (41) as desired. □

2.2 Define $X_\pm(\phi) = X(\phi) \pm iY(\phi)$. From the results of previous parts, show $X_+(\phi) = e^{i\phi}X_+$ where $X_+ = X_+(0)$. Derive the similar expression for $X_-(\phi)$.

Solution. Differentiating $X_\pm(\phi)$ and making use of (41) and (56),

$$\begin{aligned}
\frac{\mathrm{d}X_\pm(\phi)}{\mathrm{d}\phi} &= \frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} \pm i\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = -Y(\phi) \pm iX(\phi) = \pm i\left[X(\phi) \pm iY(\phi)\right] &(57)\\
&= \pm iX_\pm(\phi). &(58)
\end{aligned}$$

The differential equation (58) has solutions given by exponential functions of $\pm i\phi$. We will make the ansatz

$$X_\pm(\phi) = e^{\pm i\phi}C_\pm, \tag{59}$$

where $C_\pm$ is an operator "constant" in ϕ (that is, independent of it) and is fixed by an initial condition. Inspecting (59), clearly $X_\pm(0) = C_\pm$ where it is defined $X_\pm(0) \equiv X_\pm$. All that remains is to show that (59) obeys the relation (58), as follows:

$$\frac{\mathrm{d}X_\pm(\phi)}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(e^{\pm i\phi}\right)C_\pm = \pm ie^{\pm i\phi}C_\pm = \pm iX_\pm(\phi). \tag{60}$$

Thus, we have derived

$$X_+(\phi) = e^{i\phi}X_+, \qquad X_-(\phi) = e^{-i\phi}X_- \tag{61}$$

as desired. □

2.3 Show that $[L_3, X_+] = \hbar X_+$. Derive the similar expression for $[L_3, X_-]$.

Solution. Firstly, note that

$$X_\pm = X_\pm(0) = X(0) \pm iY(0) = U^\dagger(0)XU(0) \pm iU^\dagger(0)YU(0) = X \pm iY \tag{62}$$

because $U(0) = U^\dagger(0) = I$. Also applying the definition of L_3 in (39), we have

$$\begin{aligned}
[L_3, X_\pm] &= [XP_y - YP_x, X \pm iY] = (XP_y - YP_x)(X \pm iY) - (X \pm iY)(XP_y - YP_x) &(63)\\
&= XP_yX \pm iXP_yY - YP_xX \mp iYP_xY - XXP_y + XYP_x \mp iYXP_y \pm iYYP_x &(64)\\
&= \pm iXP_yY - YP_xX + XYP_x \mp iYXP_y = \pm iX[P_y, Y] + Y[X, P_x] &(65)\\
&= \pm\hbar X + i\hbar Y = \pm\hbar[X \pm iY] = \pm\hbar X_\pm. &(66)
\end{aligned}$$

Thus, we have shown

$$[L_3, X_+] = \hbar X_+, \qquad [L_3, X_-] = -\hbar X_- \tag{67}$$

as desired. □

3 Problem 1

Consider a particle with coordinate $x \in (-\infty, \infty)$, and momentum $p \in (-\infty, \infty)$, along with corresponding operators X and P. We have

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}. \tag{68}$$

3.1 Consider $\langle p|X|\Psi\rangle$. Express it in terms of $\langle p|\Psi\rangle$.

Solution. In the momentum space, the action of X is given by

$$\langle p|X|\Psi\rangle = i\hbar\partial_p \langle p|\Psi\rangle. \tag{69}$$

3.2 Define a state $|\Psi'\rangle$ from $|\Psi\rangle$ by $\langle p - p_0|\Psi\rangle = \langle p|\Psi'\rangle$. Construct the unitary operator $V(p_0)$ such that $|\Psi'\rangle = V(p_0)|\Psi\rangle$.

Solution. For an infinitesimal p_0,

$$V^\dagger(p_0)|p\rangle = |p - p_0\rangle = e^{-p_0\partial_p}|p\rangle \tag{70}$$

and since $\partial_p^\dagger = -\partial_p$ in the momentum basis,

$$V(p_0) = e^{p_0\partial_p} = e^{ip_0X/\hbar} \tag{71}$$

because $X = -i\hbar\partial_p$ when acting on the $|p\rangle$ basis, as given by the adjoint of (69). Then

$$\langle p|V(p_0)|\Psi\rangle = \langle p - p_0|\Psi\rangle = \langle p|\Psi'\rangle \tag{72}$$

as desired.

$V(p_0)$ has the following properties that were also required of $U(a)$:

1. In the limit $p_0 \to 0$, $V(p_0) \to I$:

$$\lim_{p_0\to 0} V(p_0) = \lim_{p_0\to 0} e^{ip_0X/\hbar} = e^0 = I. \tag{73}$$

2. Successive applications are equivalent to a single application:

$$V(p_1)V(p_2) = e^{ip_1X/\hbar}e^{ip_2X/\hbar} = e^{i(p_1+p_2)X/\hbar} = V(p_1+p_2). \tag{74}$$

3. Unitarity:

$$V(p_0)V^\dagger(p_0) = e^{ip_0X/\hbar}e^{-ip_0X/\hbar} = I, \qquad V^\dagger(p_0)V(p_0) = e^{-ip_0X/\hbar}e^{ip_0X/\hbar} = I. \tag{75}$$

3.3 Consider $|\Psi''\rangle = U(a)V(p_0)|\Psi\rangle$ where $U(a)$ is the spatial translation operator. Express $\langle x|\Psi''\rangle$ as

$$\langle x|\Psi''\rangle = \exp(i\Phi(x,a,p_0))\,\langle x''|\Psi\rangle \tag{76}$$

where the phase Φ and x'' are to be determined as part of the problem.

Solution. Using the definition of $|\Psi''\rangle$,

$$\langle x|\Psi''\rangle = \langle x|\,U(a)V(p_0)\,|\Psi\rangle = \langle x-a|\,V(p_0)\,|\Psi\rangle = \langle x-a|\,e^{-ip_0X/\hbar}\,|\Psi\rangle = e^{ip_0(x-a)/\hbar}\,\langle x-a|\Psi\rangle \tag{77}$$

which is equivalent to (76) with

$$\Phi = -\frac{p_0(x-a)}{\hbar}, \qquad x'' = x-a. \tag{78}$$

3.4 Defining $\langle X\rangle = \langle\Psi|X|\Psi\rangle$ and $\langle P\rangle = \langle\Psi|P|\Psi\rangle$, define formulas which express $\langle\Psi''|X|\Psi''\rangle$ and $\langle\Psi''|P|\Psi''\rangle$ in terms of $\langle X\rangle$, $\langle P\rangle$, and constants.

Solution. Beginning with $\langle\Psi''|V|\Psi''\rangle$, we may insert the identity operator:

$$\begin{aligned}
\langle\Psi''|X|\Psi''\rangle &= \iint \langle\Psi''|x\rangle\,\langle x|X|x'\rangle\,\langle x'|\Psi''\rangle\,\mathrm{d}x\,\mathrm{d}x' &(79)\\
&= \iint \langle\Psi|x-a\rangle\,e^{ip_0(x-a)/\hbar}x'\delta(x-x')e^{-ip_0(x'-a)/\hbar}\,\langle x'-a|\Psi\rangle\,\mathrm{d}x\,\mathrm{d}x'\,, &(80)\\
&= \int \langle\Psi|x-a\rangle\,e^{ip_0(x-a)/\hbar}xe^{-ip_0(x-a)/\hbar}\,\langle x-a|\Psi\rangle\,\mathrm{d}x &(81)\\
&= \int \langle\Psi|x-a\rangle\,x\,\langle x-a|\Psi\rangle\,\mathrm{d}x\,, &(82)
\end{aligned}$$

where in going to (80) we have substituted (77) and its adjoint. Now making the change of variable $x-a \mapsto x$, (82) becomes

$$\langle\Psi''|X|\Psi''\rangle = \int \langle\Psi|x\rangle\,(x+a)\,\langle x|\Psi\rangle\,\mathrm{d}x = \int \langle\Psi|x\rangle\,x\,\langle x|\Psi\rangle\,\mathrm{d}x + a\int \langle\Psi|x\rangle\,\langle x|\Psi\rangle\,\mathrm{d}x = \langle X\rangle + a. \tag{83}$$

Now proceeding similarly for $\langle\Psi''|P|\Psi''\rangle$,

$$\langle\Psi''|P|\Psi''\rangle = \iint \langle\Psi''|x\rangle \langle x|P|x'\rangle \langle x'|\Psi''\rangle \, \mathrm{d}x\,\mathrm{d}x' \tag{84}$$

$$= -\iint \langle\Psi|x-a\rangle\, e^{-ip_0(x-a)/\hbar} \left(i\hbar\delta(x-x')\frac{\partial}{\partial x'} e^{ip_0(x'-a)/\hbar} \langle x'-a|\Psi\rangle \right) \mathrm{d}x\,\mathrm{d}x'\,, \tag{85}$$

$$= -i\hbar \int \langle\Psi|x-a\rangle\, e^{-ip_0(x-a)/\hbar} \left(\frac{\partial}{\partial x} e^{ip_0(x-a)/\hbar} \langle x-a|\Psi\rangle \right) \mathrm{d}x\,, \tag{86}$$

$$= -i\hbar \int \langle\Psi|x-a\rangle\, e^{-ip_0(x-a)/\hbar} \left(\frac{\partial}{\partial x} e^{ip_0(x-a)/\hbar} \right) \langle x-a|\Psi\rangle \, \mathrm{d}x - i\hbar \int \langle\Psi|x-a\rangle \left(\frac{\partial}{\partial x} \langle x-a|\Psi\rangle \right) \mathrm{d}x\,, \tag{87}$$

$$= -i\hbar \frac{ip_0}{\hbar} \int \langle\Psi|x-a\rangle \langle x-a|\Psi\rangle \, \mathrm{d}x - i\hbar \int \langle\Psi|x-a\rangle \left(\frac{\partial}{\partial x} \langle x-a|\Psi\rangle \right) \mathrm{d}x\,. \tag{88}$$

Again making the change of variable $x - a \mapsto x$, (88) becomes

$$\langle\Psi''|P|\Psi''\rangle = -i\hbar \int \langle\Psi|x\rangle \left(\frac{\partial}{\partial x} \langle x|\Psi\rangle \right) \mathrm{d}x + p_0 \int \langle\Psi|x\rangle \langle x|\Psi\rangle \, \mathrm{d}x = \langle P\rangle + p_0. \tag{89}$$

In summary, we have found $\langle\Psi''|X|\Psi''\rangle = \langle X\rangle + a$ and $\langle\Psi''|P|\Psi''\rangle = \langle P\rangle - p_0$.

4 Problem 2

Suppose we have a particle moving in one dimension ($-\infty < x < \infty$), with quantum Hamiltonian given by

$$H(t) = H_0 - XF(t) \tag{90}$$

where

$$H_0 = \frac{P^2}{2m} + V(X) \tag{91}$$

where $V(X)$ is the potential and $F(t)$ is a c-number function. Consider a state ket $|\Psi(t)\rangle$ which evolves in time according to $|\Psi(t)\rangle = U(t,t')\,|\Psi(t')\rangle$, where the unitary time-evolution operator satisfies

$$i\hbar\frac{\partial}{\partial t} U(t,t') = H(t)U(t,t'). \tag{92}$$

Define the expectation values

$$\langle X\rangle(t) = \langle\Psi(t)|X|\Psi(t)\rangle\,, \qquad \langle P\rangle(t) = \langle\Psi(t)|P|\Psi(t)\rangle\,, \qquad \langle H_0\rangle(t) = \langle\Psi(t)|H_0|\Psi(t)\rangle\,. \tag{93}$$

4.1 Derive the formulas for $\partial\langle X\rangle(t)/\partial t$ and $\partial\langle P\rangle(t)/\partial t$. Your results should include other expectation values. Show that your answer reduces to a classical expression if expectation values are replaced by classical values.

Solution. Beginning with X, the product rule of differentiation yields

$$\frac{\partial}{\partial t}\langle X\rangle(t) = \frac{\partial}{\partial t}\langle\Psi(t)|X|\Psi(t)\rangle = \langle\dot\Psi(t)|X|\Psi(t)\rangle + \langle\Psi(t)|\dot X|\Psi(t)\rangle + \langle\Psi(t)|X|\dot\Psi(t)\rangle\,, \tag{94}$$

where the dots indicate $\partial/\partial t$. Obviously $\partial X/\partial t = 0$. We can find the other two terms from the Schrödinger equation (2) and its adjoint, which was found in 1.1(a):

$$i\hbar\partial_t\,|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle \implies |\dot\Psi(t)\rangle = -\frac{i}{\hbar}H(t)\,|\Psi(t)\rangle\,, \tag{95}$$

$$i\hbar\partial_t\,\langle\Psi(t)| = -\langle\Psi(t)|\,H(t) \implies \langle\dot\Psi(t)| = \frac{i}{\hbar}\langle\Psi(t)|\,H(t). \tag{96}$$

Now (94) can be written

$$\begin{aligned}\frac{\partial}{\partial t}\langle X\rangle(t) &= -\frac{i}{\hbar}\langle\Psi(t)|XH(t)|\Psi(t)\rangle + \frac{i}{\hbar}\langle\Psi(t)|H(t)X|\Psi(t)\rangle &(97)\\ &= -\frac{i}{\hbar}\langle\Psi(t)|[X,H(t)]|\Psi(t)\rangle, &(98)\end{aligned}$$

which is Ehrenfest's theorem. For the commutator,

$$[X,H(t)] = [X,P^2/(2m)] = \frac{[X,P^2]}{2m} = \frac{P[X,P]+[X,P]P}{2m} = \frac{i\hbar}{m}P, \tag{99}$$

so we find

$$\frac{\partial}{\partial t}\langle X\rangle(t) = -\frac{i}{\hbar}\frac{i\hbar}{m}\langle\Psi(t)|P|\Psi(t)\rangle = \frac{1}{m}\langle P\rangle(t). \tag{100}$$

Now for P, we have the commutator

$$\begin{aligned}[P,H(t)] &= [P,V(X)-XF(t)] = P(V(X)-XF(t)) - (V(X)-XF(t))P &(101)\\ &= PV(X) - PXF(t) - V(X)P + XF(t)P = [P,V(X)] + [X,P]F(t). &(102)\end{aligned}$$

Note that

$$\langle x|[P,V(X)]|\Psi(t)\rangle = -i\hbar\frac{\partial V(x)}{\partial x}\langle x|\Psi(t)\rangle \implies [P,V(X)] = -i\hbar\frac{\partial V(X)}{\partial X} \tag{103}$$

so (98) with $X \mapsto P$ yields

$$\begin{aligned}\frac{\partial}{\partial t}\langle P\rangle(t) &= -\frac{i}{\hbar}\langle\Psi(t)|[P,H(t)]|\Psi(t)\rangle = -\frac{i}{\hbar}\langle\Psi(t)|\left(-i\hbar\frac{\partial V(X)}{\partial X} + i\hbar F(t)\right)|\Psi(t)\rangle &(104)\\ &= -\langle\Psi(t)|\frac{\partial V(X)}{\partial X}|\Psi(t)\rangle + \langle\Psi(t)|F(t)|\Psi(t)\rangle = F(t) - \left\langle\frac{\partial V(X)}{\partial X}\right\rangle. &(105)\end{aligned}$$

However, since

$$\frac{P}{m} = \frac{\partial H_0}{\partial P} = \frac{\partial H(t)}{\partial P}, \qquad F(t) - \frac{\partial V(0)}{\partial X} = F(t) - \frac{\partial H_0}{\partial X} = -\frac{\partial H(t)}{\partial X}, \tag{106}$$

we can also write

$$\frac{\partial}{\partial t}\langle X\rangle(t) = \left\langle\frac{\partial H(t)}{\partial P}\right\rangle, \qquad \frac{\partial}{\partial t}\langle P\rangle(t) = -\left\langle\frac{\partial H(t)}{\partial X}\right\rangle, \tag{107}$$

which appear similar to Hamilton's equations.

Now we will show that (107) reduce to classical expressions when expectation values are replaced by classical values. Let $\langle X\rangle \mapsto x$, $\langle P\rangle \mapsto p$, and so on. Then (107) become

$$\begin{aligned}\frac{\partial}{\partial t}x(t) &= \frac{\partial H(t)}{\partial p} = \frac{p}{m}, &(108)\\ \frac{\partial}{\partial t}p(t) &= -\frac{\partial H(t)}{\partial x} = F(t) - \frac{\partial V(x)}{\partial x}, &(109)\end{aligned}$$

where (108) is a classical expression for velocity, and (109) is a classical expression for force. □

4.2 Derive a formula for $\partial\langle H_0\rangle/\partial t$ which involves only expectation values.

Solution. H_0 is time independent, so we may again apply (98) with $X \mapsto H_0$. For the commutator,

$$[H_0, H(t)] = [P^2/(2m) + V(X), -XF(t)] = -F(t)\left(\frac{1}{2m}[P^2, X] + [V(X), X]\right) = F(t)\frac{i\hbar}{m}P, \tag{110}$$

so

$$\frac{\partial}{\partial t}\langle H_0\rangle(t) = -\frac{i}{\hbar}\langle\Psi(t)|[H_0, H(t)]|\Psi(t)\rangle = -\frac{i}{\hbar}F(t)\frac{i\hbar}{m}\langle\Psi(t)|P|\Psi(t)\rangle = \frac{F(t)}{m}\langle P\rangle(t). \tag{111}$$

4.3 Assume that $F(t)$ vanishes for $|t| \to \infty$. In this case, it is useful to take $t' \to -\infty$. Derive a formula for the total energy put into the system by $F(t)$ over the time interval $(-\infty, \infty)$ for t. Your result will again involve expectation values. Here, the energy is defined in terms of the Hamiltonian without the external time-dependent force.

Solution. The total energy put into the system by $F(t)$ is

$$\Delta E = \langle H_0\rangle(t=\infty) - \langle H_0\rangle(t=-\infty) = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\langle H_0\rangle(t)\,\mathrm{d}t = \int_{-\infty}^{\infty}\frac{F(t)}{m}\langle P\rangle(t)\,\mathrm{d}t\,, \tag{112}$$

where we have used the fundamental theorem of calculus and (111).

5 Problem 3

Consider the harmonic oscillator described by the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}. \tag{113}$$

5.1 Consider the Heisenberg operators $X(t)$ and $P(t)$. Derive the Heisenberg equation of motion for $X(t)$ and $P(t)$.

Solution. In general, the Heisenberg equations of motion are given by

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[X(t), H], \qquad \frac{\mathrm{d}P(t)}{\mathrm{d}t} = -\frac{i}{\hbar}[P(t), H]. \tag{114}$$

Using Sakurai's partial derivative formulation for evaluating commutators,

$$[X(t), H] = i\hbar\frac{\partial H}{\partial P(t)} = i\hbar\frac{P(t)}{m}, \qquad [P(t), H] = -i\hbar\frac{\partial H}{\partial X(t)} = -i\hbar m\omega^2 X(t). \tag{115}$$

Making these substitutions into (114),

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = -\frac{i}{\hbar}i\hbar\frac{P(t)}{m} = \frac{P(t)}{m}, \qquad \frac{\mathrm{d}P(t)}{\mathrm{d}t} = \frac{i}{\hbar}i\hbar m\omega^2 X(t) = -m\omega^2 X(t) \tag{116}$$

are the Heisenberg equations of motion.

We can solve (116) by making use of the annihilation and creation operators,

$$A = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right), \qquad A^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right). \tag{117}$$

Differentiating (117) and feeding in (116), we retrieve the differential equations

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{\mathrm{d}X(t)}{\mathrm{d}t} + \frac{i}{m\omega}\frac{\mathrm{d}P(t)}{\mathrm{d}t}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{P(t)}{m} - i\omega X(t)\right) = -i\omega A(t), \tag{118}$$

$$\frac{\mathrm{d}A^\dagger(t)}{\mathrm{d}t} = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{P(t)}{m} + i\omega X(t)\right) = i\omega A^\dagger(t), \tag{119}$$

which have solutions

$$A(t) = Ae^{-i\omega t}, \qquad A^\dagger(t) = A^\dagger e^{i\omega t}, \tag{120}$$

where A and $A^\dagger$ are the Schrödinger representations, which are "constants" (that is, time independent) in the Heisenberg picture. In terms of $X(t)$ and $P(t)$, (120) become

$$X(t) + \frac{i}{m\omega}P(t) = \left(X + \frac{i}{m\omega}P\right)e^{-i\omega t}, \tag{121}$$

$$X(t) - \frac{i}{m\omega}P(t) = \left(X - \frac{i}{m\omega}P\right)e^{i\omega t}, \tag{122}$$

where X and P are the Schrödinger representations. Adding (121) and (122),

$$X(t) = X(e^{-i\omega t} + e^{i\omega t}) + \frac{i}{m\omega}P(e^{-i\omega t} - e^{i\omega t}) = X\cos(\omega t) + \frac{P}{m\omega}\sin(\omega t). \tag{123}$$

Now subtracting (122) from (121),

$$P(t) = -im\omega\left(X(e^{-i\omega t} - e^{i\omega t}) + \frac{i}{m\omega}P(e^{-i\omega t} + e^{i\omega t})\right) = P\cos(\omega t) - m\omega X\sin(\omega t). \tag{124}$$

The (solved) Heisenberg equations of motion are then

$$X(t) = X\cos(\omega t) + \frac{P}{m\omega}\sin(\omega t), \qquad P(t) = P\cos(\omega t) - m\omega X\sin(\omega t). \tag{125}$$

5.2 Consider the same oscillator classically. Derive the equations for $x(t)$ and $p(t)$ when the oscillator is released from rest at $x = b$ at $t = 0$, where b is a constant.

Solution. Using Hamilton's equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p} = \frac{p}{m}, \tag{126}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial X}{\partial x} = -m\omega^2 x \tag{127}$$

Writing (126) as $p = m\,\mathrm{d}x/\mathrm{d}t$, we can substitute into (127) to get a second-order equation in x only:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -m\omega^2 x \implies \frac{\partial^2 x}{\partial t^2} = -\omega^2 x \tag{128}$$

which has solutions

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{129}$$

$$p(t) = m\omega B\cos(\omega t) - m\omega A\sin(\omega t), \tag{130}$$

where A and B are constants. To find (130), we have applied (126). The equations are identical in form to (125).

Applying the given initial conditions, we have

$$x(0) = A = b, \qquad p(0) = 0 = m\omega B \tag{131}$$

which fixes A and implies $B = 0$. Thus

$$x(t) = b\cos(\omega t), \qquad p(t) = -m\omega b \sin(\omega t). \tag{132}$$

5.3 Take the initial wave function to be

$$\langle x|\Psi(0)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right). \tag{133}$$

This is a displaced ground wave function for the oscillator. Show that $\langle\Psi(0)|X|\Psi(0)\rangle$ and $\langle\Psi(0)|P|\Psi(0)\rangle$ agree with the classical results you found in the previous problem.

Solution. Firstly,

$$\begin{aligned}
\langle\Psi(0)|X|\Psi(0)\rangle &= \iint \langle\Psi(0)|x\rangle \left\langle x\middle|X\middle|x'\right\rangle \left\langle x'\middle|\Psi(0)\right\rangle \mathrm{d}x\,\mathrm{d}x' &(134)\\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) x'\delta(x-x') \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) \mathrm{d}x\,\mathrm{d}x' &(135)\\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \int x \exp\left(-\frac{m\omega(x-b)^2}{\hbar}\right) \mathrm{d}x\,. &(136)
\end{aligned}$$

Making the change of variable

$$u = \sqrt{\frac{m\omega}{\hbar}}(x-b) \implies x = b + u\sqrt{\frac{\hbar}{m\omega}} \implies \mathrm{d}x = \sqrt{\frac{\hbar}{m\omega}}\,\mathrm{d}u\,, \tag{137}$$

(136) becomes

$$\langle\Psi(0)|X|\Psi(0)\rangle = \frac{1}{\sqrt{\pi}} \int \left(b + u\sqrt{\frac{\hbar}{m\omega}}\right) e^{-u^2}\,\mathrm{d}u = \frac{b}{\sqrt{\pi}} \int e^{-u^2}\,\mathrm{d}u + \sqrt{\frac{\hbar}{m\pi\omega}} \int u e^{-u^2}\,\mathrm{d}u = b. \tag{138}$$

From the classical equation in (132), $x(0) = b$ as well.

Secondly,

$$\begin{aligned}
\langle\Psi(0)|P|\Psi(0)\rangle &= \iint \langle\Psi(0)|x\rangle \left\langle x\middle|P\middle|x'\right\rangle \left\langle x'\middle|\Psi(0)\right\rangle \mathrm{d}x\,\mathrm{d}x' &(139)\\
&= i\hbar\sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \delta(x-x') \frac{\partial}{\partial x'} \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) \mathrm{d}x\,\mathrm{d}x' &(140)\\
&= i\hbar\sqrt{\frac{m\omega}{\pi\hbar}} \int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \frac{\partial}{\partial x} \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \mathrm{d}x\,. &(141)
\end{aligned}$$

Again making the change of variable (137), note that $\partial/\partial x = \sqrt{m\omega/\hbar}\,\partial/\partial u$. Making these substitutions in (141),

$$\langle\Psi(0)|P|\Psi(0)\rangle = i\frac{m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \frac{\partial}{\partial u} e^{-u^2/2}\,\mathrm{d}u = -i\frac{m\omega}{\sqrt{\pi}} \int u e^{-u^2}\,\mathrm{d}u = 0. \tag{142}$$

From the classical equation in (132), $p(0) = 0$ as well. So the results agree with the classical limit for both cases, as we wanted to show. $\square$

5.4 Now consider uncertainties at $t = 0$. Define

$$\left\langle \Delta X^2 \right\rangle = \langle \Psi(0)|X^2|\Psi(0)\rangle - (\langle \Psi(0)|X|\Psi(0)\rangle)^2, \qquad \left\langle \Delta P^2 \right\rangle = \langle \Psi(0)|P^2|\Psi(0)\rangle - (\langle \Psi(0)|P|\Psi(0)\rangle)^2. \tag{143}$$

Calculate $\left\langle \Delta X^2 \right\rangle \left\langle \Delta P^2 \right\rangle$.

Solution. Once again using the change of variable (137),

$$\begin{aligned}
\langle \Psi(0)|X^2|\Psi(0)\rangle &= \iint \langle \Psi(0)|x\rangle \left\langle x\middle|X^2\middle|x'\right\rangle \left\langle x'\middle|\Psi(0)\right\rangle \mathrm{d}x\,\mathrm{d}x' & (144)\\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) x'^2 \delta(x-x') \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) \mathrm{d}x\,\mathrm{d}x' & (145)\\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \int x^2 \exp\left(-\frac{m\omega(x-b)^2}{\hbar}\right) \mathrm{d}x = \frac{1}{\sqrt{\pi}} \int \left(b + u\sqrt{\frac{\hbar}{m\omega}}\right)^2 e^{-u^2}\,\mathrm{d}u & (146)\\
&= \frac{b^2}{\sqrt{\pi}} \int e^{-u^2}\,\mathrm{d}u + \frac{2b}{\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}} \int u e^{-u^2}\,\mathrm{d}u + \frac{1}{\sqrt{\pi}}\frac{\hbar}{m\omega} \int u^2 e^{-u^2}\,\mathrm{d}u & (147)\\
&= b^2 + \frac{\hbar}{2m\omega}, & (148)
\end{aligned}$$

and

$$\begin{aligned}
\langle \Psi(0)|P^2|\Psi(0)\rangle &= \iint \langle \Psi(0)|x\rangle \left\langle x\middle|P^2\middle|x'\right\rangle \left\langle x'\middle|\Psi(0)\right\rangle \mathrm{d}x\,\mathrm{d}x' & (149)\\
&= -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \delta(x-x') \frac{\partial^2}{\partial x'^2} \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) \mathrm{d}x\,\mathrm{d}x' & (150)\\
&= -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \frac{\partial^2}{\partial x^2} \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \mathrm{d}x & (151)\\
&= -\frac{\hbar m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \frac{\partial^2}{\partial u^2} e^{-u^2/2}\,\mathrm{d}u = -\frac{\hbar m\omega}{\sqrt{\pi}} \int (u^2 - 1) e^{-u^2}\,\mathrm{d}u & (152)\\
&= -\frac{\hbar m\omega}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \sqrt{\pi}\right) = \frac{\hbar m\omega}{2} & (153)
\end{aligned}$$

. Then, using the results from (138) and (142),

$$\left\langle \Delta X^2 \right\rangle = b^2 + \frac{\hbar}{2m\omega} - b^2 = \frac{\hbar}{2m\omega}, \qquad \left\langle \Delta P^2 \right\rangle = \frac{\hbar m\omega}{2}, \qquad \left\langle \Delta X^2 \right\rangle \left\langle \Delta P^2 \right\rangle = \frac{\hbar^2}{4}. \tag{154}$$

5.5 Now consider

$$\left\langle \Delta X^2 \right\rangle(t) = \langle \Psi(t)|X^2|\Psi(t)\rangle - (\langle \Psi(t)|X|\Psi(t)\rangle)^2, \quad \left\langle \Delta P^2 \right\rangle(t) = \langle \Psi(t)|P^2|\Psi(t)\rangle - (\langle \Psi(t)|P|\Psi(t)\rangle)^2. \tag{155}$$

Calculate $\left\langle \Delta X^2 \right\rangle(t) \left\langle \Delta P^2 \right\rangle(t)$.

Solution. In the Heisenberg picture, (155) becomes

$$\begin{aligned}
\left\langle \Delta X^2 \right\rangle(t) &= \langle \Psi(0)|X(t)^2|\Psi(0)\rangle - (\langle \Psi(0)|X(t)|\Psi(0)\rangle)^2, & (156)\\
\left\langle \Delta P^2 \right\rangle(t) &= \langle \Psi(0)|P(t)^2|\Psi(0)\rangle - (\langle \Psi(0)|P(t)|\Psi(0)\rangle)^2, & (157)
\end{aligned}$$

Using the expressions for $X(t)$ and $P(t)$ in (125),

$$\langle\Psi(0)|X(t)|\Psi(0)\rangle = \langle\Psi(0)|\left(X\cos(\omega t) + \frac{P}{m\omega}\sin(\omega t)\right)|\Psi(0)\rangle = \cos(\omega t)\langle X\rangle + \frac{\sin(\omega t)}{m\omega}\langle P\rangle = b\cos(\omega t) \tag{158}$$

where we use the notation $\langle X\rangle = \langle\Psi(0)|X|\Psi(0)\rangle$ and so forth, as well as the results of (138) and (142). Continuing on,

$$\langle\Psi(0)|X(t)^2|\Psi(0)\rangle = \langle\Psi(0)|\left(X\cos(\omega t) + \frac{P}{m\omega}\sin(\omega t)\right)^2|\Psi(0)\rangle \tag{159}$$

$$= \cos^2(\omega t)\left\langle X^2\right\rangle + \frac{\sin^2(\omega t)}{m^2\omega^2}\left\langle P^2\right\rangle + \frac{\cos(\omega t)\sin(\omega t)}{m\omega}\langle XP\rangle + \frac{\cos(\omega t)\sin(\omega t)}{m\omega}\langle PX\rangle. \tag{160}$$

Again using the change of variable (137), note that

$$\langle\Psi(0)|XP|\Psi(0)\rangle = \iint \langle\Psi(0)|x\rangle\left\langle x\middle|XP\middle|x'\right\rangle\left\langle x'\middle|\Psi(0)\right\rangle \mathrm{d}x\,\mathrm{d}x' \tag{161}$$

$$= i\hbar\sqrt{\frac{m\omega}{\pi\hbar}}\iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right)x'\delta(x-x')\frac{\partial}{\partial x'}\exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right)\mathrm{d}x\,\mathrm{d}x' \tag{162}$$

$$= i\hbar\sqrt{\frac{m\omega}{\pi\hbar}}\int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right)x\frac{\partial}{\partial x}\exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right)\mathrm{d}x \tag{163}$$

$$= i\frac{m\omega}{\sqrt{\pi}}\int e^{-u^2/2}\left(b+u\sqrt{\frac{\hbar}{m\omega}}\right)\frac{\partial}{\partial u}e^{-u^2/2}\,\mathrm{d}u = -i\frac{m\omega}{\sqrt{\pi}}\int ue^{-u^2}\left(b+u\sqrt{\frac{\hbar}{m\omega}}\right)\mathrm{d}u \tag{164}$$

$$= -ib\frac{m\omega}{\sqrt{\pi}}\int ue^{-u^2/2}\,\mathrm{d}u - i\frac{m\omega}{\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\int u^2e^{-u^2}\,\mathrm{d}u = -i\frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}, \tag{165}$$

and

$$\langle\Psi(0)|PX|\Psi(0)\rangle = \iint \langle\Psi(0)|x\rangle\left\langle x\middle|PX\middle|x'\right\rangle\left\langle x'\middle|\Psi(0)\right\rangle \mathrm{d}x\,\mathrm{d}x' \tag{166}$$

$$= i\hbar\sqrt{\frac{m\omega}{\pi\hbar}}\iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right)\delta(x-x')\frac{\partial}{\partial x'}x'\exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right)\mathrm{d}x\,\mathrm{d}x' \tag{167}$$

$$= i\hbar\sqrt{\frac{m\omega}{\pi\hbar}}\int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right)\left(\frac{\partial}{\partial x}x\exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right)\right)\mathrm{d}x \tag{168}$$

$$= i\frac{m\omega}{\sqrt{\pi}}\int e^{-u^2/2}\left[\frac{\partial}{\partial u}\left(b+u\sqrt{\frac{\hbar}{m\omega}}\right)e^{-u^2/2}\right]\mathrm{d}u \tag{169}$$

$$= ib\frac{m\omega}{\sqrt{\pi}}\int e^{-u^2/2}\frac{\partial}{\partial u}e^{-u^2/2}\,\mathrm{d}u + i\frac{m\omega}{\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\int e^{-u^2/2}\left(\frac{\partial}{\partial u}ue^{-u^2/2}\right)\mathrm{d}u \tag{170}$$

$$= -ib\frac{m\omega}{\sqrt{\pi}}\int ue^{-u^2}\,\mathrm{d}u + i\frac{m\omega}{\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\int(1-u^2)e^{-u^2}\,\mathrm{d}u = i\frac{m\omega}{\sqrt{\pi}}\sqrt{\frac{\hbar}{m\omega}}\left(\sqrt{\pi}-\frac{\sqrt{\pi}}{2}\right) \tag{171}$$

$$= i\frac{m\omega}{2}\sqrt{\frac{\hbar}{m\omega}}, \tag{172}$$

so $\langle XP\rangle + \langle PX\rangle = 0$ and

$$\langle\Psi(0)|X(t)^2|\Psi(0)\rangle = \cos^2(\omega t)\left\langle X^2\right\rangle + \frac{\sin^2(\omega t)}{m^2\omega^2}\left\langle P^2\right\rangle = \cos^2(\omega t)\left(b^2+\frac{\hbar}{2m\omega}\right) + \sin^2(\omega t)\frac{\hbar}{2m\omega} \tag{173}$$

$$= b^2\cos^2(\omega t) + \frac{\hbar}{2m\omega}, \tag{174}$$

where we have substituted (148) and (153). Similarly,

$$\begin{aligned}\langle\Psi(0)|P(t)|\Psi(0)\rangle &= \langle\Psi(0)|\big(P\cos(\omega t) - m\omega X\sin(\omega t)\big)|\Psi(0)\rangle = \cos(\omega t)\langle P\rangle - m\omega\sin(\omega t)\langle X\rangle &(175)\\ &= -bm\omega\sin(\omega t), &(176)\end{aligned}$$

and

$$\begin{aligned}\langle\Psi(0)|P(t)^2|\Psi(0)\rangle &= \langle\Psi(0)|\big(P\cos(\omega t) - m\omega X\sin(\omega t)\big)^2|\Psi(0)\rangle &(177)\\ &= \cos^2(\omega t)\left\langle P^2\right\rangle + m^2\omega^2\sin^2(\omega t)\left\langle X^2\right\rangle - m\omega\sin(\omega t)\cos(\omega t)(\langle PX\rangle + \langle XP\rangle) &(178)\\ &= \frac{\hbar m\omega}{2}\cos^2(\omega t) + m^2\omega^2\sin^2(\omega t)\left(b^2 + \frac{\hbar}{2m\omega}\right) = b^2m^2\omega^2\sin^2(\omega t) + \frac{\hbar m\omega}{2}. &(179)\end{aligned}$$

Using (158), (174), (176), and (179)

$$\begin{aligned}\left\langle\Delta X^2\right\rangle(t) &= b^2\cos^2(\omega t) + \frac{\hbar}{2m\omega} - [b\cos(\omega t)]^2 = \frac{\hbar}{2m\omega}, &(180)\\ \left\langle\Delta P^2\right\rangle(t) &= b^2m^2\omega^2\sin^2(\omega t) + \frac{\hbar m\omega}{2} - [-bm\omega\sin(\omega t)]^2 = \frac{\hbar m\omega}{2}. &(181)\end{aligned}$$

Finally,

$$\left\langle\Delta X^2\right\rangle(t)\left\langle\Delta P^2\right\rangle(t) = \frac{\hbar}{4} \tag{182}$$

which is unchanged from the initial state of the system considered in 5.4.

In writing up these solutions, I consulted Sakurai's *Modern Quantum Mechanics* and Shankar's *Principles of Quantum Mechanics*.