Problem 1. Show that for an arbitrary spatially bound charge-current source, the electric dipole moment $\mathbf{p}$ satisfies

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{J}\, d^3x\,.$$

Solution. The electric dipole moment $\mathbf{p}$ is defined by Eq. (2.36),

$$\mathbf{p} = \int \mathbf{x}\,\rho(x)\, d^3x\,. \tag{1}$$

Differentiating both sides with respect to t, we find

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int \mathbf{x}\rho\, d^3x = \int \frac{d}{dt}(\mathbf{x}\rho)\, d^3x = \int \mathbf{x}\frac{\partial \rho}{\partial t}\, d^3x\,, \tag{2}$$

because $\mathbf{x}$ is simply the point at which we are evaluating the potential, and is therefore independent of time.

The charge-current conservation law is given by Eq. (5.8),

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0. \tag{3}$$

Multiplying by $\mathbf{x}$ on both sides and integrating over all space, we obtain

$$\int \mathbf{x}\frac{\partial \rho}{\partial t}\, d^3x + \int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = 0.$$

Applying (2), we have

$$\frac{d\mathbf{p}}{dt} = -\int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x\,. \tag{4}$$

It remains to be shown that the right side is equal to the integral of $\mathbf{J}$ over all space.

Vector identity (5) in Griffiths is

$$\boldsymbol{\nabla}\cdot(f\mathbf{a}) = f(\boldsymbol{\nabla}\cdot\mathbf{a}) + \mathbf{a}\cdot(\boldsymbol{\nabla}f).$$

Writing the right side of (4) in component notation and applying the identity gives us

$$-\int x_i(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = \int \mathbf{J}\cdot(\boldsymbol{\nabla}x_i)\, d^3x - \int \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x\,. \tag{5}$$

Gauss's theorem is given by Eq. (2.6),

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{v}\, d^3x = \int_S \mathbf{v}\cdot\hat{\mathbf{n}}\, dS\,,$$

Here, let $\mathcal{V}$ be a ball of radius R, with R large enough that the entire charge-current source is enclosed. Then S is the surface of $\mathcal{V}$, and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Applying Gauss's theorem to the second integral on the right side of (5), we have

$$\int \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x = \lim_{R\to\infty}\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x = \lim_{R\to\infty}\int_S x_i\mathbf{J}\cdot\hat{\mathbf{r}}\, dS = 0,$$

since $\mathbf{J}$ is bounded, and therefore $\mathbf{J}$ evaluated on S reaches zero well before x_i becomes very large.

Returning to (5), we now have

$$-\int x_i(\boldsymbol{\nabla}\cdot\mathbf{J})\,d^3x = \int \mathbf{J}\cdot(\boldsymbol{\nabla}x_i)\,d^3x = \sum_j \int J_j\partial_j x_i\,d^3x = \sum_j \int J_j\delta_{ij}\,d^3x = \int J_i\,d^3x\,,$$

where we have followed the proof in Eq. (4.24) of the course notes. Finally, (4) becomes

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{J}\,d^3x$$

as desired. □

Problem 2. A particle of charge q_1 moves with velocity v in a circular orbit of radius R about the origin in the xy plane, such that its φ coordinate varies as $\varphi = \omega t$, with $\omega = v/R$. Assume that $v \ll c$. Another particle of charge q_2 is at rest at point $\mathbf{x}$, where $|\mathbf{x}| \gg R$. To order $1/|\mathbf{x}|$, find the force $\mathbf{F}$ on the particle of charge q_2 at time t.

Solution. The Lorentz force equation, Eq. (1.25), is written

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right),$$

where $\mathbf{v}$ is the velocity of the charge q on which the force is exerted, and $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields. For this problem, we are interested in the force acting on a stationary point charge q_2, so $\mathbf{v}_2 = 0$. Additionally, we do not have to consider the self-field contribution to $\mathbf{E}$, since static charge distributions do not experience any self force. Thus we need only find the electric field due to q_1, $\mathbf{E}_1$. The multipole expansion of the electric field in electrodynamics is given by Eq. (5.70),

$$\mathbf{E}(t,\mathbf{x}) = \frac{1}{c^2|\mathbf{x}|}\left[\left(\hat{\mathbf{x}}\cdot\frac{d^2\mathbf{p}}{dt^2}\right)\hat{\mathbf{x}} - \frac{d^2\mathbf{p}}{dt^2}\right]_{\text{ret}} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right), \tag{6}$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ is the unit vector in the direction of the point at which we are evaluating the field, and $\mathbf{p}$ is the dipole moment defined by (1). In addition, (6) relies upon the assumption that the velocity of q_1, v, satisfies $v \ll c$.

The position of q_1 at time t can be expressed as

$$\mathbf{x}_1(t) = R\cos(\omega t)\,\hat{\boldsymbol{\varphi}},$$

so the charge density for q_1 everywhere is

$$\rho_1(t,\mathbf{x}) = q_1\,\delta\big(\mathbf{x}-\mathbf{x}_1(t)\big).$$

Then the dipole moment $\mathbf{p}_1(t,\mathbf{x})$ is

$$\mathbf{p}_1(t,\mathbf{x}) = \int \mathbf{x}\,\rho_1(t,\mathbf{x})\,d^3x = q_1\int \mathbf{x}\,\delta\big(\mathbf{x}-\mathbf{x}_1(t)\big)\,d^3x = q_1\mathbf{x}_1(t) = q_1R\cos(\omega t)\,\hat{\boldsymbol{\varphi}},$$

and so its second time derivative is

$$\frac{d^2\mathbf{p}_1(t)}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{p}_1}{dt}\right) = \frac{d}{dt}\Big(-q_1R\omega\sin(\omega t)\,\hat{\boldsymbol{\varphi}}\Big) = -q_1R\omega^2\cos(\omega t)\,\hat{\boldsymbol{\varphi}}.$$

The retarded time t' is defined

$$t' = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c},$$

so here

$$t' = t - \frac{|\mathbf{x} - \mathbf{x}_1(t)|}{c} = t - \frac{|\mathbf{x} - R\cos(\omega t)\,\hat{\boldsymbol{\varphi}}|}{c}.$$

Since $R \ll |\mathbf{x}|$, we can Taylor expand the second term about $R = 0$ as in Eq. (5.57),

$$\left|\mathbf{x} - \mathbf{x}'\right| = |\mathbf{x}| - \hat{\mathbf{x}} \cdot \mathbf{x}' + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right).$$

This gives us

$$|\mathbf{x} - R\cos(\omega t)\,\hat{\boldsymbol{\varphi}}| \approx |\mathbf{x}| - R\cos(\omega t)(\hat{\mathbf{x}} \cdot \hat{\boldsymbol{\varphi}}),$$

so

$$t' \approx t - \frac{|\mathbf{x}|}{c} - \frac{R\cos(\omega t)}{c}(\hat{\mathbf{x}} \cdot \hat{\boldsymbol{\varphi}}),$$

which is really not helpful.

Problem 3. An "antenna" is a segment of conducting wire in which a current flows (driven by an external power supply). Suppose an antenna of length L is placed on the z axis between $z = 0$ and $z = L$, and suppose that the current in the antenna is

$$\mathbf{J}(t, z) = I_0 \sin\left(\frac{\pi z}{L}\right) \cos(\omega t)\,\delta(x)\,\delta(y)\,\hat{\mathbf{z}}. \tag{7}$$

3.a Find the charge density $\rho(t, z)$ in the antenna.

Solution. From the charge-current conservation law (3), we have

$$\rho(t, z) = -\int \boldsymbol{\nabla} \cdot \mathbf{J}\, dt\,.$$

For $\mathbf{J}$ given by (7),

$$\boldsymbol{\nabla} \cdot \mathbf{J} = \frac{\partial J_z}{\partial z} = \frac{\pi}{L} I_0 \cos\left(\frac{\pi z}{L}\right) \cos(\omega t)\,\delta(x)\,\delta(y),$$

and so

$$\rho(t, z) = -\frac{\pi}{L} I_0 \cos\left(\frac{\pi z}{L}\right) \delta(x)\,\delta(y) \int \cos(\omega t)\, dt$$

3.b Assume that $\omega L \ll c$. Find the electric and magnetic fields, $\mathbf{E}(t, z)$ and $\mathbf{B}(t, z)$, at large distances from the antenna (valid to order $1/|\mathbf{x}|$).

In addition to the course lecture notes, I consulted Griffiths's *Introduction to Electrodynamics*, Jackson's *Classical Electrodynamics*, and K. T. McDonald's and D. K. Ghosh's notes on electromagnetism while writing up these solutions.