Problem 1.

1(a) Show by explicit computation the Lorentz invariance of the Dirac Lagrangian, by considering a Lorentz transformation of the fields.

Solution. The Dirac Lagrangian is given by Eq. (3.34) in Peskin & Schroder:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

According to their Eq. (3.33), $\bar{\psi}$ transforms as $\bar{\psi} \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}$; also, $\psi \to \Lambda_{\frac{1}{2}}\psi$. For the divergence, we refer to their Eq. (3.3): $\partial_\mu \phi(x) \to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x)$.

The Lorentz transformation of the Dirac Lagrangian is then [1, p. 42]

$$\begin{aligned}
\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) &\to \bar{\psi}(\Lambda^{-1}x)\Lambda_{\frac{1}{2}}^{-1}[i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m)]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \bar{\psi}(\Lambda^{-1}x)[i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu \Lambda_{\frac{1}{2}}(\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \bar{\psi}(\Lambda^{-1}x)[i\Lambda^\mu{}_\sigma \gamma^\sigma (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m]\psi(\Lambda^{-1}x),
\end{aligned}$$

where we have used Peskin & Schroeder (3.29), $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu$. Then

$$\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \to \bar{\psi}(\Lambda^{-1}x)[i\Lambda^\mu{}_\sigma \gamma^\sigma (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m)]\psi(\Lambda^{-1}x) = \bar{\psi}(\Lambda^{-1}x)(i\gamma^\nu \partial_\nu - m)\psi(\Lambda^{-1}x),$$

which has the same form as $\mathcal{L}_{\text{Dirac}}$. So we have shown that the Dirac Lagrangian is Lorentz invariant. □

1(b) Consider the chiral rotation of a massless Dirac field $\psi' = e^{i\alpha\gamma^5}\psi$. Find the corresponding Noether current. Show that the corresponding Noether charge measures the total helicity of a collection of massless Dirac particles, and that the addition of a mass term to the Lagrangian violates the symmetry. Find an equation that expresses the violation of current conservation by the mass.

Solution. The conserved charge is given in general by Peskin & Schroeder (2.12) and (2.13),

$$Q \equiv \int_{\text{all space}} j^0 \, d^3x \,, \qquad \text{where } j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi - J^\mu, \tag{1}$$

where J^μ is a 4-divergence that arises when transforming the Lagrangian as in Peskin & Schroeder (2.10):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x). \tag{2}$$

Under the rotation $\psi \to e^{i\alpha\gamma^5}\psi$, $\psi^\dagger \to \psi^\dagger e^{-i\alpha\gamma^5}$. Then, using $\bar{\psi} = \psi^\dagger \gamma^0$ as defined in Peskin & Schroeder (3.32),

$$\bar{\psi} \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 = -\psi^\dagger \gamma^0 e^{-i\alpha\gamma^5} = -\bar{\psi}e^{-i\alpha\gamma^5},$$

since $\{\gamma^\mu, \gamma^5\} = 0$ from Peskin & Schroeder (3.70). Then, using $m = 0$ in the Dirac Lagrangian, we have

$$\mathcal{L}_{\text{Dirac}} = i\bar{\psi}\gamma^\mu \partial_\mu \psi \to -i\bar{\psi}e^{-i\alpha\gamma^5}\gamma^\mu \partial_\mu e^{i\alpha\gamma^5}\psi = i\bar{\psi}\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu e^{i\alpha\gamma^5}\psi = i\bar{\psi}\gamma^\mu \partial_\mu \psi,$$

so the Dirac Lagrangian is indeed invariant under chiral transformations, and $J^\mu = 0$.

The infinitesimal transformations associated with $\psi \to e^{i\alpha\gamma^5}\psi$ are

$$\alpha\Delta\psi = i\alpha\gamma^5\psi, \alpha\Delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5.$$

Then we have the Noether current [1, p. 50]

$$j^\mu = -\left[\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\bar{\psi})}\Delta\bar{\psi}\right] = \bar{\psi}\gamma^\mu\gamma^5\psi,$$

where we have multiplied by an arbitrary constant [1, p. 18].

Peskin & Schroeder (3.76) defines

$$j_L^\mu = \bar{\psi}\gamma^\mu\frac{1-\gamma^5}{2}\psi, \qquad j_R^\mu = \bar{\psi}\gamma^\mu\frac{1+\gamma^5}{2}\psi,$$

as the electric current densities of left- and right-handed particles. Note that $j^\mu = j_R^\mu - j_L^\mu$. Then we have the conserved charge

$$Q = \int d^3x\, \bar{\psi}\gamma^0\gamma^5\psi = \int d^3x\, (j_R^0 - j_L^0),$$

which tells us the total helicity of a collection of massless Dirac particles. □

If $m \neq 0$ in the Dirac Lagrangian, then it transforms as

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \to -\bar{\psi}e^{-i\alpha\gamma^5}(i\gamma^\mu\partial_\mu - m)e^{i\alpha\gamma^5}\psi &= \bar{\psi}(\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu + e^{-i\alpha\gamma^5}m)e^{i\alpha\gamma^5}\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu + m)\psi,\end{aligned}$$

which is not of the same form. So the symmetry is violated for nonzero m. □

In order for the current to be conserved, we need the divergence $\partial_\mu j^\mu = 0$. Note that

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi.$$

Since ψ satisfies the Dirac equation, we can make use of the Dirac equation and its conjugate, given by Eqs. (3.31) and (3.35) in Peskin & Schroeder:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \qquad -i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0.$$

So the divergence can be written [1, p. 51]

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi = im\bar{\psi}\gamma^5\psi + \bar{\psi}\gamma^5 im\psi = 2im\bar{\psi}\gamma^5\psi,$$

which is 0 only if m is 0.

1(c) Find the Noether current related to charge conservation by considering a phase rotation of a Dirac field (of arbitrary mass) $\psi' = e^{i\alpha}\psi$.

Solution. We will once again use Eqs. (1) and (2). Under the rotation $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$. Then the Dirac Lagrangian transforms as

$$\mathcal{L}_{\text{Dirac}} \to \bar{\psi}e^{i\alpha}(i\gamma^\mu\partial_\mu - m)e^{-i\alpha}\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

so again $J^\mu = 0$.

The infinitesimal translations are

$$\alpha\Delta\psi = i\alpha\psi, \alpha\Delta\bar{\psi} \qquad = -i\alpha\bar{\psi},$$

and the Noether current is [1, p. 50]

$$j^\mu = -\left[\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\bar{\psi})}\Delta\bar{\psi}\right] = \bar{\psi}\gamma^\mu\psi.$$

Problem 2. Lorentz group (Peskin & Schroeder 3.1) Recall from Eq. (3.17) the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

2(a) Define the generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i},$$

where $i, j, k = 1, 2, 3$. An infinitesimal Lorentz transformation can then be written

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K})\Phi. \tag{3}$$

Write the commutation relations of these vector operators explicitly. (For example, $[L^i, L^j] = i\epsilon^{ijk}L^k$.) Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}), \qquad \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

commute with one another and separately satisfy the commutation relations of angular momentum.

Solution. Firstly, using Eq. (3.18),

$$\begin{aligned}
[L^i, L^j] &= \left[\frac{1}{2}\epsilon^{i\mu\nu}J^{\mu\nu}, \frac{1}{2}\epsilon^{j\rho\sigma}J^{\rho\sigma}\right] = \frac{1}{4}\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}[J^{\mu\nu}, J^{\rho\sigma}] = \frac{i}{4}\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\mu\rho}J^{\nu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\sigma}J^{\mu\rho} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\mu\sigma}J^{\nu\rho}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\nu\mu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\sigma\rho}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\nu\mu}\epsilon^{j\sigma\rho}g^{\nu\rho}J^{\mu\sigma}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma}) \\
&= i\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma},
\end{aligned}$$

where we have simply relabeled indices. Then, using $g^{ij} = -\delta^{ij}$ and $\epsilon^{ijk}\epsilon^{pqk} = \delta^{ip}\delta^{jq} - \delta^{iq}\delta^{jp}$ [2],

$$\begin{aligned}
[L^i, L^j] &= -i\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}\delta^{\nu\rho}J^{\mu\sigma} = -i\epsilon^{i\mu\nu}\epsilon^{j\nu\sigma}J^{\mu\sigma} = i\epsilon^{i\mu\nu}\epsilon^{j\sigma\nu}J^{\mu\sigma} = i(\delta^{ij}\delta^{\mu\sigma} - \delta^{i\sigma}\delta^{\mu j})J^{\mu\sigma} = i(\delta^{ij}J^{\mu\mu} - \delta^{i\sigma}J^{j\sigma}) \\
&= -iJ^{ji} = iJ^{ij},
\end{aligned}$$

where we have used the antisymmetry of J^{ij}. From $L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$, we can write

$$\epsilon^{i\rho\sigma}L^i = \frac{1}{2}\epsilon^{ijk}\epsilon^{i\rho\sigma}J^{jk} = \frac{1}{2}(\delta^{j\rho}\delta^{k\sigma} - \delta^{j\sigma}\delta^{k\rho})J^{jk} = \frac{1}{2}(\delta^{j\rho}J^{j\sigma} - \delta^{j\sigma}J^{j\rho}) = \frac{1}{2}(J^{\rho\sigma} - J^{\sigma\rho}) = J^{\rho\sigma}.$$

Then we see that

$$[L^i, L^j] = i\epsilon^{ijk}L^k.$$

Secondly,

$$[K^i, K^j] = [J^{0i}, J^{0j}] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) = -iJ^{ij} = -i\epsilon^{ijk}L^k,$$

and thirdly,

$$\begin{aligned}
[K^i, L^j] &= \left[J^{0i}, \frac{1}{2}\epsilon^{j\rho\sigma}J^{\rho\sigma}\right] = \frac{1}{2}\epsilon^{j\rho\sigma}[J^{0i}, J^{\rho\sigma}] = \frac{i}{2}\epsilon^{j\rho\sigma}(g^{i\rho}J^{0\sigma} - g^{0\rho}J^{i\sigma} - g^{i\sigma}J^{0\rho} + g^{0\sigma}J^{i\rho}) \\
&= \frac{i}{2}(\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} - \epsilon^{j\rho\sigma}g^{i\sigma}J^{0\rho}) = \frac{i}{2}(\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} - \epsilon^{j\sigma\rho}g^{i\rho}J^{0\sigma}) = i\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} = -i\epsilon^{j\rho\sigma}\delta^{i\rho}J^{0\sigma} = i\epsilon^{ij\sigma}J^{0\sigma} \\
&= i\epsilon^{ijk}K^k.
\end{aligned}$$

Next we want to show that $[\mathbf{J}_+, \mathbf{J}_-] = 0$. Note that

$$
\begin{aligned}
[J_+^i, J_-^j] &= \left[\frac{1}{2}(L^i + iK^i), \frac{1}{2}(L^j - iK^j)\right] = \frac{1}{4}\left([L^i, L^j] - i[L^i, K^j] + i[K^i, L^j] + [K^i, K^j]\right) \\
&= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{jik}K^k - \epsilon^{ijk}K^k - i\epsilon^{ijk}L^k\right) \\
&= 0,
\end{aligned}
$$

so $[\mathbf{J}_+, \mathbf{J}_-] = 0$. □

The angular momentum commutation relations are given by Peskin & Schroeder Eq. (3.12): $[J^i, J^j] = i\epsilon^{ijk}J^k$. We have

$$
\begin{aligned}
[J_\pm^i, J_\pm^j] &= \left[\frac{1}{2}(L^i \pm iK^i), \frac{1}{2}(L^j \pm iK^j)\right] = \frac{1}{4}\left([L^i, L^j] \pm i[L^i, K^j] \pm i[K^i, L^j] - [K^i, K^j]\right) \\
&= \frac{1}{4}\left(i\epsilon^{ijk}L^k \pm \epsilon^{jik}K^k \mp \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right) = \frac{1}{2}\left(i\epsilon^{ijk}L^k \mp \epsilon^{ijk}K^k\right) = \frac{1}{2}i\epsilon^{ijk}\left(L^k \pm iK^k\right) \\
&= i\epsilon^{ijk}J_\pm^k,
\end{aligned}
$$

as desired. □

2(b) The finite-dimensional representations of the rotation group correspond precisely to the allowed values for angular momentum: integers or half-integers. The result of 2(a) implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers, (j_+, j_-), corresponding to pairs of representations of the rotation group. Using the fact that $\mathbf{J} = \boldsymbol{\sigma}/2$ in the spin-1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. Show that these correspond precisely to the transformations of ψ_L and ψ_R given in (3.37).

Solution. Equation (3.37) is

$$
\psi_L \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_L, \qquad \psi_R \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_R,
$$

where ψ_L and ψ_R are the left- and right-handed Weyl spinors, respectively.

We can rewrite Eq. (3) in terms of $\mathbf{J}_+$ and $\mathbf{J}_-$:

$$
\Phi \to [1 - i\boldsymbol{\theta}\cdot(\mathbf{J}_+ + \mathbf{J}_-) - \boldsymbol{\beta}\cdot(\mathbf{J}_+ - \mathbf{J}_-)]\Phi = [1 - (i\boldsymbol{\theta} + \boldsymbol{\beta})\cdot\mathbf{J}_+ + (i\boldsymbol{\theta} - \boldsymbol{\beta})\cdot\mathbf{J}_-)]\Phi.
$$

From the final expression, we associate $\mathbf{J}_+$ and $\mathbf{J}_-$ with $\boldsymbol{\sigma}/2$ in turn, with $\mathbf{J}_+ = \boldsymbol{\sigma}/2$ corresponding to the $(\frac{1}{2}, 0)$ representation and $\mathbf{J}_- = \boldsymbol{\sigma}/2$ corresponding to the $(0, \frac{1}{2})$ representation. The transformation laws are

$$
\Phi \to \begin{cases} \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\Phi & (\frac{1}{2}, 0) \text{ representation,} \\ \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\Phi & (0, \frac{1}{2}) \text{ representation.} \end{cases}
$$

Comparing to Eq. (3.37), we see that Φ transforms as ψ_L under the $(\frac{1}{2}, 0)$ representation and as ψ_R under the $(0, \frac{1}{2})$ representation. □

2(c) The identity $\boldsymbol{\sigma}^T = -\sigma^2\boldsymbol{\sigma}\sigma^2$ allows us to rewrite the ψ_L transformations in the unitarily equivalent form

$$\psi' \to \psi'\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right),$$

where $\psi' = \psi_L^T\sigma^2$. Using this law, we can represent the object that transforms as $(\frac{1}{2},\frac{1}{2})$ as a 2×2 matrix that has the ψ_R transformations law on the left and, simultaneously, the transposed ψ_L transformation on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^3 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}.$$

Show that the object V^μ transforms as a 4-vector.

Solution. Peskin & Schroeder (3.19) shows an infinitesimal Lorentz transformation:

$$V^\alpha \to \left[\delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta\right]V^\beta,$$

where V is a 4-vector, $\omega_{\mu\nu}$ is an antisymmetric tensor that gives the infinitesimal angles, and $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha)$ from Peskin & Schroeder (3.18). Using this definition, the transformation is

$$\begin{aligned}
V^\alpha &\to \left[\delta^\alpha{}_\beta + \frac{1}{2}\omega_{\mu\nu}(\delta^{\mu\alpha}\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^{\nu\alpha})\right]V^\beta = \left[\delta^\alpha{}_\beta + \frac{1}{2}\omega_{\mu\nu}g_{\beta\gamma}(\delta^{\mu\alpha}\delta^{\nu\gamma} - \delta^{\mu\gamma}\delta^{\nu\alpha})\right]V^\beta \\
&= \left[\delta^\alpha{}_\beta + \frac{1}{2}g_{\beta\gamma}(\omega^{\alpha\gamma} - \omega^{\gamma\alpha})\right]V^\beta = (\delta^\alpha{}_\beta + g_{\beta\gamma}\omega^{\alpha\gamma})V^\beta \\
&= (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta, \qquad (4)
\end{aligned}$$

where we have used the antisymmetry of $\omega^{\mu\nu}$.

For the problem at hand, note that

$$V_\mu\sigma^\mu = \begin{pmatrix} V^0 & 0 \\ 0 & V^0 \end{pmatrix} - \begin{pmatrix} 0 & V^1 \\ V^1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -iV^2 \\ iV^2 & 0 \end{pmatrix} - \begin{pmatrix} V^3 & 0 \\ 0 & -V^3 \end{pmatrix} = \begin{pmatrix} V^0 + V^3 & V^1 - iV^3 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}.$$

Then the transformation is

$$\begin{aligned}
V_\mu\sigma^\mu &\to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)V_\mu\sigma^\mu\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right) = \left(1 + (\boldsymbol{\beta} - i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2}\right)V_\mu\sigma^\mu\left(1 + (\boldsymbol{\beta} + i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2}\right) \\
&= V_\mu\sigma^\mu + V_\mu\sigma^\mu(\boldsymbol{\beta} + i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2} + (\boldsymbol{\beta} - i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2}V_\mu\sigma^\mu,
\end{aligned}$$

where we note that θ and β are infinitesimal angles and drop terms of $\mathcal{O}(\theta^2) = \mathcal{O}(\beta^2) = \mathcal{O}(\theta\beta)$. Then

$$\begin{aligned}
V_\mu\sigma^\mu &\to V_\mu\sigma^\mu + \frac{1}{2}V_\mu\sigma^\mu(\beta_i\sigma^i + i\theta_j\sigma^j) + \frac{1}{2}V_\mu(\beta_k\sigma^k - i\theta_l\sigma^l)\sigma^\mu \\
&= V_\mu\sigma^\mu + \frac{1}{2}V_\mu\left(\beta_i\sigma^\mu\sigma^i + i\theta_j\sigma^\mu\sigma^j + \beta_k\sigma^k\sigma^\mu - i\theta_l\sigma^l\sigma^\mu\right) \\
&= V_\mu\sigma^\mu + \frac{1}{2}V_\mu\left[\beta_i(\sigma^\mu\sigma^i + \sigma^i\sigma^\mu) + i\theta_j(\sigma^\mu\sigma^j - \sigma^j\sigma^\mu)\right] = V_\mu\sigma^\mu + \frac{1}{2}V_\mu\left(\beta_i\{\sigma^\mu,\sigma^i\} + i\theta_j[\sigma^\mu,\sigma^j]\right) \\
&= V_\mu\sigma^\mu + \frac{1}{2}V_0\left(\beta_i\{\sigma^0,\sigma^i\} + i\theta_j[\sigma^0,\sigma^j]\right) + \frac{1}{2}V_k\left(\beta_i\{\sigma^k,\sigma^i\} + i\theta_j[\sigma^k,\sigma^j]\right) \\
&= V_\mu\sigma^\mu + \beta_iV_0\sigma^i + V_k\left(\beta_i\delta^{ik} - \theta_j\epsilon^{kji}\sigma^i\right) = V_\mu\sigma^\mu + \beta_iV_0\sigma^i + V_i\beta_i + V_k\theta_j\epsilon^{ijk}\sigma^i \\
&= V_\mu\sigma^\mu + V_0\beta^i\sigma_i - V_i\beta^i - V_j\epsilon^{ijk}\theta^i\sigma^k
\end{aligned}$$

where we have used $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ and $[\sigma^i,\sigma^j] = 2i\epsilon^{ijk}\sigma^k$ [3, p. 165], as well as $\{\sigma^0,\sigma^i\} = 2\sigma^i$ and $[\sigma^0,\sigma^i] = 0$.

Referring to Eq. (3.19), we define

$$\omega^{0j} = \beta^j, \qquad \omega^{ij} = \epsilon^{ijk}\theta^k.$$

Then we have

$$\begin{aligned} V_\mu\sigma^\mu &\to V_\mu\sigma^\mu + V_0\omega^{0i}\sigma_i - V_i\omega^{0i} - V_j\omega^{ij}\sigma^j = V_\mu\sigma^\mu + V^0\omega_{0i}\sigma^i - V^i\omega_{0i}\sigma^0 + V^j\omega_{ij}\sigma^j \\ &= V_\mu\sigma^\mu + V^0\omega_{0\mu}\sigma^\mu + V^i\omega_{i0}\sigma^0 + V^j\omega_{ij}\sigma^j = V_\mu\sigma^\mu + V^\nu\omega_{\nu\mu}\sigma^\mu = (\delta^\nu{}_\mu + \omega^\nu{}_\mu)V_\nu\sigma^\mu \end{aligned}$$

or

$$V^\alpha\sigma_\alpha \to (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta\sigma_\alpha \quad \Longrightarrow \quad V^\alpha \to (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta,$$

since σ^μ forms a complete basis. This is identical to Eq. (4), so we have shown that V^μ transforms as a 4-vector. □

Problem 3. Majorana fermions (Peskin & Schroeder 3.4) Recall from Eq. (3.40) that one can write a relativistic equation for a massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (ψ_L). Call such a 2-component field $\chi_a(x)$, $a = 1, 2$.

3(a) Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0. \tag{5}$$

That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

Solution. Using the result of 2(b), let

$$\Lambda_{\frac{1}{2},0} = \exp\Big(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\Big), \qquad \Lambda_{0,\frac{1}{2}} = \exp\Big(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\Big).$$

We are told that χ transforms as ψ_L, so we know $\chi \to \Lambda_{\frac{1}{2},0}\chi$. So

$$\begin{aligned} \sigma^2\chi^* &\to \sigma^2(\Lambda_{\frac{1}{2},0}\chi)^* \\ &= \sigma^2\left[\exp\Big(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\Big)\chi\right]^* = \sigma^2\left[\exp\left(i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}^*}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}^*}{2}\right)\right]\chi^* = \left[\exp\Big(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\Big)\right]\sigma^2\chi^* \\ &= \Lambda_{0,\frac{1}{2}}\sigma^2\chi^*, \end{aligned}$$

where we have used Peskin & Schroeder (3.38), $\sigma^2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma^2$.

We also need to know how to manipulate $\bar{\sigma}$. From Eqs. (3.36) and (3.37), we can write a finite Lorentz transformation as

$$\Lambda\psi = \begin{pmatrix} \Lambda_{\frac{1}{2},0} & 0 \\ 0 & \Lambda_{0,\frac{1}{2}} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \Lambda_{\frac{1}{2},0}\psi_L \\ \Lambda_{0,\frac{1}{2}}\psi_R \end{pmatrix}.$$

Then using Peskin & Schroeder (3.42),

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

we can rewrite their Eq. (3.29), $\Lambda^\mu{}_\nu\gamma^\nu = \Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}$, in matrix form. The left side is

$$\begin{pmatrix}\Lambda_{\frac{1}{2}} & 0\\ 0 & \Lambda_{\frac{1}{2}}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix} = \begin{pmatrix}0 & \Lambda^\mu{}_\nu\sigma^\nu\\ \Lambda^\mu{}_\nu\bar\sigma^\nu & 0\end{pmatrix},$$

and the right side is

$$\begin{pmatrix}\Lambda_{\frac{1}{2},0}^{-1} & 0\\ 0 & \Lambda_{0,\frac{1}{2}}^{-1}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\begin{pmatrix}\Lambda_{\frac{1}{2},0} & 0\\ 0 & \Lambda_{0,\frac{1}{2}}\end{pmatrix} = \begin{pmatrix}\Lambda_{\frac{1}{2},0}^{-1} & 0\\ 0 & \Lambda_{0,\frac{1}{2}}^{-1}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\Lambda_{\frac{1}{2},0}\\ \bar\sigma^\mu\Lambda_{0,\frac{1}{2}} & 0\end{pmatrix} = \begin{pmatrix}0 & \Lambda_{\frac{1}{2},0}^{-1}\sigma^\mu\Lambda_{0,\frac{1}{2}}\\ \Lambda_{0,\frac{1}{2}}^{-1}\bar\sigma^\mu\Lambda_{\frac{1}{2},0} & 0,\end{pmatrix}$$

So we have

$$\Lambda^\mu{}_\nu\bar\sigma^\nu = \Lambda_{0,\frac{1}{2}}^{-1}\bar\sigma^\mu\Lambda_{\frac{1}{2},0}, \qquad \bar\sigma^\nu(\Lambda^{-1})^\nu{}_\mu = \Lambda_{0,\frac{1}{2}}\bar\sigma^\mu\Lambda_{\frac{1}{2},0}^{-1}.$$

Then Eq. (5) transforms as

$$\begin{aligned}i\bar\sigma^\mu\partial_\mu\chi(x) - im\sigma^2\chi^*(x) &\to i\bar\sigma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu\Lambda_{\frac{1}{2},0}\chi(\Lambda^{-1}x) - im\Lambda_{0,\frac{1}{2}}\sigma^2\chi^*(\Lambda^{-1}x)\\ &= i\Lambda_{0,\frac{1}{2}}\bar\sigma^\mu\Lambda_{\frac{1}{2},0}^{-1}\Lambda_{\frac{1}{2},0}\partial_\nu\chi(\Lambda^{-1}x) - im\Lambda_{0,\frac{1}{2}}\sigma^2\chi^*(\Lambda^{-1}x)\\ &= \Lambda_{0,\frac{1}{2}}\left[\bar\sigma^\mu\partial_\nu\chi(\Lambda^{-1}x) - im\sigma^2\chi^*(\Lambda^{-1}x)\right]\\ &= 0,\end{aligned}$$

so this equation is relativistically invariant. □

The Klein-Gordon equation does not include χ^*, so we want to eliminate it from the Majorana equation. From Eq. (5),

$$i\bar\sigma\cdot\partial\chi = im\sigma^2\chi^* \quad\implies\quad \chi^* = \frac{\sigma^2}{m}\bar\sigma\cdot\partial\chi,$$

since $\sigma^i\sigma^i = 1$ [3, p. 164]. We also need a ∂^2 term in the Klein-Gordon equation, so we can feed this expression into the complex conjugate of the Majorana equation:

$$\begin{aligned}0 &= -i\bar\sigma^*\cdot\partial\chi^* + im\sigma^{*2}\chi = -i\bar\sigma^{*\mu}\partial_\mu\left(\frac{\sigma^2}{m}\bar\sigma\cdot\partial\chi\right) + im\sigma^{*2}\chi = -i\bar\sigma^{*\mu}\frac{\sigma^2}{m}\bar\sigma^\nu\partial_\mu\partial_\nu\chi + im\sigma^{*2}\chi\\ &= \bar\sigma^{*\mu}\sigma^2\bar\sigma^\nu\partial_\mu\partial_\nu\chi - m^2\sigma^{*2}\chi.\end{aligned} \tag{6}$$

Since $\bar\sigma^\mu = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$, then $\bar\sigma^{*\mu} = (\sigma^0, -\sigma^1, \sigma^2, -\sigma^3)$, and so

$$\bar\sigma^{*\mu}\sigma^2 = (\sigma^2, -\sigma^1\sigma^2, \sigma^2\sigma^2, \sigma^3\sigma^2) = (\sigma^2, \sigma^2\sigma^1, \sigma^2\sigma^2, -\sigma^2\sigma^3) = \sigma^2\sigma^\mu, \tag{7}$$

where we have used $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ [3, p. 165]. Using this in Eq. (6),

$$0 = \sigma^2\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu\chi - m^2\sigma^{*2}\chi = \sigma^2\sigma^2\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu\chi - m^2\sigma^2\sigma^{*2}\chi = \sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu\chi + m^2\chi \tag{8}$$

since $\sigma^{*2} = -\sigma^2$. The anticommutation relation also implies that $\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2g^{\mu\nu}$. Note that

$$\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu = \frac{1}{2}(\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu + \sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu) = \frac{1}{2}(\sigma^\mu\bar\sigma^\nu\partial_\mu\partial_\nu + \sigma^\nu\bar\sigma^\mu\partial_\nu\partial_\mu) = \frac{1}{2}(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu,$$

where we have just relabeled indices. Then we have

$$0 = \partial_\mu\partial^\mu\chi + m^2\chi,$$

which is the Klein-Gordon equation. □

3(b) Does the Majorana equation follow from a Lagrangian? The mass term would seem to be the variation of $(\sigma^2)_{ab}\chi_a^*\chi_b^*$; however, since σ^2 is antisymmetric, this expression would vanish if $\chi(x)$ were an ordinary c-number field. When we go to quantum field theory, we know that $\chi(x)$ will become an anticommuting quantum field. Therefore, it makes sense to develop its classical theory by considering $\chi(x)$ as a classical anticommuting field, that is, as a field that takes as values *Grassmann numbers* which satisfy

$$\alpha\beta = -\beta\alpha, \qquad \text{for any } \alpha, \beta.$$

Note that this relation implies that $\alpha^2 = 0$. A Grassmann field $\xi(x)$ can be expanded in a basis of functions as

$$\xi(x) = \sum_n \alpha_n \phi_n(x),$$

where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a profuct of Grassmann numbers to reverse the order:

$$(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^*.$$

This rule imitates the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right], \tag{9}$$

(where $\chi^\dagger = (\chi^*)^T$) is real ($S^* = S$), and that varying this S with respect to χ and χ^* yields the Majorana equation.

Solution. To show that S is real, we take its complex conjugate using the rules for Grassman numbers:

$$\begin{aligned}
S^* &= \int d^4x \left[-i\left(\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi\right)^* - \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)^*\right] = \int d^4x \left[-i\chi^T\bar{\sigma}^{*\mu}\partial_\mu\chi^* + \frac{im}{2}\left(\chi^\dagger\sigma^{*2}\chi^* - \chi^T\sigma^{*2}\chi\right)\right] \\
&= \int d^4x \left[-i\partial_\mu\chi^\dagger\bar{\sigma}^{\dagger\mu}\chi - \frac{im}{2}\left(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\right)\right] = \int d^4x \left[-i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right],
\end{aligned}$$

where we have transposed the first term. This is allowed because it is a c-number, and the sum of the matrix elements is the same if the matrices are transposed. We have also used the fact that σ^μ is Hermitian, since all σ^i are [3, p. 165]. Then, integrating by parts, note that

$$\int d^4x\, \partial_\mu\chi^\dagger\bar{\sigma}^\mu\chi = \left[\chi^\dagger\bar{\sigma}^\mu\chi\right]_{-\infty}^{\infty} - \int d^4x\, \chi^\dagger\partial_\mu\chi = \int d^4x\, \chi^\dagger\partial_\mu\chi,$$

so we have

$$S^* = \int d^4x \left[i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right] = S,$$

as desired. □

As usual, we may treat χ and χ^* independently. Firstly, let $\delta\chi$ be a variation of the field χ that vanishes at the boundaries, and that $\delta S = S[\chi + \delta\chi] - S[\chi] = 0$. Then

$$\begin{aligned}
\delta S &= \int d^4x \left\{i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\chi+\delta\chi) + \frac{im}{2}\left[(\chi^T+\delta\chi^T)\sigma^2(\chi+\delta\chi) - \chi^\dagger\sigma^2\chi^*\right] - i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right\} \\
&= \int d^4x \left[i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\delta\chi + \frac{im}{2}\left(\delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi\right)\right] = \int d^4x \left(i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\delta\chi + im\chi^T\sigma^2\delta\chi\right) \\
&= \int d^4x\, \delta\chi\left(-i\bar{\sigma}^\mu\partial_\mu\chi^\dagger + im\chi^T\sigma^2\right),
\end{aligned}$$

where we have integrated by parts and used the fact that $\delta\chi$ vanishes at the boundary. This gives us

$$-i\bar{\sigma}^\mu\partial_\mu\chi^\dagger + im\chi^T\sigma^2 = 0 \implies 0 = (-i\bar{\sigma}^\mu\partial_\mu\chi^\dagger + im\chi^T\sigma^2)^\dagger = i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^*,$$

which is the Majorana equation.

Secondly, let $\delta\chi^*$ be a variation of the field χ^* that vanishes at the boundaries, and that $\delta S = S[\chi+\delta\chi]-S[\chi] = 0$. Then

$$\begin{aligned}\delta S &= \int d^4x \left\{ i(\chi^\dagger + \delta\chi^\dagger)\bar{\sigma}^\mu\partial_\mu\chi + \frac{im}{2}\left[\chi^T\sigma^2\chi - (\chi^\dagger+\delta\chi^\dagger)\sigma^2(\chi^*+\delta\chi^*)\right] - \chi^\dagger i\bar{\sigma}\cdot\partial\chi - \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right\} \\ &= \int d^4x \left[i\delta\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi - \frac{im}{2}\left(\delta\chi^\dagger\sigma^2\chi^* + \chi^\dagger\sigma^2\delta\chi^*\right)\right] = \int d^4x\, \delta\chi^\dagger\left(i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^*\right),\end{aligned}$$

which implies

$$i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^* = 0,$$

which is the Majorana equation. □

3(c) Let us write a 4-component Dirac field as

$$\psi(x) = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix},$$

and recall that the lower components of ψ transform in a way equivalent by a unitary transformation to the complex conjugate of the representation ψ_L. In this way, we can rewrite the 4-component Dirac field in terms of two 2-component spinors:

$$\psi_L(x) = \chi_1(x), \qquad \psi_R(x) = i\sigma^2\chi_2^*(x).$$

Rewrite the Dirac Lagrangian in terms of χ_1 and χ_2 and note the form of the mass term.

Solution. Beginning from the Dirac Lagrangian in Peskin & Schroeder (3.34),

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \psi^\dagger\gamma^0(i\gamma^\mu\partial_\mu - m)\psi.$$

where we have used Eq. (3.32), $\bar{\psi} = \psi^\dagger\gamma^0$. From the problem statement,

$$\psi = \begin{pmatrix}\chi_1 \\ i\sigma^2\chi_2^*\end{pmatrix}, \qquad \psi^\dagger = \begin{pmatrix}\chi_1^\dagger & i\sigma^{*2}\chi_2^T\end{pmatrix} = \begin{pmatrix}\chi_1^\dagger & -i\sigma^2\chi_2^T\end{pmatrix}.$$

Additionally, from Peskin & Schroeder (3.25),

$$\gamma^0 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix},$$

and from (3.42),

$$0 = (i\gamma^\mu\partial_\mu - m)\psi(x) = \begin{pmatrix}-m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m\end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}.$$

So we can write

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} &= \begin{pmatrix}\chi_1^\dagger & -i\sigma^2\chi_2^T\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}-m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m\end{pmatrix}\begin{pmatrix}\chi_1 \\ i\sigma^2\chi_2^*\end{pmatrix} = \begin{pmatrix}\chi_1^\dagger & -i\sigma^2\chi_2^T\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}-\sigma^\mu\partial_\mu\sigma^2\chi_2^* - m\chi_1 \\ i\bar{\sigma}^\mu\partial_\mu\chi_1 - im\sigma^2\chi_2^*\end{pmatrix} \\ &= \begin{pmatrix}\chi_1^\dagger & -i\sigma^2\chi_2^T\end{pmatrix}\begin{pmatrix}i\bar{\sigma}^\mu\partial_\mu\chi_1 - im\sigma^2\chi_2^* \\ -\sigma^\mu\partial_\mu\sigma^2\chi_2^* - m\chi_1\end{pmatrix} = i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + im\chi_1^\dagger\sigma^2\chi_2^* + i\sigma^2\chi_2^T\sigma^\mu\partial_\mu\sigma^2\chi_2^* + im\sigma^2\chi_2^T\chi_1 \\ &= i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + i\chi_2^T\bar{\sigma}^{*\mu}\partial_\mu\chi_2^* + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*) = i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + i\chi_2\bar{\sigma}^{\dagger\mu}\partial_\mu\chi_2^\dagger + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*) \\ &= i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 - i\chi_2\bar{\sigma}^\mu\partial_\mu\chi_2^\dagger + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*),\end{aligned}$$

where we have applied Eq. (7) and transposed the second term. Since $\mathcal{L}_{\text{Dirac}}$ is a Lagrangian density, we may integrate this term by parts to obtain

$$\mathcal{L}_{\text{Dirac}} = i\chi_1^\dagger \bar{\sigma}^\mu \partial_\mu \chi_1 + i\chi_2^\dagger \bar{\sigma}^\mu \partial_\mu \chi_2 + im(\chi_2^T \sigma^2 \chi_1 - \chi_1^\dagger \sigma^2 \chi_2^*).$$

The mass term has a form similar to the integrand in Eq. (9). If $\chi_1 = \chi_2 = \chi$, then the mass term in the Dirac Lagrangian is twice the mass term in the Lagrangian of Eq. (9). In fact, the entirety of the Dirac Lagrangian is twice that Lagrangian in the case $\chi_1 = \chi_2 = \chi$.

3(e) Quantize the Majorana theory of 3(a) and 3(b). That is, promote $\chi(x)$ to a quantum field satisfying the canonical anticommutation relation

$$\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}),$$

construct a Hermitian Hamiltonian, and find a representation of the canonical anticommutation relations that diagonalizes the Hamiltonian in terms of a set of creation and annihilation operators. (Hint: Compare $\chi(x)$ to the top two components of the quantized Dirac field.)

Solution. As was pointed out in the solution of 3(c), the Majorana Lagrangian is half the Dirac Lagrangian if $\chi_1 = \chi_2 = \chi$. Using the definition in the statement of 3(c), this means $\psi_L = \chi$. From Peskin & Schroeder (3.99),

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^s u^s(p) e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip\cdot x} \right),$$

and Eqs. (3.50) and (3.62) give

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \qquad s = 1, 2.$$

From $\chi = \psi_L$,

$$\chi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^s \sqrt{p\cdot\sigma}\xi^s e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} \sqrt{p\cdot\sigma}\eta^s e^{ip\cdot x} \right) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^s \xi^s e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} \eta^s e^{ip\cdot x} \right).$$

The Dirac Hamiltonian is given by Eq. (3.104),

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s \right).$$

Since the Majorana Lagrangian is half the Dirac Lagrangian, the Majorana Hamiltonian is

$$H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s \right),$$

which is Hermitian.

Since we know χ obeys the Majorana equation, we can use it to search for a constraint on the creation and annihilation operators that will further distinguish the Majorana Hamiltonian from the Dirac Hamiltonian. For the mass term of the Majorana equation,

$$\begin{aligned} im\sigma^2\chi^* = im\sigma^2\chi^\dagger &= im\sigma^2 \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{p\cdot\sigma^*}{2E_{\mathbf{p}}}} \sum_s \left(a_{\mathbf{p}}^{s\dagger} \xi^s e^{ip\cdot x} + b_{\mathbf{p}}^s \eta^{s*} e^{ip\cdot x} \right) \\ &= im \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{p\cdot\bar{\sigma}}{2E_{\mathbf{p}}}} \sigma^2 \sum_s \left(a_{\mathbf{p}}^{s\dagger} \xi^s e^{ip\cdot x} + b_{\mathbf{p}}^s \eta^{s*} e^{ip\cdot x} \right), \end{aligned}$$

where we have used

$$\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^2\sigma^0, -\sigma^2\sigma^1, -\sigma^2\sigma^2, -\sigma^2\sigma^3)\sigma^2 = (\sigma^0\sigma^2, \sigma^1\sigma^2, -\sigma^2\sigma^2, \sigma^3\sigma^2)\sigma^2 = \sigma^{*\mu}\sigma^2\sigma^2 = \sigma^{*\mu}. \tag{10}$$

For the divergence term,

$$\begin{aligned}
i\bar{\sigma}^\mu\partial_\mu\chi &= i\bar{\sigma}^\mu \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\xi^s\partial_\mu e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}}\eta^s\partial_\mu e^{ip\cdot x}\right)\\
&= \bar{\sigma}^\mu \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\xi^s p_\mu e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}}\eta^s p_\mu e^{ip\cdot x}\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\bar{\sigma}^\mu p_\mu\sqrt{\frac{p\cdot\sigma}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\xi^s e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{(p\cdot\bar{\sigma})(p\cdot\bar{\sigma})(p\cdot\sigma)}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\xi^s e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right)\\
&= m\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar{\sigma}}{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}\xi^s e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right),
\end{aligned}$$

where we have used Peskin & Schroeder (3.51), $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2$.

A relation between $u^s(p)$ and $v^s(p)$ is given by Eq. (3.144),

$$u^s(p) = -i\gamma^2[v^s(p)]^*, \qquad\qquad v^s(p) = -i\gamma^2[u^s(p)]^*.$$

This can be written as [1, p. 70]

$$\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix} = -i\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma^*}\eta^{s*}\\ -\sqrt{p\cdot\bar{\sigma}^*}\eta^{s*}\end{pmatrix} = i\sigma^2\begin{pmatrix}\sqrt{p\cdot\bar{\sigma}^*}\\ \sqrt{p\cdot\sigma^*}\end{pmatrix}\eta^{s*} = i\begin{pmatrix}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\sigma^2\eta^{s*}$$

where we have used Eq. (10) and

$$\sigma^2\sigma^\mu\sigma^2 = (\sigma^2\sigma^0, \sigma^2\sigma^1, \sigma^2\sigma^2, \sigma^2\sigma^3)\sigma^2 = (\sigma^0\sigma^2, -\sigma^1\sigma^2, \sigma^2\sigma^2, -\sigma^3\sigma^2)\sigma^2 = \bar{\sigma}^{*\mu}\sigma^2\sigma^2 = \bar{\sigma}^{*\mu}.$$

This implies

$$\xi^s = i\sigma^2\eta^{s*}.$$

Then the full Majorana equation can be written

$$\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar{\sigma}}{2E_{\mathbf{p}}}}\sum_s \left(i\sigma^2 b^s_{\mathbf{p}}\eta^{s*}e^{ip\cdot x} - a^{s\dagger}_{\mathbf{p}}\eta^{s*}e^{ip\cdot x}\right) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\bar{\sigma}}{2E_{\mathbf{p}}}}\sum_s \left(i\sigma^2 a^s_{\mathbf{p}}\eta^{s*}e^{-ip\cdot x} - b^{s\dagger}_{\mathbf{p}}\eta^s e^{ip\cdot x}\right),$$

which implies

$$a^s_{\mathbf{p}} = b^s_{\mathbf{p}};$$

that is, the fermion is its own antiparticle. So the Majorana Hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s E_{\mathbf{p}} a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}},$$

which is diagonal. The canonical anticommutation relation is the same as for the Dirac creation and annihilation operators, since the additional restriction on the Majorana operators does not change their commutation properties. It is given by Peskin & Schroeder (3.101),

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta^{rs}.$$

Problem 4. (Peskin & Schroeder 3.7) This problem concerns the discrete symmetries P, C, and T.

4(a) Compute the transformation properties under P, C, and T of the antisymmetric tensor fermion bilinears, $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. This completes the table of the transformation properties of bilinears at the end of the chapter.

Solution. For the transformation under P,

$$\begin{aligned} P\bar{\psi}\sigma^{\mu\nu}\psi P = P\bar{\psi}PP\sigma^{\mu\nu}PP\psi P &= \eta_a^*\bar{\psi}(t,-\mathbf{x})\gamma^0 P\sigma^{\mu\nu}P\eta_a\gamma^0\psi(t,-\mathbf{x}) \\ &= \frac{i}{2}|\eta_a|^2\bar{\psi}(t,-\mathbf{x})\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\psi(t,-\mathbf{x}) = \frac{i}{2}\bar{\psi}(t,-\mathbf{x})\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\psi(t,-\mathbf{x}), \end{aligned}$$

where we have used Peskin & Schroeder (3.125), $\eta\eta_b = -\eta_a\eta^s = -1$ and $\eta_b^* = -\eta_a$ [1, p. 66]. We have also used their Eqs. (3.126) and (3.128),

$$P\psi(t,\mathbf{x})P = \eta_a\gamma^0\psi(t,-\mathbf{x}), \qquad P\bar{\psi}(t,\mathbf{x})P = \eta_a^*\bar{\psi}(t,-\mathbf{x})\gamma^0.$$

From Eqs. (3.26) and (3.27),

$$\sigma^{0i} = \frac{i}{2}[\gamma^0,\gamma^i] = -i\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}, \qquad \sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j] = \epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}.$$

Note also that

$$\gamma^0\begin{pmatrix}\sigma^i & 0\\ 0 & \pm\sigma^i\end{pmatrix}\gamma^0 = \begin{pmatrix}0 & \sigma^0\\ \sigma^0 & 0\end{pmatrix}\begin{pmatrix}\sigma^i & 0\\ 0 & \pm\sigma^i\end{pmatrix}\begin{pmatrix}0 & \sigma^0\\ \sigma^0 & 0\end{pmatrix} = \begin{pmatrix}0 & \sigma^0\\ \sigma^0 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ \pm\sigma^i & 0\end{pmatrix} = \begin{pmatrix}\pm\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}.$$

for any σ^i. Then

$$\begin{aligned} &P\bar{\psi}\sigma^{00}\psi P = 0, \\ &P\bar{\psi}\sigma^{0i}\psi P = i\bar{\psi}(t,-\mathbf{x})\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}\psi(t,-\mathbf{x}) = -\bar{\psi}(t,-\mathbf{x})\sigma^{0i}\psi(t,-\mathbf{x}), \\ &P\bar{\psi}\sigma^{i0}\psi P = -i\bar{\psi}(t,-\mathbf{x})\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}\psi(t,-\mathbf{x}) = -\bar{\psi}(t,-\mathbf{x})\sigma^{i0}\psi(t,-\mathbf{x}), \\ &P\bar{\psi}\sigma^{ij}\psi P = \epsilon^{ijk}\bar{\psi}(t,-\mathbf{x})\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\psi(t,-\mathbf{x}) = \bar{\psi}(t,-\mathbf{x})\bar{\psi}\sigma^{ij}\psi\psi(t,-\mathbf{x}), \end{aligned}$$

or

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = \begin{cases} 0 & \mu=\nu=0; \\ -\bar{\psi}(t,-\mathbf{x})\sigma^{\mu\nu}\psi(t,-\mathbf{x}) & \mu=0,\nu\neq 0; \\ -\bar{\psi}(t,-\mathbf{x})\sigma^{\mu\nu}\psi(t,-\mathbf{x}) & \mu\neq 0,\nu=0; \\ \bar{\psi}(t,-\mathbf{x})\sigma^{\mu\nu}\psi(t,-\mathbf{x}) & \mu\neq 0,\nu\neq 0. \end{cases}$$

For the transformation under T,

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = T\bar{\psi}TT\sigma^{\mu\nu}TT\psi T = -\bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3 T\sigma^{\mu\nu}T\gamma^1\gamma^3\psi(-t,\mathbf{x}) = -\bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3\sigma^{*\mu\nu}\gamma^1\gamma^3\psi(-t,\mathbf{x}),$$

where we have used Peskin & Schroeder (3.139) and (3.140),

$$T\psi(t,\mathbf{x})T = -\gamma^1\gamma^3\psi(t,-\mathbf{x}), \qquad T\bar{\psi}(t,\mathbf{x})T = \bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3,$$

and Eq. (3.133), $T(\text{c-number}) = (\text{c-number})^* T$. Note that

$$\gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = -\begin{pmatrix} \sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3 \end{pmatrix},$$

and that

$$\gamma^1\gamma^3\begin{pmatrix} \sigma^i & 0 \\ 0 & \pm\sigma^i \end{pmatrix}\gamma^1\gamma^3 = \begin{pmatrix} \sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & \pm\sigma^i \end{pmatrix}\begin{pmatrix} \sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3 \end{pmatrix} = \begin{pmatrix} \sigma^1\sigma^3\sigma^i\sigma^1\sigma^3 & 0 \\ 0 & \pm\sigma^1\sigma^3\sigma^i\sigma^1\sigma^3 \end{pmatrix},$$

Taking each case of the matrix elements,

$$\begin{aligned}
&\sigma^1\sigma^3\sigma^1\sigma^1\sigma^3 = \sigma^1\sigma^3\sigma^3 = \sigma^1, \\
&\sigma^1\sigma^3\sigma^2\sigma^1\sigma^3 = -\sigma^1\sigma^3\sigma^1\sigma^2\sigma^3 = \sigma^1\sigma^3\sigma^1\sigma^3\sigma^2 = -\sigma^1\sigma^1\sigma^3\sigma^3\sigma^2 = -\sigma^2, \\
&\sigma^1\sigma^3\sigma^3\sigma^1\sigma^3 = \sigma^1\sigma^1\sigma^3 = \sigma^3.
\end{aligned}$$

The case of $T\bar{\psi}\sigma^{\mu\nu}\psi T$ where $\mu = \nu = 0$ is trivially 0. Now we consider the case where either μ or ν is 0. Since $\sigma^{02} = \sigma^{*02}$ and $\sigma^{0i} = -\sigma^{*0i}$ for $i \in \{1, 3\}$,

$$\begin{aligned}
T\bar{\psi}\sigma^{02}\psi T &= -\bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3\sigma^{02}\gamma^1\gamma^3\psi(-t,\mathbf{x}) = \bar{\psi}(-t,\mathbf{x})\sigma^{02}\psi(-t,\mathbf{x}), \\
T\bar{\psi}\sigma^{0i}\psi T &= \bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3\sigma^{0i}\gamma^1\gamma^3\psi(-t,\mathbf{x}) = \bar{\psi}(-t,\mathbf{x})\sigma^{0i}\psi(-t,\mathbf{x}).
\end{aligned}$$

Finally we consider the case where $\mu \neq 0$ and $\nu \neq 0$. If $i, j \in \{1, 3\}$, $\sigma^{*ij} = -\sigma^{ij}$ and $\sigma^{i2} = \sigma^{*i2} = \sigma^{2i} = \sigma^{*2i}$. Then

$$\begin{aligned}
T\bar{\psi}\sigma^{ij}\psi T &= \bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3\sigma^{ij}\gamma^1\gamma^3\psi(-t,\mathbf{x}) = -\bar{\psi}(-t,\mathbf{x})\sigma^{ij}\psi(-t,\mathbf{x}), \\
T\bar{\psi}\sigma^{i2}\psi T &= -\bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3\sigma^{i2}\gamma^1\gamma^3\psi(-t,\mathbf{x}) = \bar{\psi}(-t,\mathbf{x})\sigma^{i2}\psi(-t,\mathbf{x}), \\
T\bar{\psi}\sigma^{2i}\psi T &= -\bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3\sigma^{2i}\gamma^1\gamma^3\psi(-t,\mathbf{x}) = \bar{\psi}(-t,\mathbf{x})\sigma^{2i}\psi(-t,\mathbf{x}).
\end{aligned}$$

In summary,

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = \begin{cases} 0 & \mu = \nu = 0; \\ \bar{\psi}(-t,\mathbf{x})\sigma^{\mu\nu}\psi(-t,\mathbf{x}) & \mu = 0, \nu \neq 0; \\ \bar{\psi}(-t,\mathbf{x})\sigma^{\mu\nu}\psi(-t,\mathbf{x}) & \mu \neq 0, \nu = 0; \\ -\bar{\psi}(-t,\mathbf{x})\sigma^{\mu\nu}\psi(-t,\mathbf{x}) & \mu \neq 0, \nu \neq 0. \end{cases}$$

For the transformation under C,

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = C\bar{\psi}CC\sigma^{\mu\nu}CC\psi C = -(\gamma^0\gamma^2\psi)^T C\sigma^{\mu\nu}C(\bar{\psi}\gamma^0\gamma^2)^T = -(\gamma^0\gamma^2\psi)^T\sigma^{\mu\nu}(\bar{\psi}\gamma^0\gamma^2)^T,$$

where we have used Peskin & Schroeder (3.145) and (3.146),

$$C\psi(x)C = -i(\bar{\psi}\gamma^0\gamma^2)^T, \qquad C\bar{\psi}(x)C = -i(\gamma^0\gamma^2\psi)^T.$$

Since $\gamma^{0T} = \gamma^0$ and $\gamma^{iT} = -\gamma^i$,

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = -\psi^T\gamma^2\gamma^0\sigma^{\mu\nu}\gamma^2\gamma^0\bar{\psi}^T.$$

Note that

$$\gamma^2\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix},$$

and that

$$\gamma^2\gamma^0\begin{pmatrix} \sigma^i & 0 \\ 0 & \pm\sigma^i \end{pmatrix}\gamma^2\gamma^0 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}\begin{pmatrix} \sigma^i & 0 \\ 0 & \pm\sigma^i \end{pmatrix}\begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}\begin{pmatrix} 0 & \mp\sigma^i\sigma^2 \\ \sigma^i\sigma^2 & 0 \end{pmatrix} = -\begin{pmatrix} \sigma^i & 0 \\ 0 & \pm\sigma^i \end{pmatrix},$$

so

$$\begin{aligned}
C\bar{\psi}\sigma^{0i}\psi C &= \psi^T\sigma^{0i}\bar{\psi}^T = -\bar{\psi}\sigma^{0i}\psi, \\
C\bar{\psi}\sigma^{i0}\psi C &= \psi^T\sigma^{i0}\bar{\psi}^T = -\bar{\psi}\sigma^{i0}\psi, \\
C\bar{\psi}\sigma^{ij}\psi C &= \psi^T\sigma^{ij}\bar{\psi}^T = -\bar{\psi}\sigma^{ij}\psi,
\end{aligned}$$

where we have used anticommutation of fermions [1, p. 70]. In summary,

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = \begin{cases} 0 & \mu = \nu = 0; \\ -\bar{\psi}\sigma^{ij}\psi & \text{otherwise.} \end{cases}$$

4(b) Let $\phi(x)$ be a complex-valued Klein-Gordon field, such as we considered in Problem 2.2. Find unitary operators P, C, and an antiunitary operator T (all defined in terms of their action on the annihilation operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ for the Klein-Gordon particles and antiparticles) that give the following transformations of the Klein-Gordon field:

$$P\phi(t,\mathbf{x})P = \phi(t,-\mathbf{x}), \qquad T\phi(t,\mathbf{x})T = \phi(-t,\mathbf{x}), \qquad C\phi(t,\mathbf{x})C = \phi^*(t,\mathbf{x}).$$

Find the transformation properties of the components of the current

$$J^\mu = i(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi)$$

under P, C, and T.

Solution. Referring to Peskin & Schroeder (2.47) and 2(b) of Homework 1, the complex-valued Klein-Gordon field can be written

$$\begin{aligned}
\phi(t,\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{ip\cdot x} + b^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right)\Big|_{p_0=E_{\mathbf{p}}}, \\
\phi^*(t,\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{ip\cdot x} + a^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right)\Big|_{p_0=E_{\mathbf{p}}}.
\end{aligned}$$

From here the evaluation at $p_0 = E_{\mathbf{p}}$ will be understood.

Beginning with P, adapting Eq. (3.123) to the spin-0 Klein-Gordon field,

$$Pa_{\mathbf{p}}P = a_{-\mathbf{p}}, \qquad Pb_{\mathbf{p}}P = b_{-\mathbf{p}},$$

where we choose the phase to be 1 without loss of generality [1, p. 66]. These imply

$$Pa^\dagger_{\mathbf{p}}P = a^\dagger_{-\mathbf{p}}, \qquad Pb^\dagger_{\mathbf{p}}P = b^\dagger_{-\mathbf{p}}.$$

Then

$$\begin{aligned}
P\phi(t,\mathbf{x})P &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}P\left(a_{\mathbf{p}}e^{ip\cdot x} + b^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(Pa_{\mathbf{p}}Pe^{ip\cdot x} + Pb^\dagger_{\mathbf{p}}Pe^{-ip\cdot x}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{-\mathbf{p}}e^{ip\cdot x} + b^\dagger_{-\mathbf{p}}e^{-ip\cdot x}\right).
\end{aligned}$$

Changing variables to $\tilde{p} = (p_0, -\mathbf{p})$ [1, p. 65],

$$P\phi(t,\mathbf{x})P = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{\mathbf{p}}}}}\left(a_{\tilde{\mathbf{p}}}e^{i\tilde{p}\cdot x} + b^\dagger_{\tilde{\mathbf{p}}}e^{-i\tilde{p}\cdot x}\right) = \phi(t,-\mathbf{x})$$

as required.

For T, we adapt Eq. (3.138) as follows:

$$Ta_{\mathbf{p}}T = a_{-\mathbf{p}}, \qquad Tb_{\mathbf{p}}T = b_{-\mathbf{p}},$$

and

$$Ta^\dagger_{\mathbf{p}}T = a^\dagger_{-\mathbf{p}}, \qquad Tb^\dagger_{\mathbf{p}}T = b^\dagger_{-\mathbf{p}}.$$

Then

$$\begin{aligned}
T\psi(t,\mathbf{x})T &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}T\left(a_{\mathbf{p}}e^{ip\cdot x} + b^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right)T = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(Ta_{\mathbf{p}}Te^{-ip\cdot x} + Tb^\dagger_{\mathbf{p}}Te^{ip\cdot x}\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{-\mathbf{p}}e^{-ip\cdot x} + a_{-\mathbf{p}}e^{ip\cdot x}\right) = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{\mathbf{p}}}}}\left(a_{\tilde{\mathbf{p}}}e^{-i\tilde{p}\cdot x} + a_{\tilde{\mathbf{p}}}e^{i\tilde{p}\cdot x}\right)\\
&= \phi(-t,\mathbf{x}),
\end{aligned}$$

where we have used the antiunitarity of T and the fact that $\tilde{p}\cdot(t,-\mathbf{x}) = -\tilde{p}\cdot(-t,\mathbf{x})$ [1, p. 69].

For C, we adapt Eq. (3.143) as follows:

$$Ca_{\mathbf{p}}C = b_{\mathbf{p}}, \qquad Cb_{\mathbf{p}}C = a_{\mathbf{p}},$$

which imply

$$Ca^\dagger_{\mathbf{p}}C = b^\dagger_{\mathbf{p}}, \qquad Cb^\dagger_{\mathbf{p}}C = a^\dagger_{\mathbf{p}}.$$

So

$$\begin{aligned}
C\psi(t,\mathbf{x})C &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}C\left(a_{\mathbf{p}}e^{ip\cdot x} + b^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right)C = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(Ca_{\mathbf{p}}Ce^{ip\cdot x} + Cb^\dagger_{\mathbf{p}}Ce^{-ip\cdot x}\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{ip\cdot x} + a^\dagger_{\mathbf{p}}e^{-ip\cdot x}\right)\\
&= \phi^*(t,\mathbf{x})
\end{aligned}$$

as required.

For the current, note that

$$P\phi^*(t,\mathbf{x})P = \phi^*(t,-\mathbf{x}), \qquad T\phi^*(t,\mathbf{x})T = \phi^*(-t,\mathbf{x}), \qquad C\phi^*(t,\mathbf{x})C = \phi(t,\mathbf{x}),$$

and that

$$P\partial_\mu P = (-1)^\mu\partial_\mu, \qquad T\partial_\mu T = -(-1)^\mu\partial_\mu C\partial_\mu C \qquad = \partial_\mu,$$

where $(-1)^0 = 1$ and $(-1)^i = -1$ for $i \in \{1,2,3\}$ [1, p. 71]. Then we have

$$\begin{aligned}
PJ^\mu(t,\mathbf{x})P &= iP[\phi^*(t,\mathbf{x})\partial^\mu\phi(t,\mathbf{x}) - \partial^\mu\phi^*(t,\mathbf{x})\phi(t,\mathbf{x})]P\\
&= i[P\phi^*(t,\mathbf{x})PP\partial^\mu PP\phi(t,\mathbf{x})P - P\partial^\mu PP\phi^*(t,\mathbf{x})P\phi(t,\mathbf{x})P]\\
&= i(-1)^\mu[\phi^*(t,-\mathbf{x})\partial^\mu\phi(t,-\mathbf{x}) - \partial^\mu\phi^*(t,-\mathbf{x})\phi(t,-\mathbf{x})]\\
&= (-1)^\mu J^\mu(t,-\mathbf{x}),
\end{aligned}$$

$$\begin{aligned}
TJ^\mu(t,\mathbf{x})T &= -iT[\phi^*(t,\mathbf{x})\partial^\mu\phi(t,\mathbf{x}) - \partial^\mu\phi^*(t,\mathbf{x})\phi(t,\mathbf{x})]T \\
&= -i[T\phi^*(t,\mathbf{x})TT\partial^\mu TT\phi(t,\mathbf{x})T - T\partial^\mu TT\phi^*(t,\mathbf{x})TT\phi(t,\mathbf{x})T] \\
&= i(-1)^\mu[\phi^*(-t,\mathbf{x})\partial^\mu\phi(-t,\mathbf{x}) - \partial^\mu\phi^*(-t,\mathbf{x})\phi(-t,\mathbf{x})] \\
&= (-1)^\mu J^\mu(-t,\mathbf{x}),
\end{aligned}$$

$$\begin{aligned}
CJ^\mu(t,\mathbf{x})C &= iC[\phi^*(t,\mathbf{x})\partial^\mu\phi(t,\mathbf{x}) - \partial^\mu\phi^*(t,\mathbf{x})\phi(t,\mathbf{x})]C \\
&= i[C\phi^*(t,\mathbf{x})CC\partial^\mu CC\phi(t,\mathbf{x})C - C\partial^\mu CC\phi^*(t,\mathbf{x})CC\phi(t,\mathbf{x})C] \\
&= i[\phi(t,\mathbf{x})\partial^\mu\phi^*(t,\mathbf{x}) - \partial^\mu\phi(t,\mathbf{x})\phi^*(t,\mathbf{x})] = -i[\phi^*(t,\mathbf{x})\partial^\mu\phi(t,\mathbf{x}) - \partial^\mu\phi^*(t,\mathbf{x})\phi(t,\mathbf{x})] \\
&= -J^\mu(t,\mathbf{x}),
\end{aligned}$$

where we have used $[\phi(\mathbf{x}),\phi^*(\mathbf{y})] = 0$ from 2(b) of Homework 1.

4(c) Show that any Hermitian Lorentz-scalar local operator built from $\psi(x)$, $\phi(x)$, and their conjugates has $CPT = +1$.

Solution. The only Hermitian Lorentz-scalar local operator that can be built from these $\psi(x)$ and its conjugate is $\bar\psi\psi$. From $\psi(x)$ and its conjugate, we have $\phi^*\phi = \phi\phi^*$ since $[\phi(\mathbf{x}),\phi^*(\mathbf{y})] = 0$. We can combine these to build the Hermitian Lorentz-scalar local operator

$$(\bar\psi\psi)^N(\phi^*\phi)^M,$$

where N and M are arbitrary integers. Since

$$[\psi(\mathbf{x}),\phi(\mathbf{y})] = [\bar\psi(\mathbf{x}),\phi(\mathbf{y})] = [\psi(\mathbf{x}),\phi^*(\mathbf{y})] = [\bar\psi(\mathbf{x}),\phi^*(\mathbf{y})] = 0,$$

other orderings of the fields are equivalent as long as $\bar\psi$ and ψ are in the correct order (for if they are not, the object is not a Lorentz scalar).

We know from the table in Peskin & Schroder that $\bar\psi\psi$ has $CPT = +1$ [1, p. 71]. Using the results of 4(b),

$$\begin{aligned}
CPT\phi^*(t,\mathbf{x})\phi(t,\mathbf{x})TPC = CP\phi^*(-t,\mathbf{x})\phi(-t,\mathbf{x})PC = C\phi^*(-t,-\mathbf{x})\phi(-t,-\mathbf{x})C = \phi(-t,-\mathbf{x})\phi^*(-t,-\mathbf{x}) \\
= \phi^*(-t,-\mathbf{x})\phi(-t,-\mathbf{x}),
\end{aligned}$$

so $\phi^*\phi$ has $CPT = +1$ also. Finally, $(\bar\psi\psi)^N(\phi^*\phi)^M$ has

$$CPT = (+1)^N(+1)^M = (+1)(+1) = +1$$

as we wanted to show. □

References

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