

1 Problem 1

A particle of mass m is moving on a sphere of radius a . Its wave function is given by $\psi(\theta, \phi)$ where θ and ϕ parameterize the sphere $(x, y, z) = a(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The Hamiltonian of the system is $H = \mathbf{L}^2/2ma^2$, where \mathbf{L}^2 is the square of the angular momentum operator, and is given by

$$\mathbf{L}^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right).$$

The eigenfunctions of H are spherical harmonics Y_m^l with energies

$$E_l = \frac{\hbar^2 l(l+1)}{2ma^2}. \quad (1)$$

1.1 The wave function of the system at $t = 0$ is given by

$$\psi(\theta, \phi, 0) = A \sin^2 \theta \cos^2 \phi,$$

where A is a constant. This wave function can be expanded in spherical harmonics:

$$\psi(\theta, \phi, 0) = \sum_{l,m} a_m^l Y_m^l(\theta, \phi).$$

Find all nonzero a_m^l .

Solution. We will look for nonzero a_m^l by comparing the θ and ϕ dependence of Y_m^l and $\psi(\theta, \phi, 0)$. From (3.6.36) and (3.6.37) in Sakurai, the spherical harmonic functions are given by

$$Y_m^l(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} \frac{e^{im\phi}}{\sin^m \theta} \frac{d^{l-m}(\sin \theta)^{2l}}{d(\cos \theta)^{l-m}}, \quad Y_{-m}^l(\theta, \phi) = (-1)^m Y_m^{l*}(\theta, \phi)$$

for $m \geq 0$. Beginning with the ϕ dependence of $\psi(\theta, \phi, 0)$, note that

$$\psi(\theta, \phi, 0) \propto \cos^2 \phi = \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right)^2 = \frac{1}{2} + \frac{e^{i2\phi}}{4} + \frac{e^{-i2\phi}}{4}, \quad (2)$$

which implies that the only nonzero a_m^l correspond to $m \in \{0, \pm 2\}$.

For the θ dependence, we have $\psi(\theta, \phi, 0) \propto \sin^2 \theta$. Looking at Y_m^l , note that $(\sin \theta)^{2l} = (1 - \cos^2 \theta)^l$, so

$$Y_m^l \propto \frac{1}{\sin^m \theta} \frac{d^{l-m}(1 - \cos^2 \theta)^l}{d(\cos \theta)^{l-m}}.$$

Plugging in $m = 0$ and the first few values of l ,

$$\begin{aligned} Y_0^0 &\propto \frac{d^0(1 - \cos^2 \theta)^0}{d(\cos \theta)^0} = 1, \\ Y_0^1 &\propto \frac{d(1 - \cos^2 \theta)}{d(\cos \theta)} = -2 \cos \theta, \\ Y_0^2 &\propto \frac{d^2(1 - 2 \cos^2 \theta + \cos^4 \theta)}{d(\cos \theta)^2} = \frac{d(-4 \cos \theta + 4 \cos^3 \theta)}{d(\cos \theta)} = -4 + 12 \cos^2 \theta = 8 - 12 \sin^2 \theta, \end{aligned}$$

so we know $a_0^1 = 0$. Inspecting the above, we deduce that Y_0^l with $l > 2$ contain mixed terms of $\sin \theta$ and $\cos \theta$ and higher powers of $\sin \theta$, so $a_0^l = 0$ for $l > 2$.

Plugging in $m = \pm 2$ and $l = 2$,

$$Y_{\pm 2}^2 \propto \frac{1}{\sin^2 \theta} \frac{d^0(1 - \cos^2 \theta)^2}{d(\cos \theta)^0} = \frac{\sin^4 \theta}{\sin^2 \theta} = \sin^2 \theta.$$

Again, by inspection $Y_{\pm 2}^l$ with $l > 2$ contain terms that are not in $\psi(\theta, \phi, 0)$, so $a_{\pm 2}^l = 0$ for $l > 2$ as well.

Thus, only a_0^0 , a_0^2 , and $a_{\pm 2}^2$ are nonzero; that is,

$$\psi(\theta, \phi, 0) = a_0^0 Y_0^0 + a_0^2 Y_0^2 + a_2^2 Y_2^2 + a_{-2}^2 Y_{-2}^2.$$

The relevant spherical harmonics are

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_0^2 = \sqrt{\frac{5}{16\pi}}(2 - 3\sin^2 \theta), \quad Y_{\pm 2}^2 = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}. \quad (3)$$

Expanding out $\psi(\theta, \phi, 0)$ as in (2),

$$\psi(\theta, \phi, 0) = \frac{A}{2} \sin^2 \theta + \frac{A}{4} \sin^2 \theta e^{i2\phi} + \frac{A}{4} \sin^2 \theta e^{-i2\phi}.$$

Then we can deduce the nonzero a_m^l :

$$\begin{aligned} \frac{A}{4} \sin^2 \theta e^{\pm i2\phi} &= a_{\pm 2}^2 \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi} \implies a_{\pm 2}^2 = A \sqrt{\frac{2\pi}{15}}, \\ \frac{A}{2} \sin^2 \theta &= a_0^0 \sqrt{\frac{1}{4\pi}} + a_0^2 \sqrt{\frac{5}{16\pi}}(2 - 3\sin^2 \theta) \implies a_0^2 = -\frac{2}{3} A \sqrt{\frac{\pi}{5}}, \quad a_0^0 = \frac{2}{3} A \sqrt{\pi}. \end{aligned}$$

1.2 Now consider the wave function at nonzero time t . Use your results from 1.1 and the expressions for spherical harmonics to derive an explicit expression in terms of sines and cosines of θ and ϕ for $\psi(\theta, \phi, t)$.

Solution. From 1.1, we have

$$\psi(\theta, \phi, 0) = \frac{2}{3} A \sqrt{\pi} Y_0^0 - \frac{2}{3} A \sqrt{\frac{\pi}{5}} Y_0^2 + A \sqrt{\frac{2\pi}{15}} Y_2^2 + A \sqrt{\frac{2\pi}{15}} Y_{-2}^2. \quad (4)$$

We can evaluate the time evolution for each spherical harmonic term in (4) individually, and sum them up to find $\psi(\theta, \phi, t)$:

$$\psi(\theta, \phi, t) = U(t)\psi(\theta, \phi, 0) = \frac{2}{3} A \sqrt{\pi} U(t)Y_0^0 - \frac{2}{3} A \sqrt{\frac{\pi}{5}} U(t)Y_0^2 + A \sqrt{\frac{2\pi}{15}} U(t)Y_2^2 + A \sqrt{\frac{2\pi}{15}} U(t)Y_{-2}^2$$

The time evolution operator is given by $U(t) = e^{-iHt/\hbar}$. From (1), the relevant eigenvalues are

$$E_0 = 0, \quad E_2 = 3 \frac{\hbar^2}{ma^2},$$

so

$$U(t)Y_0^0 = \exp\left(-\frac{i}{\hbar} E_0 t\right) Y_0^0 = Y_0^0, \quad U(t)Y_m^2 = \exp\left(-\frac{i}{\hbar} E_2 t\right) Y_m^2 = \exp\left(-3i \frac{\hbar}{ma^2} t\right) Y_m^2.$$

Then, using the explicit Y_m^l from (3),

$$\begin{aligned}
 \psi(\theta, \phi, t) &= \frac{2}{3}A\sqrt{\pi}\sqrt{\frac{1}{4\pi}} - \frac{2}{3}A\sqrt{\frac{\pi}{5}}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sqrt{\frac{5}{16\pi}}(2 - 3\sin^2\theta) + A\sqrt{\frac{2\pi}{15}}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi} \\
 &\quad + A\sqrt{\frac{2\pi}{15}}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\phi} \\
 &= \frac{A}{3} - \frac{A}{6}\exp\left(-3i\frac{\hbar}{ma^2}t\right)(2 - 3\sin^2\theta) + \frac{A}{4}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta e^{2i\phi} + \frac{A}{4}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta e^{-2i\phi} \\
 &= \frac{A}{3} - \frac{A}{3}\exp\left(-3i\frac{\hbar}{ma^2}t\right) + \frac{A}{2}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta + \frac{A}{2}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta\cos 2\phi \\
 &= \frac{A}{3}\left[1 - \exp\left(-3i\frac{\hbar}{ma^2}t\right)\right] + A\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta\cos^2\phi. \tag{5}
 \end{aligned}$$

1.3 Use your results from 1.2 to derive expressions for the expected values of L_x , L_y , and L_z as functions of time.

Solution. From (3.6.23) in Sakurai, $\langle\theta, \phi|l, m\rangle = Y_m^l(\theta, \phi)$ and therefore $\psi(\theta, \phi, t) = \langle\theta, \phi|\psi(t)\rangle$. Using the result of 1.2, this implies

$$|\psi(t)\rangle = a_0^0|0, 0\rangle + a_0^2\exp\left(-3i\frac{\hbar}{ma^2}t\right)|2, 0\rangle + a_2^2\exp\left(-3i\frac{\hbar}{ma^2}t\right)|2, 2\rangle + a_{-2}^2\exp\left(-3i\frac{\hbar}{ma^2}t\right)|2, -2\rangle.$$

Then the time-dependent expectation value of an operator O is given by

$$\begin{aligned}
 \langle\psi(t)|O|\psi(t)\rangle &= a_0^{02}\langle 0, 0|O|0, 0\rangle + a_0^0a_0^2U(t)\langle 0, 0|O|2, 0\rangle + a_0^0a_2^2U(t)\langle 0, 0|O|2, 2\rangle + a_0^0a_{-2}^2U(t)\langle 0, 0|O|2, -2\rangle \\
 &\quad + a_0^0a_0^2U^\dagger(t)\langle 2, 0|O|0, 0\rangle + a_0^{22}\langle 2, 0|O|2, 0\rangle + a_0^2a_2^2\langle 2, 0|O|2, 2\rangle + a_0^2a_{-2}^2\langle 2, 0|O|2, -2\rangle \\
 &\quad + a_0^0a_2^2U^\dagger(t)\langle 2, 2|O|0, 0\rangle + a_0^2a_2^2\langle 2, 2|O|2, 0\rangle + a_2^{22}\langle 2, 2|O|2, 2\rangle + a_2^2a_{-2}^2\langle 2, 2|O|2, -2\rangle \\
 &\quad + a_0^0a_{-2}^2U^\dagger(t)\langle 2, -2|O|0, 0\rangle + a_0^2a_{-2}^2\langle 2, -2|O|2, 0\rangle + a_2^2a_{-2}^2\langle 2, -2|O|2, 2\rangle + a_{-2}^{22}\langle 2, -2|O|2, -2\rangle,
 \end{aligned}$$

where $U(t) = e^{-3i\hbar t/ma^2}$ and $U^\dagger(t) = e^{3i\hbar t/ma^2}$.

From the results of 3.3 on the previous homework,

$$\begin{aligned}
 0 &= \langle 2, -2|L_i|2, -2\rangle = \langle 2, -2|L_i|2, 0\rangle = \langle 2, -2|L_i|2, 2\rangle \\
 &= \langle 2, 0|L_i|2, -2\rangle = \langle 2, 0|L_i|2, 0\rangle = \langle 2, 0|L_i|2, 2\rangle \\
 &= \langle 2, 2|L_i|2, -2\rangle = \langle 2, 2|L_i|2, 0\rangle = \langle 2, 2|L_i|2, 2\rangle
 \end{aligned}$$

for $i \in \{x, y, z\}$. For $(l, m) = (0, 0)$, a similar procedure to the one used for 3.3 yields

$$\begin{aligned}
 \langle l', m'|L_x|0, 0\rangle &= \langle 0, 0|L_x|l', m'\rangle = \frac{\hbar}{2}\delta_{0,l'}\delta_{1,m'}\sqrt{l^2+l} = 0, \\
 \langle l', m'|L_y|0, 0\rangle &= \langle 0, 0|L_y|l', m'\rangle = -\frac{i\hbar}{2}\delta_{0,l'}\delta_{1,m'}\sqrt{l^2+l} = 0, \\
 \langle l', m'|L_z|0, 0\rangle &= \langle 0, 0|L_z|l', m'\rangle = 0,
 \end{aligned}$$

where the last result comes from the eigenvalues of L_z being $\hbar m$. Thus, we find

$$\langle\psi(t)|L_x|\psi(t)\rangle = \langle\psi(t)|L_y|\psi(t)\rangle = \langle\psi(t)|L_z|\psi(t)\rangle = 0.$$

2 Problem 2

In this problem, we are working in the basis that diagonalizes the z component of the spin.

2.1 Consider $\mathbf{n} \cdot \boldsymbol{\sigma}$, where \mathbf{n} is a three-dimensional unit vector and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ represents the Pauli matrices. Compute the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $|\lambda_1\rangle, |\lambda_2\rangle$ of $\mathbf{n} \cdot \boldsymbol{\sigma}$. Use them to obtain the spectrum decomposition of $\mathbf{n} \cdot \boldsymbol{\sigma}$.

Solution. From (3.2.32) in Sakurai, the Pauli matrices are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (6)$$

Let $\mathbf{n} = (n_x, n_y, n_z)$. Then

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \begin{bmatrix} 0 & n_x \\ n_x & 0 \end{bmatrix} + i \begin{bmatrix} 0 & -n_y \\ n_y & 0 \end{bmatrix} + \begin{bmatrix} n_z & 0 \\ 0 & -n_z \end{bmatrix} = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}. \quad (7)$$

The eigenvalues of $\mathbf{n} \cdot \boldsymbol{\sigma}$ are the solutions to the characteristic polynomial equation

$$0 = \det(\mathbf{n} \cdot \boldsymbol{\sigma} - \lambda I) = \begin{vmatrix} n_z - \lambda & n_x - in_y \\ n_x + in_y & -(n_z + \lambda) \end{vmatrix} = -(n_z - \lambda)(n_z + \lambda) - (n_x - in_y)(n_x + in_y) = \lambda^2 - n_x^2 - n_y^2 - n_z^2.$$

Since $|\mathbf{n}|^2 = n_x^2 + n_y^2 + n_z^2$, we have $\lambda = \pm|\mathbf{n}| = \pm 1$. Let $\lambda_1 = 1$ and $\lambda_2 = -1$.

For the eigenvectors, let $|\lambda_+\rangle$ and $|\lambda_-\rangle$ be the non-normalized eigenkets corresponding to $|\lambda_1\rangle$ and $|\lambda_2\rangle$, respectively. Let the elements of $|\lambda_+\rangle$ be $\lambda_{+1}, \lambda_{+2}$ and the elements of $|\lambda_-\rangle$ be $\lambda_{-1}, \lambda_{-2}$. Then

$$\begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} \begin{bmatrix} \lambda_{\pm 1} \\ \lambda_{\pm 2} \end{bmatrix} = \pm \begin{bmatrix} \lambda_{\pm 1} \\ \lambda_{\pm 2} \end{bmatrix},$$

which is equivalent to the system of equations

$$n_z \lambda_{\pm 1} + (n_x - in_y) \lambda_{\pm 2} = \pm \lambda_{\pm 1}, \quad (n_x + in_y) \lambda_{\pm 1} - n_z \lambda_{\pm 2} = \pm \lambda_{\pm 2}.$$

We may fix $\lambda_{\pm 2} = n_x + in_y$ without loss of generality. Then $\lambda_{\pm 1} = n_z \pm 1$, so

$$|\lambda_+\rangle = \begin{bmatrix} n_z + 1 \\ n_x + in_y \end{bmatrix}, \quad |\lambda_-\rangle = \begin{bmatrix} n_z - 1 \\ n_x + in_y \end{bmatrix}.$$

For the normalization,

$$\begin{aligned} \langle \lambda_+ | \lambda_+ \rangle &= (n_z + 1)^2 + (n_x - in_y)(n_x + in_y) = n_z^2 + 2n_z + 1 + n_x^2 + n_y^2 = 2(1 + n_z), \\ \langle \lambda_- | \lambda_- \rangle &= (n_z - 1)^2 + (n_x - in_y)(n_x + in_y) = n_z^2 - 2n_z + 1 + n_x^2 + n_y^2 = 2(1 - n_z), \end{aligned}$$

so the normalized eigenkets are

$$|\lambda_1\rangle = \frac{1}{\sqrt{2(1 + n_z)}} \begin{bmatrix} n_z + 1 \\ n_x + in_y \end{bmatrix}, \quad |\lambda_2\rangle = \frac{1}{\sqrt{2(1 - n_z)}} \begin{bmatrix} n_z - 1 \\ n_x + in_y \end{bmatrix}.$$

Finally, the spectrum decomposition of $\mathbf{n} \cdot \boldsymbol{\sigma}$ is

$$\begin{aligned} \mathbf{n} \cdot \boldsymbol{\sigma} &= \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = |\lambda_1\rangle \langle \lambda_1| - |\lambda_2\rangle \langle \lambda_2| \\ &= \frac{1}{2(1 + n_z)} \begin{bmatrix} n_z + 1 \\ n_x + in_y \end{bmatrix} \begin{bmatrix} n_z + 1 & n_x - in_y \end{bmatrix} - \frac{1}{2(1 - n_z)} \begin{bmatrix} n_z - 1 \\ n_x + in_y \end{bmatrix} \begin{bmatrix} n_z - 1 & n_x - in_y \end{bmatrix}. \end{aligned} \quad (8)$$

2.2 Express the matrix $e^{i\alpha\mathbf{n}\cdot\boldsymbol{\sigma}}$ in terms of σ_x , σ_y , and σ_z and the 2×2 unit matrix.

Solution. Denote the 2×2 unit matrix as I . From (7), note that

$$(\mathbf{n} \cdot \boldsymbol{\sigma})^2 = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}^2 = \begin{bmatrix} n_x^2 + n_y^2 + n_z^2 & (n_z - n_z)(n_x - in_y) \\ (n_z - n_z)(n_x + in_y) & n_x^2 + n_y^2 + n_z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \quad (9)$$

Using the power series expansion,

$$\begin{aligned} e^{i\alpha\mathbf{n}\cdot\boldsymbol{\sigma}} &= \sum_{n=0}^{\infty} \frac{(i\alpha\mathbf{n}\cdot\boldsymbol{\sigma})^n}{n!} = i\alpha\mathbf{n}\cdot\boldsymbol{\sigma} - \frac{\alpha^2}{2}I - \frac{i\alpha^3}{6}\mathbf{n}\cdot\boldsymbol{\sigma} + \frac{\alpha^4}{24}I + \cdots = i\mathbf{n}\cdot\boldsymbol{\sigma} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\alpha^{2n-1}}{(2n-1)!} + I \sum_{n=0}^{\infty} \frac{(-1)^n\alpha^{2n}}{(2n)!} \\ &= i\sin\alpha\mathbf{n}\cdot\boldsymbol{\sigma} + \cos\alpha I = i\sin\alpha(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z) + \cos\alpha I. \end{aligned}$$

2.3 Consider two spin 1/2 degrees of freedom. The total spin is $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. Consider the state $|j, m\rangle = |1, 1\rangle$, where j is the total spin and m is the z component of the total spin. Compute $e^{-i\theta S_y/\hbar}|1, 1\rangle$ and express it as a linear superposition of $|j, m\rangle$.

Solution. In the S_z eigenbasis, which we will call $\{|s_z\rangle\}$ where $s_z \in \{\uparrow = 1/2, \downarrow = -1/2\}$,

$$S_{y1} \sim S_{y2} = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

In the basis $\{|s_{z1} s_{z2}\rangle\}$,

$$S_y = S_{y2} \otimes I + I \otimes S_{y2} = \frac{\hbar}{2} \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} + \frac{\hbar}{2} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} = \frac{\hbar}{2} \begin{matrix} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \\ \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix} \end{matrix},$$

where we have labeled the columns corresponding to (s_{z1}, s_{z2}) .

We will solve the problem in the basis $\{|s_{z1} s_{z2}\rangle\}$, and then express it in terms of $|j, m\rangle$. Proceeding similarly to 2.2, note that

$$\begin{aligned} S_y^2 &= \frac{\hbar^2}{4} \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix}^2 = \frac{\hbar^2}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \equiv \hbar K, \\ S_y^3 &= \frac{\hbar^3}{8} \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix}^3 = \frac{\hbar^3}{2} \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix} = \hbar^2 S_y, \\ S_y^4 &= \frac{\hbar^4}{16} \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix}^4 = \frac{\hbar^4}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \hbar^3 K, \end{aligned}$$

where we have defined K .

Then, once more using the power series expansion,

$$\begin{aligned} e^{-i\theta S_y/\hbar} &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i\theta S_y}{\hbar} \right)^n = -\frac{i\theta}{\hbar} S_y - \frac{\theta^2}{2\hbar} K + \frac{i\theta^3}{6\hbar} S_y + \frac{\theta^4}{24\hbar} K + \cdots \\ &= -\frac{i}{\hbar} S_y \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \theta^{2n-1}}{(2n-1)!} + \frac{1}{\hbar} K \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} = \frac{1}{\hbar} (-i \sin \theta S_y + \cos \theta K). \end{aligned}$$

From (3.7.15) in Sakurai,

$$|1, 1\rangle = |\uparrow\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |1, 0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad |0, 0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad |1, -1\rangle = |\downarrow\downarrow\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note also that

$$|\uparrow\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \frac{|1, 0\rangle + |0, 0\rangle}{\sqrt{2}}, \quad |\downarrow\uparrow\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \frac{|1, 0\rangle - |0, 0\rangle}{\sqrt{2}}.$$

Finally,

$$\begin{aligned} e^{-i\theta S_y/\hbar} |1, 1\rangle &= -\frac{i}{\hbar} \sin \theta S_y |1, 1\rangle + \frac{1}{\hbar} \cos \theta K |1, 1\rangle \\ &= -\frac{i}{2} \sin \theta \begin{bmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \cos \theta \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{i}{2} \sin \theta \begin{bmatrix} 0 \\ i \\ i \\ 0 \end{bmatrix} + \frac{1}{2} \cos \theta \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \cos \theta \\ \sin \theta \\ \sin \theta \\ -\cos \theta \end{bmatrix} = \frac{1}{2} \left(\cos \theta |1, 1\rangle + \sin \theta \frac{|1, 0\rangle + |0, 0\rangle}{\sqrt{2}} + \sin \theta \frac{|1, 0\rangle - |0, 0\rangle}{\sqrt{2}} - \cos \theta |1, -1\rangle \right) \\ &= \frac{\cos \theta}{2} |1, 1\rangle + \frac{\sin \theta}{\sqrt{2}} |1, 0\rangle - \frac{\cos \theta}{2} |1, -1\rangle. \end{aligned}$$

3 Problem 3

Consider a spin 1/2 state $|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the S_z eigenstates with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

3.1 Consider the operator $\rho = |\psi\rangle\langle\psi|$. Write down the matrix elements of ρ in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

Solution. From the definition of $|\psi\rangle$,

$$\langle\uparrow|\psi\rangle = c_1, \quad \langle\psi|\uparrow\rangle = c_1^*, \quad \langle\downarrow|\psi\rangle = c_2, \quad \langle\psi|\downarrow\rangle = c_2^*.$$

Using these,

$$\langle\uparrow|\rho|\uparrow\rangle = \langle\uparrow|\psi\rangle\langle\psi|\uparrow\rangle = c_1 c_1^* = |c_1|^2, \quad \langle\uparrow|\rho|\downarrow\rangle = \langle\uparrow|\psi\rangle\langle\psi|\downarrow\rangle = c_1 c_2^*,$$

$$\langle \downarrow | \rho | \uparrow \rangle = \langle \downarrow | \psi \rangle \langle \psi | \uparrow \rangle = c_1^* c_2,$$

$$\langle \downarrow | \rho | \downarrow \rangle = \langle \downarrow | \psi \rangle \langle \psi | \downarrow \rangle = c_2 c_2^* = |c_2|^2.$$

In matrix form,

$$\rho = \begin{bmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{bmatrix}. \quad (10)$$

3.2 In the S_z eigenbasis, express ρ by using the Pauli matrices. That is, write ρ as

$$\rho = \frac{s_0}{2} I + \frac{1}{2} \mathbf{s} \cdot \boldsymbol{\sigma}, \quad (11)$$

and express s_0, s_1, s_2, s_3 in terms of c_1 and c_2 .

Solution. Expanding (11) using the Pauli matrices given by (6),

$$\rho = \frac{1}{2} \begin{bmatrix} s_0 & 0 \\ 0 & s_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & s_1 \\ s_1 & 0 \end{bmatrix} + \frac{i}{2} \begin{bmatrix} 0 & -s_2 \\ s_2 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} s_3 & 0 \\ 0 & -s_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} s_0 + s_3 & s_1 - is_2 \\ s_1 + is_2 & s_0 - s_3 \end{bmatrix}.$$

Matching up the matrix elements with those in (10), we have a system of four equations:

$$\frac{s_0 + s_3}{2} = |c_1|^2 \quad \frac{s_1 - is_2}{2} = c_1 c_2^* \quad \frac{s_1 + is_2}{2} = c_1^* c_2 \quad \frac{s_0 - s_3}{2} = |c_2|^2.$$

Solving these for s_0, s_1, s_2, s_3 ,

$$s_0 = \frac{s_0 + s_3}{2} + \frac{s_0 - s_3}{2} = |c_1|^2 + |c_2|^2, \quad s_1 = \frac{s_1 - is_2}{2} + \frac{s_1 + is_2}{2} = c_1 c_2^* + c_1^* c_2, \quad (12)$$

$$s_2 = i \left(\frac{s_1 - is_2}{2} - \frac{s_1 + is_2}{2} \right) = i(c_1 c_2^* - c_1^* c_2), \quad s_3 = \frac{s_0 + s_3}{2} - \frac{s_0 - s_3}{2} = |c_1|^2 - |c_2|^2. \quad (13)$$

Note that s_0 is the norm of $|\psi\rangle$, so $s_0 = 1$ assuming $|\psi\rangle$ is normalized. Then (11) can be written

$$\begin{aligned} \rho &= \frac{|c_1|^2 + |c_2|^2}{2} I + \frac{c_1 c_2^* + c_1^* c_2}{2} \sigma_x + i \frac{c_1 c_2^* - c_1^* c_2}{2} \sigma_y + \frac{|c_1|^2 - |c_2|^2}{2} \sigma_z \\ &= \frac{1}{2} I + \frac{c_1 c_2^* + c_1^* c_2}{2} \sigma_x + i \frac{c_1 c_2^* - c_1^* c_2}{2} \sigma_y + \frac{|c_1|^2 - |c_2|^2}{2} \sigma_z. \end{aligned}$$

3.3 Compute ρ^2 and \mathbf{s}^2 .

Solution. Using the representation of ρ in (11),

$$\rho^2 = \frac{s_0^2}{4} I^2 + \frac{s_0}{4} I(\mathbf{s} \cdot \boldsymbol{\sigma}) + \frac{s_0}{4} (\mathbf{s} \cdot \boldsymbol{\sigma}) I + \frac{1}{4} (\mathbf{s} \cdot \boldsymbol{\sigma})^2 = \frac{1}{4} (s_0^2 + |\mathbf{s}|^2) I + \frac{s_0}{2} \mathbf{s} \cdot \boldsymbol{\sigma} = \frac{1}{4} (1 + |\mathbf{s}|^2) I + \frac{1}{2} \mathbf{s} \cdot \boldsymbol{\sigma},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, and in the final equality we have assumed $|\psi\rangle$ is normalized. Using (12) and (13),

$$\begin{aligned} \mathbf{s}^2 &= \mathbf{s} \cdot \mathbf{s} = s_1^2 + s_2^2 + s_3^2 = (c_1 c_2^* + c_1^* c_2)^2 - (c_1 c_2^* - c_1^* c_2)^2 + (|c_1|^2 - |c_2|^2)^2 \\ &= c_1^2 c_2^{*2} + 2|c_1|^2 |c_2|^2 + c_1^{*2} c_2^2 - c_1^2 c_2^{*2} + 2|c_1|^2 |c_2|^2 - c_1^{*2} c_2^2 + |c_1|^4 - 2|c_1|^2 |c_2|^2 + |c_2|^4 \\ &= |c_1|^4 + 2|c_1|^2 |c_2|^2 + |c_2|^4 = (|c_1|^2 + |c_2|^2)^2 \\ &= s_0^2 = 1. \end{aligned}$$

We can use this result to simplify the expression for ρ^2 :

$$\rho^2 = \frac{1}{2} s_0^2 I + \frac{s_0}{2} \mathbf{s} \cdot \boldsymbol{\sigma} = \rho.$$

3.4 Using the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle, \quad (14)$$

derive the equation of motion obeyed by ρ .

Solution. The adjoint of (14) is

$$-i\hbar \frac{\partial}{\partial t} \langle\psi| = \langle\psi| H \iff i\hbar \frac{\partial}{\partial t} \langle\psi| = -\langle\psi| H.$$

Then

$$i\hbar \frac{d\rho}{dt} = i\hbar \frac{d}{dt} |\psi\rangle\langle\psi| = H |\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| H = H\rho - \rho H = [H, \rho],$$

so the equation of motion is

$$\frac{d\rho}{dt} = \frac{i}{\hbar} [\rho, H].$$

3.5 When the Hamiltonian is given by

$$H = \frac{\hbar}{2} \mathbf{\Omega} \cdot \boldsymbol{\sigma}, \quad (15)$$

write down the equation of motion obeyed by $(\psi_\uparrow(t), \psi_\downarrow(t))$, where $\psi_\uparrow(t), \psi_\downarrow(t)$ are the (complex) expansion coefficients of the state ket

$$|\psi(t)\rangle = \psi_\uparrow(t) |\uparrow\rangle + \psi_\downarrow(t) |\downarrow\rangle.$$

Solution. Using (14),

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \frac{\hbar}{2} \mathbf{\Omega} \cdot \boldsymbol{\sigma} |\psi(t)\rangle \iff i\hbar \frac{d}{dt} (\psi_\uparrow(t) |\uparrow\rangle + \psi_\downarrow(t) |\downarrow\rangle) = \frac{\hbar}{2} \mathbf{\Omega} \cdot \boldsymbol{\sigma} (\psi_\uparrow(t) |\uparrow\rangle + \psi_\downarrow(t) |\downarrow\rangle),$$

which can be separated into two equations:

$$\frac{d}{dt} \psi_\uparrow(t) |\uparrow\rangle = -\frac{i}{2} \psi_\uparrow(t) \mathbf{\Omega} \cdot \boldsymbol{\sigma} |\uparrow\rangle, \quad \frac{d}{dt} \psi_\downarrow(t) |\downarrow\rangle = -\frac{i}{2} \psi_\downarrow(t) \mathbf{\Omega} \cdot \boldsymbol{\sigma} |\downarrow\rangle. \quad (16)$$

Generalizing the results of 2.1, let $\mathbf{\Omega} = \Omega \mathbf{n}$ where \mathbf{n} is a unit vector. Using (7),

$$\mathbf{\Omega} \cdot \boldsymbol{\sigma} |\uparrow\rangle = \Omega \begin{bmatrix} n_z \\ n_x + in_y \end{bmatrix} = \Omega_z |\uparrow\rangle + (\Omega_x + i\Omega_y) |\downarrow\rangle, \quad \mathbf{\Omega} \cdot \boldsymbol{\sigma} |\downarrow\rangle = \Omega \begin{bmatrix} n_x - in_y \\ -n_z \end{bmatrix} = (\Omega_x - i\Omega_y) |\uparrow\rangle - \Omega_z |\downarrow\rangle,$$

where $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$. Then, explicitly, we have the equations of motion

$$\frac{d}{dt} \psi_\uparrow(t) |\uparrow\rangle = -\frac{i}{2} \psi_\uparrow(t) [\Omega_z |\uparrow\rangle + (\Omega_x + i\Omega_y) |\downarrow\rangle], \quad \frac{d}{dt} \psi_\downarrow(t) |\downarrow\rangle = -\frac{i}{2} \psi_\downarrow(t) [(\Omega_x - i\Omega_y) |\uparrow\rangle - \Omega_z |\downarrow\rangle]. \quad (17)$$

This can also be written in matrix form in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis:

$$\frac{d}{dt} \begin{bmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{bmatrix} = -\frac{i}{2} \begin{bmatrix} \Omega_z & \Omega_x - i\Omega_y \\ \Omega_x + i\Omega_y & -\Omega_z \end{bmatrix} \begin{bmatrix} \psi_\uparrow(t) \\ \psi_\downarrow(t) \end{bmatrix}.$$

3.6 When the Hamiltonian is given by (15), write down the (Heisenberg) equation of motion obeyed by the spin operators \mathbf{S} .

Solution. From (3.2.35) in Sakurai,

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k.$$

Let $\mathbf{S} = (S_1, S_2, S_3)$ and $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$. Then the Heisenberg equation of motion is

$$\frac{dS_i}{dt} = -\frac{i}{\hbar}[S_i, H] = -\frac{i\hbar}{4}[\sigma_i, \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}] = -\frac{i\hbar}{4} \sum_{j=1}^3 \Omega_j [\sigma_i, \sigma_j] = \frac{\hbar}{2} \sum_{j=1}^3 \Omega_j \varepsilon_{ijk} \sigma_k.$$

Explicitly, the equations of motion are

$$\frac{dS_x}{dt} = \frac{\hbar}{2}(\Omega_y \sigma_z - \Omega_z \sigma_y), \quad \frac{dS_y}{dt} = \frac{\hbar}{2}(\Omega_z \sigma_x - \Omega_x \sigma_z), \quad \frac{dS_z}{dt} = \frac{\hbar}{2}(\Omega_x \sigma_y - \Omega_y \sigma_x).$$

While writing up these solutions, I consulted Sakurai's *Modern Quantum Mechanics* and Shankar's *Principles of Quantum Mechanics*.