Problem 1 (Jackson 11.3). Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}.$$

This is an alternative way to derive the parallel-velocity addition law.

Solution. The general expression for a Lorentz transformation in the x_1 direction is given by Jackson (11.16),

$$x_0' = \gamma(x_0 - \beta x_1), \qquad x_1' = \gamma(x_1 - \beta x_0), \qquad x_2' = x_2, \qquad x_3' = x_3. \tag{1}$$

Define β, β_1, and β_2 by

$$\beta = \frac{v}{c}, \qquad \beta_1 = \frac{v_1}{c}, \qquad \beta_2 = \frac{v_2}{c},$$

and define γ, γ_1, and γ_2 correspondingly.

First applying the boost corresponding to β_1 yields

$$x_0' = \gamma_1(x_0 - \beta_1 x_1), \qquad x_1' = \gamma_1(x_1 - \beta_1 x_0). \tag{2}$$

Successively applying the boost corresponding to β_2 yields

$$x_0'' = \gamma_2(x_0' - \beta_2 x_1'), \qquad x_1'' = \gamma_2(x_1' - \beta_2 x_0'). \tag{3}$$

Substituting (2) into (3) gives us

$$\begin{aligned} x_0'' &= \gamma_1\gamma_2[(x_0 - \beta_1 x_1) - \beta_2(x_1 - \beta_1 x_0)] = \gamma_1\gamma_2[(1+\beta_1\beta_2)x_0 - (\beta_1+\beta_2)x_1] \\ x_1'' &= \gamma_1\gamma_2[(x_1 - \beta_1 x_0) - \beta_2(x_0 - \beta_1 x_1)] = \gamma_1\gamma_2[(1+\beta_1\beta_2)x_1 - (\beta_1+\beta_2)x_0], \end{aligned}$$

or

$$x_0'' = (1+\beta_1\beta_2)\gamma_1\gamma_2\left(x_0 - \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}x_1\right), \qquad x_1'' = (1+\beta_1\beta_2)\gamma_1\gamma_2\left(x_1 - \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}x_0\right). \tag{4}$$

Note that

$$\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} = \frac{1}{c}\frac{v_1+v_2}{1+v_1v_2/c^2} = \frac{v}{c} = \beta,$$

and that

$$\begin{aligned} (1+\beta_1\beta_2)\gamma_1\gamma_2 &= \frac{1+\beta_1\beta_2}{\sqrt{(1-\beta_1^2)(1-\beta_2^2)}} = \sqrt{\frac{(1-\beta_1^2)(1-\beta_2^2)}{(1+\beta_1\beta_2)^2}}^{-1} = \sqrt{\frac{1-\beta_1^2-\beta_2^2+\beta_1^2\beta_2^2}{(1+\beta_1\beta_2)^2}}^{-1} \\ &= \sqrt{\frac{1+2\beta_1\beta_2+\beta_1^2\beta_2^2-(\beta_1^2+2\beta_1\beta_2+\beta_2^2)}{(1+\beta_1\beta_2)^2}}^{-1} = \sqrt{\frac{(1+\beta_1\beta_2)^2-(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}}^{-1} \\ &= \sqrt{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right)^2}^{-1} = \sqrt{1-\beta^2}^{-1} = \gamma. \end{aligned}$$

Then (4) becomes

$$x_0'' = \gamma\left(x_0 - \beta x_1\right), \qquad x_1'' = \gamma\left(x_1 - \beta x_0\right),$$

which is a single Lorentz transformation with velocity v, as we wanted to show. $\square$

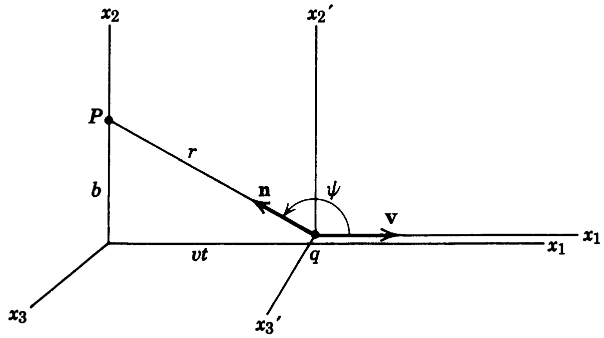

Figure 1: (Jackson Fig. 11.8) Particle of charge q moving at constant velocity $\mathbf{v}$ passes an observation point P at impact parameter b.

Problem 2 (Jackson 11.17). The electric and magnetic fields (6) of a charge in uniform motion can be obtained from Coulomb's law in the charge's rest frame and the fact that the field strength $F^{\alpha\beta}$ is an antisymmetric tensor of rank 2 without considering *explicitly* the Lorentz transformation. The idea is the following. For a charge in uniform motion the only relevant variables are the charge's 4-velocity U^α and the relative coordinate $X^\alpha = x_p^\alpha - x_q^\alpha$, where x_p^α and x_q^α are the 4-vector coordinates of the observation point and the charge, respectively. The only antisymmetric tensor that can be formed is $(X^\alpha U^\beta - X^\beta U^\alpha)$. Thus the electromagnetic field $F^{\alpha\beta}$ must be this tensor multiplied by some scalar function of the possible scalar products, $X_\alpha X^\alpha$, $X_\alpha U^\alpha$, $U_\alpha U^\alpha$.

2.a For the geometry of Fig. 1 the coordinates of P and q at a common time in K can be written $x_p^\alpha = (ct, \mathbf{b})$, $x_q^\alpha = (ct, \mathbf{v}t)$, with $\mathbf{b} \cdot \mathbf{v} = 0$. By considering the general form of $F^{\alpha\beta}$ in the rest frame of the charge, show that

$$F^{\alpha\beta} = \frac{q}{c} \frac{X^\alpha U^\beta - X^\beta U^\alpha}{[(U_\alpha X^\alpha / c)^2 - X_\alpha X^\alpha]^{3/2}}. \tag{5}$$

Verify that this yields the expressions

$$E_1 = E_1' = -\frac{q\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_2 = \gamma E_2' = \frac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad B_3 = \gamma\beta E_2' = \beta E_2, \tag{6}$$

with all other components vanishing, in the inertial frame K.

Solution. From Jackson (11.137),

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{7}$$

and from the equation immediately preceding Jackson (11.151),

$$E_1' = -\frac{qvt'}{r'^3}, \qquad E_2' = \frac{qb}{r'^3}, \qquad E_3' = 0, \qquad B_1' = 0, \qquad B_2' = 0, \qquad B_3' = 0,$$

in the rest frame of the charge. Here, $r' = \sqrt{b^2 + v^2 t'^2}$. Then, in K',

$$F'^{\alpha\beta} = \frac{q}{(b^2 + v^2 t'^2)^{3/2}} \begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Now we will boost into the frame K. From Jackson (11.147), $F' = \Lambda F \tilde{\Lambda}$, although we need $F = \Lambda F' \tilde{\Lambda}$, where we boost in the direction opposite the particle's motion. According to Jackson (11.113), the Lorentz boost in the $-x^{1'}$ direction is

$$\Lambda = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

Then

$$\begin{aligned} F^{\alpha\beta} &= \frac{q}{(b^2 + v^2t'^2)^{3/2}} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \frac{q}{(b^2 + v^2t'^2)^{3/2}} \begin{bmatrix} -\gamma\beta vt' & \gamma vt' & -\gamma b & 0 \\ -\gamma vt' & \gamma\beta vt' & -\gamma\beta b & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{q}{(b^2 + v^2t'^2)^{3/2}} \begin{bmatrix} 0 & vt' & -\gamma b & 0 \\ -vt' & 0 & -\gamma\beta b & 0 \\ \gamma b & \gamma\beta b & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Note that $x^1 = 0$, so from (1), $t' = \gamma t$. Finally,

$$F^{\alpha\beta} = \frac{\gamma q}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

Now we will begin from (5) and find $F^{\alpha\beta}$ directly in K. In accordance with Fig. 1,

$$X^\alpha = (0, \mathbf{b} - \mathbf{v}t) = (0, -vt, b, 0), \qquad U^\alpha = \gamma(c, \mathbf{v}) = \gamma(c, v, 0, 0),$$

and so

$$X^\alpha U^\beta - X^\beta U^\alpha = \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ -cvt & -v^2t & 0 & 0 \\ cb & vb & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \gamma \begin{bmatrix} 0 & -cvt & cb & 0 \\ 0 & -v^2t & vb & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \gamma \begin{bmatrix} 0 & cvt & -cb & 0 \\ -cvt & 0 & -vb & 0 \\ cb & vb & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Additionally,

$$U_\alpha X^\alpha = \gamma \begin{bmatrix} c & -v & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -vt \\ b \\ 0 \end{bmatrix} = \gamma v^2 t, \qquad X_\alpha X^\alpha = \begin{bmatrix} 0 & vt & -b & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -vt \\ b \\ 0 \end{bmatrix} = -v^2t^2 - b^2.$$

Then, applying (5),

$$F^{\alpha\beta} = \frac{\gamma q}{(\gamma^2 v^4 t^2/c^2 + v^2t^2 + b^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that

$$v^2t^2 + \frac{\gamma^2 v^4 t^2}{c^2} = v^2t^2 \left(1 + \gamma^2 \frac{v^2}{c^2}\right) = v^2t^2 \left(1 + \frac{\beta^2}{1-\beta^2}\right) = v^2t^2 \frac{1 - \beta^2 + \beta^2}{1 - \beta^2} = \gamma^2 v^2 t^2,$$

so we have again arrived at (10). Thus, we have proven (5).

In addition, comparing (10) with (7), we see that

$$E_1 = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2 = \frac{\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad B_3 = \frac{\gamma\beta qb}{(b^2+\gamma^2v^2t^2)^{3/2}} = \beta E_2.$$

Comparing (8) with (7) as well, and making the substitution $t' = \gamma t$, yields

$$E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2' = \frac{qb}{(b^2+\gamma^2v^2t^2)^{3/2}},$$

so we have also verified (6).

2.b Repeat the calculation, using as the starting point the common-time coordinates in the rest frame, $x'^{\alpha}_{p} = (ct', \mathbf{b}-\mathbf{v}t')$ and $x'^{\alpha}_{q} = (ct', 0)$. Show that

$$F^{\alpha\beta} = \frac{q}{c}\frac{Y^\alpha U^\beta - Y^\beta U^\alpha}{(-Y_\alpha Y^\alpha)^{3/2}}, \tag{11}$$

where $Y'^\alpha = x'^{\alpha}_{p} - x'^{\alpha}_{q}$. Verify that the fields are the same as in 2.a. Note that to obtain the results of (6) it is necessary to use the time t of the observation point P in K as the time parameter.

Solution. Firstly, note that

$$Y'^\alpha = (0, \mathbf{b}-\mathbf{v}t') = (0, -vt', b, 0), \qquad U'^\alpha = (c, \mathbf{0}) = (c, 0, 0, 0),$$

Then

$$Y'^\alpha U'^\beta - Y'^\beta U'^\alpha = c\begin{bmatrix} 0 & 0 & 0 & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - c\begin{bmatrix} 0 & -vt' & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = c\begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Y'_\alpha Y'^\alpha = \begin{bmatrix} 0 & vt' & -b & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -vt' \\ b \\ 0 \end{bmatrix} = -v^2t'^2 - b^2,$$

so, from (11), in K' we have

$$F'^{\alpha\beta} = \frac{q}{(b^2+v^2t'^2)^{3/2}}\begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is identical to (8). We know that boosting into K yields (10).

Now we will find $F^{\alpha\beta}$ directly in K by boosting Y'^α and U'^α. From Jackson (11.84), $x' = \Lambda x$, and we once again use Λ given by (9) to perform $x = \Lambda x'$. We obtain

$$Y = \Lambda Y' = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -vt' \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} -\gamma\beta vt' \\ -\gamma vt' \\ b \\ 0 \end{bmatrix}, \qquad U = \Lambda U' = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \gamma c\begin{bmatrix} 1 \\ \beta \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$Y^\alpha U^\beta - Y^\beta U^\alpha = \gamma c \begin{bmatrix} -\gamma\beta vt' & -\gamma\beta^2 vt' & 0 & 0 \\ -\gamma vt' & -\gamma\beta vt' & 0 & 0 \\ b & \beta b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \gamma c \begin{bmatrix} -\gamma\beta vt' & -\gamma vt' & b & 0 \\ -\gamma\beta^2 vt' & -\gamma\beta vt' & \beta b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = c \begin{bmatrix} 0 & vt' & -\gamma b & 0 \\ -vt' & 0 & -\gamma\beta b & 0 \\ \gamma b & \gamma\beta b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Y_\alpha Y^\alpha = \begin{bmatrix} -\gamma\beta vt' & \gamma vt' & -b & 0 \end{bmatrix} \begin{bmatrix} -\gamma\beta vt' \\ -\gamma vt' \\ b \\ 0 \end{bmatrix} = \gamma^2\beta^2 v^2 t'^2 - \gamma^2 v^2 t'^2 - b^2 = -v^2 t'^2 - b^2.$$

Making these substitutions into (11), and using $t' = \gamma t$

$$F^{\alpha\beta} = \frac{q}{(b^2 + v^2 t'^2)^{3/2}} \begin{bmatrix} 0 & vt' & -\gamma b & 0 \\ -vt' & 0 & -\gamma vb/c & 0 \\ \gamma b & \gamma vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{\gamma q}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is identical to (10), and therefore gives the fields from (6) as in 2.a. Thus, we have proven (11). □

2.c Finally, consider the coordinate $x_p^\alpha = (ct, \mathbf{b})$ and the "retarded-time" coordinate $x_q^\alpha = [ct - R, \boldsymbol{\beta}(ct - R)]$ where R is the distance between P and q at the retarded time. Define the difference as $Z^\alpha = [R, \mathbf{b} - \boldsymbol{\beta}(ct - R)]$. Show that in terms of Z^α and U^α the field is

$$F^{\alpha\beta} = \frac{q}{c} \frac{Z^\alpha U^\beta - Z^\beta U^\alpha}{(U_\alpha Z^\alpha / c)^3}. \tag{12}$$

Solution. Referring to Fig 1,

$$Z^\alpha = (R, \mathbf{b} - \mathbf{v}t + \mathbf{v}R/c) = [R, -v(t - R/c), b, 0], \qquad U^\alpha = \gamma(c, \mathbf{v}) = \gamma(c, v, 0, 0).$$

Then

$$\begin{aligned} Z^\alpha U^\beta - Z^\beta U^\alpha &= \gamma \begin{bmatrix} cR & Rv & 0 & 0 \\ -v(ct - R) & -v^2(t - R/c) & 0 & 0 \\ cb & bv & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \gamma \begin{bmatrix} cR & -v(ct - R) & cb & 0 \\ Rv & -v^2(t - R/c) & bv & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \gamma c \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$U_\alpha Z^\alpha = \gamma \begin{bmatrix} c & -v & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ -v(t - R/c) \\ b \\ 0 \end{bmatrix} = \gamma c R + \gamma v^2 (t - R/c),$$

so

$$\frac{U_\alpha Z^\alpha}{c} = \gamma R + \gamma\beta^2 ct - \gamma\beta^2 R = (1-\beta^2)\gamma R + \gamma\beta^2 ct = \frac{R}{\gamma} + \gamma\beta^2 ct.$$

Note that x_p^α and x_q^α, as they are defined here, have lightlike separation. Then

$$0 = Z_\alpha Z^\alpha = \begin{bmatrix} R & v(t-R/c) & -b & 0 \end{bmatrix} \begin{bmatrix} R \\ -v(t-R/c) \\ b \\ 0 \end{bmatrix} = R^2 - v^2(t-R/c)^2 - b^2,$$

which implies

$$R^2 = b^2 + v^2(t-R/c)^2.$$

This is corroborated by the geometry of Fig. 1, since $t - R/c$ is the retarded time. Then, referring to the denominator of (10), we find

$$\begin{aligned} b^2 + \gamma^2 v^2 t^2 &= R^2 - v^2(t-R/c)^2 + \gamma^2 v^2 t^2 = R^2 - \beta^2(c^2t^2 - 2Rct + R^2) + \gamma^2\beta^2c^2t^2 \\ &= (1-\beta^2)R^2 + 2R\beta^2 ct + (\gamma^2-1)\beta^2c^2t^2 = \frac{R^2}{\gamma^2} + 2R\beta^2 ct + \gamma^2\beta^4c^2t^2 = \left(\frac{R}{\gamma} + \gamma\beta^2 ct\right)^2 \\ &= \left(\frac{U_\alpha Z^\alpha}{c}\right)^2. \end{aligned}$$

In summary, we have found

$$F^{\alpha\beta} = \frac{\gamma q}{(b^2+\gamma^2v^2t^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is identical to (10). Thus, we have proven (12). □

Problem 3 (Jackson 11.18). The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by (6), become more and more concentrated as $\beta \to 1$. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\mathbf{r}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

3.a Show that the fields can be written as

$$\mathbf{E} = 2q\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct-z), \qquad \mathbf{B} = 2q\frac{\hat{\mathbf{v}} \times \mathbf{r}_\perp}{r_\perp^2}\delta(ct-z),$$

where $\hat{\mathbf{v}}$ is a unit vector in the direction of the particle's velocity.

Solution. According to Jackson (11.154), the electric field can be written

$$\mathbf{E} = \frac{q\mathbf{r}}{r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}.$$

Here $\psi = \cos^{-1}(\hat{\mathbf{n}} \cdot \hat{\mathbf{v}})$ is shown in Fig. 1, and $\hat{\mathbf{n}}$ is a unit vector pointing from the current position of the charge to the observation point $\mathbf{r}$. Note that

$$\lim_{\beta \to 1} \mathbf{E} \sim \lim_{\beta \to 1} \frac{1}{\gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}} = \lim_{\beta \to 1} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \psi)^{3/2}} = \begin{cases} \infty & \text{if } \sin \psi = 1, \\ 0 & \text{if } \sin \psi \neq 1. \end{cases}$$

Obviously $\sin \psi = 1$ if and only if $\cos \psi = 0$.

Let $\mathbf{r}'(t)$ be the position of the charge, so $\hat{\mathbf{n}} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$. We know $\hat{\mathbf{v}} = \hat{\mathbf{z}}$, and $\mathbf{r}' = \mathbf{v}t \to ct\,\hat{\mathbf{z}}$ as $\beta \to 1$. Then

$$\cos \psi = \hat{\mathbf{n}} \cdot \hat{\mathbf{v}} = \frac{(\mathbf{r} - \mathbf{r}') \cdot \hat{\mathbf{z}}}{|\mathbf{r} - \mathbf{r}'|} = \frac{z - ct}{|\mathbf{r} - \mathbf{r}'|},$$

so $\cos \psi = 0$ if and only if $z = ct$. Thus, we have shown that

$$\lim_{\beta \to 1} \frac{1}{\gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}} = \delta(ct - z).$$

Now we have

$$\lim_{\beta \to 1} \mathbf{E} = \frac{q\mathbf{r}}{r^3} \delta(ct - z)$$

3.b Show that by substitution into the Maxwell equations that these fields are consistent with a 4-vector source density,

$$J^\alpha = qcv^\alpha \delta^2(\mathbf{r}_\perp) \delta(ct - z),$$

where the 4-vector $v^\alpha = (1, \hat{\mathbf{v}})$.

Solution. According to Jackson (11.141), the inhomogeneous Maxwell equations can be written

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta.$$

Here,

$$F^{\alpha\beta} = \frac{2q}{{r_\perp}^2} \delta(ct - z) \begin{bmatrix} 0 & -x & -y & 0 \\ x & 0 & 0 & x \\ y & 0 & 0 & y \\ 0 & -x & -y & 0 \end{bmatrix}$$

and so

$$\partial_\alpha F^{\alpha\beta} = \begin{bmatrix} \partial_t & -\partial_x & -\partial_y & -\partial_z \end{bmatrix} \frac{2q}{{r_\perp}^2} \delta(ct - z) \begin{bmatrix} 0 & -x & -y & 0 \\ x & 0 & 0 & x \\ y & 0 & 0 & y \\ 0 & -x & -y & 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

3.c Show that the fields of (3.a) are derivable from either of the following 4-vector potentials,

$$A^0 = A^z = -2q\delta(ct - z) \ln(\lambda r_\perp), \qquad \mathbf{A}_\perp = 0,$$

or

$$A^0 = 0 = A^z, \qquad \mathbf{A}_\perp = -2q\Theta(ct - z) \boldsymbol{\nabla}_\perp \ln(\lambda r_\perp),$$

where λ is an irrelevant parameter setting the scale of the logarithm.

Show that the two potentials differ by a gauge transformation and find the gauge function, χ.

Problem 4 (Jackson 11.20). The lambda particle (Λ) is a neutral baryon of mass $M = 1115\,\mathrm{MeV}$ that decays with a lifetime of $\tau = 2.9 \times 10^{-10}\,\mathrm{s}$ into a nucleon of mass $m_1 \approx 939\,\mathrm{MeV}$ and a pi-meson of mass $m_2 \approx 140\,\mathrm{MeV}$. It was first observed in flight by its charged decay mode $\Lambda \to p + \pi^-$ in cloud chambers. The charged tracks originate from a single point and have the appearance of an inverted vee or lambda. The particles' identities and momenta can be inferred from their ranges and curvature in the magnetic field of the chamber.

4.a Using conservation of momentum and energy and the invariance of scalar products of 4-vectors show that, if the opening angle θ between the two tracks is measured, the mass of the decaying particle can be found from the formula

$$M^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta,$$

where here p_1 and p_2 are the magnitudes of the 3-momenta.

Solution. The general momentum 4-vector for a particle is

$$P^\mu = (E/c, \mathbf{p}),$$

where E is the energy of the particle and $\mathbf{p}$ its three-dimensional momentum. Let ${P_1}^\mu$ and ${P_1}^\mu$ be the momentum 4-vectors for the two particles, and define

$$P^\mu = {P_1}^\mu + {P_1}^\mu.$$

Firstly, note that

$$P^\mu P_\mu = \frac{E^2}{c^2} - p^2.$$

According to Jackson (11.55),

$$E = \sqrt{c^2p^2 + m^2c^4}, \tag{13}$$

so we have

$$P^\mu P_\mu = \frac{c^2p^2 + M^2c^4}{c^2} - p^2 = M^2c^2. \tag{14}$$

Note also that

$$P^\mu P_\mu = ({P_1}^\mu + {P_1}^\mu)(P_{1\mu} + P_{1\mu}) = \frac{(E_1+E_2)^2}{c^2} - (\mathbf{p}_1 + \mathbf{p}_2)^2 = \frac{E_1^2 + 2E_1E_2 + E_2^2}{c^2} - p_1^2 - 2\mathbf{p}_1\cdot\mathbf{p}_2 - p_2^2,$$

and once again making use of (13),

$$P^\mu P_\mu = \frac{c^2p_1^2 + m_1^2c^4 + 2E_1E_2 + c^2p_2^2 + m_2^2c^4}{c^2} - p_1^2 - 2\mathbf{p}_1\cdot\mathbf{p}_2 - p_2^2 = m_1^2c^2 + m_2^2c^2 + 2E_1E_2/c^2 - 2p_1p_2\cos\theta. \tag{15}$$

Equating (14) and (17), we have

$$M^2c^2 = m_1^2c^2 + m_2^2c^2 + 2E_1E_2/c^2 - 2p_1p_2\cos\theta.$$

Taking $c = 1$, this becomes

$$M^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta$$

as desired. $\square$

4.b A lambda particle is created with a total energy of 10 GeV in a collision in the top plate of a cloud chamber. How far on the average will it travel in the chamber before decaying? What range of opening angles will occur for a 10 GeV lambda if the decay is more or less isotropic in the lambda's rest frame?

Solution. According to Jackson (11.51),

$$E = \gamma mc^2.$$

For the Λ, this gives us $\gamma = E/M$ in natural units. Substituting this into the expression for time dilation, we find

$$\Delta t = \gamma\tau = \frac{E}{M}\tau$$

as the average lifetime in the lab frame. We also find

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{M^2}{E^2}}.$$

Then the average distance traveled before decaying is

$$d = v\,\Delta t = c\tau\sqrt{1 - \frac{M^2}{E^2}}\frac{E}{M} = (3.00 \times 10^8\,\mathrm{m\,s^{-1}})\sqrt{1 - \frac{(1115\,\mathrm{MeV})^2}{(10\,\mathrm{GeV})^2}}\frac{10\,\mathrm{GeV}}{1115\,\mathrm{MeV}}(2.9 \times 10^{-10}\,\mathrm{s}) = 77\,\mathrm{cm}.$$

Let K' denote the rest frame of the Λ. Using the result of 4.a and the fact that M, m_1, and m_2 are Lorentz invariant, we can write

$$E_1'E_2' - p_1'p_2'\cos\theta' = E_1E_2 - p_1p_2\cos\theta, \tag{16}$$

where θ' is the angle between the daughter particles in K'. We know from conservation of momentum that they must be "back to back" in this frame, meaning $\theta' = \pi$. Taking this into account and rearranging, we find

$$\cos\theta = \frac{E_1E_2 - E_1'E_2' - p_1'p_2'}{p_1p_2}. \tag{17}$$

Say that the Λ is moving in the z direction in the lab frame K. Then the opening angle θ will be maximized when the daughter particles are emitted transverse to the z' axis in K'.

Using natural units, in the K' frame we have

$$P'^{\mu} = (M, \mathbf{0}), \qquad P_1'^{\mu} = (E_1', \mathbf{p}_1'), \qquad P_2'^{\mu} = (E_2', \mathbf{p}_2').$$

Conservation of momentum and energy stipulates that

$$M = E_1' + E_1' = \gamma_1'm_1 + \gamma_2'm_2, \qquad \mathbf{0} = \mathbf{p}_1' + \mathbf{p}_2' = \gamma_1'\mathbf{p}_1' + \gamma_2'\mathbf{p}_2' = \gamma_1'm_1\mathbf{v}_1' + \gamma_2'm_2\mathbf{v}_2',$$

where γ_1' and γ_2' are associated with boosting to the rest frames of m_1 and m_2, respectively, from K'. Note that

$$\gamma_1'm_1\mathbf{v}_1' = -\gamma_2'm_2\mathbf{v}_2' \quad\implies\quad {\gamma_1'}^2m_1^2{\mathbf{v}_1'}^2 = {\gamma_2'}^2m_2^2{\mathbf{v}_2'}^2,$$

and that, in natural units,

$$\gamma^2v^2 = \gamma^2\beta^2 = \gamma^2\left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 - 1.$$

Then

$$({\gamma_1'}^2 - 1)m_1^2 = ({\gamma_2'}^2 - 1)m_2^2 \quad\implies\quad {E_1'}^2 - m_1^2 = {E_2'}^2 - m_2^2 \quad\implies\quad m_1^2 - m_2^2 = {E_1'}^2 - {E_2'}^2,$$

which implies

$$m_1^2 - m_2^2 = {E_1'}^2 - (M - E_1')^2 = -M^2 - 2ME_1' \quad\implies\quad E_1' = \frac{M^2 + m_1^2 - m_2^2}{2M},$$

and similarly for E_2. Let m_1 denote the nucleon and m_2 the pion. Then we have

$$E_1' = \frac{1}{2}\frac{(1115\,\mathrm{MeV})^2 + (939\,\mathrm{MeV})^2 - (140\,\mathrm{MeV})^2}{1115\,\mathrm{MeV}} \approx 944\,\mathrm{MeV},$$
$$E_2' = \frac{1}{2}\frac{(1115\,\mathrm{MeV})^2 - (939\,\mathrm{MeV})^2 + (140\,\mathrm{MeV})^2}{1115\,\mathrm{MeV}} \approx 171\,\mathrm{MeV}.$$

For the momenta, note that

$$p_1' = m_1\sqrt{{\gamma_1'}^2 - 1} = m_2\sqrt{{\gamma_2'}^2 - 1} = p_2',$$

where

$$\gamma_1' = \frac{E_1'}{m_1} = \frac{944\,\mathrm{MeV}}{939\,\mathrm{MeV}} \approx 1.005, \qquad \gamma_2' = \frac{E_2'}{m_2} = \frac{171\,\mathrm{MeV}}{140\,\mathrm{MeV}} \approx 1.221.$$

Then

$$p_1' = p_2' = (140\,\mathrm{MeV})\sqrt{1.221^2 - 1} \approx 98\,\mathrm{MeV}.$$

For the left side of (16), note that

$$\gamma = \frac{E}{M} = \frac{10\,\mathrm{GeV}}{1115\,\mathrm{MeV}} \approx 8.969 \qquad \Longrightarrow \qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 0.994,$$

whereas

$$\beta_1' = \sqrt{1 - \frac{1}{{\gamma_1'}^2}} \approx 0.104, \qquad \beta_2' = \sqrt{1 - \frac{1}{{\gamma_2'}^2}} \approx 0.573.$$

Since these are not nearly as relativistic as β, and we have stipulated that $\boldsymbol{\beta}_1'$ and $\boldsymbol{\beta}_1'$ are transverse to the z' axis, it may be sufficient to approximate $\gamma_1, \gamma_2 \approx \gamma$. Then we have

$$E_1 \approx \gamma m_1 = 8.969(939\,\mathrm{MeV}) \approx 8422\,\mathrm{MeV}, \qquad E_2 \approx \gamma m_2 = 8.969(140\,\mathrm{MeV}) \approx 1256\,\mathrm{MeV},$$
$$p_1 \approx m_1\sqrt{\gamma^2 - 1} = (939\,\mathrm{MeV})\sqrt{8.969^2 - 1} \approx 8369\,\mathrm{MeV}, \qquad p_2 \approx m_2\sqrt{\gamma^2 - 1} \approx 1248\,\mathrm{MeV}.$$

Making these substitutions into (17), we have

$$\cos\theta = \frac{(8422\,\mathrm{MeV})(1256\,\mathrm{MeV}) - (944\,\mathrm{MeV})(171\,\mathrm{MeV}) - (98\,\mathrm{MeV})^2}{(8369\,\mathrm{MeV})(1248\,\mathrm{MeV})} \approx 0.996$$

which implies

$$\theta \approx \cos^{-1}(0.996) \approx 5.13^\circ.$$

Problem 5. Show that $\mathbf{E}\cdot\mathbf{B}$ is Lorentz invariant. You can do this either by using the Lorentz transformation laws for $\mathbf{E}$ and $\mathbf{B}$ derived in class, or by writing $\mathbf{E}\cdot\mathbf{B}$ in a manifestly Lorentz invariant (and gauge invariant) form.

Solution. From Jackson (11.140),

$$\tilde{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix}.$$

Then, applying (7) as well,

$$\begin{aligned} F_{\alpha\beta}\tilde{F}^{\alpha\beta} &= \sum_{\alpha,\beta} \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix} = \sum_{\alpha,\beta} \begin{bmatrix} \mathbf{E}\cdot\mathbf{B} & 0 & 0 & 0 \\ 0 & \mathbf{E}\cdot\mathbf{B} & 0 & 0 \\ 0 & 0 & \mathbf{E}\cdot\mathbf{B} & 0 \\ 0 & 0 & 0 & \mathbf{E}\cdot\mathbf{B} \end{bmatrix} \\ &= 4\mathbf{E}\cdot\mathbf{B}. \end{aligned}$$

Here we have shown that $\mathbf{E}\cdot\mathbf{B}$ is directly proportional to the inner product of two 4-tensors. Thus, it is Lorentz invariant. $\square$

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics* while writing these solutions.