Problem 1. (Jackson 9.8)

1(a) Show that a classical oscillating electric dipole $\mathbf{p}$ with fields given by

$$\mathbf{H} = \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right), \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}\frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}, \tag{1}$$

radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\mathbf{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^* \times \mathbf{p}].$$

Solution. According to Jackson (9.20), the time-averaged angular momentum density is

$$\mathbf{l} = \frac{\mathrm{Re}[\mathbf{x} \times (\mathbf{E} \times \mathbf{H}^*)}{2c^2}.$$

One of the vector identities on the inside cover of Jackson is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, so

$$\mathbf{l} = \frac{(\mathbf{x} \cdot \mathbf{H}^*)\mathbf{E} - (\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{2c^2}. \tag{2}$$

From Eq. (**??**), note that

$$\mathbf{x} \cdot \mathbf{H}^* \propto \mathbf{x} \cdot (\hat{\mathbf{n}} \times \mathbf{p}^*) = \mathbf{p}^* \cdot (\mathbf{x} \times \hat{\mathbf{n}}) = \mathbf{0},$$

where we have used the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ and the fact that $\hat{\mathbf{n}}$ points in the $\mathbf{x}$ direction. For $\mathbf{x} \cdot \mathbf{E}$, note that

$$\begin{aligned}
\mathbf{x} \cdot [(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}] &= -\mathbf{x} \cdot [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{p})] = -\mathbf{x} \cdot [(\hat{\mathbf{n}} \cdot \mathbf{p})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{p}] = -(\hat{\mathbf{n}} \cdot \mathbf{p})(\mathbf{x} \cdot \hat{\mathbf{n}}) + \mathbf{x} \cdot \mathbf{p} \\
&= -r(\hat{\mathbf{n}} \cdot \mathbf{p}) + \mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{p} = 0,
\end{aligned}$$

$$\mathbf{x} \cdot [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] = 3(\mathbf{x} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3r(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3(\mathbf{x} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 2(\mathbf{x} \cdot \mathbf{p}),$$

since $|\mathbf{x}| = r$ and $\mathbf{x} = r\,\hat{\mathbf{n}}$. Then

$$\mathbf{x} \cdot \mathbf{E} = \frac{1}{2\pi\epsilon_0}(\mathbf{x} \cdot \mathbf{p})\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} = \frac{1}{2\pi\epsilon_0}(\hat{\mathbf{n}} \cdot \mathbf{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}.$$

With these substitutions, Eq. (**??**) becomes

$$\begin{aligned}
\mathbf{l} &= -\frac{(\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{c^2} = -\frac{1}{4\pi\epsilon_0 c^2}(\hat{\mathbf{n}} \cdot \mathbf{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}\frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p}^*)\frac{e^{-ikr}}{r}\left(1 + \frac{1}{ikr}\right) \\
&= -\frac{k^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{r^2} - \frac{ik}{r}\right)\left(1 - \frac{i}{kr}\right) = -\frac{k^2}{16\pi^2\epsilon_0 c}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{r^2} - \frac{i}{kr^3} - \frac{ik}{r} - \frac{1}{r^2}\right) \\
&= -\frac{ik^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{kr^3} + \frac{k}{r^2}\right) = \frac{ik^3}{16\pi^2\epsilon_0 cr^2}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{k^2r^2} + 1\right).
\end{aligned}$$

Let $\mathbf{L}$ be the angular momentum radiated to a distance R. Then

$$\mathbf{L} = \int_R \mathbf{l}(r)\,d^3x = \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r)\,r^2 \sin\theta\,dr\,d\phi\,d\theta,$$

and the time derivative is

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\mathbf{l}(r)\,r^2\sin\theta\,dr\,d\phi\,d\theta\right) = \frac{dr}{dt}\frac{d}{dr}\left(\int_0^{\pi}\int_0^{2\pi}\int_0^{R}\mathbf{l}(r)\,r^2\sin\theta\,dr\,d\phi\,d\theta\right)\\ &= c\int_0^{\pi}\int_0^{2\pi}\mathbf{l}(r)\,r^2\sin\theta\,d\phi\,d\theta = \frac{ik^3}{16\pi^2\epsilon_0}\left(\frac{1}{k^2r^2}+1\right)\int_0^{\pi}\int_0^{2\pi}(\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\times\mathbf{p}^*)\sin\theta\,d\phi\,d\theta. \qquad (3)\end{aligned}$$

Note that

$$[(\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\times\mathbf{p}^*)]_i = \sum_{j=1}^{3} n_j p_j(\hat{\mathbf{n}}\times\mathbf{p}^*)_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\epsilon_{ikl}n_j p_j n_k p_l^*,$$

so

$$\frac{dL_i}{dt} \propto \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\epsilon_{ikl}p_j p_l^*\int n_j p_k\,d\Omega = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\epsilon_{ikl}p_j p_l^*\frac{4\pi}{3}\delta_{jk} = \frac{4\pi}{3}\epsilon_{ikl}p_k p_l^* = \frac{4\pi}{3}(\mathbf{p}\times\mathbf{p}^*)_i,$$

where we have used Jackson (9.47), $\int n_\beta n_\gamma\,d\Omega = 4\pi\delta_{\beta\gamma}/3$. Making this substitution into Eq. (**??**),

$$\frac{d\mathbf{L}}{dt} = \frac{ik^3}{6\pi\epsilon_0}\left(\frac{1}{k^2r^2}+1\right)(\mathbf{p}\times\mathbf{p}^*).$$

Taking the limit as $r\to\infty$, we find

$$\frac{d\mathbf{L}}{dt} = \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0}(\mathbf{p}\times\mathbf{p}^*)\right] = \mathrm{Re}\left[-\frac{ik^3}{12\pi\epsilon_0}(\mathbf{p}^*\times\mathbf{p})\right] = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}], \qquad (4)$$

as desired. □

1(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

Solution. According to Jackson (9.24), the total power radiated by an oscillating electric dipole $\mathbf{p}$ is

$$P = \frac{dE}{dt} = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2.$$

Then the ratio of angular momentum radiated to energy radiated is

$$\frac{d\mathbf{L}/dt}{dE/dt} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}]\frac{12\pi}{c^2Z_0k^4|\mathbf{p}|^2} = \frac{1}{\epsilon_0}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}]\frac{1}{c^2Z_0k|\mathbf{p}|^2} = \frac{\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}]}{\omega|\mathbf{p}|^2},$$

where we have used $Z_0 = \sqrt{\mu_0/\epsilon_0} = 1/\sqrt{\epsilon_0^2c^2} = 1/\epsilon_0 c$, $c^2 = 1/(\epsilon_0\mu_0)$, and $\omega = kc$.

In the limit of high frequency, $(d\mathbf{L}/dt)/(dE/dt)\to 0$. In this scenario, the energy radiated dominates over the angular momentum radiated. Likewise, in the limit of low frequency, $(d\mathbf{L}/dt)/(dE/dt)\to\infty$, meaning that angular momentum radiation dominates. This is sensible because rotational kinetic energy $E\propto\omega^2$, while angular momentum $L\propto\omega$.

1(c) For a charge e rotating in the xy plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^2k^3a^2/6\pi\epsilon_0$. What about a charge oscillating along the z axis?

Solution. We know from Homework 5 that the position of a point charge rotating counterclockwise in the xy plane is

$$\mathbf{x}(t) = a\cos(\omega t)\,\mathbf{x} + a\sin(\omega t)\,\hat{\mathbf{y}}.$$

Then the charge distribution is

$$\rho(\mathbf{x},t) = e\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z).$$

According to Jackson (4.8), the dipole moment is defined

$$\mathbf{p} = \int \mathbf{x}'\,\rho(\mathbf{x}')\,d^3x'\,.$$

The components of $\mathbf{p}$ for the point charge are then

$$\begin{aligned}
p_x &= e\iiint x\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z)\,dx\,dy\,dz = ea\cos(\omega t),\\
p_y &= e\iiint y\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z)\,dx\,dy\,dz = ea\sin(\omega t),\\
p_z &= e\iiint z\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z)\,dx\,dy\,dz = 0,
\end{aligned}$$

so we can write $\mathbf{p} = ea\,e^{-i\omega t}(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}})$. Substituting into Eq. (**??**),

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0}e^2a^2e^{-i\omega t}e^{i\omega t}[(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}})\times(\hat{\mathbf{x}} - i\,\hat{\mathbf{y}})]\right] = \mathrm{Re}\left[\frac{ie^2k^3a^2}{12\pi\epsilon_0}(-2i\,\hat{\mathbf{x}}\times\hat{\mathbf{y}})\right] = \mathrm{Re}\left[\frac{e^2k^3a^2}{6\pi\epsilon_0}\,\hat{\mathbf{z}}\right]\\
&= \frac{e^2k^3a^2}{6\pi\epsilon_0}\cos(\omega t)\,\hat{\mathbf{z}},
\end{aligned}$$

as desired. □

A charge oscillating along the z axis with amplitude a has the charge density

$$\rho(\mathbf{x},t) = ea\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)],$$

which gives the dipole moment

$$\begin{aligned}
p_x &= ea\iiint x\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)]\,dx\,dy\,dz = 0,\\
p_y &= ea\iiint y\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)]\,dx\,dy\,dz = 0,\\
p_z &= ea\iiint z\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)]\,dx\,dy\,dz = ea\cos(\omega t).
\end{aligned}$$

In complex notation, $\mathbf{p} = ea\,e^{-i\omega t}\,\hat{\mathbf{z}}$. Substituting into Eq. (**??**), we find

$$\frac{d\mathbf{L}}{dt} = \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0}e^2a^2e^{-i\omega t}e^{i\omega t}(\hat{\mathbf{z}}\times\hat{\mathbf{z}})\right] = \mathbf{0}.$$

So we see that a charge undergoing linear motion does not lead to a radiated angular momentum, which is sensible.

1(d) What are the results corresponding to Probs. 1(a) and 1(b) for magnetic dipole radiation?

Solution. The radiation fields for a magnetic dipole are given by Jackson (19.35–36),

$$\mathbf{H} = \frac{1}{4\pi}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{m})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{m})-\mathbf{m}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}, \quad \mathbf{E} = -\frac{Z_0}{4\pi}k^2(\hat{\mathbf{n}}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right).$$

Comparing with Eq. (**??**), we see that $\mathbf{H}\to-\mathbf{E}/Z_0$, $\mathbf{E}\to Z_0\mathbf{H}$, and $\mathbf{p}\to\mathbf{m}/c$ as stated in the book [**?** , p. 413]. Making these substitutions, the results of Probs. 1.1(a) and (b) become

$$\frac{d\mathbf{L}}{dt} = \frac{\mu_0 k^3}{12\pi}\,\mathrm{Im}[\mathbf{m}^*\times\mathbf{m}], \qquad \frac{d\mathbf{L}/dt}{dE/dt} = \frac{\mathrm{Im}[\mathbf{m}^*\times\mathbf{m}]}{\omega|\mathbf{m}|^2}$$

where we have used $\mu = 1/\epsilon_0 c^2$.

Problem 2. (Jackson 10.1)

2(a) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{4} - |\boldsymbol{\epsilon}_0\cdot\hat{\mathbf{n}}|^2 - \frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}\right],$$

where $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{n}}$ are the directions of the incident and scattered radiations, respectively, while $\boldsymbol{\epsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\boldsymbol{\epsilon}_0^*\cdot\boldsymbol{\epsilon}_0 = 1$; $\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0 = 0$).

Solution. Jackson (10.14) gives the differential cross section for scattering off a small, perfectly conducting sphere with initial polarization $\boldsymbol{\epsilon}_0$ and outgoing polarization $\boldsymbol{\epsilon}$:

$$\frac{d\sigma}{d\Omega}\hat{\mathbf{n}},\boldsymbol{\epsilon};\hat{\mathbf{n}}_0,\boldsymbol{\epsilon}_0) = k^4a^6\left|(\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0 - \frac{1}{2}(\hat{\mathbf{n}}\times\boldsymbol{\epsilon}^*)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2. \tag{5}$$

We will use the polarization vectors $\boldsymbol{\epsilon}^{(1)}$ and $\boldsymbol{\epsilon}^{(2)}$, which are defined in Fig. (**??**) [**?** , p. 458]. According to the figure,

$$\boldsymbol{\epsilon}^{(2)} = \frac{\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0}{|\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0|} = \frac{\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}},$$
$$\boldsymbol{\epsilon}^{(1)} = \boldsymbol{\epsilon}^{(2)}\times\hat{\mathbf{n}} = \frac{-\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0)}{\sin\theta} = \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sin\theta} = \frac{\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}},$$

which are both real. In the denominator, we have used $\sin^2\theta = 1+\cos^2\theta = 1+(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2$. We also note that $\hat{\mathbf{n}}_0$, $\hat{\mathbf{n}}$, and $\boldsymbol{\epsilon}^{(1)}$ are in the same plane, and that $\hat{\mathbf{n}}\perp\boldsymbol{\epsilon}^{(1)}$.

The cross section summed over outgoing polarizations is then found by plugging $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(1)}$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(2)}$ into Eq. (**??**), and taking the sum. For the first term,

$$\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}},\boldsymbol{\epsilon}^{(1)};\hat{\mathbf{n}}_0,\boldsymbol{\epsilon}_0) = k^4a^6\left|\boldsymbol{\epsilon}^{(1)^*}\cdot\boldsymbol{\epsilon}_0 - \frac{1}{2}(\hat{\mathbf{n}}\times\boldsymbol{\epsilon}^{(1)^*})\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2$$

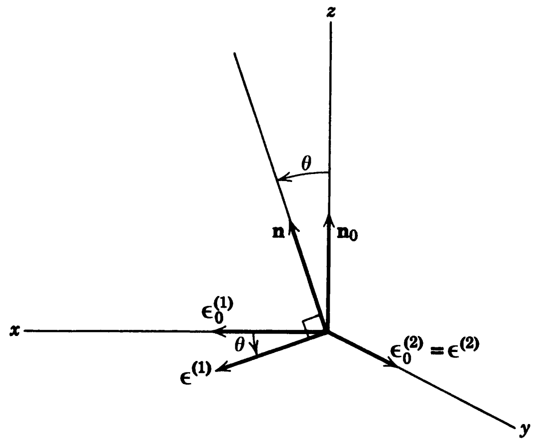

Figure 1: (Jackson 10.1) Polarization and propagation vectors for the incident and scattered radiation.

$$= k^4a^6\left|\frac{\hat{\mathbf{n}}_0-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}}\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}\left(\hat{\mathbf{n}}\times\frac{\hat{\mathbf{n}}_0-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}}\right)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2$$
$$= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)(\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0)-\frac{1}{2}(\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2.$$

One of the vector identities on the inside cover of Jackson is $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. Applying this, we have

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}},\boldsymbol{\epsilon}^{(1)};\hat{\mathbf{n}}_0,\boldsymbol{\epsilon}_0) &= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)(\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0)+\frac{1}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)(\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0)-\frac{1}{2}(\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0)(\hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}_0)\right|^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)(\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0)-\frac{1}{2}(\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0)\right|^2=\frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\left[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)-\frac{1}{2}\right]^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\left[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)+\frac{1}{4}\right].\end{aligned}$$

For the second term,

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}},\boldsymbol{\epsilon}^{(2)};\hat{\mathbf{n}}_0,\boldsymbol{\epsilon}_0) &= k^4a^6\left|\boldsymbol{\epsilon}^{(2)*}\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}(\hat{\mathbf{n}}\times\boldsymbol{\epsilon}^{(2)*})\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2\\
&= k^4a^6\left|\frac{\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}}\cdot\boldsymbol{\epsilon}_0-\frac{1}{2}\left(\hat{\mathbf{n}}\times\frac{\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}}\right)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)-\frac{1}{2}[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}_0]\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)-\frac{1}{2}[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}-\hat{\mathbf{n}}_0]\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)-\frac{1}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)+\frac{1}{2}\epsilon_0\cdot(\hat{\mathbf{n}}_0\times\hat{\mathbf{n}}_0)\right|^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|\left(1-\frac{1}{2}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\right)\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2\\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)+\frac{1}{4}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right]|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2.\end{aligned}$$

Summing the two terms, we find

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(1)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) + \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(2)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left\{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\left[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0) + \frac{1}{4}\right] + \left[1 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0) + \frac{1}{4}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right]|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2\right\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\Bigg\{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2 - |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0) + \frac{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2}{4} + |\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 \\
&\qquad\qquad - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4}\Bigg\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\Bigg\{\frac{5|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + \frac{5|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2}{4} - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2 \\
&\qquad\qquad - \frac{|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} - \frac{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2}{4} + \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\Bigg\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left\{\left[\frac{5}{4} - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right]\left[|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right] - \left[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right]\left[\frac{|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right]\right\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left[\frac{5}{4} - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right]\left[|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right] - k^4a^6\left[\frac{|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right]. \qquad (6)
\end{aligned}$$

Since $\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0 = 0$, we note that

$$\hat{\mathbf{n}} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\,\hat{\mathbf{n}}_0 + (\hat{\mathbf{n}}\cdot\epsilon_0)\,\boldsymbol{\epsilon}_0 + [\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)]\,(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0) \quad\Longrightarrow\quad 1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2 + |\hat{\mathbf{n}}\cdot\epsilon_0|^2 + |\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0)|^2. \qquad (7)$$

Substituting into Eq. (**??**),

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left[\frac{5}{4} - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right]\left[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right] - k^4a^6\left[\frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right] \\
&= k^4a^6\left[\frac{5}{4} - |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2 - \frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right], \qquad (8)
\end{aligned}$$

as we sought to prove. □

2(b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi)\right],$$

where $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of linear polarization.

Solution. We choose coordinates as in Fig. **??**, such that the direction of linear polarization $\boldsymbol{\epsilon}_0$ points along the x axis and $\hat{\mathbf{n}}_0$ points along the z axis. Then $\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0$ points along the y axis. In spherical coordinates, Eq. (**??**) becomes

$$\hat{\mathbf{n}} = \cos\phi\sin\theta\,\boldsymbol{\epsilon}_0 + \sin\phi\sin\theta\,(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0) + \cos\theta\,\hat{\mathbf{n}}_0, \qquad (9)$$

which implies

$$\cos\phi\sin\theta = \hat{\mathbf{n}}\cdot\epsilon_0, \qquad \sin\phi\sin\theta = \hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0), \qquad \cos\theta = \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0.$$

Making these substitutions in Eq. (**??**), we obtain

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta,\phi) &= k^4a^6\left[\frac{5}{4} - \cos^2\phi\sin^2\theta - \frac{1}{4}\sin^2\phi\sin^2\theta - \cos\theta\right] \\ &= k^4a^6\left[\frac{5}{4} - \frac{1}{2}[1+\cos(2\phi)]\sin^2\theta - \frac{1}{8}[1-\cos(2\phi)]\sin^2\theta - \cos\theta\right] \\ &= k^4a^6\left[\frac{5}{4} - \frac{1}{2}(1-\cos^2\theta) - \frac{1}{2}\cos(2\phi)\sin^2\theta - \frac{1}{8}(1-\cos^2\theta) + \frac{1}{8}\cos(2\phi)\sin^2\theta - \cos\theta\right] \\ &= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi)\right],\end{aligned}$$

where we have used the identities $2\sin^2\phi = 1-\cos(2\phi)$, $2\cos^2\phi = 1+\cos(2\phi)$, and $\cos^2\theta + \sin^2\theta = 1$. $\square$

2(c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

Solution. Firstly, note that

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\pi/2,0) &= k^4a^6\left[\frac{5}{8} - \frac{3}{8}\right] = \frac{k^4a^6}{4}, \\ \frac{d\sigma}{d\Omega}(\pi/2,\pi/2) &= k^4a^6\left[\frac{5}{8} + \frac{3}{8}\right] = k^4a^6,\end{aligned}$$

so the ratio is

$$\frac{d\sigma/d\Omega\ (\pi/2,0)}{d\sigma/d\Omega\ (\pi/2,\pi/2)} = \frac{1}{4}.$$

According to Jackson (10.12–13), the electric and magnetic dipole moments of a perfectly conducting sphere are, respectively,

$$\mathbf{p} = 4\pi\epsilon_0 a^3\mathbf{E}_{\text{inc}}, \qquad \mathbf{m} = 2\pi a^3\mathbf{H}_{\text{inc}},$$

where $\mathbf{E}_{\text{inc}}$ and $\mathbf{H}_{\text{inc}}$ are the incident fields. They are given by Jackson (10.1), wherein

$$\mathbf{E}_{\text{inc}} = \boldsymbol{\epsilon}_0 E_o e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}}, \qquad \mathbf{H}_{\text{inc}} = \hat{\mathbf{n}}_0 \times \mathbf{E}_{\text{inc}}/Z_0.$$

The scattered fields are given by Jackson (10.2),

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}} - \hat{\mathbf{n}}\times\frac{\mathbf{m}}{c}\right], \qquad \mathbf{H}_{\text{sc}} = \hat{\mathbf{n}}\times\frac{\mathbf{E}_{\text{sc}}}{Z_0}.$$

When $\phi = 0$, Eq. (**??**) indicates that $\hat{\mathbf{n}} = \boldsymbol{\epsilon}_0$. Applying the relations above, $\hat{\mathbf{n}}$ and $\mathbf{p}$ therefore point in the same direction. This means $\hat{\mathbf{n}}\times\mathbf{p} = \mathbf{0}$, so $\mathbf{E}_{\text{sc}}$ only has a contribution from the magnetic dipole. However, When $\phi = \pi/2$, $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0$ and therefore $\hat{\mathbf{n}}$ points in the same direction as $\mathbf{m}$. This means $\hat{\mathbf{n}}\times\mathbf{m} = \mathbf{0}$, so $\mathbf{E}_{\text{sc}}$ only has a contribution from the electric dipole. The ratio 1/4 indicates that the strength of radiation from a purely electric dipole is four times that from a purely magnetic dipole.

Problem 3. **(Jackson 12.15)** Consider the Proca equation for a localized steady-state distribution of current that has only a static magnetic moment. This model can be used to study the observable effects of a finite photon mass on the earth's magnetic field. Note that if the magnetization is $\boldsymbol{\mathcal{M}}(\mathbf{x})$ the current density can be written as $\mathbf{J} = c\,(\boldsymbol{\nabla} \times \boldsymbol{\mathcal{M}})$.

3(a) Show that if $\boldsymbol{\mathcal{M}} = \mathbf{m}\,f(\mathbf{x})$, where $\mathbf{m}$ is a fixed vector and $f(\mathbf{x})$ is a localized scalar function, the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m} \times \boldsymbol{\nabla} \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x'\,.$$

Solution. The Proca equations of motion in the static limit are given by the equation immediately following Jackson (12.93),

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha,$$

which implies

$$\nabla^2 \mathbf{A} - \mu^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{10}$$

Substituting Eq. (**??**) into the left side of Eq. (**??**), we have

$$\begin{aligned}
\nabla^2 \mathbf{A} - \mu^2 \mathbf{A} &= -(\nabla^2 - \mu^2)\mathbf{m} \times \left(\boldsymbol{\nabla} \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \right) \\
&= -(\nabla^2 - \mu^2)\mathbf{m} \times \left(\int \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \boldsymbol{\nabla} f(\mathbf{x}')\, d^3x' + \int f(\mathbf{x}') \boldsymbol{\nabla} \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \right) \\
&= (\nabla^2 - \mu^2)\mathbf{m} \times \int f(\mathbf{x}') \boldsymbol{\nabla}' \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \\
\\
&= 4\pi[\boldsymbol{\nabla} \times \mathbf{m}\, f(\mathbf{x})]
\end{aligned}$$

3(b) If the magnetic dipole is a point dipole at the origin $[f(\mathbf{x}) = \delta(\mathbf{x})]$, show that the magnetic field away from the origin is

$$\mathbf{B}(\mathbf{x}) = [3\,\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}] \left(1 + \mu r + \frac{\mu^2 r^2}{3}\right) \frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2 \mathbf{m} \frac{e^{-\mu r}}{r}.$$

3(c) The result of Prob. 3(b) shows that at fixed $r = R$ (on the surface of the earth), the earth's magnetic field will appear as a dipole angular distribution, plus an added constant magnetic field (an apparently external field) antiparallel to $\mathbf{m}$. Satellite and surface observations lead to the conclusion that the "external" field is less than 4×10^{-3} times the dipole field at the magnetic equator. Estimate a lower limit on μ^{-1} in earth radii and an upper limit on the photon mass in grams from this datum.