1 Problem 1

Consider operators J and K acting in a three-dimensional space as

$$J|e_1\rangle = i|e_2\rangle, \qquad J|e_2\rangle = -i|e_1\rangle, \qquad J|e_3\rangle = 0, \tag{1}$$
$$K|e_1\rangle = 0, \qquad K|e_2\rangle = i|e_3\rangle, \qquad K|e_3\rangle = -i|e_2\rangle, \tag{2}$$

where $|e_1\rangle, |e_2\rangle, |e_3\rangle$ for a complete orthonormal basis.

1.1

Compute the matrix elements of J and K.

Solution. The matrix elements of J are

$$J_{11} = \langle e_1|J|e_1\rangle = i\langle e_1|e_2\rangle = 0, \tag{3}$$
$$J_{12} = \langle e_1|J|e_2\rangle = -i\langle e_1|e_1\rangle = -i, \tag{4}$$
$$J_{13} = \langle e_1|J|e_3\rangle = 0, \tag{5}$$
$$J_{21} = \langle e_2|J|e_1\rangle = i\langle e_2|e_2\rangle = i, \tag{6}$$
$$J_{22} = \langle e_2|J|e_2\rangle = -i\langle e_2|e_1\rangle = 0, \tag{7}$$
$$J_{23} = \langle e_2|J|e_3\rangle = 0, \tag{8}$$
$$J_{31} = \langle e_3|J|e_1\rangle = i\langle e_3|e_2\rangle = 0, \tag{9}$$
$$J_{32} = \langle e_3|J|e_2\rangle = -i\langle e_3|e_1\rangle = 0, \tag{10}$$
$$J_{33} = \langle e_3|J|e_3\rangle = 0. \tag{11}$$

The matrix elements of K are

$$K_{11} = \langle e_1|K|e_1\rangle = 0, \tag{12}$$
$$K_{12} = \langle e_1|K|e_2\rangle = i\langle e_1|e_3\rangle = 0, \tag{13}$$
$$K_{13} = \langle e_1|K|e_3\rangle = -i\langle e_1|e_2\rangle = 0, \tag{14}$$
$$K_{21} = \langle e_2|K|e_1\rangle = 0, \tag{15}$$
$$K_{22} = \langle e_2|K|e_2\rangle = i\langle e_2|e_3\rangle = 0, \tag{16}$$
$$K_{23} = \langle e_2|K|e_3\rangle = -i\langle e_2|e_2\rangle = -i, \tag{17}$$
$$K_{31} = \langle e_3|K|e_1\rangle = 0, \tag{18}$$
$$K_{32} = \langle e_3|K|e_2\rangle = i\langle e_3|e_3\rangle = i, \tag{19}$$
$$K_{33} = \langle e_3|K|e_3\rangle = -i\langle e_3|e_2\rangle = 0. \tag{20}$$

1.2

Consider $O = AJ + BK$ where A, B are real numbers. Show that O is Hermitian.

Solution. Using (**??**)–(**??**), the matrix elements of O are

$$O_{11} = 0, \tag{21}$$
$$O_{12} = -iA, \tag{22}$$
$$O_{13} = 0, \tag{23}$$
$$O_{21} = iA, \tag{24}$$
$$O_{22} = 0, \tag{25}$$
$$O_{23} = -iA, \tag{26}$$
$$O_{31} = 0, \tag{27}$$
$$O_{32} = iB, \tag{28}$$
$$O_{33} = 0. \tag{29}$$