Problem 1. **(Peskin & Schroeder 2.1)** Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

1(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying

$$E^i = -F^{0i}; \qquad \epsilon^{ijk} B^k = -F^{ij}. \tag{2}$$

Solution. We want to extremize the action,

$$S[A_\mu] = \int d^4x \, \mathcal{L}(A_\mu, \partial_\mu A_\mu),$$

where $\mathcal{L}$ is the integrand of Eq. (1). Let δA_μ denote some arbitrary variation that vanishes at the boundaries of spacetime. The action for $A_\mu + \delta A_\mu$ is

$$S[A_\mu + \delta A_\mu] = \int d^4x \, \mathcal{L}(A_\mu + \delta A_\mu, \partial_\nu A_\mu + \partial_\nu \delta A_\mu).$$

Then, to first order in δA_μ, the variation of the action is

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu],$$

which we want to vanish for all δA_μ. Let $\delta F_{\mu\nu} = \partial_\mu \, \delta A_\nu - \partial_\nu \, \delta A_\mu$. Then, applying the definition of $F_{\mu\nu}$ given in Eq. (1),

$$\begin{aligned}
\delta S &= \int d^4x \left(-\frac{1}{4}(F_{\mu\nu} + \delta F_{\mu\nu})(F^{\mu\nu} + \delta F^{\mu\nu}) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\
&\approx \int d^4x \left(-\frac{1}{4}(F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \, \delta F^{\mu\nu} + \delta F_{\mu\nu} \, F^{\mu\nu}) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\
&= \int d^4x \left(-\frac{1}{4}(F_{\mu\nu} \, \delta F^{\mu\nu} + \delta F_{\mu\nu} \, F^{\mu\nu}) \right) \\
&= \int d^4x \left(-\frac{1}{2} \delta F_{\mu\nu} \, F^{\mu\nu} \right),
\end{aligned} \tag{3}$$

where we have discarded terms of $\mathcal{O}\big((\delta A^\mu)^2\big)$ and swapped covariant and contravariant indices in going to the final equality.

Note that

$$\begin{aligned}
\delta F_{\mu\nu} \, F^{\mu\nu} &= (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \delta A_\nu \, \partial^\mu A^\nu - \partial_\mu \delta A_\nu \, \partial^\nu A^\mu - \partial_\nu \delta A_\mu \, \partial^\mu A^\nu + \partial_\nu \delta A_\mu \, \partial^\nu A^\mu.
\end{aligned} \tag{4}$$

Integrating the first term of Eq. (4) by parts, we have

$$\int d^4x \, \frac{\partial \, \delta A_\nu}{\partial x^\mu} \frac{\partial A^\nu}{\partial x_\mu} = \left[\delta A_\nu \frac{\partial A^\nu}{\partial x_\mu} \right]_{-\infty}^{\infty} - \int d^4x \, \delta A_\nu \frac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} = -\int d^4x \, \delta A_\nu \, \partial_\mu \partial^\mu A^\nu,$$

because δA^ν vanishes at $\pm\infty$. The other terms follow similarly. Then we find

$$\begin{aligned}\int d^4x\,\delta F_{\mu\nu}\,F^{\mu\nu} &= -\int d^4x\,(\delta A_\nu\,\partial_\mu\partial^\mu A^\nu - \delta A_\nu\,\partial_\mu\partial^\nu A^\mu - \delta A_\mu\,\partial_\nu\partial^\mu A^\nu + \delta A_\mu\,\partial_\nu\partial^\nu A^\mu)\\ &= -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\mu\,\partial_\nu F^{\nu\mu}) = -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\nu\,\partial_\mu F^{\mu\nu})\\ &= -2\int d^4x\,\delta A_\nu\,\partial_\mu F^{\mu\nu},\end{aligned}$$

where in going to the second-to-last equality we have simply swapped the indices.

Making this substitution in Eq. (3), we obtain

$$\delta S = \delta A_\nu \int d^4x\,\partial_\mu F^{\mu\nu}.$$

In order for the action to be at a local extremum, we need $\delta S = 0$ for any δA_ν. This implies that the integrand is 0. Thus, we obtain

$$\partial_\mu F^{\mu\nu} = 0, \tag{5}$$

which is the covariant form of the inhomogeneous Maxwell equations in a source-free region [1, p. 557], as we sought to derive. □

From Eq. (2) and the knowledge that $F^{\mu\nu}$ is antisymmetric [1, p. 556], we can write

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}. \tag{6}$$

The first equation of Eq. (2) is equivalent to $E^i = F^{i0}$. Then the zeroth component of Eq. (5) can be written

$$\partial_\mu F^{\mu 0} = \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = \boldsymbol{\nabla}\cdot\mathbf{E} = 0,$$

which is the differential form of Gauss's law.

For the remaining components of Eq. (5), we apply the second equation of Eq. (5) to find

$$\partial_\mu F^{\mu i} = -\frac{\partial E^i}{\partial t} + \epsilon^{ijk}\frac{\partial B^k}{\partial x^j} = 0.$$

In vector form, this is

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0},$$

the differential form of Ampère's law.

1(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{7}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu, \tag{8}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{E^2 + B^2}{2}; \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

Solution. We want to evaluate Eq. (2.17) of Peskin & Schroeder,

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu \quad \Longrightarrow \quad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - \mathcal{L} g^{\mu\nu}, \tag{9}$$

where we have associated the field ϕ with A^λ. In order to evaluate the derivatives, we can use the variational method to calculate $\partial\mathcal{L}/\partial(\partial_\alpha A_\beta)$ by letting $\partial_\alpha A_\beta \to \partial_\alpha A_\beta + \delta \partial_\alpha A_\beta$ [2, p. 81]. Let

$$\delta\mathcal{L} = \mathcal{L}(\partial_\alpha A_\beta) - \mathcal{L}(\partial_\alpha A_\beta + \delta\partial_\alpha A_\beta).$$

Note that

$$\mathcal{L}(\partial_\alpha A_\beta + \delta\partial_\alpha A_\beta) = -\frac{1}{4}(F_{\alpha\beta} + \delta F_{\alpha\beta})(F^{\alpha\beta} + \delta F^{\alpha\beta}) \approx -\frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta} + F_{\alpha\beta}\, \delta F^{\alpha\beta} + \delta F_{\alpha\beta}\, F^{\alpha\beta}),$$

so

$$\begin{aligned}\delta\mathcal{L} &= -\frac{1}{4}(F_{\alpha\beta}\, \delta F^{\alpha\beta} + \delta F_{\alpha\beta}\, F^{\alpha\beta}) = -\frac{1}{2}\delta F_{\alpha\beta}\, F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha\, \delta A_\beta - \partial_\beta\, \delta A_\alpha)F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha\, \delta A_\beta + \partial_\alpha\, \delta A_\beta)F^{\alpha\beta} \\ &= -\partial_\alpha\, \delta A_\beta\, F^{\alpha\beta},\end{aligned}$$

where we have used the antisymmetry of $F^{\alpha\beta}$. This gives us

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta} \quad \Longrightarrow \quad \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\beta)} = -F^\alpha{}_\beta,$$

and then we find

$$T^{\mu\nu} = -F^\mu{}_\lambda\, \partial^\nu A^\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda\, \partial^\nu A^\lambda. \tag{10}$$

Adding $K^{\lambda\mu\nu}$ as defined in Eq. (8), Eq. (7) becomes

$$\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda\, \partial^\nu A^\lambda + \partial_\lambda(F^{\mu\lambda}A^\nu). \tag{11}$$

Applying the product rule to the third term, we find

$$\partial_\lambda(F^{\mu\lambda}A^\nu) = A^\nu \partial_\lambda F^{\mu\lambda} + F^{\mu\lambda}\partial_\lambda A^\nu = -A^\nu \partial_\lambda F^{\lambda\mu} + F^{\mu\lambda}\partial_\lambda A^\nu = F^{\mu\lambda}\partial_\lambda A^\nu,$$

where we have applied the antisymmetry of $F^{\mu\nu}$ and Eq. (5). Making this substitution in Eq. (11),

$$\begin{aligned}\hat{T}^{\mu\nu} &= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda\, \partial^\nu A^\lambda + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} + F^\mu{}_\lambda \partial^\lambda A^\nu - F^\mu{}_\lambda\, \partial^\nu A^\lambda = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} + F^\mu{}_\lambda F^{\lambda\nu} \\ &= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^\nu{}_\lambda. \end{aligned} \tag{12}$$

To show that $\hat{T}^{\mu\nu}$ is symmetric, note that

$$\hat{T}^{\nu\mu} = \frac{1}{4} g^{\nu\mu} F_{\alpha\beta}F^{\alpha\beta} - F^{\nu\lambda}F^\mu{}_\lambda = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda F^{\nu\lambda} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^\nu{}_\lambda = \hat{T}^{\mu\nu}$$

as desired. $\square$

For the energy and momentum densities, from Eq. (12) we have

$$\hat{T}^{00} = \frac{1}{4}g^{00}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^{0}{}_{\lambda} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + F^{0\lambda}F_{\lambda}{}^{0}, \tag{13}$$

$$\hat{T}^{0i} = \frac{1}{4}g^{0i}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^{i}{}_{\lambda} + = F^{0\lambda}F_{\lambda}{}^{i}. \tag{14}$$

Using Eq. (6),

$$F_{\mu\nu}F^{\mu\nu} = -E^{x2} - E^{y2} - E^{z2} - E^{x2} + B^{z2} + B^{y2} - E^{y2} + B^{z2} + B^{x2} - E^{z2} + B^{y2} + B^{x2} = 2(\mathbf{B}^2 - \mathbf{E}^2).$$

Note also from Eq. (6) that

$$F_{\lambda}{}^{\nu} = g_{\lambda\mu}F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{bmatrix},$$

so

$$F^{0\lambda}F_{\lambda}{}^{0} = (-\mathbf{E})\cdot(-\mathbf{E}) = \mathbf{E}^2, \qquad F^{0\lambda}F_{\lambda}{}^{i} = B_jE_k - E_kB_j = (\mathbf{E}\times\mathbf{B})_i.$$

Equations (13–14) are then

$$\hat{T}^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \mathcal{E}, \qquad \hat{T}^{0i} = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i = \mathbf{S},$$

as we sought to show. $\square$

Problem 2. The complex scalar field (Peskin & Schroeder 2.2) Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x\,(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi). \tag{15}$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

2(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x\,(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi). \tag{16}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

Solution. The momentum density conjugate to $\phi(x)$ is defined in Peskin & Schroeder (2.4):

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}$$

Here, $\mathcal{L}$ is the integrand of Eq. (15). Expanding its first term yields

$$\mathcal{L} = \dot{\phi}\dot{\phi}^* - \nabla\phi\cdot\nabla\phi^*, \tag{17}$$

so then

$$\pi(x) = \dot{\phi}^*, \qquad\qquad \pi^*(x) = \dot{\phi}, \tag{18}$$

where $\pi^*(x)$ is the momentum conjugate to $\phi^*(x)$. The canonical commutation relations follow from Peskin & Schroeder (2.20):

$$\begin{aligned}
[\phi(\mathbf{x}), \pi(\mathbf{y})] &= [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\,\delta^3(\mathbf{x}-\mathbf{y}),\\
[\phi(\mathbf{x}), \phi(\mathbf{y})] &= [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = 0,\\
[\pi(\mathbf{x}), \pi(\mathbf{y})] &= [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0,\\
[\phi(\mathbf{x}), \pi^*(\mathbf{y})] &= [\phi(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi^*(\mathbf{y})] = 0.
\end{aligned}$$

The Hamiltonian is given in general for a single field by Peskin & Schroeder (2.5),

$$H = \int d^3x \left(\pi(x)\,\dot{\phi}(x) - \mathcal{L}\right).$$

For the two fields $\phi(x)$ and $\phi^*(x)$, this becomes

$$\begin{aligned}
H &= \int d^3x \left(\pi(x)\,\dot{\phi}(x) + \pi^*(x)\,\dot{\phi}^*(x) - \mathcal{L}\right)\\
&= \int d^3x \left(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \dot{\phi}\dot{\phi}^* + \nabla\phi\cdot\nabla\phi^* + m^2\phi^*\phi\right)\\
&= \int d^3x \left(\pi\pi^* + \dot{\phi}\dot{\phi}^* - \dot{\phi}\dot{\phi}^* + \nabla\phi\cdot\nabla\phi^* + m^2\phi^*\phi\right)\\
&= \int d^3x \left(\pi^*\pi + \nabla\phi\cdot\nabla\phi^* + m^2\phi^*\phi\right),
\end{aligned}$$

where we have used Eqs. (17) and (18) as well as the commutation relations. So we have proven Eq. (16). □

The Heisenberg equation of motion is Peskin & Schroeder (2.44),

$$i\frac{\partial\mathcal{O}}{\partial t} = [\mathcal{O}, H],$$

where $\mathcal{O}$ is an arbitrary operator. Then

$$\begin{aligned}
i\frac{\partial\phi(x)}{\partial t} &= [\phi(x), H]\\
&= \left[\phi(\mathbf{x},t), \int d^3x'\,\pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] + \left[\phi(\mathbf{x},t), \int d^3x'\,\nabla'\phi(\mathbf{x}',t)\cdot\nabla'\phi^*(\mathbf{x}',t)\right]\\
&\qquad + m^2\left[\phi(\mathbf{x},t), \int d^3x'\,\phi^*(\mathbf{x}',t)\,\phi(\mathbf{x}',t)\right]\\
&= \left[\phi(\mathbf{x},t), \int d^3x'\,\pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] = i\int d^3x'\,\delta^3(\mathbf{x}-\mathbf{x}')\,\pi^*(\mathbf{x}',t) = i\pi^*(x),
\end{aligned}$$

$$\begin{aligned}
i\frac{\partial\phi^*(x)}{\partial t} &= [\phi^*(x), H]\\
&= \left[\phi^*(\mathbf{x},t), \int d^3x'\,\pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] = i\int d^3x'\,\delta^3(\mathbf{x}-\mathbf{x}')\,\pi(\mathbf{x}',t) = i\pi(x),
\end{aligned}$$

$$
\begin{aligned}
i\frac{\partial \pi(x)}{\partial t} &= [\pi(x), H] \\
&= \left[\pi(\mathbf{x},t), \int d^3x'\, \pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] + \left[\pi(\mathbf{x},t), \int d^3x'\, \boldsymbol{\nabla}'\phi(\mathbf{x}',t)\cdot\boldsymbol{\nabla}'\phi^*(\mathbf{x}',t)\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + m^2\left[\pi(\mathbf{x},t), \int d^3x'\, \phi^*(\mathbf{x}',t)\,\phi(\mathbf{x}',t)\right] \\
&= -i\int d^3x'\left[\boldsymbol{\nabla}'\delta^3(\mathbf{x}-\mathbf{x}')\cdot\boldsymbol{\nabla}'\phi^*(\mathbf{x}',t) + m^2\delta^3(\mathbf{x}-\mathbf{x}')\,\phi^*(\mathbf{x}',t)\right] \\
&= -i\int d^3x'\left[-\delta^3(\mathbf{x}-\mathbf{x}')\cdot\nabla'^2\phi^*(\mathbf{x}',t) + m^2\delta^3(\mathbf{x}-\mathbf{x}')\,\phi^*(\mathbf{x}',t)\right] = -i(-\nabla^2+m^2)\,\phi^*(x),
\end{aligned}
$$

$$
\begin{aligned}
i\frac{\partial \pi^*(x)}{\partial t} &= [\pi^*(x), H] \\
&= -i\int d^3x'\left[\boldsymbol{\nabla}'\phi(\mathbf{x}',t)\cdot\boldsymbol{\nabla}'\delta(\mathbf{x}-\mathbf{x}') + m^2\delta^3(\mathbf{x}-\mathbf{x}')\,\phi(\mathbf{x}',t)\right] \\
&= -i\int d^3x'\left[-\delta^3(\mathbf{x}-\mathbf{x}')\cdot\nabla'^2\phi(\mathbf{x}',t) + m^2\delta^3(\mathbf{x}-\mathbf{x}')\,\phi(\mathbf{x}',t)\right] = -i(-\nabla^2+m^2)\,\phi(x).
\end{aligned}
$$

Thus we have obtained

$$
\frac{\partial\phi(x)}{\partial t} = \pi^*(x), \qquad \frac{\partial\phi^*(x)}{\partial t} = \pi(x), \qquad \frac{\partial\pi(x)}{\partial t} = (\nabla^2 - m^2)\,\phi^*(x), \qquad \frac{\partial\pi^*(x)}{\partial t} = (\nabla^2 - m^2)\,\phi(x).
$$

Combining these results yields

$$
\frac{\partial^2\phi}{\partial t^2} = (\nabla^2 - m^2)\phi, \qquad\qquad \frac{\partial^2\phi^*}{\partial t^2} = (\nabla^2 - m^2)\phi^*,
$$

which is the Klein-Gordon equation and its complex conjugate, as we sought to show. □

2(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

Solution. Peskin & Schroeder (2.21) gives the Klein-Gordon equation in the momentum basis,

$$
\left(\frac{\partial^2}{\partial t^2} + \mathbf{p}^2 + m^2\right)\phi(\mathbf{p},t) = 0.
$$

This is the same as the harmonic oscillator equation of motion. It has solutions [3]

$$
\phi(\mathbf{p},t) = A(\mathbf{p})\,e^{i\omega_{\mathbf{p}}t} + B(\mathbf{p})\,e^{-i\omega_{\mathbf{p}}t},
$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ as in Peskin & Schroeder Eq. (2.22), and $A(\mathbf{p})$ and $B(\mathbf{p})$ are arbitrary functions of $\mathbf{p}$. The complex conjugate of this solution is

$$
\phi^*(\mathbf{p},t) = B^*(\mathbf{p})\,e^{i\omega_{\mathbf{p}}t} + A^*(\mathbf{p})\,e^{-i\omega_{\mathbf{p}}t}.
$$

The field ϕ in the position basis can be expanded as [4, p. 20],

$$
\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\,\phi(\mathbf{p},t).
$$

so we can write [**?** , p. 33]

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \qquad \phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right),$$

where $a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{p}}$ $(a_{\mathbf{p}}, b_{\mathbf{p}})$ are creation (annihilation) operators. By analogy to Eq. (2.26) of Peskin & Schroeder, we can also write

$$\pi(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \qquad \pi^*(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - b^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right).$$

Simplifying these expressions as in their Eqs. (2.27) and (2.28), we have

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(b_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{19}$$

$$\pi(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(b_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \pi^*(\mathbf{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}} - b^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{20}$$

Also generalizing their Eq. (2.24),

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{p}'), \qquad [a_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = [b_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = 0.$$

Feeding Eqs. (19) and (20) into Eq. (16) yields

$$H = \int d^3x \int \frac{d^3p\,d^3p'}{(2\pi)^6}\,e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\left[-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2}\left(a_{\mathbf{p}} - b^\dagger_{-\mathbf{p}}\right)\left(b_{\mathbf{p}'} - a^\dagger_{-\mathbf{p}'}\right) + \frac{-\mathbf{p}\cdot\mathbf{p}'+m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)\left(b_{\mathbf{p}'} + a^\dagger_{-\mathbf{p}'}\right)\right].$$

Using the delta function identity [5]

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-a)}\,dp\,,$$

this becomes

$$\begin{aligned}
H &= \int \frac{d^3p\,d^3p'}{(2\pi)^3}\,\delta^3(\mathbf{p}+\mathbf{p}')\left[-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2}\left(a_{\mathbf{p}} - b^\dagger_{-\mathbf{p}}\right)\left(b_{\mathbf{p}'} - a^\dagger_{-\mathbf{p}'}\right) + \frac{-\mathbf{p}\cdot\mathbf{p}'+m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)\left(b_{\mathbf{p}'} + a^\dagger_{-\mathbf{p}'}\right)\right]\\
&= \int \frac{d^3p}{(2\pi)^3}\left[-\frac{\omega_{\mathbf{p}}}{2}\left(a_{\mathbf{p}} - b^\dagger_{-\mathbf{p}}\right)\left(b_{-\mathbf{p}} - a^\dagger_{\mathbf{p}}\right) + \frac{\mathbf{p}^2+m^2}{2\omega_{\mathbf{p}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)\left(b_{-\mathbf{p}} + a^\dagger_{\mathbf{p}}\right)\right]\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{2}\left[a_{\mathbf{p}}b_{-\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}b_{-\mathbf{p}} + b^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}} - \left(a_{\mathbf{p}}b_{-\mathbf{p}} - a_{\mathbf{p}}a^\dagger_{\mathbf{p}} - b^\dagger_{-\mathbf{p}}b_{-\mathbf{p}} + b^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}}\right)\right]\\
&= \int \frac{d^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}b_{-\mathbf{p}}\right) = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]\right).
\end{aligned}$$

Ignoring the infinite constant term [4, p. 21], we have

$$H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right). \tag{21}$$

To show that the theory contains two sets of particles of mass m, we evaluate the commutators [4, p. 22]:

$$[H, a^\dagger_{\mathbf{p}}] = \left[\int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}a^\dagger_{\mathbf{p}'}a_{\mathbf{p}'}, a^\dagger_{\mathbf{p}}\right] = \omega_{\mathbf{p}}a^\dagger_{\mathbf{p}}, \qquad [H, a_{\mathbf{p}}] = \left[\int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}a^\dagger_{\mathbf{p}'}a_{\mathbf{p}'}, a_{\mathbf{p}}\right] = -\omega_{\mathbf{p}}a_{\mathbf{p}},$$

$$[H, b^\dagger_{\mathbf{p}}] = \left[\int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}b^\dagger_{\mathbf{p}'}b_{\mathbf{p}'}, b^\dagger_{\mathbf{p}}\right] = \omega_{\mathbf{p}}b^\dagger_{\mathbf{p}}, \qquad [H, b_{\mathbf{p}}] = \left[\int \frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}b^\dagger_{\mathbf{p}'}b_{\mathbf{p}'}, b_{\mathbf{p}}\right] = -\omega_{\mathbf{p}}b_{\mathbf{p}}.$$

Then we can define the eigenstates of the Hamiltonian by

$$(a_{\mathbf{p}}^\dagger)^{n_a}\,(b_{\mathbf{p}}^\dagger)^{n_b}\,|0,0\rangle \equiv |n_a,n_b\rangle\,,$$

which have eigenvalues $(n_a+n_b)\omega_{\mathbf{p}}$. So the expression for the Hamiltonian in Eq. (21) is diagonal in the occupation number basis $\{|n_a,n_b\rangle\}$, where n_a indicates the number of particles created with $a_{\mathbf{p}}^\dagger$ and n_b the number created with $b_{\mathbf{p}}^\dagger$. The ground state is $|0,0\rangle$; it has zero energy since its eigenvalue is zero. Since each operation of $a_{\mathbf{p}}^\dagger$ or $b_{\mathbf{p}}^\dagger$ imparts energy $\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$ to the system, and each operation of $a_{\mathbf{p}}$ or $b_{\mathbf{p}}$ removes energy $\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$ from the system, we can conclude that each of the two sets of operators corresponds to a set of particles of mass m. □

2(c) Rewrite the conserved charge

$$Q=\int d^3x\,\frac{i}{2}(\phi^*\pi^*-\pi\phi) \tag{22}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

Solution. Applying Eqs. (19) and (20), we find

$$\begin{aligned}
Q&=\frac{1}{4}\int d^3x\int\frac{d^3p\,d^3p'}{(2\pi)^6}\,e^{i\mathbf{p}\cdot\mathbf{x}}\left[\left(b_{\mathbf{p}}+a_{-\mathbf{p}}^\dagger\right)\left(a_{\mathbf{p}'}-b_{-\mathbf{p}'}^\dagger\right)-\left(b_{\mathbf{p}}-a_{-\mathbf{p}}^\dagger\right)\left(a_{\mathbf{p}'}+b_{-\mathbf{p}'}^\dagger\right)\right]\\
&=\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\left[\left(b_{\mathbf{p}}+a_{-\mathbf{p}}^\dagger\right)\left(a_{-\mathbf{p}}-b_{\mathbf{p}}^\dagger\right)-\left(b_{\mathbf{p}}-a_{-\mathbf{p}}^\dagger\right)\left(a_{-\mathbf{p}}+b_{\mathbf{p}}^\dagger\right)\right]\\
&=\frac{1}{4}\int\frac{d^3p}{(2\pi)^3}\left[\left(b_{\mathbf{p}}a_{-\mathbf{p}}-b_{\mathbf{p}}b_{\mathbf{p}}^\dagger+a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}-a_{-\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger\right)-\left(b_{\mathbf{p}}a_{-\mathbf{p}}+b_{\mathbf{p}}b_{\mathbf{p}}^\dagger-a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}-a_{-\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger\right)\right]\\
&=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}-b_{\mathbf{p}}b_{\mathbf{p}}^\dagger\right)=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}-b_{\mathbf{p}}^\dagger b_{\mathbf{p}}-[b_{\mathbf{p}},b_{\mathbf{p}}^\dagger]\right)\\
&=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}-b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right).
\end{aligned}$$

The particles associated with $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ must have positive charge, since $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ represents their number and has the same sign as the conserved charge. Similarly, the particles associated with $b_{\mathbf{p}}^\dagger b_{\mathbf{p}}$ must have negative charge.

2(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a=1,2$. Show that there are now four conserved charges, one given by the generalization of part 2(c), and the other three given by

$$Q^i=\int d^3x\,\frac{i}{2}(\phi_a^*\sigma^i{}_{ab}\pi_b^*-\pi_a\sigma^i{}_{ab}\phi_b),$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum $(SU(2))$. Generalize these results to the case of n identical complex scalar fields.

Solution. Generalizing Eq. (15), the Lagrangian for the two Klein-Gordon fields is

$$\mathcal{L}=\partial_\mu\phi_1^*\partial^\mu\phi_1-m^2\phi_1^*\phi_1+\partial_\mu\phi_2^*\partial^\mu\phi_2-m^2\phi_2^*\phi_2. \tag{23}$$

The conserved charge is given in general by Peskin & Schroeder (2.12) and (2.13),

$$Q\equiv\int_{\text{all space}} j^0\,d^3x\,,\qquad\qquad \text{where } j^\mu(x)=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi-J^\mu,$$

where J^μ is a 4-divergence that arises when transforming the Lagrangian as in Peskin & Schroeder (2.10):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu J^\mu(x).$$

For the first conserved charge, we note that the Lagrangian in Eq. (23) is invariant under the transformations $\phi_a \to e^{i\alpha}\phi_a$ [4, p. 18]:

$$\mathcal{L} \to \sum_a \left[\partial_\mu(e^{-i\alpha}\phi_a^*)\,\partial^\mu(e^{i\alpha}\phi_a) - m^2 e^{-i\alpha}\phi_a^*\,e^{i\alpha}\phi_a\right] = \mathcal{L},$$

so $J^\mu(x) = 0$. The relevant infinitesimal transformations are found by generalizing Peskin & Schroeder (2.15):

$$\alpha\,\Delta\phi_a = i\alpha\phi_a, \qquad\qquad \alpha\,\Delta\phi_a^* = -i\alpha\phi_a^*. \tag{24}$$

These transformations yield the conserved current

$$j^\mu = -\frac{1}{2}\sum_a \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a^*)}\Delta\phi_a^*\right) = -\frac{i}{2}\sum_a \left(\phi_a\partial^\mu\phi_a^* - \phi_a^*\partial^\mu\phi_a\right),$$

where we have arbitrarily chosen the overall constant [4, p. 18]. Then, generalizing Eq. (18), the corresponding conserved charge is

$$\begin{aligned}Q^0 &= \int d^3x\, j^0 = -\frac{i}{2}\int d^3x \sum_a \left(\phi_a\dot\phi_a^* - \phi_a^*\dot\phi_a\right) = -\frac{i}{2}\int d^3x \sum_a \left(\phi_a\pi_a - \phi_a^*\pi_a^*\right) \\ &= \int d^3x\,\frac{i}{2}\left(\phi_1^*\pi_1^* - \phi_1\pi_1 + \phi_2^*\pi_2^* - \phi_2\pi_2\right),\end{aligned}$$

which is the generalization of Eq. (22) for two fields.

From the problem statement, we make the ansatz that $\mathcal{L}$ is also invariant under rotations, $\phi \to e^{i\alpha^i\sigma^i/2}\phi$ where $\phi = (\phi_1, \phi_2)$ is a two-component spinor, from Peskin & Schroeder (15.19) and (15.20). To verify,

$$\mathcal{L} \to \sum_a \left[\partial_\mu(e^{-i\alpha^i\sigma^i/2}\phi_a^*)\,\partial^\mu(e^{i\alpha^i\sigma^i/2}\phi_a) - m^2 e^{-i\alpha^i\sigma^i/2}\phi_a^*\,e^{i\alpha^i\sigma^i/2}\phi_a\right] = \mathcal{L}.$$

Again, $J^\mu = 0$. By analogy with Eq. (24), the infinitesimal transformations are

$$\alpha^i\,\Delta\phi = \frac{i}{2}\phi\alpha^i\sigma^i, \qquad\qquad \alpha^i\,\Delta\phi_a = -\frac{i}{2}\phi_a\alpha^i\sigma^i.$$

We have the conserved currents

$$j_i^\mu = -\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*\right) = \frac{i}{2}\left(\phi^*\sigma^i\,\partial^\mu\phi - \phi\sigma^i\,\partial^\mu\phi^*\right),$$

where we have chosen a different overall constant factor. Then the corresponding conserved charges are

$$Q^i = \int d^3x\, j_i^0 = \int d^3x\,\frac{i}{2}\left(\phi^*\sigma^i\dot\phi - \phi\sigma^i\dot\phi^*\right) = \int d^3x\,\frac{i}{2}\left(\phi^*\sigma^i\pi^* - \phi\sigma^i\pi\right) = \int d^3x\,\frac{i}{2}(\phi_a^*{\sigma^i}_{ab}\pi_b^* - \pi_a{\sigma^i}_{ab}\phi_b),$$

as desired. $\square$

References

[1] J. D. Jackson, “Classical Electrodynamics”. Wiley, 3rd edition, 1999.

[2] L. D. Landau and E. M. Lifshitz, “The Classical Theory of Fields”, volume 2. Pergamon Press, 3rd edition, 1971.

[3] E. W. Weisstein, “Simple Harmonic Motion.” From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/SimpleHarmonicMotion.html`.

[4] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”. Perseus Books Publishing, 1995.

[5] W. contributors, “Dirac delta function.” From Wikipedia, the Free Encyclodia. `https://en.wikipedia.org/wiki/Dirac_delta_function`.