Problem 1. A fellow student with a mathematical bent tells you that the wave function of a traveling wave on a thin rope is

$$y(x,t) = (2.30\,\mathrm{mm})\cos[(6.98\,\mathrm{rad\,m^{-1}})x + (742\,\mathrm{rad\,s^{-1}})t]. \tag{1}$$

Being more practical, you measure the rope to have a length of 1.35 m and a mass of 0.003 38 kg. You are then asked to determine the following:

(a) amplitude;

(b) frequency;

(c) wavelength;

(d) wave speed;

(e) direction the wave is traveling;

(f) tension in the rope;

(g) average power transmitted by the wave.

Solution.

(a) A standing wave has the general form

$$y(x,t) = y_0 \sin(kx - \omega t),$$

where y_0 is the amplitude of the wave, k its wavenumber, and ω its angular frequency. However, since sine and cosine differ only by a phase, we might as well write

$$y(x,t) = y_0 \cos(kx - \omega t), \tag{2}$$

which is the form given in the problem, Eq. (1). Then we can easily read off the amplitude:

$$y_0 = 2.30\,\mathrm{mm}.$$

(b) Once again referring to Eq. (2), we can read off the *angular* frequency $\omega = 742\,\mathrm{rad\,s^{-1}}$ from Eq. (1). Then we can easily solve for the frequency f:

$$f = \frac{\omega}{2\pi} = \frac{742\,\mathrm{rad\,s^{-1}}}{2\pi\,\mathrm{rad}} = 118\,\mathrm{Hz}.$$

(c) Reading off the wave number from Eq. (1), we find $k = 6.98\,\mathrm{rad\,m^{-1}}$. Solving for the wavelength λ, we find

$$\lambda = \frac{2\pi}{k} = \frac{2\pi\,\mathrm{rad}}{6.98\,\mathrm{rad\,m^{-1}}} = 0.90\,\mathrm{m} = 90\,\mathrm{cm}.$$

(d) The wave speed is defined as $v = \omega/k$. Plugging in the values of ω and k that we found in (b) and (c),

$$v = \frac{\omega}{k} = \frac{742\,\mathrm{rad\,s^{-1}}}{6.98\,\mathrm{rad\,m^{-1}}} = 106\,\mathrm{m\,s^{-1}}.$$

(e) Equation (2) gives the general expression for a wave traveling in the $+x$ direction. Here, the argument of the cosine function is $kx + \omega t$. However, in the given expression of Eq. (1), the argument has the form $kx - \omega t$. This means that the wave is traveling in the $-x$ direction.

(f) Another expression for the wave speed is

$$v = \sqrt{\frac{T}{\mu}},$$

where μ is the mass density of the rope, and T is the tension in the rope. Solving this definition for T and substituting in $\mu = m/L$ gives us

$$T = \mu v^2 = \frac{mv^2}{L}.$$

Plugging in the given values of m and L, and our result for v from (d), we find

$$T = \frac{(0.003\,38\,\mathrm{kg})(106\,\mathrm{m\,s^{-1}})^2}{1.35\,\mathrm{m}} = 28.3\,\mathrm{N}.$$

(g) The average power $\langle P \rangle$ transmitted by the wave is given by

$$\langle P \rangle = \frac{1}{2}\mu\omega^2 y_0^2 v = \frac{m\omega^2 y_0^2 v}{2L},$$

so plugging in known quantities and previous results gives us

$$\langle P \rangle = \frac{(0.003\,38\,\mathrm{kg})(742\,\mathrm{rad\,s^{-1}})^2(0.0023\,\mathrm{m})^2(106\,\mathrm{m\,s^{-1}})}{2(1.35\,\mathrm{m})} = 0.39\,\mathrm{W}.$$

Problem 3. A transverse sinusoidal wave with wavelength 15 cm and wave speed $20\,\mathrm{m\,s^{-1}}$ is traveling on a 5 m-long string of mass 2 g. The average power of the wave is 35 W. What is the amplitude of the wave? What is the average power if the wave speed is tripled?

Solution. The average power $\langle P \rangle$ of a wave is given by

$$\langle P \rangle = \frac{1}{2}\mu\omega^2 z_0^2 v, \tag{3}$$

where $\mu = m/L$ is the mass density of the string, ω is the wave's angular frequency, z_0 is its amplitude, and v is the wave speed. Solving for the amplitude, we find

$$z_0 = \sqrt{\frac{2\langle P \rangle}{\mu\omega^2 v}}. \tag{4}$$

We need to find ω in terms of given quantities. We know $\omega = kv$ and $k = 2\pi/\lambda$, where k is the wave number and λ the wavelength. Thus,

$$\omega = \frac{2\pi v}{\lambda}.$$

Substituting this and $\mu = m/L$ into Eq. (4) gives us

$$z_0 = \sqrt{\frac{2L\langle P \rangle}{mv}\frac{\lambda^2}{4\pi^2 v^2}} = \frac{1}{\pi}\sqrt{\frac{L\lambda^2\langle P \rangle}{2mv^3}}.$$

Substituting in the given quantities, and recalling that $1\,\mathrm{W} = 1\,\mathrm{J\,s^{-1}} = 1\,\mathrm{kg\,m^2\,s^{-3}}$, we have

$$z_0 = \frac{1}{\pi}\sqrt{\frac{(5\,\mathrm{m})(15\times 10^{-2}\,\mathrm{m})^2(35\,\mathrm{kg\,m^2\,s^{-3}})}{2(2\times 10^{-3}\,\mathrm{kg})(20\,\mathrm{m\,s^{-1}})^3}} = \frac{1}{\pi}\sqrt{\frac{(5)(15)^2(35)\times 10^{-4}}{2(2)(20)^3\times 10^{-3}}\mathrm{m}^2} = \frac{1}{\pi}\sqrt{\frac{39375}{32000}\times 10^{-1}\,\mathrm{m}^2} = \frac{\sqrt{0.123}}{\pi}\mathrm{m}$$
$$= 0.11\,\mathrm{m} = 11\,\mathrm{cm}.$$

When we change the amplitude, we will hold all quantities fixed other than the wave speed. Referring back to Eq. (3), we can write

$$\langle P \rangle \propto v \quad\Longrightarrow\quad \frac{\langle P \rangle_f}{\langle P \rangle_i} = \frac{v_f}{v_i},$$

where v_f and v_i are the wave speeds before and after tripling, respectively, and $\langle P \rangle_i$ and $\langle P \rangle_f$ are the corresponding average powers. We know $v_f/v_i = 3$. Plugging in the given average power for the original amplitude, we find

$$\langle P \rangle_f = 3\langle P \rangle_i = 3(35\,\mathrm{W}) = 105\,\mathrm{W}.$$

If we instead allow the frequency vary as well, $\omega = kv$ tells us that $\omega_f/\omega_i = 3$ as well. Then we will get

$$\frac{\langle P \rangle_f}{\langle P \rangle_i} = \left(\frac{\omega_f}{\omega_i}\right)^2\frac{v_f}{v_i} = (3^2)(3) = 27,$$

and so

$$\langle P \rangle_f = 27\langle P \rangle_i = 27(35\,\mathrm{W}) = 945\,\mathrm{W}.$$