Problem 1. **(Peskin & Schroeder 2.1)** Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

1(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying

$$E^i = -F^{0i}; \qquad \epsilon^{ijk}B^k = -F^{ij}. \tag{2}$$

Solution. We want to extremize the action,

$$S[A_\mu] = \int d^4x\, \mathcal{L}(A_\mu, \partial_\mu A_\mu),$$

where $\mathcal{L}$ is the integrand of Eq. (1). Let δA_μ denote some arbitrary variation that vanishes at the boundaries of spacetime. The action for $A_\mu + \delta A_\mu$ is

$$S[A_\mu + \delta A_\mu] = \int d^4x\, \mathcal{L}(A_\mu + \delta A_\mu, \partial_\nu A_\mu + \partial_\nu \delta A_\mu).$$

Then, to first order in δA_μ, the variation of the action is

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu],$$

which we want to vanish for all δA_μ. Let $\delta F_{\mu\nu} = \partial_\mu\, \delta A_\nu - \partial_\nu\, \delta A_\mu$. Then, applying the definition of $F_{\mu\nu}$ given in Eq. (1),

$$\begin{aligned}
\delta S &= \int d^4x \left(-\frac{1}{4}(F_{\mu\nu} + \delta F_{\mu\nu})(F^{\mu\nu} + \delta F^{\mu\nu}) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\
&\approx \int d^4x \left(-\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\,\delta F^{\mu\nu} + \delta F_{\mu\nu}\, F^{\mu\nu}) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\
&= \int d^4x \left(-\frac{1}{4}(F_{\mu\nu}\,\delta F^{\mu\nu} + \delta F_{\mu\nu}\, F^{\mu\nu})\right) \\
&= \int d^4x \left(-\frac{1}{2}\delta F_{\mu\nu}\, F^{\mu\nu}\right),
\end{aligned} \tag{3}$$

where we have discarded terms of $\mathcal{O}\big((\delta A^\mu)^2\big)$ and swapped covariant and contravariant indices in going to the final equality.

Note that

$$\begin{aligned}
\delta F_{\mu\nu}\, F^{\mu\nu} &= (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \delta A_\nu\, \partial^\mu A^\nu - \partial_\mu \delta A_\nu\, \partial^\nu A^\mu - \partial_\nu \delta A_\mu\, \partial^\mu A^\nu + \partial_\nu \delta A_\mu\, \partial^\nu A^\mu.
\end{aligned} \tag{4}$$

Integrating the first term of Eq. (4) by parts, we have

$$\int d^4x\, \frac{\partial\, \delta A_\nu}{\partial x^\mu}\frac{\partial A^\nu}{\partial x_\mu} = \left[\delta A_\nu \frac{\partial A^\nu}{\partial x_\mu}\right]_{-\infty}^{\infty} - \int d^4x\, \delta A_\nu \frac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} = -\int d^4x\, \delta A_\nu\, \partial_\mu \partial^\mu A^\nu,$$

because δA^ν vanishes at $\pm\infty$. The other terms follow similarly. Then we find

$$\begin{aligned}\int d^4x\,\delta F_{\mu\nu}\,F^{\mu\nu} &= -\int d^4x\,(\delta A_\nu\,\partial_\mu\partial^\mu A^\nu - \delta A_\nu\,\partial_\mu\partial^\nu A^\mu - \delta A_\mu\,\partial_\nu\partial^\mu A^\nu + \delta A_\mu\,\partial_\nu\partial^\nu A^\mu)\\ &= -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\mu\,\partial_\nu F^{\nu\mu}) = -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\nu\,\partial_\mu F^{\mu\nu})\\ &= -2\int d^4x\,\delta A_\nu\,\partial_\mu F^{\mu\nu},\end{aligned}$$

where in going to the second-to-last equality we have simply swapped the indices.

Making this substitution in Eq. (3), we obtain

$$\delta S = \delta A_\nu \int d^4x\,\partial_\mu F^{\mu\nu}.$$

In order for the action to be at a local extremum, we need $\delta S = 0$ for any δA_ν. This implies that the integrand is 0. Thus, we obtain

$$\partial_\mu F^{\mu\nu} = 0, \tag{5}$$

which is the covariant form of the inhomogeneous Maxwell equations in a source-free region [**?** , p. 557], as we sought to derive. □

From Eq. (2) and the knowledge that $F^{\mu\nu}$ is antisymmetric [**?** , p. 556], we can write

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}. \tag{6}$$

The first equation of Eq. (2) is equivalent to $E^i = F^{i0}$. Then the zeroth component of Eq. (5) can be written

$$\partial_\mu F^{\mu 0} = \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = \boldsymbol{\nabla}\cdot\mathbf{E} = 0,$$

which is the differential form of Gauss's law.

For the remaining components of Eq. (5), we apply the second equation of Eq. (5) to find

$$\partial_\mu F^{\mu i} = -\frac{\partial E^i}{\partial t} + \epsilon^{ijk}\frac{\partial B^k}{\partial x^j} = 0.$$

In vector form, this is

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{0},$$

the differential form of Ampère's law.

1(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{7}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu, \tag{8}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{E^2 + B^2}{2}; \qquad\qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

Solution. We want to evaluate Eq. (2.17) of Peskin & Schroeder,

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \mathcal{L}\delta^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)}\partial^\nu A^\lambda - \mathcal{L}\delta^\mu{}_\nu, \tag{9}$$

where we have associated the field ϕ with A^λ. In order to evaluate the derivatives, we can use the variational method to calculate $\partial\mathcal{L}/\partial(\partial_\alpha A_\beta)$ by letting $\partial_\alpha A_\beta \to \partial_\alpha A_\beta + \delta\partial_\alpha A_\beta$ [**?** , p. 81]. Let

$$\delta\mathcal{L} = \mathcal{L}(\partial_\alpha A_\beta) - \mathcal{L}(\partial_\alpha A_\beta + \delta\partial_\alpha A_\beta).$$

Note that

$$\mathcal{L}(\partial_\alpha A_\beta + \delta\partial_\alpha A_\beta) = -\frac{1}{4}(F_{\alpha\beta} + \delta F_{\alpha\beta})(F^{\alpha\beta} + \delta F^{\alpha\beta}) \approx -\frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta} + F_{\alpha\beta}\,\delta F^{\alpha\beta} + \delta F_{\alpha\beta}\,F^{\alpha\beta}),$$

so

$$\begin{aligned}\delta\mathcal{L} &= -\frac{1}{4}(F_{\alpha\beta}\,\delta F^{\alpha\beta} + \delta F_{\alpha\beta}\,F^{\alpha\beta}) = -\frac{1}{2}\delta F_{\alpha\beta}\,F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha\,\delta A_\beta - \partial_\beta\,\delta A_\alpha)F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha\,\delta A_\beta + \partial_\alpha\,\delta A_\beta)F^{\alpha\beta}\\ &= -\partial_\alpha\,\delta A_\beta\,F^{\alpha\beta},\end{aligned}$$

where we have used the antisymmetry of $F^{\alpha\beta}$. This gives us

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta} \quad\implies\quad \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\beta)} = -F^\alpha{}_\beta,$$

and then we find

$$T^{\mu\nu} = -F^\mu{}_\lambda\,\partial^\nu A^\lambda + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda\,\partial^\nu A^\lambda. \tag{10}$$

Adding $K^{\lambda\mu\nu}$ as defined in Eq. (8), Eq. (7) becomes

$$\hat{T}^{\mu\nu} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda\,\partial^\nu A^\lambda + \partial_\lambda(F^{\mu\lambda}A^\nu). \tag{11}$$

Applying the product rule to the third term, we find

$$\partial_\lambda(F^{\mu\lambda}A^\nu) = A^\nu\partial_\lambda F^{\mu\lambda} + F^{\mu\lambda}\partial_\lambda A^\nu = -A^\nu\partial_\lambda F^{\lambda\mu} + F^{\mu\lambda}\partial_\lambda A^\nu = F^{\mu\lambda}\partial_\lambda A^\nu,$$

where we have applied the antisymmetry of $F^{\mu\nu}$ and Eq. (5). Making this substitution in Eq. (11),

$$\begin{aligned}\hat{T}^{\mu\nu} &= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda\,\partial^\nu A^\lambda + F^{\mu\lambda}\partial_\lambda A^\nu\\ &= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^\mu{}_\lambda\partial^\lambda A^\nu - F^\mu{}_\lambda\,\partial^\nu A^\lambda = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^\mu{}_\lambda F^{\lambda\nu}\\ &= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^\nu{}_\lambda.\end{aligned} \tag{12}$$

To show that $\hat{T}^{\mu\nu}$ is symmetric, note that

$$\hat{T}^{\nu\mu} = \frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta} - F^{\nu\lambda}F^\mu{}_\lambda = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\lambda F^{\nu\lambda} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^\nu{}_\lambda = \hat{T}^{\mu\nu}$$

as desired. □

For the energy and momentum densities, from Eq. (12) we have

$$\hat{T}^{00} = \frac{1}{4}g^{00}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^{0}{}_{\lambda} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + F^{0\lambda}F_{\lambda}{}^{0}, \tag{13}$$

$$\hat{T}^{0i} = \frac{1}{4}g^{0i}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^{i}{}_{\lambda} + = F^{0\lambda}F_{\lambda}{}^{i}. \tag{14}$$

Using Eq. (6),

$$F_{\mu\nu}F^{\mu\nu} = -E^{x2} - E^{y2} - E^{z2} - E^{x2} + B^{z2} + B^{y2} - E^{y2} + B^{z2} + B^{x2} - E^{z2} + B^{y2} + B^{x2} = 2(\mathbf{B}^2 - \mathbf{E}^2).$$

Note also from Eq. (6) that

$$F_{\lambda}{}^{\nu} = g_{\lambda\mu}F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{bmatrix},$$

so

$$F^{0\lambda}F_{\lambda}{}^{0} = (-\mathbf{E})\cdot(-\mathbf{E}) = \mathbf{E}^2, \qquad F^{0\lambda}F_{\lambda}{}^{i} = B_jE_k - E_kB_j = (\mathbf{E}\times\mathbf{B})_i.$$

Equations (13–14) are then

$$\hat{T}^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \mathcal{E}, \qquad \hat{T}^{0i} = \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})_i = \mathbf{S},$$

as we sought to show. □

Problem 2. The complex scalar field (Peskin & Schroeder 2.2) Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x\,(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi).$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

2(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x\,(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.