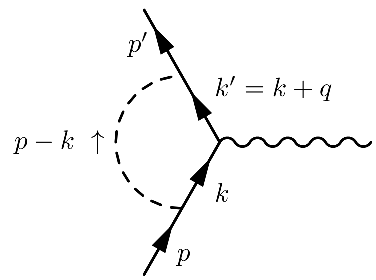
Problem 1. Exotic contributions to g − 2 (Peskin & Schroeder 6.3) Any particles that couples to the electron can produce a correction to the electron-photon form factors and, in particular, a correction to $g-2$. Because the electron $g-2$ agrees with QED to high accuracy, these corrections allow us to constrain the properties of hypothetical new particles.

1(a) The unified theory of weak and electromagnetic interactions contains a scalar particle h called the *Higgs boson*, which couples to the electron according to

$$H_{\text{int}} = \int d^3x \, \frac{\lambda}{\sqrt{2}} h\bar{\psi}\psi.$$

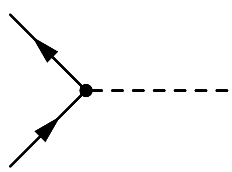
Compute the contribution of a virtual Higgs boson to the electron $(g-2)$, in terms of λ and the mass m_h of the Higgs boson.

Solution. We are interested in the diagram

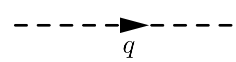


(1)

The Higgs field is a scalar Yukawa field, so we can use the form of the Yukawa interaction Hamiltonian of Peskin & Schroeder (4.112) and the appropriate Feynman rules to write [1, p. 118]



$$= -i\frac{\lambda}{\sqrt{2}},$$



$$= \frac{i}{q^2 - m_h^2 + i\epsilon}.$$

Our diagram is similar to the one on p. 189 in Peskin & Schroeder. We can then adapt (6.38) to write

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m_h^2 + i\epsilon} \bar{u}(p') \left(-i\frac{\lambda}{\sqrt{2}}\right) \frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon} \left(-i\frac{\lambda}{\sqrt{2}}\right) u(p) \\
&= i\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)}{[(k-p)^2 - m_h^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} u(p).
\end{aligned} \tag{2}$$

To evaluate the integral, we use Peskin & Schroeder (6.41) with $n = 3$:

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^n}.$$

Applying this to the denominator of the integrand of Eq. (2) gives us

$$\frac{1}{[(k-p)^2 - m_h^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3}, \tag{3}$$

where [1, pp. 190–191]

$$\begin{aligned}D &= x(k^2 - m_e^2) + y(k'^2 - m_e^2) + z[(k-p)^2 - m_h^2] + (x+y+z)i\epsilon \\ &= x(k^2 - m_e^2) + y(k^2 + 2kq + q^2 - m_e^2) + z(k^2 - 2kp + p^2 - m_h^2) + i\epsilon \\ &= (x+y+z)k^2 - (x+y)m_e^2 + 2k(qy - pz) + z(p^2 - m_h^2) + i\epsilon \\ &= k^2 + 2k(qy - pz) + z(p^2 - m_h^2) - (1-z)m_e^2 + i\epsilon.\end{aligned}$$

Here we have used $x + y + z = 1$ and $k' = k + q$. Let $\ell \equiv k + yq - zp$ [1, p. 191]. Then

$$D = \ell^2 + xyq^2 - (1-z)^2 m_e^2 - m_h^2 z + i\epsilon \equiv \ell^2 - \Delta + i\epsilon, \tag{4}$$

where we have defined $\Delta \equiv -xyq^2 + (1-z)^2 m_e^2 + z m_h^2$ [1, p. 191].

For the numerator of Eq. (2), let $N \equiv \bar{u}(p')(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)u(p)$. Then using $k' = k + q$ and $\ell \equiv k + yq - zp$ [1, p. 191],

$$N = \bar{u}(p')(\not{k} + \not{q} + m_e)\gamma^\mu(\not{k} + m_e)u(p) = \bar{u}(p')[\not{\ell} + (1-y)\not{q} + z\not{p} + m_e]\gamma^\mu(\not{\ell} - y\not{q} + z\not{p} + m_e)u(p). \tag{5}$$

We should be able to write this as an expression of the form given in (6.31) of Peskin & Schroeder [1, p. 191],

$$\Gamma^\mu = \gamma^\mu \cdot A + (p^{\mu\prime} + p^\mu) \cdot B + (p^{\mu\prime} - p^\mu) \cdot C = \gamma^\mu \cdot A + (p^{\mu\prime} + p^\mu) \cdot B + q^\mu \cdot C. \tag{6}$$

But from (6.45),

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0.$$

This means we can discard terms of $\mathcal{O}(\ell)$. We also know from (6.33) that

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m_e} F_2(q^2). \tag{7}$$

Since the correction to $g - 2$ is given by $F_2(q^2 = 0) = 0 + \delta F_2(q^2 = 0)$ (since $F_2 = 0$ to lowest order), we can discard terms of $\mathcal{O}(\gamma^\mu)$ in Eq. (5) [1, pp. 186, 196]. So Eq. (5) becomes

$$\begin{aligned}N &= \bar{u}(p')[\not{\ell} + (1-y)\not{q} + z\not{p} + m_e]\gamma^\mu(\not{\ell} - y\not{q} + z\not{p} + m_e)u(p) \\ &= \bar{u}(p')[\not{\ell}\gamma^\mu\not{\ell} - y(1-y)\not{q}\gamma^\mu\not{q} + z(1-y)\not{q}\gamma^\mu\not{p} + m_e(1-y)\not{q}\gamma^\mu - yz\not{p}\gamma^\mu\not{q} \\ &\qquad + z^2\not{p}\gamma^\mu\not{p} + m_e z\not{p}\gamma^\mu - m_e y\gamma^\mu\not{q} + m_e z\gamma^\mu\not{p}]u(p).\end{aligned} \tag{8}$$

To simplify these terms, we use Peskin & Schroeder (6.46):

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{g^{\mu\nu}\ell^2}{4D^3},$$

as well as [1, pp. 191–192]

$$\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}, \qquad \not{p}u(p) = m_e u(p), \qquad \bar{u}(p')\not{p}' = \bar{u}(p')m_e$$

and [2, 3]

$$\not{a}\not{b} = a \cdot b, \qquad \not{a}\not{b} + \not{b}\not{a} = 2a \cdot b.$$

We find

$$
\begin{aligned}
\not{\ell}\gamma^\mu\not{\ell} &= (2\ell^\mu - \gamma^\mu\not{\ell})\not{\ell} = 2\ell^\mu\ell^\nu\gamma_\nu - \gamma^\mu\not{\ell}\not{\ell} \to \frac{\ell^2 g^{\mu\nu}\gamma_\nu}{2} - \gamma^\mu\ell^2 = -\frac{\ell^2\gamma^\mu}{2}\\
&\to 0,\\
\not{q}\gamma^\mu\not{q} &= (2q^\mu - \gamma^\mu\not{q})\not{q} \to -\gamma^\mu\not{q}\not{q} = -q^2\gamma^\mu\\
&\to 0,\\
\not{q}\gamma^\mu\not{p} &\to \not{q}\gamma^\mu m_e = (\not{p}' - \not{p})\gamma^\mu m_e \to (m_e - \not{p})\gamma^\mu m_e = m_e^2\gamma^\mu - 2m_e p^\mu + m_e\gamma^\mu\not{p}\\
&\to -2m_e p^\mu,\\
\not{q}\gamma^\mu &= (\not{p}' - \not{p})\gamma^\mu \to m_e\gamma^\mu - \not{p}\gamma^\mu \to -2p^\mu + \gamma^\mu\not{p} \to -2p^\mu + \gamma^\mu m_e\\
&\to -2p^\mu,\\
\not{p}\gamma^\mu\not{q} &= (2p^\mu - \gamma^\mu\not{p})\not{q} \to -\gamma^\mu\not{p}\not{q} = -2\gamma^\mu p\cdot q + \gamma^\mu\not{q}\not{p} \to -2\gamma^\mu p\cdot q + \gamma^\mu(\not{p}' - \not{p})m_e\\
&\to \gamma^\mu q^2 + m_e\gamma^\mu\not{p}' - m_e^2\gamma^\mu = \gamma^\mu q^2 + m_e(2p^{\mu\prime} - \not{p}'\gamma^\mu) - m_e^2\gamma^\mu \to \gamma^\mu q^2 + 2m_e p^{\mu\prime} - 2m_e^2\gamma^\mu\\
&\to 2m_e p^{\mu\prime},
\end{aligned}
$$

where we have used

$$
2p\cdot q = p\cdot q + q\cdot p = p\cdot q + p'\cdot q - q^2 = p'^2 + p'\cdot p - p'\cdot p - p^2 - q^2 \to m_e^2 - m_e^2 - q^2 = -q^2,
$$

and

$$
\begin{aligned}
&\not{p}\gamma^\mu\not{p} \to m_e\not{p}\gamma^\mu = m_e(2p^\mu - \gamma^\mu\not{p}) \to 2m_e p^\mu - m_e^2\gamma^\mu \to 2m_e p^\mu,\\
&\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p} = 2p^\mu - \gamma^\mu m_e \to 2p^\mu,\\
&\gamma^\mu\not{q} = \gamma^\mu(\not{p}' - \not{p}) \to \gamma^\mu\not{p}' - \gamma^\mu m_e \to 2p^{\mu\prime} - \not{p}'\gamma^\mu \to 2p^{\mu\prime} - m_e\gamma^\mu \to 2p^{\mu\prime},\\
&\gamma^\mu\not{p} \to 0.
\end{aligned}
$$

Feeding these into Eq. (8), we obtain

$$
\begin{aligned}
N &= \bar{u}(p')[-2m_e z(1-y)p^\mu - 2m_e(1-y)p^\mu - 2m_e yz p^{\mu\prime} + 2m_e z^2 p^\mu + 2m_e z p^\mu - 2m_e y p^{\mu\prime}]u(p)\\
&= 2m_e\bar{u}(p')\{[z^2 + z - z(1-y) - (1-y)]p^\mu - y(1+z)p^{\mu\prime}\}u(p)\\
&= 2m_e\bar{u}(p')\{[z^2 + y(1+z) - 1]p^\mu - y(1+z)p^{\mu\prime}\}u(p)\\
&= 2m_e\bar{u}(p')[(z^2-1)p^\mu + y(1+z)(p^\mu - p^{\mu\prime})]u(p)\\
&= m_e\bar{u}(p')[(z^2-1)p^\mu + (z^2-1)p^\mu + 2y(1+z)(p^\mu - p^{\mu\prime}) + (z^2-1)p^{\mu\prime} - (z^2-1)p^{\mu\prime}]u(p)\\
&= m_e\bar{u}(p')[(z^2-1)(p^\mu + p^{\mu\prime}) + (z^2-1)(p^\mu - p^{\mu\prime}) + 2y(1+z)(p^\mu - p^{\mu\prime})]u(p)\\
&= m_e\bar{u}(p')[(z^2-1)(p^\mu + p^{\mu\prime}) - (z^2 + 2y(1+z) - 1)(p^{\mu\prime} - p^\mu)]u(p),
\end{aligned} \tag{9}
$$

which has the form of the second two terms of Eq. (6). According to the Ward identity, the coefficient of $q^\mu = p^\mu - p^\mu$ vanishes [1, p. 192]. Further, according to the Gordon identity given by Peskin & Schroeder (6.32),

$$
\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{p^{\mu\prime} + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p).
$$

So Eq. (9) becomes

$$
N = im_e\bar{u}(p')(1-z^2)\sigma^{\mu\nu}g^{\mu\nu}u(p). \tag{10}
$$

Feeding Eqs. (3), (4), and (10) into Eq. (2), we have (ignoring the $\mathcal{O}(\gamma^\mu)$ term)

$$\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) \to i\frac{\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p')\frac{2im_e(1-z^2)\sigma^{\mu\nu}g^{\mu\nu}}{(\ell^2-\Delta+i\epsilon)^3}u(p).$$

From Eq. (7), we can write

$$\delta F_2(q^2) = i\frac{\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2m_e^2(1-z^2)}{(\ell^2-\Delta+i\epsilon)^3}.$$

Computing the integral using Peskin & Schroeder (6.49),

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}, \tag{11}$$

we find

$$\delta F_2(q^2) = \frac{\lambda^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{m_e^2(1-z^2)}{\Delta}.$$

So, using Eq. (4),

$$\begin{aligned}
\delta F_2(q^2=0) &= \frac{\lambda^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{m_e^2(1-z^2)}{(1-z)^2m_e^2+zm_h^2}\\
&= \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\,\frac{m_e^2(1-z^2)}{(1-z)^2m_e^2+zm_h^2}\\
&= \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\,\frac{m_e^2(1-z^2)(1-z)}{(1-z)^2m_e^2+zm_h^2}\\
&= \frac{\lambda^2 m_e^2}{(4\pi)^2}\int_0^1 dz\,\frac{1-z-z^2+z^3}{(1-z)^2m_e^2+zm_h^2}\\
&= \frac{\lambda^2 m_e^2}{(4\pi)^2}\left(\int_0^1 dz\,\frac{1}{(1-z)^2m_e^2+zm_h^2} - \int_0^1 dz\,\frac{z(1+z-z^2)}{(1-z)^2m_e^2+zm_h^2}\right)\\
&\approx \frac{\lambda^2}{(4\pi)^2}\frac{m_e^2}{m_h^2}\left(\int_{m_e^2}^{m_h^2}\frac{du}{u} - \int_0^1 dz\,(1+z-z^2)\right)\\
&= \frac{\lambda^2}{(4\pi)^2}\frac{m_e^2}{m_h^2}\left(\Big[\ln u\Big]_{m_e^2}^{m_h^2} - \left[z+\frac{z^2}{2}-\frac{z^3}{3}\right]_0^1\right),
\end{aligned}$$

where we have used the substitution $u = (1-z)^2m_e^2 + zm_h^2$, and

$$du = 2(z-1)m_e^2 + m_h^2 \approx m_h^2$$

in the limit $m_e \ll m_h$. Referring to Peskin & Schroeder (6.58) and (6.59), we have

$$F_2(q^2=0) = \frac{\lambda^2}{(4\pi)^2}\frac{m_e^2}{m_h^2}\left[\ln\left(\frac{m_h^2}{m_e^2}\right) - \frac{7}{6}\right] \tag{12}$$

as the correction to the electron g factor from a virtual Higgs boson.

1(b) QED accounts extremely well for the electron's anomalous magnetic moment. If $a = (g-2)/2$,

$$|a_{\text{expt.}} - a_{\text{QED}}| < 1 \times 10^{-10}.$$

What limits does this place on λ and m_h? In the simplest version of the electroweak theory, $\lambda = 3 \times 10^{-6}$ and $m_h > 60\,\text{GeV}$. Show that these values are not excluded. The coupling of the Higgs boson to the muon is larger by a factor (m_μ/m_e): $\lambda = 6 \times 10^{-4}$. Thus, although our experimental knowledge of the muon anomalous magnetic moment is not as precise,

$$|a_{\text{expt.}} - a_{\text{QED}}| < 3 \times 10^{-8},$$

one can still obtain a stronger limit on m_h. Is it strong enough?

Solution. Feeding $m_e = 0.511\,\text{MeV}$ into Eq. (12), the limit becomes

$$\frac{\lambda^2}{m_h^2}\left[\ln\left(\frac{m_h^2}{0.261\,\text{MeV}^2}\right) - \frac{7}{6}\right] < 6.05 \times 10^{-8}\,\text{MeV}^{-2}.$$

For $\lambda = 3 \times 10^{-6}$ and $m_h > 60\,\text{GeV}$,

$$|a_{\text{expt.}} - a_{\text{QED}}| < \frac{(3\times 10^{-6})^2}{(4\pi)^2}\frac{(0.511\,\text{MeV})^2}{(60\times 10^3\,\text{MeV})^2}\left[\ln\left(\frac{(60\times 10^3\,\text{MeV})^2}{(0.511\,\text{MeV})^2}\right) - \frac{7}{6}\right] \approx 9\times 10^{-23} < 1\times 10^{-10},$$

so these values are not excluded. □

Now using $\lambda = 6 \times 10^{-4}$ and $m_\mu = 106\,\text{MeV}$ [4],

$$|a_{\text{expt.}} - a_{\text{QED}}| < \frac{(6\times 10^{-4})^2}{(4\pi)^2}\frac{(106\,\text{MeV})^2}{(60\times 10^3\,\text{MeV})^2}\left[\ln\left(\frac{(60\times 10^3\,\text{MeV})^2}{(106\,\text{MeV})^2}\right) - \frac{7}{6}\right] \approx 8\times 10^{-14} < 1\times 10^{-10},$$

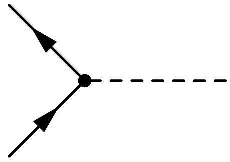
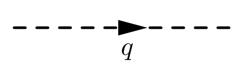
so these values are not excluded either, although they come closer to the upper limit.

1(c) Some more complex versions of this theory contain a pseudoscalar particle called the *axion*, which couples to the electron according to

$$H_{\text{int}} = \int d^3x\, \frac{i\lambda}{\sqrt{2}} a\bar{\psi}\gamma^5\psi.$$

The axion may be as light as the electron, or lighter, and may couple more strongly than the Higgs boson. Compute the contribution of a virtual axion to the $g-2$ of the electron, and work out the excluded values of λ and m_a.

Solution. The diagram we are interested in is the same as Eq. (1), but now the dashed line represents the pseudoscalar axion rather than the scalar Higgs boson. The Feynman rules are now [5, pp. 24–25]

 $= \dfrac{\lambda}{\sqrt{2}}\gamma^5,$ 

$= \dfrac{i}{q^2 - m_h^2 + i\epsilon}.$

Again, we adapt Peskin & Schroeder (6.38) to write

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{(k-p)^2 - m_h^2 + i\epsilon}\bar{u}(p')\left(\frac{\lambda}{\sqrt{2}}\gamma^5\right)\frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon}\gamma^\mu\frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon}\left(\frac{\lambda}{\sqrt{2}}\gamma^5\right)u(p) \\
&= i\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\bar{u}(p')\frac{\gamma^5(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)\gamma^5}{[(k-p)^2 - m_h^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)}u(p). \qquad (13)
\end{aligned}$$

The denominator is the same as in 1(a). For the numerator,

$$\begin{aligned} N &= \bar{u}(p')\gamma^5(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)\gamma^5 u(p) \\ &= \bar{u}(p')(-\not{k}' + m_e)\gamma^\mu(-\not{k} + m_e)u(p) \\ &= -\bar{u}(p')(\not{k}' - m_e)\gamma^\mu(\not{k} - m_e)u(p) \end{aligned}$$

since $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = 1$ by Peskin & Schroeder (3.70) and (3.71). Feeding in $k' = k + q$ and $\ell \equiv k + yq - zp$, and using the calculations from 1(a),

$$\begin{aligned} N &= -\bar{u}(p')[\not{\ell} + (1-y)\not{q} + z\not{p} - m_e]\gamma^\mu(\not{\ell} - y\not{q} + z\not{p} - m_e)u(p) \\ &= -\bar{u}(p')[\not{\ell}\gamma^\mu\not{\ell} - y(1-y)\not{q}\gamma^\mu\not{q} + z(1-y)\not{q}\gamma^\mu\not{p} - m_e(1-y)\not{q}\gamma^\mu - yz\not{p}\gamma^\mu\not{q} \\ &\qquad + z^2\not{p}\gamma^\mu\not{p} - m_e z\not{p}\gamma^\mu + m_e y\gamma^\mu\not{q} - m_e z\gamma^\mu\not{p}]u(p) \\ &\to -\bar{u}(p')[-2m_e z(1-y)p^\mu + 2m_e(1-y)p^\mu - 2m_e yzp^{\mu\prime} + 2m_e z^2 p^\mu - 2m_e zp^\mu + 2m_e yp^{\mu\prime}]u(p) \\ &= -2m_e\bar{u}(p')\{[(1-z)^2 - y(1-z)]p^\mu + y(1-z)p^{\mu\prime}\}u(p) \\ &= -2m_e\bar{u}(p')[(1-z)^2 p^\mu + y(1-z)(p^\mu - p^\mu)]u(p) \\ &= -m_e\bar{u}(p')[(1-z)^2p^\mu + (1-z)^2p^\mu + (1-z)^2p^{\mu\prime} - (1-z)^2p^{\mu\prime} + 2y(1-z)(p^{\mu\prime} - p^\mu)]u(p) \\ &= -m_e\bar{u}(p')[(1-z)^2(p^\mu + p^{\mu\prime}) - (1-z)^2(p^{\mu\prime} - p^\mu) + 2y(1-z)(p^{\mu\prime} - p^\mu)]u(p) \\ &= m_e\bar{u}(p')[(z-1)(2y+z-1)(p^{\mu\prime} - p^\mu) - (1-z)^2(p^\mu + p^{\mu\prime})]u(p). \end{aligned}$$

Feeding this as well as Eqs. (3) and (4) into Eq. (2), we have (again ignoring the $\mathcal{O}(\gamma^\mu)$ term)

$$\bar{u}(p')\delta\Gamma^\mu(p', p)u(p) \to i\frac{\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p')\frac{2im_e(1-z)^2\sigma^{\mu\nu}g^{\mu\nu}}{(\ell^2 - \Delta + i\epsilon)^3}u(p).$$

Once again using Eq. (11), we find

$$\delta F_2(q^2) = \frac{\lambda^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{m_e^2(1-z)^2}{\Delta},$$

which implies

$$\begin{aligned} \delta F_2(q^2 = 0) &= \frac{\lambda^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{m_e^2(1-z)^2}{(1-z)^2m_e^2 + zm_a^2} \\ &= \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z} dy\,\frac{m_e^2(1-z)^2}{(1-z)^2m_e^2 + zm_a^2} \\ &= \frac{\lambda^2 m_e^2}{(4\pi)^2}\int_0^1 dz\,\frac{(1-z)^3}{(1-z)^2m_e^2 + zm_a^2}. \end{aligned}$$

In the limit $m_a \ll m_e$, which is consistent with current limits on axion masses [4]

$$\delta F_2(q^2 = 0) \approx \frac{\lambda^2 m_e^2}{(4\pi)^2}\int_0^1 dz\,\frac{(1-z)^3}{(1-z)^2m_e^2} = \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\,(1-z) = \frac{\lambda^2}{(4\pi)^2}\left[1 - \frac{z^2}{2}\right] = \frac{\lambda^2}{32\pi^2}$$

is the contribution of a virtual axion to the electron $g-2$. For the excluded values of λ, then,

$$\frac{\lambda^2}{32\pi^2} < 1 \times 10^{-10} \qquad \Longrightarrow \qquad \lambda < 1.78 \times 10^{-4}.$$

This means $\lambda > 1.78 \times 10^{-4}$ is excluded for $m_a \ll m_e$.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] Wikipedia contributors, "Gamma matrices." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Gamma_matrices`.

[3] Wikipedia contributors, "Feynman slash notation." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Feynman_slash_notation`.

[4] P. D. Group, "Review of particle physics", *Progress of Theoretical and Experimental Physics* **2020** (8, 2020) `doi:10.1093/ptep/ptaa104`, `arXiv:https://academic.oup.com/ptep/article-pdf/2020/8/083C01/33653179/ptaa104.pdf`. 083C01.

[5] Chris Blair, "Quantum field theory—useful formulae and feynman rules", May, 2010. `https://www.maths.tcd.ie/~cblair/notes/list.pdf`.