Problem 1. Consider the following probabalistic game: There are four doors (Q, R, S, T). Behind each door is a device which displays ± 1 randomly according to the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$. Alice and Bob are on the same team. Alice has to choose either Q and R, and then Bob has to choose either S and T. When the numbers match, they get $+1$ point; when the numbers do not match, they get -1 point. However, when they open Q and T, it's an exception. When the numbers (do not) match, they get -1 $(+1)$.

1.1 Let's assume Alice and Bob open the doors completely randomly. When all numbers are $+1$ with probability 1, what is the expectation value of the point they get?

Solution. Let $\mathbf{E}$ be the expectation value of the number of points. In this case, the numbers behind the two doors will always match. So

$$\mathbf{E} = \frac{QS + RS + RT - QT}{4} = \frac{1 + 1 + 1 - 1}{4} = \frac{1}{2}.$$

1.2 As it turns out, irrespective of how hard you fine tune the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$, the expectation value of the point Alice and Bob get cannot exceed a certain value Max:

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} \leq \text{Max}.$$

Here, $\mathbf{E}(QS)$, etc. is the expectation value of the point when Alice opens Q and Bob opens S. This is a Bell inequality. Determine Max.

Hint: For a given realization of the numbers $Q = \pm 1$, $R = \pm 1$, $S = \pm 1$, $T = \pm 1$, which occurs with probability $P(Q, R, S, T)$, note that $QS + RS + RT - QT = (Q + R)S + (R - Q)S$, where one of $\{(R + Q), (R - Q)\}$ is 2 and the other 0.

Solution. The possiblities are listed in the following table:

Q	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	$+1$	-1
R	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	$+1$	-1	$+1$	-1	-1	-1	$+1$	-1	-1
S	$+1$	$+1$	-1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	$+1$	-1	$+1$	-1	-1	-1
T	$+1$	-1	$+1$	$+1$	$+1$	-1	-1	-1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1
$\mathbf{E}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Clearly, $\text{Max} = 1/2$.

1.3 Frustrated by the upper bound set by the Bell inequality, Bob decides to cheat. He now changes the value of T after Alice chooses Q or R. Assume Q, R, S are set to be $+1$ with probability 1. To make the expectation value of the point they get equal to $+1$, what values should Bob set after Alice chooses Q or R?

Solution. If Alice chooses R, Bob should set $T = 1$. If Alice chooses Q, Bob should set $T = -1$. This way,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1 + 1 + 1 + 1}{4} = 1.$$

1.4 Now consider a quantum mechanical version of the game. There are quantum states of two spin-1/2 degrees of freedom shared by Alice and Bob. Alice can measure the z component or x components of the first spin $\mathbf{S}^A$. (This corresponds to $Q = \pm 1$ or $R = \pm 1$.) Bob can measure the $-(z+x)$ component or the $(z-x)$ component of the first spin $\mathbf{S}^B$. (This corresponds to $S = \pm 1$ or $T = \pm 1$.)

More specifically, Alice and Bob share the quantum state

$$|\psi\rangle = \frac{|\uparrow_z\rangle \otimes |\downarrow_z\rangle - |\downarrow_z\rangle \otimes |\uparrow_z\rangle}{\sqrt{2}}.$$

The operators to be measured are

$$Q = S_z^A, \qquad R = S_x^A, \qquad S = -\frac{S_z^B + S_x^B}{\sqrt{2}}, \qquad T = \frac{S_x^B - S_z^B}{\sqrt{2}}.$$

Let us consider the case when Alice measures Q and Bob measures T. Calculate the probability $P(Q,T)$ for Alice and Bob getting the measurement outcomes $(Q,T) = (\pm 1, \pm 1)$.

Solution. All of the operators have eigenvalues $\pm\hbar/2$. In the S_z^A basis, Q and its eigenvectors are

$$Q = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad |Q_+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |Q_-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In the S_z^B basis, T can be written

$$T = \frac{\hbar}{2\sqrt{2}}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{\hbar}{2\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

The eigenvectors corresponding to eigenvalues $\pm\hbar/2$ can be found by

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \pm\sqrt{2}\begin{bmatrix} u \\ v \end{bmatrix},$$

which is satisfied when

$$v = (1 \mp \sqrt{2})u, \qquad u = -(1 \pm \sqrt{2})v.$$

We will fix $u = 1$. Then the normalization constants $A_\pm$ are found by

$$1 = |\langle T_\pm | T_\pm \rangle|^2 = A_\pm^2 \begin{bmatrix} 1 & 1 \mp \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \mp \sqrt{2} \end{bmatrix} = A_\pm^2 (4 \mp 2\sqrt{2}) \implies A_\pm = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}}.$$

The normalized eigenvectors are

$$|T_+\rangle = \frac{1}{\sqrt{4 - 2\sqrt{2}}}\begin{bmatrix} 1 \\ 1 - \sqrt{2} \end{bmatrix}, \qquad |T_-\rangle = \frac{1}{\sqrt{4 + 2\sqrt{2}}}\begin{bmatrix} 1 \\ 1 + \sqrt{2} \end{bmatrix}.$$

The probability for Alice and Bob's obtaining $(Q,T) = (+1,+1)$ is

$$P(Q=+1, T=+1) = |\langle Q_-, T_- | \psi \rangle|^2 = \left| \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{4 - 2\sqrt{2}}}\begin{bmatrix} 1 & 1 - \sqrt{2} \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \otimes -\frac{1}{\sqrt{4 - 2\sqrt{2}}} \right|^2 = \frac{1}{2}\frac{1}{4 - 2\sqrt{2}} = \frac{1}{8 - 4\sqrt{2}},$$

and the probability for their obtaining $(Q, T) = (-1, -1)$ is

$$P(Q=-1, T=-1) = |\langle Q_+, T_+|\psi\rangle|^2 = \left|\begin{bmatrix}0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1 & 1+\sqrt{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right|^2$$

$$= \left|-\frac{1}{\sqrt{2}} \otimes \frac{1}{\sqrt{4+2\sqrt{2}}}\right|^2 = \frac{1}{2}\frac{1}{4+2\sqrt{2}} = \frac{1}{8+4\sqrt{2}}.$$

1.5 Similarly, consider the case when Alice measures R and Bob measures T. Calculate the probability $P(R, T)$ for Alice and Bob getting the measurement outcomes $(R, T) = (\pm 1, \pm 1)$.

1.6 Compute the expectation values $\mathbf{E}(QT)$, $\mathbf{E}(RT)$, $\mathbf{E}(QS)$, and $\mathbf{E}(RS)$. Compute

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4}.$$

I consulted Sakurai's *Modern Quantum Mechanics*, Shankar's *Principles of Quantum Mechanics*, and Wolfram MathWorld while writing up these solutions.