Problem 1. (Jackson 9.8) *Hint:* The electromagnetic angular momentum density comes from more than the transverse (radiation zone) components of the fields.

1(a) Show that a classical oscillating electric dipole $\mathbf{p}$ with fields given by

$$\mathbf{H} = \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right), \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}\frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}, \tag{1}$$

radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\mathbf{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^* \times \mathbf{p}].$$

Solution. From Prob. 3 of Homework 4 for Physics 322, the angular momentum density is $\mathbf{l} = \mathbf{x} \times \mathbf{g}$, where $\mathbf{g}$ is the linear momentum density in SI units. According to Jackson 6.118, $\mathbf{g} = (\mathbf{E} \times \mathbf{H})/c^2$. However, according to Jackson (9.20), the time-averaged power radiated per unit solid angle by $\mathbf{p}$ is given by

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}[r^2\,\hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^*],$$

which indicates that we should send $\mathbf{H} \to \mathbf{H}^*$ for a sensible result. Thus,

$$\mathbf{l} = \frac{\mathbf{E} \times \mathbf{H}^*}{2c^2}.$$

One of the vector identities on the inside cover of Jackson is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, so

$$\mathbf{l} = \frac{(\mathbf{x} \cdot \mathbf{H}^*)\mathbf{E} - (\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{2c^2}. \tag{2}$$

From Eq. (1), note that

$$\mathbf{x} \cdot \mathbf{H}^* \propto \mathbf{x} \cdot (\hat{\mathbf{n}} \times \mathbf{p}^*) = \mathbf{p}^* \cdot (\mathbf{x} \times \hat{\mathbf{n}}) = \mathbf{0},$$

where we have used the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ and the fact that $\hat{\mathbf{n}}$ points in the $\mathbf{x}$ direction. For $\mathbf{x} \cdot \mathbf{E}$, note that

$$\begin{aligned}\mathbf{x} \cdot [(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}] &= -\mathbf{x} \cdot [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{p})] = -\mathbf{x} \cdot [(\hat{\mathbf{n}} \cdot \mathbf{p})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{p}] = -(\hat{\mathbf{n}} \cdot \mathbf{p})(\mathbf{x} \cdot \hat{\mathbf{n}}) + \mathbf{x} \cdot \mathbf{p} \\ &= -r(\hat{\mathbf{n}} \cdot \mathbf{p}) + \mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{p} = 0,\end{aligned}$$

$$\mathbf{x} \cdot [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] = 3(\mathbf{x} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3r(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3(\mathbf{x} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 2(\mathbf{x} \cdot \mathbf{p}),$$

since $|\mathbf{x}| = r$ and $\mathbf{x} = r\,\hat{\mathbf{n}}$. Then

$$\mathbf{x} \cdot \mathbf{E} = \frac{1}{2\pi\epsilon_0}(\mathbf{x} \cdot \mathbf{p})\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} = \frac{1}{2\pi\epsilon_0}(\hat{\mathbf{n}} \cdot \mathbf{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}.$$

With these substitutions, Eq. (2) becomes

$$\begin{aligned}\mathbf{l} &= -\frac{(\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{c^2} = -\frac{1}{4\pi\epsilon_0 c^2}(\hat{\mathbf{n}} \cdot \mathbf{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}\frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p}^*)\frac{e^{-ikr}}{r}\left(1 + \frac{1}{ikr}\right) \\ &= -\frac{k^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{r^2} - \frac{ik}{r}\right)\left(1 - \frac{i}{kr}\right) = -\frac{k^2}{16\pi^2\epsilon_0 c}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{r^2} - \frac{i}{kr^3} - \frac{ik}{r} - \frac{1}{r^2}\right) \\ &= -\frac{ik^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{kr^3} + \frac{k}{r^2}\right) = \frac{ik^3}{16\pi^2\epsilon_0 cr^2}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{k^2r^2} + 1\right).\end{aligned}$$

Let $\mathbf{L}$ be the angular momentum radiated to a distance R. Then

$$\mathbf{L} = \int_R \mathbf{l}(r)\, d^3x = \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r)\, r^2 \sin\theta \, dr \, d\phi \, d\theta,$$

and the time derivative is

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r)\, r^2 \sin\theta \, dr \, d\phi \, d\theta\right) = \frac{dr}{dt}\frac{d}{dr}\left(\int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r)\, r^2 \sin\theta \, dr \, d\phi \, d\theta\right) \\ &= c\int_0^\pi \int_0^{2\pi} \mathbf{l}(r)\, r^2 \sin\theta \, d\phi \, d\theta = \frac{ik^3}{16\pi^2\epsilon_0 c}\left(\frac{1}{k^2r^2}+1\right)\int_0^\pi \int_0^{2\pi} (\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\times\mathbf{p}^*)\sin\theta \, d\phi \, d\theta. \qquad (3)\end{aligned}$$

Note that

$$[(\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\times\mathbf{p}^*)]_i = \sum_{j=1}^3 n_j p_j (\hat{\mathbf{n}}\times\mathbf{p}^*)_i = \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \epsilon_{ikl} n_j p_j n_k p_l^*,$$

so

$$\frac{dL_i}{dt} \propto \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \int n_j p_k \, d\Omega = \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \frac{4\pi}{3}\delta_{jk} = \frac{4\pi}{3}\epsilon_{ikl} p_k p_l^* = \frac{4\pi}{3}(\mathbf{p}\times\mathbf{p}^*)_i,$$

where we have used Jackson (9.47), $\int n_\beta n_\gamma \, d\Omega = 4\pi\delta_{\beta\gamma}/3$. Making this substitution into Eq. (3),

$$\frac{d\mathbf{L}}{dt} = \frac{ik^3}{6\pi\epsilon_0 c}\left(\frac{1}{k^2r^2}+1\right)(\mathbf{p}\times\mathbf{p}^*).$$

Taking the limit as $r\to\infty$, we find

$$\frac{d\mathbf{L}}{dt} = \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0 c}(\mathbf{p}\times\mathbf{p}^*)\right] = \mathrm{Re}\left[-\frac{ik^3}{12\pi\epsilon_0 c}(\mathbf{p}^*\times\mathbf{p})\right] = \frac{k^3}{12\pi\epsilon_0 c}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}],$$

as desired. □

1(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

1(c) For a charge e rotating in the xy plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^2k^3a^2/6\pi\epsilon_0$. What about a charge oscillating along the z axis?

1(d) What are the results corresponding to Probs. 1(a) and 1(b) for magnetic dipole radiation?

Problem 2. (Jackson 10.1)

2(a) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{4} - |\boldsymbol{\epsilon}_0\cdot\hat{\mathbf{n}}|^2 - \frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}\right],$$

where $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{n}}$ are the directions of the incident and scattered radiations, respectively, while $\boldsymbol{\epsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\boldsymbol{\epsilon}_0^*\cdot\boldsymbol{\epsilon}_0 = 1$; $\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0 = 0$).

2(b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi)\right],$$

where $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of linear polarization.

2(c) What is the ratio of scattered intensities at $\theta = \pi/2$, $\phi = 0$ and $\theta = \pi/2$, $\phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

Problem 3. (Jackson 12.15) Consider the Proca equation for a localized steady-state distribution of current that has only a static magnetic moment. This model can be used to study the observable effects of a finite photon mass on the earth's magnetic field. Note that if the magnetization is $\boldsymbol{\mathcal{M}}(\mathbf{x})$ the current density can be written as $\mathbf{J} = c(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}})$.

3(a) Show that if $\boldsymbol{\mathcal{M}} = \mathbf{m}f(\mathbf{x})$, where $\mathbf{m}$ is a fixed vector and $f(\mathbf{x})$ is a localized scalar function, the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m}\times\boldsymbol{\nabla}\int f(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'.$$

3(b) If the magnetic dipole is a point dipole at the origin [$f(\mathbf{x}) = \delta(\mathbf{x})$], show that the magnetic field away from the origin is

$$\mathbf{B}(\mathbf{x}) = [3\,\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}]\left(1 + \mu r + \frac{\mu^2r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2\mathbf{m}\frac{e^{-\mu r}}{r}.$$

3(c) The result of Prob. 3(b) shows that at fixed $r = R$ (on the surface of the earth), the earth's magnetic field will appear as a dipole angular distribution, plus an added constant magnetic field (an apparently external field) antiparallel to $\mathbf{m}$. Satellite and surface observations lead to the conclusion that the "external" field is less than 4×10^{-3} times the dipole field at the magnetic equator. Estimate a lower limit on μ^{-1} in earth radii and an upper limit on the photon mass in grams from this datum.