1

Find the Euler-Lagrange equation associated with the functional

$$J[u(x,y,z)] = \int_R \sqrt{1+u_x^2+u_y^2+u_z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \tag{1}$$

where R is a region in three-dimensional space.

Solution. We will assume $u(x,y,z)$ has explicit values on the boundary of R, ∂R. By the definition of the action,

$$J[u] = \int_R \mathcal{L}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \implies \mathcal{L} = \sqrt{1+u_x^2+u_y^2+u_z^2}. \tag{2}$$

In general, the Euler-Lagrange equation is

$$0 = \frac{\partial\mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\partial u_z}. \tag{3}$$

Here,

$$\frac{\partial\mathcal{L}}{\partial u} = 0, \qquad \frac{\partial\mathcal{L}}{\partial u_x} = \frac{\partial\mathcal{L}}{\partial u_x^2}\frac{\partial u_x^2}{\partial u_x} = \frac{u_x}{\sqrt{1+u_x^2+u_y^2+u_z^2}} = \frac{u_x}{\mathcal{L}} \qquad \frac{\partial\mathcal{L}}{\partial u_y} = \frac{u_y}{\mathcal{L}}, \qquad \frac{\partial\mathcal{L}}{\partial u_z} = \frac{u_z}{\mathcal{L}}. \tag{4}$$

For the $\partial/\partial x$ term of (3),

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x} = \frac{\partial}{\partial x}\frac{u_x}{\mathcal{L}} = \frac{\partial u_x}{\partial x}\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} + \frac{\partial u_y}{\partial x}\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} + \frac{\partial u_z}{\partial x}\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}} \tag{5}$$

where

$$\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} = \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_x}\mathcal{L} - u_x\frac{\partial\mathcal{L}}{\partial u_x}\right) = \frac{1}{\mathcal{L}^2}\left(\mathcal{L} - u_x\frac{u_x}{\mathcal{L}}\right) = \frac{\mathcal{L}^2-u_x^2}{\mathcal{L}^3}, \tag{6}$$

$$\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} = \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_y}\mathcal{L} - u_x\frac{\partial\mathcal{L}}{\partial u_y}\right) = -\frac{u_xu_y}{\mathcal{L}^3}, \tag{7}$$

$$\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}} = -\frac{u_xu_z}{\mathcal{L}^3}, \tag{8}$$

so, generalizing this result to the $\partial/\partial y$ and $\partial/\partial z$ terms,

$$\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x} = u_{xx}\frac{\mathcal{L}^2-u_x^2}{\mathcal{L}^3} - u_{yx}\frac{u_xu_y}{\mathcal{L}^3} - u_{zx}\frac{u_xu_z}{\mathcal{L}^3}, \tag{9}$$

$$\frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial u_y} = u_{yy}\frac{\mathcal{L}^2-u_y^2}{\mathcal{L}^3} - u_{xy}\frac{u_xu_y}{\mathcal{L}^3} - u_{zy}\frac{u_yu_z}{\mathcal{L}^3}, \tag{10}$$

$$\frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\partial u_z} = u_{zz}\frac{\mathcal{L}^2-u_z^2}{\mathcal{L}^3} - u_{xz}\frac{u_xu_z}{\mathcal{L}^3} - u_{yz}\frac{u_yu_z}{\mathcal{L}^3}. \tag{11}$$

Then, assuming $u_{xy} = u_{yx}$ and so on, (3) becomes

$$\begin{aligned} 0 &= u_{xx}(\mathcal{L}^4-u_x^2) + u_{yy}(\mathcal{L}^4-u_y^2) + u_{zz}(\mathcal{L}^4-u_z^2) - 2u_{xy}u_xu_y - 2u_{yz}u_yu_z - 2u_{xz}u_xu_z && (12)\\ &= (u_{xx}+u_{yy}+u_{zz})(1+u_x^2+u_y^2+u_z^2) - u_{xx}u_x^2 - u_{yy}u_y^2 - u_{zz}u_z^2 - 2u_{xy}u_xu_y - 2u_{yz}u_yu_z - 2u_{xz}u_xu_z && (13)\\ &= u_{xx}(1+u_y^2+u_z^2) + u_{yy}(1+u_x^2+u_z^2) + u_{zz}(1+u_x^2+u_y^2) - 2u_{xy}u_xu_y - 2u_{yz}u_yu_z - 2u_{xz}u_xu_z. && (14) \end{aligned}$$

2 Plate vibrations (preliminiaries)

Start from Green's theorem

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial R} (P \, \mathrm{d}x + Q \, \mathrm{d}y), \tag{15}$$

where R is the region in the xy plane spanned by the plate, and ∂R its boundary.

2.a Show that

$$\int_R \phi \frac{\partial^2 \psi}{\partial x^2} \, \mathrm{d}x \, \mathrm{d}y = \int_R \psi \frac{\partial^2 \phi}{\partial x^2} \, \mathrm{d}x \, \mathrm{d}y + \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x}\right) \mathrm{d}y \, . \tag{16}$$

Solution. Let