

Problem 1. A spherical shell of radius R has a total charge Q uniformly spread over the shell. The shell is now put into uniform rotation about the z axis with angular velocity ω . Find the vector potential $\mathbf{A}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ everywhere, i.e., both inside and outside of the shell.

Solution. Let $\rho(\mathbf{x})$ be the charge density everywhere in space, so

$$\rho(\mathbf{x}) = \frac{1}{4\pi} \frac{Q}{R^2} \delta(r - R).$$

The linear velocity of the moving charge everywhere is

$$\mathbf{v}(\mathbf{x}) = \boldsymbol{\omega} \times \mathbf{x} = \omega r \delta(r - R) \hat{\boldsymbol{\phi}},$$

where $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$ is the angular velocity vector. Then the current density \mathbf{J} is simply the product of charge density and the linear velocity of the charge:

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x}) \mathbf{v}(\mathbf{x}) = \frac{Q\omega}{4\pi} \frac{r}{R^2} \delta(r - R) \hat{\boldsymbol{\phi}}.$$

From Eq. (4.21) in the lecture notes, $\mathbf{A}(\mathbf{x})$ everywhere is given by

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'.$$

The integral we need to evaluate is then

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi c} \frac{Q\omega}{R^2} \int \frac{r' \delta(r' - R)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \quad (1)$$

The problem is azimuthally symmetric, so we will rotate our coordinate system such that \mathbf{x} points along the z axis. Then, in the new coordinate system,

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{\sqrt{\mathbf{x}^2 - 2\mathbf{x} \cdot \mathbf{x}' + \mathbf{x}'^2}} = \frac{1}{\sqrt{r^2 - 2rr' \cos \theta' + r'^2}}.$$

Let $\boldsymbol{\omega}$ lie in the xz plane, and let α be the angle between $\boldsymbol{\omega}$ and the z axis. Then the linear velocity of the moving charge is

$$\begin{aligned} \mathbf{v}(\mathbf{x}') &= \boldsymbol{\omega} \times \mathbf{x}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin \alpha & 0 & \omega \cos \alpha \\ r' \sin \theta' \cos \varphi' & r' \sin \theta' \sin \varphi' & r' \cos \theta' \end{vmatrix} \\ &= -\omega r' (\cos \alpha \sin \theta' \sin \varphi') \hat{\mathbf{x}} + \omega r' (\cos \alpha \sin \theta' \cos \varphi' - \sin \alpha \cos \theta') \hat{\mathbf{y}} + \omega r' (\sin \alpha \sin \theta' \sin \varphi') \hat{\mathbf{z}}, \end{aligned}$$

so in the new coordinate system,

$$\mathbf{J}(\mathbf{x}') = \frac{Q}{4\pi} \frac{\boldsymbol{\omega} \times \mathbf{x}'}{R^2} \delta(r' - R) = \frac{Q\omega}{4\pi} \frac{r'}{R^2} (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}') \delta(r' - R),$$

where

$$\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}' = -(\cos \alpha \sin \theta' \sin \varphi') \hat{\mathbf{x}} + (\cos \alpha \sin \theta' \cos \varphi' - \sin \alpha \cos \theta') \hat{\mathbf{y}} + (\sin \alpha \sin \theta' \sin \varphi') \hat{\mathbf{z}}. \quad (2)$$

Then (1) becomes

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi c} \frac{Q\omega}{R^2} \int_0^{2\pi} \int_{-1}^1 \int_0^\infty \frac{r'^3 (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}') \delta(r' - R)}{\sqrt{r^2 - 2rr' \cos \theta' + r'^2}} dr' d(\cos \theta') d\varphi.$$

Evaluating the radial integral, we have

$$\mathbf{A}(\mathbf{x}) = \frac{Q\omega R}{4\pi c} \int_0^{2\pi} \int_{-1}^1 \frac{\hat{\omega} \times \hat{\mathbf{x}}'}{\sqrt{r^2 - 2Rr \cos \theta' + R^2}} d(\cos \theta') d\varphi. \quad (3)$$

For the angular integrals, the $\hat{\mathbf{x}}$ term in (??) gives us

$$-\cos \alpha \hat{\mathbf{x}} \int_{-1}^1 \frac{\sin \theta'}{\sqrt{r^2 - 2Rr \cos \theta' + R^2}} d(\cos \theta') \int_0^{2\pi} \sin \varphi' d\varphi \propto \left[-\cos \varphi' \right]_0^{2\pi} = 0.$$

Similarly, the $\hat{\mathbf{z}}$ term is

$$\sin \alpha \hat{\mathbf{z}} \int_{-1}^1 \frac{\sin \theta'}{\sqrt{r^2 - 2Rr \cos \theta' + R^2}} d(\cos \theta') \int_0^{2\pi} \sin \varphi' d\varphi \propto \left[-\cos \varphi' \right]_0^{2\pi} = 0.$$

There are two $\hat{\mathbf{y}}$ terms. For the first,

$$\cos \alpha \hat{\mathbf{y}} \int_{-1}^1 \frac{\sin \theta'}{\sqrt{r^2 - 2Rr \cos \theta' + R^2}} d(\cos \theta') \int_0^{2\pi} \cos \varphi' d\varphi \propto \left[\sin \varphi' \right]_0^{2\pi} = 0.$$

For the second,

$$\begin{aligned} & -\sin \alpha \hat{\mathbf{y}} \int_{-1}^1 \frac{\cos \theta'}{\sqrt{r^2 - 2Rr \cos \theta' + R^2}} d(\cos \theta') \int_0^{2\pi} d\varphi = -2\pi \sin \alpha \hat{\mathbf{y}} \int_{-1}^1 \frac{\cos \theta'}{\sqrt{r^2 - 2Rr \cos \theta' + R^2}} d(\cos \theta') \\ & = -2\pi \sin \alpha \hat{\mathbf{y}} \left(\left[-\frac{\cos \theta' \sqrt{r^2 - 2Rr \cos \theta' + R^2}}{Rr} \right]_{-1}^1 + \frac{1}{Rr} \int_{-1}^1 \sqrt{r^2 - 2Rr \cos \theta' + R^2} d(\cos \theta') \right) \\ & = -2\pi \sin \alpha \hat{\mathbf{y}} \left(\left[-\frac{\cos \theta' \sqrt{r^2 - 2Rr \cos \theta' + R^2}}{Rr} \right]_{-1}^1 + \frac{1}{Rr} \left[-\frac{(r^2 - 2Rr \cos \theta' + R^2)^{3/2}}{3Rr} \right]_{-1}^1 \right) \\ & = -2\pi \sin \alpha \hat{\mathbf{y}} \left(-\frac{\sqrt{r^2 + 2Rr + R^2}}{Rr} + \frac{\sqrt{r^2 - 2Rr + R^2}}{Rr} - \frac{(r^2 - 2Rr + R^2)^{3/2}}{3R^2 r^2} + \frac{(r^2 + 2Rr + R^2)^{3/2}}{3R^2 r^2} \right) \\ & = 2\pi \sin \alpha \frac{3Rr \sqrt{(r+R)^2} - 3Rr \sqrt{(r-R)^2} + [(r-R)^2]^{3/2} - [(r+R)^2]^{3/2}}{3R^2 r^2} \hat{\mathbf{y}} \\ & = 2\pi \sin \alpha \frac{3Rr|r+R| - 3Rr|r-R| + (r-R)^2|r-R| - (r+R)^2|r+R|}{3R^2 r^2} \hat{\mathbf{y}} \\ & = 2\pi \sin \alpha \frac{(r^2 + Rr + R^2)|r-R| - (r^2 - Rr + R^2)(r+R)}{3R^2 r^2} \hat{\mathbf{y}} \\ & = \frac{2\pi \sin \alpha \hat{\mathbf{y}}}{3R^2 r^2} \begin{cases} (r^2 + Rr + R^2)(R-r) - (r^2 - Rr + R^2)(r+R) & r < R, \\ (r^2 + Rr + R^2)(r-R) - (r^2 - Rr + R^2)(r+R) & r > R \end{cases} \\ & = -\frac{4}{3}\pi \sin \alpha \hat{\mathbf{y}} \begin{cases} \frac{r}{R^2} & r < R, \\ \frac{R}{r^2} & r > R. \end{cases} \end{aligned}$$

Finally, in the new coordinate system (3) is

$$\mathbf{A}(\mathbf{x}) = -\frac{Q\omega}{3c} \sin \alpha \hat{\mathbf{y}} \begin{cases} \frac{r}{R} & r < R, \\ \frac{R^2}{r^2} & r > R. \end{cases}$$

Transforming back to the old coordinate system, $\sin \alpha \rightarrow -\sin \theta$, and $\hat{\mathbf{y}} \rightarrow \hat{\boldsymbol{\varphi}}$ since the original system is azimuthally symmetric. Thus we have

$$\mathbf{A}(\mathbf{x}) = \frac{Q\omega}{3c} \sin \theta \hat{\boldsymbol{\varphi}} \begin{cases} \frac{r}{R} & r < R, \\ \frac{R^2}{r^2} & r > R. \end{cases}$$

The magnetic field is given by Eq. (1.7),

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (4)$$

In spherical coordinates,

$$\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\varphi}},$$

so

$$\mathbf{B} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial r} \hat{\boldsymbol{\theta}}.$$

For $r < R$,

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{Q\omega}{3c} \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{r}{R} \sin^2 \theta \right) \hat{\mathbf{r}} - \frac{\partial}{\partial r} \left(\frac{r^2}{R} \sin \theta \right) \hat{\boldsymbol{\theta}} \right] = \frac{Q\omega}{3c} \frac{1}{r} \left(\frac{r}{R} \frac{2 \cos \theta \sin \theta}{\sin \theta} \hat{\mathbf{r}} - \frac{2r}{R} \sin \theta \hat{\boldsymbol{\theta}} \right) \\ &= \frac{2}{3} \frac{Q\omega}{cR} (\cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}) = \frac{2}{3} \frac{Q\omega}{cR} \hat{\mathbf{z}}. \end{aligned}$$

For $r > R$,

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{Q\omega}{3c} \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{R^2}{r^2} \sin^2 \theta \right) \hat{\mathbf{r}} - \frac{\partial}{\partial r} \left(\frac{R^2}{r} \sin \theta \right) \hat{\boldsymbol{\theta}} \right] = \frac{Q\omega}{3c} \frac{1}{r} \left(\frac{R^2}{r^2} \frac{2 \cos \theta \sin \theta}{\sin \theta} \hat{\mathbf{r}} + 2 \frac{R^2}{r^2} \sin \theta \hat{\boldsymbol{\theta}} \right) \\ &= \frac{2}{3} \frac{Q\omega}{c} \frac{R^2}{r^3} (\cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}). \end{aligned}$$

In summary,

$$\mathbf{B}(\mathbf{x}) = \frac{2}{3} \frac{Q\omega}{c} \begin{cases} \frac{\hat{\mathbf{z}}}{R} & r < R, \\ \frac{R^2}{r^3} (\cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}) & r > R. \end{cases}$$

Problem 2. If an electric and magnetic field are both present, the momentum density carried by the electromagnetic field is given by Poynting's formula

$$\mathcal{P} = \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B}).$$

Consider a bounded distribution of time-independent charges and currents, i.e., $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are time independent and vanish when $|\mathbf{x}| > R$ for some R .

2.a Show that the total momentum can be written as

$$\mathbf{P} \equiv \int \mathcal{P}(\mathbf{x}) d^3x = \int \phi(\mathbf{x}) \mathbf{J}(\mathbf{x}) d^3x.$$

Solution. Applying (4),

$$\mathbf{E} \times \mathbf{B} = \mathbf{E} \times (\nabla \times \mathbf{A}).$$

Vector identity (4) in Griffiths is

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a},$$

which allows us to write

$$\mathbf{E} \times \mathbf{B} = \nabla(\mathbf{A} \cdot \mathbf{E}) - \mathbf{A} \times (\nabla \times \mathbf{E}) - (\mathbf{A} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{A} = \nabla(\mathbf{A} \cdot \mathbf{E}) - (\mathbf{A} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{A}, \quad (5)$$

since $\nabla \times \mathbf{E} = 0$ in electrostatics by Eq. (1.4) in the lecture notes. Now using component notation with implied sums,

$$(\mathbf{A} \cdot \nabla)E_i = A_j \frac{\partial E_i}{\partial x_j} = \frac{\partial}{\partial x_j} - E_i \frac{\partial A_j}{\partial x_j} = \frac{\partial}{\partial x_j}.$$

Here we have used the product rule in addition to Eq. (4.20), which states that $\nabla \cdot \mathbf{A} = 0$ in the Coulomb gauge, which we may choose without loss of generality. Similarly,

$$(\mathbf{E} \cdot \nabla)A_i = E_j \frac{\partial A_i}{\partial x_j} = \frac{\partial}{\partial x_j} - A_i \frac{\partial E_j}{\partial x_j} = \frac{\partial}{\partial x_j} + A_i \nabla^2 \phi,$$

where we have used Eq. (2.2), $\mathbf{E} = -\nabla\phi$, which holds in the electrostatic case. Putting this all together, (5) gives us

$$(\mathbf{E} \times \mathbf{B})_i = \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} - A_i \nabla^2 \phi,$$

and so

$$\int (\mathbf{E} \times \mathbf{B})_i d^3x = \int \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} - A_i \nabla^2 \phi \right) d^3x. \quad (6)$$

Let $L \geq R$. Note that

$$\int f(\mathbf{x}) d^3x = \lim_{L \rightarrow \infty} \int_{-L}^L \int_{-L}^L \int_{-L}^L f(\mathbf{x}) dx dy dz.$$

Then for the first term of (6), integrating with respect to x_i by parts yields

$$\lim_{L \rightarrow \infty} \int_{-L}^L \frac{\partial}{\partial x_i} dx_i = \left[A_j E_j \right]_{-L}^L = 0,$$

since both $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ vanish for $|\mathbf{x}| > R$. This means $\mathbf{E} \rightarrow 0$ and $\mathbf{A} \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$. Applying similar logic to the second and third terms,

$$\lim_{L \rightarrow \infty} \int_{-L}^L \frac{\partial}{\partial x_j} dx_j = \left[A_j E_i \right]_{-L}^L = 0, \quad \lim_{L \rightarrow \infty} \int_{-L}^L \frac{\partial}{\partial x_j} dx_j = \left[E_j A_i \right]_{-L}^L = 0,$$

where there are no implied sums over the derivatives. Now (6)

$$\int (\mathbf{E} \times \mathbf{B})_i d^3x = - \int A_i \nabla^2 \phi d^3x. \quad (7)$$

Green's theorem is given by Eq. (2.96),

$$\int_S \hat{\mathbf{n}} \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) dS = -4\pi \int_V (\phi_1 \rho_2 - \phi_2 \rho_1) d^3x = \int_V (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) d^3x,$$

where the final equality comes from the proof in Eq. (2.97). Let \mathcal{V} be a cube of side length $2L$ centered at the origin, and let S be the surface of the cube. Applying Green's theorem gives us

$$\int_{\mathcal{V}} (\mathbf{E} \times \mathbf{B})_i d^3x = \int_S \hat{\mathbf{n}} \cdot (\phi \nabla A_i - A_i \nabla \phi) dS - \int_{\mathcal{V}} \phi \nabla^2 A_i d^3x.$$

Note that

$$\lim_{L \rightarrow \infty} \int_S \hat{\mathbf{n}} \cdot (\phi \nabla A_i - A_i \nabla \phi) dS \sim \lim_{L \rightarrow \infty} \int_S \frac{1}{|\mathbf{x}|^3} dS = 0$$

since $\phi, A_i \sim 1/|\mathbf{x}|$ for bounded ρ, \mathbf{J} and therefore $\nabla \phi, \nabla A_i \sim 1/|\mathbf{x}|^2$. Now (7) is

$$\int (\mathbf{E} \times \mathbf{B}) d^3x = - \int \phi \nabla^2 \mathbf{A} d^3x.$$

Vector identity (11) in Griffiths states that

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a},$$

which gives us

$$\int (\mathbf{E} \times \mathbf{B}) d^3x = \int \phi [\nabla \times (\nabla \times \mathbf{A}) - \nabla(\nabla \cdot \mathbf{A})] d^3x = \frac{4\pi}{c} \int \phi \mathbf{J} d^3x,$$

where we have once again used the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$, and that $\nabla \times (\nabla \times \mathbf{A}) = 4\pi \mathbf{J}/c$ from Eq. (4.4). Thus, we have proven

$$\mathbf{P} = \int \mathcal{P}(\mathbf{x}) d^3x = \frac{1}{4\pi c} \int (\mathbf{E} \times \mathbf{B}) d^3x = \frac{1}{c^2} \int \phi(\mathbf{x}) \mathbf{J}(\mathbf{x}) d^3x$$

as desired, except for the factor of $1/c^2$. □

2.b Give an example of a stationary, bounded charge and current distribution for which $\mathbf{P} \neq 0$.

Solution. Consider a toroid in the xy plane centered on the origin with current I through its wire, which has N total turns. Let \mathcal{V} be the interior volume of the toroid. In cylindrical coordinates (s, φ, z) , the magnetic field due to the toroid is given by Eq. (5.60) in Griffiths (which I have converted into Gaussian units):

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \frac{2NI}{s} \hat{\varphi} & \mathbf{x} \in \mathcal{V}, \\ 0 & \mathbf{x} \notin \mathcal{V}. \end{cases}$$

Consider also a point charge of charge Q at the origin, so in the center of the toroid. The electric field due to the point charge is simply

$$\mathbf{E} = \frac{Q}{s^2 + z^2} \hat{\mathbf{r}}.$$

Note that $\mathbf{E} \times \mathbf{B} \neq 0$ only within \mathcal{V} . Then

$$\mathbf{P} = \frac{1}{4\pi c} \int (\mathbf{E} \times \mathbf{B}) d^3x = \frac{2NIQ}{4\pi c} \int_{\mathcal{V}} \frac{\hat{\mathbf{s}} \times \hat{\varphi}}{s(s^2 + z^2)} d^3x = \frac{NIQ}{2\pi c} \hat{\mathbf{z}} \int_{\mathcal{V}} \frac{d^3x}{s(s^2 + z^2)},$$

which is nonzero because \mathcal{V} is bounded.

Problem 3. The angular momentum density of the electromagnetic field is given by

$$\mathbf{l} = \mathbf{x} \times \mathcal{P} = \frac{c}{4\pi} \mathbf{x} \times (\mathbf{E} \times \mathbf{B}).$$

Consider a source free ($\rho = 0$, $\mathbf{J} = 0$) solution to Maxwell's equations in electrodynamics with \mathbf{E} and \mathbf{B} vanishing rapidly as $|\mathbf{x}| \rightarrow \infty$, so the total angular momentum

$$\mathbf{L} = \int \mathbf{l} d^3x$$

is well defined. Show that \mathbf{L} is conserved, i.e., independent of time.

Solution. We want to show that

$$\frac{d\mathbf{L}}{dt} = 0.$$

From Eq. (5.18), the failure of linear momentum conservation to hold for the electromagnetic field alone, in general, is

$$\frac{\partial \mathcal{P}_i}{\partial t} - \sum_{j=1}^3 \partial_j T_{ij} = - \left[\rho E_i + \frac{1}{c} (\mathbf{J} \times \mathbf{B})_i \right],$$

where the stress-energy tensor T_{ij} is defined in Eq. (5.11),

$$T_{ij} = \frac{1}{4\pi} \left[E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (|\mathbf{E}|^2 + |\mathbf{B}|^2) \right],$$

and Eq. (5.19) defines

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B}$$

as the force meter unit volume that the electromagnetic field exerts on matter.

We can use the distributive property of the cross product to write

$$\mathbf{x} \times \frac{\partial \mathcal{P}}{\partial t} - \mathbf{x} \times (\nabla \cdot \mathbf{T}) = -\mathbf{x} \times \mathbf{f} = 0,$$

where we use the notation $(\nabla \cdot \mathbf{T})_i = \sum_{j=1}^3 \partial_j T_{ij}$, and we have used the fact that we are working with a source-free solution to write $\mathbf{f} = 0$. We are free to move the time derivative since \mathbf{x} represents the point at which we are evaluating the angular momentum, and is not time dependent. Thus, we have

$$\mathbf{x} \times (\nabla \cdot \mathbf{T}) = \frac{\partial}{\partial t} \mathbf{l} = \frac{\partial \mathbf{l}}{\partial t}. \quad (8)$$

The y component of this vector is

$$\frac{\partial l_y}{\partial t} = z(\nabla \cdot \mathbf{T})_x - x(\nabla \cdot \mathbf{T})_z,$$

where

$$\begin{aligned} (\nabla \cdot \mathbf{T})_x &= \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \right) \\ &= \frac{1}{4\pi} \left[\frac{\partial}{\partial x} \left(E_x^2 + B_x^2 - \frac{E^2 + B^2}{2} \right) + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right]. \end{aligned}$$

Note that $(\nabla \cdot \mathbf{T})_y$ and $(\nabla \cdot \mathbf{T})_y$ have similar forms.

Integrating (8) over all of space,

$$\int \frac{\partial l_y}{\partial t} d^3x = \int [z(\nabla \cdot \mathbf{T})_x - x(\nabla \cdot \mathbf{T})_z] d^3x. \quad (9)$$

Note that

$$\int \frac{\partial \mathbf{l}}{\partial t} d^3x = \lim_{L \rightarrow \infty} \int_{-L}^L \int_{-L}^L \int_{-L}^L \frac{\partial \mathbf{l}}{\partial t} dx dy dz,$$

so the first term of (9) becomes

$$\begin{aligned} \int z(\nabla \cdot \mathbf{T})_x d^3x &= \lim_{L \rightarrow \infty} \frac{1}{4\pi} \int_{-L}^L \int_{-L}^L \int_{-L}^L z \frac{\partial}{\partial x} \left(E_x^2 + B_x^2 - \frac{E^2 + B^2}{2} \right) dx dy dz \\ &\quad + \lim_{L \rightarrow \infty} \frac{1}{4\pi} \int_{-L}^L \int_{-L}^L \int_{-L}^L z \frac{\partial}{\partial y} dx dy dz \\ &\quad + \lim_{L \rightarrow \infty} \frac{1}{4\pi} \int_{-L}^L \int_{-L}^L \int_{-L}^L z \frac{\partial}{\partial z} dx dy dz. \end{aligned}$$

For the first term of this integral, integrating by parts with respect to x yields

$$\lim_{L \rightarrow \infty} \int_{-L}^L \frac{\partial}{\partial x} \left(E_x^2 + B_x^2 - \frac{E^2 + B^2}{2} \right) dx = \lim_{L \rightarrow \infty} \left[E_x^2 + B_x^2 - \frac{E^2 + B^2}{2} \right]_{-L}^L = 0,$$

since \mathbf{E} and \mathbf{B} vanish rapidly as $|\mathbf{x}| \rightarrow \infty$. For the second term,

$$\int_{-L}^L \frac{\partial}{\partial y} dy = \lim_{L \rightarrow \infty} \left[E_x E_y + B_x B_y \right]_{-L}^L = 0.$$

For the third term,

$$\int_{-L}^L z \frac{\partial}{\partial z} dz = \lim_{L \rightarrow \infty} \left[z(E_x E_z + B_x B_z) \right]_{-L}^L - \lim_{L \rightarrow \infty} \int_{-L}^L (E_x E_z + B_x B_z) dz.$$

Since \mathbf{E} and \mathbf{B} fall off “rapidly,” we assume they overtake $|\mathbf{x}|$ as $|\mathbf{x}| \rightarrow \infty$. That is, $\mathbf{E}, \mathbf{B} \sim 1/|\mathbf{x}|^n$ where $n > 1$. Then

$$\lim_{L \rightarrow \infty} \left[z(E_x E_z + B_x B_z) \right]_{-L}^L \sim \lim_{L \rightarrow \infty} \left[\frac{1}{z^{n-1}} \right]_{-L}^L = 0, \quad \lim_{L \rightarrow \infty} \int_{-L}^L (E_x E_z + B_x B_z) dz \sim \lim_{L \rightarrow \infty} \left[\frac{1}{z^{n-1}} \right]_{-L}^L = 0$$

Thus, (9) becomes

$$\int \frac{\partial l_y}{\partial t} d^3x = 0 \implies \int \frac{\partial \mathbf{l}}{\partial t} d^3x = 0$$

by symmetry. We may move the derivative out of the integral to obtain

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \int \mathbf{l} d^3x = \int \frac{\partial \mathbf{l}}{\partial t} d^3x = 0,$$

as we sought to prove. \square

In addition to the course lecture notes, I consulted Griffiths’s *Introduction to Electrodynamics*, Jackson’s *Classical Electrodynamics*, and K. T. McDonald’s and D. K. Ghosh’s notes on electromagnetism while writing up these solutions.