1

Find the Euler-Lagrange equation associated with the functional

$$J[u(x,y,z)] = \int_R \sqrt{1+u_x^2+u_y^2+u_z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,,$$

where R is a region in three-dimensional space.

Solution. We will assume $u(x,y,z)$ has explicit values on the boundary of R, ∂R. By the definition of the action,

$$J[u] = \int_R \mathcal{L}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \implies \mathcal{L} = \sqrt{1+u_x^2+u_y^2+u_z^2}.$$

In general, the Euler-Lagrange equation is

$$0 = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z}. \tag{1}$$

Here,

$$\frac{\partial \mathcal{L}}{\partial u} = 0, \qquad \frac{\partial \mathcal{L}}{\partial u_x} = \frac{\partial \mathcal{L}}{\partial u_x^2}\frac{\partial u_x^2}{\partial u_x} = \frac{u_x}{\sqrt{1+u_x^2+u_y^2+u_z^2}} = \frac{u_x}{\mathcal{L}} \qquad \frac{\partial \mathcal{L}}{\partial u_y} = \frac{u_y}{\mathcal{L}}, \qquad \frac{\partial \mathcal{L}}{\partial u_z} = \frac{u_z}{\mathcal{L}}.$$

For the $\partial/\partial x$ term of (1),

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} = \frac{\partial}{\partial x}\frac{u_x}{\mathcal{L}} = \frac{\partial u_x}{\partial x}\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} + \frac{\partial u_y}{\partial x}\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} + \frac{\partial u_z}{\partial x}\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}}$$

where

$$\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} = \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_x}\mathcal{L} - u_x\frac{\partial \mathcal{L}}{\partial u_x}\right) = \frac{1}{\mathcal{L}^2}\left(\mathcal{L} - u_x\frac{u_x}{\mathcal{L}}\right) = \frac{\mathcal{L}^2-u_x^2}{\mathcal{L}^3}, \tag{2}$$

$$\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} = \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_y}\mathcal{L} - u_x\frac{\partial \mathcal{L}}{\partial u_y}\right) = -\frac{u_x u_y}{\mathcal{L}^3}, \tag{3}$$

$$\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}} = -\frac{u_x u_z}{\mathcal{L}^3}, \tag{4}$$

Generalizing (2)–(4) to the $\partial/\partial y$ and $\partial/\partial z$ terms,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} &= u_{xx}\frac{\mathcal{L}^2-u_x^2}{\mathcal{L}^3} - u_{yx}\frac{u_x u_y}{\mathcal{L}^3} - u_{zx}\frac{u_x u_z}{\mathcal{L}^3}, \\
\frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} &= u_{yy}\frac{\mathcal{L}^2-u_y^2}{\mathcal{L}^3} - u_{xy}\frac{u_x u_y}{\mathcal{L}^3} - u_{zy}\frac{u_y u_z}{\mathcal{L}^3}, \\
\frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z} &= u_{zz}\frac{\mathcal{L}^2-u_z^2}{\mathcal{L}^3} - u_{xz}\frac{u_x u_z}{\mathcal{L}^3} - u_{yz}\frac{u_y u_z}{\mathcal{L}^3}.
\end{aligned}$$

Then, assuming $u_{xy}=u_{yx}$, $u_{yz}=u_{zy}$, and $u_{xz}=u_{zx}$, (1) becomes

$$\begin{aligned}
0 &= u_{xx}(\mathcal{L}^4-u_x^2) + u_{yy}(\mathcal{L}^4-u_y^2) + u_{zz}(\mathcal{L}^4-u_z^2) - 2u_{xy}u_xu_y - 2u_{yz}u_yu_z - 2u_{xz}u_xu_z \\
&= (u_{xx}+u_{yy}+u_{zz})(1+u_x^2+u_y^2+u_z^2) - u_{xx}u_x^2 - u_{yy}u_y^2 - u_{zz}u_z^2 - 2u_{xy}u_xu_y - 2u_{yz}u_yu_z - 2u_{xz}u_xu_z \\
&= u_{xx}(1+u_y^2+u_z^2) + u_{yy}(1+u_x^2+u_z^2) + u_{zz}(1+u_x^2+u_y^2) - 2u_{xy}u_xu_y - 2u_{yz}u_yu_z - 2u_{xz}u_xu_z.
\end{aligned}$$

2 Plate vibrations (preliminaries)

Start from Green's theorem

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{\partial R} (P\,\mathrm{d}x + Q\,\mathrm{d}y), \tag{5}$$

where R is the region in the xy plane spanned by the plate, and ∂R its boundary.

2.a Show that

$$\int_R \phi \frac{\partial^2 \psi}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_R \psi \frac{\partial^2 \phi}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial R} \left(\phi\frac{\partial \psi}{\partial x} - \psi\frac{\partial \phi}{\partial x}\right) \mathrm{d}y\,.$$

Solution. In (5), let

$$Q = \phi\frac{\partial \psi}{\partial x} - \psi\frac{\partial \phi}{\partial x}, \qquad P = 0.$$

Then

$$\frac{\partial Q}{\partial x} = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \phi\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial x} - \psi\frac{\partial^2 \phi}{\partial x^2} = \phi\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \phi}{\partial x^2}, \qquad P = 0.$$

Making these substitutions into (5) gives

$$\begin{aligned} \int_R \left(\phi\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \phi}{\partial x^2}\right) \mathrm{d}x\,\mathrm{d}y &= \int_{\partial R} \left(\phi\frac{\partial \psi}{\partial x} - \psi\frac{\partial \phi}{\partial x}\right) \mathrm{d}y \\ \iff \int_R \phi\frac{\partial^2 \psi}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_R \psi\frac{\partial^2 \phi}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial R} \left(\phi\frac{\partial \psi}{\partial x} - \psi\frac{\partial \phi}{\partial x}\right) \mathrm{d}y \end{aligned}$$

as desired. □

2.b Work out analogous expressions for

$$\int_R \psi \frac{\partial^2 \psi}{\partial x^2}\,\mathrm{d}x\,\mathrm{d}y\,, \tag{6}$$

$$\int_R \phi \frac{\partial^2 \psi}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{7}$$

Solution. For (6), ???

$$\int_R \phi \frac{\partial^2 \psi}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_R \psi \frac{\partial^2 \psi}{\partial y^2}\,\mathrm{d}x\,\mathrm{d}y - \int_{\partial R} \left(\phi\frac{\partial \psi}{\partial y} - \psi\frac{\partial \phi}{\partial y}\right) \mathrm{d}x$$

For (7), let

$$2Q = \phi\frac{\partial \psi}{\partial y} - \psi\frac{\partial \phi}{\partial y}, \qquad 2P = \psi\frac{\partial \phi}{\partial x} - \phi\frac{\partial \psi}{\partial x}.$$

Then

$$\begin{aligned} 2\frac{\partial Q}{\partial x} &= \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} + \phi\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial y} - \psi\frac{\partial^2 \phi}{\partial x \partial y} = \phi\frac{\partial^2 \psi}{\partial x \partial y} - \psi\frac{\partial^2 \phi}{\partial x \partial y}, \\ 2\frac{\partial P}{\partial y} &= \frac{\partial \psi}{\partial y}\frac{\partial \phi}{\partial x} + \psi\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x} - \phi\frac{\partial^2 \psi}{\partial x \partial y} = \psi\frac{\partial^2 \phi}{\partial x \partial y} - \phi\frac{\partial^2 \psi}{\partial x \partial y}. \end{aligned}$$

Substituting into (5), we have

$$\frac{1}{2}\int_R\left(\phi\frac{\partial^2\psi}{\partial x\partial y}-\psi\frac{\partial^2\phi}{\partial x\partial y}-\psi\frac{\partial^2\phi}{\partial x\partial y}+\phi\frac{\partial^2\psi}{\partial x\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\int_{\partial R}\left(\psi\frac{\partial\phi}{\partial x}-\phi\frac{\partial\psi}{\partial x}\right)\mathrm{d}x+\frac{1}{2}\int_{\partial R}\left(\phi\frac{\partial\psi}{\partial y}-\psi\frac{\partial\phi}{\partial y}\right)\mathrm{d}y$$
$$\iff\int_R\phi\frac{\partial^2\psi}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y=\int_R\psi\frac{\partial^2\phi}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2}\int_{\partial R}\left(\psi\frac{\partial\phi}{\partial x}-\phi\frac{\partial\psi}{\partial x}\right)\mathrm{d}x+\frac{1}{2}\int_{\partial R}\left(\phi\frac{\partial\psi}{\partial y}-\psi\frac{\partial\phi}{\partial y}\right)\mathrm{d}y\,.$$

3 Plate vibrations

Start with the action for a vibrating plate whose potential energy is dominated by bending,

$$S[u(x,y,t)]=\epsilon\int_{t_0}^{t_1}\int_R\left\{\rho u_t^2-\kappa_1\left[(u_{xx}^2+u_{yy}^2)-2(1-\mu)(u_{xx}u_{yy}-u_{xy}^2)\right]\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \tag{8}$$

where ρ is the mass density per unit area, κ_1 has the dimension of energy and is sometimes called flexural rigidity, and μ is a dimensionless material constant called Poisson's ratio. For isotropic material, $\rho = 1/4$. Notice that there is *no* external bending moment applied to the plate boundary. There is also *no* external forcing.

3.a Using the results of problem 2, show that the variation generated by going from a solution u_0 to $u_0+\epsilon\psi$ has the form

$$\delta S=\epsilon\int_{t_0}^{t_1}\int_R\left(\rho u_{tt}-\kappa_1\nabla^4 u\right)\psi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\epsilon\int_{t_0}^{t_1}\int_{\partial R}\left(P(u)\psi+M(u)\frac{\partial\psi}{\partial n}\right)\mathrm{d}l\,\mathrm{d}t\,.$$

Specify $P(u)$ and $M(u)$.

Solution. Define $\mathcal{L}$ as the integrand in (8), and $\psi=\psi(x,y,t)$ as some variatoin of u. Then in general,

$$\delta S=\epsilon\int_{t_0}^{t_1}\int_R\left[\left(\frac{\partial\mathcal{L}}{\partial u}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial\mathcal{L}}{\partial u_y}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial u_t}\right)\psi+\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial u_x}\psi\right)+\frac{\partial}{\partial y}\left(\frac{\partial\mathcal{L}}{\partial u_y}\psi\right)+\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial u_t}\psi\right)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

3.b Finally, derive the Euler-Lagrange equation and the associated boundary conditions.

4 Vibrations of a circular disk

The only scenario in which plate vibrations can be described analytically in terms of known functions is a circular disk. Work with polar coordinates (r,θ), the Euler-Lagrange equation

$$u_{tt}-\lambda\nabla^4 u=0, \tag{9}$$

and the boundary conditions

$$u=0, \qquad \frac{\partial u}{\partial n}=0.$$

4.a Show that this problem reduces to an eigenvalue problem if we assume that $u(r,\theta,t)$ is separable:

$$u = v(r,\theta)\, g(t). \tag{10}$$

Write down the general form of $g(t)$.

Solution. Substituting the ansatz (10) into (9), we have

$$v\frac{\partial^2 g}{\partial t^2} - \lambda g\,\nabla^4 v = 0 \implies \frac{1}{g}\frac{\partial^2 g}{\partial t^2} = \lambda\frac{1}{v}\nabla^4 v \equiv -\mu \tag{11}$$

where we have defined some constant μ. We may then separate (11) into two differential equations,

$$\lambda\nabla^4 v + \mu v = 0, \tag{12}$$

$$\frac{\partial^2 g}{\partial t^2} + \mu g = 0. \tag{13}$$

The eigenvalue problem is (12), which we may solve for the eigenvalues μ_n and obtain the eigenfunctions $v_n(r,\theta)$. Then we simply feed μ_n into (13) to obtain $g_n(t)$, which have the general form

$$g(t) = C_1 e^{\sqrt{\mu}x} + C_2 e^{-\sqrt{\mu}x}, \tag{14}$$

where we note that $\sqrt{\mu}$ may be imaginary. If so, (14) may be written in terms of sines and cosines. Finally, the solutions to (9) are $u_n(r,\theta,t) = v_n(r,\theta)\, g_n(t)$.

4.b Now consider the eigenvalue problem

$$(\nabla^4 - k^4)v(r,\theta) = 0, \tag{15}$$

with λ set to be k^4. Notice that it factors into

$$(\nabla^2 - k^2)(\nabla^2 + k^2)v(r,\theta) = 0, \tag{16}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Since the disk is circular, we expect the vibration modes to be periodic in θ. This suggests the ansatz

$$v = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{in\theta}. \tag{17}$$

Obtain the ODE governing $f_n(r)$.

Solution. Firstly, note that

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial\theta^4}.$$

Substituting the ansatz of (17) into (15) yields

$$\begin{aligned}
k^4 f_n(r)\, e^{in\theta} &= -\nabla^4 f_n(r)\, e^{in\theta} \\
&= \left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial\theta^4}\right) f_n(r)\, e^{in\theta} \\
&= e^{in\theta}\left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2n^2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2n^2}{r^3}\frac{\partial}{\partial r} + \frac{n^4}{r^4}\right) f_n(r).
\end{aligned}$$

Dividing out $e^{in\theta}$, we have

$$k^4 f_n(r) = \frac{\partial^4 f_n(r)}{\partial r^4} + \frac{2}{r}\frac{\partial^3 f_n(r)}{\partial r^3} + \frac{1-2n^2}{r^2}\frac{\partial^2 f_n(r)}{\partial r^2} - \frac{2n^2}{r^3}\frac{\partial f_n(r)}{\partial r} + \frac{n^4}{r^4} f_n(r)$$

as the ODE governing $f_n(r)$.

4.c What are the appropriate boundary conditions on $f_n(r)$?

Solution. Firstly, note that (15) may be separated into the two eigenvalue problems

$$0 = \nabla^4 v - k^2 v, \tag{18}$$

$$0 = \nabla^4 v + k^2 v. \tag{19}$$

Any k that corresponds to a nontrivial solution of (15) must also correspond to a nontrivial solution of (18) and of (19). We will proceed by solving (18) for k_m and (19) for k_p. Then, any $k_n \in k_m \cap k_p$ that nontrivially solves both (19) and (18) must also nontrivially solve (15).

Beginning with (18), we have

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f_m(r)\, e^{im\theta} - k_m^2 f_m(r)\, e^{im\theta} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) f_m\, e^{im\theta} - \frac{1}{r^2} m^2 f_m\, e^{im\theta} - k_m^2 f_m\, e^{im\theta}$$

where we have substituted the ansatz (17), here $v_m = f_m(r)\, e^{im\theta}$. Dividing out $e^{im\theta}$, this becomes

$$0 = r^2 \frac{\partial^2 f_m}{\partial r^2} + r\frac{\partial f_m}{\partial r} - (k_m^2 r^2 + m^2) f_m, \tag{20}$$

which is the modified Bessel equation of order m. It has solutions

$$f_m(r) = C_1 I_m(kr) + C_2 K_p(kr),$$

where C_1 and C_2 are constants, I_m is the modified Bessel function of the first kind, and K_m is the modified Bessel function of the second kind. Both functions are of order m.

Proceeding similarly for (19), we obtain

$$0 = r^2 \frac{\partial^2 f_p}{\partial r^2} + r\frac{\partial f_p}{\partial r} + (k_m^2 r^2 - p^2) f_p, \tag{21}$$

which is the Bessel equation of order p, and has solutions

$$f_p(r) = D_1 J_p(kr) + D_2 Y_p(kr),$$

where D_1 and D_2 are constants, J_p is the Bessel function of the first kind, Y_p is the Bessel function of the second kind, and both are of order p.

Both Y_n and K_n diverge as $r \to 0$ for all n, so we do not want them in our solution.

In writing these solutions, I consulted Gelfand and Fomin's *Calculus of Variations*, Olmstead and Volpert's *Differential Equations in Applied Mathemtics.*