1. **Problem.** Suppose we have a mechanical system with $n$ degrees of freedom. Let $q_1(t), q_2(t), \ldots, q_n(t)$ be its generalized coordinates. Now consider a time-dependent coordinate transformation

$$Q_i = Q_i(t, q_1, q_2, \ldots, q_n) \qquad i = 1, 2, \ldots, n.$$

Show that if $q_i(t)$ solve a system of Euler-Lagrange equations involving a Lagrangian $L(t, q_i, \dot{q}_i)$, then $Q_i(t)$ solves the Euler-Lagrange equations involving $L(t, Q_i, \dot{Q}_i)$ provided the time-dependent coordinate transformation fulfills some minimal standard of good behavior. Specify this "minimal standard of good behavior."

**Solution.** Suppose that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0; \tag{1}$$

that is, $q_i(t)$ solve a system of Euler-Lagrange equations. We want to show that

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = 0. \tag{2}$$

Beginning with the first term of (2), we can use the chain rule to write

$$\frac{\partial L}{\partial Q_i} = \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial Q_i}. \tag{3}$$

However, $\partial q_j/\partial Q_i$ and $\partial \dot{q}_j/\partial Q_i$ are only guaranteed to exist if there is an inverse transformation

$$q_i = q_i(t, Q_1, Q_2, \ldots, Q_n) \qquad i = 1, 2, \ldots, n. \tag{4}$$

This is only possible if there is a one-to-one correspondence between $q_i(t)$ and $Q_i(t)$, which is the "minimal standard of good behavior" for the transformation.

Assuming this is the case, we can write

$$\dot{q}_i = \frac{\partial q_i}{\partial Q_j}\dot{Q}_j + \frac{\partial q_i}{\partial t} \tag{5}$$

so (3) becomes

$$\frac{\partial L}{\partial Q_i} = \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j}\left(\frac{\partial^2 q_j}{\partial Q_i\,\partial Q_k}\dot{Q}_k + \frac{\partial^2 q_j}{\partial t\,\partial Q_i}\right). \tag{6}$$

For the second term of (2), we have

$$\frac{\partial L}{\partial \dot{Q}_i} = \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \dot{Q}_i} = \frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial Q_i} \tag{7}$$

where the right-hand side comes from applying (5). Then, using the product rule to take the time derivative,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = \frac{\partial q_j}{\partial Q_i}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\frac{\partial q_j}{\partial Q_i}. \tag{8}$$

For the second term of (8), the chain rule gives

$$\frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}\frac{\partial q_j}{\partial Q_i} = \frac{\partial L}{\partial \dot{q}_j}\left(\frac{\partial^2 q_j}{\partial t\,\partial Q_i} + \frac{\partial^2 q_j}{\partial Q_i\,\partial Q_k}\dot{Q}_k\right). \tag{9}$$

The second term on the right side also appeared in (6), so substituting back into (8) we now have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = \frac{\partial q_j}{\partial Q_i}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial Q_i} - \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_i}. \tag{10}$$

Rearranged, this is

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = \frac{\partial q_j}{\partial Q_i}\left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right). \tag{11}$$

Finally, substituting the original assumption (1), we have

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = 0 \tag{12}$$

which is what we sought to prove. □