Problem 1. Beta functions in Yukawa theory (P&S 12.1) In the pseudoscalar Yukawa theory studied in Problem 10.2, with masses set to zero,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \bar{\psi}(i\not{\partial})\psi - ig\bar{\psi}\gamma^5\psi\phi, \tag{1}$$

compute the Callan-Symanzik β functions for λ and g:

$$\beta_\lambda(\lambda, g), \qquad\qquad \beta_g(\lambda, g),$$

to leading order in coupling constants, assuming that λ and g^2 are of the same order. Sketch the coupling constant flows in the λ-g plane.

Solution. The β function of a generic dimensionless coupling constant g, associated with an n-point vertex, is given by P&S (12.53),

$$\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}g\sum_i \delta_{Z_i}\right), \tag{2}$$

where the sum is over the external legs. This expression for β implies P&S (12.54),

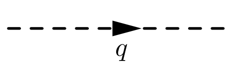
$$\beta(g) = -2B - g\sum_i A_i, \tag{3}$$

where [1, pp. 414–415]

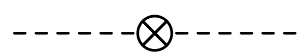
$$\delta_Z = A\ln\left(\frac{\Lambda^2}{M^2}\right) + \text{finite}, \qquad\qquad \delta_g = -B\ln\left(\frac{\Lambda^2}{M^2}\right) + \text{finite}; \tag{4}$$

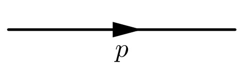
Λ being a momentum cutoff and M being the renormalization scale at which we define the theory [1, p. 408]. Both Eq. (2) and Eq. (3) hold for the β function of Yukawa theory [1, p. 415]. It also holds for our pseudoscalar Yukawa theory, since having a pseudoscalar field as opposed to a scalar one only changes the values of the counterterms.

We computed the Feynman rules for a Lagrangian like Eq. (1) in Problem 1 of Homework 3. Setting the masses to zero, we have

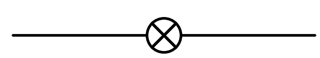


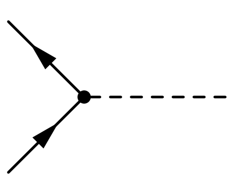
$= \dfrac{i}{q^2 + i\epsilon}$

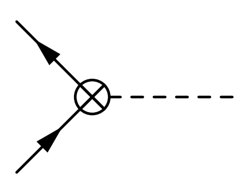

$= ip^2\delta_{Z_1}$

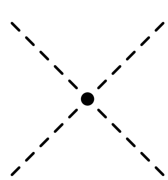


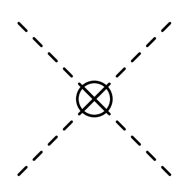
$= \dfrac{i\not{p}}{p^2 + i\epsilon}$


$= i\not{p}\delta_{Z_2}$


$= g\gamma^5$


$= \delta_g\gamma^5$


$= -i\lambda$


$= -i\delta_\lambda$

where the counterterms are (omitting the finite parts)

$$\delta_{Z_1} = -\frac{g^2}{32\pi^2}\frac{2}{\epsilon}, \qquad \delta_{Z_2} = -\frac{g^2}{8\pi^2}\frac{2}{\epsilon}, \qquad \delta_g = \frac{g^3}{16\pi^2}\frac{2}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2 - 48g^4}{32\pi^2}\frac{2}{\epsilon}. \tag{5}$$

When computing these counterterms, we used dimensional regularization. However, in order to use Eq. (3) to find the β function, we need the counterterms to have the form of Eq. (4). This requires switching to the modified minimal subtraction scheme with renormalization scale M. We can find the M dependence by simply making the replacement $2/\epsilon \to -\ln(M^2)$ in Eq. (5).

We check that this is true by comparing the δ_λ counterterm for ϕ^4 theory using the two schemes. With dimensional regularization, it is given by P&S (10.24),

$$\delta_\lambda = \frac{3\lambda^2}{32\pi^2}\frac{2}{\epsilon} + \text{finite}$$

in the limit $d \to 4$. With renormalization scale M, it is given by P&S (12.45):

$$\delta_\lambda = \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{2-d/2} - \ln(M^2) + \text{finite}\right)$$

in the limit $d \to 4$.

With similar replacements, Eq. (5) becomes

$$\delta_{Z_1} = \frac{g^2}{32\pi^2}\ln(M^2), \qquad \delta_{Z_2} = \frac{g^2}{8\pi^2}\ln(M^2), \qquad \delta_g = -\frac{g^3}{16\pi^2}\ln(M^2), \qquad \delta_\lambda = -\frac{3\lambda^2 - 48g^4}{32\pi^2}\ln(M^2).$$

Referring to Eq. (4), this implies

$$A_{Z_1} = -\frac{g^2}{32\pi^2}, \qquad A_{Z_2} = -\frac{g^2}{8\pi^2}, \qquad B_g = -\frac{g^3}{16\pi^2}, \qquad B_\lambda = -\frac{3\lambda^2 - 48g^4}{32\pi^2}.$$

Applying Eq. (3) for g, we have

$$\beta_g(\lambda, g) = 2B_g - g(A_{Z_2} + 2A_{Z_1}) = 2\frac{g^3}{16\pi^2} - g\left(\frac{g^2}{8\pi^2} + 2\frac{g^2}{32\pi^2}\right) = \frac{g^3 + 2g^3 + g^3}{16\pi^2} = \frac{5g^3}{16\pi^2},$$

where the factors of 1 and 2 come from the numbers of external pseudoscalar and fermion legs, respectively, in the Feynman diagram with vertex g.

Now applying Eq. (3) for λ, we have

$$\beta_\lambda(\lambda, g) = 2B_\lambda - \lambda(4A_{Z_1}) = 2\frac{3\lambda^2 - 48g^4}{32\pi^2} - 4\lambda\frac{g^2}{32\pi^2} = \frac{3\lambda^2 - 48g^4 + 2\lambda g^2}{16\pi^2}$$

where the factor of 4 comes from the number of external scalar legs in the 4ϕ vertex.

We know from Lecture 13 that the β functions of the components of the vector field tangent to the renormalization group flows of the coupling constants. Figure 1 shows the coupling constant flows in the λ-g plane. This figure was created by plotting the streamlines for the vector field (β_λ, β_g) in Mathematica.

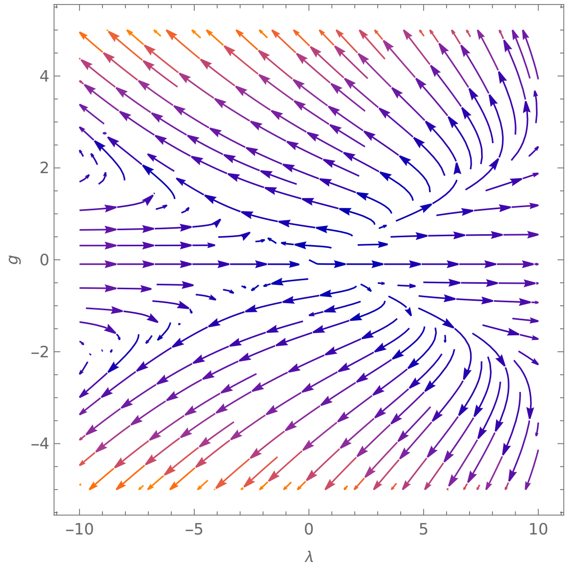


Figure 1: Coupling constant flow in the λ-g plane.

Problem 2. Beta function of the Gross-Neveu model (P&S 12.2) Compute $\beta(g)$ in the two-dimensional Gross-Neveu model studied in Problem 11.3,

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2,$$

with $i = 1, \dots, N$. You should find that this model is asymptotically free. How was that fact reflected in the solution to Problem 11.3?

Solution. We saw in Problem 2 of Homework 4 that this Lagrangian can be written as

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2,$$

where σ is a new scalar field with no kinetic energy terms. In the modified minimal subtraction scheme, we found the effective potential was

$$V_{\text{eff}} = \sigma^2\left\{\frac{1}{2g^2} + \frac{N}{4\pi}\left[\ln\left(\frac{\sigma^2}{M^2}\right) - 1\right]\right\}. \tag{6}$$

Since $\Gamma[\phi_{\text{cl}}] = -(VT)V_{\text{eff}}(\phi)$ by P&S (11.50), we have

$$\Gamma[\sigma_{\text{cl}}] = -(VT)\sigma^2\left\{\frac{1}{2g^2} + \frac{N}{4\pi}\left[\ln\left(\frac{\sigma^2}{M^2}\right) - 1\right]\right\}. \tag{7}$$

Referring to p. 3 of Lecture 11, we can apply the Callan-Symanzik equation to Γ. The Callan-Symanzik equation is P&S (12.41),

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right]G^{(n)}(\{x_i\}; M, \lambda) = 0.$$

For our problem, γ is 0 because there are no field insertions. That is, we have

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right]\Gamma[\phi_{\text{cl}}] = 0.$$

Using Eq. (7), note that

$$\frac{\partial \Gamma}{\partial M} = (VT)\frac{N\sigma^2}{2\pi M}, \qquad \frac{\partial \Gamma}{\partial g} = (VT)\frac{\sigma^2}{g^3}.$$

Then

$$0 = (VT)\left(\frac{N\sigma^2}{2\pi} + \beta(g)\frac{\sigma^2}{g^3}\right) \quad \Longrightarrow \quad \beta_g = -\frac{Ng^3}{2\pi}.$$

This model is asymptotically free because the β function is proportional to $-g^3$ [1, pp. 424–425].

In 2(e) of Homework 4, we found that the vacuum expectation value of σ was

$$\sigma = \pm M e^{-\pi/Ng^2} = \pm v.$$

We showed that the vacuum expectation value does not depend on the renormalization condition chosen. This means that we can increase $M \to 0$ while holding σ constant, and see that $g \to 0$ logarithmically. This is indicative of an asymptotically-free theory [1, p. 425]. □

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.