Problem 1. Connection coefficients for spherical polar coordinates (MCP 24.9)

1(a) Consider spherical polar coordinates in 3-dimensional space, and verify that the nonzero connection coefficients, assuming an orthonormal basis, are given by Eq. (11.71).

Solution. We follow the procedure on pp. 1171–1172 of MCP for computing the connection coefficients. We first evaluate the commutation coefficients $c_{\alpha\beta}{}^{\rho}$ using MCP (24.38a),

$$c_{\alpha\beta}{}^{\rho} = \vec{e}^{\rho} \cdot [\vec{e}_\alpha, \vec{e}_\beta], \tag{1}$$

We lower the last index using (24.38b),

$$c_{\alpha\beta\gamma} = c_{\alpha\beta}{}^{\rho} \boldsymbol{g}_{\rho\gamma}.$$

Then we use (24.38c) to compute

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\boldsymbol{g}_{\alpha\beta,\gamma} + \boldsymbol{g}_{\alpha\gamma,\beta} - \boldsymbol{g}_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}), \tag{2}$$

and raise the first index using (24.38d),

$$\Gamma^{\mu}{}_{\beta\gamma} = \boldsymbol{g}^{\mu\alpha}\Gamma_{\alpha\beta\gamma}. \tag{3}$$

From (24.40), the commutator is given by

$$[\vec{A}, \vec{B}] = \nabla_{\vec{A}}\vec{B} - \nabla_{\vec{B}}\vec{A}. \tag{4}$$

We also note that $\boldsymbol{g}_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$ [1, p. 1161].

For an orthonormal basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$, $\boldsymbol{g}$ is the Kronecker delta [1, p. 614]. In spherical coordinates, the gradient is

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial \phi},$$

and its components are [2]

$$\begin{aligned}
&\nabla_r\hat{\mathbf{r}} = \mathbf{0}, && \nabla_\theta\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\theta}}, && \nabla_\phi\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\phi}},\\
&\nabla_r\hat{\boldsymbol{\theta}} = \mathbf{0}, && \nabla_\theta\hat{\boldsymbol{\theta}} = -\frac{1}{r}\hat{\mathbf{r}}, && \nabla_\phi\hat{\boldsymbol{\theta}} = \frac{1}{r\tan\theta}\hat{\boldsymbol{\phi}},\\
&\nabla_r\hat{\boldsymbol{\phi}} = \mathbf{0}, && \nabla_\theta\hat{\boldsymbol{\phi}} = \mathbf{0}, && \nabla_\phi\hat{\boldsymbol{\phi}} = -\frac{1}{r\tan\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r}\hat{\mathbf{r}}.
\end{aligned}$$

Applying Eq. (4) and the above, we find

$$\begin{aligned}
&[\hat{\mathbf{r}}, \hat{\mathbf{r}}] = \nabla_r\hat{\mathbf{r}} - \nabla_r\hat{\mathbf{r}} = \mathbf{0}, && [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = \nabla_r\hat{\boldsymbol{\theta}} - \nabla_\theta\hat{\mathbf{r}} = -\frac{1}{r}\hat{\boldsymbol{\theta}}, && [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = \nabla_r\hat{\boldsymbol{\phi}} - \nabla_\phi\hat{\mathbf{r}} = -\frac{1}{r}\hat{\boldsymbol{\phi}},\\
&[\hat{\boldsymbol{\theta}}, \hat{\mathbf{r}}] = -[\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = \frac{1}{r}\hat{\boldsymbol{\theta}}, && [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] = \nabla_\theta\hat{\boldsymbol{\theta}} - \nabla_\theta\hat{\boldsymbol{\theta}} = \mathbf{0}, && [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = \nabla_\theta\hat{\boldsymbol{\phi}} - \nabla_\phi\hat{\boldsymbol{\theta}} = -\frac{1}{r\tan\theta}\hat{\boldsymbol{\phi}},\\
&[\hat{\boldsymbol{\phi}}, \hat{\mathbf{r}}] = -[\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = \frac{1}{r}\hat{\boldsymbol{\phi}}, && [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}] = -[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = \frac{1}{r\tan\theta}\hat{\boldsymbol{\phi}}, && [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] = \nabla_\phi\hat{\boldsymbol{\phi}} - \nabla_\phi\hat{\boldsymbol{\phi}} = \mathbf{0}.
\end{aligned}$$

Since $\boldsymbol{g}$ is the Kronecker delta, we can immediately write from Eq. (1)

$$\begin{aligned}
&c_{rrr} = [\hat{\mathbf{r}}, \hat{\mathbf{r}}] \cdot \hat{\mathbf{r}} = 0, && c_{r\theta r} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\mathbf{r}} = 0, && c_{r\phi r} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\mathbf{r}} = 0,\\
&c_{\theta rr} = -c_{r\theta r} = 0, && c_{\theta\theta r} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\mathbf{r}} = 0, && c_{\theta\phi r} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\mathbf{r}} = 0,\\
&c_{\phi rr} = -c_{r\phi r} = 0, && c_{\phi\theta r} = -c_{\theta\phi r} = 0, && c_{\phi\phi r} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\mathbf{r}} = 0,
\end{aligned}$$

$$c_{rr\theta} = [\hat{\mathbf{r}}, \hat{\mathbf{r}}] \cdot \hat{\boldsymbol{\theta}} = 0, \qquad c_{r\theta\theta} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{r}, \qquad c_{r\phi\theta} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\theta}} = 0,$$
$$c_{\theta r\theta} = -c_{r\theta\theta} = \frac{1}{r}, \qquad c_{\theta\theta\theta} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\theta}} = 0, \qquad c_{\theta\phi\theta} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\theta}} = 0,$$
$$c_{\phi r\theta} = -c_{r\phi\theta} = 0, \qquad c_{\phi\theta\theta} = -c_{\theta\phi\theta} = 0, \qquad c_{\phi\phi\theta} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\theta}} = 0,$$

$$c_{rr\phi} = [\hat{\mathbf{r}}, \hat{\mathbf{r}}] \cdot \hat{\boldsymbol{\phi}} = 0, \qquad c_{r\theta\phi} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\phi}} = 0, \qquad c_{r\phi\phi} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\phi}} = -\frac{1}{r},$$
$$c_{\theta r\phi} = -c_{r\theta\phi} = 0, \qquad c_{\theta\theta\phi} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\phi}} = 0, \qquad c_{\theta\phi\phi} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\phi}} = -\frac{1}{r\tan\theta},$$
$$c_{\phi r\phi} = -c_{r\phi\phi} = \frac{1}{r}, \qquad c_{\phi\theta\phi} = -c_{\theta\phi\phi} = \frac{1}{r\tan\theta}, \qquad c_{\phi\phi\phi} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\phi}} = 0.$$

From Eq. (2) we again use the fact that $\boldsymbol{g}$ is the identity to write

$$\Gamma_{rrr} = \frac{c_{rrr} + c_{rrr} - c_{rrr}}{2} = 0, \qquad \Gamma_{rr\theta} = \frac{c_{rr\theta} + c_{r\theta r} - c_{r\theta r}}{2} = 0, \qquad \Gamma_{rr\phi} = \frac{c_{rr\phi} + c_{r\phi r} - c_{r\phi r}}{2} = 0,$$
$$\Gamma_{r\theta r} = \frac{c_{r\theta r} + c_{rr\theta} - c_{\theta rr}}{2} = 0, \qquad \Gamma_{r\theta\theta} = \frac{c_{r\theta\theta} + c_{r\theta\theta} - c_{\theta\theta r}}{2} = -\frac{1}{r}, \qquad \Gamma_{r\theta\phi} = \frac{c_{r\theta\phi} + c_{r\phi\theta} - c_{\theta\phi r}}{2} = 0,$$
$$\Gamma_{r\phi r} = \frac{c_{r\phi r} + c_{rr\phi} - c_{\phi rr}}{2} = 0, \qquad \Gamma_{r\phi\theta} = \frac{c_{r\phi\theta} + c_{r\theta\phi} - c_{\phi\theta r}}{2} = 0, \qquad \Gamma_{r\phi\phi} = \frac{c_{r\phi\phi} + c_{r\phi\phi} - c_{\phi\phi r}}{2} = -\frac{1}{r},$$

$$\Gamma_{\theta rr} = \frac{c_{\theta rr} + c_{\theta rr} - c_{rr\theta}}{2} = 0, \qquad \Gamma_{\theta r\theta} = \frac{c_{\theta r\theta} + c_{\theta\theta r} - c_{r\theta\theta}}{2} = \frac{1}{r}, \qquad \Gamma_{\theta r\phi} = \frac{c_{\theta r\phi} + c_{\theta\phi r} - c_{r\phi\theta}}{2} = 0,$$
$$\Gamma_{\theta\theta r} = \frac{c_{\theta\theta r} + c_{\theta r\theta} - c_{\theta r\theta}}{2} = 0, \qquad \Gamma_{\theta\theta\theta} = \frac{c_{\theta\theta\theta} + c_{\theta\theta\theta} - c_{\theta\theta\theta}}{2} = 0, \qquad \Gamma_{\theta\theta\phi} = \frac{c_{\theta\theta\phi} + c_{\theta\phi\theta} - c_{\theta\phi\theta}}{2} = 0,$$
$$\Gamma_{\theta\phi r} = \frac{c_{\theta\phi r} + c_{\theta r\phi} - c_{\phi r\theta}}{2} = 0, \qquad \Gamma_{\theta\phi\theta} = \frac{c_{\theta\phi\theta} + c_{\theta\theta\phi} - c_{\phi\theta\theta}}{2} = 0, \qquad \Gamma_{\theta\phi\phi} = \frac{c_{\theta\phi\phi} + c_{\theta\phi\phi} - c_{\phi\phi\theta}}{2} = -\frac{1}{r\tan\theta},$$

$$\Gamma_{\phi rr} = \frac{c_{\phi rr} + c_{\phi rr} - c_{rr\phi}}{2} = 0, \qquad \Gamma_{\phi r\theta} = \frac{c_{\phi r\theta} + c_{\phi\theta r} - c_{r\theta\phi}}{2} = 0, \qquad \Gamma_{\phi r\phi} = \frac{c_{\phi r\phi} + c_{\phi\phi r} - c_{r\phi\phi}}{2} = \frac{1}{r},$$
$$\Gamma_{\phi\theta r} = \frac{c_{\phi\theta r} + c_{\phi r\theta} - c_{\theta r\phi}}{2} = 0, \qquad \Gamma_{\phi\theta\theta} = \frac{c_{\phi\theta\theta} + c_{\phi\theta\theta} - c_{\theta\theta\phi}}{2} = 0, \qquad \Gamma_{\phi\theta\phi} = \frac{c_{\phi\theta\phi} + c_{\phi\phi\theta} - c_{\theta\phi\phi}}{2} = \frac{1}{r\tan\theta},$$
$$\Gamma_{\phi\phi r} = \frac{c_{\phi\phi r} + c_{\phi r\phi} - c_{\phi r\phi}}{2} = 0, \qquad \Gamma_{\phi\phi\theta} = \frac{c_{\phi\phi\theta} + c_{\phi\theta\phi} - c_{\phi\theta\phi}}{2} = 0, \qquad \Gamma_{\phi\phi\phi} = \frac{c_{\phi\phi\phi} + c_{\phi\phi\phi} - c_{\phi\phi\phi}}{2} = 0.$$

In summary, we have the nonzero connection coefficients

$$\Gamma_{r\theta\theta} = \Gamma_{r\phi\phi} = -\frac{1}{r}, \qquad \Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = \frac{1}{r}, \qquad \Gamma_{\theta\phi\phi} = -\frac{1}{r\tan\theta}, \qquad \Gamma_{\phi\theta\phi} = \frac{1}{r\tan\theta}.$$

This is in agreement with MCP (11.71), which gives the nonzero connection coefficients as

$$\Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = -\Gamma_{r\theta\theta} = -\Gamma_{r\phi\phi} = \frac{1}{r}, \qquad \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = \frac{\cot\theta}{r}. \quad \square$$

1(b) Repeat the exercise in 1(a) assuming a coordinate basis with

$$\mathbf{e}_r \equiv \frac{\partial}{\partial r}, \qquad \mathbf{e}_\theta \equiv \frac{\partial}{\partial \theta}, \qquad \mathbf{e}_\phi \equiv \frac{\partial}{\partial \phi}.$$

Solution. In a coordinate basis, it is always true that $[\vec{e}_\alpha, \vec{e}_\beta] = 0$ [1, p. 1168]. In this case, the nonzero elements of $\boldsymbol{g}$ are [2]

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2\sin^2\theta,$$

which implies

$$g^{rr} = 1, \qquad g^{\theta\theta} = \frac{1}{r^2}, \qquad g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta},$$

since the matrix of contravariant components of the metric is inverse to that of the covariant components [1, p. 1162]. The only nonzero derivatives are

$$g_{\theta\theta,r} = 2r, \qquad g_{\phi\phi,r} = 2r\sin^2\theta, \qquad g_{\phi\phi,\theta} = 2r^2\sin\theta\cos\theta.$$

From Eq. (2), the $\Gamma_{\alpha\beta\gamma}$ are

$$\begin{aligned}
\Gamma_{rrr} &= \frac{g_{rr,r} + g_{rr,r} - g_{rr,r}}{2} = 0, & \Gamma_{rr\theta} &= \frac{g_{rr,\theta} + g_{r\theta,r} - g_{r\theta,r}}{2} = 0,\\
\Gamma_{rr\phi} &= \frac{g_{rr,\phi} + g_{r\phi,r} - g_{r\phi,r}}{2} = 0, & \Gamma_{r\theta r} &= \frac{g_{r\theta,r} + g_{rr,\theta} - g_{rr,\theta}}{2} = 0,\\
\Gamma_{r\theta\theta} &= \frac{g_{r\theta,\theta} + g_{r\theta,\theta} - g_{\theta\theta,r}}{2} = -r, & \Gamma_{r\theta\phi} &= \frac{g_{r\theta,\phi} + g_{r\phi,\theta} - g_{\theta\phi,r}}{2} = 0,\\
\Gamma_{r\phi r} &= \frac{g_{r\phi,r} + g_{rr,\phi} - g_{\phi r,r}}{2} = 0, & \Gamma_{r\phi\theta} &= \frac{g_{r\phi,\theta} + g_{r\theta,\phi} - g_{\phi\theta,r}}{2} = 0,\\
\Gamma_{r\phi\phi} &= \frac{g_{r\phi,\phi} + g_{r\phi,\phi} - g_{\phi\phi,r}}{2} = -r\sin^2\theta,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\theta rr} &= \frac{g_{\theta r,r} + g_{\theta r,r} - g_{rr,\theta}}{2} = 0, & \Gamma_{\theta r\theta} &= \frac{g_{\theta r,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}}{2} = r,\\
\Gamma_{\theta r\phi} &= \frac{g_{\theta r,\phi} + g_{\theta\phi,r} - g_{r\phi,\theta}}{2} = 0, & \Gamma_{\theta\theta r} &= \frac{g_{\theta\theta,r} + g_{\theta r,\theta} - g_{\theta r,\theta}}{2} = r,\\
\Gamma_{\theta\theta\theta} &= \frac{g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}}{2} = 0, & \Gamma_{\theta\theta\phi} &= \frac{g_{\theta\theta,\phi} + g_{\theta\phi,\theta} - g_{\theta\phi,\theta}}{2} = 0,\\
\Gamma_{\theta\phi r} &= \frac{g_{\theta\phi,r} + g_{\theta r,\phi} - g_{\phi r,\theta}}{2} = 0, & \Gamma_{\theta\phi\theta} &= \frac{g_{\theta\phi,\theta} + g_{\theta\theta,\phi} - g_{\phi\theta,\theta}}{2} = 0,\\
\Gamma_{\theta\phi\phi} &= \frac{g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta}}{2} = -r^2\sin\theta\cos\theta,
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\phi rr} &= \frac{g_{\phi r,r} + g_{\phi r,r} - g_{rr,\phi}}{2} = 0, & \Gamma_{\phi r\theta} &= \frac{g_{\phi r,\theta} + g_{\phi\theta,r} - g_{r\theta,\phi}}{2} = 0,\\
\Gamma_{\phi r\phi} &= \frac{g_{\phi r,\phi} + g_{\phi\phi,r} - g_{r\phi,\phi}}{2} = r\sin^2\theta, & \Gamma_{\phi\theta r} &= \frac{g_{\phi\theta,r} + g_{\phi r,\theta} - g_{\theta r,\phi}}{2} = 0,\\
\Gamma_{\phi\theta\theta} &= \frac{g_{\phi\theta,\theta} + g_{\phi\theta,\theta} - g_{\theta\theta,\phi}}{2} = 0, & \Gamma_{\phi\theta\phi} &= \frac{g_{\phi\theta,\phi} + g_{\phi\phi,\theta} - g_{\theta\phi,\phi}}{2} = r^2\sin\theta\cos\theta,\\
\Gamma_{\phi\phi r} &= \frac{g_{\phi\phi,r} + g_{\phi r,\phi} - g_{\phi r,\phi}}{2} = r\sin^2\theta, & \Gamma_{\phi\phi\theta} &= \frac{g_{\phi\phi,\theta} + g_{\phi\theta,\phi} - g_{\phi\theta,\phi}}{2} = r^2\sin\theta\cos\theta,\\
\Gamma_{\phi\phi\phi} &= \frac{g_{\phi\phi,\phi} + g_{\phi\phi,\phi} - g_{\phi\phi,\phi}}{2} = 0.
\end{aligned}$$

Now applying Eq. (3),

$$\begin{aligned}
\Gamma^r{}_{rr} &= g^{rr}\Gamma_{rrr} = 0, & \Gamma^r{}_{r\theta} &= g^{rr}\Gamma_{rr\theta} = 0, & \Gamma^r{}_{r\phi} &= g^{rr}\Gamma_{rr\phi} = 0,\\
\Gamma^r{}_{\theta r} &= g^{rr}\Gamma_{r\theta r} = 0, & \Gamma^r{}_{\theta\theta} &= g^{rr}\Gamma_{r\theta\theta} = -r, & \Gamma^r{}_{\theta\phi} &= g^{rr}\Gamma_{r\theta\phi} = 0,\\
\Gamma^r{}_{\phi r} &= g^{rr}\Gamma_{r\phi r} = 0, & \Gamma^r{}_{\phi\theta} &= g^{rr}\Gamma_{r\phi r} = 0, & \Gamma^r{}_{\phi\phi} &= g^{rr}\Gamma_{r\phi\phi} = -r\sin^2\theta,
\end{aligned}$$

$$\begin{aligned}
\Gamma^\theta{}_{rr} &= g^{\theta\theta}\Gamma_{\theta rr} = 0, & \Gamma^\theta{}_{r\theta} &= g^{\theta\theta}\Gamma_{\theta r\theta} = \frac{1}{r}, & \Gamma^\theta{}_{r\phi} &= g^{\theta\theta}\Gamma_{\theta r\phi} = 0,\\
\Gamma^\theta{}_{\theta r} &= g^{\theta\theta}\Gamma_{\theta\theta r} = \frac{1}{r}, & \Gamma^\theta{}_{\theta\theta} &= g^{\theta\theta}\Gamma_{\theta\theta\theta} = 0, & \Gamma^\theta{}_{\theta\phi} &= g^{\theta\theta}\Gamma_{\theta\theta\phi} = 0,\\
\Gamma^\theta{}_{\phi r} &= g^{\theta\theta}\Gamma_{\theta\phi r} = 0, & \Gamma^\theta{}_{\phi\theta} &= g^{\theta\theta}\Gamma_{\theta\phi\theta} = 0, & \Gamma^\theta{}_{\phi\phi} &= g^{\theta\theta}\Gamma_{\theta\phi\phi} = -\sin\theta\cos\theta,
\end{aligned}$$

$$\Gamma^{\phi}{}_{rr} = g^{\phi\phi}\Gamma_{\phi rr} = 0, \qquad \Gamma^{\phi}{}_{r\theta} = g^{\phi\phi}\Gamma_{\phi r\theta} = 0, \qquad \Gamma^{\phi}{}_{r\phi} = g^{\phi\phi}\Gamma_{\phi r\phi} = \frac{1}{r},$$
$$\Gamma^{\phi}{}_{\theta r} = g^{\phi\phi}\Gamma_{\phi\theta r} = 0, \qquad \Gamma^{\phi}{}_{\theta\theta} = g^{\phi\phi}\Gamma_{\phi\theta\theta} = 0, \qquad \Gamma^{\phi}{}_{\theta\phi} = g^{\phi\phi}\Gamma_{\phi\theta\phi} = \frac{1}{\tan\theta},$$
$$\Gamma^{\phi}{}_{\phi r} = g^{\phi\phi}\Gamma_{\phi\phi r} = \frac{1}{r}, \qquad \Gamma^{\phi}{}_{\phi\theta} = g^{\phi\phi}\Gamma_{\phi\phi\theta} = \frac{1}{\tan\theta}, \qquad \Gamma^{\phi}{}_{\phi\phi} = g^{\phi\phi}\Gamma_{\phi\phi\phi} = 0.$$

Thus we have found that the nonzero connection coeffients are

$$\Gamma^{r}{}_{\theta\theta} = -r, \qquad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta, \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{1}{\tan\theta},$$
$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}.$$

1(c) Repeat both computations in 1(a) and 1(b) using symbolic manipulation software on a computer.

Solution. For 1(b), we use the Mathematica notebook from Ref. [3] with $r \to 1$, $\theta \to 2$, and $\phi \to 3$:

```
In[∘]:= n = 3
Out[∘]= 3

In[∘]:= coord = {r, θ, ϕ}
Out[∘]= {r, θ, ϕ}

In[∘]:= metric = {{1, 0, 0}, {0, r^2, 0}, {0, 0, r^2 Sin[θ]^2}}
Out[∘]= {{1, 0, 0}, {0, r^2, 0}, {0, 0, r^2 Sin[θ]^2}}

In[∘]:= inversemetric = Simplify[Inverse[metric]]
Out[∘]= {{1, 0, 0}, {0, 1/r^2, 0}, {0, 0, Csc[θ]^2/r^2}}

In[∘]:= affine := affine = Simplify[Table[(1/2)*Sum[(inversemetric[[i, s]])*
            (D[metric[[s, j]], coord[[k]]] +
              D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]), {s, 1, n}],
         {i, 1, n}, {j, 1, n}, {k, 1, n}]]

In[∘]:= listaffine :=
        Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[i, j, k]], affine[[i, j, k]]}] ,
         {i, 1, n}, {j, 1, n}, {k, 1, j}]

In[∘]:= TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
        TableSpacing → {2, 2}]

Out[∘]//TableForm=
        Γ[1, 2, 2]   -r
        Γ[1, 3, 3]   -r Sin[θ]^2
        Γ[2, 2, 1]   1/r
        Γ[2, 3, 3]   -Cos[θ] Sin[θ]
        Γ[3, 3, 1]   1/r
        Γ[3, 3, 2]   Cot[θ]
```

Taking into account that in a coordinate basis $\Gamma_{\alpha\beta\gamma}$ is symmetric in its last two indices [1, p. 1172], these match our result from 1(b).

For 1(a), I wrote a Mathematica notebook on my own (taking some inspiration from Ref. [3]):

```
In[∘ ]:= vr = ( Sin[θ] Cos[ϕ] ; Sin[θ] Sin[ϕ] ; Cos[θ] ); vθ = ( Cos[θ] Cos[ϕ] ; Cos[θ] Sin[ϕ] ; -Sin[θ] ); vϕ = ( -Sin[ϕ] ; Cos[ϕ] ; 0 );

In[∘ ]:= coords = {r, θ, ϕ}; vecs = {vr, vθ, vϕ}; grad = {1, 1/r, 1/(r Sin[θ])};

In[∘ ]:= comm[i_, j_] := grad[[i]] * D[vecs[[j]], coords[[i]]] - grad[[j]] * D[vecs[[i]], coords[[j]]]

In[∘ ]:= commcoeff[i_, j_, k_] := Simplify[Transpose[comm[i, j]].vecs[[k]]]

In[∘ ]:= conncoeff[i_, j_, k_] :=
  First[First[Simplify[1/2 (commcoeff[i, j, k] + commcoeff[i, k, j] - commcoeff[j, k, i])]]]

In[∘ ]:= table :=
  Table[If[UnsameQ[conncoeff[i, j, k], 0], {ToString[Γ[i, j, k]], conncoeff[i, j, k]}],
   {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]

In[∘ ]:= TableForm[Partition[DeleteCases[Flatten[table], Null], 2], TableSpacing → {2, 2}]

Out[∘ ]//TableForm=
 Γ[1, 2, 2]   -1/r
 Γ[1, 3, 3]   -1/r
 Γ[2, 1, 2]   1/r
 Γ[2, 3, 3]   -Cot[θ]/r
 Γ[3, 1, 3]   1/r
 Γ[3, 2, 3]   Cot[θ]/r
```

For the result, we again have $r \to 1$, $\theta \to 2$, and $\phi \to 3$. These match our result from 1(a).

Problem 2. Let V be a vector field. Prove the covariant divergence formula valid in a coordinate basis

$$\nabla_\alpha V^\alpha = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} V^\alpha),$$

where g is the determinant of the metric.

Solution. From Lecture 7, the covariant derivative can be written

$$\nabla_\beta V^\beta = \partial_\beta V^\beta + \Gamma^\gamma{}_{\beta\gamma} V^\beta. \tag{5}$$

Applying Eqs. (3) and (2),

$$\Gamma^\gamma{}_{\beta\gamma} = g^{\gamma\alpha}\Gamma_{\alpha\beta\gamma} = \frac{1}{2} g^{\gamma\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}) = \frac{1}{2} g^{\gamma\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}),$$

where we have used the fact that $[\vec{e}_\alpha, \vec{e}_\beta] = 0$ in a coordinate basis [1, p. 1168], rendering all of the commutation coefficients zero. Then

$$\begin{aligned}
\Gamma^\gamma{}_{\beta\gamma} &= \frac{1}{2} g^{\gamma\alpha}(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) \\
&= \frac{1}{2}(g^{\gamma\alpha}\partial_\gamma g_{\alpha\beta} + g^{\gamma\alpha}\partial_\beta g_{\alpha\gamma} - g^{\gamma\alpha}\partial_\gamma g_{\alpha\beta}) \\
&= \frac{1}{2} g^{\gamma\alpha}\partial_\beta g_{\alpha\gamma}, \qquad (6)
\end{aligned}$$

where we have used the symmetry of the metric. Since $\mathrm{tr}(AB) = A_{ij}B_{ji}$ [4], we can write Eq. (6) as

$$\Gamma^{\gamma}{}_{\beta\gamma} = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{g}\partial_\beta \boldsymbol{g}). \tag{7}$$

We now apply the identity [5, p. 106]

$$\mathrm{tr}\big[M^{-1}(x)\partial_\lambda M(x)\big] = \partial_\lambda[\ln\det M(x)].$$

Using also the fact that $\boldsymbol{g}^{\mu\beta}\boldsymbol{g}_{\beta\nu} = \delta^{\mu}{}_{\nu}$ by MCP (24.10), Eq. (7) becomes [5, p. 107]

$$\Gamma^{\gamma}{}_{\beta\gamma} = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{g}^{\gamma\alpha}\partial_\beta \boldsymbol{g}_{\alpha\gamma}) = \frac{1}{2}\partial_\beta(\ln\det \boldsymbol{g}_{\alpha\gamma}) = \frac{1}{2}\partial_\beta(\ln g) = \partial_\beta(\ln\sqrt{|g|}) = \frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}).$$

Feeding this into Eq. (5) and integrating by parts, we have

$$\nabla_\beta V^\beta = \partial_\beta V^\beta + V^\beta\frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}) = \frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}V^\beta)$$

as we wanted to show. □

Problem 3. In this problem you will explore the geometry of a sphere S^2 of radius R.

3(a) A vector $\vec{V} = V^\theta\vec{e}_\theta + V^\phi\vec{e}_\phi$ is defined at a point (θ,ϕ) on the sphere. It is then parallel transported around the circle of constant θ with $\phi \to \phi + 2\pi$. What are its resulting components? What is its length?

3(b) Write the geodesic equation in (θ,ϕ) angular coordinates. Show that the solutions are *great circles*, i.e. circles on the sphere of largest diameter.

3(c) Consider a disk of radius ϵ on the sphere. Working in the limit of small ϵ, compute the area of the disk to order ϵ^4. Compare your results to $\mathbb{R}^2$ with the flat metric.

3(d) A spherical triangle is made from three points on the sphere pairwise connected by geodesics. Let the angles on the triangle be α, β, and γ. By drawing pictures, show that $\alpha+\beta+\gamma$ can be larger than π.

3(e) Define the excess angle E of a spherical triangle by $E = \alpha+\beta+\gamma-\pi$. Prove that the area of the triangle is R^2E.

Problem 4. In this problem you will explore the geometry on the space of possible inertial velocities.

4(a) Suppose two inertial frames move with 3-velocities $\vec{v}_1$ and $\vec{v}_2$ relative to a fixed inertial frame. Show that their relative velocity $\vec{v}$ has magnitude v given by

$$v^2 = \frac{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}{(1 - \vec{v}_1 \cdot \vec{v}_2)^2}.$$

4(b) We define a metric on the space of all possible 3-velocities by defining the distance between two nearby velocities to be their relative velocity. Using the result from 4(a), show that this metric is

$$ds^2 = d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)\, d\phi^2),$$

where χ is the rapidity $v = \tanh(\chi)$, and θ, ϕ are polar and azimuthal angles defined relative to $\vec{v}$.

4(c) Show that the geodesics of this metric are paths of minimum fuel use for a rocket ship changing its velocity.

4(d) A rocket ship in interstellar travel with velocity $\vec{v}_1$ relative to earth changes to a new velocity $\vec{v}_2$ in a manner that uses the least amount of fuel. What is the ship's smallest velocity relative to earth during the change?

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