1 Reduced three-body problem

The problem of three point particles interacting gravitationally has a particularly simple limit: let the third body $m_3 \ll m_2, m_1$ so that its effect on the motions of m_1 and m_2 is negligible. Assume in addition that m_3 moves in the same orbital plane as m_1 and m_2. For simplicity, consider only the case of m_1 and m_2 in circular orbit about their center of mass.

1.1 Switch into a reference frame rotating with angular velocity ω associated with the circular orbit for the two-body problem. Choose the center of mass of the two-body problem to be the origin. Choose the x axis to go through m_1 and m_2. Show that the (now stationary) m_1 and m_2 are located at $-r_c\mu/m_1$ and $r_c\mu/m_2$.

Solution. Call the stationary coordinate system $\mathbf{R} = (X, Y, Z)$, and choose (X, Y) as the orbital plane. Call the rotating coordinate system $\mathbf{r} = (x, y, z)$, which is rotated about the Z axis by angle ωt. This gives us the transformation

$$x = X\cos\omega t + Y\sin\omega t, \tag{1}$$
$$y = Y\cos\omega t - X\sin\omega t, \tag{2}$$
$$z = Z. \tag{3}$$

From our choice of orbital plane, there is no motion in the z direction. Let the locations of m_1 and m_2 be given by $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in the rotating frame.

From our choice of x axis, we know that $y_1 = y_2 = 0$. From our choice of the origin as the center of mass, we have

$$m_1x_1 + m_2x_2 = 0. \tag{4}$$

By construction, m_1 and m_2 are stationary in the rotating frame, so $\mathrm{d}x_1/\mathrm{d}t = \mathrm{d}x_2/\mathrm{d}t = 0$. In other words, x_1 and x_2 must both be constant. Therefore, let

$$r_c = x_2 - x_1 \tag{5}$$

be the constant distance between m_1 and m_2. Now we have the system of two equations (4) and (5), so we can solve for x_1 and x_2. Substituting (5) as $x_2 = r_c + x_1$ into (4),

$$m_1x_1 + m_2(r_c + x_1) = 0 \implies x_1 = -\frac{m_2}{m_1 + m_2}r_c. \tag{6}$$

Now substituting (6) back into (4),

$$r_c = x_2 + \frac{m_2}{m_1 + m_2}r_c \implies x_2 = \frac{m_1}{m_1 + m_2}r_c. \tag{7}$$

Note that the reduced mass $\mu = m_1m_2/(m_1 + m_2)$. Substituting μ into (6) and (7) yields

$$\mathbf{r}_1 = x_1 = -r_c\mu/m_1, \mathbf{r}_2 = x_2 = r_c\mu/m_2 \tag{8}$$

as desired. □

1.2 Show that the Lagrangian governing the equation of motion of m_3 at location $(x(t), y(t))$ is

$$L_3 = \frac{m_3}{2}\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right] - U_{13} - U_{23}, \tag{9}$$

where $U_13(x, y)$ is the gravitational interaction of m_3 with m_1, while $U_{23}(x, y)$ is associated with m_3 and m_2.

Solution. In general, the Lagrangian for m_3 is given by

$$L_3 = T_3 - U_3, \tag{10}$$

where T_3 is the kinetic energy of m_3 and U_3 is its potential energy.

Beginning with U_3, the only forces acting upon m_3 are the gravitational interactions with m_1 and m_2. We know from the problem statement that these interactions are independent of each other; m_3 has a negligible effect on the motions of each m_1 and m_2, so it cannot couple them in any way. Thus, we can write

$$U_3 = -G\frac{m_1 m_3}{r_{13}} - G\frac{m_2 m_3}{r_{23}} \equiv U_{13} + U_{23}, \tag{11}$$

where r_{13} ($r_2 3$) is the separation between m_3 and m_1 (m_2), and we have defined U_{13} and U_{23}.

Now we will find an expression for T_3. Let $\mathbf{R}_3 = (X(t), Y(t))$ be the position of m_3 in the stationary coordinate system. Then

$$T_3 = \frac{m_3}{2}\dot{\mathbf{R}}_3^2 = \frac{m_3}{2}(\dot{X} + \dot{Y})^2. \tag{12}$$

We can define an inverse transformation back to the old coordinate system by simply rotating the (x, y) plane about the z axis by angle $-\omega t$. This inverse transformation is

$$X = x\cos\omega t - y\sin\omega t, \tag{13}$$
$$Y = x\sin\omega t + y\cos\omega t, \tag{14}$$
$$Z = z. \tag{15}$$

It follows from (13) and (14) that

$$\dot{X} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial X}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = -\omega x\sin\omega t - \omega y\cos\omega t + \dot{x}\cos\omega t - \dot{y}\sin\omega t \tag{16}$$
$$= (\dot{x} - \omega y)\cos\omega t - (\dot{y} + \omega x)\sin\omega t, \tag{17}$$
$$\dot{Y} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial Y}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \omega x\cos\omega t - \omega y\sin\omega t + \dot{x}\sin\omega t + \dot{y}\cos\omega t \tag{18}$$
$$= (\dot{y} + \omega x)\cos\omega t + (\dot{x} - \omega y)\sin\omega t. \tag{19}$$

Now using the forms of (16) and (18),

$$\dot{X}^2 = (\dot{x} - \omega y)^2\cos^2\omega t - 2(\dot{x} - \omega y)(\dot{y} + \omega x)\cos\omega t\sin\omega t + (\dot{y} + \omega x)^2\sin^2\omega t, \tag{20}$$
$$\dot{Y}^2 = (\dot{y} + \omega x)^2\cos^2\omega t + 2(\dot{x} - \omega y)(\dot{y} + \omega x)\cos\omega t\sin\omega t + (\dot{x} - \omega y)^2\sin^2\omega t, \tag{21}$$

which implies

$$\dot{X}^2 + \dot{Y}^2 = (\dot{x} - \omega y)^2(\cos^2 \omega t + \sin^2 \omega t) + (\dot{y} + \omega x)^2(\cos^2 \omega t + \sin^2 \omega t) \tag{22}$$

$$= (\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2. \tag{23}$$

Substituting (23) into (12), we have

$$T_3 = \frac{m_3}{2}\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right]. \tag{24}$$

Finally, substituting (11) and (24) into (10) yields (9) as desired. □

1.3 Show that the mechanical system described by L_3 has five locations in mechanical equilibrium. These are known as *Lagrange points*. (Hint: the graphical method is perfectly good for demonstrating a real root exists in a particular instance.)

Solution. The Euler-Lagrange equations for m_3 are given by

$$0 = \frac{\partial L_3}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_3}{\partial \dot{x}}, \tag{25}$$

$$0 = \frac{\partial L_3}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_3}{\partial \dot{y}}. \tag{26}$$

We will attack each term of the Lagrangian in (10) separately. Beginning with T_3, note that

$$\frac{\partial T_3}{\partial x} = m_3(\omega\dot{y} + \omega^2 x), \qquad \frac{\partial T_3}{\partial y} = m_3(-\omega\dot{x} + \omega^2 y), \tag{27}$$

$$\frac{\partial T_3}{\partial \dot{x}} = m_3(\dot{x} - \omega y), \qquad \frac{\partial T_3}{\partial \dot{y}} = m_3(\dot{y} + \omega x). \tag{28}$$

In turn, (28) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_3}{\partial \dot{x}} = m_3(\ddot{x} - \omega\dot{y}), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_3}{\partial \dot{y}} = m_3(\ddot{y} + \omega\dot{x}). \tag{29}$$

Now for U_3, we can find explicitly the r_{13} and r_{23} appearing in (11) using the positions of m_1 and m_2 on the x axis given by (8). These are

$$r_{13} = \sqrt{\left(x + \frac{r_c\mu}{m_1}\right)^2 + y^2}, \qquad r_{23} = \sqrt{\left(x - \frac{r_c\mu}{m_2}\right)^2 + y^2}. \tag{30}$$

It follows from (30) that

$$\frac{\partial r_{13}}{\partial x} = \frac{1}{r_{13}}\left(x + \frac{r_c\mu}{m_1}\right), \qquad \frac{\partial r_{23}}{\partial x} = \frac{1}{r_{23}}\left(x - \frac{r_c\mu}{m_2}\right), \tag{31}$$

$$\frac{\partial r_{13}}{\partial y} = \frac{y}{r_{13}}, \qquad \frac{\partial r_{23}}{\partial y} = \frac{y}{r_{23}}, \tag{32}$$

$$\frac{\partial r_{13}}{\partial \dot{x}} = \frac{\partial r_{23}}{\partial \dot{x}} = \frac{\partial r_{13}}{\partial \dot{y}} = \frac{\partial r_{23}}{\partial \dot{y}} = 0. \tag{33}$$

Then

$$\frac{\partial U_{13}}{\partial x} = \frac{\partial U_{13}}{\partial r_{13}}\frac{\partial r_{13}}{\partial x} = G\frac{m_1 m_3}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right), \qquad \frac{\partial U_{23}}{\partial x} = G\frac{m_2 m_3}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right), \tag{34}$$

$$\frac{\partial U_{13}}{\partial y} = \frac{\partial U_{13}}{\partial r_{13}}\frac{\partial r_{13}}{\partial y} = G\frac{m_1 m_3}{r_{13}^3}y, \qquad \frac{\partial U_{23}}{\partial y} = G\frac{m_2 m_3}{r_{23}^3}y, \tag{35}$$

$$\frac{\partial U_{13}}{\partial \dot{x}} = \frac{\partial U_{23}}{\partial \dot{x}} = \frac{\partial U_{13}}{\partial \dot{y}} = \frac{\partial U_{23}}{\partial \dot{y}} = 0. \tag{36}$$

Making the appropriate substitutions, (25) becomes

$$0 = m_3(\omega\dot{y} + \omega^2 x) - G\frac{m_1 m_3}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) - G\frac{m_2 m_3}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right) - m_3(\ddot{x} - \omega\dot{y}) \tag{37}$$

$$\implies \ddot{x} = 2\omega\dot{y} + \omega^2 x - G\frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) - G\frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right), \tag{38}$$

and (26) becomes

$$0 = m_3(-\omega\dot{x} + \omega^2 y) - G\frac{m_1 m_3}{r_{13}^3}y - G\frac{m_2 m_3}{r_{23}^3}y - m_3(\ddot{y} + \omega\dot{x}) \tag{39}$$

$$\implies \ddot{y} = -2\omega\dot{x} + \omega^2 y - Gy\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right). \tag{40}$$

The system is in mechanical equilibrium at points where $\dot{x} = \dot{y} = 0$. The equilibrium behavior persists over time, implying $\ddot{x} = \ddot{y} = 0$. With these restrictions, (38) and (40) become

$$x = \frac{G}{\omega^2}\frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{G}{\omega^2}\frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right), \tag{41}$$

$$y = \frac{G}{\omega^2}y\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right). \tag{42}$$

The real roots of (41) and (42) are the Lagrange points.

Inspection of (42) indicates that there is at least one solution where $y = 0$. In this case (42) is eliminated. Additionally, (30) becomes

$$r_{13} = \left|x + \frac{r_c\mu}{m_1}\right|, \qquad r_{23} = \left|x - \frac{r_c\mu}{m_2}\right|, \tag{43}$$

and thus (41) reduces to

$$x = \frac{G}{\omega^2}\frac{m_1}{|x + r_c\mu/m_1|^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{G}{\omega^2}\frac{m_2}{|x - r_c\mu/m_2|^3}\left(x - \frac{r_c\mu}{m_2}\right) \equiv f(x), \tag{44}$$

where we have defined $f(x)$ as the right-hand side of the equation. Note the following observations about $f(x)$:

- $f(x)$ has singularities at $x = -r_c\mu/m_1$ and $x = r_c\mu/m_2$;

- $f(x) < 0$ in the regime $x < -r_c\mu/m_1$;
- $f(x) > 0$ in the regime $x > r_c\mu/m_2$;
- $f(x)$ crosses the x axis somewhere in the regime $-r_c\mu/m_1 < x < r_c\mu/m_2$;
- $\mathrm{d}f/\mathrm{d}x < 0$ for all defined values of x because it is dominated by negative powers of x.

Based on these observations, we can sketch $f(x)$ and x as shown in Fig. 1. The three intersection points indicate that there are three real roots of (44). These are the first three Lagrange points.

In the case $y \neq 0$, (42) may be written

$$\frac{\omega^2}{G} = \frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}. \tag{45}$$

Substituting (45) into (41),

$$\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right) x = \frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right) \implies \frac{m_1}{r_{13}^3}\frac{r_c\mu}{m_1} = \frac{m_2}{r_{23}^3}\frac{r_c\mu}{m_2} \implies r_{13} = r_{23}. \tag{46}$$

Applying this condition to (30),

$$\left(x + \frac{r_c\mu}{m_1}\right)^2 + y^2 = \left(x - \frac{r_c\mu}{m_2}\right)^2 + y^2 \implies \tag{47}$$

But why does this imply they equal r?. Geometrically, this is only possible at two locations in the (x, y) plane as shown in Fig. 1. These are the final two Lagrange points, for a total of five as desired. $\square$

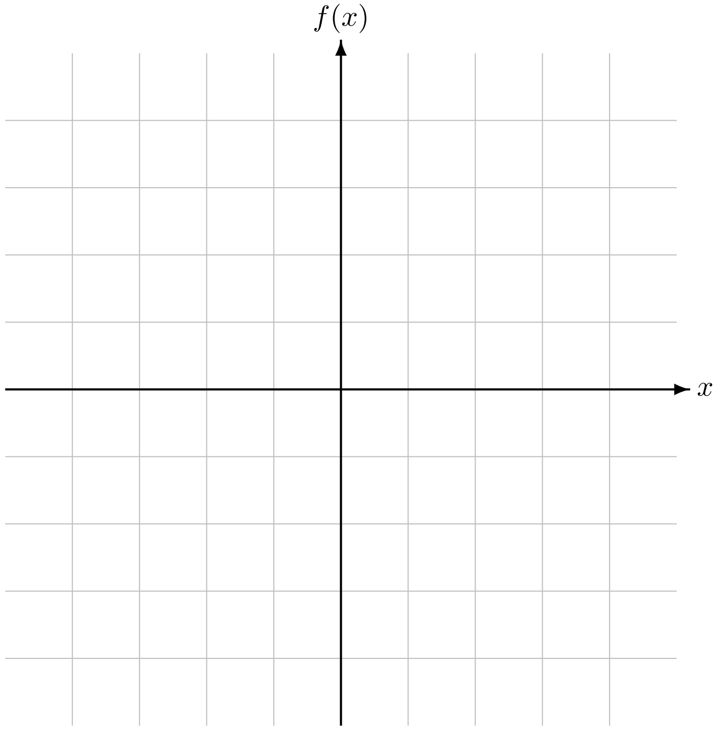

Figure 1: Three Lagrange points, indicated by roots of (44).

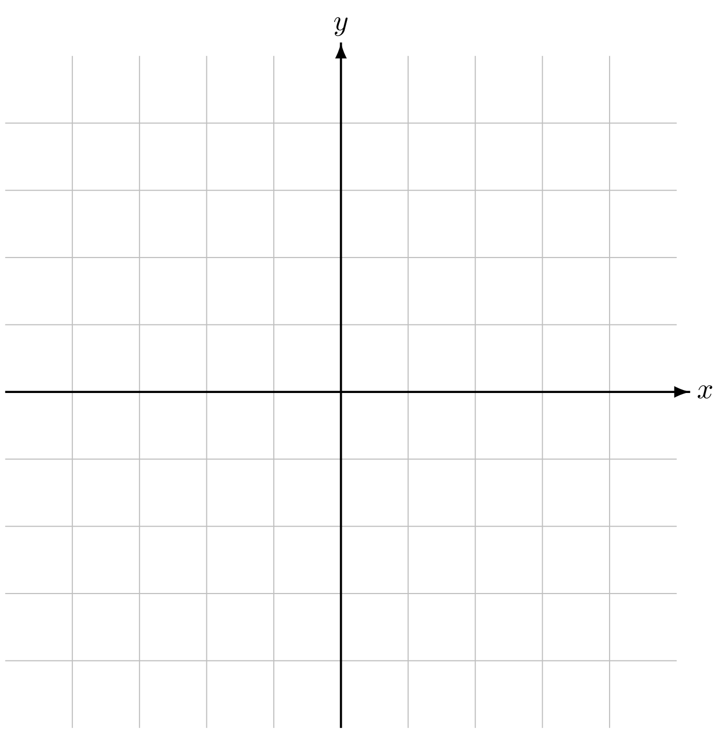

Figure 2: Two more Lagrange points, found by the geometrical argument implied by (47).