Problem 1. Ocean tides (MCP 25.9)

1(a) Place a local Lorentz frame at the center of Earth, and let $\mathcal{E}_{jk}$ be the tidal field there, produced by the Newtonian gravitational fields of the Sun and the Moon. For simplicity, treat Earth as precisely spherical. Show that the gravitational acceleration (relative to Earth's center) at some location on or near Earth's surface (radius r) is

$$g_j = -\frac{GM}{r^2} n^j - \mathcal{E}^j{}_k r n^k, \tag{1}$$

where M is Earth's mass, and n^j is a unit vector pointing from Earth's center to the location at which g_j is evaluated.

Solution. First we find the gravitational acceleration for a spherical body with no tidal field. We know that the Newtonian potential outside a spherical body with weak Newtonian gravity is

$$\Phi = -\frac{GM}{r},$$

where G is Newton's gravitation constant, M is the Earth's mass, and r is the distance from its center [1, p. 1211]. Then the acceleration due to this potential is, in spherical coordinates,

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi = -\frac{d}{dr}\left(-\frac{GM}{r}\right)\hat{\mathbf{r}} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \tag{2}$$

where $\hat{\mathbf{r}}$ is the radial unit vector. In this problem, $\hat{\mathbf{r}} \to n^j$.

For the tidal field, we use the Newtonian description of tidal gravity. Let $\boldsymbol{\xi}$ be the vector separation of two particles in Euclidean 3-space. The relative acceleration of the relative separation is given by MCP (25.23),

$$\frac{d^2\xi^j}{dt^2} = -\mathcal{E}^j{}_k \xi^k, \tag{3}$$

where $\boldsymbol{\mathcal{E}}$ is the Newtonian tidal gravitational field [1, p. 1208]. In this problem, $\xi^k \to rn^k$ since our vector separation is from the center of the Earth to some point a distance r away. Combining Eqs. (2) and (3), then, we have

$$g_j = -\frac{GM}{r^2} n^j - \mathcal{E}^j{}_k r n^k$$

as we wanted to show. $\square$

1(b) Show that this gravitational acceleration is minus the gradient of the Newtonian potential

$$\Phi = -\frac{GM}{r} + \frac{1}{2}\mathcal{E}_{jk} r^2 n^j n^k.$$

Solution. We have

$$\begin{aligned}
g^i &= -\nabla_i \Phi \\
&= -\left[\frac{d}{dr}\left(\frac{GM}{r}\right) + \nabla_i(\frac{1}{2}\mathcal{E}_{jk} r^2 n^j n^k)\right] n^i \\
&= -\left[\frac{GM}{r^2} + \nabla_i\left(\frac{1}{2}\mathcal{E}_{jk} x^j x^k\right)\right] n^i \\
&= -\left[\frac{GM}{r^2} + \frac{1}{2}(\partial_i \mathcal{E}_{jk}) x^j x^k + \frac{1}{2}\mathcal{E}_{jk}\delta^i_j x^k + \frac{1}{2}\mathcal{E}_{jk} x^j \delta^i_k\right] n^i \\
&= -\frac{GM}{r^2} n^i - \mathcal{E}_{ik} r n^i n^k,
\end{aligned}$$

where $x^j = rn^j$, and we have used $\partial_i \mathcal{E}_{jk} = 0$. This is true because $\mathcal{E}_{jk}$ is evaluated at the center of the Earth, and is therefore constant. $\square$

1(c) Consider regions of Earth's oceans that are far from any coast and have ocean depth large compared to the heights of ocean tides. If Earth were nonrotating, then explain why the analysis of Sec. 13.3 predicts that the ocean surface in these regions would be a surface of constant Φ. Explain why this remains true to good accuracy also for the rotating Earth.

Solution. The analysis of a fluid at hydrostatic equilibrium in Sec. 13.3 results in MCP (13.7),

$$\boldsymbol{\nabla}\Phi \times \boldsymbol{\nabla}\rho = 0, \tag{4}$$

where Φ is the Newtonian gravitational potential and ρ is the density of the fluid. This means that the contours of constant density correspond with the equipotential surfaces [1, p. 682]. We expect that, in deep areas of the still ocean, the density gradient points radially outward. This is because the stronger gravitational force near the center of the Earth creates a higher density there. Then the surfaces of constant density are spherical shells, and one of these shells is the surface of the ocean [1, p. 690]. Thus the surface of the ocean is also a surface of constant Φ.

The case of a hydrostatic fluid with a uniform angular velocity $\boldsymbol{\Omega}$ is discussed in Sec. 13.3.3. It is stated that in the presence of uniform rotation, all hydrostatic theorems remain valid in a corotating reference frame with Φ replaced by $\Phi + \Phi_{\text{cen}}$. The centrifugal potential Φ_{cen} is defined by MCP (13.27),

$$\Phi_{\text{cen}} = -\frac{1}{2}(\boldsymbol{\Omega} \times \mathbf{r})^2.$$

Thus Eq. (4) becomes

$$\boldsymbol{\nabla}(\Phi + \Phi_{\text{cen}}) \times \boldsymbol{\nabla}\rho = 0.$$

However, Φ_{cen} is small because the Earth rotates slowly [2]. So it can be neglected, and Eq. (4) is still a good approximation.

1(d) Show that in these ocean regions, the Moon creates high tides pointing toward and away from itself and low tides in the transverse directions on Earth; and similarly for the Sun. Compute the difference between high and low tides produced by the Moon and by the Sun, and the difference of the total tide when the Moon and the Sun are in approximately the same direction in the sky. Your answers are reasonably accurate for deep-ocean regions far from any coast, but near a coast, the tides are typically larger and sometimes far larger, and they are shifted in phase relative to the positions of the moon and Sun. Why?

Solution. I don't have enough time to figure out how to do this part.

Problem 2. Components of the Riemann tensor in an arbitrary basis (MCP 25.11) By evaluating expressions (25.30) in an arbitrary basis (which might not even be a coordinate basis), derive Eq. (25.50) for the components of the Riemann tensor. In your derivation keep in mind that commas denote partial derivations *only* in a coordinate basis; in an arbitrary basis they denote the result of letting a basis vector act as a differential operator.

Solution. MCP (25.30) is

$$p^{\alpha}{}_{;\gamma\delta} - p^{\alpha}{}_{;\delta\gamma} = -R^{\alpha}{}_{\beta\gamma\delta}p^{\beta}, \tag{5}$$

where $\vec{p}$ is any vector field. MCP (25.50) is

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu}c_{\gamma\delta}{}^{\mu}, \tag{6}$$

where $\Gamma^{\alpha}{}_{\beta\gamma}$ are the connection coefficients in the chosen basis, $\Gamma^{\alpha}{}_{\beta\gamma,\delta}$ is the result of letting the basis vector $\vec{e}_\delta$ act as a differential operator on $\Gamma^{\alpha}{}_{\beta\gamma}$, and $c_{\gamma\delta}{}^{\mu}$ are the basis vectors' commutation coefficients.

Beginning from Eq. (5) [3, p. 122],

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta}p^{\beta} &= p^{\alpha}{}_{;\delta\gamma} - p^{\alpha}{}_{;\gamma\delta} \\
&= \nabla_{\gamma}p^{\alpha}{}_{;\delta} - \nabla_{\delta}p^{\alpha}{}_{;\gamma} \\
&= p^{\alpha}{}_{;\delta,\gamma} + p^{\nu}{}_{;\delta}\Gamma^{\alpha}{}_{\nu\gamma} - p^{\alpha}{}_{;\gamma,\delta} - p^{\mu}{}_{;\gamma}\Gamma^{\alpha}{}_{\mu\delta} \\
&= (p^{\alpha}{}_{,\delta} + p^{\sigma}\Gamma^{\alpha}{}_{\sigma\delta})_{,\gamma} + (p^{\nu}{}_{,\delta} + p^{\tau}\Gamma^{\nu}{}_{\tau\delta})\Gamma^{\alpha}{}_{\nu\gamma} - (p^{\alpha}{}_{,\gamma} + p^{\lambda}\Gamma^{\alpha}{}_{\lambda\gamma})_{,\delta} - (p^{\mu}{}_{,\gamma} + p^{\rho}\Gamma^{\mu}{}_{\rho\gamma})\Gamma^{\alpha}{}_{\mu\delta},
\end{aligned} \tag{7}$$

where we have applied MCP (24.35),

$$A^{\mu}{}_{;\beta} = A^{\mu}{}_{,\beta} + A^{\alpha}\Gamma^{\mu}{}_{\alpha\beta}. \tag{8}$$

Relabeling indices as needed, Eq. (13) becomes

$$\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta}p^{\beta} &= p^{\alpha}{}_{,\delta\gamma} + p^{\sigma}{}_{,\gamma}\Gamma^{\alpha}{}_{\sigma\delta} + p^{\sigma}\Gamma^{\alpha}{}_{\sigma\delta,\gamma} + p^{\nu}{}_{,\delta}\Gamma^{\alpha}{}_{\nu\gamma} + p^{\tau}\Gamma^{\nu}{}_{\tau\delta}\Gamma^{\alpha}{}_{\nu\gamma} \\
&\qquad - p^{\alpha}{}_{,\gamma\delta} - p^{\lambda}{}_{,\delta}\Gamma^{\alpha}{}_{\lambda\gamma} - p^{\lambda}\Gamma^{\alpha}{}_{\lambda\gamma,\delta} - p^{\mu}{}_{,\gamma}\Gamma^{\alpha}{}_{\mu\delta} - p^{\rho}\Gamma^{\mu}{}_{\rho\gamma}\Gamma^{\alpha}{}_{\mu\delta} \\
&= p^{\alpha}{}_{,\delta\gamma} + p^{\sigma}\Gamma^{\alpha}{}_{\sigma\delta,\gamma} + p^{\tau}\Gamma^{\nu}{}_{\tau\delta}\Gamma^{\alpha}{}_{\nu\gamma} - p^{\alpha}{}_{,\gamma\delta} - p^{\lambda}\Gamma^{\alpha}{}_{\lambda\gamma,\delta} - p^{\rho}\Gamma^{\mu}{}_{\rho\gamma}\Gamma^{\alpha}{}_{\mu\delta} \\
&= (\Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma})p^{\beta} + p^{\alpha}{}_{,\delta\gamma} - p^{\alpha}{}_{,\gamma\delta}.
\end{aligned} \tag{9}$$

Using MCP (24.38a), $[\vec{e}_\alpha, \vec{e}_\beta] = c_{\alpha\beta}{}^{\rho}\vec{e}_\rho$, note that

$$p^{\alpha}{}_{,\delta\gamma} - p^{\alpha}{}_{,\gamma\delta} = (\vec{e}_\gamma\vec{e}_\delta - \vec{e}_\delta\vec{e}_\gamma)p^{\alpha} = [\vec{e}_\gamma, \vec{e}_\delta]p^{\alpha} = c_{\gamma\delta}{}^{\mu}\vec{e}_\mu p^{\alpha} = c_{\gamma\delta}{}^{\mu}p^{\alpha}{}_{,\mu}.$$

With another application of Eq. (8), this becomes

$$p^{\alpha}{}_{,\delta\gamma} - p^{\alpha}{}_{,\gamma\delta} = c_{\gamma\delta}{}^{\mu}(p^{\alpha}{}_{;\mu} - p^{\beta}\Gamma^{\alpha}{}_{\beta\mu}) = -c_{\gamma\delta}{}^{\mu}\Gamma^{\alpha}{}_{\beta\mu}p^{\beta},$$

where the first term disappears because $c_{\gamma\delta}{}^{\mu}$ is antisymmetric in its first two indices, but $p^{\alpha}{}_{;\mu}$ is a vector. Making this substitution, Eq. (9) implies

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu}c_{\gamma\delta}{}^{\mu}$$

as we wanted to show. □

Problem 3. Weyl curvature tensor (MCP 25.12) Show that the Weyl curvature tensor (25.48) has vanishing contraction on all its slots and has the same symmetries as Riemann: Eqs. (25.45). From these properties, show that Weyl has just 10 independent components. Write the Riemann tensor in terms of the Weyl tensor, the Ricci tensor, and the scalar curvature.

Solution. MCP (25.48) is

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - 2g^{[\mu}{}_{[\rho}R^{\nu]}{}_{\sigma]} + \frac{1}{3}g^{[\mu}{}_{[\rho}g^{\nu]}{}_{\sigma]}R, \tag{10}$$

where the square brackets denote antisymmetrization, $A_{[\alpha\beta]} = (A_{\alpha\beta} - A_{\beta\alpha})/2$, and $R = R^{\alpha}{}_{\alpha}$. Lowering the indices and expanding out the antisymmetrizations gives us

$$\begin{aligned} C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \left(g_{\mu\ [\rho}R_{\nu\ \sigma]} - g_{\nu\ [\rho}R_{\mu\ \sigma]}\right) + \frac{1}{6}\left(g_{\mu\ [\rho}g_{\nu\ \sigma]} - g_{\nu\ [\rho}g_{\mu\ \sigma]}\right)R \\ &= R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)R. \end{aligned} \tag{11}$$

We begin by interchanging indices since it makes the other proofs easier. Interchanging the first two indices,

$$\begin{aligned} C_{\nu\mu\rho\sigma} &= R_{\nu\mu\rho\sigma} - \frac{1}{2}\left(g_{\nu\rho}R_{\mu\sigma} - g_{\nu\sigma}R_{\mu\rho} - g_{\mu\rho}R_{\nu\sigma} + g_{\mu\sigma}R_{\nu\rho}\right) + \frac{1}{12}\left(g_{\nu\rho}g_{\mu\sigma} - g_{\nu\sigma}g_{\mu\rho} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)R \\ &= -\left[R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)R\right] \\ &= -C_{\mu\nu\rho\sigma}, \end{aligned}$$

where we have used $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$ from MCP (25.45a).

Interchanging the last two indices,

$$\begin{aligned} C_{\mu\nu\sigma\rho} &= R_{\mu\nu\sigma\rho} - \frac{1}{2}\left(g_{\mu\sigma}R_{\nu\rho} - g_{\mu\rho}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\rho} + g_{\nu\rho}R_{\mu\sigma}\right) + \frac{1}{12}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} - g_{\nu\sigma}g_{\mu\rho} + g_{\nu\rho}g_{\mu\sigma}\right)R \\ &= -\left[R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)R\right] \\ &= -C^{\mu\nu}{}_{\rho\sigma}, \end{aligned}$$

where we have used $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$ from MCP (25.45a).

Interchanging the first and second pair of indices,

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{1}{2}\left(g_{\rho\mu}R_{\sigma\nu} - g_{\rho\nu}R_{\sigma\mu} - g_{\sigma\mu}R_{\rho\nu} + g_{\sigma\nu}R_{\rho\mu}\right) + \frac{1}{12}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} - g_{\sigma\mu}g_{\rho\nu} + g_{\sigma\nu}g_{\rho\mu}\right)R \\ &= R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)R \\ &= C^{\mu\nu}{}_{\rho\sigma}, \end{aligned}$$

where we have used the symmetry of the metric, $g^{\mu}{}_{\nu} = g^{\nu}{}_{\mu}$, and $R_{\alpha\beta\gamma\delta} = +R_{\gamma\delta\alpha\beta}$ from MCP (25.45a).

Note also that

$$\begin{aligned} C_{\mu\rho\sigma\nu} &= R_{\mu\rho\sigma\nu} - \frac{1}{2}\left(g_{\mu\sigma}R_{\rho\nu} - g_{\mu\nu}R_{\rho\sigma} - g_{\rho\sigma}R_{\mu\nu} + g_{\rho\nu}R_{\mu\sigma}\right) + \frac{1}{12}\left(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right)R, \\ C_{\mu\sigma\nu\rho} &= R_{\mu\sigma\nu\rho} - \frac{1}{2}\left(g_{\mu\nu}R_{\sigma\rho} - g_{\mu\rho}R_{\sigma\nu} - g_{\sigma\nu}R_{\mu\rho} + g_{\sigma\rho}R_{\mu\nu}\right) + \frac{1}{12}\left(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu} - g_{\sigma\nu}g_{\mu\rho} + g_{\sigma\rho}g_{\mu\nu}\right)R. \end{aligned}$$

Then for the equivalent to MCP (25.45b),

$$\begin{aligned}
C_{\mu\nu\rho\sigma} + C_{\mu\rho\sigma\nu} + C_{\mu\sigma\nu\rho} = R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &- \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) \\
&- \frac{1}{2}\left(g_{\mu\sigma}R_{\rho\nu} - g_{\mu\nu}R_{\rho\sigma} - g_{\rho\sigma}R_{\mu\nu} + g_{\rho\nu}R_{\mu\sigma}\right) \\
&- \frac{1}{2}\left(g_{\mu\nu}R_{\sigma\rho} - g_{\mu\rho}R_{\sigma\nu} - g_{\sigma\nu}R_{\mu\rho} + g_{\sigma\rho}R_{\mu\nu}\right) \\
&+ \frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)R \\
&+ \frac{1}{12}\left(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right)R \\
&+ \frac{1}{12}\left(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu} - g_{\sigma\nu}g_{\mu\rho} + g_{\sigma\rho}g_{\mu\nu}\right)R \\
= 0,
\end{aligned}$$

where we have used $R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$ from MCP (25.45b), and the symmetry of the metric tensor. Thus we have shown that the Weyl curvature tensor has the same symmetries as Riemann in MCP (25.45):

$$C_{\nu\mu\rho\sigma} = -C_{\mu\nu\rho\sigma}, \qquad C_{\mu\nu\sigma\rho} = -C_{\mu\nu\rho\sigma}, \qquad C_{\rho\sigma\mu\nu} = +C_{\mu\nu\rho\sigma}, \qquad C_{\mu\nu\rho\sigma} + C_{\mu\rho\sigma\nu} + C_{\mu\sigma\nu\rho} = 0. \tag{12}$$

Now for the contractions. Contracting the indices within each pair,

$$g^{\mu\nu}C_{\mu\nu\rho\sigma} = C_{\mu}{}^{\mu}{}_{\rho\sigma} = -C_{\mu}{}^{\mu}{}_{\rho\sigma} = 0, \qquad\qquad g^{\rho\sigma}C_{\mu\nu\rho\sigma} = C_{\mu\nu\rho}{}^{\rho} = -C_{\mu\nu\rho}{}^{\rho} = 0, \tag{13}$$

where we have used $C_{\nu\mu\rho\sigma} = -C_{\mu\nu\rho\sigma}$ and our ability to swap covariant and contravariant for summed indices.

Contracting the first and third indices,

$$\begin{aligned}
g^{\mu\rho}C_{\mu\nu\rho\sigma} &= g^{\mu\rho}\left[R_{\mu\nu\rho\sigma} - \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) + \frac{1}{12}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - g_{\nu\rho}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu\rho}\right)R\right] \\
&= R_{\mu\nu}{}^{\mu}{}_{\sigma} - \frac{1}{2}\left(g_{\mu}{}^{\mu}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu}{}^{\mu} - g_{\nu}{}^{\mu}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu}{}^{\mu}\right) + \frac{1}{12}\left(g_{\mu}{}^{\mu}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu}{}^{\mu} - g_{\nu}{}^{\mu}g_{\mu\sigma} + g_{\nu\sigma}g_{\mu}{}^{\mu}\right)R \\
&= R_{\nu\sigma} - \frac{1}{2}\left(4R_{\nu\sigma} - R_{\nu\sigma} - R_{\nu\sigma} + g_{\nu\sigma}R\right) + \frac{1}{12}\left(4g_{\nu\sigma} - g_{\nu\sigma} - g_{\nu\sigma} + 4g_{\nu\sigma}\right)R \\
&= R_{\nu\sigma} - R_{\nu\sigma} - \frac{1}{2}g_{\nu\sigma}R + \frac{1}{2}g_{\nu\sigma}R \\
&= 0,
\end{aligned}$$

where we have used MCP (24.10), $g^{\mu\beta}g_{\beta\nu} = \delta^{\mu}{}_{\nu}$, to find $g^{\mu}{}_{\mu} = 4$. Then, using Eq. (12), we have

$$0 = C_{\mu\nu}{}^{\mu}{}_{\sigma} = -C_{\mu\nu\sigma}{}^{\mu} = -C_{\nu\mu}{}^{\mu}{}_{\sigma} = C_{\nu\mu\sigma}{}^{\mu}.$$

Rewriting these as contractions, and incorporating our results in Eq. (13), we have shown

$$0 = g^{\mu\nu}C_{\mu\nu\rho\sigma} = g^{\mu\rho}C_{\mu\nu\rho\sigma} = g^{\mu\sigma}C_{\mu\nu\rho\sigma} = g^{\nu\rho}C_{\mu\nu\rho\sigma} = g^{\nu\sigma}C_{\mu\nu\rho\sigma} = g^{\rho\sigma}C_{\mu\nu\rho\sigma};$$

that is, the Weyl curvature tensor has vanishing contraction on all its slots.

To show that Weyl has just 10 independent components, we refer to the discussion at the end of Lecture 11. If there were no symmetries, in $n = 4$ spacetime dimensions $C_{\alpha\beta\gamma\delta}$ would have $n^4 = 4^4$ independent components.

Each symmetry gives rise to some number of independent components:

$$\begin{aligned} \alpha \leftrightarrow \beta \text{ antisymmetry} &\quad\Longrightarrow\quad \frac{n(n-1)}{2} = 6 \text{ components}, \\ \gamma \leftrightarrow \delta \text{ antisymmetry} &\quad\Longrightarrow\quad \frac{n(n-1)}{2} = 6 \text{ components}, \\ \alpha\beta \leftrightarrow \gamma\delta \text{ symmetry} &\quad\Longrightarrow\quad \text{components reduced by } \frac{1}{2}. \end{aligned}$$

Combining the three (anti)symmetries, we have a total of

$$\left(\frac{n(n-1)}{2}\right)\left(\frac{n(n-1)}{2}+1\right)\frac{1}{2} = 21 \text{ components}.$$

Now we account for the constraints imposed by $C_{\mu\nu\rho\sigma} + C_{\mu\rho\sigma\nu} + C_{\mu\sigma\nu\rho} = 0$. This expression is redundant if any of the indices are the same; for example,

$$C_{\mu\mu\rho\sigma} + C_{\mu\rho\sigma\mu} + C_{\mu\sigma\mu\rho} = 0 + C_{\sigma\mu\mu\rho} + C_{\mu\sigma\mu\rho} = C_{\sigma\mu\mu\rho} - C_{\sigma\mu\mu\rho} = 0,$$

which we already knew from the other symmetries. Since there are four indices, this requirement gives us

$$\binom{n}{4} = 1 \text{ constraint}.$$

Finally we account for the constraints imposed by the vanishing contractions. Only one of the contractions is not redundant under the other symmetries; say, $C^{\mu}{}_{\nu\mu\sigma} = 0$. This imposes [4, p. 146]

$$\frac{n(n+1)}{2} = 10 \text{ constraints}.$$

Finally, we find the number of independent components by

$$N_{\text{components}} - N_{\text{constraints}} = 21 - (1+10) = 10 \text{ independent components}$$

as we wanted to show. □

Rewriting Eq. (10),

$$R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} - 2g^{[\mu}{}_{[\rho}R^{\nu]}{}_{\sigma]} + \frac{1}{3}g^{[\mu}{}_{[\rho}g^{\nu]}{}_{\sigma]}R,$$

which makes it clear that 10 of the 20 independent components of the Riemann tensor come from the Weyl tensor and the remaining 10 come from the Ricci tensor.

Problem 4. Curvature of the surface of a sphere (MCP 25.13) On the surface of a sphere, such as Earth, introduce spherical polar coordinates in which the metric, written as a line element, takes the form

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{14}$$

where a is the sphere's radius.

4(a) Show (first by hand and then by computer) that the connection coefficients for the coordinate basis $\{\partial/\partial\theta\,,\,\partial/\partial\phi\}$ are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta, \qquad \text{all others vanish.} \tag{15}$$

Solution. We follow the procedure on pp. 1171–1172 of MCP for computing the connection coefficients. We use (24.38c) to compute

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}), \tag{16}$$

where we know the commutation coefficients $c_{\alpha\beta\gamma}$ vanish in a coordinate basis [1, p. 1171]. We raise the first index using (24.38d),

$$\Gamma^{\mu}{}_{\beta\gamma} = g^{\mu\alpha}\Gamma_{\alpha\beta\gamma}. \tag{17}$$

According to MCP (24.7), the invariant interval is defined

$$ds^2 = g_{\alpha\beta}\Delta x^{\alpha}\Delta x^{\beta}.$$

This means the metric tensor components can be read off of Eq. (14) as

$$g_{\theta\theta} = a^2, \qquad g_{\phi\phi} = a^2\sin^2\theta. \tag{18}$$

The only nonzero derivative is

$$g_{\phi\phi,\theta} = 2a^2\sin\theta\cos\theta.$$

so only connection coefficients with some combination of $\{\phi, \phi, \theta\}$ can be nonzero. Applying Eq. (16) to the possible candidates, we find

$$\begin{aligned}
\Gamma_{\theta\phi\phi} &= \frac{1}{2}(g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta}) = -a^2\sin\theta\cos\theta, \\
\Gamma_{\phi\theta\phi} &= \frac{1}{2}(g_{\phi\theta,\phi} + g_{\phi\phi,\theta} - g_{\theta\phi,\theta}) = a^2\sin\theta\cos\theta, \\
\Gamma_{\phi\phi\theta} &= \frac{1}{2}(g_{\phi\phi,\theta} + g_{\phi\theta,\phi} - g_{\phi\theta,\phi}) = a^2\sin\theta\cos\theta.
\end{aligned}$$

The components of the inverse metric tensor are

$$g^{\theta\theta} = \frac{1}{a^2}, \qquad g^{\phi\phi} = \frac{1}{a^2\sin^2\theta}.$$

Now applying Eq. (17), we have

$$\Gamma^{\theta}{}_{\phi\phi} = g^{\theta\theta}\Gamma_{\theta\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}{}_{\theta\phi} = g^{\phi\phi}\Gamma_{\phi\theta\phi} = \cot\theta, \qquad \Gamma^{\phi}{}_{\phi\theta} = g^{\phi\phi}\Gamma_{\phi\phi\theta} = \cot\theta.$$

as we wanted to show. □

By computer, we use a Mathematica notebook adapted from Ref. [5]:

```
In[ ]:= n = 2
Out[ ]= 2

In[ ]:= coord = {θ, ϕ}
Out[ ]= {θ, ϕ}

In[ ]:= metric = a^2 {{1, 0}, {0, Sin[θ]^2}}
Out[ ]= {{a^2, 0}, {0, a^2 Sin[θ]^2}}

In[ ]:= inversemetric = Simplify[Inverse[metric]]
Out[ ]= {{1/a^2, 0}, {0, Csc[θ]^2/a^2}}

In[ ]:= affine := affine = Simplify[Table[(1/2)*Sum[(inversemetric[[i, s]])*
          (D[metric[[s, j]], coord[[k]]] +
            D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]), {s, 1, n}],
        {i, 1, n}, {j, 1, n}, {k, 1, n}]]

In[ ]:= listaffine :=
         Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[i, j, k]], affine[[i, j, k]]}] ,
          {i, 1, n}, {j, 1, n}, {k, 1, j}]

In[ ]:= TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
         TableSpacing → {2, 2}]
Out[ ]//TableForm=
        Γ[1, 2, 2]   -Cos[θ] Sin[θ]
        Γ[2, 2, 1]   Cot[θ]
```

Here $\theta \to 1$ and $\phi \to 2$. Taking into account that in a coordinate basis $\Gamma_{\alpha\beta\gamma}$ is symmetric in its last two indices [1, p. 1172], these match our result.

4(b) Show that the symmetries (25.45) of the Riemann tensor guarantee that its only nonzero components in the above coordinate basis are

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi}.$$

Solution. According to MCP (25.45a),

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}, \qquad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}, \qquad R_{\alpha\beta\gamma\delta} = +R_{\gamma\delta\alpha\beta}. \tag{19}$$

From the first two equations here, all permutations of four like indices and of three like indices vanish. That is,

$$R_{\theta\theta\theta\theta} = -R_{\theta\theta\theta\theta} = 0, \qquad R_{\theta\phi\phi\phi} = R_{\phi\theta\phi\phi} = R_{\phi\phi\theta\phi} = R_{\phi\phi\phi\theta} = 0,$$

and likewise for $\theta \leftrightarrow \phi$. The remaining possibilities are permutations of $\{\theta, \theta, \phi, \phi\}$. Applying Eq. (19), we have

$$R_{\theta\theta\phi\phi} = -R_{\theta\theta\phi\phi} = 0, \qquad R_{\phi\phi\theta\theta} = -R_{\phi\phi\theta\theta} = 0, \qquad R_{\theta\phi\phi\theta} = -R_{\phi\theta\phi\theta} = -R_{\theta\phi\theta\phi} = R_{\phi\theta\theta\phi} \neq 0,$$

as we wanted to show. □

4(c) Show, first by hand and then by computer, that

$$R_{\theta\phi\theta\phi} = a^2 \sin^2\theta.$$

Solution. From Eq. (6),

$$R^{\theta}{}_{\phi\theta\phi} = \Gamma^{\theta}{}_{\phi\phi,\theta} - \Gamma^{\theta}{}_{\phi\theta,\phi} + \Gamma^{\theta}{}_{\mu\theta}\Gamma^{\mu}{}_{\phi\phi} - \Gamma^{\theta}{}_{\mu\phi}\Gamma^{\mu}{}_{\phi\theta} - \Gamma^{\theta}{}_{\phi\mu}c_{\theta\phi}{}^{\mu}.$$

From Eq. (15),

$$\Gamma^{\theta}{}_{\phi\phi,\theta} = \sin^2\theta - \cos^2\theta, \qquad \Gamma^{\phi}{}_{\theta\phi,\theta} = \Gamma^{\phi}{}_{\phi\theta,\theta} = -\csc^2\theta, \qquad \Gamma^{\theta}{}_{\phi\phi,\phi} = \Gamma^{\phi}{}_{\theta\phi,\phi} = \Gamma^{\phi}{}_{\phi\theta,\phi} = 0.$$

We also know that all $c_{\alpha\beta\gamma} = 0$. Equation (**??**) is then

$$R^{\theta}{}_{\phi\theta\phi} = \Gamma^{\theta}{}_{\phi\phi,\theta} - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta} = \sin^2\theta - \cos^2\theta - (-\sin\theta\cos\theta)(\cot\theta) = \sin^2\theta.$$

Lowering the first index using Eq. (18), we find

$$R_{\theta\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta$$

as we wanted to show. □

Continuing the notebook from 4(a),

```
In[∘]:= riemann := riemann = Simplify[Table[
        D[affine[[i, j, l]], coord[[k]]] - D[affine[[i, j, k]], coord[[l]]] +
         Sum[affine[[s, j, l]] × affine[[i, k, s]] - affine[[s, j, k]] × affine[[i, l, s]],
          {s, 1, n}],
        {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}]]

In[∘]:= listriemann :=
       Table[If[UnsameQ[riemann[[i, j, k, l]], 0],
         {ToString[R[i, j, k, l]], riemann[[i, j, k, l]]}] , {i, 1, n}, {j, 1, n},
        {k, 1, n}, {l, 1, k - 1}]

In[∘]:= TableForm[Partition[DeleteCases[Flatten[listriemann], Null], 2],
       TableSpacing → {2, 2}]

Out[∘]//TableForm=
        R[1, 2, 2, 1]   -Sin[θ]^2
        R[2, 1, 2, 1]   1

In[∘]:= riemannLow := riemannLow = Simplify[Table[
        Sum[metric[[m, i]], {m, 1, n}] × riemann[[i, j, k, l]],
        {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}]]

In[∘]:= listriemannLow :=
       Table[If[UnsameQ[riemannLow[[i, j, k, l]], 0],
         {ToString[R[i, j, k, l]], riemannLow[[i, j, k, l]]}] , {i, 1, n}, {j, 1, n},
        {k, 1, n}, {l, 1, k - 1}]

In[∘]:= TableForm[Partition[DeleteCases[Flatten[listriemannLow], Null], 2],
       TableSpacing → {2, 2}]

Out[∘]//TableForm=
        R[1, 2, 2, 1]   -a^2 Sin[θ]^2
        R[2, 1, 2, 1]   a^2 Sin[θ]^2
```

Here the second list of `R[_,_,_,_]` has the first index lowered. By Eq. (19), `R[1,2,2,1]` is equivalent to `R[1,2,1,2]` $= R_{\theta\phi\theta\phi}$, which agrees with our calculation.

4(d) Show that in the basis

$$\{\vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}}\} = \left\{\frac{1}{a}\frac{\partial}{\partial\theta}, \frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}\right\}, \tag{20}$$

the components of the metric, the Riemann tensor, the Ricci tensor, the curvature scalar, and the Weyl tensor are

$$g_{\hat{j}\hat{k}} = \delta_{jk}, \qquad R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^2}, \qquad R_{\hat{j}\hat{k}} = \frac{1}{a^2}g_{\hat{j}\hat{k}}, \qquad R = \frac{2}{a^2}, \tag{21}$$

respectively. The first of these implies that the basis is orthonormal; the rest imply that the curvature is independent of location on the sphere, as it should be by spherical symmetry.

Solution. To find the components of the metric, we use MCP (24.16),

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta\,.$$

Comparing this with Eq. (14), we have

$$ds^2 = a^2\,d\theta^2 + a^2\sin^2\theta\,d\phi^2 = g_{\hat{\theta}\hat{\theta}}\,d\hat{\theta}^2 + g_{\hat{\phi}\hat{\phi}}\,d\hat{\phi}^2$$

since $d\hat{\theta} = a\,d\theta$ and $d\hat{\phi} = a\sin\theta\,d\phi$ from Eq. (20). We conclude from the above that

$$g_{\hat{\theta}\hat{\theta}} = g_{\hat{\phi}\hat{\phi}} = 1, \qquad g_{\hat{\theta}\hat{\phi}} = g_{\hat{\phi}\hat{\theta}} = 0;$$

that is, $g_{\hat{j}\hat{k}} = \delta_{jk}$. □

To find $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$, we can adapt the connection coefficients from Problem 1(a) of Homework 3; we simply ignore the coefficients with indices $\hat{r}$ and let $r \to a$ in the ones that remain. So the only nonzero coefficients are

$$\Gamma_{\hat{\phi}\hat{\theta}\hat{\phi}} = -\Gamma_{\hat{\theta}\hat{\phi}\hat{\phi}} = \frac{\cot\theta}{a}. \tag{22}$$

From Eq. (6),

$$\begin{aligned}
R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} &= g_{\hat{\theta}\mu}R^\mu{}_{\hat{\phi}\hat{\theta}\hat{\phi}} \\
&= \Gamma_{\hat{\theta}\hat{\phi}\hat{\phi},\hat{\theta}} - \Gamma_{\theta\hat{\phi}\hat{\theta},\hat{\phi}} + \Gamma_{\hat{\theta}\mu\hat{\theta}}\Gamma^\mu{}_{\hat{\phi}\hat{\phi}} - \Gamma_{\hat{\theta}\mu\hat{\phi}}\Gamma^\mu{}_{\hat{\phi}\hat{\theta}} - \Gamma_{\hat{\theta}\hat{\phi}\mu}c_{\hat{\theta}\hat{\phi}}{}^\mu \\
&= \Gamma_{\hat{\theta}\hat{\phi}\hat{\phi},\hat{\theta}} - \Gamma_{\hat{\theta}\hat{\phi}\hat{\phi}}c_{\hat{\theta}\hat{\phi}}{}^{\hat{\phi}} \\
&= \frac{1}{a}\frac{d\theta}{d\Gamma_{\hat{\theta}\hat{\phi}\hat{\phi}}} + \frac{\cot\theta}{a}\Gamma_{\hat{\theta}\hat{\phi}\hat{\phi}} \\
&= \frac{1}{a}\frac{d}{d\theta}\left(-\frac{\cot\theta}{a}\right) - \frac{\cot\theta}{a}\frac{\cot\theta}{a} \\
&= \frac{\csc^2\theta - \cot^2\theta}{a^2} \\
&= \frac{1}{a^2},
\end{aligned}$$

where we have used $c_{\hat{\theta}\hat{\phi}}{}^{\hat{\phi}} = -\cot\theta/a$ from Problem 1(a) of Homework 3. □

It is also useful to note that

$$\begin{aligned}
R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\theta}} &= \Gamma_{\hat{\theta}\hat{\theta}\hat{\theta},\hat{\theta}} - \Gamma_{\hat{\theta}\hat{\theta}\hat{\theta},\hat{\theta}} + \Gamma_{\hat{\theta}\mu\hat{\theta}}\Gamma^\mu{}_{\hat{\theta}\hat{\theta}} - \Gamma_{\hat{\theta}\mu\hat{\theta}}\Gamma^\mu{}_{\hat{\theta}\hat{\theta}} - \Gamma_{\hat{\theta}\hat{\theta}\mu}c_{\hat{\theta}\hat{\theta}}{}^\mu = 0, \\
R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} &= \Gamma_{\hat{\phi}\hat{\phi}\hat{\phi},\hat{\phi}} - \Gamma_{\hat{\phi}\hat{\phi}\hat{\phi},\hat{\phi}} + \Gamma_{\hat{\phi}\mu\hat{\phi}}\Gamma^\mu{}_{\hat{\phi}\hat{\phi}} - \Gamma_{\hat{\phi}\mu\hat{\phi}}\Gamma^\mu{}_{\hat{\phi}\hat{\phi}} - \Gamma_{\hat{\phi}\hat{\phi}\mu}c_{\hat{\phi}\hat{\phi}}{}^\mu = 0, \\
R_{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\theta}} &= \Gamma_{\hat{\phi}\hat{\theta}\hat{\theta},\hat{\theta}} - \Gamma_{\hat{\phi}\hat{\theta}\hat{\theta},\hat{\theta}} + \Gamma_{\hat{\phi}\mu\hat{\theta}}\Gamma^\mu{}_{\hat{\theta}\hat{\theta}} - \Gamma_{\hat{\phi}\mu\hat{\theta}}\Gamma^\mu{}_{\hat{\theta}\hat{\theta}} - \Gamma_{\hat{\phi}\hat{\theta}\mu}c_{\hat{\theta}\hat{\theta}}{}^\mu = R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\phi}} = 0, \\
R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\phi}} &= \Gamma_{\hat{\theta}\hat{\phi}\hat{\phi},\hat{\phi}} - \Gamma_{\hat{\theta}\hat{\phi}\hat{\phi},\hat{\phi}} + \Gamma_{\hat{\theta}\mu\hat{\phi}}\Gamma^\mu{}_{\hat{\phi}\hat{\phi}} - \Gamma_{\hat{\theta}\mu\hat{\phi}}\Gamma^\mu{}_{\hat{\phi}\hat{\phi}} - \Gamma_{\hat{\theta}\hat{\phi}\mu}c_{\hat{\phi}\hat{\phi}}{}^\mu = R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\phi}} = 0, \\
R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} &= R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} = \frac{1}{a^2},
\end{aligned}$$

where we have used $c_{\hat{\theta}\hat{\theta}}{}^{\hat{\phi}} = c_{\hat{\phi}\hat{\phi}}{}^{\hat{\phi}} = 0$ from Problem 1(a) of Homework 3. From MCP (25.46),

$$R_{\hat{j}\hat{k}} = R^{\mu}{}_{\hat{j}\mu\hat{k}} = R^{\hat{\theta}}{}_{\hat{j}\hat{\theta}\hat{k}} + R^{\hat{\phi}}{}_{\hat{j}\hat{\phi}\hat{k}}.$$

Then

$$\begin{aligned}
R_{\hat{\theta}\hat{\theta}} &= R^{\mu}{}_{\hat{\theta}\mu\hat{\theta}} = R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{\theta}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\theta}} = \frac{1}{a^2}, \\
R_{\hat{\theta}\hat{\phi}} &= R^{\mu}{}_{\hat{\theta}\mu\hat{\phi}} = R^{\hat{\theta}}{}_{\hat{\theta}\hat{\theta}\hat{\phi}} + R^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}\hat{\phi}} = 0, \\
R_{\hat{\phi}\hat{\theta}} &= R_{\hat{\theta}\hat{\phi}} = 0, \\
R_{\hat{\phi}\hat{\phi}} &= R^{\mu}{}_{\hat{\phi}\mu\hat{\phi}} = R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} + R^{\hat{\phi}}{}_{\hat{\phi}\hat{\phi}\hat{\phi}} = \frac{1}{a^2},
\end{aligned}$$

where we have applied MCP (25.47), $R_{\alpha\beta} = R_{\beta\alpha}$. Summarizing these results, we have $R_{\hat{j}\hat{k}} = g_{\hat{j}\hat{k}}/a^2$ as we wanted to show. □

From MCP (25.49), $R = R^{\alpha}{}_{\alpha}$. Then

$$R = R_{\theta\theta} + R_{\hat{\phi}\hat{\phi}} = \frac{1}{a^2} + \frac{1}{a^2} = \frac{2}{a^2}$$

as desired. □

Problem 5. Geodesic deviation on a sphere (MCP 25.14) Consider two neighboring geodesics (great circles) on a sphere of radius a, one the equator and the other a geodesic slightly displaced from the equator (by $\Delta\theta = b$) and parallel to it at $\phi = 0$. Let $\vec{\xi}$ be the separation vector between the two geodesics, and note that at $\phi = 0$, $\vec{\xi} = b\,\partial/\partial\theta$. Let l be proper distance along the equatorial geodesic, so $d/dl = \vec{u}$ is its tangent vector.

5(a) Show that $l = a\phi$ along the equatorial geodesic.

Solution. We know from Eq. (14) that, on the equator where $\theta = \pi/2$,

$$ds^2 = a^2 \sin^2\left(\frac{\pi}{2}\right) d\phi^2 = a^2\, d\phi^2\,.$$

Integrating both sides of the equation,

$$\int_0^l ds = \int_0^\phi a\, d\phi \quad \Longrightarrow \quad l = a\phi.$$

This is merely the expression for the arc length of a circle of radius a for a segment of angle θ. □

5(b) Show that the equation of geodesic deviation (25.31) reduces to

$$\frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta, \qquad \frac{d^2\xi^\phi}{d\phi^2} = 0. \tag{23}$$

Solution. The equation of geodesic deviation is

$$(\xi^\alpha{}_{;\beta}p^\beta)_{;\gamma}p^\gamma = -R^\alpha{}_{\beta\gamma\delta}p^\beta\xi^\gamma p^\delta, \tag{24}$$

where $\vec{p}$ is a 4-momentum tangent problem and is equal to $\vec{u}$ in this problem. By definition, $\vec{u}$ has only a component in the ϕ direction, which is equal to 1. Then Eq. (24) reduces to

$$(\xi^\alpha{}_{;\phi})_{;\phi}p^\phi = -R^\alpha{}_{\phi\gamma\phi}\xi^\gamma = -R^\alpha{}_{\phi\theta\phi}\xi^\theta, \tag{25}$$

since the only nonzero $R_{\alpha\beta\gamma\delta}$ have two θ and two ϕ indices. Applying Eq. (8) to the left-hand side of 25,

$$(\xi^\alpha{}_{;\phi})_{;\phi} = (\xi^\alpha{}_{,\phi} + \xi^\mu\Gamma^\alpha{}_{\mu\phi})_{,\phi} + (\xi^\nu{}_{,\phi} + \xi^\mu\Gamma^\nu{}_{\mu\phi})\Gamma^\alpha{}_{\nu\phi}. \tag{26}$$

For $\alpha = \theta$,

$$\begin{aligned}
(\xi^\theta{}_{;\phi})_{;\phi} &= (\xi^\theta{}_{,\phi} + \xi^\theta\Gamma^\theta{}_{\theta\phi} + \xi^\phi\Gamma^\theta{}_{\phi\phi})_{,\phi} + (\xi^\theta{}_{,\phi} + \xi^\theta\Gamma^\theta{}_{\theta\phi} + \xi^\phi\Gamma^\theta{}_{\phi\phi})\Gamma^\theta{}_{\theta\phi} + (\xi^\phi\Gamma^\phi{}_{\theta\phi} + \xi^\phi\Gamma^\phi{}_{\phi\phi})\Gamma^\theta{}_{\phi\phi} \\
&= (\xi^\theta{}_{,\phi} + \xi^\phi\Gamma^\theta{}_{\phi\phi})_{,\phi} + \xi^\phi\Gamma^\phi{}_{\theta\phi}\Gamma^\theta{}_{\phi\phi} \\
&= \left(\xi^\theta{}_{,\phi} - \frac{\cot\theta}{a}\xi^\phi\right)_{,\phi} - \frac{\cot^2\theta}{a^2}\xi^\phi \\
&= \frac{1}{a^2}\frac{d^2\xi^\theta}{d\phi^2},
\end{aligned}$$

where we have used Eq. (22) and $\theta = \pi/2$. Combining with the right-hand side of Eq. (25) then gives us

$$\frac{1}{a^2}\frac{d^2\xi^\theta}{d\phi^2} = -R^\theta{}_{\phi\theta\phi}\xi^\theta = -\frac{1}{a^2}\xi^\theta \quad \Longrightarrow \quad \frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta,$$

where we have used Eq. (21).

For $\alpha = \phi$, Eq. (26) gives us

$$\begin{aligned}
(\xi^\phi{}_{;\phi})_{;\phi} &= (\xi^\phi{}_{,\phi} + \xi^\theta\Gamma^\phi{}_{\theta\phi} + \xi^\phi\Gamma^\phi{}_{\phi\phi})_{,\phi} + (\xi^\theta{}_{,\phi} + \xi^\theta\Gamma^\theta{}_{\theta\phi} + \xi^\phi\Gamma^\theta{}_{\phi\phi})\Gamma^\phi{}_{\theta\phi} + (\xi^\phi{}_{,\phi} + \xi^\theta\Gamma^\phi{}_{\theta\phi} + \xi^\phi\Gamma^\phi{}_{\phi\phi})\Gamma^\phi{}_{\phi\phi} \\
&= (\xi^\phi{}_{,\phi} + \xi^\theta\Gamma^\phi{}_{\theta\phi})_{,\phi} + (\xi^\theta{}_{,\phi} + \xi^\phi\Gamma^\theta{}_{\phi\phi})\Gamma^\phi{}_{\theta\phi} \\
&= \left(\xi^\phi{}_{,\phi} + \frac{\cot\theta}{r}\xi^\theta\right)_{,\phi} + \left(\xi^\theta{}_{,\phi} - \frac{\cot\theta}{r}\xi^\phi\right)\frac{\cot\theta}{r} \\
&= \frac{1}{a^2}\frac{d^2\xi^\phi}{d\phi^2},
\end{aligned}$$

and combining with the right-hand side of Eq. (25),

$$\frac{1}{a^2}\frac{d^2\xi^\phi}{d\phi^2} = 0 \quad \Longrightarrow \quad \frac{d^2\xi^\phi}{d\phi^2} = 0,$$

so we have proven Eq. (23). □

5(c) Solve Eq. (23), subject to the above initial conditions, to obtain

$$\xi^\theta = b\cos\phi, \qquad\qquad \xi^\phi = 0.$$

Verify, by drawing a picture, that this is precisely what one would expect for the separation vector between two great circles.

Solution. The differential equations have the solutions [6, pp. 206–207]

$$\frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta \quad\Longrightarrow\quad \xi^\theta(\phi) = A\sin\phi + B\cos\phi, \qquad\qquad \frac{d^2\xi^\phi}{d\phi^2} = 0 \quad\Longrightarrow\quad \xi^\phi(\phi) = C + D\phi.$$

For the initial conditions, as stated $\vec{\xi}(\phi = 0) = b\vec{e}_\theta$, so $\xi^\theta(0) = b$ and $\xi^\phi(0) = 0$. This means

$$b = B, \qquad\qquad 0 = C.$$

It is also clear that the great circles intersect at $\xi = \pm\pi/2$. This gives us

$$\xi^\theta(\pi/2) = A = 0, \qquad\qquad \xi^\phi(\pi/2) = D\frac{\pi}{2} = 0 \quad\Longrightarrow\quad D = 0.$$

So we have the solutions

$$\xi^\theta = b\cos\phi, \qquad\qquad \xi^\phi = 0$$

as we wanted to show. □

Figure (1) shows $\vec{\xi}$ at a few points on the sphere, with the coordinates labeled for the given initial conditions. At any given point $\mathcal{P}$ on the equator (red line), the closest point on the second great circle (blue line) will always be directly "above" or "below" $\mathcal{P}$. That is to say, if ξ^θ is nonzero at any $\mathcal{P}$, $\vec{\xi}$ would not be the separation vector between the two geodesics because it would not indicate the shortest distance between them. The points at which the great circles intersect for the given initial condition make ξ^ϕ obvious.

Figure 1: Figure showing $\vec{\xi}$ on the sphere of Problem 5. The equator is indicated by the red line, and the second great circle by the blue line.

Problem 6. Curvature-coupling torque (MCP 25.16)

6(a) In the Newtonian theory of gravity, consider an axisymmetric, spinning body (e.g., Earth) with spin angular momentum S_j and time-independent mass distribution $\rho(\mathbf{x})$, interacting with an externally produced tidal gravitational field $\mathcal{E}_{jk}$ (e.g., that of the Sun and the Moon). Show that the torque around the body's center of mass, exerted by the tidal field, and the resulting evolution of the body's spin are

$$\frac{dS_i}{dt} = -\epsilon_{ijk}\mathcal{I}_{jl}\mathcal{E}_{kl}. \tag{27}$$

Here

$$\mathcal{I}_{kl} = \int \rho \left(x_k x_l - \frac{1}{3} r^2 \delta_{kl} \right) dV$$

is the body's mass quadrupole moment, with $r = \sqrt{\delta_{ij} x_i x_j}$ the distance from the center of mass.

Solution. Torque is defined by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \quad \implies \quad \tau_i = \epsilon_{ijk} x_j F_k.$$

We take $\boldsymbol{\tau} = d\mathbf{S}/dt$ as given. Then for a mass density [7, p. 110], applying Newton's second law yields

$$\frac{dS_i}{dt} = \epsilon_{ijk} \int \rho x_j g_k \, dV \,, \tag{28}$$

where the acceleration $\mathbf{g}$ is due only to the tidal field. From Eq. (1), this gives us

$$g_k = -\mathcal{E}_{kl} r n_l = -\mathcal{E}_{kl} x_l,$$

where we have used the symmetry of $\boldsymbol{\mathcal{E}}$ [1, p. 1208]. Making this substitution in Eq. (28),

$$\frac{dS_i}{dt} = -\epsilon_{ijk}\mathcal{E}_{kl} \int \rho x_j x_l \, dV \,. \tag{29}$$

The Taylor expansion of an arbitrary function $v(\mathbf{r} - \mathbf{R})$ about the origin $\mathbf{r} = \mathbf{0}$ can be written

$$r_\alpha r_\beta v_{\alpha\beta}(\mathbf{R}) = \frac{1}{3}(3 r_\alpha r_\beta - \delta_{\alpha\beta} r^2) v_{\alpha\beta}(\mathbf{R}),$$

so long as $v_{\alpha\alpha}(\mathbf{R}) = 0$; that is, $v(\mathbf{R})$ satisfies the Laplace equation [8]. We would need $\epsilon_{i\alpha k}\mathcal{E}_{k\alpha} = 0$, which is true:

$$\begin{aligned}
\epsilon_{i\alpha k}\mathcal{E}_{k\alpha} &= \epsilon_{i12}\mathcal{E}_{21} + \epsilon_{i13}\mathcal{E}_{31} + \epsilon_{i21}\mathcal{E}_{12} + \epsilon_{i23}\mathcal{E}_{32} + \epsilon_{i31}\mathcal{E}_{13} + \epsilon_{i32}\mathcal{E}_{23} \\
&= \epsilon_{i12}\mathcal{E}_{12} + \epsilon_{i13}\mathcal{E}_{13} - \epsilon_{i12}\mathcal{E}_{12} + \epsilon_{i23}\mathcal{E}_{23} - \epsilon_{i13}\mathcal{E}_{13} - \epsilon_{i23}\mathcal{E}_{23} \\
&= 0,
\end{aligned}$$

where we have again used the symmetry of $\boldsymbol{\mathcal{E}}$. Now we can write Eq. (29) as

$$\frac{dS_i}{dt} = -\epsilon_{ijk}\mathcal{E}_{kl} \int \rho \left(x_j x_l - \frac{1}{3} r^2 \delta_{jl} \right) dV \equiv -\epsilon_{ijk}\mathcal{I}_{jl}\mathcal{E}_{kl},$$

where we have defined

$$\mathcal{I}_{kl} \equiv \int \rho \left(x_k x_l - \frac{1}{3} r^2 \delta_{kl} \right) dV \,,$$

which is what we wanted to show. □

6(b) For the centrifugally flattened Earth interacting with the tidal fields of the Moon and the Sun, estimate in order of magnitude the spin-precession period produced by this torque. (The observed precession period is 26,000 years.)

Solution. I couldn't figure out how to do this one either.

6(c) Show that when rewritten in the language of general relativity, and in frame-independent, geometric language, Eq. (27) takes the form (25.59) discussed in the text. As part of showing this, explain the meaning of $\mathcal{I}_{\beta\mu}$ in that equation.

Solution. MCP (25.59) is

$$S^{\alpha}{}_{;\mu}u^{\mu} = \epsilon^{\alpha\beta\gamma\delta}\mathcal{I}_{\beta\mu}R^{\mu}{}_{\nu\gamma\zeta}u_{\delta}u^{\nu}u^{\zeta}.$$

In the proper rest frame of the Earth, $\vec{u} = (1, \mathbf{0})$. So the left-hand side of Eq. (27) is

$$\frac{dS^i}{dt} = \nabla_0 S^i \to (\nabla_\mu S^i)u^\mu = S^i{}_{;\mu}u^\mu. \tag{30}$$

By MCP (25.25), in the local rest frame of two particles with a relative acceleration,

$$R^j{}_{0k0} = \mathcal{E}_{jk},$$

and by symmetry $R^j{}_{0k0} = R^k{}_{0j0}$. Given that $\vec{u} = (1, \mathbf{0})$ in the Earth's proper rest frame, we have

$$\mathcal{E}_{jk} = R^k{}_{0j0} \to R^k{}_{\nu j\zeta}u^\nu u^\zeta. \tag{31}$$

Similarly,

$$\epsilon^{ijk} = \epsilon^{0ijk} = \epsilon^{ijk0} \to \epsilon^{ijk\delta}u_\delta. \tag{32}$$

Feeding Eqs. (30), (31), and (32) into Eq. (27), we have

$$S^{\alpha}{}_{;\mu}u^{\mu} = -\epsilon^{\alpha\beta\gamma\delta}\mathcal{I}_{\beta\mu}R^{\mu}{}_{\nu\gamma\zeta}u_{\delta}u^{\nu}u^{\zeta},$$

which is a minus sign away from what we wanted to show.

References

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