Problem 1. Consider the charge density $\rho(\mathbf{x})$ given by

$$\rho(\mathbf{x}) = \begin{cases} (R-r)(1-\cos\theta)^2 & \text{for } |\mathbf{x}| \le R, \\ 0 & \text{for } |\mathbf{x}| \ge R. \end{cases} \tag{1}$$

Find the electrostatic potential, $\phi(\mathbf{x})$, of this charge distribution at all $\mathbf{x}$ with $|\mathbf{x}| \ge R$.

Solution. The multipole expansion in spherical harmonics is given by Eq. (2.79) in the course notes,

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi), \tag{2}$$

where the spherical multipole moments q_{lm} are defined in Eq. (2.80),

$$q_{lm} \equiv \int \rho(\mathbf{x}')\, r'^l\, Y^*_{lm}(\theta', \phi')\, d^3x' \,.$$

Note that (2) is valid only for $|\mathbf{x}| \ge R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only within $|\mathbf{x}'| \le R$, which is the regime we are interested in here.

The spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi},$$

and the Lagrange polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l,$$

although in practice I am taking all spherical harmonics from the table in Jackson.

We can write the angular component of $\rho(\mathbf{x})$ as an expansion of spherical harmonics. Inspecting (1), we will only have terms of $l = 0, 1, 2$ and $m = 0$. The relevant spherical harmonics are

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \qquad Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta, \qquad Y_{20}(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right).$$

Then we have

$$\begin{aligned} \rho(r, \theta,\ vph) &= (R-r)(1 - 2\cos\theta + \cos^2\theta) \\ &= (R-r) \left(\frac{2}{3}\sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) - 2\sqrt{\frac{4\pi}{3}} Y_{10}(\theta, \phi) + 4\frac{\sqrt{4\pi}}{3} Y_{00}(\theta, \phi) \right). \end{aligned}$$

The only nonzero q_{lm} are q_{00}, q_{10}, and q_{20}:

$$\begin{aligned} q_{00} &= \int_0^{2\pi} \int_{-1}^{1} \int_0^R \rho(\mathbf{x}')\, r'^0\, Y^*_{00}(\theta', \phi')\, r'\, dr'\, d(\cos\theta')\, d\varphi' \\ &= 4\frac{\sqrt{4\pi}}{3} \int_0^{2\pi} \int_{-1}^{1} Y^*_{00}(\theta', \phi') Y_{00}(\theta', \phi')\, d(\cos\theta')\, d\varphi' \int_0^R (R-r') r'\, dr' \\ &= 4\frac{\sqrt{4\pi}}{3} \left[\frac{Rr'^2}{2} - \frac{r'^3}{3} \right]_0^R = 4\frac{\sqrt{4\pi}}{3}\frac{R^3}{6} = \frac{4\sqrt{\pi}}{9} R^3, \end{aligned}$$

$$\begin{aligned}
q_{10} &= \int_0^{2\pi}\int_{-1}^{1}\int_0^R \rho(\mathbf{x}')\, r'^1\, Y_{10}^*(\theta',\phi')\, r'\, dr'\, d(\cos\theta')\, d\varphi' \\
&= -2\sqrt{\frac{4\pi}{3}}\int_0^{2\pi}\int_{-1}^{1} Y_{10}^*(\theta',\phi')Y_{10}(\theta',\phi')\, d(\cos\theta')\, d\varphi' \int_0^R (R-r')r'^2\, dr' \\
&= -2\sqrt{\frac{4\pi}{3}}\left[\frac{Rr'^3}{3} - \frac{r'^4}{4}\right]_0^R = -2\sqrt{\frac{4\pi}{3}}\frac{R^4}{12} = -\frac{1}{3}\sqrt{\frac{\pi}{3}}R^4,
\end{aligned}$$

$$\begin{aligned}
q_{20} &= \int_0^{2\pi}\int_{-1}^{1}\int_0^R \rho(\mathbf{x}')\, r'^2\, Y_{20}^*(\theta',\phi')\, r'\, dr'\, d(\cos\theta')\, d\varphi' \\
&= \frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^{2\pi}\int_{-1}^{1} Y_{20}^*(\theta',\phi')Y_{20}(\theta',\phi')\, d(\cos\theta')\, d\varphi' \int_0^R (R-r')r'^3\, dr' \\
&= \frac{2}{3}\sqrt{\frac{4\pi}{5}}\left[\frac{Rr'^4}{4} - \frac{r'^5}{5}\right]_0^R = \frac{2}{3}\sqrt{\frac{4\pi}{5}}\frac{R^5}{20} = \frac{1}{15}\sqrt{\frac{\pi}{5}}R^5.
\end{aligned}$$

Then ϕ is given by

$$\begin{aligned}
\phi(\mathbf{x}) &= \frac{4\pi}{1}\frac{q_{00}}{r^1}Y_{00}(\theta,\phi) + \frac{4\pi}{2+1}\frac{q_{10}}{r^2}Y_{10}(\theta,\phi) + \frac{4\pi}{5}\frac{q_{20}}{r^3}Y_{20}(\theta,\phi) \\
&= (4\pi)\frac{4\sqrt{\pi}}{9}\frac{R^3}{r}\frac{1}{\sqrt{4\pi}} - \frac{4\pi}{3}\frac{1}{3}\sqrt{\frac{\pi}{3}}\frac{R^4}{r^2}\sqrt{\frac{3}{4\pi}}\cos\theta + \frac{4\pi}{5}\frac{1}{15}\sqrt{\frac{\pi}{5}}\frac{R^5}{r^3}\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \\
&= \frac{8\pi}{9}\frac{R^3}{r} - \frac{2\pi}{9}\frac{R^4}{r^2}\cos\theta + \frac{\pi}{75}\frac{R^5}{r^3}(2\cos^2\theta - 1).
\end{aligned}$$

Problem 2. Let $\mathcal{V}$ be an arbitrary bounded region of space and suppose that a total charge Q is to be distributed in $\mathcal{V}$ in an arbitrary way, with $\rho = 0$ outside of $\mathcal{V}$. Show that the total energy is minimized if the charge is distributed the way that it would be if $\mathcal{V}$ were a conductor, so that $\phi = \text{const.}$ within $\mathcal{V}$ (and thus, in particular, all of the charge lies on the boundary of $\mathcal{V}$).

Hint: Let $\phi_0(\mathbf{x})$ be the potential one would obtain if $\mathcal{V}$ were filled by a conducting body. Consider the energy of $\phi_0 + \phi'$, where the source ρ' of ϕ' vanishes outside of $\mathcal{V}$ and has no net charge within $\mathcal{V}$.

Solution. Let $S = \partial\mathcal{V}$ denote the boundary of $\mathcal{V}$. We separate space into three mutually exclusive regions: $\mathcal{V}$, S, and the region outside (in which we are not interested). By the superposition principle, we may write

$$\rho = \rho_0 + \rho', \qquad \phi = \phi_0 + \phi',$$

where ρ_0 is the charge of a conducting body filling $\mathcal{V}$, ϕ_0 is the electrostatic potential due to ρ_0, ρ' is the charge distribution within $\mathcal{V}$, and ϕ' is the electrostatic potential due to ρ'. In order to eliminate ambiguity on the boundary, we require

$$\rho_0|_{\mathcal{V}} = 0, \qquad \rho'|_S = 0. \tag{3}$$

That is, $\rho_0 = 0$ inside the conductor by definition, and ρ' vanishes on the boundary where ρ_0 is nonzero. For the entire body to have charge Q, we need

$$\int \rho_0\, d^3x = \int_{\mathcal{V}} \rho'\, d^3x + \int_S \rho_0\, dS = Q.$$

From (3), it follows that

$$\phi_0|_{\mathcal{V}} = \phi_0|_S = \text{const.}$$

The total energy is given by Eq. (2.25) in the course notes,

$$\mathscr{E} = \frac{1}{2}\int \phi\rho\, d^3x\,.$$

So

$$\begin{aligned}\mathscr{E} &= \frac{1}{2}\int (\phi_0+\phi')(\rho_0+\rho')\, d^3x = \frac{1}{2}\left(\int \phi_0(\rho_0+\rho')\, d^3x + \int \phi'(\rho_0+\rho')\, d^3x\right)\\ &= \frac{1}{2}\left(\phi_0 Q + \int_{\mathcal{V}} \phi'\rho'\, d^3x + \int_S \phi'\rho_0\, dS\right).\end{aligned}$$

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Problem 3. Charge is distributed on a (nonconducting) sphere of radius R, i.e., the charge density throughout space is of the form $\rho(\mathbf{x}) = \sigma(\theta,\varphi)\,\delta(r-R)$. The surface charge distribution σ on the sphere is chosen in such a way that the electrostatic potential on the sphere is $\phi(r=R,\theta,\varphi) = \alpha\cos\theta$, where α is a constant.

3.a Find the electrostatic potential $\phi(\mathbf{x})$ at all $r \le R$.

Solution. This is a Dirichlet boundary value problem. We are seeking the solution to Poisson's equation $\nabla^2\phi = -4\pi\rho$ subject to $\phi|_S = \psi = \alpha\cos\theta$. Equation (2.100) in the lecture notes gives the general solution,

$$\phi(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x},\mathbf{x}')\,\rho(\mathbf{x}')\,d^3x' - \frac{1}{4\pi}\int_S \psi(\mathbf{x}')\,\hat{\mathbf{n}}'\cdot\nabla_{x'}G_D(\mathbf{x}',\mathbf{x})\,dS_{x'}\,.$$

The Green's function for a sphere is given by Eq. (2.91),

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\beta}{|\mathbf{x}-\mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}'\frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \beta = -\frac{R}{|\mathbf{x}'|}.$$

Spherical harmonic expansion (2.78)?

3.b Find the electrostatic potential $\phi(\mathbf{x})$ at all $r \ge R$.

3.c Find the surface charge density $\sigma(\theta,\varphi)$ that was required in order to produce this potential ϕ.

3.d Find the total electrostatic energy.

Problem 4. A point charge of charge q is placed at point $\mathbf{x}'$ inside a conducting spherical shell of radius R. There is no net charge on the conductor. The potential inside the sphere is thus given by $q\,G_D(\mathbf{x},\mathbf{x}')$, where the explicit formula for $G_D(\mathbf{x},\mathbf{x}')$ for a spherical cavity is given in the lecture notes.

4.a Find the surface charge density $\sigma(\theta,\varphi)$ on the conducting shell.

Solution. The Green's function for a spherical cavity is given by Eq. (2.91),

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\alpha}{|\mathbf{x}-\mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}'\frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \alpha = -\frac{R}{|\mathbf{x}'|}.$$

The surface charge density can be found from Eq. (2.86),

$$\mathbf{E}\cdot\hat{\mathbf{n}} = 4\pi\sigma, \tag{4}$$

where $\mathbf{E} = -\nabla\phi$ in electrostatics.

We will begin by finding $\mathbf{E}$. We will orient our coordinate system such that $\mathbf{x}'$ (and consequently $\mathbf{x}''$) points along the z axis. Note that

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{R}{|\mathbf{x}'|\left|\mathbf{x}-\frac{R^2}{|\mathbf{x}'|^2}\mathbf{x}'\right|} = \frac{1}{\sqrt{\mathbf{x}^2 - 2\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'^2}} - \frac{R}{|\mathbf{x}'|\sqrt{\mathbf{x}^2 - 2\frac{R^2}{\mathbf{x}'^2}\mathbf{x}\cdot\mathbf{x}' + \frac{R^4}{\mathbf{x}'^4}\mathbf{x}'^2}}.$$

In spherical coordinates, we have

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{r^2 - 2rr'\cos\theta + r'^2}} - \frac{R}{r'}\frac{1}{\sqrt{r^2 - 2R^2 r\cos\theta/r' + R^4/r'^2}},$$

where we note that θ is the angle between $\mathbf{x}$ and the z axis. The gradient in spherical coordinates is given by

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\boldsymbol{\varphi}}.$$

The r component of the electric field inside the conductor is then

$$E_r(\mathbf{x}) = -q\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial r} = q\left(\frac{r - r'\cos\theta}{(r^2 - 2rr'\cos\theta + r'^2)^{3/2}} - \frac{R}{r'}\frac{r - R^2\cos\theta/r'}{(r^2 - 2R^2 r\cos\theta/r' + R^4/r'^2)^{3/2}}\right).$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ for the inner surface of a sphere, we are interested in only the r component of the field. On the surface of the sphere, the field is $E_r(r=R)\,\hat{\mathbf{r}}$. So we have

$$\begin{aligned}
E_r(r=R) &= q\left(\frac{R - r'\cos\theta}{(R^2 - 2Rr'\cos\theta + r'^2)^{3/2}} - \frac{R}{r'}\frac{R - R^2\cos\theta/r'}{(R^2 - 2R^3\cos\theta/r' + R^4/r'^2)^{3/2}}\right)\\
&= q\left(\frac{R - r'\cos\theta}{r'^3(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}} - \frac{R}{r'}\frac{R - R^2\cos\theta/r'}{R^3(1 - 2R\cos\theta/r' + R^2/r'^2)^{3/2}}\right)\\
&= \frac{q}{r'}\frac{R^3 - R^2 r'\cos\theta - Rr'^2 + R^2 r'\cos\theta}{R^2 r'^2(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}} = \frac{q}{Rr'^3}\frac{R^2 - r'^2}{(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}}.
\end{aligned}$$

Finally, feeding this into (4),

$$\sigma = -\frac{\mathbf{E}\cdot\hat{\mathbf{r}}}{4\pi} = \frac{q}{4\pi Rr'^3}\frac{r'^2 - R^2}{(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}} = \frac{q}{4\pi R|\mathbf{x}'|^3}\frac{|\mathbf{x}'|^2 - R^2}{(R^2/|\mathbf{x}'|^2 - 2R\cos\theta/|\mathbf{x}'| + 1)^{3/2}}.$$

4.b Find the force $\mathbf{F}$ that must be exerted on the point charge in order to hold it in place.

Solution. The total force on a charge distribution arises only from the external electric field $\mathbf{E}_0$, and is given by Eq. (2.42) in the lecture notes:

$$\mathbf{F} = \int \rho(\mathbf{x})\, \mathbf{E}_0(\mathbf{x})\, d^3x\,.$$

We now need the θ component of the field inside the conductor, which is

$$E_\theta(\mathbf{x}) = -\frac{q}{r}\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial\theta} = -q\left(\frac{r'\sin\theta}{(r^2-2rr'\cos\theta+r'^2)^{3/2}} - \frac{R^3\sin\theta}{r'^2(r^2-2R^2r\cos\theta/r'+R^4/r'^2)^{3/2}}\right).$$

The charge density for a point charge located at $\mathbf{x}'$ is given by $\rho(\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}')$. Evaluating the integral, we have

$$\mathbf{F} = \int q\,\delta(\mathbf{x}-\mathbf{x}')\,\mathbf{E}(\mathbf{x})\,d^3x = q\mathbf{E}(\mathbf{x}').$$

Recall that we chose $\mathbf{x}'$ to point along the z axis, so $\theta' = 0$. The θ component of $\mathbf{F}$ is then 0, and the r component is

$$\begin{aligned} F_r &= q^2\left(\frac{r'-r'}{(r'^2-2r'^2+r'^2)^{3/2}} - \frac{R}{r'}\frac{r'-R^2/r'}{(r'^2-2R^2+R^4/r'^2)^{3/2}}\right) = q^2Rr'^2\frac{R^2/r'-r'}{(r'^4-2R^2r'^2+R^4)^{3/2}} \\ &= q^2Rr'^2\frac{(R^2-r'^2)/r'}{(R^2-r'^2)^3} = q^2\frac{Rr'}{(R^2-r'^2)^2}. \end{aligned}$$

Since only the r component of $\mathbf{F}$ is nonzero, it points in the z direction, which we chose to be equivalent to the unit vector $\mathbf{x}'/|\mathbf{x}'|$. Therefore,

$$\mathbf{F} = q^2\frac{R|\mathbf{x}'|}{(R^2-|\mathbf{x}'|^2)^2}\frac{\mathbf{x}'}{|\mathbf{x}'|} = q^2\frac{R}{(R^2-|\mathbf{x}'|^2)^2}\mathbf{x}'.$$

Problem 5. The "mean value theorem" is stated as follows: For any solution ϕ to $\nabla^2\phi = 0$, the value of ϕ at $\mathbf{x}$ is equal to the average value of ϕ on a sphere of radius R (for any R) centered at $\mathbf{x}$.

5.a Prove the mean value theorem. Hint: Apply Green's theorem to ϕ and $1/|\mathbf{x}-\mathbf{x}'|$ for a suitable choice of region and a suitable choice of $\mathbf{x}'$.

5.b Use this result to show that a point charge can never be in stable equilibrium if placed in an electric field $\mathbf{E}$ that is source free in a neighborhood of the charge—and, indeed, it can be in neutral equilibrium only if $\mathbf{E} = 0$ in this neighborhood.