Problem 1. Verify that the functional

$$J[u] = \int_R \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy \tag{1}$$

is invariant under a rotation of the coordinate axis:

$$\tilde{x} = x \cos\epsilon + y \sin\epsilon, \qquad \tilde{y} = -x \sin\epsilon + y \cos\epsilon. \tag{2}$$

Solution. The functional is invariant if $J[u(x,y)] = J[u(\tilde{x},\tilde{y})]$. By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = \cos\epsilon \frac{\partial}{\partial \tilde{x}} - \sin\epsilon \frac{\partial}{\partial \tilde{y}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = \sin\epsilon \frac{\partial}{\partial \tilde{x}} + \cos\epsilon \frac{\partial}{\partial \tilde{y}}.$$

The Jacobian for (2) is

$$A = \begin{bmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial y \\ \partial\tilde{y}/\partial x & \partial\tilde{y}/\partial y \end{bmatrix} = \begin{bmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{bmatrix} \implies \det(A) = \cos^2\epsilon + \sin^2\epsilon = 1,$$

and therefore

$$\int_R dx\, dy \mapsto \int_{\tilde{R}} d\tilde{x}\, d\tilde{y}\,.$$

Making these substitutions into (1), we have

$$\begin{aligned}
J[u(x,y)] &= \int_R \left[\left(\cos\epsilon \frac{\partial u}{\partial \tilde{x}} - \sin\epsilon \frac{\partial u}{\partial \tilde{y}} \right)^2 + \left(\sin\epsilon \frac{\partial u}{\partial \tilde{x}} + \cos\epsilon \frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy \\
&= \int_R \left(\cos^2\epsilon \frac{\partial^2 u}{\partial \tilde{x}^2} - 2\cos\epsilon \sin\epsilon \frac{\partial^2 u}{\partial \tilde{x}\partial \tilde{y}} + \sin^2\epsilon \frac{\partial^2 u}{\partial \tilde{y}^2} + \sin^2\epsilon \frac{\partial^2 u}{\partial \tilde{x}^2} + 2\cos\epsilon \sin\epsilon \frac{\partial^2 u}{\partial \tilde{x}\partial \tilde{y}} + \cos^2\epsilon \frac{\partial^2 u}{\partial \tilde{y}^2} \right) dx\, dy \\
&= \int_R \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy = \int_{\tilde{R}} \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] d\tilde{x}\, d\tilde{y} \\
&= J[u(\tilde{x},\tilde{y})]
\end{aligned}$$

as desired. □

Problem 2. Consider the real-valued Lagrangian density $\mathcal{L}$ depending on a complex-valued function $\phi(t,x,y)$:

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt}\phi \right) - \nabla\phi^* \cdot \nabla\phi - m^2 \phi^* \phi, \tag{3}$$

where * is complex conjugation, and $\nabla\phi = (\partial\phi/\partial x\,,\, \partial\phi/\partial y)$. Treating ϕ and ϕ^* as independent objects, derive the Euler-Lagrange equations.

Solution. We will have two Euler-Lagrange equations; one for ϕ and one for ϕ^*. In general, they are given by

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x} - \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y}, \qquad 0 = \frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t^*} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x^*} - \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y^*}.$$

Expanding out $\nabla\phi^* \cdot \nabla\phi$, (3) becomes

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt}\phi \right) - \frac{\partial \phi^*}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial \phi^*}{\partial y}\frac{\partial \phi}{\partial y} - m^2 \phi^* \phi.$$

Then

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^*, \qquad \frac{\partial\mathcal{L}}{\partial\phi_t} = \frac{i}{2}\phi^*, \qquad \frac{\partial\mathcal{L}}{\partial\phi_x} = -\frac{\partial\phi^*}{\partial x}, \qquad \frac{\partial\mathcal{L}}{\partial\phi_y} = -\frac{\partial\phi^*}{\partial y},$$
$$\frac{\partial\mathcal{L}}{\partial\phi^*} = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi_t^*} = -\frac{i}{2}\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi_x^*} = -\frac{\partial\phi}{\partial x}, \qquad \frac{\partial\mathcal{L}}{\partial\phi_y^*} = -\frac{\partial\phi}{\partial y},$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi_t} = \frac{i}{2}\frac{d\phi^*}{dt}, \qquad \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi_x} = -\frac{\partial^2\phi^*}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi_y} = -\frac{\partial^2\phi^*}{\partial y^2},$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi_t^*} = -\frac{i}{2}\frac{d\phi}{dt}, \qquad \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi_x^*} = -\frac{\partial^2\phi}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi_y^*} = -\frac{\partial^2\phi}{\partial y^2}.$$

Then the Euler-Lagrange equations are

$$0 = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^* - \frac{i}{2}\frac{d\phi^*}{dt} + \frac{\partial^2\phi^*}{\partial x^2} + \frac{\partial^2\phi^*}{\partial y^2}, \qquad 0 = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi + \frac{i}{2}\frac{d\phi}{dt} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2},$$

which simplify to

$$0 = i\frac{d\phi^*}{dt} - \nabla^2\phi^* + m^2\phi^*, \qquad 0 = i\frac{d\phi}{dt} + \nabla^2\phi^* - m^2\phi^*.$$

Problem 3. The nondimensionalized, multidimensional Sine-Gordon equation,

$$\theta_{xx} + \theta_{yy} - \theta_{tt} = \sin\theta$$

for $\theta(x, y, t)$, is the Euler-Lagrange equation for the action integral

$$S[\theta] = \int_R \left\{\frac{1}{2}\left[\theta_t^2 - (\nabla\theta)^2\right] - \sin\theta\right\} dx\,dy\,dt$$

with $\nabla\theta = (\partial\theta/\partial x\,,\,\partial\theta/\partial y)$. The functional $S[\theta]$ is invariant under translation of x, y, and t. Find the associated energy-momentum tensor and energy-momentum vector.

Solution. Expanding out $(\nabla\theta)^2$, the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) - \sin\theta. \tag{4}$$

The energy-momentum tensor is defined by

$$T_{ij} = \frac{\partial\mathcal{L}}{\partial\theta_{x_i}}\frac{\partial\theta}{\partial x_j} - \mathcal{L}\,\delta_{ij},$$

where $x_i \in \{x_0, x_1, x_2\} = \{t, x, y\}$. The diagonal elements of T are then

$$T_{00} = \frac{\partial\mathcal{L}}{\partial\theta_t}\frac{\partial\theta}{\partial t} - \mathcal{L} = \theta_t^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = \frac{1}{2}(\theta_t^2 + \theta_x^2 + \theta_y^2) + \sin\theta,$$
$$T_{11} = \frac{\partial\mathcal{L}}{\partial\theta_x}\frac{\partial\theta}{\partial x} - \mathcal{L} = -\theta_x^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 + \theta_x^2 - \theta_y^2) + \sin\theta,$$
$$T_{22} = \frac{\partial\mathcal{L}}{\partial\theta_y}\frac{\partial\theta}{\partial y} - \mathcal{L} = -\theta_y^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 - \theta_x^2 + \theta_y^2) + \sin\theta,$$

and the nondiagonal elements are

$$T_{01} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial x} = \theta_t\theta_x, \qquad T_{02} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial y} = \theta_t\theta_y, \qquad T_{12} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial y} = -\theta_x\theta_y,$$
$$T_{10} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial t} = -\theta_t\theta_x, \qquad T_{20} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial t} = -\theta_t\theta_y, \qquad T_{21} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial x} = -\theta_x\theta_y.$$

In matrix form, we have

$$T = \begin{bmatrix} (\theta_t^2+\theta_x^2+\theta_y^2)/2+\sin\theta & \theta_t\theta_x & \theta_t\theta_y \\ -\theta_t\theta_x & -(\theta_t^2+\theta_x^2-\theta_y^2)/2+\sin\theta & -\theta_x\theta_y \\ -\theta_t\theta_y & -\theta_x\theta_y & -(\theta_t^2-\theta_x^2+\theta_y^2)/2+\sin\theta \end{bmatrix}.$$

The energy-momentum vector is defined by

$$P_j = \int T_{0j}\, dx_1\, dx_2\,.$$

Its components are then

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2+\theta_x^2+\theta_y^2)+\sin\theta\right] dx\, dy\,, \qquad P_1 = \int \theta_t\theta_x\, dx\, dy\,, \qquad P_2 = \int \theta_t\theta_y\, dx\, dy\,.$$

Problem 4. Extra credit

4.a Verify that the nondimensionalized, one-dimensional Sine-Gordon equation,

$$\theta_{xx} - \theta_{tt} = \sin\theta, \tag{5}$$

is also invariant under a Lorentz transformation on $(x_0 = t, x_1 = x)$. The transformation is given by

$$\begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix},$$

where $\gamma = 1/\sqrt{1-\nu^2}$.

Solution. Define $(\tilde{t}, \tilde{x})$ as the transformed coordinates. (5) is invariant if it has the same form under the substitution $\theta(t,x) \mapsto \theta(\tilde{t},\tilde{x})$. The new coordinates are given by

$$\begin{bmatrix} \tilde{t} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix}\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma(t-\nu x) \\ \gamma(x-\nu t) \end{bmatrix},$$

or

$$\tilde{t} = \gamma(t-\nu x), \qquad \tilde{x} = \gamma(x-\nu t).$$

Proceeding similarly to problem 1, the chain rule gives us

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} + \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial t} = \gamma\left(\frac{\partial}{\partial \tilde{t}} - \nu\frac{\partial}{\partial \tilde{x}}\right), \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial x} + \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} = \gamma\left(\frac{\partial}{\partial \tilde{x}} - \nu\frac{\partial}{\partial \tilde{t}}\right).$$

For the second derivatives,

$$\begin{aligned}\frac{\partial^2}{\partial t^2} &= \gamma^2\left(\frac{\partial}{\partial \tilde{t}} - \nu\frac{\partial}{\partial \tilde{x}}\right)^2 = \gamma^2\left(\frac{\partial^2}{\partial \tilde{t}^2} - 2\nu\frac{\partial^2}{\partial \tilde{t}\partial \tilde{x}} + \nu^2\frac{\partial^2}{\partial \tilde{x}^2}\right), \\ \frac{\partial^2}{\partial x^2} &= \gamma^2\left(\frac{\partial}{\partial \tilde{x}} - \nu\frac{\partial}{\partial \tilde{t}}\right)^2 = \gamma^2\left(\frac{\partial^2}{\partial \tilde{x}^2} - 2\nu\frac{\partial^2}{\partial \tilde{t}\partial \tilde{x}} + \nu^2\frac{\partial^2}{\partial \tilde{t}^2}\right).\end{aligned}$$

Making these substitutions, (5) becomes

$$\begin{aligned}\sin\theta &= \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial t^2} \\ &= \gamma^2\left(\frac{\partial^2\theta}{\partial \tilde{x}^2} - 2\nu\frac{\partial^2\theta}{\partial \tilde{t}\partial \tilde{x}} + \nu^2\frac{\partial^2\theta}{\partial \tilde{t}^2}\right) - \gamma^2\left(\frac{\partial^2\theta}{\partial \tilde{t}^2} - 2\nu\frac{\partial^2\theta}{\partial \tilde{t}\partial \tilde{x}} + \nu^2\frac{\partial^2\theta}{\partial \tilde{x}^2}\right) \\ &= \gamma^2\left[(1-\nu^2)\frac{\partial^2\theta}{\partial \tilde{x}^2} - (1-\nu^2)\frac{\partial^2\theta}{\partial \tilde{t}^2}\right] \\ &= \frac{\partial^2\theta}{\partial \tilde{x}^2} - \frac{\partial^2\theta}{\partial^2},\end{aligned}$$

because $\gamma^2 = 1/(1-\nu^2)$. Thus, we have demonstrated the invariance of (5). □

4.b Find the associated conserved quantity. Is it analogous to a common conserved quantity in classical mechanics?

Solution. By analogy to problem 3, the Lagrangian for this system is given by

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2) - \sin\theta$$

which is like (4), but with only one spatial dimension. Continuing the analogy, the components of the energy-momentum vector are

$$P_0 = \int\left[\frac{1}{2}(\theta_t^2 + \theta_x^2) + \sin\theta\right]dx\,, \qquad P_1 = \int \theta_t\theta_x\,dx\,.$$

These are the conserved quantitites, or "currents." The component P_0 is analagous to the calssical Hamiltonian, or the total energy of the system. This corresponds to $\mathcal{L}$'s having no explicit t dependence. The component P_1 is like the momentum conjugate to x, since it corresponds to $\mathcal{L}$'s having no explicit x dependence. Since we are concerned with only one spatial dimension, P_1 is analogous to the classical total (linear) momentum of the system.

Problem 5. Interacting line vortices

A system of n vortices moving on a two-dimensional plane has the Hamiltonian

$$H = \sum_{j=1}^{n}\sum_{i=1}^{j-1} -\gamma^{(i)}\gamma^{(j)}\ln|\mathbf{r}_i - \mathbf{r}_j|,$$

where $\gamma^{(i)}$ is the strength of the ith line vortex, and $\mathbf{r}_i = (x_i, y_i)$ its position in the plane. Using the Poisson bracket structure

$$[f,g] = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right),$$

Hamilton's equations for the vortex system simplify to

$$\dot{x}_i = [x_i, H], \qquad \dot{y}_i = [y_i, H].$$

Consider two vortices. Show that the equations of motion can be solved explicitly. Most importantly, show that the solution tells us the two vortices orbit each other with a frequency that is inversely proportional to the square of their separation.

Solution. For two vortices, the Hamiltonian reduces to

$$H = -\gamma^{(1)}\gamma^{(2)} \ln |\mathbf{r}_1 - \mathbf{r}_2|.$$

For $i, j \in \{1, 2\}$, note that

$$\frac{\partial x_i}{\partial x_j} = \frac{\partial y_i}{\partial y_j} = \delta_{ij}, \qquad \frac{\partial x_i}{\partial y_j} = \frac{\partial y_i}{\partial x_j} = 0.$$

Note also that

$$|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2},$$

and define $R \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ as the separation of the vortices. Define also

$$u \equiv \ln R, \qquad v \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 = R^2.$$

Then

$$\begin{aligned}
\frac{\partial H}{\partial x_i} &= \frac{\partial H}{\partial u}\frac{\partial u}{\partial R}\frac{\partial R}{\partial v}\frac{\partial v}{\partial x_i} = -\gamma^{(i)}\gamma^{(j)}\frac{1}{R}\frac{1}{2R}(2x_i - 2x_j) = -\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{(x_1 - x_2)^2 + (y_1 - y_2)^2} = -\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{R^2}, \\
\frac{\partial H}{\partial y_i} &= \frac{\partial H}{\partial u}\frac{\partial u}{\partial R}\frac{\partial R}{\partial v}\frac{\partial v}{\partial y_i} = -\gamma^{(i)}\gamma^{(j)}\frac{y_i - y_j}{R^2},
\end{aligned}$$

where $i \neq j$.

Combining the above, and again fixing $i \neq j$,

$$\begin{aligned}
\dot{x}_i &= \frac{1}{\gamma^{(i)}}\left(\frac{\partial x_i}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial x_i}{\partial y_i}\frac{\partial H}{\partial x_i}\right) + \frac{1}{\gamma^{(j)}}\left(\frac{\partial x_i}{\partial x_j}\frac{\partial H}{\partial y_j} - \frac{\partial x_i}{\partial y_j}\frac{\partial H}{\partial x_j}\right) = \frac{1}{\gamma^{(i)}}\frac{\partial H}{\partial y_i} = -\gamma^{(j)}\frac{y_i - y_j}{R^2}, \\
\dot{y}_i &= \frac{1}{\gamma^{(i)}}\left(\frac{\partial y_i}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial y_i}{\partial y_i}\frac{\partial H}{\partial x_i}\right) + \frac{1}{\gamma^{(j)}}\left(\frac{\partial y_i}{\partial x_j}\frac{\partial H}{\partial y_j} - \frac{\partial y_i}{\partial y_j}\frac{\partial H}{\partial x_j}\right) = -\frac{1}{\gamma^{(i)}}\frac{\partial H}{\partial x_i} = \gamma^{(j)}\frac{x_i - x_j}{R^2}.
\end{aligned}$$

Explicitly, the four equations of motion are

$$\dot{x}_1 = -\gamma^{(2)}\frac{y_1 - y_2}{R^2}, \qquad \dot{x}_2 = \gamma^{(1)}\frac{y_1 - y_2}{R^2}, \qquad \dot{y}_1 = \gamma^{(2)}\frac{x_1 - x_2}{R^2}, \qquad \dot{y}_2 = -\gamma^{(1)}\frac{x_1 - x_2}{R^2}. \tag{6}$$

Note that

$$\frac{\partial R}{\partial x_i} = \frac{\partial R}{\partial v}\frac{\partial v}{\partial x_i} = \frac{x_i - x_j}{R^2}, \qquad \frac{\partial R}{\partial y_i} = \frac{y_i - y_j}{R^2}$$

so

$$[H, R] = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(\frac{\partial H}{\partial x_i}\frac{\partial v}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial v}{\partial x_i}\right) = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(-\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{R^2}\frac{y_i - y_j}{R^2} + \gamma^{(i)}\gamma^{(j)}\frac{y_i - y_j}{R^2}\frac{x_i - x_j}{R^2}\right) = 0.$$

This means R is a constant.

Define $\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2 = (X, Y)$, where $|\mathbf{R}| = R$. Now we have two generalized coordinates X and Y, where $X = x_1 - x_2$ and $Y = y_1 - y_2$. This gives us the two equations of motion

$$\dot{X} = \dot{x}_1 - \dot{x}_2 = -(\gamma^{(1)} + \gamma^{(2)})\frac{y_1 - y_2}{R^2} = -\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}Y, \tag{7}$$

$$\dot{Y} = \dot{y}_1 - \dot{y}_2 = (\gamma^{(1)} + \gamma^{(2)})\frac{x_1 - x_2}{R^2} = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}X. \tag{8}$$

We can differentiate these to obtain two uncoupled second-order equations:

$$\ddot{X} = -\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}\dot{Y} = -\frac{(\gamma^{(1)} + \gamma^{(2)})^2}{R^4}X, \qquad \ddot{Y} = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}\dot{X} = -\frac{(\gamma^{(1)} + \gamma^{(2)})^2}{R^4}Y.$$

These equations have the solutions

$$X(t) = C_1 \cos\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right) + C_2 \sin\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right), \quad Y(t) = D_1 \cos\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right) + D_2 \sin\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right),$$

where C_1, C_2, D_1, D_2 are constants. Define

$$\omega \equiv \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2} \tag{9}$$

as the angular frequency. To solve for the constants, we apply (7) and (8):

$$\dot{X}(t) = -C_1\omega \sin(\omega t) + C_2\omega\cos(\omega t) = -\omega Y, \qquad \dot{Y}(t) = -D_1\omega\sin(\omega t) + D_2\omega\cos(\omega t) = \omega X.$$

This implies $C_1 = D_2$ and $C_2 = -D_1$. We can fix $C_1 = D_2 = 0$ without loss of generality, which implies $C_2 = R$ and $D_1 = -R$. We now have

$$X(t) = R\sin(\omega t) = x_1(t) - x_2(t), \qquad Y(t) = -R\cos(\omega t) = y_1(t) - y_2(t). \tag{10}$$

We can now find the solutions to the original four equations by integrating (6) with respect to t:

$$\begin{aligned}
x_1(t) &= \frac{\gamma^{(2)}}{R}\int \cos(\omega t)\,dt = \frac{\gamma^{(2)}}{\omega R}\sin(\omega t) = \frac{\gamma^{(2)}}{\gamma^{(1)} + \gamma^{(2)}}R\sin(\omega t),\\
x_2(t) &= -\frac{\gamma^{(1)}}{R}\int \cos(\omega t) = -\frac{\gamma^{(1)}}{\gamma^{(1)} + \gamma^{(2)}}R\sin(\omega t),\\
y_1(t) &= \gamma^{(2)}\int \sin(\omega t)\,dt = -\frac{\gamma^{(2)}}{\gamma^{(1)} + \gamma^{(2)}}R\cos(\omega t),\\
y_2(t) &= -\gamma^{(1)}\int \sin(\omega t)\,dt = \frac{\gamma^{(1)}}{\gamma^{(1)} + \gamma^{(2)}}R\cos(\omega t),
\end{aligned}$$

where we have taken the constants of integration to be zero without loss of generality.

Thus, we have shown that the equations of motion can be solved explicitly. Since $\mathbf{R} = (X, Y)$ is the vector separating the vortices, and (10) show that it rotates in a circle, we have also shown that the vortices orbit each other. The orbital frequency ω given by (9) is clearly inversely proportional to R^2, where $R = |\mathbf{R}|$ is the magnitude of the vortices' separation.

Problem 6. Conserved quantities for a system of line vortices

Now consider the general case of n vortices.

6.a Verify that the "total linear momentum along x" and the "total linear momentum along y,"

$$P_x = \sum_{i=1}^{n} \gamma^{(i)} y_i, \qquad P_y = \sum_{i=1}^{n} -\gamma^{(i)} x_i,$$

are conserved.

6.b Verify that $[P_x, P_y]$ gives a conserved quantity.

Problem 7. Charged particle in a magnetic field

Suppose a charged particle moves in a two-dimensional plane while experiencing a magnetic field $\mathbf{B} = (0, 0, B)$. Use the vector potential $\mathbf{A} = (-By, 0, 0)$. The Hamiltonian for the particle is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{p_y^2}{2m}.$$

7.a Write down Hamilton's equations. Verify that by appropriate manipulation we have

$$p_y + \frac{eB}{c}x = a, \qquad p_x = m\dot{x} - \frac{eB}{c}y = b,$$

where a and b are constants.

Solution. Note that

$$H = \frac{1}{2m}\left(p_x^2 + 2\frac{eB}{c}p_x y + \frac{e^2B^2}{c^2}y^2\right) + \frac{p_y^2}{2m} = \frac{p_x^2}{2m} + \frac{eB}{c}\frac{p_x y}{m} + \frac{e^2B^2}{c^2}\frac{y^2}{2m} + \frac{p_y^2}{2m}$$

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{eB}{c}y\right), \tag{11}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \tag{12}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \tag{13}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{eB}{c}\frac{1}{m}\left(p_x + \frac{eB}{c}y\right). \tag{14}$$

Substituting (11) into (14),

$$\dot{p}_y = -\frac{eB}{c}\dot{x}.$$

By integrating with respect to t, we obtain

$$p_y = -\frac{eB}{c}\int \dot{x}\,dt = -\frac{eB}{c}x + a,$$

where a is some constant. Therefore, we have

$$p_y + \frac{eB}{c}x = a, \tag{15}$$

as desired.

From (11),

$$m\dot{x} = p_x + \frac{eB}{c}y \iff p_x = m\dot{x} - \frac{eB}{c}y,$$

and from (12),

$$p_x = \int 0\,dt = b,$$

where b is some constant. Combining these, we have

$$p_x = m\dot{x} - \frac{eB}{c}y = b \tag{16}$$

as desired. $\square$

7.b Using the relations above and the equations of motion, verify that the charged particle moves in a circle in the (x, y) plane and that the circling frequency ω is given by

$$\omega = \frac{eB}{mc}.$$

This is called the *Larmor frequency*.

Solution. Substituting (16) into (11) yields

$$\dot{x} = \frac{1}{m}\left(b + \frac{eB}{c}y\right) = \frac{eB}{mc}\left(\frac{c}{eB}b + y\right). \tag{17}$$

Similarly, solving (15) for p_y and substituting into (13) gives us

$$\dot{y} = \frac{1}{m}\left(a - \frac{eB}{c}x\right) = \frac{eB}{mc}\left(\frac{c}{eB}a - x\right). \tag{18}$$

Differentiating (17) and (18) by t, we obtain two uncoupled second-order equations:

$$\ddot{x} = \frac{eB}{mc}\dot{y} = -\frac{e^2B^2}{m^2c^2}\left(x - \frac{c}{eB}a\right), \qquad \ddot{y} = -\frac{eB}{mc}\dot{x} = -\frac{e^2B^2}{m^2c^2}\left(y + \frac{c}{eB}b\right). \tag{19}$$

Let $\tilde{x}$ and $\tilde{y}$ be new coordinates such that

$$\tilde{x} \equiv x - \frac{c}{eB}a, \qquad \tilde{y} \equiv y + \frac{c}{eB}b.$$

Then

$$\frac{d\tilde{x}}{dt} = \dot{x}, \qquad \frac{d^2\tilde{x}}{dt^2} = \ddot{x}, \qquad \frac{d\tilde{y}}{dt} = \dot{y}, \qquad \frac{d^2\tilde{y}}{dt^2} = \ddot{y},$$

and the equations (19) can be rewritten in terms of $\tilde{x}$ and $\tilde{y}$:

$$\frac{d^2\tilde{x}}{dt^2} = -\frac{e^2B^2}{m^2c^2}\tilde{x}, \qquad \frac{d^2\tilde{y}}{dt^2} = -\frac{e^2B^2}{m^2c^2}\tilde{y}.$$

These equations have the solutions

$$\tilde{x}(t) = C_1 \cos\left(\frac{eB}{mc}t\right) + C_2 \sin\left(\frac{eB}{mc}t\right) \equiv C_1 \cos(\omega t) + C_2 \sin(\omega t),$$
$$\tilde{y}(t) = D_1 \cos\left(\frac{eB}{mc}t\right) + D_2 \sin\left(\frac{eB}{mc}t\right) \equiv D_1 \cos(\omega t) + D_2 \sin(\omega t),$$

where C_1, C_2, D_1, D_2 are constants, and we have defined

$$\omega \equiv \frac{eB}{mc}. \tag{20}$$

Applying (17) and (18), we have

$$\frac{d\tilde{x}}{dt} = -C_1\omega \sin(\omega t) + C_2\omega \cos(\omega t) = \omega\tilde{y}, \qquad \frac{d\tilde{y}}{dt} = -D_1\omega \sin(\omega t) + D_2\omega \cos(\omega t) = -\omega\tilde{x}.$$

This implies $C_1 = -D_2$ and $C_2 = D_1$. We may fix $C_1 = D_2 = 0$ and $C_2 = D_1 = R$ without loss of generality, where R is some constant. Transforming back to the original coordinates, we have

$$x(t) = R\sin(\omega t) + \frac{c}{eB}a, \qquad y(t) = R\cos(\omega t) - \frac{c}{eB}b.$$

These solutions show that the particle moves in a circle with angular frequency ω defined by (20), as desired. $\square$

7.c Now consider the limit where the B field can be made arbitrarily strong. Compare the Poisson bracket $[x, p_x]$ for the charged particle with the Poisson bracket relation

$$[x_i, y_i] = \frac{\delta_{ij}}{\gamma^{(i)}}$$

for the system of line vortices described in problems 5 and 6.

Solution. The Poisson bracket for the charged particle is given by

$$[x, p_x] = \frac{\partial x}{\partial x}\frac{\partial p_x}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial p_x}{\partial x} + \frac{\partial x}{\partial y}\frac{\partial p_x}{\partial p_y} - \frac{\partial x}{\partial p_y}\frac{\partial p_x}{\partial y}.$$

Note that

$$\frac{\partial x}{\partial x} = \frac{\partial p_x}{\partial p_x} = 1, \qquad \frac{\partial x}{\partial p_x} = \frac{\partial p_x}{\partial x} = \frac{\partial x}{\partial y} = \frac{\partial p_x}{\partial p_y} = 0, \qquad \frac{\partial x}{\partial p_y} = -\frac{c}{eB}, \qquad \frac{\partial p_x}{\partial y} = -\frac{eB}{c}.$$

Then

$$[x, p_x] = 1 - 1 = 0.$$

It's different?

While writing up these solutions, I consulted Gelfand and Fomin's *Calculus of Variations*, Goldstein's *Classical Mechanics*, Tong's *Classical Dynamics*, and Riley, Hobson, and Bence's *Mathematical Methods for Physics and Engineering*.