1 Reduced three-body problem

The problem of three point particles interacting gravitationally has a particularly simple limit: let the third body $m_3 \ll m_2, m_1$ so that its effect on the motions of m_1 and m_2 is negligible. Assume in addition that m_3 moves in the same orbital plane as m_1 and m_2. For simplicity, consider only the case of m_1 and m_2 in circular orbit about their center of mass.

1.1 Switch into a reference frame rotating with angular velocity ω associated with the circular orbit for the two-body problem. Choose the center of mass of the two-body problem to be the origin. Choose the x axis to go through m_1 and m_2. Show that the (now stationary) m_1 and m_2 are located at $-r_c\mu/m_1$ and $r_c\mu/m_2$.

Solution. Call the stationary coordinate system (r, θ, ϕ). We showed in Prob. 4 of Homework 1, that the motion for m_1, m_2 is confined to a plane, which we may choose to be (r, θ). Then m_1 and m_2 are located at $\mathbf{R}_1 = \mathbf{R}_1(t, r, \theta)$ and $\mathbf{R}_2 = \mathbf{R}_2(t, r, \theta)$. Let $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2$ denote the separation between the masses. Using the method discussed in class, the motion of m_1 and m_2 is governed by the equation

$$\mu\ddot{\mathbf{R}} = -\frac{\partial}{\partial r} V_{\text{eff}} \tag{1}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass, $\mathbf{R}_{12} = \mathbf{R}_1 - \mathbf{R}_2$ is the separation between the two masses, and the effective potential

$$V_{\text{eff}}(R) = V(R) + \frac{J^2}{2\mu R^2} = -\frac{Gm_1 m_2}{R} + \frac{J^2}{2\mu R^2} \tag{2}$$

where $J = \mu R^2 \dot{\theta}$ is the magnitude of the total angular momentum, which is conserved.

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Call the stationary coordinate system (X, Y, Z), and choose (X, Y) as the orbital plane. Let the locations of m_1 and m_2 be given by $\mathbf{R}_1 = \mathbf{R}_1(t, X, Y)$ and $\mathbf{R}_2 = \mathbf{R}_2(t, X, Y)$. Let r_c be the radius of the orbit about the Z axis. We know that m_1, m_2 both have circular orbits about the Z axis with frequency ω, so we can write

$$\mathbf{R}_1(t) = \tag{3}$$

Call the rotating coordinate system (x, y, z), in which the locations of m_1 and m_2 are $\mathbf{r}_1 = \mathbf{r}_1(t, x, y, z)$ and $\mathbf{r}_2 = \mathbf{r}_2(x, y, z)$.

Since we have chosen the x axis to go through m_1 and m_2, we will choose our reference frame to be rotating about the z axis. We thus have the transformation

$$x = X\cos\omega t + Y\sin\omega t, \tag{4}$$
$$y = Y\cos\omega t - X\sin\omega t, \tag{5}$$
$$z = Z. \tag{6}$$

1.2 Show that the Lagrangian governing the wquation of motion of m_3 at location $(x(t), y(t))$ is

$$L_3 = \frac{m_3}{2}\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right] - U_{13} - U_{23}, \tag{7}$$

where $U_1 3(x, y)$ is the gravitaional interaction of m_3 with m_1, while U_{23} is associated with m_3 and m_2.