1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1}$$

1.1 Verify the following:

a. $i\hbar\partial_t \langle \Psi(t)|x\rangle = -\langle \Psi(t)|H|x\rangle$,

b. $i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle - \langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle$,

c. $i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle = -\dfrac{\hbar^2}{2m}\big(\langle \Phi(t)|x\rangle\, \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle)\, \langle x|\Psi(t)\rangle\big)$,

d.

Solution.

a. Beginning with Schrödinger's equation, note that

$$\begin{aligned} i\hbar\partial_t |\Psi(t)\rangle &= H\,|\Psi(t)\rangle && (2)\\ i\hbar\partial_t \langle x|\Psi(t)\rangle &= \langle x|H|\Psi(t)\rangle && (3)\\ (i\hbar\partial_t \langle x|\Psi(t)\rangle)^* &= (\langle x|H|\Psi(t)\rangle)^* && (4)\\ -i\hbar\partial_t \langle \Psi(t)|x\rangle &= \langle \Psi(t)|H|x\rangle && (5)\\ i\hbar\partial_t \langle \Psi(t)|x\rangle &= -\langle \Psi(t)|H|x\rangle, && (6) \end{aligned}$$

where in going to (5) we have used the fact that H is Hermitian. □

b. Beginning with what was proven in (a),

$$\begin{aligned} i\hbar\partial_t \langle \Phi(t)|x\rangle &= -\langle \Phi(t)|H|x\rangle && (7)\\ i\hbar(\partial_t \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle &= -\langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle. && (8) \end{aligned}$$

From (3), we can write

$$\langle \Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle. \tag{9}$$

Adding (8) and (9) yields

$$\begin{aligned} \langle \Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle + i\hbar(\partial_t \langle \Phi(t)|x\rangle)\langle x|\Psi(t)\rangle &= \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle - \langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle && (10)\\ i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle &= \langle \Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle - \langle \Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle, && (11) \end{aligned}$$

where in going to (11) we have used the product rule of differentiation. □