1. **Problem.** Suppose we have a mechanical system with n degrees of freedom. Let $q_1(t), q_2(t), \ldots, q_n(t)$ be its generalized coordinates. Now consider a time-dependent coordinate transformation

$$Q_i = Q_i(t, q_1, q_2, \ldots, q_n) \qquad i = 1, 2, \ldots, n.$$

Show that if $q_i(t)$ solves a system of Euler-Lagrange equations involving a Lagrangian $L(t, q_i, \dot{q}_i)$, then $Q_i(t)$ solves the Euler-Lagrange equations involving $L(t, Q_i, \dot{Q}_i)$ provided the time-dependent coordinate transformation fulfills some minimal standard of good behavior. Specify this "minimal standard of good behavior."

Solution. Suppose that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0; \tag{1}$$

that is, $q_i(t)$ solve a system of Euler-Lagrange equations. We want to show that

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_i} = 0. \tag{2}$$

Beginning with the first term of (1), we can use the chain rule for $L(t, Q_i, \dot{Q}_i)$ to write

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial q_i}, \tag{3}$$

provided there exists an inverse transformation

$$q_i = q_i(t, Q_1, Q_2, \ldots, Q_n) \qquad i = 1, 2, \ldots, n \tag{4}$$

that allows us to write $L(t, q_i, \dot{q}_i)$ in terms of t, Q_i, and $\dot{Q}_i$. This is only possible if there is a one-to-one correspondence between $q_i(t)$ and $Q_i(t)$, which is the "minimal standard of good behavior" for the transformation. We will assume the transformation is so well behaved.

Again using the chain rule for $Q_j = Q_j(t, q_1, q_2, \ldots, q_n)$, note that

$$\dot{Q}_j = \frac{\partial Q_j}{\partial t} + \frac{\partial Q_j}{\partial q_i}\dot{q}_i \tag{5}$$

so (3) becomes

$$\frac{\partial L}{\partial q_i} = \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial q_i\,\partial t} + \frac{\partial^2 Q_j}{\partial q_i\,\partial q_k}\dot{q}_k\right). \tag{6}$$

Applying the chain rule now to the second term of (1), we have

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial Q_j}{\partial q_i} \tag{7}$$

where the right-hand side comes from (5). Then, using the product rule to take the time derivative,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{d}{dt}\frac{\partial Q_j}{\partial q_i}\right). \tag{8}$$

For the second term of (8), the chain rule for $Q_j = Q_j(t, q_1, q_2, \dots, q_n)$ gives

$$\frac{d}{dt}\frac{\partial Q_j}{\partial q_i} = \frac{\partial^2 Q_j}{\partial t\,\partial q_i} + \frac{\partial^2 Q_j}{\partial q_i\,\partial q_k}\dot{q}_k. \tag{9}$$

Substituting (9) into (8), we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j}\left(\frac{\partial^2 Q_j}{\partial t\,\partial q_i} + \frac{\partial^2 Q_j}{\partial q_k\,\partial q_i}\dot{q}_k\right) \tag{10}$$

where the terms on the far right appeared in (6). Making this substitution,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right)\frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial q_i}, \tag{11}$$

and rearranging,

$$\frac{\partial Q_j}{\partial q_i}\left(\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j}\right) = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}, \tag{12}$$

Finally, substituting the original assumption (1), we find

$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}_j} = 0 \tag{13}$$

which is what we sought to prove. □

2. **Problem.** Look at the Lagrangian

$$L = e^{\sigma t}\left(\frac{m\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

for one-dimensional motion.

(a) Write down the associated Euler-Lagrange ODE.

(b) Now perform a point transformation

$$Q = e^{\sigma t/2} q$$

where the new position coordinate Q is a function of t and q. What is the equation of motion for $Q(t)$? Are there conserved quantities?

Solution.

(a) Beginning from the general expression for the Euler-Lagrange equations,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -e^{\sigma t}kq - \frac{d}{dt}\left(e^{\sigma t}m\dot{q}\right) = -me^{\sigma t}\left(\ddot{q} + \sigma\dot{q} + \frac{k}{m}q\right) \tag{14}$$

so the ODE is

$$0 = \ddot{q} + \sigma\dot{q} + \frac{k}{m}q. \tag{15}$$

(b) It is possible to invert this transformation and write $q = q(t, Q)$. Explicitly, this is

$$q = Qe^{-\sigma t/2} \tag{16}$$

so

$$\dot{q} = e^{-\sigma t/2}\left(\dot{Q} - \frac{\sigma}{2}Q\right). \tag{17}$$

Rewriting the Lagrangian such that $L = L(t, Q, \dot{Q})$ results in

$$\begin{aligned} L &= e^{\sigma t}\left(\frac{m}{2}\left(e^{-\sigma t/2}\left(\dot{Q} - \frac{\sigma}{2}Q\right)\right)^2 - \frac{k}{2}\left(Qe^{-\sigma t/2}\right)^2\right) && (18) \\ &= \frac{m}{2}\left(\dot{Q} - \frac{\sigma}{2}Q\right)^2 - \frac{k}{2}Q^2 && (19) \\ &= \frac{m}{2}\left(\dot{Q}^2 - \sigma Q\dot{Q} + \left(\frac{\sigma^2}{4} - \frac{k}{m}\right)Q^2\right). && (20) \end{aligned}$$

Then the Euler-Lagrange equations are given by

$$0 = \frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = \frac{m}{2}\left(-\sigma\dot{Q} + 2\left(\frac{\sigma^2}{4} - \frac{k}{m}\right)Q - \frac{d}{dt}\left(2\dot{Q} - \sigma Q\right)\right) \tag{21}$$

which simplifies to

$$0 = \ddot{Q} + \left(\frac{k}{m} - \frac{\sigma^2}{4}\right)Q. \tag{22}$$

The solutions to (22) have the form

$$Q(t) = \begin{cases} A_1 \sin\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\,t + A_2 \cos\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\,t & \text{if } \frac{k}{m} > \frac{\sigma^2}{4}, \\ B_1 + B_2 t & \text{if } \frac{k}{m} = \frac{\sigma^2}{4}, \\ C_1 \exp\left\{-\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\,t\right\} + C_2 \exp\left\{\sqrt{\frac{k}{m} - \frac{\sigma^2}{4}}\,t\right\} & \text{if } \frac{k}{m} < \frac{\sigma^2}{4}, \end{cases} \tag{23}$$

where A_i, B_i, C_i are real constants.

The Lagrangian in (20) does not explicitly depend on time. Thus, the total energy of the system

$$E = \dot{Q}\frac{\partial L}{\partial \dot{Q}} - L \tag{24}$$

is conserved.