

Problem 1. (Peskin & Schroeder 2.1) Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

1(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying

$$E^i = -F^{0i}; \quad \epsilon^{ijk} B^k = -F^{ij}. \quad (2)$$

Solution. We want to extremize the action,

$$S[A_\mu] = \int d^4x \mathcal{L}(A_\mu, \partial_\mu A_\mu),$$

where \mathcal{L} is the integrand of Eq. (1). Let δA_μ denote some arbitrary variation that vanishes at the boundaries of spacetime. The action for $A_\mu + \delta A_\mu$ is

$$S[A_\mu + \delta A_\mu] = \int d^4x \mathcal{L}(A_\mu + \delta A_\mu, \partial_\nu A_\mu + \partial_\nu \delta A_\mu).$$

Then, to first order in δA_μ , the variation of the action is

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu],$$

which we want to vanish for all δA_μ . Let $\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$. Then, applying the definition of $F_{\mu\nu}$ given in Eq. (1),

$$\begin{aligned} \delta S &= \int d^4x \left(-\frac{1}{4} (F_{\mu\nu} + \delta F_{\mu\nu})(F^{\mu\nu} + \delta F^{\mu\nu}) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\ &\approx \int d^4x \left(-\frac{1}{4} (F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \delta F^{\mu\nu} + \delta F_{\mu\nu} F^{\mu\nu}) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\ &= \int d^4x \left(-\frac{1}{4} (F_{\mu\nu} \delta F^{\mu\nu} + \delta F_{\mu\nu} F^{\mu\nu}) \right) \\ &= \int d^4x \left(-\frac{1}{2} \delta F_{\mu\nu} F^{\mu\nu} \right), \end{aligned} \quad (3)$$

where we have discarded terms of $\mathcal{O}((\delta A^\mu)^2)$ and swapped covariant and contravariant indices in going to the final equality.

Note that

$$\begin{aligned} \delta F_{\mu\nu} F^{\mu\nu} &= (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \partial_\mu \delta A_\nu \partial^\mu A^\nu - \partial_\mu \delta A_\nu \partial^\nu A^\mu - \partial_\nu \delta A_\mu \partial^\mu A^\nu + \partial_\nu \delta A_\mu \partial^\nu A^\mu. \end{aligned} \quad (4)$$

Integrating the first term of Eq. (4) by parts, we have

$$\int d^4x \frac{\partial \delta A_\nu}{\partial x^\mu} \frac{\partial A^\nu}{\partial x_\mu} = \left[\delta A_\nu \frac{\partial A^\nu}{\partial x_\mu} \right]_{-\infty}^{\infty} - \int d^4x \delta A_\nu \frac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} = - \int d^4x \delta A_\nu \partial_\mu \partial^\mu A^\nu,$$

because δA^ν vanishes at $\pm\infty$. The other terms follow similarly. Then we find

$$\begin{aligned}\int d^4x \delta F_{\mu\nu} F^{\mu\nu} &= - \int d^4x (\delta A_\nu \partial_\mu \partial^\mu A^\nu - \delta A_\nu \partial_\mu \partial^\nu A^\mu - \delta A_\mu \partial_\nu \partial^\mu A^\nu + \delta A_\mu \partial_\nu \partial^\nu A^\mu) \\ &= - \int d^4x (\delta A_\nu \partial_\mu F^{\mu\nu} + \delta A_\mu \partial_\nu F^{\nu\mu}) = - \int d^4x (\delta A_\nu \partial_\mu F^{\mu\nu} + \delta A_\nu \partial_\mu F^{\mu\nu}) \\ &= -2 \int d^4x \delta A_\nu \partial_\mu F^{\mu\nu},\end{aligned}$$

where in going to the second-to-last equality we have simply swapped the indices.

Making this substitution in Eq. (3), we obtain

$$\delta S = \delta A_\nu \int d^4x \partial_\mu F^{\mu\nu}.$$

In order for the action to be at a local extremum, we need $\delta S = 0$ for any δA_ν . This implies that the integrand is 0. Thus, we obtain

$$\partial_\mu F^{\mu\nu} = 0, \quad (5)$$

which is the covariant form of the inhomogeneous Maxwell equations in a source-free region [?, p. 557], as we sought to derive. \square

From Eq. (2) and the knowledge that $F^{\mu\nu}$ is antisymmetric [?, p. 556], we can write

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}. \quad (6)$$

The first equation of Eq. (2) is equivalent to $E^i = F^{i0}$. Then the zeroth component of Eq. (5) can be written

$$\partial_\mu F^{\mu 0} = \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = \nabla \cdot \mathbf{E} = 0,$$

which is the differential form of Gauss's law.

For the remaining components of Eq. (5), we apply the second equation of Eq. (5) to find

$$\partial_\mu F^{\mu i} = -\frac{\partial E^i}{\partial t} + \epsilon^{ijk} \frac{\partial B^k}{\partial x^j} = 0.$$

In vector form, this is

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0},$$

the differential form of Ampère's law.

1(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad (7)$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu, \quad (8)$$

leads to an energy-momentum tensor \hat{T} that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{E^2 + B^2}{2}; \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

Solution. We want to evaluate Eq. (2.17) of Peskin & Schroeder,

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu \implies T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - \mathcal{L} g^{\mu\nu}, \quad (9)$$

where we have associated the field ϕ with A^λ . In order to evaluate the derivatives, we can use the variational method to calculate $\partial \mathcal{L} / \partial(\partial_\alpha A_\beta)$ by letting $\partial_\alpha A_\beta \rightarrow \partial_\alpha A_\beta + \delta \partial_\alpha A_\beta$ [?, p. 81]. Let

$$\delta \mathcal{L} = \mathcal{L}(\partial_\alpha A_\beta) - \mathcal{L}(\partial_\alpha A_\beta + \delta \partial_\alpha A_\beta).$$

Note that

$$\mathcal{L}(\partial_\alpha A_\beta + \delta \partial_\alpha A_\beta) = -\frac{1}{4}(F_{\alpha\beta} + \delta F_{\alpha\beta})(F^{\alpha\beta} + \delta F^{\alpha\beta}) \approx -\frac{1}{4}(F_{\alpha\beta} F^{\alpha\beta} + F_{\alpha\beta} \delta F^{\alpha\beta} + \delta F_{\alpha\beta} F^{\alpha\beta}),$$

so

$$\begin{aligned} \delta \mathcal{L} &= -\frac{1}{4}(F_{\alpha\beta} \delta F^{\alpha\beta} + \delta F_{\alpha\beta} F^{\alpha\beta}) = -\frac{1}{2} \delta F_{\alpha\beta} F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha) F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha \delta A_\beta + \partial_\alpha \delta A_\beta) F^{\alpha\beta} \\ &= -\partial_\alpha \delta A_\beta F^{\alpha\beta}, \end{aligned}$$

where we have used the antisymmetry of $F^{\alpha\beta}$. This gives us

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta} \implies \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\beta)} = -F^\alpha{}_\beta,$$

and then we find

$$T^{\mu\nu} = -F^\mu{}_\lambda \partial^\nu A^\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^\mu{}_\lambda \partial^\nu A^\lambda. \quad (10)$$

Adding $K^{\lambda\mu\nu}$ as defined in Eq. (8), Eq. (7) becomes

$$\hat{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^\mu{}_\lambda \partial^\nu A^\lambda + \partial_\lambda (F^{\mu\lambda} A^\nu). \quad (11)$$

Applying the product rule to the third term, we find

$$\partial_\lambda (F^{\mu\lambda} A^\nu) = A^\nu \partial_\lambda F^{\mu\lambda} + F^{\mu\lambda} \partial_\lambda A^\nu = -A^\nu \partial_\lambda F^{\lambda\mu} + F^{\mu\lambda} \partial_\lambda A^\nu = F^{\mu\lambda} \partial_\lambda A^\nu,$$

where we have applied the antisymmetry of $F^{\mu\nu}$ and Eq. (5). Making this substitution in Eq. (11),

$$\begin{aligned} \hat{T}^{\mu\nu} &= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^\mu{}_\lambda \partial^\nu A^\lambda + F^{\mu\lambda} \partial_\lambda A^\nu \\ &= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^\mu{}_\lambda \partial^\lambda A^\nu - F^\mu{}_\lambda \partial^\nu A^\lambda = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F^\mu{}_\lambda F^{\lambda\nu} \\ &= \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F^\nu{}_\lambda. \end{aligned} \quad (12)$$

To show that $\hat{T}^{\mu\nu}$ is symmetric, note that

$$\hat{T}^{\nu\mu} = \frac{1}{4} g^{\nu\mu} F_{\alpha\beta} F^{\alpha\beta} - F^{\nu\lambda} F^\mu{}_\lambda = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^\mu{}_\lambda F^{\nu\lambda} = \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F^{\mu\lambda} F^\nu{}_\lambda = \hat{T}^{\mu\nu}$$

as desired. \square

For the energy and momentum densities, from Eq. (12) we have

$$\hat{T}^{00} = \frac{1}{4}g^{00}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^0_{\lambda} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + F^{0\lambda}F_{\lambda}^0, \quad (13)$$

$$\hat{T}^{0i} = \frac{1}{4}g^{0i}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^i_{\lambda} = F^{0\lambda}F_{\lambda}^i. \quad (14)$$

Using Eq. (6),

$$F_{\mu\nu}F^{\mu\nu} = -E^{x2} - E^{y2} - E^{z2} - E^{x2} + B^{z2} + B^{y2} - E^{y2} + B^{z2} + B^{x2} - E^{z2} + B^{y2} + B^{x2} = 2(\mathbf{B}^2 - \mathbf{E}^2).$$

Note also from Eq. (6) that

$$F_{\lambda}^{\nu} = g_{\lambda\mu}F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{bmatrix},$$

so

$$F^{0\lambda}F_{\lambda}^0 = (-\mathbf{E}) \cdot (-\mathbf{E}) = \mathbf{E}^2, \quad F^{0\lambda}F_{\lambda}^i = B_j E_k - E_k B_j = (\mathbf{E} \times \mathbf{B})_i.$$

Equations (13–14) are then

$$\hat{T}^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \mathcal{E}, \quad \hat{T}^{0i} = \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})_i = \mathbf{S},$$

as we sought to show. \square

Problem 2. The complex scalar field (Peskin & Schroeder 2.2) Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x (\partial_{\mu}\phi^* \partial^{\mu}\phi - m^2\phi^*\phi). \quad (15)$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

2(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x (\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi). \quad (16)$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

Solution. The momentum density conjugate to $\phi(x)$ is defined in Peskin & Schroeder (2.4):

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$$

Here, \mathcal{L} is the integrand of Eq. (15). Expanding its first term yields

$$\mathcal{L} = \dot{\phi}\dot{\phi}^* - \nabla\phi \cdot \nabla\phi^*, \quad (17)$$

so then

$$\pi(x) = \dot{\phi}^*, \quad \pi^*(x) = \dot{\phi}, \quad (18)$$

where $\pi^*(x)$ is the momentum conjugate to $\phi^*(x)$. The canonical commutation relations follow from Peskin & Schroeder (2.20):

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i \delta^3(\mathbf{x} - \mathbf{y}), \\ [\phi(\mathbf{x}), \phi(\mathbf{y})] &= [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = 0, \\ [\pi(\mathbf{x}), \pi(\mathbf{y})] &= [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0, \\ [\phi(\mathbf{x}), \pi^*(\mathbf{y})] &= [\phi(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi^*(\mathbf{y})] = 0. \end{aligned}$$

The Hamiltonian is given in general for a single field by Peskin & Schroeder (2.5),

$$H = \int d^3x \left(\pi(x) \dot{\phi}(x) - \mathcal{L} \right).$$

For the two fields $\phi(x)$ and $\phi^*(x)$, this becomes

$$\begin{aligned} H &= \int d^3x \left(\pi(x) \dot{\phi}(x) + \pi^*(x) \dot{\phi}^*(x) - \mathcal{L} \right) \\ &= \int d^3x \left(\pi \dot{\phi} + \pi^* \dot{\phi}^* - \dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi \right) \\ &= \int d^3x \left(\pi \pi^* + \dot{\phi} \dot{\phi}^* - \dot{\phi} \dot{\phi}^* + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi \right) \\ &= \int d^3x \left(\pi^* \pi + \nabla \phi \cdot \nabla \phi^* + m^2 \phi^* \phi \right), \end{aligned}$$

where we have used Eqs. (17) and (18) as well as the commutation relations. So we have proven Eq. (16). \square

The Heisenberg equation of motion is Peskin & Schroeder (2.44),

$$i \frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H],$$

where \mathcal{O} is an arbitrary operator. Then

$$\begin{aligned} i \frac{\partial \phi(x)}{\partial t} &= [\phi(x), H] \\ &= \left[\phi(\mathbf{x}, t), \int d^3x' \pi^*(\mathbf{x}', t) \pi(\mathbf{x}', t) \right] + \left[\phi(\mathbf{x}, t), \int d^3x' \nabla' \phi(\mathbf{x}', t) \cdot \nabla' \phi^*(\mathbf{x}', t) \right] \\ &\quad + m^2 \left[\phi(\mathbf{x}, t), \int d^3x' \phi^*(\mathbf{x}', t) \phi(\mathbf{x}', t) \right] \\ &= \left[\phi(\mathbf{x}, t), \int d^3x' \pi^*(\mathbf{x}', t) \pi(\mathbf{x}', t) \right] = i \int d^3x' \delta^3(\mathbf{x} - \mathbf{x}') \pi^*(\mathbf{x}', t) = i \pi^*(x), \end{aligned}$$

$$\begin{aligned} i \frac{\partial \phi^*(x)}{\partial t} &= [\phi^*(x), H] \\ &= \left[\phi^*(\mathbf{x}, t), \int d^3x' \pi^*(\mathbf{x}', t) \pi(\mathbf{x}', t) \right] = i \int d^3x' \delta^3(\mathbf{x} - \mathbf{x}') \pi(\mathbf{x}', t) = i \pi(x), \end{aligned}$$

$$\begin{aligned}
i \frac{\partial \pi(x)}{\partial t} &= [\pi(x), H] \\
&= \left[\pi(\mathbf{x}, t), \int d^3 x' \pi^*(\mathbf{x}', t) \pi(\mathbf{x}', t) \right] + \left[\pi(\mathbf{x}, t), \int d^3 x' \nabla' \phi(\mathbf{x}', t) \cdot \nabla' \phi^*(\mathbf{x}', t) \right] \\
&\quad + m^2 \left[\pi(\mathbf{x}, t), \int d^3 x' \phi^*(\mathbf{x}', t) \phi(\mathbf{x}', t) \right] \\
&= -i \int d^3 x' [\nabla' \delta^3(\mathbf{x} - \mathbf{x}') \cdot \nabla' \phi^*(\mathbf{x}', t) + m^2 \delta^3(\mathbf{x} - \mathbf{x}') \phi^*(\mathbf{x}', t)] \\
&= -i \int d^3 x' [-\delta^3(\mathbf{x} - \mathbf{x}') \cdot \nabla'^2 \phi^*(\mathbf{x}', t) + m^2 \delta^3(\mathbf{x} - \mathbf{x}') \phi^*(\mathbf{x}', t)] = -i(-\nabla^2 + m^2) \phi^*(x),
\end{aligned}$$

$$\begin{aligned}
i \frac{\partial \pi^*(x)}{\partial t} &= [\pi^*(x), H] \\
&= -i \int d^3 x' [\nabla' \phi(\mathbf{x}', t) \cdot \nabla' \delta^3(\mathbf{x} - \mathbf{x}') + m^2 \delta^3(\mathbf{x} - \mathbf{x}') \phi(\mathbf{x}', t)] \\
&= -i \int d^3 x' [-\delta^3(\mathbf{x} - \mathbf{x}') \cdot \nabla'^2 \phi(\mathbf{x}', t) + m^2 \delta^3(\mathbf{x} - \mathbf{x}') \phi(\mathbf{x}', t)] = -i(-\nabla^2 + m^2) \phi(x).
\end{aligned}$$

Thus we have obtained

$$\frac{\partial \phi(x)}{\partial t} = \pi^*(x), \quad \frac{\partial \phi^*(x)}{\partial t} = \pi(x), \quad \frac{\partial \pi(x)}{\partial t} = (\nabla^2 - m^2) \phi^*(x), \quad \frac{\partial \pi^*(x)}{\partial t} = (\nabla^2 - m^2) \phi(x).$$

Combining these results yields

$$\frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - m^2) \phi, \quad \frac{\partial^2 \phi^*}{\partial t^2} = (\nabla^2 - m^2) \phi^*,$$

which is the Klein-Gordon equation and its complex conjugate, as we sought to show. \square

2(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m .

Solution. Peskin & Schroeder (2.21) gives the Klein-Gordon equation in the momentum basis,

$$\left(\frac{\partial^2}{\partial t^2} + \mathbf{p}^2 + m^2 \right) \phi(\mathbf{p}, t) = 0.$$

This is the same as the harmonic oscillator equation of motion. It has solutions [?]]

$$\phi(\mathbf{p}, t) = A(\mathbf{p}) e^{i\omega_{\mathbf{p}} t} + B(\mathbf{p}) e^{-i\omega_{\mathbf{p}} t},$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ as in Peskin & Schroeder Eq. (2.22), and $A(\mathbf{p})$ and $B(\mathbf{p})$ are arbitrary functions of \mathbf{p} . The complex conjugate of this solution is

$$\phi^*(\mathbf{p}, t) = B^*(\mathbf{p}) e^{i\omega_{\mathbf{p}} t} + A^*(\mathbf{p}) e^{-i\omega_{\mathbf{p}} t}.$$

The field ϕ in the position basis can be expanded as [?, p. 20],

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \phi(\mathbf{p}, t).$$

so we can write [?, p. 33]

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \quad \phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}}^* e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

where $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$ are creation and annihilation operators. By analogy to Eq. (2.26) of Peskin & Schroeder, we can also write

$$\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}}^* e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \quad \pi^*(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right).$$

Simplifying these expressions as in their Eqs. (2.27) and (2.28), we have

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}}^* + a_{-\mathbf{p}}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (19)$$

$$\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}}^* - a_{-\mathbf{p}}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \pi^*(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger} \right) e^{i\mathbf{p}\cdot\mathbf{x}}. \quad (20)$$

Also generalizing their Eq. (2.24),

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}] = [a_{\mathbf{p}}^*, a_{\mathbf{p}'}^{\dagger}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'), \quad [a_{\mathbf{p}}, a_{\mathbf{p}'}^*] = [a_{\mathbf{p}}^*, a_{\mathbf{p}'}^{\dagger}] = 0.$$

Feeding Eqs. (19) and (20) into Eq. (16),

$$H = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \left[-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger} \right) \left(a_{\mathbf{p}'}^* - a_{-\mathbf{p}'}^{\dagger} \right) + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger} \right) \left(a_{\mathbf{p}'}^* + a_{-\mathbf{p}'}^{\dagger} \right) \right]$$