Problem 1. Consider the path integral for a single point particle, with the action

$$S = \int_0^1 dt \left[p_\mu(t)\dot{x}^\mu(t) + \frac{N(t)}{2}[p^2(t) - m^2 - i\epsilon] \right]. \tag{1}$$

This represents the quantization of the coordinates and momenta of the particle, subject to the mass shell constraint $p^2 = m^2$ (together with the $i\epsilon$ prescription) imposed by the Lagrange multiplier N. This action admits the reparametrization symmetry $\delta x = \alpha p$, $\delta p = 0$, $\delta N = -\partial_t \alpha$ where $\alpha(t)$ is any function. This symmetry allows us to fix the gauge condition $N(t) = T$; the constant T must still be integrated over, however.

1(a) Path integrate over $x(t)$, subject to the boundary conditions $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$, yielding a delta function $\delta(\dot{p})$ along the path. Solve this constraint (find the set of functions that solve it) and path integrate over those $p(t)$ to find the quantum mechanical propagation amplitude

$$\langle y|x\rangle = D_F(x-y) = \int_0^\infty dT\,(2\pi i T)^{-d/2} \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T} \right) \right], \tag{2}$$

where d is the number of spacetime dimensions.

1(b) Use this integral representation to show that D_F satisfies

$$(\partial^2 + m^2)D_F = i\delta^{(d)}(x-y). \tag{3}$$

Solution. Feeding Eq. (2) into the left-hand side of Eq. (3), we have

$$(\partial^2 + m^2)D_F = m^2 D_F + \int_0^\infty dT\,(2\pi i T)^{-d/2}\partial^2 \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right],$$

where

$$\begin{aligned}
\partial^2 \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] &= \partial\left\{-\frac{i}{2}\partial\left(\frac{(x-y)^2}{T}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\right\} \\
&= -\frac{i}{2}\partial^2\left(\frac{(x-y)^2}{T}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\
&\qquad - \frac{i}{2}\partial\left(\frac{(x-y)^2}{T}\right)\partial\left\{\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\right\} \\
&= -\frac{i}{2}\partial^2\left(\frac{(x-y)^2}{T}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\
&\qquad - \frac{i}{2}\partial\left(\frac{(x-y)^2}{T}\right)\left\{-\frac{i}{2}\partial\left(\frac{(x-y)^2}{T}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right]\right\} \\
&= -\frac{i}{2}\partial^2\left(\frac{(x-y)^2}{T}\right)\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\
&\qquad - \frac{1}{4}\left[\partial\left(\frac{(x-y)^2}{T}\right)\right]^2\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\
&= -\left[\frac{i}{2T}\partial^2(x-y)^2 + \frac{1}{4T^2}\left[\partial(x-y)^2\right]^2\right]\exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T}\right)\right].
\end{aligned}$$

Note that

$$\partial(x-y)^2 = \partial_\nu(x_\mu x^\mu - 2x_\mu y^\mu + y_\mu y^\mu),$$

so

$$\left[\partial(x-y)^2\right]^2 = \partial_\nu(x_\mu x^\mu - 2x_\mu y^\mu + y_\mu y^\mu)\partial^\nu(x_\mu x^\mu - 2x_\mu y^\mu + y_\mu y^\mu).$$

Note also that

$$\partial^2(x-y)^2 = \partial_\nu\partial^\nu(x_\mu x^\mu - 2x_\mu y^\mu + y_\mu y^\mu)$$

1(c) Evaluate the T integral in terms of Bessel functions.

Solution. The integral in Eq. (2) has the form [, p. 368]

$$\int_0^\infty x^{\nu-1} e^{-\beta/x-\gamma x}\, dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}),$$

where K_ν is the modified Bessel function of the second kind. To evaluate Eq. (2)m we note that

$$x \to T, \qquad \nu \to 1-\frac{d}{2}, \qquad \beta \to \frac{i}{2}(x-y)^2, \qquad \gamma \to \frac{i}{2}(m^2-i\epsilon).$$

So we have

$$\begin{aligned}
\langle y|x\rangle &= (2\pi i)^{-d/2}\int_0^\infty dT\, T^{d/2} \exp\left[-\frac{i}{2}\left((m^2-i\epsilon)T + \frac{(x-y)^2}{T}\right)\right] \\
&= 2(2\pi i)^{-d/2}\left(\frac{(x-y)^2}{m^2-i\epsilon}\right)^{1/2-d/4} K_{1-d/2}\left(2\sqrt{-\frac{(x-y)^2(m^2-i\epsilon)}{4}}\right) \\
&= 2^{1-d/2}(\pi i)^{-d/2}\left(\frac{(x-y)^2}{m^2-i\epsilon}\right)^{1/2-d/4} K_{1-d/2}\left(i\sqrt{(x-y)^2(m^2-i\epsilon)}\right).
\end{aligned}$$

Problem 2. Quantum statistical mechanics (Peskin & Schroeder 9.2)

2(a) Evaluate the quantum statistical partition function

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

(where $\beta = 1/kT$) using the strategy of Section 9.1 for evaluating the matrix elements of e^{-iHt} in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

2(b) Evaluate this integral for a simple harmonic oscillator,

$$L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2,$$

by introducing a Fourier decomposition of $x(t)$:

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that, up to a (possibly divergent and β-dependent) constant, the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You may find the identity

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right)$$

useful.]

2(c) Generalize this construction to field theory. Show that the quantum statistical partition function for a free scalar field can be written in terms of a functional integral. The value of this integral is given formally by

$$[\det\left(-\partial^2 + m^2\right)]^{-1/2},$$

where the operator acts on functions on Euclidean space that are periodic in the time direction with periodicity β. As before, the β dependence of this expression is difficult to compute directly. However, the dependence on m^2 is unambiguous. Show that the determinant indeed reproduces the partition function for relativistic scalar particles.