Problem 25.58 A resistor with resistance R is connected to a battery that has emf $12.0\,\mathrm{V}$ and internal resistance $r = 0.40\,\Omega$. For what two values of R will the power in the resistor be $80.0\,\mathrm{W}$?

Solution. The power P delivered to a resistor is

$$P = I^2 R, \tag{25.18}$$

where I is the current through the resistor and R its resistance. We can find the current from

$$V_{ab} = \mathcal{E} - Ir, \tag{25.17}$$

where V_{ab} is the voltage difference across the resistor, $\mathcal{E}$ is the emf of the battery, and r its internal resistance. We also know that

$$V_{ab} = IR. \tag{25.11}$$

Substituting (25.11) into (25.17), we get

$$IR = \mathcal{E} - Ir \implies \mathcal{E} = I(R + r) \implies I = \frac{\mathcal{E}}{R + r}.$$

Now we can substitute this result into (25.18) and solve for R:

$$\begin{aligned} P = \frac{\mathcal{E}^2}{(R+r)^2}R &\implies \mathcal{E}^2 R = P(R^2 + 2Rr + r^2) \implies 0 = PR^2 + (2Pr - \mathcal{E}^2)R + Pr^2 \\ &\implies R = \frac{\mathcal{E}^2 - 2Pr \pm \sqrt{(2Pr - \mathcal{E}^2)^2 - 4P^2r^2}}{2P} \end{aligned}$$

Plugging in our numerical values for r, P, and $\mathcal{E}$, and recalling that $1\,\mathrm{W} = 1\,\mathrm{V}^2\,\Omega^{-1}$, we get

$$\begin{aligned} R &= \frac{(12.0\,\mathrm{V})^2 - 2(80.0\,\mathrm{W})(0.40\,\Omega) \pm \sqrt{[2(80.0\,\mathrm{W})(0.40\,\Omega) - (12.0\,\mathrm{V})^2]^2 - 4(80.0\,\mathrm{W})^2(0.40\,\Omega)^2}}{2(80.0\,\mathrm{W})} \\ &= \frac{80.0\,\mathrm{V}^2 - \pm\sqrt{(80\,\mathrm{V}^2)^2 - (64\,\mathrm{V}^2)}}{160\,\mathrm{V}^2\,\Omega^{-1}} = \frac{80.0\,\mathrm{V}^2 \pm \sqrt{2306\,\mathrm{V}^4}}{160\,\mathrm{V}^2\,\Omega^{-1}} = \frac{80.0 \pm 48.0}{160}\,\Omega = (0.50 \pm 0.30)\,\Omega. \end{aligned}$$

So the two possible resistances are

$$R = 0.80\,\Omega, \qquad R = 0.20\,\Omega.$$

Exercise 26.26 In the circuit shown in **Fig. E26.26**, find

(a) the current in each branch, and

(b) the potential difference V_{ab} of point a relative to point b.

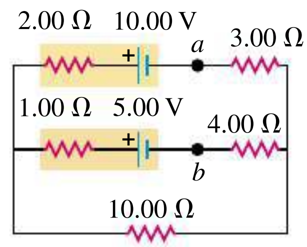


Figure E26.26

Solution. For this problem we need to use Kirchhoff's rules:

$$\sum I = 0 \qquad \text{(junction rule)}, \tag{26.5}$$

$$\sum V = 0 \qquad \text{(loop rule)}. \tag{26.6}$$

(a) Let I_1 be the current through the top branch, I_2 the current through the middle branch, and I_3 the current through the bottom branch. Let's choose all three currents to be flowing to the left. Then applying the junction rule to the junction on the left gives us

$$0 = I_1 + I_2 + I_3.$$

Now we will apply the loop rule to the top and the bottom loops, and move through each counterclockwise, starting at the battery. For the top loop,

$$0 = 10\,\mathrm{V} - (2\,\Omega)I_1 + (1\,\Omega)I_2 - 5\,\mathrm{V} + (4\,\Omega)I_2 - (3\,\Omega)I_1 \implies 5\,\mathrm{V} = (5\,\Omega)I_1 - (5\,\Omega)I_2 \implies 1\,\mathrm{A} = I_1 - I_2,$$

where in going to the final equation we have simply divided by $5\,\Omega$, since $1\,\mathrm{V} = 1\,\Omega\,\mathrm{A}$. For the bottom loop,

$$0 = 5\,\mathrm{V} - (1\,\Omega)I_2 + (10\,\Omega)I_3 - (4\,\Omega)I_2 \implies 5\,\mathrm{V} = (5\,\Omega)I_2 - (10\,\Omega)I_3 \implies 1\,\mathrm{A} = I_2 - 2I_3.$$

Now we have three equations in three unknowns, which you can solve using your favorite method. I like to do Gaussian elimination with an augmented matrix. The matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

which can be solved as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3/5 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/5 \\ 0 & 1 & 0 & -1/5 \\ 0 & 0 & 1 & -3/5 \end{array}\right].$$

So we have

$$I_1 = 0.800\,\mathrm{A}\ \text{(to the left)}, \qquad I_2 = -0.200\,\mathrm{A}\ \text{(to the right)}, \qquad I_3 = -0.600\,\mathrm{A}\ \text{(to the right)}.$$

(b) We can find the potential difference between points a and b by moving counterclockwise through the top loop, similar to applying the loop rule in (a). But now we start at b and end on a:

$$V_{ab} = (4\,\Omega)I_2 - (3\,\Omega)I_1 = (4.00\,\Omega)(-0.200\,\mathrm{A}) - (3.00\,\Omega)(0.800\,\mathrm{A}) = 32.0\,\mathrm{V}.$$

Exercise 26.29 In the circuit shown in **Fig. E26.29** the batteries have negligible internal resistance and the meters are both idealized. With the switch S open, the voltmeter reads 15.0 V.

(a) Find the emf $\mathcal{E}$ of the battery.

(b) What will the ammeter read when the switch is closed?

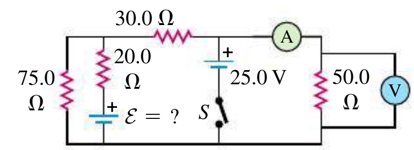


Figure E26.29

Exercise 26.41 In the circuit shown in **Fig. E26.41** both capacitors are initially charged to 45.0 V.

(a) How long after closing the switch S will the potential across each capacitor be reduced to 10.0 V, and

(b) what will be the current at that time?

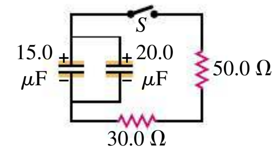


Figure E26.41

Solution. This problem is about discharging capacitors. The charge q on a capacitor at a given time t after an RC circuit is closed is given by

$$q = Q_0 e^{-t/RC}, \tag{26.16}$$

where Q_0 is the initial charge on the capacitor, C is its capacitance, and R is the resistance of the resistor connected in series.

(a) The definition of capacitance C is

$$C = \frac{Q}{V_{ab}}, \tag{24.1}$$

where Q is the charge on the capacitor and V_{ab} is the potential across it. C is a property of a given capacitor and does not change with time. So we can use $Q = CV$ and the fact that C is constant to write (26.16) in terms of C and V:

$$v = V_0 e^{-t/RC}, \tag{$*$}$$

where v is the potential across the capacitor at time t after the circuit is closed.

The capacitors in this problem are connected in parallel, so their equivalent capacitance is simply the sum of their individual capacitances:

$$C_{\text{eq}} = 15.0\,\mu\text{F} + 20.0\,\mu\text{F} = 35.0\,\mu\text{F}.$$

The resistors are connected in series, so their equivalent resistance is also just the sum:

$$R_{\text{eq}} = 30.0\,\Omega + 50.0\,\Omega = 80.0\,\Omega.$$

We can plug these values and our intended value of $v = 10.0\,\text{V}$ into ($*$) to solve for the time t:

$$10.0\,\text{V} = (45.0\,\text{V})\exp\left(-\frac{t}{(80.0\,\Omega)(35.0\,\mu\text{F})}\right) \implies \frac{2}{9} = \exp\left(-\frac{t}{2800\,\mu\text{s}}\right) \implies -1.50 = -\frac{t}{2800\,\mu\text{s}}$$
$$\implies t = 4210\,\mu\text{s} = 4.21\,\text{ms},$$

where we have used the fact that $1\,\text{s} = 1\,\Omega\,\text{F}$.

(b) The current i through a capacitor at a given time t after the circuit is closed is given by

$$i = \frac{Q_0}{RC} e^{-t/RC}. \tag{26.13}$$

From (a), we know that $Q_0/C = V_0$, so we can write this as

$$i = \frac{V_0}{R} e^{-t/RC}.$$

Plugging in our V_0, R_{eq}, C_{eq}, and t that we found in (a), we have

$$i = \frac{45.0\,\text{V}}{80.0\,\Omega}\exp\left(-\frac{4210\,\mu\text{s}}{2800\,\mu\text{s}}\right) = \frac{9}{16}e^{-1.50}\,\text{A} = \frac{9}{16}\frac{2}{9}\,\text{A} = \frac{1}{8}\,\text{A} = 0.125\,\text{A}.$$

Exercise 26.47 In the circuit shown in **Fig. E26.47** the capacitors are initially uncharged, the battery has no internal resistance, and the ammeter is idealized. Find the ammeter reading

(a) just after the switch S is closed, and

(b) after S has been closed for a very long time.

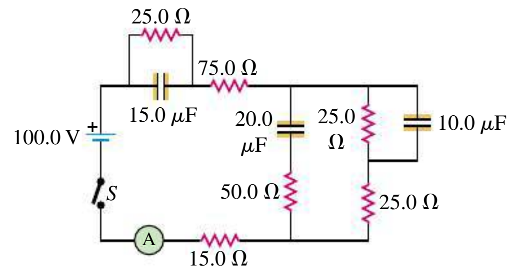


Figure E26.47

Problem 26.53 A capacitor with capacitance C is connected in series to a resistor of resistance R and a battery with emf $\mathcal{E}$. The circuit is completed at time $t = 0$.

(a) In terms of $\mathcal{E}$, R, and C, how much energy is stored in the capacitor when it is fully charged?

(b) The power output of the battery is $P_{\mathcal{E}} = \mathcal{E}i$, with i given by Eq. (26.13). The electrical energy supplied in an infinitesimal time dt is $P_{\mathcal{E}}\,dt$. Integrate from $t = 0$ to $t \to \infty$ to find the total energy supplied by the battery.

(c) The rate of consumption of electrical energy in the resistor is $P_R = i^2R$. In an infinitesimal time interval dt, the amount of electrical energy consumed by the resistor is $P_R\,dt$. Integrate from $t = 0$ to $t \to \infty$ to find the total energy consumed by the resistor.

(d) What fraction of the total energy supplied by the battery is stored in the capacitor? What fraction is consumed in the resistor?

Solution.

(a) In general, the potential energy U stored in a capacitor is given by

$$U = \frac{1}{2}CV^2, \tag{24.9}$$

where C is the capacitor's capacitance and V the potential difference between its plates. When the capacitor its fully charged, the potential difference between its plates is the emf $\mathcal{E}$. So we have

$$U_C = \frac{1}{2}C\mathcal{E}^2.$$

(b) In terms of the given quantities $\mathcal{E}$, R, and C, the instantaneous current i is given by

$$i = \frac{\mathcal{E}}{R}e^{-t/RC}. \tag{26.13}$$

Plugging this in and integrating, the total energy supplied by the battery is

$$U_{\mathcal{E}} = \int_0^\infty P_{\mathcal{E}}\,dt = \int_0^\infty \mathcal{E}i\,dt = \int_0^\infty \mathcal{E}\frac{\mathcal{E}}{R}e^{-t/RC}\,dt = \frac{\mathcal{E}^2}{R}\Big[-RCe^{-t/RC}\Big]_0^\infty = \frac{\mathcal{E}^2}{R}RC = C\mathcal{E}^2.$$

(c) We just need to plug (26.13) in again and integrate to find the total energy consumed by the resistor:

$$U_R = \int_0^\infty P_R\,dt = \int_0^\infty i^2R\,dt = \int_0^\infty \left(\frac{\mathcal{E}}{R}e^{-t/RC}\right)^2 R\,dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-2t/RC}\,dt = \frac{\mathcal{E}^2}{R}\left[-\frac{RCe^{-2t/RC}}{2}\right]_0^\infty$$
$$= \frac{\mathcal{E}^2}{R}\frac{RC}{2} = \frac{1}{2}C\mathcal{E}^2.$$

(d) From (a) and (b), $U_C/U_{\mathcal{E}} = 1/2$, so half of the total energy supplied by the battery is stored in the capacitor. From (b) and (c), $U_R/U_{\mathcal{E}} = 1/2$, so the other half of the total energy is consumed by the resistor.

Problem 26.59 Calculate the currents I_1, I_2, and I_3 indicated in the circuit diagram shown in **Fig. P26.59**.

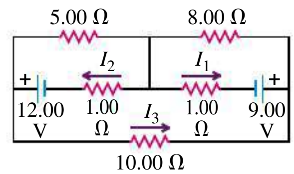
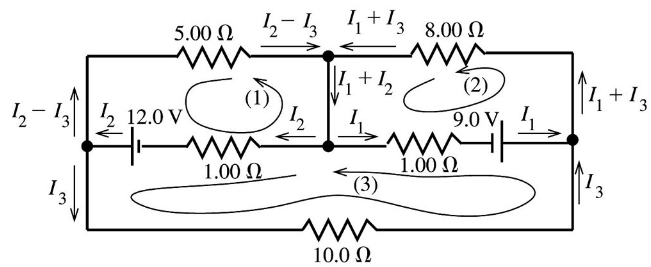


Figure P26.59

Solution. This is another problem for Kirchhoff's rules:



The circuit has three loops, so we will need to use the loop rule three times as indicated in the diagram above. But first we need to use the junction rule to find the current through the $5\,\Omega$ and $8\,\Omega$ resistors. At the junction on the left, we have

$$0 = I_2 - I_3 + I_{5\,\Omega} \implies I_{5\,\Omega} = I_3 - I_2,$$

meaning it is pointing away from the junction if $I_2 > I_3$. At the junction on the right, we have

$$0 = I_1 + I_3 + I_{8\,\Omega} \implies I_{8\,\Omega} = -I_1 - I_3,$$

meaning it is pointing away from the junction.

Now we can apply the loop rule, moving counterclockwise from each battery. For loop (1), we have

$$0 = -12\,\mathrm{V} + (1\,\Omega)I_2 + (5\,\Omega)(I_2 - I_3) \implies 12\,\mathrm{A} = 6I_2 - 5I_3.$$

For loop (2),

$$0 = 9\,\mathrm{V} - (8\,\Omega)(I_1 - I_3) + (1\,\Omega)I_1 \implies 9\,\mathrm{A} = 9I_1 + 8I_3.$$

For loop (3), let's start at the 12 V battery:

$$0 = 12\,\mathrm{V} - (10\,\Omega)I_3 - 9\,\mathrm{V} + (1\,\Omega)I_1 - (1\,\Omega)I_2 \implies 3\,\mathrm{A} = -I_1 + I_2 + 10I_3.$$

The matrix equation is

$$\begin{bmatrix} 0 & 6 & -5 \\ 9 & 0 & 8 \\ -1 & 1 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 3 \end{bmatrix},$$

which can be solved as follows:

$$\left[\begin{array}{ccc|c} 0 & 6 & -5 & 12 \\ 9 & 0 & 8 & 9 \\ -1 & 1 & 10 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 8/9 & 1 \\ 0 & 1 & -5/6 & 2 \\ -1 & 1 & 10 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 8/9 & 1 \\ 0 & 1 & -5/6 & 2 \\ 0 & 1 & 98/9 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 8/9 & 1 \\ 0 & 1 & -5/6 & 2 \\ 0 & 0 & 211/18 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 179/211 \\ 0 & 1 & 0 & 452/211 \\ 0 & 0 & 1 & 36/211 \end{array}\right].$$

So we have

$$I_1 = 0.848\,\mathrm{A}\ \text{(to the right)}, \qquad I_2 = 2.14\,\mathrm{A}\ \text{(to the left)}, \qquad I_3 = 0.171\,\mathrm{A}\ \text{(to the right)}.$$