Problem 1.

1(a) Show that the Maxwell equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

can be obtained by varying the Lagrangian

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu .$$

Solution. We want to extremize the action,

$$S[A_\mu] = \int \mathcal{L}(A_\mu, \partial_\mu A_\mu)\; d^4x\,.$$

Let δA_μ denote some arbitrary variation that vanishes at the boundaries of spacetime. The action for $A_\mu + \delta A_\mu$ is

$$S[A_\mu + \delta A_\mu] = \int \mathcal{L}(A_\mu + \delta A_\mu, \partial_\nu A_\mu + \partial_\nu \delta A_\mu)\, d^4x\,.$$

Then, to first order in δA_μ, the variation of the action is

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu],$$

which we want to vanish for all δA_μ. From Jackson (11.136), $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Let $\delta F^{\mu\nu} = \partial^\mu \delta A^\nu - \partial^\nu \delta A^\mu$. Then

$$\begin{aligned}
\delta S &= \int \left(-\frac{1}{16\pi}(F_{\mu\nu} + \delta F_{\mu\nu})(F^{\mu\nu} + \delta F^{\mu\nu}) - \frac{1}{c} J_\mu (A^\mu + \delta A^\mu) + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu \right) d^4x \\
&\approx \int \left(-\frac{1}{16\pi}(F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu}\, \delta F^{\mu\nu} + \delta F_{\mu\nu}\, F^{\mu\nu}) - \frac{1}{c} J_\mu (A^\mu + \delta A^\mu) + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu \right) d^4x \\
&= \int \left(-\frac{1}{16\pi}(F_{\mu\nu}\, \delta F^{\mu\nu} + \delta F_{\mu\nu}\, F^{\mu\nu}) - \frac{1}{c} J_\mu\, \delta A^\mu \right) d^4x\,, \qquad (1)
\end{aligned}$$

where we have discarded terms of $\mathcal{O}\big((\delta A^\mu)^2\big)$.

Note that

$$\delta F_{\mu\nu}\, F^{\mu\nu} = (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \delta A_\nu\, \partial^\mu A^\nu - \partial_\mu \delta A_\nu\, \partial^\nu A^\mu - \partial_\nu \delta A_\mu\, \partial^\mu A^\nu + \partial_\nu \delta A_\mu\, \partial^\nu A^\mu.$$

Integrating the first term of the expansion by parts, we have

$$\int \frac{\partial\, \delta A_\nu}{\partial x_\mu} \frac{\partial A^\nu}{\partial x^\mu}\, d^4x = \left[\delta A_\nu \frac{\partial A^\nu}{\partial x^\mu} \right]_{-\infty}^{\infty} - \int \delta A_\nu \frac{\partial}{\partial x_\mu} \frac{\partial A^\nu}{\partial x^\mu}\, d^4x = -\int \delta A_\nu\, \partial^\mu \partial_\mu A_\nu\, d^4x\,,$$

because δA^ν vanishes at $\pm\infty$. Performing similar integrations for the other terms, we find

$$\begin{aligned}
\int \delta F_{\mu\nu}\, F^{\mu\nu}\, d^4x &= -\int (\delta A_\nu\, \partial_\mu \partial^\mu A^\nu - \delta A_\nu\, \partial_\mu \partial^\nu A^\mu - \delta A_\mu\, \partial_\nu \partial^\mu A^\nu + \delta A_\mu\, \partial_\nu \partial^\nu A^\mu)\, d^4x \\
&= -\int (\delta A_\nu\, \partial_\mu F^{\mu\nu} + \delta A_\mu\, \partial_\nu F^{\nu\mu})\, d^4x = -\int (\delta A_\nu\, \partial_\mu F^{\mu\nu} + \delta A_\nu\, \partial_\mu F^{\mu\nu})\, d^4x\,,
\end{aligned}$$

where in going to the final equality we have simply swapped the indices.

Likewise,

$$\begin{aligned}\int F_{\mu\nu}\,\delta F^{\mu\nu}\,d^4x &= -\int (\delta A^\nu\,\partial^\mu\partial_\mu A_\nu - \delta A^\mu\,\partial^\nu\partial_\mu A_\nu - \delta A^\nu\,\partial^\mu\partial_\nu A_\mu + \delta A^\mu\,\partial^\nu\partial_\nu A_\mu)\,d^4x\\ &= -\int (\delta A^\nu\,\partial^\mu F_{\mu\nu} + \delta A^\mu\,\partial^\nu F_{\nu\mu})\,d^4x = -\int (\delta A^\nu\,\partial^\mu F_{\mu\nu} + \delta A^\nu\,\partial^\mu F_{\mu\nu})\,d^4x\\ &= -\int (\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\nu\,\partial_\mu F^{\mu\nu})\,d^4x\,,\end{aligned}$$

where in going to the final equality we have swapped contravariant and covariant.

Making these substitutions in Eq. (1), we obtain

$$\delta S = \int \left(\frac{1}{16\pi}(4\,\delta A_\nu\,\partial_\mu F^{\mu\nu}) - \frac{1}{c}J_\nu\,\delta A^\nu\right)d^4x = \delta A_\nu \int \left(\frac{1}{4\pi}\partial_\mu F^{\mu\nu} - \frac{1}{c}J^\nu\right)d^4x\,,$$

where we have changed indices and swapped contravariant and covariant in the final term. In order for the action to be at a local extremum, we need $\delta S = 0$ for any δA_ν. This implies that the integrand is 0. Finally, we obtain

$$\frac{1}{4\pi}\partial_\mu F^{\mu\nu} - \frac{1}{c}J^\nu = 0 \quad\Longrightarrow\quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu,$$

as we sought to prove. □

1(b) Suppose we add to $\mathcal{L}$ the term $\delta\mathcal{L} = \theta F_{\mu\nu}\tilde{F}^{\mu\nu}$, where θ is some constant. How do the equations of motion of $\mathcal{L} + \delta\mathcal{L}$ differ from those of $\mathcal{L}$? Can you think of a reason for this?

Solution. With this extra term, Eq. (1) becomes

$$\begin{aligned}\delta S &= \int \left(-\frac{1}{16\pi}(F_{\mu\nu}\,\delta F^{\mu\nu} + \delta F_{\mu\nu}\,F^{\mu\nu}) - \frac{1}{c}J_\mu\,\delta A^\mu + \theta(F_{\mu\nu} + \delta F_{\mu\nu})(\tilde{F}^{\mu\nu} + \delta\tilde{F}^{\mu\nu}) - \theta F_{\mu\nu}\tilde{F}^{\mu\nu}\right)d^4x\\ &\approx \int \left(-\frac{1}{16\pi}(F_{\mu\nu}\,\delta F^{\mu\nu} + \delta F_{\mu\nu}\,F^{\mu\nu}) - \frac{1}{c}J_\mu\,\delta A^\mu + \theta(F_{\mu\nu}\,\delta\tilde{F}^{\mu\nu} + \delta F_{\mu\nu}\,\tilde{F}^{\mu\nu})\right)d^4x\,. \qquad (2)\end{aligned}$$

From Jackson (11.140), $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$. Then

$$\begin{aligned}\delta F_{\mu\nu}\,\tilde{F}^{\mu\nu} &= (\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu)\frac{\epsilon^{\mu\nu\alpha\beta}}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\\ &= \frac{\partial_\mu\delta A_\nu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha A_\beta - \partial_\mu\delta A_\nu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\beta A_\alpha - \partial_\nu\delta A_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha A_\beta + \partial_\nu\delta A_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\beta A_\alpha}{2}.\end{aligned}$$

Integrating by parts as in Prob. 1(a),

$$\begin{aligned}\int \delta F_{\mu\nu}\,\tilde{F}^{\mu\nu}\,d^4x &= -\frac{1}{2}\int (\delta A_\nu\,\partial_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha A_\beta - \delta A_\nu\,\partial_\mu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\beta A_\alpha - \delta A_\mu\,\partial_\nu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha A_\beta + \delta A_\mu\,\partial_\nu\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\beta A_\alpha)\,d^4x\\ &= -\frac{1}{2}\int (\delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu} - \delta A_\mu\,\partial_\nu\tilde{F}^{\mu\nu})\,d^4x = -\frac{1}{2}\int (\delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu} - \delta A_\nu\,\partial_\mu\tilde{F}^{\nu\mu})\,d^4x\\ &= -\frac{1}{2}\int (\delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu} + \delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu})\,d^4x\,,\end{aligned}$$

where we have made use of the antisymmetry of $\tilde{F}^{\mu\nu}$.

Similarly,

$$\begin{aligned}\int F_{\mu\nu}\,\delta\tilde{F}^{\mu\nu}\,d^4x &= -\frac{1}{2}\int(\delta A_\beta\,\partial_\alpha\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\mu A_\nu - \delta A_\alpha\,\partial_\beta\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\mu A_\nu - \delta A_\beta\,\partial_\alpha\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\mu + \delta A_\alpha\,\partial_\beta\,\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\mu)\,d^4x\\ &= -\frac{1}{2}\int(\delta A_\beta\,\partial_\alpha\,\epsilon^{\alpha\beta\mu\nu}\,\partial_\mu A_\nu - \delta A_\alpha\,\partial_\beta\,\epsilon^{\alpha\beta\mu\nu}\,\partial_\mu A_\nu - \delta A_\beta\,\partial_\alpha\,\epsilon^{\alpha\beta\mu\nu}\,\partial_\nu A_\mu + \delta A_\alpha\,\partial_\beta\,\epsilon^{\alpha\beta\mu\nu}\,\partial_\nu A_\mu)\,d^4x\\ &= -\frac{1}{2}\int(\delta A_\beta\,\partial_\alpha\tilde{F}^{\alpha\beta} - \delta A_\alpha\,\partial_\beta\,\tilde{F}^{\alpha\beta})\,d^4x = -\frac{1}{2}\int(\delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu} + \delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu})\,d^4x\,,\end{aligned}$$

where we have used the fact that $\epsilon^{\alpha\beta\mu\nu} = \epsilon^{\mu\nu\alpha\beta}$.

Substituting into Eq. (2),

$$\delta S = \int\left(\frac{1}{16\pi}(4\,\delta A_\nu\,\partial_\mu F^{\mu\nu}) - \frac{1}{c}J_\nu\,\delta A^\nu + \theta(4\delta A_\nu\,\partial_\mu\tilde{F}^{\mu\nu})\right)d^4x = \delta A_\nu\int\left(\frac{1}{4\pi}\partial_\mu F^{\mu\nu} + 4\theta\partial_\mu\tilde{F}^{\mu\nu} - \frac{1}{c}J^\nu\right)d^4x\,,$$

so we find the equations of motion

$$\partial_\mu F^{\mu\nu} + 16\pi\theta\partial_\mu\tilde{F}^{\mu\nu} - \frac{4\pi}{c}J^\nu = 0 \quad\Longrightarrow\quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu,$$

where we have applied the homogeneous Maxwell equations $\partial_\mu\tilde{F}^{\mu\nu} = 0$, according to Jackson (11.142). So we have once again recovered the inhomogeneous Maxwell equations. Therefore, the equations of motion of $\mathcal{L}+\delta\mathcal{L}$ do not differ from those of $\mathcal{L}$.

The mathematical reason for this is that $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, as mentioned in the lecture notes on p. 103 of the lecture notes. This means there exists some quantity $f = f(t, A_\mu, \partial_\mu A_\mu)$ such that $F_{\mu\nu}\tilde{F}^{\mu\nu} = df/dt$, and therefore $\delta\mathcal{L}$ trivially satisfies the Euler-Lagrange equations.

A more physical argument is related to the solution of Prob. 5 of the previous homework, in which we showed that $F_{\mu\nu}\tilde{F}^{\mu\nu} \propto \mathbf{E}\cdot\mathbf{B}$. Since $\mathbf{E}$ and $\mathbf{B}$ are both determined completely by A^μ and its derivatives, adding a term proportional to $\mathbf{E}\cdot\mathbf{B}$ to the Lagrangian cannot provide any new information or stipulations, and thus should not alter the equations of motion.

Problem 2. In this problem we will derive the form of the stress tensor

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^i)}\partial^\nu\phi^i - \eta^{\mu\nu}\mathcal{L}, \tag{3}$$

for a system of fields $\phi_i(x^\mu)$, governed by an action

$$S = \int\mathcal{L}(\phi_i, \partial_\mu\phi_i)\,d^4x\,.$$

The fields ϕ_i transform under translations as $\phi'_i(x') = \phi_i(x)$, where $x'_\mu = x_\mu + a_\mu$ and a_μ is an arbitrary four-vector, the amount by which we translate.

2(a) For an infinitesimal translation a^μ, compute $\delta\phi_i(x) = \phi'_i(x) - \phi_i(x)$.

Solution. We know $\phi'_i(x') = \phi'_i(x+a) = \phi_i(x)$, which implies $\phi'_i(x) = \phi_i(x-a)$. Then $\delta\phi_i(x) = \phi_i(x-a) - \phi_i(x)$. We can perform a Taylor series expansion about $a = 0$:

$$\phi_i(x-a) = \phi_i(x) + a\left[\frac{\partial\phi_i}{\partial x}\right]_{a=0} + \frac{a^2}{2}\left[\frac{\partial^2\phi_i}{\partial x^2}\right]_{a=0} + \mathcal{O}(a^3).$$

For the purposes of varying the action, we need only concern ourselves with terms of $\mathcal{O}(a)$. So we have

$$\delta\phi_i(x) = a^\mu \partial_\mu \phi_i(x).$$

2(b) Compute the variation of the action S under the transformation $\phi_i \to \phi_i + \delta\phi_i$. What is K^μ for this case?

Solution. From p. 97 in the lecture notes, the variation of the action is

$$\delta S = \int \frac{\delta S}{\delta \phi_i} \delta\phi_i = \int \left(\delta\phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + (\partial_\mu \delta\phi_i) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) d^4x\,.$$

Note that

$$\partial_\mu \delta\phi_i = a^\nu \partial_\nu \partial_\mu \phi_i + \partial_\mu a^\nu\, \partial_\nu \phi_i.$$

Now we will vary the action, stipulating that ϕ_i is a solution of the Euler-Lagrange equations; that is, it extremizes the action for an *arbitrary* variation. This means $\delta S = 0$. Then, substituting $\delta\phi_i = a^\mu \partial_\mu \phi_i$,

$$\begin{aligned} \delta S &= \int \left(a^\mu \partial_\mu \phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + (a^\nu \partial_\nu \partial_\mu \phi_i + \partial_\mu a^\nu\, \partial_\nu \phi_i) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) d^4x \\ &= \int \left(a^\nu \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + a^\nu \partial_\nu \partial_\mu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} + \partial_\mu a^\nu\, \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) d^4x\,. \end{aligned} \tag{4}$$

Note that [? , p. 82]

$$\partial_\mu \mathcal{L} = \partial_\mu \phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + \partial_\mu \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)}.$$

Substituting into Eq. (4), we have

$$\delta S = \int \left(a^\nu \partial_\nu \mathcal{L} + \partial_\mu a^\nu\, \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) d^4x \tag{5}$$

Integrating the second term by parts,

$$\int \partial_\mu a^\nu\, \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\, d^4x = \left[a^\nu\, \partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right]_{-\infty}^{\infty} - \int a^\nu \partial_\mu \left(\partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) d^4x = -\int a^\nu \partial_\mu \left(\partial_\nu \phi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) d^4x\,.$$

Finally, Eq. (5) becomes

$$\begin{aligned} \delta S &= \int a^\nu \left[\partial_\nu \mathcal{L} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i \right) \right] d^4x = \int a^\nu \left[\delta^\mu{}_\nu \partial_\mu \mathcal{L} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i \right) \right] d^4x \\ &= \int a^\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i - \delta^\mu{}_\nu \mathcal{L} \right) d^4x\,, \end{aligned} \tag{6}$$

where in going to the second equality we have inserted a factor of $\delta^\mu{}_\nu$ [? , p. 83]. In the final equality, we have multiplied by -1 since $\delta S = 0$. According to Jackson (11.71), $\eta_{\mu\alpha}\eta^{\alpha\nu} = \delta^\mu{}_\nu$.

According to p. 114.8 in the lecture notes,

$$\int \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta_s q_i - K \right) dt = 0.$$

For a field, this becomes

$$\int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta_s \phi_i - K^\mu \right) dt = 0.$$

Reading off Eq. (6), we find

$$K^\mu = a_\nu \eta^{\mu\nu} \mathcal{L}.$$

2(c) Use our general result for the conserved current,

$$J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \delta_s \phi_i - K^\mu,$$

to find the conserved current associated to translational symmetry. You should reproduce Eq. (3). Explain how the fact that translations are four continuous symmetries is related to the fact that $T^{\mu\nu}$ is a two-index tensor.

Solution. From Eq. (6),

$$J^\mu = a_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \partial^\nu \phi^i - \eta^{\mu\nu} \mathcal{L} \right).$$

We see that $J^\mu = a_\nu T^{\mu\nu}$, where

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \partial^\nu \phi^i - \eta^{\mu\nu} \mathcal{L},$$

as in Eq. (3). □

For a single continuous symmetry θ as we discussed in lecture, we found the conserved current J^μ, which is a four-vector. Instead of writing a^μ as a vector, we could have considered it as four single continuous symmetries: a^0, a^1, a^2, and a^3. We would have found four conserved four-vector currents: $J^{\mu 0}$, $J^{\mu 1}$, $J^{\mu 2}$, and $J^{\mu 3}$. Together, these currents are specified by sixteen elements. A more compact way of writing these is as a two-index tensor $T^{\mu\nu}$, which also has sixteen elements.

Problem 3.

3(a) Apply the Noether procedure for constructing the energy-momentum tensor to the source-free electromagnetic field and show that the resulting tensor $T^{\mu\nu}$ satisfies the conservation equation $\partial_\mu T^{\mu\nu} = 0$.

3(b) Show that the "improvement" of this tensor discussed in class, that leads to

$$T^{\mu\nu} = \frac{1}{4\pi} \left(F^{\mu\lambda} F_\lambda{}^\lambda + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right),$$

does not spoil conservation.

3(c) Evaluate T^{00} and T^{0i} in terms of electric and magnetic fields. What is the physical interpretation of these quantities?

3(d) Calculate the correction to the conservation quantity $\partial_\mu T^{\mu\nu} = 0$ in the presence of a nonzero current J^μ.