Problem 25.58 A resistor with resistance R is connected to a battery that has emf $12.0\,\mathrm{V}$ and internal resistance $r = 0.40\,\Omega$. For what two values of R will the power in the resistor be $80.0\,\mathrm{W}$?

Solution. We can think of the battery's internal resistance as another resistor of resistance r connected in series.

The power P delivered to the resistor of resistance R is

$$P = I^2 R, \tag{25.18}$$

where I is the current through the resistor. We have a one-loop circuit, so the current through all elements is the same. We can easily find I using Kirchhoff's loop rule,

$$\sum V = 0. \tag{26.6}$$

Let's start at the battery and work counterclockwise as indicated in the diagram. We get

$$0 = \mathcal{E} - Ir - IR \quad\implies\quad \mathcal{E} = I(R+R) \quad\implies\quad I = \frac{\mathcal{E}}{R+r}.$$

Now we can substitute this result into (25.18) and solve for R:

$$P = \frac{\mathcal{E}^2}{(R+r)^2}R \quad\implies\quad \mathcal{E}^2 R = P(R^2 + 2Rr + r^2) \quad\implies\quad 0 = PR^2 + (2Pr - \mathcal{E}^2)R + Pr^2$$
$$\implies\quad R = \frac{\mathcal{E}^2 - 2Pr \pm \sqrt{(2Pr - \mathcal{E}^2)^2 - 4P^2r^2}}{2P}.$$

Plugging in our numerical values for r, P, and $\mathcal{E}$, and recalling that $1\,\mathrm{W} = 1\,\mathrm{V}^2\,\Omega^{-1}$, we get

$$R = \frac{(12.0\,\mathrm{V})^2 - 2(80.0\,\mathrm{W})(0.40\,\Omega) \pm \sqrt{[2(80.0\,\mathrm{W})(0.40\,\Omega) - (12.0\,\mathrm{V})^2]^2 - 4(80.0\,\mathrm{W})^2(0.40\,\Omega)^2}}{2(80.0\,\mathrm{W})}$$
$$= \frac{80.0\,\mathrm{V}^2 - \pm\sqrt{(80\,\mathrm{V}^2)^2 - (64\,\mathrm{V}^2)}}{160\,\mathrm{V}^2\,\Omega^{-1}} = \frac{80.0\,\mathrm{V}^2 \pm \sqrt{2306\,\mathrm{V}^4}}{160\,\mathrm{V}^2\,\Omega^{-1}} = \frac{80.0 \pm 48.0}{160}\,\Omega = (0.50 \pm 0.30)\,\Omega.$$

So the two possible resistances are

$$R = 0.80\,\Omega, \qquad R = 0.20\,\Omega.$$

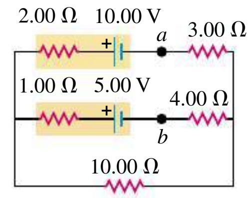


Figure E26.26

Exercise 26.26 In the circuit shown in **Fig. E26.26**, find

(a) the current in each branch, and

(b) the potential difference V_{ab} of point a relative to point b.

Solution. For this problem we need to use Kirchhoff's rules:

$$\sum I = 0 \qquad \text{(junction rule)}, \qquad (26.5)$$

$$\sum V = 0 \qquad \text{(loop rule)}. \qquad (26.6)$$

(a) Let I_1 be the current through the top branch, I_2 the current through the middle branch, and I_3 the current through the bottom branch. Let's choose all three currents to be flowing to the left.

Then applying the junction rule to the junction on the left (shown in red) gives us

$$0 = I_1 + I_2 + I_3, \qquad \text{(A)}$$

which would not make sense if we had chosen the directions of all three currents correctly. But the currents whose direction we have chosen incorrectly will just end up being negative in our solution.

Now let's apply the loop rule to the top and the bottom loops, and move through each counterclockwise, starting at the battery. For the top loop,

$$0 = 10\,\mathrm{V} - (2\,\Omega)I_1 + (1\,\Omega)I_2 - 5\,\mathrm{V} + (4\,\Omega)I_2 - (3\,\Omega)I_1 \quad \Longrightarrow \quad 5\,\mathrm{V} = (5\,\Omega)I_1 - (5\,\Omega)I_2,$$

which simplifies to

$$1\,\mathrm{A} = I_1 - I_2, \qquad \text{(B)}$$

where we have just divided by $5\,\Omega$, since $1\,\mathrm{V} = 1\,\Omega\,\mathrm{A}$. For the bottom loop,

$$0 = 5\,\mathrm{V} - (1\,\Omega)I_2 + (10\,\Omega)I_3 - (4\,\Omega)I_2 \quad \Longrightarrow \quad 5\,\mathrm{V} = (5\,\Omega)I_2 - (10\,\Omega)I_3,$$

which simplifies to

$$1\,\mathrm{A} = I_2 - 2I_3. \qquad \text{(C)}$$

Now we have the three equations (A), (B), and (C) in three unknowns, which you can solve using your favorite method. If we want to solve them algebraically, we can begin by solving (A) for I_1, which gives us

$$I_1 = -I_2 - I_3.$$

Now we can feed this into (B) and solve for I_2:

$$1\,\mathrm{A} = -I_2 - I_3 - I_2 = -2I_2 - I_3 \quad \Longrightarrow \quad I_2 = -\frac{I_3 + 1\,\mathrm{A}}{2} = -\frac{I_3}{2} - \frac{1}{2}\,\mathrm{A}.$$

Substituting this expression into (C), we get

$$1\,\mathrm{A} = -\frac{I_3}{2} - \frac{1}{2}\,\mathrm{A} - 2I_3 \quad \Longrightarrow \quad \frac{5I_3}{2} = -\frac{3}{2}\,\mathrm{A} \quad \Longrightarrow \quad I_3 = \frac{3}{5}\,\mathrm{A}.$$

Now we can plug this result for I_3 into (C) to find I_2:

$$1\,\mathrm{A} = I_2 + 2\frac{3}{5}\,\mathrm{A} = I_2 + \frac{6}{5}\,\mathrm{A} \Longrightarrow I_2 = -\frac{1}{5}\,\mathrm{A}.$$

Finally, we can plug this result for I_2 into (B) to get I_1:

$$1\,\mathrm{A} = I_1 + \frac{1}{5}\,\mathrm{A} \quad \Longrightarrow \quad I_1 = \frac{4}{5}\,\mathrm{A}.$$

As it turns out, both I_2 and I_3 are in the direction opposite of what we guessed. Gathering all of our results, we have

Top branch: 0.800 A (to the left),
Middle branch: 0.200 A (to the right),
Bottom branch: 0.600 A (to the right).

For a tricky system of three or more equations, it might be easier to use Gaussian elimination instead. The matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

which can be solved as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/5 \\ 0 & 1 & 0 & -1/5 \\ 0 & 0 & 1 & -3/5 \end{array}\right],$$

which is the same as what we got through algebraic substitution.

(b) We can find the potential difference between points a and b by moving counterclockwise through the top loop, similar to applying the loop rule in (a). But now we start at b and end on a:

$$V_{ab} = (4\,\Omega)I_2 - (3\,\Omega)I_1 = (4.00\,\Omega)(-0.200\,\mathrm{A}) - (3.00\,\Omega)(0.800\,\mathrm{A}) = 32.0\,\mathrm{V}.$$

Exercise 26.29 In the circuit shown in **Fig. E26.29** the batteries have negligible internal resistance and the meters are both idealized. With the switch S open, the voltmeter reads 15.0 V.

(a) Find the emf $\mathcal{E}$ of the battery.

(b) What will the ammeter read when the switch is closed?

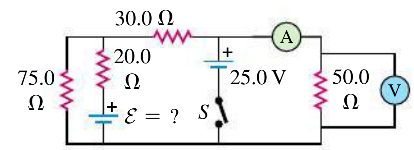


Figure E26.29

Solution.

(a) We need to simplify the circuit to the point that we know the current through the battery and the single equivalent resistance connected to it. When the switch is open, we can ignore the branch of the circuit with the 25 V battery.

Let's begin with what we are given, which is the voltmeter reading. Since the voltmeter is connected in parallel to the 50 Ω resistor, we know that the potential across this resistor is 15 V. So we can use Ohm's law to find the current through this resistor:

$$V = IR \qquad \Longrightarrow \qquad I_{50\,\Omega} = \frac{15\,\text{V}}{50\,\Omega} = \frac{3}{10}\,\text{A} = 0.300\,\text{A}.$$

This is the current through any components connected in series with the 50 Ω resistor, so it is also the current through their equivalent resistance. The 30 Ω resistor is in series with the 50 Ω resistor, and their equivalent resistance is

$$R_{\text{eq}_1} = 50\,\Omega + 30\,\Omega = 80.0\,\Omega.$$

The potential difference across R_{eq_1} can be found by using Ohm's law once more:

$$V_{80\,\Omega} = I_{50\,\Omega} R_{\text{eq}_1} = (0.300\,\text{A})(80.0\,\Omega) = 24.0\,\text{V}.$$

This is also the potential difference across any components connected in parallel to R_{eq_1}. The 75 Ω resistor is connected in parallel, so we can find the current through it using Ohm's law:

$$I_{75\,\Omega} = \frac{V_{80\,\Omega}}{75\,\Omega} = \frac{24.0\,\text{V}}{75.0\,\Omega} = 0.320\,\text{A}.$$

Applying Kirchhoff's junction rule to the top junction in the diagram just above, we can find the current through the branch of the circuit with the unknown emf $\mathcal{E}$:

$$0 = I_{\mathcal{E}} - I_{75\,\Omega} - I_{80\,\Omega} \qquad \Longrightarrow \qquad I_{\mathcal{E}} = 0.320\,\text{A} + 0.300\,\text{A} = 0.620\,\text{A}.$$

Now that we know the current through the battery, we can simplify the circuit to just one equivalent resistance. First adding the $75\,\Omega$ and $80\,\Omega$ resistors in parallel, we find

$$\frac{1}{R_{\text{eq}2}} = \frac{1}{75\,\Omega} + \frac{1}{R_{\text{eq}1}} = \frac{1}{75\,\Omega} + \frac{1}{80\,\Omega} = \frac{31}{1200}\,\Omega^{-1} \quad \Longrightarrow \quad R_{\text{eq}2} = \frac{1200}{31}\,\Omega = 38.7\,\Omega.$$

Adding this in series with the $20\,\Omega$ resistor, we find

$$R_{\text{tot}} = R_{\text{eq}2} + 20\,\Omega = 38.7\,\Omega + 20\,\Omega = 58.7\,\Omega.$$

Finally, we know that the potential across R_{tot} is the same as across the battery. So Ohm's law tells us

$$\mathcal{E} = I_{\mathcal{E}} R_{\text{tot}} = (0.620\,\text{A})(58.7\,\Omega) = 36.4\,\text{V}.$$

(b) When the switch is closed, the $50\,\Omega$ resistor is connected in parallel to the $25\,\text{V}$ battery, so the potential across the $50\,\Omega$ resistor is equal to $25\,\text{V}$.

We can apply Ohm's law once more to find the current through the $50\,\Omega$ resistor, which is the same as the current through the ammeter connected in series:

$$I_A = \frac{25.0\,\text{V}}{50.0\,\Omega} = 0.500\,\text{A}.$$

Exercise 26.41 In the circuit shown in **Fig. E26.41** both capacitors are initially charged to 45.0 V.

(a) How long after closing the switch S will the potential across each capacitor be reduced to 10.0 V, and

(b) what will be the current at that time?

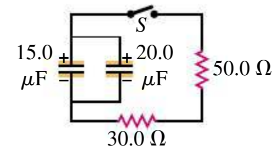


Figure E26.41

Solution. This problem is about discharging capacitors. The charge q on a capacitor at time t after closing an RC circuit is given by

$$q = Q_0 e^{-t/RC}, \tag{26.16}$$

where Q_0 is the initial charge on the capacitor, C is its capacitance, and R is the resistance of the resistor connected to it in series.

(a) The definition of capacitance is

$$C = \frac{Q}{V},$$

where Q is the charge on the capacitor and V is the potential across it. C is a property of a given capacitor and does not change with time. So we can use $Q = CV$ and the fact that C is constant to write (26.16) in terms of C and V:

$$v = V_0 e^{-t/RC}, \tag{$*$}$$

where v is the potential across the capacitor at time t.

The capacitors in this problem are connected in parallel, so their equivalent capacitance is simply the sum of their individual capacitances:

$$C_{\text{eq}} = 15.0\,\mu\text{F} + 20.0\,\mu\text{F} = 35.0\,\mu\text{F}.$$

The resistors are connected in series, so their equivalent resistance is also just the sum:

$$R_{\text{eq}} = 30.0\,\Omega + 50.0\,\Omega = 80.0\,\Omega.$$

We can plug these values and our intended value of $v = 10.0\,\text{V}$ into ($*$) to solve for the time t:

$$10.0\,\text{V} = (45.0\,\text{V}) \exp\left(-\frac{t}{(80.0\,\Omega)(35.0\,\mu\text{F})}\right) \quad\Longrightarrow\quad \frac{2}{9} = \exp\left(-\frac{t}{2800\,\mu\text{s}}\right) \quad\Longrightarrow\quad -1.50 = -\frac{t}{2800\,\mu\text{s}}$$

$$\Longrightarrow \quad t = 4210\,\mu\text{s} = 4.21\,\text{ms},$$

where we have used the fact that $1\,\text{s} = 1\,\Omega\,\text{F}$.

(b) The current i through a capacitor at a given time t after the circuit is closed is given by

$$i = \frac{Q_0}{RC} e^{-t/RC}. \tag{26.13}$$

From (a), we know that $Q_0/C = V_0$, so we can write this as

$$i = \frac{V_0}{R} e^{-t/RC}.$$

Plugging in our V_0, R_{eq}, C_{eq}, and t that we found in (a), we have

$$i = \frac{45.0\,\text{V}}{80.0\,\Omega} \exp\left(-\frac{4210\,\mu\text{s}}{2800\,\mu\text{s}}\right) = \frac{9}{16} e^{-1.50}\,\text{A} = \frac{9}{16}\frac{2}{9}\,\text{A} = \frac{1}{8}\,\text{A} = 0.125\,\text{A}.$$

Exercise 26.47 In the circuit shown in **Fig. E26.47** the capacitors are initially uncharged, the battery has no internal resistance, and the ammeter is idealized. Find the ammeter reading

(a) just after the switch S is closed, and

(b) after S has been closed for a very long time.

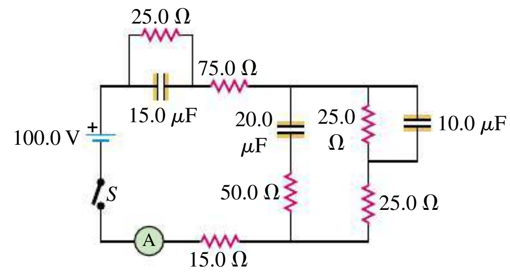


Figure E26.47

Solution.

(a) Just after the switch is closed, there is no charge on any of the capacitors. This means they act like a short in the circuit, so we can ignore any resistor that is in parallel with a capacitor.

The $50\,\Omega$ and $25\,\Omega$ resistors are in parallel, and have the equivalent resistance R_{eq} given by

$$\frac{1}{R_{\text{eq}}} = \frac{1}{50\,\Omega} + \frac{1}{25\,\Omega} = \frac{3}{50}\,\Omega^{-1} \qquad \Longrightarrow \qquad R_{\text{eq}} = \frac{50}{3}\,\Omega = 16.7\,\Omega.$$

The three resistances in series now give us the total resistance,

$$R_{\text{tot}} = 75.0\,\Omega + 16.7\,\Omega + 15.0\,\Omega = 106.7\,\Omega.$$

We can use Ohm's law to find the current through the equivalent resistance, which is the same as the ammeter reading:

$$\mathcal{E} = IR_{\text{tot}} \qquad \Longrightarrow \qquad I = \frac{\mathcal{E}}{R_{\text{tot}}} = \frac{100.0\,\text{V}}{106.7\,\Omega} = 0.937\,\text{A}.$$

(b) After S has been closed for a very long time, the capacitors are completely charged and current cannot flow through them. Therefore every capacitor acts like a break in the circuit, and we can adjust our diagram accordingly.

This gives us five resistors in series, which have the equivalent total resistance

$$R_{\text{tot}} = 25.0\,\Omega + 75.0\,\Omega + 25.0\,\Omega + 25.0\,\Omega + 15.0\,\Omega = 165.0\,\Omega.$$

Once again, we use Ohm's law to find the ammeter reading:

$$I = \frac{\mathcal{E}}{R_{\text{tot}}} = \frac{100.0\,\text{V}}{165.0\,\Omega} = 0.606\,\text{A}.$$

Problem 26.53 A capacitor with capacitance C is connected in series to a resistor of resistance R and a battery with emf $\mathcal{E}$. The circuit is completed at time $t = 0$.

(a) In terms of $\mathcal{E}$, R, and C, how much energy is stored in the capacitor when it is fully charged?

(b) The power output of the battery is $P_{\mathcal{E}} = \mathcal{E}i$, with i given by Eq. (26.13). The electrical energy supplied in an infinitesimal time dt is $P_{\mathcal{E}}\,dt$. Integrate from $t = 0$ to $t \to \infty$ to find the total energy supplied by the battery.

(c) The rate of consumption of electrical energy in the resistor is $P_R = i^2R$. In an infinitesimal time interval dt, the amount of electrical energy consumed by the resistor is $P_R\,dt$. Integrate from $t = 0$ to $t \to \infty$ to find the total energy consumed by the resistor.

(d) What fraction of the total energy supplied by the battery is stored in the capacitor? What fraction is consumed in the resistor?

Solution.

(a) In general, the potential energy U stored in a capacitor is given by

$$U = \frac{1}{2}CV^2, \tag{24.9}$$

where C is the capacitor's capacitance and V the potential difference between its plates. When the capacitor its fully charged, the potential difference between its plates is the emf $\mathcal{E}$. So we have

$$U_C = \frac{1}{2}C\mathcal{E}^2.$$

(b) In terms of the given quantities $\mathcal{E}$, R, and C, the instantaneous current i is given by

$$i = \frac{\mathcal{E}}{R}e^{-t/RC}. \tag{26.13}$$

Plugging this in and integrating, the total energy supplied by the battery is

$$U_{\mathcal{E}} = \int_0^\infty P_{\mathcal{E}}\,dt = \int_0^\infty \mathcal{E}i\,dt = \int_0^\infty \mathcal{E}\frac{\mathcal{E}}{R}e^{-t/RC}\,dt = \frac{\mathcal{E}^2}{R}\Big[-RCe^{-t/RC}\Big]_0^\infty = \frac{\mathcal{E}^2}{R}RC = C\mathcal{E}^2.$$

(c) We just need to plug (26.13) in again and integrate to find the total energy consumed by the resistor:

$$U_R = \int_0^\infty P_R\,dt = \int_0^\infty i^2R\,dt = \int_0^\infty \left(\frac{\mathcal{E}}{R}e^{-t/RC}\right)^2 R\,dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-2t/RC}\,dt = \frac{\mathcal{E}^2}{R}\left[-\frac{RCe^{-2t/RC}}{2}\right]_0^\infty$$
$$= \frac{\mathcal{E}^2}{R}\frac{RC}{2} = \frac{1}{2}C\mathcal{E}^2.$$

(d) From (a) and (b), $U_C/U_{\mathcal{E}} = 1/2$, so half of the total energy supplied by the battery is stored in the capacitor. From (b) and (c), $U_R/U_{\mathcal{E}} = 1/2$, so the other half of the total energy is consumed by the resistor.

Problem 26.59 Calculate the currents I_1, I_2, and I_3 indicated in the circuit diagram shown in **Fig. P26.59**.

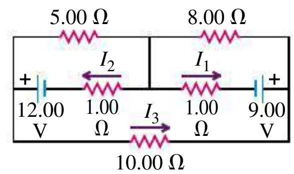
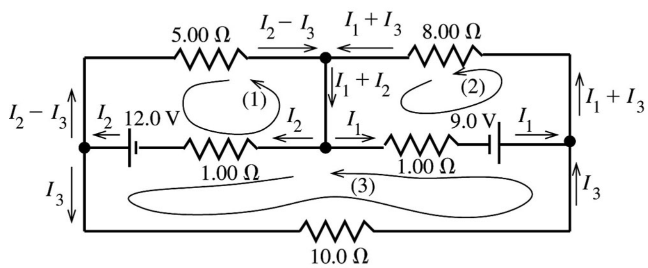


Figure P26.59

Solution. This is another problem we can solve using Kirchhoff's rules.



The circuit has three loops, so we will need to use the loop rule three times. But first we need to use the junction rule to find the current through the $5\,\Omega$ and $8\,\Omega$ resistors. At the junction on the left (shown in red), we have

$$0 = I_2 - I_3 + I_{5\,\Omega},$$

which tells us the current through the $8\,\Omega$ resistor is

$$I_{5\,\Omega} = I_3 - I_2, \tag{A}$$

meaning it is pointing away from the junction (as shown in the diagram) if $I_2 > I_3$.

At the junction on the right, we have

$$0 = I_1 + I_3 + I_{8\,\Omega},$$

which tells us the current through the $8\,\Omega$ resistor is

$$I_{8\,\Omega} = -I_1 - I_3, \tag{B}$$

meaning it is pointing away from the junction as shown.

Now we can apply the loop rule, moving counterclockwise from each battery. For loop (1), we have

$$0 = -12\,\text{V} + (1\,\Omega)I_2 + (5\,\Omega)(I_2 - I_3),$$

which simplifies to

$$12\,\text{A} = 6I_2 - 5I_3. \tag{1}$$

For loop (2),

$$0 = 9\,\text{V} - (8\,\Omega)(I_1 - I_3) + (1\,\Omega)I_1,$$

which simplifies to

$$9\,\text{A} = 9I_1 + 8I_3. \tag{2}$$

For loop (3), let's start at the $12\,\text{V}$ battery. We find

$$0 = 12\,\text{V} - (10\,\Omega)I_3 - 9\,\text{V} + (1\,\Omega)I_1 - (1\,\Omega)I_2,$$

which simplifies to

$$3\,\text{A} = -I_1 + I_2 + 10I_3. \tag{3}$$

The loop rule has given us the system of three equations (1), (2), and (3), which you can solve using whatever method you like. Since this system is more complicated than the system in **Exercise 26.26**, I prefer to use Gaussian elimination. The matrix equation is

$$\begin{bmatrix} 0 & 6 & -5 \\ 9 & 0 & 8 \\ -1 & 1 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 3 \end{bmatrix},$$

which can be solved as follows:

$$\left[\begin{array}{ccc|c} 0 & 6 & -5 & 12 \\ 9 & 0 & 8 & 9 \\ -1 & 1 & 10 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 8/9 & 1 \\ 0 & 1 & -5/6 & 2 \\ -1 & 1 & 10 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 8/9 & 1 \\ 0 & 1 & -5/6 & 2 \\ 0 & 1 & 98/9 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 8/9 & 1 \\ 0 & 1 & -5/6 & 2 \\ 0 & 0 & 211/18 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 179/211 \\ 0 & 1 & 0 & 452/211 \\ 0 & 0 & 1 & 36/211 \end{array}\right].$$

So we have

$$I_1 = 0.848\,\mathrm{A}\ \text{(to the right)}, \qquad I_2 = 2.14\,\mathrm{A}\ \text{(to the left)}, \qquad I_3 = 0.171\,\mathrm{A}\ \text{(to the right)}.$$