Problem 1. Spin-wave theory (P&S 11.1)

1(a) Prove the following wonderful formula: Let $\phi(x)$ be a free scalar field with propagator $\langle T\phi(x)\phi(0)\rangle = D(x)$. Then

$$\left\langle T e^{i\phi(x)} e^{-i\phi(0)} \right\rangle = e^{[D(x)-D(0)]}. \tag{1}$$

(The factor $D(0)$ gives a formally divergent adjustment of the overall normalization.)

Solution. According to P&S (9.18),

$$\langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle = \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp\left[i\int d^4x\, \mathcal{L}\right]}{\int \mathcal{D}\phi\, \exp\left[i\int d^4x\, \mathcal{L}\right]}.$$

We use this expression to write the left-hand side of Eq. (1):

$$\left\langle T e^{i\phi(x)} e^{-i\phi(0)} \right\rangle = \frac{\int \mathcal{D}\phi\, e^{i\phi(x)} e^{-i\phi(0)} \exp\left[i\int d^4y\, \mathcal{L}\right]}{\int \mathcal{D}\phi\, \exp\left[i\int d^4y\, \mathcal{L}\right]} = \frac{\int \mathcal{D}\phi\, \exp\left[i\phi(x) - i\phi(0) + i\int d^4y\, \mathcal{L}\right]}{\int \mathcal{D}\phi\, \exp\left[i\int d^4y\, \mathcal{L}\right]}. \tag{2}$$

For a free Klein-Gordon (i.e., scalar) field, Eq. (9.39) tells us that the generating functional $Z[J]$ is given by

$$Z[J] = Z_0 \exp\left[-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right],$$

where $Z_0 = Z[0]$. Thus, we want to find some $J(y)$ such that

$$\left\langle T e^{i\phi(x)} e^{-i\phi(0)} \right\rangle = \frac{Z[J]}{Z_0} \tag{3}$$

where in general

$$Z[J] = \int \mathcal{D}\phi\, \exp\left[i\int d^4x\, [\mathcal{L} + J(x)\phi(x)]\right]$$

by (9.34). Inspecting Eq. (2), we recognize the denominator as Z_0 and see that if

$$J(y) = \delta^{(4)}(y-x) - \delta^{(4)}(y)$$

we have an expression like Eq. (3). Collecting these findings, we have

$$\begin{aligned}
\left\langle T e^{i\phi(x)} e^{-i\phi(0)} \right\rangle &= \frac{Z[J]}{Z_0} \\
&= \exp\left[-\frac{1}{2}\int d^4y\, d^4z\, J(y) D_F(y-z) J(z)\right] \\
&= \exp\left[-\frac{1}{2}\int d^4y\, d^4z\, J(y) D_F(y-z) [\delta^{(4)}(z-x) - \delta^{(4)}(z)]\right] \\
&= \exp\left[-\frac{1}{2}\int d^4y\, [\delta^{(4)}(y-x) - \delta^{(4)}(y)][D_F(y-x) - D_F(y)]\right] \\
&= \exp\left[-\frac{1}{2}[D_F(0) - D_F(x) - D_F(-x) + D_F(0)]\right] \\
&= \exp[D_F(x) - D_F(0)] \\
&= e^{[D(x)-D(0)]},
\end{aligned}$$

as we wanted to show. □

1(b) We can use this formula in Euclidean field theory to discuss correlation functions in a theory with spontaneously broken symmetry for $T < T_C$. Let us consider only the simplest case of a broken $O(2)$ or $U(1)$ symmetry. We can write the local spin density as a complex variable

$$s(x) = s^1(x) + is^2(x).$$

The global symmetry is the transformation

$$s(x) \to e^{-i\alpha} s(x).$$

If we assume that the physics freezes the modulus of $s(x)$, we can parameterize

$$s(x) = Ae^{i\phi(x)} \tag{4}$$

and write an effective Lagrangian for the field $\phi(x)$. The symmetry of the theory becomes the translation symmetry

$$\phi(x) \to \phi(x) - \alpha. \tag{5}$$

Show that (for $d > 0$) the most general renormalizable Lagrangian consistent with this symmetry is the free field theory

$$\mathcal{L} = \frac{1}{2}\rho(\vec{\nabla}\phi)^2. \tag{6}$$

In statistical mechanics, the constant ρ is called the *spin wave modulus*. A reasonable hypothesis for ρ is that it is finite for $T < T_C$ and tends to 0 as $T \to T_C$ from below.

Solution. In accordance with the Klein-Gordon Lagrangian in P&S (2.6), we interpret $(\vec{\nabla}\phi)^2$ as $(\partial\phi)^2$.

The Lagrangian cannot have terms of $\mathcal{O}(\phi^n)$ for any $n \neq 0$ since $\phi(x)$ is not invariant under Eq. (5). Any combination of derivatives of ϕ is invariant, however, since α is a constant and does not contribute to any derivative. Thus, only terms like $\partial^n \phi^m$ (where n denotes a power of ∂) for integers $n, m > 0$ and $n \geq m$ are consistent with the symmetry of Eq. (5) for d an integer.

Now we must determine which of these terms are renormalizable. We know that the Lagrangian must have dimension d, and that ϕ has dimension $(d-2)/2$. Taking a derivative adds a mass dimension. The theory is renormalizable if the coupling constant ρ has dimension greater than or equal to 0 [1, p. 322]. Let p be the dimension of ρ, which must be an integer. The dimension of our allowed term is then

$$[\rho\partial^n\phi^m] = p + n + m\frac{d-2}{2},$$

which we require to be equal to d. Thus we seek solutions to the system of equations

$$d = p + n + m\frac{d-2}{2}, \qquad n \geq m, \qquad p \geq 0.$$

Solving with Mathematica, we find that this system has only one solution, $n = m = 2$ and $p = 0$. This means that ρ must be dimensionless and the only allowed terms in the Lagrangian are proportional to $\partial^2\phi^2 = (\partial\phi)^4$, which is consistent with Eq. (6). $\square$

1(c) Compute the correlation function $\langle s(x)s^*(0)\rangle$. Adjust A to give a physically sensible normalization (assuming that the system has a physical cutoff at the scale of one atomic spacing) and display the dependence of this correlation function on x for $d = 1, 2, 3, 4$. Explain the significance of your results.

Solution. Applying Eq. (4),

$$\langle s(x)s^*(0)\rangle = \langle Ae^{i\phi(x)}A^*e^{-i\phi(0)}\rangle = \langle |A|^2\rangle \langle e^{i\phi(x)}e^{-i\phi(0)}\rangle .$$

Now we can apply Eq. (1) to find

$$\langle s(x)s^*(0)\rangle = \langle |A|^2\rangle \exp[D(x) - D(0)], \tag{7}$$

where $D(x-y)$ is a Green's function. It is similar to the Green's function of the Klein-Gordon operator, which is given by P&S (2.56):

$$(\partial^2 + m^2)D(x-y) = -i\delta^{(4)}(x-y).$$

The Feynman prescription for this Green's function is given by (2.59),

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{ip\cdot(x-y)}. \tag{8}$$

For the Lagrangian in Eq. (6), we set $m = 0$ and insert a factor of ρ:

$$\rho\partial^2 D(x-y) = -i\delta^{(d)}(x-y),$$

so adapting Eq. (8) for this situation yields

$$D_F(x-y) = \frac{1}{\rho}\int \frac{d^dp}{(2\pi)^d}\frac{i}{p^2 + i\epsilon}e^{ip\cdot(x-y)}.$$

We see that $D_F(0)$ diverges, so we absorb it into the normalization to make it physically sensible. Define A' such that

$$A'^2 = \langle |A|^2\rangle e^{-D(0)}.$$

Then Eq. (7) can be written

$$\langle s(x)s^*(0)\rangle = A'^2 D(x).$$

By analogy with P&S (9.48), after Wick rotation the Green's function is

$$D_F(x_E - y_E) = \int \frac{d^dk_E}{(2\pi)^d}\frac{e^{ik_E\cdot(x_E-y_E)}}{k_E^2}$$

how to evaluate this integral???

Problem 2. The Gross-Neveu model (P&S 11.3) The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2 \tag{9}$$

with $i = 1, \ldots, N$. The kinetic term of two-dimensional fermions is built from matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma^0 = \sigma^2, \qquad \gamma^1 = i\sigma^1,$$

where σ^i are Pauli sigma matrices. Define

$$\gamma^5 = \gamma^0\gamma^1 = \sigma^3;$$

this matrix anticommutes with the γ^μ.

2(a) Show that this theory is invariant with respect to

$$\psi_i \to \gamma^5\psi_i, \tag{10}$$

and that this symmetry forbids the appearance of a fermion mass.

Solution. Under this transformation, the Lagrangian Eq. (9) is unchanged:

$$\begin{aligned}
\mathcal{L} &= i\psi_i^\dagger\gamma^0\gamma^\mu\partial_\mu\psi_i + \frac{1}{2}g^2(\psi_i^\dagger\gamma^0\psi_i)^2 \\
&\to i\psi_i^\dagger\gamma^5\gamma^0\gamma^\mu\partial_\mu\gamma^5\psi_i + \frac{1}{2}g^2(\psi_i^\dagger\gamma^5\gamma^0\gamma^5\psi_i)^2 \\
&= -i\psi_i^\dagger\gamma^5\gamma^0\gamma^5\gamma^\mu\partial_\mu\psi_i + \frac{1}{2}g^2(-\psi_i^\dagger\gamma^0\psi_i)^2 \\
&= i\psi_i^\dagger\gamma^0\gamma^\mu\partial_\mu\psi_i + \frac{1}{2}g^2(\psi_i^\dagger\gamma^0\psi_i)^2 \\
&= \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2.
\end{aligned}$$

Here we have used $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ [1, p. 49], $\bar{\psi} = \psi^\dagger\gamma^0$ by P&S (3.32), and (3.69)–(3.71):

$$(\gamma^5)^\dagger = \gamma^5, \qquad (\gamma^5)^2 = 1, \qquad \{\gamma^5, \gamma^\mu\} = 0,$$

which also hold for the 2-dimensional Dirac algebra. Thus we have shown that the theory is invariant with respect to the transformation in Eq. (10) because it does not change the Lagrangian. □

If the theory had a mass, the first term would take the form of the Dirac Lagrangian in (3.34):

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}_i(i\partial\!\!\!/ - m)\psi_i. \tag{11}$$

However, this term is not invariant under Eq. (10):

$$\begin{aligned}
\mathcal{L}_{\text{Dirac}} &= i\psi_i^\dagger\gamma^0\gamma^\mu\partial_\mu\psi_i - m\psi_i^\dagger\gamma^0\psi_i \\
&\to i\psi_i^\dagger\gamma^5\gamma^0\gamma^\mu\partial_\mu\gamma^5\psi_i - m\psi_i^\dagger\gamma^5\gamma^0\gamma^5\psi_i \\
&= i\psi_i^\dagger\gamma^0\gamma^\mu\partial_\mu\psi_i + m\psi_i^\dagger\gamma^0\psi_i \\
&= \bar{\psi}_i(i\partial\!\!\!/ + m)\psi_i.
\end{aligned}$$

Since the sign of the mass term changes under the transformation, a nonzero fermion mass m is forbidden. □

2(b) Show that this theory is renormalizable in 2 dimensions (at the level of dimensional analysis).

Solution. We need to find the dimension of the coupling constant g. As in 1(b), the Lagrangian must have dimension $d = 2$. The γ^μ are dimensionless, and ∂ adds one mass dimension. We can find the dimension of ψ_i by requiring that the dimension of the first term of Eq. (9) is 2. Let $[\psi_i] = n$. Then

$$2 = [\bar{\psi}_i \not{\partial} \psi_i] = n + 1 + n \qquad \Longrightarrow \qquad n = \frac{1}{2}.$$

We may now use this result in the second term to find the dimension of g. Let $[g] = m$. Then

$$2 = [g^2(\bar{\psi}_i\psi_i)^2] = 2m + 2(n+n) = 2(m+1) \qquad \Longrightarrow \qquad m = 0,$$

meaning g is dimensionless. Therefore the theory is indeed renormalizable [1, p. 322]. $\square$

2(c) Show that the functional integral for this theory can be represented in the following form:

$$\int \mathcal{D}\psi\, e^{i\int d^2x \mathcal{L}} = \int \mathcal{D}\psi\, \mathcal{D}\sigma\, \exp\left[i\int d^2x \left\{\bar{\psi}_i i\not{\partial}\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right], \tag{12}$$

where $\sigma(x)$ (not to be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

Solution. Completing the square in the exponent of the right-hand side of Eq. (12) yields

$$\int \mathcal{D}\psi\, \mathcal{D}\sigma\, \exp\left[i\int d^2x \left\{-\frac{1}{2g^2}(\sigma + g^2\bar{\psi}_i\psi_i)^2 + \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2 + \bar{\psi}_i i\not{\partial}\psi_i\right\}\right].$$

Pulling out the integral over σ, note that

$$\int \mathcal{D}\sigma\, \exp\left[-i\int d^2x\, \frac{1}{2g^2}(\sigma + g^2\bar{\psi}_i\psi_i)^2\right] \propto \frac{1}{\sqrt{2\det(g^2)}} = \text{const.} \tag{13}$$

This is obtained from P&S (9.24),

$$\left(\prod_k \int d\xi_k\right) \exp[-\xi_i B_{ij}\xi_j] = \text{const} \times [\det B]^{-1/2}, \tag{14}$$

where in Eq. (13) we obtain an expression like this if we transform variables to $u = \sigma + g^2\bar{\psi}_i\psi_i$. At any rate, the result is that we obtain

$$\begin{aligned}\int \mathcal{D}\psi\, \mathcal{D}\sigma\, \exp\left[i\int d^2x \left\{\bar{\psi}_i i\not{\partial}\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right] &= \text{const} \times \int \mathcal{D}\psi\, \exp\left[i\int d^2x \left\{\frac{g^2}{2}(\bar{\psi}_i\psi_i)^2 + \bar{\psi}_i i\not{\partial}\psi_i\right\}\right] \\ &= \text{const} \times \int \mathcal{D}\psi\, e^{i\int d^2x \mathcal{L}},\end{aligned}$$

where $\mathcal{L}$ is given by Eq. (9). The overall constant can be disregarded, so we have proven Eq. (12). $\square$

2(d) Compute the leading correction to the effective potential for σ by integrating over the fermion fields ψ_i. You will encounter the determinant of a Dirac operator; to evaluate this determinant, diagonalize the operator by first going to Fourier components and then diagonalizing the 2×2 Pauli matrix associated with each Fourier mode. (Alternatively, you might just take the determinant of this 2×2 matrix.) This 1-loop contribution requires a renormalization proportional to σ^2 (that is, a renormalization of g^2). Renormalize by minimal subtraction.

Solution. The right-hand side of Eq. (12) can be written

$$\int \mathcal{D}\sigma \, \exp\left[-i\int d^2x \, \frac{1}{2g^2}\sigma^2\right]\int \mathcal{D}\psi \, \exp\left[i\int d^2x \, \bar{\psi}_i(i\partial\!\!\!/ - \sigma)\psi_i\right]. \tag{15}$$

The integral of the exponential argument is the Dirac Lagrangian Eq. (11) with $m \to \sigma$. A similar integral is given by P&S (9.76):

$$\int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left[i\int d^2x \, \bar{\psi}(iD\!\!\!\!/ - m)\psi\right] = \det\bigl(iD\!\!\!\!/ - m\bigr).$$

For the integral over ψ in Eq. (15), we have $D\!\!\!\!/ \to \partial\!\!\!/$. The result is then

$$\int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left[i\int d^2x \, \bar{\psi}_i(i\partial\!\!\!/ - \sigma)\psi_i\right] = [\det\bigl(i\partial\!\!\!/ - \sigma\bigr)]^N.$$

Here we get a power of N because we are integrating over $\bar{\psi}_i, \psi_i$ for $i = 1, \ldots, N$. Now we apply (9.77),

$$\det B = \exp[\mathrm{tr}(\log B)].$$

Now we follow a similar procedure as in (11.71),

$$\mathrm{tr}\bigl[\ln\bigl(\partial^2 + m^2\bigr)\bigr] = (VT)\int \frac{d^4k}{(2\pi)^4}\ln\bigl(-k^2 + m^2\bigr),$$

where (VT) is the four-dimensional volume of the functional integral. Adapting this for the Dirac operator in two dimensions, we have

$$\mathrm{tr}\bigl[\ln\bigl(i\partial\!\!\!/ - \sigma\bigr)\bigr] = (VT)\int \frac{d^2k}{(2\pi)^2}\ln\bigl(-k^2 + \sigma^2\bigr).$$

Here we have used the result

$$\begin{aligned}
\det(i\gamma^\mu\partial_\mu - \sigma) &= \det\bigl(i\gamma^0\partial_0 + i\gamma^1\partial_1 - \sigma\bigr) \\
&= \det\bigl(i\sigma^2\partial_0 - \sigma^1\partial_1 - \sigma\bigr) \\
&= \det\left(i\begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}\partial_0 - \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\partial_1 - \sigma\right) \\
&= \det\begin{bmatrix}-\sigma & \partial_0 - \partial_1 \\ -(\partial_0 + \partial_1) & -\sigma\end{bmatrix} \\
&= \sigma^2 + \partial_0^2 + \partial_1^2 \\
&= \partial^\mu\partial_\mu + \sigma^2.
\end{aligned}$$

Now we may apply (11.72),

$$\int \frac{d^4k}{(2\pi)^4}\ln\bigl(-k^2 + m^2\bigr) = -i\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\frac{1}{(m^2)^{-d/2}}.$$

Letting $d = 2 - \epsilon$ and expanding about $\epsilon = 0$ (using Mathematica), we find

$$\int \frac{d^2k}{(2\pi)^2}\ln\bigl(-k^2 + \sigma^2\bigr) = -i\frac{\Gamma(\epsilon/2 - 1)}{(4\pi)^{1-\epsilon/2}}\sigma^{2-\epsilon} \approx \frac{i\sigma^2}{4\pi}\left(\frac{2}{\epsilon} + 1 - \gamma + \ln(4\pi) - 2\ln(\sigma)\right).$$

Now we renormalize by minimal subtraction. Following the examples in (11.77) and (11.78), we make the replacement

$$\int \frac{d^2k}{(2\pi)^2}\ln\bigl(-k^2 + \sigma^2\bigr) \to \frac{i\sigma^2}{4\pi}\left[1 - \ln\left(\frac{\sigma^2}{M^2}\right)\right],$$

where M is an arbitrary mass parameter.

Applying our work to Eq. (12), we have

$$\begin{aligned}\int \mathcal{D}\psi\, e^{i\int d^2x \mathcal{L}} &= \int \mathcal{D}\sigma\, \exp\left[-i\int d^2x\, \frac{1}{2g^2}\sigma^2\right]\left[\det\left(i\partial\!\!\!/ - \sigma\right)\right]^N \\ &= \int \mathcal{D}\sigma\, \exp\left[\int d^2x \left(\frac{-i}{2g^2}\sigma^2 + N\int \frac{d^2k}{(2\pi)^2}\ln\left(-k^2+\sigma^2\right)\right)\right] \\ &= \int \mathcal{D}\sigma\, \exp\left[-i\int d^2x\,\sigma^2\left\{\frac{1}{2g^2}+\frac{N}{4\pi}\left[\ln\left(\frac{\sigma^2}{M^2}\right)-1\right]\right\}\right].\end{aligned}$$

Then the leading correction to the effective potential is

$$V_{\text{eff}} = \sigma^2\left\{\frac{1}{2g^2}+\frac{N}{4\pi}\left[\ln\left(\frac{\sigma^2}{M^2}\right)-1\right]\right\},$$

where we have referred to (11.79).

2(e) Ignoring two-loop and higher-order contributions, minimize this potential. Show that the σ field acquires a vacuum expectation value which breaks the symmetry of 2(a). Convince yourself that this result does not depend on the particular renormalization condition chosen.

2(f) Note that the effective potential derived in 2(e) depends on g and N according to the form

$$V_{\text{eff}}(\sigma_{\text{cl}}) = N\cdot f(g^2N).$$

(The overall factor of N is expected in a theory with N fields.) Construct a few of the higher-order contributions to the effective potential and show that they contain additional factors of N^{-1} which suppress them if we take the limit $N\to\infty$, (g^2N) fixed. In this limit, the result of 2(e) is unambiguous.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.