

**Problem 1. Alternative regulators in QED (Peskin & Schroeder 7.2)** In Section. 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity  $Z_1 = Z_2$ , to order  $\alpha$ , for several choices of the regulator. We have already verified that the relation holds for Paul-Villars regularization.

**1(a)** Recompute  $\delta Z_1$  and  $\delta Z_2$ , defining the integrals (6.49) and (6.50) by simply placing an upper limit  $\Lambda$  on the integration over  $\ell_E$ . Show that, with this definition,  $\delta Z_1 \neq \delta Z_2$ .

**Solution.** From (7.47) in Peskin & Schroeder,

$$\Gamma^\mu(q=0) = \frac{1}{Z_1} \gamma^\mu,$$

we can find an expression for  $\delta Z_1$ , similar to how (7.31) is obtained from (7.26). Roughly,

$$\frac{1}{Z_1 + \delta Z_1} \gamma^\mu \approx Z_1(1 - \delta Z_1) \gamma^\mu = \Gamma^\mu(q=0) + \delta \Gamma^\mu(q=0) \implies \delta \Gamma^\mu(q=0) = -\delta Z_1 \gamma^\mu. \quad (1)$$

According to (6.33),

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2).$$

We note that  $\Gamma^\mu = \gamma^\mu$ ,  $F_1 = 1$ , and  $F_2 = 0$  to lowest order [1, pp. 185–186]. Then we can write

$$\delta \Gamma^\mu(q=0) = \gamma^\mu \delta F_1(0) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \delta F_2(0). \quad (2)$$

Using this equation and the identity  $\gamma^\mu \gamma_\mu = 4$  [2], Eq. (1) can be written

$$\delta Z_1 = -\frac{1}{4} \gamma_\mu \delta \Gamma^\mu(q=0) = -\delta F_1(0) - \gamma_\mu \frac{i\sigma^{\mu\nu} q_\nu}{8m} \delta F_2(0). \quad (3)$$

In order to find  $\delta \Gamma^\mu$  we use (6.47):

$$\begin{aligned} \bar{u}(p') \delta \Gamma^\mu(p', p) u(p) &= 2ie^2 \int \frac{d^4 \ell}{(2\pi)^4} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3} \\ &\times \bar{u}(p') \left\{ \gamma^\mu \left[ -\frac{\ell^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2 \right] \right. \\ &\quad \left. + \frac{i\sigma^{\mu\nu} q_\nu}{2m} [2m^2 z(1-z)] \right\} u(p), \end{aligned} \quad (4)$$

where  $\Delta \equiv -xyq^2 + (1-z)^2 m^2$  by (6.44),  $\ell \equiv k + yq - zp$ , and  $D = \ell^2 - \Delta + i\epsilon$  [1, p. 191]. The momenta  $k$  and  $p$  are assigned in the Feynman diagram on p. 189 of Peskin & Schroeder, and  $x, y$  are Feynman parameters [1, p. 190].

The forms of the two integrals we need to compute are given by Peskin & Schroeder (6.49) and (6.50). Equation (6.49) is

$$\int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^m} = \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{\Delta^{m-2}}. \quad (5)$$

Here  $m = 3$  because we have  $D^{-3}$  in Eq. (4). We can evaluate the left-hand side using the Euclidian 4-momentum defined in (6.48),

$$\ell^0 \equiv i\ell_E^0, \quad \ell = \ell_E. \quad (6)$$

Following the steps on p. 193, we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^3} = \frac{i}{(-1)^3} \frac{1}{(2\pi)^4} \int \frac{d^4\ell_E}{(\ell_E^2 + \Delta)^3} = \frac{i(-1)^3}{(2\pi)^4} \int d\Omega_4 \int_0^\Lambda d\ell_E \frac{\ell_E^3}{(\ell_E^2 + \Delta)^3},$$

where we have replaced the upper (infinite) bound of integration by a finite number  $\Lambda$ . Evaluating this integral using Mathematica and using  $\int d\Omega_4 = 2\pi^2$  [1, p. 193], we find

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^3} &= \frac{i(-1)^3}{(2\pi)^4} (2\pi^2) \frac{\Lambda^4}{4\Delta(\Delta + \Lambda^2)^2} \\ &= -\frac{i}{32\pi^2} \frac{\Lambda^4}{\Delta(\Delta + \Lambda^2)^2} \\ &\approx -\frac{i}{32\pi^2} \frac{1}{\Delta} \equiv \alpha, \end{aligned} \quad (7)$$

where we have taken the limit  $\Lambda \gg \Delta$  [1, p. 218] and defined  $\alpha$ . Equation (6.50) in Peskin & Schroeder is

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - \Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}}.$$

Following similar steps as for Eq. (4), the left-hand side is

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - \Delta)^3} &= \frac{i}{(-1)^3} \frac{1}{(2\pi)^4} \int d^4\ell_E \frac{\ell_E^2}{(\ell_E^2 + \Delta)^3} \\ &= \frac{i(-1)^3}{(2\pi)^4} \int d\Omega_4 \int_0^\Lambda d\ell_E \frac{\ell_E^5}{(\ell_E^2 + \Delta)^3} \\ &= \frac{i(-1)^3}{(2\pi)^4} (2\pi^2) \frac{1}{4} \left[ \frac{\Delta(3\Delta + 4\Lambda^2)}{(\Delta + \Lambda^2)^2} + 2\ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 3 \right] \\ &= -\frac{i}{32\pi^2} \left[ \frac{\Delta(3\Delta + 4\Lambda^2)}{(\Delta + \Lambda^2)^2} + 2\ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 3 \right] \\ &\approx -\frac{i}{16\pi^2} \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - \frac{3}{2} \right] \equiv \beta, \end{aligned} \quad (8)$$

where we have defined  $\beta$  and ignored terms of  $\mathcal{O}(\Lambda^{-2})$  [1, p. 218].

We now set  $q^2 = 0$ , and define  $\Delta_0 = (1 - z)^2 m^2$ . Then  $\Delta \rightarrow \Delta_0$  in our expression and  $\alpha \rightarrow \alpha_0, \beta \rightarrow \beta_0$  (which are functions of  $\Delta_0$ ). Feeding in Eqs. (7) and (8), Eq. (4) can be written

$$\bar{u}(p') \delta\Gamma^\mu(q=0) u(p) = 2ie^2 \int_0^1 dx dy dz \delta(x+y+z-1) \bar{u}(p') \int \left\{ \gamma^\mu [-\beta_0 + 2(1-4z+z^2)m^2\alpha_0] \right\} u(p).$$

Then

$$\begin{aligned} \delta F_1(0) &= 2ie^2 \int_0^1 dx dy dz \delta(x+y+z-1) [-\beta_0 + 2(1-4z+z^2)m^2\alpha_0] \\ &= 2ie^2 \int_0^1 dz (1-z) [-\beta_0 + 2m^2(1-4z+z^2)\alpha_0], \\ \delta F_2(0) &= 8ie^2 \int_0^1 dx dy dz \delta(x+y+z-1) m^2 z(1-z)\alpha_0 \\ &= 8ie^2 \int_0^1 dz m^2 z(1-z)^2\alpha_0. \end{aligned}$$

We ignore  $\delta F_2(0)$  since it is not affected by the divergence [1, p. 196]. In order to avoid issues coming from the divergence in  $\delta F_1(0)$ , we add a  $z\mu^2$  term to  $\Delta_0$  [1, p. 195]. So, feeding these results into Eq. (3), we obtain

$$\begin{aligned}\delta Z_1 &= -2ie^2 \int_0^1 dz (1-z) [-\beta_0 + 2(1-4z+z^2)m^2\alpha_0] \\ &= -2ie^2 \int_0^1 dz (1-z) \left\{ \frac{i}{16\pi^2} \left[ \ln\left(\frac{\Delta_0 + \Lambda^2}{\Delta_0}\right) - \frac{3}{2} \right] - 2(1-4z+z^2)m^2 \frac{i}{32\pi^2} \frac{1}{\Delta_0} \right\} \\ &= \frac{e^2}{8\pi^2} \int_0^1 dz (1-z) \left[ \ln\left(\frac{\Delta_0 + \Lambda^2}{\Delta_0}\right) - \frac{3}{2} - \frac{m^2(1-4z+z^2)}{\Delta_0} \right],\end{aligned}\tag{9}$$

where

$$\Delta_0 = (1-z)^2 m^2 + z\mu^2.\tag{10}$$

For  $\delta Z_2$ , we can use the first part of Peskin & Schroeder (7.31),

$$\delta Z_2 = \left. \frac{d\Sigma_2}{d\mathbf{p}} \right|_{\mathbf{p}=m},\tag{11}$$

where  $\Sigma_2$  is given by (7.17),

$$-i\Sigma_2(p) = -e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-2x\not{p} + 4m_0}{(\ell^2 - \Delta + i\epsilon)^2},\tag{12}$$

where  $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$ . We may once again follow the steps on p. 193 to evaluate the integral, now with  $m = 2$ . Changing the upper bound of integration to  $\Lambda$  once more, we have

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^2} &= \frac{i}{(-1)^2} \frac{1}{(2\pi)^4} \int d^4\ell_E \frac{1}{(\ell_E^2 + \Delta)^2} \\ &= \frac{i(-1)^2}{(2\pi)^4} \int d\Omega_4 \int_0^\Lambda d\ell_E \frac{\ell_E^3}{(\ell_E^2 + \Delta)^2} \\ &= \frac{i}{(2\pi)^4} (2\pi^2) \frac{1}{2} \left[ \frac{\Delta}{\Delta + \Lambda^2} + \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \\ &= \frac{i}{16\pi^2} \left[ \frac{\Delta}{\Delta + \Lambda^2} + \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \\ &\approx \frac{i}{16\pi^2} \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right],\end{aligned}$$

where we have evaluated the integral using Mathematica and dropped terms of  $\mathcal{O}(\Lambda^{-2})$ . Substituting back into Eq. (12), we find

$$\Sigma_2(p) = -ie^2 \int_0^1 dx (-2x\not{p} + 4m_0) \frac{i}{16\pi^2} \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right].$$

Note that

$$\begin{aligned}\frac{d\Sigma_2}{d\mathbf{p}} &= \frac{e^2}{16\pi^2} \frac{d}{d\mathbf{p}} \left\{ \int_0^1 dx (-2x\not{p} + 4m_0) \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \right\} \\ &= \frac{e^2}{16\pi^2} \int_0^1 dx \left\{ \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \frac{d}{d\mathbf{p}} (-2x\not{p} + 4m_0) + (-2x\not{p} + 4m_0) \frac{d}{d\mathbf{p}} \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \right\} \\ &= \frac{e^2}{16\pi^2} \int_0^1 dx \left\{ \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \frac{d}{d\mathbf{p}} (-2x\not{p} + 4m_0) + (-2x\not{p} + 4m_0) \frac{d}{d\Delta} \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] \frac{d\Delta}{d\mathbf{p}} \right\}.\end{aligned}\tag{13}$$

Using  $p^2 = \not{p}^2$  [1, p. 220], note that

$$\frac{d\Delta}{d\not{p}} = \frac{d}{d\not{p}}[-x(1-x)\not{p}^2 + x\mu^2 + (1-x)m_0^2] = -2x(1-x)\not{p}. \quad (14)$$

Also, ignoring terms of  $\mathcal{O}(\Lambda^{-2})$ ,

$$\frac{d}{d\not{p}}(-2x\not{p} + 4m_0) = -2x, \quad (15)$$

$$\frac{d}{d\Delta} \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] = \frac{d}{d\Delta} [\ln(\Delta + \Lambda^2) - \ln(\Delta) - 1] = \frac{1}{\Delta + \Lambda^2} - \frac{1}{\Delta} \approx -\frac{1}{\Delta}. \quad (16)$$

Making these substitutions in Eq. (13),

$$\frac{d\Sigma_2}{d\not{p}} = \frac{e^2}{16\pi^2} \int_0^1 dx \left[ -2x \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - 1 \right] - \frac{(2x\not{p} - 4m_0)[2x(1-x)\not{p}]}{\Delta} \right].$$

We now define

$$\Delta_m \equiv -x(1-x)m^2 + x\mu^2 + (1-x)m_0^2 \approx (1-x)^2 m^2 + x\mu^2, \quad (17)$$

since  $m \approx m_0$ . Then Eq. (11) becomes

$$\begin{aligned} \delta Z_2 &= \frac{e^2}{16\pi^2} \int_0^1 dx \left[ -x \left[ \ln\left(\frac{\Delta_m + \Lambda^2}{\Delta_m}\right) - 1 \right] + \frac{(2xm - 4m)[2x(1-x)m]}{\Delta_m} \right] \\ &= -\frac{e^2}{8\pi^2} \int_0^1 dx \left[ x \ln\left(\frac{\Delta_m + \Lambda^2}{\Delta_m}\right) - x - \frac{2xm^2(2-x)(1-x)}{\Delta_m} \right] \end{aligned} \quad (18)$$

Now we rename  $x \rightarrow z$  in  $\delta Z_2$ . This means  $\Delta_0 = \Delta_m$  from Eqs. (10) and (17). Naming  $\Delta \equiv \Delta_0 = \Delta_m$  in Eqs. (9) and (18), we have

$$\begin{aligned} \delta Z_1 &= \frac{e^2}{8\pi^2} \int_0^1 dz (1-z) \left[ \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - \frac{3}{2} - \frac{m^2(1-4z+z^2)}{\Delta} \right], \\ \delta Z_2 &= -\frac{e^2}{8\pi^2} \int_0^1 dz \left[ x \ln\left(\frac{\Delta + \Lambda^2}{\Delta}\right) - x - \frac{2xm^2(2-x)(1-x)}{\Delta} \right], \end{aligned}$$

where  $\Delta = (1-z)^2 m^2 + z\mu^2$ . We see that  $\delta Z_1 \neq \delta Z_2$ , as we wanted to show.  $\square$

**1(b)** Recompute  $\delta Z_1$  and  $\delta Z_2$ , defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be  $4 \times 4$  as usual, but note that, in  $d$  dimensions,

$$g_{\mu\nu} \gamma^\mu \gamma^\nu = d. \quad (19)$$

Show that, with this definition,  $\delta Z_1 = \delta Z_2$ .

**Solution.** To find  $\delta Z_1$ , we need to start at Peskin & Schroeder (6.38) for arbitrary dimension:

$$\begin{aligned} \bar{u}(p') \delta \Gamma^\mu(p', p) u(p) &= \int \frac{d^d k}{(2\pi)^d} \frac{-ig_{\nu\rho}}{(k-p)^2 + i\epsilon} \bar{u}(p') (-ie\gamma^\nu) \frac{i(\not{k}' + m)}{k'^2 - m^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} (-ie\gamma^\rho) u(p) \\ &= -ie^2 \int \frac{d^d k}{(2\pi)^d} \bar{u}(p') \frac{\gamma^\nu (\not{k}' + m) \gamma^\mu (\not{k} + m) \gamma_\nu}{[(k-p)^2 + i\epsilon](k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} u(p). \end{aligned}$$

Let  $N$  be the numerator of the integrand. Then

$$N = \bar{u}(p') \gamma^\nu (\not{k}' + m) \gamma^\mu (\not{k} + m) \gamma_\nu u(p) = \bar{u}(p') \gamma^\nu (\not{k}' \gamma^\mu \not{k} + \not{k}' \gamma^\mu m + m \gamma^\mu \not{k} + m^2 \gamma^\mu) \gamma_\nu u(p).$$

using the p identity thing, Note that

$$\gamma^\nu \not{k}' \gamma^\mu \not{k} \gamma_\nu = \gamma^\nu (2k'^\mu - \gamma^\mu \not{k}') (2k_\nu - \gamma_\nu \not{k}) = 4\gamma^\nu k'^\mu k_\nu - 2\gamma^\nu k'^\mu \gamma_\nu \not{k} - 2\gamma^\nu \gamma^\mu \not{k}' k_\nu + \gamma^\nu \gamma^\mu \gamma_\nu \not{k}' \not{k}$$

We need to fix Peskin & Schroeder (7.17) so it has arbitrary  $d$  instead of  $d = 4$ . We begin from (7.16), changing  $4 \rightarrow d$ :

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^d k}{(2\pi)^d} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 + m_0^2 + i\epsilon} \gamma^\mu \frac{i}{(p-k)^2 - \mu^2 + i\epsilon}.$$

Following the procedure on pp. 217–218, we introduce the Feynman parameter  $x$  to combine the denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon} \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx \frac{1}{[k^2 - 2xk \cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}.$$

Let  $\ell = k - xp$  and  $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$ . Then

$$\begin{aligned} -i\Sigma_2(p) &= (-ie)^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \gamma^\mu \frac{i^2(\not{k} + m_0)}{[\ell^2 - \Delta + i\epsilon]^2} \gamma^\mu \\ &= -(-ie)^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \gamma^\mu \frac{\not{\ell} + x\not{p} + m_0}{[\ell^2 - \Delta + i\epsilon]^2} \gamma^\mu \\ &= -e^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \gamma^\mu \frac{2\ell^\mu - \gamma_\mu \not{\ell} + x(2p_\mu - \gamma_\mu \not{p}) + m_0 \gamma_\mu}{[\ell^2 - \Delta + i\epsilon]^2} \\ &= -e^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{(2-d)x\not{p} + dm_0}{[\ell^2 - \Delta + i\epsilon]^2}, \end{aligned} \quad (20)$$

where we have applied Eq. (19) and  $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$  [1, p. 191], and we have dropped terms linear in  $\ell$  [cite].

To evaluate the integral, we can write it in terms of the Euclidean 4-momentum defined in Eq. (6), as on p. 193 in Peskin & Schroeder:

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(-1)^2} \frac{1}{(2\pi)^d} \int d^d \ell_E \frac{1}{(\ell_E^2 + \Delta)^2} = i \int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2}.$$

Then we can apply (7.84), which takes the limit as  $d \rightarrow 4$ :

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} \rightarrow \frac{1}{(4\pi)^2} \left( \frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right),$$

where  $\epsilon = 4 - d$  [1, p. 250]. Making these substitutions into Eq. (20), we find

$$\Sigma_2(p) = \frac{e^2}{16\pi^2} \int_0^1 dx [(2-d)x\not{p} + dm_0] \left( \frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right).$$

Then

$$\begin{aligned} \frac{d\Sigma_2}{d\not{p}} &= \frac{e^2}{16\pi^2} \int_0^1 dx \left( \frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) \frac{d}{d\not{p}} [(2-d)x\not{p} + dm_0] \\ &\quad + [(2-d)x\not{p} + dm_0] \frac{d}{d\not{p}} \left( \frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) \\ &= \frac{e^2}{16\pi^2} \int_0^1 dx \left[ \left( \frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) (2-d)x + [(2-d)x\not{p} + dm_0] \frac{2x(1-x)\not{p}}{\Delta} \right] \end{aligned}$$

where

$$\frac{d}{d\not{p}} (\ln \Delta) = \frac{d\Delta}{d\not{p}} \frac{d}{d\Delta} (\ln \Delta) = -\frac{2x(1-x)\not{p}}{\Delta}$$

from Eqs. (14) and (15). Then applying  $m \approx m_0$ ,

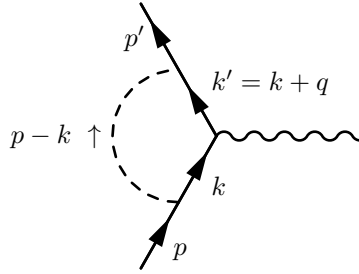
$$\delta Z_2 = \frac{e^2}{16\pi^2} \int_0^1 dx \left[ (2-d)x \left( \frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) + \frac{2x(1-x)[(2-d)x + d]m^2}{\Delta_m} \right].$$

**Problem 2. (Peskin & Schroeder 7.3)** Consider a theory of elementary fermions that couple both to QED and to a Yukawa field  $\phi$ :

$$H_{\text{int}} = \int d^3x \frac{\lambda}{\sqrt{2}} \phi \bar{\psi} \psi + \int d^3x e A_\mu \bar{\psi} \gamma^\mu \psi.$$

**2(a)** Verify that the contribution to  $Z_1$  from the vertex diagram with a virtual  $\phi$  equals the contribution to  $Z_2$  from the diagram with a virtual  $\phi$ . Use dimensional regularization. Is the Ward identity generally true in this theory?

**Solution.** We are interested in the diagram



We also considered this diagram in Homework 6, with the Yukawa field identified as a Higgs field. There, we adapted Peskin & Schroeder (6.38) to write

$$\begin{aligned} \bar{u}(p') \delta \Gamma^\mu(p, p') u(p) &= \int \frac{d^d k}{(2\pi)^d} \frac{i}{(k-p)^2 - m_\phi^2 + i\epsilon} \bar{u}(p') \left( -i \frac{\lambda}{\sqrt{2}} \right) \frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon} \left( -i \frac{\lambda}{\sqrt{2}} \right) u(p) \\ &= i \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \bar{u}(p') \frac{(\not{k}' + m_e) \gamma^\mu (\not{k} + m_e)}{[(k-p)^2 - m_\phi^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} u(p). \end{aligned} \quad (21)$$

We used Peskin & Schroeder (6.41) to write

$$\frac{1}{[(k-p)^2 - m_\phi^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} = \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3}, \quad (22)$$

where [1, pp. 190–191]

$$D = k^2 + 2k(qy - pz) + z(p^2 - m_\phi^2) - (1-z)m_e^2 + i\epsilon \equiv \ell^2 - \Delta + i\epsilon,$$

where we used  $x + y + z = 1$  and  $k' = k + q$ , and defined  $\ell \equiv k + yq - zp$  [1, p. 191], and

$$\Delta \equiv -xyq^2 + (1-z)^2 m_e^2 + zm_\phi^2.$$

$$D = \ell^2 + xyq^2 - (1-z)^2 m_e^2 - m_\phi^2 z + i\epsilon. \quad (23)$$

For the numerator of Eq. (21), we used  $k' = k + q$  and  $\ell \equiv k + yq - zp$  [1, p. 191], and defined

$$N \equiv \bar{u}(p') [\not{\ell} + (1-y)\not{q} + z\not{p} + m_e] \gamma^\mu (\not{\ell} - y\not{q} + z\not{p} + m_e) u(p) \quad (24)$$

To simplify  $N$ , we can discard terms that are linear in  $\ell$  by (6.45),

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu}{D^3} = 0.$$

Then Eq. (24) simplifies to

$$N = \bar{u}(p')[\ell\gamma^\mu\ell - y(1-y)\not{q}\gamma^\mu\not{q} + z(1-y)\not{q}\gamma^\mu\not{p} + m_e(1-y)\not{q}\gamma^\mu - yz\not{p}\gamma^\mu\not{q} + z^2\not{p}\gamma^\mu\not{p} + m_e z\not{p}\gamma^\mu - m_e y\gamma^\mu\not{q} + m_e z\gamma^\mu\not{p} + m_e^2\gamma^\mu]u(p) \quad (25)$$

To simplify each of these terms we apply (7.87),

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu\ell^\nu}{D^3} = \int \frac{d^d\ell}{(2\pi)^d} \frac{g^{\mu\nu}\ell^2}{gD^3},$$

as well as [1, pp. 191–192]

$$\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}, \quad \not{p}u(p) = m_e u(p), \quad \bar{u}(p')\not{p}' = \bar{u}(p')m_e.$$

We know from Eqs. (2) and (3) that we only care about terms in Eq. (21) that are linear in  $\gamma^\mu$ . So the terms in Eq. (25) reduce to

$$\begin{aligned} \ell\gamma^\mu\ell &= (2\ell^\mu - \gamma^\mu\ell)\ell = 2\ell^\mu\ell^\nu\gamma_\nu - \gamma^\mu\ell\ell \rightarrow \frac{2}{d}\ell^2 g^{\mu\nu}\gamma_\nu - \gamma^\mu\ell^2 = \frac{2-d}{d}\ell^2\gamma^\mu, \\ \not{q}\gamma^\mu\not{q} &= (2q^\mu - \gamma^\mu\not{q})\not{q} \rightarrow -\gamma^\mu\not{q}\not{q} = -q^2\gamma^\mu, \\ \not{q}\gamma^\mu\not{p} &\rightarrow \not{q}\gamma^\mu m_e = (\not{p}' - \not{p})\gamma^\mu m_e \rightarrow (m_e - \not{p})\gamma^\mu m_e = m_e^2\gamma^\mu - 2m_e p^\mu + m_e\gamma^\mu\not{p} \rightarrow 2m_e^2\gamma^\mu, \\ \not{q}\gamma^\mu &= (\not{p}' - \not{p})\gamma^\mu \rightarrow m_e\gamma^\mu - \not{p}\gamma^\mu \rightarrow -2p^\mu + \gamma^\mu\not{p} \rightarrow -2p^\mu + \gamma^\mu m_e \rightarrow \gamma^\mu m_e, \\ \not{p}\gamma^\mu\not{q} &= (2p^\mu - \gamma^\mu\not{p})\not{q} \rightarrow -\gamma^\mu\not{p}\not{q} = -2\gamma^\mu p \cdot q + \gamma^\mu\not{p}\not{q} \rightarrow -2\gamma^\mu p \cdot q + \gamma^\mu(\not{p}' - \not{p})m_e \rightarrow \gamma^\mu q^2 + m_e\gamma^\mu\not{p}' - m_e^2\gamma^\mu \\ &= \gamma^\mu q^2 + m_e(2p^{\mu'} - \not{p}'\gamma^\mu) - m_e^2\gamma^\mu \rightarrow \gamma^\mu q^2 + 2m_e p^{\mu'} - 2m_e^2\gamma^\mu \rightarrow \gamma^\mu q^2 - 2m_e^2\gamma^\mu, \end{aligned}$$

where we have used

$$2p \cdot q = p \cdot q + q \cdot p = p \cdot q + p' \cdot q - q^2 = p'^2 + p' \cdot p - p' \cdot p - p^2 - q^2 \rightarrow m_e^2 - m_e^2 - q^2 = -q^2,$$

and

$$\begin{aligned} \not{p}\gamma^\mu\not{p} &\rightarrow m_e\not{p}\gamma^\mu = m_e(2p^\mu - \gamma^\mu\not{p}) \rightarrow 2m_e p^\mu - m_e^2\gamma^\mu \rightarrow -m_e^2\gamma^\mu, \\ \not{p}\gamma^\mu &= 2p^\mu - \gamma^\mu\not{p} = 2p^\mu - \gamma^\mu m_e \rightarrow -\gamma^\mu m_e, \\ \gamma^\mu\not{q} &= \gamma^\mu(\not{p}' - \not{p}) \rightarrow \gamma^\mu\not{p}' - \gamma^\mu m_e \rightarrow 2p^{\mu'} - \not{p}'\gamma^\mu \rightarrow 2p^{\mu'} - m_e\gamma^\mu \rightarrow -m_e\gamma^\mu, \\ \gamma^\mu\not{p} &\rightarrow m_e\gamma^\mu. \end{aligned}$$

Feeding these into Eq. (25), we obtain

$$N =$$

where we have set  $q^2 = 0$  as per Eq. (2).

**2(b)** Now consider the renormalization of the  $\phi\bar{\psi}\psi$  vertex. Show that the rescaling of this vertex at  $q^2 = 0$  is *not* canceled by the correction to  $Z_2$ . (It suffices to compute the ultraviolet-divergent parts of the diagrams.) In this theory, the vertex and field-strength rescaling give additional shifts of the observable coupling constant relative to its bare value.

## References

- [1] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”. Perseus Books Publishing, 1995.
- [2] Wikipedia contributors, “Gamma matrices.” From Wikipedia, the Free Encyclopedia.  
[https://en.wikipedia.org/wiki/Gamma\\_matrices](https://en.wikipedia.org/wiki/Gamma_matrices).