Problem 3. Consider a sinusoidal electromagnetic wave propagating in the $+x$ direction, whose electric field is parallel to the y axis. The wave has wavelength 475 nm, and the electric field has amplitude $3.20 \times 10^{-3}\,\mathrm{V\,m^{-1}}$. What is the frequency of the wave? What is the amplitude of the magnetic field? What are the vector equations for $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$?

Solution. Frequency is related to wavelength by $v = \lambda f$, where the wave speed $v = c$ for an electromagnetic wave in vacuum. So

$$f = \frac{c}{\lambda} = \frac{3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}}{475 \times 10^{-9}\,\mathrm{m}} = \frac{3.00 \times 10^{8}}{4.75 \times 10^{-7}}\,\mathrm{Hz} = 6.32 \times 10^{14}\,\mathrm{Hz}.$$

The amplitudes of the fields are related by $E_0 = cB_0$, so

$$B_0 = \frac{E_0}{c} = \frac{3.20 \times 10^{-3}\,\mathrm{V\,m^{-1}}}{3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}} = \frac{3.20}{3.00} \times 10^{-11}\,\mathrm{T} = 1.07 \times 10^{-11}\,\mathrm{T}.$$

where we have used $1\,\mathrm{T} = 1\,\mathrm{V\,s\,m^{-2}}$.

The direction of propagation of the wave is $\mathbf{E} \times \mathbf{B}$. We know from the problem statement that the wave is propagating in the $\hat{\mathbf{x}}$ direction, and that the electric field points in the $\hat{\mathbf{y}}$ direction. Since $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$, the magnetic field must point in the $\hat{\mathbf{z}}$ direction. Then the vector equations are

$$\mathbf{E}(x,t) = E_0 \cos(kx - \omega t)\,\hat{\mathbf{y}}, \qquad \mathbf{B}(x,t) = B_0 \cos(kx - \omega t)\,\hat{\mathbf{z}}.$$

We can find the wave number k and angular frequency ω as follows:

$$\begin{aligned} k &= \frac{2\pi}{\lambda} = \frac{2\pi\,\mathrm{rad}}{4.75 \times 10^{-7}\,\mathrm{m}} = 1.32 \times 10^{7}\,\mathrm{rad\,m^{-1}}, \\ \omega &= 2\pi f = (2\pi\,\mathrm{rad})(6.32 \times 10^{14}\,\mathrm{Hz}) = 3.97 \times 10^{15}\,\mathrm{rad\,s^{-1}}. \end{aligned}$$

Then we have

$$\begin{aligned} \mathbf{E}(x,t) &= (3.20 \times 10^{-3}\,\mathrm{V\,m^{-1}}) \cos[(1.32 \times 10^{7}\,\mathrm{rad\,m^{-1}})x - (3.97 \times 10^{15}\,\mathrm{rad\,s^{-1}})t]\,\hat{\mathbf{y}}, \\ \mathbf{B}(x,t) &= (1.07 \times 10^{-11}\,\mathrm{T}) \cos[(1.32 \times 10^{7}\,\mathrm{rad\,m^{-1}})x - (3.97 \times 10^{15}\,\mathrm{rad\,s^{-1}})t]\,\hat{\mathbf{z}}. \end{aligned}$$