Problem 1. Consider a spin-1 particle. The unperturbed Hamiltonian is $H_0 = AS_z^2$, where A is a constant. Consider the perturbation $V = B(S_x^2 - S_y^2)$, where $|A| \gg |B|$. Note that S_i are the 3×3 spin matrices.

1.1 Calculate the first-order correction to the energies.

Solution. Firstly, note that

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad S_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then

$$H_0 = A\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad V = B\frac{\hbar^2}{2} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right) = B\hbar^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of H_0 are $E_1^{(0)} = E_3^{(0)} = A\hbar^2$ and $E_2^{(0)} = 0$, so the problem is degenerate. The eigenkets are the S_z basis kets:

$$|1^{(0)}\rangle = |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad |2^{(0)}\rangle = |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad |3^{(0)}\rangle = |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We will begin with the correction to $E_2^{(0)}$, which is nondegenerate. From (5.1.20) and (5.1.37) in Sakurai, the first-order energy corrections in the unperturbed case are given by

$$\Delta_n^{(1)} \equiv E_n^{(1)} - E_n^{(0)} = \langle n^{(0)}|V|n^{(0)}\rangle .$$

This gives us

$$\Delta_2^{(1)} = \langle 2^{(0)}|V|2^{(0)}\rangle = \langle 2|V|2\rangle = 0.$$

For $E_1^{(0)}$ and $E_2^{(0)}$, consider the degenerate subspace spanned by $\{|1\rangle, |3\rangle\}$. Let P_0 be a projection onto this subspace, and let

$$V_0 = P_0 V P_0 = B\hbar^2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = B\hbar^2 \sigma_x,$$

where σ_x is the Pauli matrix. Therefore, we know that V_0 has eigenvalues $v_\pm = \pm B\hbar^2$ and eigenvectors

$$|v_+\rangle = \frac{B\hbar^2}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{B\hbar^2}{\sqrt{2}}(|1\rangle + |3\rangle), \qquad |v_-\rangle = \frac{B\hbar^2}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{B\hbar^2}{\sqrt{2}}(|3\rangle - |1\rangle).$$

In this basis, V_0 is diagonal.

1.2 Solve the problem exactly, and compare your result to the perturbation theory result.

Problem 2. Consider the Stark effect for the $n = 3$ states of hydrogen. There are initially nine degenerate states $|3, l, m\rangle$ (neglect spin), and an electric field E is turned on in the z direction.

2.1 Construct the 9×9 matrix representing the perturbing Hamiltonian in this case. Show your work when deriving the nonzero matrix elements, and provide an explanation as to why the other elements are zero.

2.2 Determine the first order corrections, $E^{(1)}$, to the energies due to this perturbation, and write down the degeneracies of these energies.

Problem 3. Consider the Hamiltonian H_0 acting on a three-dimensional Hilbert space spanned by the orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$. $H_0 = \sum_{i=3}^{3} E_i |i\rangle\langle i|$, with energy eigenvalues E_1, E_2, E_3. Assume $E_1 = E_2 = E$. To H_0, we add a perturbation

$$V = v_1 |1\rangle\langle 3| + v_1^* |3\rangle\langle 1| + v_2 |2\rangle\langle 3| + v_2^* |3\rangle\langle 2| .$$

Here, v_1 and v_2 are complex constants and small compared to E_3.

3.1 To second order in V, write down the explicit form of the effective Hamiltonian acting on the subspace spanned by $\{|1\rangle, |2\rangle\}$.

3.2 By solving the effective Hamiltonian, construct the approximate solution for the eigenvalues and eigenfunctions of $H_0 + V$. (The eigenkets only need to be constructed within the degenerate subspace.)

While writing up these solutions, I consulted Sakurai's *Modern Quantum Mechanics* and Shankar's *Principles of Quantum Mechanics*.