Problem 1. Linear sigma model (Peskin & Schroeder 4.3) The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Psi^i(x)$, $i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\boldsymbol{\nabla}\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \Phi \cdot \Phi$, and

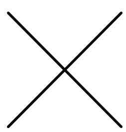
$$V(\Phi^2) = \frac{1}{2}m^2(\Phi^i)^2 + \frac{\lambda}{4}\left((\Phi^i)^2\right)^2$$

is a function symmetric under rotations of Φ. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

1(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$\contraction{}{\Phi}{^i(x)}{\Phi}\Phi^i(x)\Phi^j(y) = \delta^{ij} D_F(x-y),$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

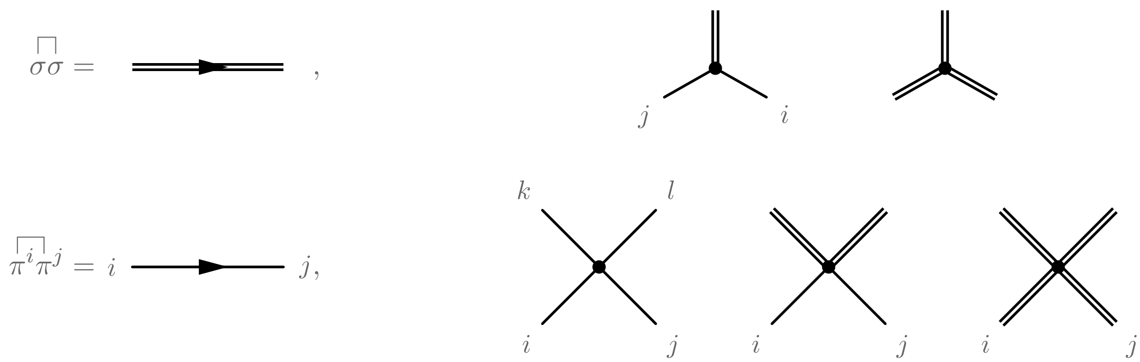
$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

1(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\Phi^i(x) = \pi(x), \qquad i = 1, \ldots, N-1, \qquad \Phi^N(x) = v + \sigma(x)$$

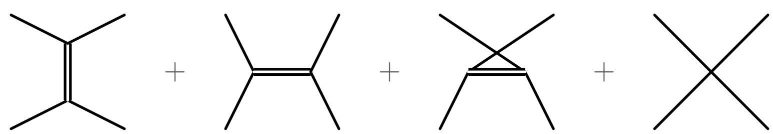
where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:



1(c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. Show that, in the special case $N = 2$ (1 species of pion), the term $\mathcal{O}(p^2)$ also cancels.

1(d) Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. Find the new value of v that minimizes V, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to a.

Problem 2. Rutherford scattering (Peskin & Schroeder 4.4) The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu,$$

where $\psi(x)$ is the usual quantized Dirac field.

2(a) Show that the T-matrix element for electron scattering off a localized classical potential is, to lowest order,

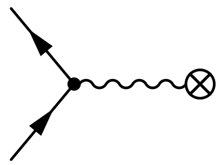
$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\cdot\tilde{A}_\mu(p'-p),$$

where $\tilde{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

2(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p' | iT | p \rangle \equiv i\mathcal{M} \cdot (2\pi)\delta(E_f - E_i),$$

where E_i and E_f are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:



$$= -ie\gamma^\mu \tilde{A}_\mu(\mathbf{q}),$$

where $\tilde{A}_\mu(\mathbf{q})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-independent, localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|\mathcal{M}(p_i \to p_f)\right|^2 (2\pi)\delta(E_f - E_i),$$

where v_i is the particle's initial velocity. This formula is a natural modification of (4.79). Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.

2(c) Specialize to the case of electron scattering from a Coulomb potential ($A^0 = Ze/4\pi r$). Working in the nonrelativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}.$$