Problem 1. Consider the following probabalistic game: There are four doors (Q, R, S, T). Behind each door is a device which displays ± 1 randomly according to the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$. Alice and Bob are on the same team. Alice has to choose either Q and R, and then Bob has to choose either S and T. When the numbers match, they get $+1$ point; when the numbers do not match, they get -1 point. However, when they open Q and T, it's an exception. When the numbers (do not) match, they get -1 $(+1)$.

1.1 Let's assume Alice and Bob open the doors completely randomly. When all numbers are $+1$ with probability 1, what is the expectation value of the point they get?

Solution. Let $\mathbf{E}$ be the expectation value of the number of points. In this case, the numbers behind the two doors will always match. So

$$\mathbf{E} = \frac{QS + RS + RT - QT}{4} = \frac{1+1+1-1}{4} = \frac{1}{2}.$$

1.2 As it turns out, irrespective of how hard you fine tune the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$, the expectation value of the point Alice and Bob get cannot exceed a certain value Max:

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} \leq \text{Max}.$$

Here, $\mathbf{E}(QS)$, etc. is the expectation value of the point when Alice opens Q and Bob opens S. This is a Bell inequality. Determine Max.

Hint: For a given realization of the numbers $Q = \pm 1$, $R = \pm 1$, $S = \pm 1$, $T = \pm 1$, which occurs with probability $P(Q, R, S, T)$, note that $QS + RS + RT - QT = (Q + R)S + (R - Q)S$, where one of $\{(R + Q), (R - Q)\}$ is 2 and the other 0.

Solution. The possiblities are listed in the following table:

Q	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	$+1$	-1
R	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	$+1$	-1	$+1$	-1	-1	-1	$+1$	-1	-1
S	$+1$	$+1$	-1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	$+1$	-1	$+1$	-1	-1	-1
T	$+1$	-1	$+1$	$+1$	$+1$	-1	-1	-1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1
$\mathbf{E}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Clearly,

$$\text{Max} = \frac{1}{2}.$$

1.3 Frustrated by the upper bound set by the Bell inequality, Bob decides to cheat. He now changes the value of T after Alice chooses Q or R. Assume Q, R, S are set to be $+1$ with probability 1. To make the expectation value of the point they get equal to $+1$, what values should Bob set after Alice chooses Q or R?

Solution. If Alice chooses R, Bob should set $T = 1$. If Alice chooses Q, Bob should set $T = -1$. This way,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1+1+1+1}{4} = 1.$$

1.4 Now consider a quantum mechanical version of the game. There are quantum states of two spin-1/2 degrees of freedom shared by Alice and Bob. Alice can measure the z component or x components of the first spin $\mathbf{S}^A$. (This corresponds to $Q = \pm 1$ or $R = \pm 1$.) Bob can measure the $-(z + x)$ component or the $(z - x)$ component of the first spin $\mathbf{S}^B$. (This corresponds to $S = \pm 1$ or $T = \pm 1$.)

More specifically, Alice and Bob share the quantum state

$$|\psi\rangle = \frac{|{\uparrow_z}\rangle \otimes |{\downarrow_z}\rangle - |{\downarrow_z}\rangle \otimes |{\uparrow_z}\rangle}{\sqrt{2}}.$$

The operators to be measured are

$$Q = S_z^A, \qquad R = S_x^A, \qquad S = -\frac{S_z^B + S_x^B}{\sqrt{2}}, \qquad T = \frac{S_x^B - S_z^B}{\sqrt{2}}.$$

Let us consider the case when Alice measures Q and Bob measures T. Calculate the probability $P(Q, T)$ for Alice and Bob getting the measurement outcomes $(Q, T) = (\pm 1, \pm 1)$.

1.5 Similarly, consider the case when Alice measures R and Bob measures T. Calculate the probability $P(R, T)$ for Alice and Bob getting the measurement outcomes $(R, T) = (\pm 1, \pm 1)$.

1.6 Compute the expectation values $\mathbf{E}(QT)$, $\mathbf{E}(RT)$, $\mathbf{E}(QS)$, and $\mathbf{E}(RS)$. Compute

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4}.$$

I consulted Sakurai's *Modern Quantum Mechanics*, Shankar's *Principles of Quantum Mechanics*, and Wolfram MathWorld while writing up these solutions.