Problem 1. (Jackson 9.1) A common textbook example of a radiating system is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is *not* of the form

$$\rho(\mathbf{x},t) = \rho(\mathbf{x})\,e^{-i\omega t}, \qquad \mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x})\,e^{-i\omega t}. \tag{1}$$

1(a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\mathbf{x},t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of Eq. (1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

Solution. The multipole moments are given by Wald (2.80),

$$q_{lm} = \int \rho(\mathbf{x}')\,r'^{l}\,Y^*_{lm}(\theta',\phi')\,d^3x'\,, \tag{2}$$

where the spherical harmonics Y_{lm} are given by Wald (2.58),

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)\,e^{im\phi}, \tag{3}$$

and P_{lm} are the associated Legendre polynomials.

We will choose coordinates for a frame K such that the charge distribution rotates counterclockwise about the z axis with angular velocity ω. Note that a point charge orbiting counterclockwise about z axis in the xy plane has the position vector

$$\mathbf{x}(t) = R\cos(\omega t)\,\hat{\mathbf{x}} + R\sin(\omega t)\,\hat{\mathbf{y}} = R\,\hat{\mathbf{r}} + \omega t\,\hat{\boldsymbol{\phi}}, \tag{4}$$

where R is its distance from the origin. This means that, if we take a stationary charge distribution $\rho(\mathbf{x})$ and set it rotating in this way, we are making the change of variable $\phi \to \phi + \omega t$. Hence $\rho(\mathbf{x},t) = \rho(r,\theta,\phi+\omega t)$.

Making this substitution into Eq. (2), we obtain the time-dependent multipole moments

$$q_{lm}(t) = \int \rho(\mathbf{x}',t)\,r'^{l}\,Y^*_{lm}(\theta',\phi')\,d^3x' = \int \rho(r',\theta',\phi'+\omega t)\,r'^{l}\,Y^*_{lm}(\theta',\phi')\,d^3x'\,.$$

To find something that resembles Eq. (1), we transform to a rotating coordinate frame $\tilde{K}$ with $(\tilde{r},\tilde{\theta},\tilde{\phi}) = (r,\theta,\phi+\omega t)$. In this frame, $\rho(\mathbf{x},t)$ is stationary and the time dependence is in Y_{lm}. In this frame, the time-dependent multipole moments are

$$\tilde{q}_{lm}(t) = \int \rho(\tilde{r}',\tilde{\theta}',\tilde{\phi}')\,r'^{l}\,Y^*_{lm}(\tilde{\theta}',\tilde{\phi}'-\omega t)\,d^3x'\,.$$

From Eq. (3),

$$Y^*_{lm}(\theta,\phi-\omega t) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)\,e^{-im(\phi-\omega t)} = e^{im\omega t}\,Y^*_{lm}(\theta,\phi),$$

so we have

$$\tilde{q}_{lm}(t) = e^{im\omega t}\int \rho(\tilde{r}',\tilde{\theta}',\tilde{\phi}')\,r'^{l}\,Y^*_{lm}(\tilde{\theta}',\tilde{\phi}')\,d^3x' = e^{im\omega t}\,\tilde{q}_{lm}.$$

Since the charge distribution is stationary in $\tilde{K}$, $\tilde{q}_{lm}(t) = q_{lm}$ and $q_{lm}(t) = \tilde{q}_{lm}$. Switching back to K, then, gives us

$$q_{lm}(t) = q_{lm}\,e^{-im\omega t},$$

which has the form of Eq. (1). Here, we have multiple frequencies $m\omega$, where $m \in [-l,l]$ for integer m. However, a frequency $m\omega$ is unphysical for $m < 0$.

To form real linear combinations, we note that $q_{lm}Y_{lm}$ is real since the scalar potential defined by Wald (2.79),

$$\Phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta,\phi),$$

is a linear combination of $q_{lm}Y_{lm}$. According to Jackson (3.54) and (4.7), $Y_{l\,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$ and $q_{l\,-m} = (-1)^m q^*_{lm}$. So

$$q_{l\,-m}(t)\, Y_{l\,-m}(\theta,\phi) = (-1)^{2m} q^*_{lm}(t)\, Y^*_{lm}(\theta,\phi) = q^*_{lm}(t)\, Y^*_{lm}(\theta,\phi).$$

Let $q_{lm}(t)\, Y_{lm}(\theta,\phi) = a + ib$, so $\mathrm{Re}[q_{lm}(t)\, Y_{lm}(\theta,\phi)] = a$ and $\mathrm{Im}[q_{lm}(t)\, Y_{lm}(\theta,\phi)] = b$. Then

$$q_{lm}(t)\, Y_{lm}(\theta,\phi) + q_{l\,-m}(t)\, Y_{l\,-m}(\theta,\phi) = a + ib + a - ib = 2a = 2\,\mathrm{Re}[q_{lm}(t)\, Y_{lm}(\theta,\phi)]. \tag{5}$$

This expression gives the real time-dependent multipole moments, which let us avoid negative frequencies.

Finally, let q_{lm} be the multipole moment corresponding to the harmonic frequency $m\omega$. These multipole moments are given by

$$q_{lm} = \begin{cases} 2\,\mathrm{Re}\left[\int \rho(\mathbf{x}',0)\, r'^{l}\, Y^*_{lm}(\theta',\phi')\, d^3x'\right] & m > 0, \\ \int \rho(\mathbf{x}',0)\, r'^{l}\, Y^*_{lm}(\theta',\phi')\, d^3x' & m = 0, \\ 0 & m < 0, \end{cases} \tag{6}$$

where $\rho = \rho(\mathbf{x}, t)$.

1(b) Consider a charge density $\rho(\mathbf{x}, t)$ that is periodic in time with period $T = 2\pi/\omega$. By making a Fourier *series* expansion, show that it can be written as

$$\rho(\mathbf{x}, t) = \rho_0(x) + \sum_{n=1}^{\infty} \mathrm{Re}[2\rho_n(\mathbf{x})\, e^{-in\omega_0 t}],$$

where

$$\rho_n(\mathbf{x}) = \frac{1}{T} \int_0^T \rho(\mathbf{x}, t)\, e^{in\omega_0 t}\, dt\,.$$

This shows explicitly how to establish connection with Eq. (1).

Solution. The Fourier series is, according to Jackson (2.36–37),

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right)\right]$$

where the interval in x is $(-a/2, a/2)$, and

$$A_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos\left(\frac{2\pi m x}{a}\right) dx\,, \qquad B_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin\left(\frac{2\pi m x}{a}\right) dx\,.$$

Making the variables changes $x \to t$, $a \to T$, and $m \to n$, we have

$$\begin{aligned} \rho(\mathbf{x}, t) &= \frac{A_0}{2} + \sum_{m=1}^{\infty} [A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)] = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} + B_n \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}\right) \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{A_n - iB_n}{2} e^{in\omega_0 t} + \frac{A_n + iB_n}{2} e^{-in\omega_0 t}\right). \end{aligned} \tag{7}$$

For the coefficients, since the integrands are T-periodic, we can freely shift the bounds of integration. Doing so,

$$A_0 = \frac{2}{T}\int_0^T \rho(\mathbf{x},t)\,dt\,, \qquad A_n = \frac{2}{T}\int_0^T \rho(\mathbf{x},t)\cos(n\omega_0 t)\,dt\,, \qquad B_n = \frac{2}{T}\int_0^T \rho(\mathbf{x},t)\sin(n\omega_0 t)\,dt\,.$$

Then the "new" coefficients are

$$\begin{aligned}
\frac{A_n - iB_n}{2} &= \frac{1}{T}\int_0^T \rho(\mathbf{x},t)[\cos(n\omega_0 t) - i\sin(n\omega_0 t)]\,dt = \frac{1}{T}\int_0^T \rho(\mathbf{x},t)\,e^{-in\omega_0 t}\,dt \equiv \rho_{-n}(\mathbf{x},t),\\
\frac{A_0}{2} &= \frac{1}{T}\int_0^T \rho(\mathbf{x},t)\,dt \equiv \rho_0(\mathbf{x},t),\\
\frac{A_n + iB_n}{2} &= \frac{1}{T}\int_0^T \rho(\mathbf{x},t)[\cos(n\omega_0 t) + i\sin(n\omega_0 t)]\,dt = \frac{1}{T}\int_0^T \rho(\mathbf{x},t)\,e^{in\omega_0 t}\,dt \equiv \rho_n(\mathbf{x},t),
\end{aligned}$$

where we have identified each with $\rho_n(\mathbf{x},t)$. We note that $\rho_{-n}(\mathbf{x},t) = \rho_n^*(\mathbf{x},t)$, and that $\rho_n(\mathbf{x},t)\,e^{in\omega_0 t}$ is real. Then, similarly to Eq. (5),

$$\rho_n(\mathbf{x},t)\,e^{in\omega_0 t} + \rho_{-n}(\mathbf{x},t)\,e^{-in\omega_0 t} = 2\,\mathrm{Re}[\rho_n(\mathbf{x},t)\,e^{-in\omega_0 t}].$$

Making these substitutions in Eq. (7), we have

$$\rho(\mathbf{x},t) = \rho_0(\mathbf{x},t) + \sum_{n=1}^{\infty}\mathrm{Re}[2\rho_n(\mathbf{x})\,e^{-in\omega_0 t}], \qquad \text{where} \quad \rho_n(\mathbf{x}) = \frac{1}{T}\int_0^T \rho(\mathbf{x},t)\,e^{in\omega_0 t}\,dt\,, \tag{8}$$

as desired. □

1(c) For a single charge q rotating about the origin in the xy plane in a circle of radius R at constant angular speed ω_0, calculate the $l = 0$ and $l = 1$ multipole moments by the methods of Probs. 1(a) and 1(b) and compare. In method (b) express the charge density $\rho_n(\mathbf{x})$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

Solution. From Eq. (4), the charge distribution for the orbiting point charge is

$$\rho(\mathbf{x},t) = \frac{q}{R^2}\delta(r-R)\,\delta(\theta-\pi/2)\,\delta(\phi-\omega_0 t) \quad\Longrightarrow\quad \rho(\mathbf{x},0) = \frac{q}{R^2}\delta(r-R)\,\delta(\theta-\pi/2)\,\delta(\phi). \tag{9}$$

We want to find q_{00}, q_{10}, q_{11}, and $q_{1\,-1}$. According to the table on p. 109 in Jackson, the relevant spherical harmonics are

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi}. \tag{10}$$

Using the method of Prob. 1(a), Eq. (6) gives us

$$\begin{aligned}
q_{00} &= \int \rho(\mathbf{x}',0)\,r'^0\,Y_{00}^*(\theta',\phi')\,d^3x' = \frac{1}{\sqrt{4\pi}}\frac{q}{R^2}\int_0^\infty \delta(r'-R)\,r'^2\,dr'\int_0^\pi \delta(\theta'-\pi/2)\,\sin\theta\,d\theta'\int_0^{2\pi}\delta(\phi')\,d\phi'\\
&= \frac{1}{\sqrt{4\pi}}\frac{q}{R^2}r^2\sin(\pi/2) = \frac{q}{\sqrt{4\pi}},
\end{aligned}$$

$$\begin{aligned}
q_{10} &= \int \rho(\mathbf{x}',0)\,r'\,Y_{10}^*(\theta',\phi')\,d^3x' = \sqrt{\frac{3}{4\pi}}\frac{q}{R^2}\int \delta(r'-R)\,r'^3\,dr'\int_0^\pi \delta(\theta'-\pi/2)\cos\theta\sin\theta\,d\theta'\int_0^{2\pi}\delta(\phi')\,d\phi'\\
&= \sqrt{\frac{3}{4\pi}}\frac{q}{R^2}R^3\cos(\pi/2)\sin(\pi/2) = 0,
\end{aligned}$$

$$q_{1\,-1} = 0,$$

$$\begin{aligned} q_{11} &= 2\,\mathrm{Re}\left[\int \rho(\mathbf{x}',0)\, r'\, Y_{11}^*(\theta',\phi')\, d^3x'\right] \\ &= -2\sqrt{\frac{3}{8\pi}}\frac{q}{R^2}\,\mathrm{Re}\left[\int_0^\infty \delta(r'-R)\, r'^3\, dr' \int_0^\pi \delta(\theta'-\pi/2)\, \sin^2\theta'\, d\theta' \int_0^{2\pi} \delta(\phi')\, e^{i\phi'}\, d\phi'\right] \\ &= -2\sqrt{\frac{3}{8\pi}}\frac{q}{R^2} R^3 \sin^2(\pi/2) = -qR\sqrt{\frac{3}{2\pi}}. \end{aligned}$$

In cylindrical coordinates (r,ϕ,z),

$$\rho(\mathbf{x},t) = \frac{q}{R^2}\delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z). \tag{11}$$

Using the method of Prob. 1(b), the right side of Eq. (8) becomes

$$\begin{aligned} \rho_n(\mathbf{x}) &= \frac{1}{T}\frac{q}{R^2}\int_0^T \delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\, e^{in\omega_0 t}\, dt = \frac{1}{T}\frac{q}{R^2}\int_0^T \delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\, e^{in\omega_0 t}\frac{d(\omega_0 t)}{\omega_0} \\ &= \frac{1}{2\pi}\frac{q}{R^2}\delta(r-R)\,\delta(z)\, e^{in\phi}. \end{aligned} \tag{12}$$

To transform the spherical harmonics to cylindrical coordinates, note that ϕ is the same in both and $z=\cos\theta$ for the unit sphere. Then Eq. (10) becomes

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}z, \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sqrt{1-z^2}\, e^{i\phi},$$

where we have used $\cos^2\theta + \sin^2\theta = 1$. Since the Y_{lm} are mutually orthogonal, the q_{lm} corresponding to $\rho_n(\mathbf{x})$ is 0 if $n \neq m$. Then, applying the left side of Eq. (8),

$$\begin{aligned} q_{00} &= \int \rho(\mathbf{x}')\, r'^0\, Y_{00}^*(\theta',\phi')\, d^3x' = \int \rho_0(\mathbf{x}')\, r'^0\, Y_{00}^*(\theta',\phi')\, d^3x' \\ &= \frac{1}{2\pi}\frac{1}{\sqrt{4\pi}}\frac{q}{R^2}\int_0^\infty \delta(r'-R)\, r'^2\, dr' \int_0^{2\pi} d\phi' \int_0^\infty \delta(z')\, dz' = \frac{1}{\sqrt{4\pi}}\frac{1}{2\pi}\frac{q}{R^2}R^2(2\pi) = \frac{q}{\sqrt{4\pi}}, \end{aligned}$$

$$\begin{aligned} q_{10} &= \int \rho(\mathbf{x}')\, r'\, Y_{10}^*(\theta',\phi')\, d^3x' = \int \rho_0(\mathbf{x}')\, r'\, Y_{10}^*(\theta',\phi')\, d^3x' \\ &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}}\frac{q}{R^2}\int_0^\infty \delta(r'-R)\, r'^3\, dr' \int_0^{2\pi} d\phi' \int_0^\infty \delta(z')\, z'\, dz' = 0, \end{aligned}$$

$$q_{1\,-1} = \int \rho(\mathbf{x}')\, r'\, Y_{1-1}^*(\theta',\phi')\, d^3x' = 0,$$

$$\begin{aligned} q_{11} &= \int \rho(\mathbf{x}')\, r'\, Y_{11}^*(\theta',\phi')\, d^3x' = \int \mathrm{Re}[2\rho_1(\mathbf{x}')]\, r'\, Y_{11}^*(\theta',\phi')\, d^3x' \\ &= -\frac{2}{\pi}\sqrt{\frac{3}{8\pi}}\frac{q}{R^2}\int_0^\infty \delta(r'-R)\, r'^3\, dr' \int_0^{2\pi} \cos\phi' e^{i\phi'}\, d\phi' \int_0^\infty \delta(z')\, \sqrt{1-z'^2}\, dz' \\ &= -\frac{2}{\pi}\sqrt{\frac{3}{8\pi}}\frac{1}{R^2}R^3\sqrt{1}\int_0^{2\pi}(\cos^2\phi' + i\cos\phi'\sin\phi')\, d\phi' = -2\frac{qR}{\pi}\sqrt{\frac{3}{8\pi}}(\pi+0) = -qR\sqrt{\frac{3}{2\pi}}, \end{aligned}$$

which are the same as those calculated using method (a).

To determine whether there are higher multipoles, we will examine Eq. (6). We know from the fact that $\rho_n(\mathbf{x},t)$ exists for all $n = m \geq 0$, so multipoles for a given m would not exist only if $P_{lm}(\cos\theta) = 0$ for all $l \geq m$.

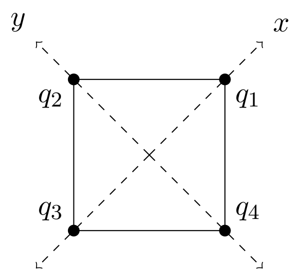
However, the associated Legendre polynomials P_{lm} are given by [1]

$$P_{lm}(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l.$$

By inspection of the derivative, there always exists some allowed (l, m) pair such that $P_{lm}(x) \neq 0$. Thus, higher multipoles exist for all frequencies $m\omega_0$ such that $m \geq 0$.

Problem 2. (Jackson 9.2) A radiating quadrupole consists of a square of side a with charges $\pm q$ at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of the radiation?

Solution. We will define the x and y axes such that the positive x axis points toward the upper-right corner of the square, and label the charges as shown below.



Note that each charge is a distance $a/\sqrt{2}$ from the origin. For a square rotating counter-clockwise, the locations of each of the point charges are

$$\mathbf{x}_1(t) = \frac{a}{\sqrt{2}}(\cos\omega t\,\hat{\mathbf{x}} + \sin\omega t\,\hat{\mathbf{y}}), \qquad \mathbf{x}_2(t) = \frac{a}{\sqrt{2}}(-\sin\omega t\,\hat{\mathbf{x}} + \cos\omega t\,\hat{\mathbf{y}}),$$
$$\mathbf{x}_3(t) = \frac{a}{\sqrt{2}}(-\cos\omega t\,\hat{\mathbf{x}} - \sin\omega t\,\hat{\mathbf{y}}), \qquad \mathbf{x}_4(t) = \frac{a}{\sqrt{2}}(\sin\omega t\,\hat{\mathbf{x}} - \cos\omega t\,\hat{\mathbf{y}}).$$

Define the charge distributions for each of the point charges as

$$\rho_1(\mathbf{x},t) = -q\,\delta(\mathbf{x}-\mathbf{x}_1), \qquad \rho_2(\mathbf{x},t) = q\,\delta(\mathbf{x}-\mathbf{x}_2), \qquad \rho_3(\mathbf{x},t) = -q\,\delta(\mathbf{x}-\mathbf{x}_3), \qquad \rho_4(\mathbf{x},t) = q\,\delta(\mathbf{x}-\mathbf{x}_4),$$

so the total charge distribution is

$$\rho(\mathbf{x},t) = \sum_i \rho_i(\mathbf{x},t).$$

The quadrupole moment is defined by Wald (2.47),

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho\, d^3x\,.$$

Note that Q_{ij} is symmetric. Its elements are

$$\begin{aligned}
Q_{11} &= \int (3x^2 - x^2 - y^2 - z^2)[\rho_1(\mathbf{x},t) + \rho_2(\mathbf{x},t) + \rho_3(\mathbf{x},t) + \rho_4(\mathbf{x},t)]\, d^3x \\
&= q\int (2x^2 - y^2 - z^2)[-\delta(\mathbf{x}-\mathbf{x}_1) + \delta(\mathbf{x}-\mathbf{x}_2) - \delta(\mathbf{x}-\mathbf{x}_3) + \delta(\mathbf{x}-\mathbf{x}_4)]\, d^3x \\
&= \frac{qa^2}{2}[-2\cos^2(\omega t) + \sin^2(\omega t) + 2\sin^2(\omega t) - \cos^2(\omega t) - 2\cos^2(\omega t) + \sin^2(\omega t) + 2\sin^2(\omega t) - \cos^2(\omega t)] \\
&= 3qa^2[\sin^2(\omega) - \cos^2(\omega t)] = -3qa^2\cos(2\omega t),
\end{aligned}$$

$$\begin{aligned}
Q_{12} &= 3\int xy[\rho_1(\mathbf{x},t)+\rho_2(\mathbf{x},t)+\rho_3(\mathbf{x},t)+\rho_4(\mathbf{x},t)]\,d^3x \\
&= 3q\int[-\delta(\mathbf{x}-\mathbf{x}_1)+\delta(\mathbf{x}-\mathbf{x}_2)-\delta(\mathbf{x}-\mathbf{x}_3)+\delta(\mathbf{x}-\mathbf{x}_4)]\,d^3x = -6qa^2\sin\omega t\cos\omega t = -3qa^2\sin 2\omega t \\
&= Q_{21},
\end{aligned}$$

$$\begin{aligned}
Q_{22} &= q\int(2y^2-x^2-z^2)[-\delta(\mathbf{x}-\mathbf{x}_1)+\delta(\mathbf{x}-\mathbf{x}_2)-\delta(\mathbf{x}-\mathbf{x}_3)+\delta(\mathbf{x}-\mathbf{x}_4)]\,d^3x = 3qa^2[\sin^2(\omega t)-\cos^2(\omega t)] \\
&= -Q_{11},
\end{aligned}$$

$$Q_{13} = Q_{23} = Q_{33} = 0.$$

So the quadrupole moment tensor is

$$Q = -3qa^2\begin{bmatrix}\cos(2\omega t) & \sin(2\omega t) & 0\\ \sin(2\omega t) & -\cos(2\omega t) & 0\\ 0 & 0 & 0\end{bmatrix} = \mathrm{Re}\left\{-3qa^2e^{-2i\omega t}\begin{bmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\right\}.$$

The magnetic induction field for quadrupole radiation is given by Jackson (9.44),

$$\mathbf{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}}),$$

where k is the wave number, $\hat{\mathbf{n}}$ is a unit vector in the direction of the observtion point, and the vector $\mathbf{Q}(\hat{\mathbf{n}})$ is defined by Jackson (9.43) with components

$$Q_i = \sum_j Q_{ij}n_j.$$

Note that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, the radial unit vector in spherical coordinates. So in Cartesian coordinates,

$$\hat{\mathbf{n}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}.$$

Also,

$$\begin{aligned}
Q_x &= -3qa^2\sin\theta e^{-2i\omega t}(\cos\phi + i\sin\phi) = -3qa^2\sin\theta\, e^{-2i\omega t}\,e^{i\phi},\\
Q_y &= -3qa^2\sin\theta e^{-2i\omega t}[i\cos\phi - \sin\phi] = -3iqa^2\sin\theta\, e^{-2i\omega t}\,e^{i\phi},\\
Q_z &= 0.
\end{aligned} \tag{13}$$

Then

$$\begin{aligned}
\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}}) &= (n_yQ_z - n_zQ_y)\,\hat{\mathbf{x}} + (n_zQ_x - n_xQ_z)\,\hat{\mathbf{y}} + (n_xQ_y - n_yQ_x)\,\hat{\mathbf{z}}\\
&= -3qa^2\sin\theta\, e^{-2i\omega t}\,e^{i\phi}[-i\cos\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}} + \sin\theta(i\cos\phi - \sin\phi)\,\hat{\mathbf{z}}\\
&= -3qa^2\sin\theta\, e^{-2i\omega t}\,e^{i\phi}(-i\cos\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}} + i\sin\theta\, e^{i\phi}\,\hat{\mathbf{z}})\\
&= -\frac{3}{2}qa^2\,e^{-2i\omega t}\,e^{i\phi}\{-i\sin(2\theta)\,\hat{\mathbf{x}} + \sin(2\theta)\,\hat{\mathbf{y}} + i[1-\cos(2\theta)]\,e^{i\phi}\,\hat{\mathbf{z}}\},
\end{aligned}$$

so

$$\mathbf{H} = \frac{ick^3}{16\pi}\frac{qa^2}{r}\exp[i(kr+\phi-2\omega t)]\{-i\sin(2\theta)\,\hat{\mathbf{x}} + \sin(2\theta)\,\hat{\mathbf{y}} + i[1-\cos(2\theta)]\,e^{i\phi}\,\hat{\mathbf{z}}\}.$$

The electric field for quadrupole radiation is given on p. 280 in the lecture notes,

$$\mathbf{E} = \frac{ik}{\mu_0}Z_0(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}} = Z_0\mathbf{H}\times\hat{\mathbf{n}} = -\frac{ick^3}{24\pi}Z_0\frac{e^{ikr}}{r}[\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}})]\times\hat{\mathbf{n}}.$$

One of the vector identities on the inside cover of Jackson is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$

so

$$[\hat{\mathbf{n}} \times \mathbf{Q}(\hat{\mathbf{n}})] \times \hat{\mathbf{n}} = -\hat{\mathbf{n}} \times [\hat{\mathbf{n}} \times \mathbf{Q}(\hat{\mathbf{n}})] = \mathbf{Q}(\hat{\mathbf{n}}) - [\hat{\mathbf{n}} \cdot \mathbf{Q}(\hat{\mathbf{n}})]\hat{\mathbf{n}}.$$

Note that

$$\hat{\mathbf{n}} \cdot \mathbf{Q}(\hat{\mathbf{n}}) = -3qa^2 \sin\theta\, e^{-2i\omega t}\, e^{i\phi}(\sin\theta\cos\phi + i\sin\theta\sin\phi) = -3qa^2\sin^2\theta\, e^{-2i\omega t}\, e^{2i\phi}$$

and

$$\begin{aligned}\mathbf{Q}(\hat{\mathbf{n}}) - [\hat{\mathbf{n}} \cdot \mathbf{Q}(\hat{\mathbf{n}})]\hat{\mathbf{n}} &= -3qa^2 \sin\theta\, e^{-2i\omega t}\, e^{i\phi}(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}}) + 3qa^2\sin^2\theta\, e^{-2i\omega t}\, e^{2i\phi}(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\ &= 3qa^2\sin\theta\, e^{-2i\omega t}\, e^{i\phi}[(\sin^2\theta\cos\phi\, e^{i\phi} - 1)\,\hat{\mathbf{x}} + (\sin^2\theta\sin\phi\, e^{i\phi} - i)\,\hat{\mathbf{y}} + \sin\theta\cos\theta\, e^{i\phi}\,\hat{\mathbf{z}}].\end{aligned}$$

Then

$$\mathbf{E} = -Z_0 \frac{ick^3}{8\pi}\frac{qa^2}{r}\sin\theta\, \exp[i(kr + \phi - 2\omega t)][(\sin^2\theta\cos\phi\, e^{i\phi} - 1)\,\hat{\mathbf{x}} + (\sin^2\theta\sin\phi\, e^{i\phi} - i)\,\hat{\mathbf{y}} + \sin\theta\cos\theta\, e^{i\phi}\,\hat{\mathbf{z}}].$$

The angular distribution of radiation for a quadrupole is given by Jackson (9.45),

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 |[\hat{\mathbf{n}} \times \mathbf{Q}(\hat{\mathbf{n}})] \times \hat{\mathbf{n}}|^2.$$

According to (9.46),

$$\begin{aligned}|[\hat{\mathbf{n}} \times \mathbf{Q}(\hat{\mathbf{n}})] \times \hat{\mathbf{n}}|^2 &= |\mathbf{Q}(\hat{\mathbf{n}})|^2 - |\hat{\mathbf{n}} \cdot \mathbf{Q}(\hat{\mathbf{n}})|^2 \\ &= \left|-3qa^2\sin\theta\, e^{-2i\omega t}\, e^{i\phi}\right|^2 + \left|-3iqa^2\sin\theta\, e^{-2i\omega t}\, e^{i\phi}\right|^2 - \left|-3qa^2\sin^2\theta\, e^{-2i\omega t}\, e^{2i\phi}\right|^2 \\ &= 9q^2q^4\sin^2\theta(2 - \sin^2\theta) = 9q^2q^4(1 - \cos^2\theta)(1 + \cos^2\theta) = 9q^2q^4(1 - \cos^4\theta),\end{aligned}$$

so

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{128\pi^2} k^6 q^2 q^4 (1 - \cos^4\theta).$$

The total radiated power for a quadrupole is given by Jackson (9.49),

$$P = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{i,j} |Q_{ij}|^2.$$

Note that

$$\sum_{i,j} |Q_{ij}|^2 = \left|-3qa^2 e^{-2i\omega t}\right|^2 [1^2 + |i|^2 + |i|^2 + (-1)^2] = 36q^2a^4,$$

so

$$P = \frac{c^2 Z_0}{40\pi} k^6 q^2 a^4.$$

Since $Q_{ij} \propto e^{2i\omega t}$, the frequency of radiation is 2ω. This means $k = 2\omega/c$.

Writing the previous results in terms of the frequency, we have

$$\mathbf{H} = \frac{i\omega^3}{2\pi c^2}\frac{qa^2}{r}\exp[i(kr + \phi - 2\omega t)]\{-i\sin(2\theta)\,\hat{\mathbf{x}} + \sin(2\theta)\,\hat{\mathbf{y}} + i[1 - \cos(2\theta)]\, e^{i\phi}\,\hat{\mathbf{z}}\},$$

$$\mathbf{E} = -Z_0 \frac{i\omega^3}{\pi c^2}\frac{qa^2}{r}\sin\theta\, \exp[i(kr + \phi - 2\omega t)][(\sin^2\theta\cos\phi\, e^{i\phi} - 1)\,\hat{\mathbf{x}} + (\sin^2\theta\sin\phi\, e^{i\phi} - i)\,\hat{\mathbf{y}} + \sin\theta\cos\theta\, e^{i\phi}\,\hat{\mathbf{z}}],$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2\pi^2 c^4}\omega^6 q^2 q^4 (1 - \cos^4\theta), \qquad P = \frac{8Z_0}{5\pi c^4}\omega^6 q^2 a^4.$$

Problem 3. **(Jackson 9.4)** Apply the approach of Prob. 1(b) to the current and magnetization densities of the particle of charge q rotating about the origin in the xy plane in a circle of radius R at constant angular speed ω_0. The motion is such that $\omega_0 R \ll c$.

3(a) Find $(J_x)_n$, $(J_y)_n$, and $(J_z)_n$ in terms of cylindrical coordinates for all n. Also determine the components of the orbital "magnetization," $(\mathbf{x}\times\mathbf{J}_n)/2$, and its divergence (which plays the role of a magnetic charge density for magnetic multipoles, as in

$$M_{lm} = -\frac{1}{l+1}\int r^l\, Y^*_{lm}(\theta,\phi)\,\boldsymbol{\nabla}\cdot(\mathbf{x}\cdot\mathbf{J})\,d^3x). \tag{14}$$

Solution. The current density is given by $\mathbf{J} = \rho\mathbf{v}$, where $\mathbf{v}$ is the velocity of the particle. Using Eq. (4),

$$\mathbf{v} = \frac{d\mathbf{x}(t)}{dt} = \frac{d}{dt}[R\cos(\omega_0 t)\,\hat{\mathbf{x}} + R\sin(\omega_0 t)\,\hat{\mathbf{y}}] = -R\omega_0\sin(\omega_0 t)\,\hat{\mathbf{x}} + R\omega\cos(\omega_0 t)\,\hat{\mathbf{y}} = R\omega_0\,\mathrm{Re}[ie^{i\omega_0 t}\,\hat{\mathbf{x}} + e^{i\omega_0 t}\,\hat{\mathbf{y}}].$$

Using the approach of Prob. 1(b) and adapting the right side of Eq. (8),

$$\begin{aligned}\mathbf{J}_n(\mathbf{x}) &= \frac{1}{T}\int_0^T \mathbf{J}(\mathbf{x},t)\,e^{in\omega_0 t}\,dt = \frac{1}{T}\int_0^T \rho(\mathbf{x},t)\,\mathbf{v}(t)\,e^{in\omega_0 t}\,dt\\ &= \frac{1}{2\pi}\frac{q\omega_0^2}{R}\int_0^T \delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\,e^{i(n+1)\omega_0 t}[i\,\hat{\mathbf{x}}+\hat{\mathbf{y}}]\,dt = \frac{1}{2\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\,e^{i(n+1)\phi}[i\,\hat{\mathbf{x}}+\hat{\mathbf{y}}]\end{aligned}$$

where $\rho(\mathbf{x},t)$ is given by Eq. (11). So the components are

$$\begin{aligned}(J_x)_n &= \frac{i}{2\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\,e^{i(n+1)\phi} = -\frac{1}{2\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\,e^{in\phi}\sin\phi,\\ (J_y)_n &= \frac{1}{2\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\,e^{i(n+1)\phi} = \frac{1}{2\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(\phi-\omega_0 t)\,\delta(z)\,e^{in\phi}\cos\phi,\\ (J_z)_n &= 0.\end{aligned}$$

Let $\mathbf{M}_n = (\mathbf{x}\times\mathbf{J}_n)/2$, and note that

$$\begin{aligned}\mathbf{M}_n &= \frac{[y\,(J_z)_n - z\,(J_y)_n]\,\hat{\mathbf{x}} + [z\,(J_x)_n - x\,(J_z)_n]\,\hat{\mathbf{y}} + [x\,(J_y)_n - y\,(J_x)_n]\,\hat{\mathbf{z}}}{2}\\ &= \frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(z)\,e^{i(n+1)\phi}[-z\,\hat{\mathbf{x}} + iz\,\hat{\mathbf{y}} + (x-iy)\,\hat{\mathbf{z}}].\end{aligned}$$

Transforming to cylindrical coordinates, in which $x - iy = r\cos\phi - ir\sin\phi = re^{-i\phi}$, the components are

$$\begin{aligned}(M_x)_n &= -\frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,z\,\delta(z)\,e^{i(n+1)\phi} = -\frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,z\,\delta(z)\,e^{in\phi}\cos\phi,\\ (M_y)_n &= \frac{i}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,z\,\delta(z)\,e^{i(n+1)\phi} = -\frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,z\,\delta(z)\,e^{in\phi}\sin\phi,\\ (M_z)_n &= \frac{1}{4\pi}\frac{q\omega_0^2}{R}r\,\delta(r-R)\,\delta(z)\,e^{in\phi}.\end{aligned}$$

To find the divergence, we will write $\mathbf{M}$ in cylindrical coordinates. According to the inside cover of Griffiths, $\hat{\mathbf{r}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$. So we have

$$\mathbf{M}_n = -\frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(z)\,e^{in\phi}(z\,\cos\phi\,\hat{\mathbf{x}} + z\,\sin\phi\,\hat{\mathbf{y}} + r\,\hat{\mathbf{z}}) = -\frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(z)\,e^{in\phi}(z\,\hat{\mathbf{r}} - r\,\hat{\mathbf{z}}).$$

The divergence in cylindrical coordinates is on the inside cover of Jackson:

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial \phi}{\partial A_\phi} + \frac{\partial A_z}{\partial z}.$$

Note that

$$\begin{aligned}
\frac{\partial}{\partial r}[r(M_r)_n] &\propto \frac{\partial}{\partial r}[r\,\delta(r-R)] = r\frac{\partial}{\partial r}[\delta(r-R)] + \delta(r-R) = \delta(r-R) - \delta(r-R) = 0,\\
\frac{\partial}{\partial z}[(M_z)_n] &\propto \frac{\partial}{\partial z}[\delta(z)] = -\delta(z)\frac{\partial}{\partial z},
\end{aligned}$$

where we have used $\delta'(x) = -\delta(x)\,d/dx$ and $x\,\delta'(x) = -\delta(x)$ [2]. So the divergence is

$$\boldsymbol{\nabla}\cdot\mathbf{M}_n = -\frac{1}{4\pi}\frac{q\omega_0^2}{R} r\,\delta(r-R)\,e^{in\phi}\,\delta(z)\frac{\partial}{\partial z}. \tag{15}$$

3(b) What long-wavelength magnetic multipoles (l, m) occur and at what frequencies? (Remember that the multipole order l does not necessarily equal the harmonic number n.)

Solution. The long-wavelength magnetic multipole moments are given by Eq. (14). (This system has no intrinsic magnetization, so this is the only relevant equation.) Writing Eq. (15), in spherical coordinates with

$$r \to r\sin\theta, \qquad z \to r\cos\theta, \qquad \frac{\partial}{\partial z} \to \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

we have

$$\boldsymbol{\nabla}\cdot\mathbf{M}_n = \frac{1}{4\pi}\frac{q\omega_0^2}{R}\delta(r-R)\,\delta(\theta-\pi/2)\,e^{in\phi}\sin^2\theta\frac{\partial}{\partial\theta}.$$

Making this substitution in Eq. (14), and noting that

$$\sin\theta\,d\theta = -\,d(\cos\theta) \qquad\Longrightarrow\qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta} = -\frac{\partial}{\partial(\cos\theta)},$$

we find

$$\begin{aligned}
M_{lm} &= -\frac{1}{l+1}\int \boldsymbol{\nabla}\cdot\mathbf{M}_n\, r^l\, Y_{lm}^*(\theta,\phi)\,d^3x\\
&= \frac{1}{4\pi(l+1)}\frac{q\omega_0^2}{R}\int \delta(r-R)\,\delta(\theta-\pi/2)\,e^{in\phi}\left[r\sin\theta\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\frac{\partial}{\partial\theta}\right] r^l\, Y_{lm}^*(\theta,\phi)\,d^3x\\
&= \frac{1}{4\pi(l+1)}\frac{q\omega_0^2}{R}\int \delta(r-R)\,\delta(\theta-\pi/2)\,e^{in\phi}\left[r\sin\theta\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\frac{\partial}{\partial\theta}\right] r^l\, Y_{lm}^*(\theta,\phi)\,d^3x\\
&= \frac{1}{4\pi(l+1)}\frac{q\omega_0^2}{R}\left[l\int \delta(r-R)\,\delta(\theta-\pi/2)\,e^{in\phi}\sin\theta\cos\theta\, r^l\, Y_{lm}^*(\theta,\phi)\,d^3x\right.\\
&\qquad\qquad\left. + \int \delta(r-R)\,\delta(\theta-\pi/2)\,e^{in\phi}\sin^3\theta\, r^l\frac{\partial Y_{lm}^*(\theta,\phi)}{\partial(\cos\theta)}\,d^3x\right].
\end{aligned}$$

Since the spherical harmonics are mutually orthogonal, all of the integrals will be nonzero only for $m = n$. By inspection, we can also associate the first integral with Y_{21} and the third with Y_{43}, where

$$Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{i\phi}, \qquad Y_{43}(\theta,\phi) = -\frac{3}{8}\sqrt{\frac{35}{\pi}}\cos\theta\sin^3\theta\,e^{3i\phi}.$$

We know from Prob. 1 that the radiation frequencies are $m\omega_0$. This means that the long-wavelength magnetic multipoles $(2,1)$ and $(4,3)$ occur with respective frequencies ω_0 and $3\omega_0$.

3(c) Use linear superposition to generalize your argument to the four charges rotating in Prob. 2 at radius $R = a/\sqrt{2}$. What harmonics occur, and what magnetic multipoles at each harmonic? Is there a magnetic multipole contribution at the $E2$ frequency of Prob. 2? Is it significant relative to the $E2$ radiation?

Solution. The $E2$ frequency is the electric quadrupole frequency [3].

References

[1] E. W. Weisstein, "Associated Legendre Polynomial." From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/AssociatedLegendrePolynomial.html`.

[2] E. W. Weisstein, "Delta Function." From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/DeltaFunction.html`.

[3] Wikipedia contributors, "Selection rule." From Wikipedia, the Free Encyclodia. `https://en.wikipedia.org/w/index.php?title=Selection_rule`.

[4] J. D. Jackson, "Classical Electrodynamics". Wiley, 3rd edition, 1999.

[5] L. D. Landau and E. M. Lifshitz, "The Classical Theory of Fields", volume 2. Butterworth Heinemann, 4th edition, 1975.

[6] R. M. Wald, "Introduction to Advanced Classical Electrodynamics". March, 2020.

[7] D. J. Griffiths, "Introduction to Electrodynamics". Prentice Hall, 3rd edition, 1999.