Problem 1. Electron-phonon interaction Write short notes explaining the physical effects that may be produced by the electron-phonon interaction in metals.

Solution. Phonons cause the crystal lattice to distort on a local scale, which moves the ions from their equilibrium positions. Since the ions carry charge, this disturbance creates an electric potential that is screened by nearby conduction electrons. The potential scatters electrons from state $\mathbf{k}$ to state $\mathbf{k}'$, which alters the density distribution of the electron gas. The disturbance in the electron density caused by the scattering may in turn create a new phonon or lattice distortion, the degree of which is determined by the phonon susceptibility of the crystal [lecture notes, p. 129–130][1, pp. 671–672][2, p. 512].

The lattice distortion created by an electron density fluctuation lasts longer than the fluctuation itself, and creates more local electron density fluctuations over its lifetime. This retarded interaction creates an effective "attraction" between conduction electrons in the metal, which can lead to superconductivity and the creation of Cooper pairs. In addition, the interaction between phonons and electrons causes electrons to effectively carry polarized lattice distortions with them as they move. This decreases their effective velocity and increases their effective mass [lecture notes, p. 130–134][1, pp. 672].

Problem 2. Electronic mass enhancement The integral in Eq. (7.10) can be approximated by neglecting the momentum dependence of the coupling constant g, and replacing the phonon frequency by the characteristic scale ω_D. Show that in this case the integral becomes

$$g^2 \int_{-\infty}^{\mu} d\epsilon' \, \frac{N(\epsilon')}{(\epsilon' - \epsilon_{\mathbf{k}})^2 - \omega_D^2}$$

where $N(\epsilon)$ is the density of states in energy.

Solution. Equation (7.10) is

$$\epsilon_{\mathbf{k}} - \mu = \epsilon^0_{\mathbf{k}} - \mu - \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{|g_{\mathbf{k}-\mathbf{k}'}|^2 n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega(\mathbf{k}-\mathbf{k}')^2}. \tag{1}$$

Applying $\omega \to \omega_D$ and neglecting the momentum dependence of g,

$$\int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{|g_{\mathbf{k}-\mathbf{k}'}|^2 n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega(\mathbf{k}-\mathbf{k}')^2} \approx g^2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D(\mathbf{k}-\mathbf{k}')^2}.$$

Since the mass enhancement only exists for states whose energies are the same within $\hbar\omega_D$ (lecture notes p. 132), $(\mathbf{k}-\mathbf{k}')^2 \approx \omega_D$. Thus

$$g^2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D(\mathbf{k}-\mathbf{k}')^2} \approx g^2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D^2}. \tag{2}$$

The definition of $n_{\mathbf{k}}$ is given on p. 91 of the lecture notes:

$$n(k) = \begin{cases} 1 & |k| < k_F, \\ 0 & \text{otherwise.} \end{cases}$$

Feeding this into Eq. (2), we can change the limits of integration to $(-k_F, k_F)$:

$$g^2 \int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D^2} = g^2 \int_{-k_F}^{k_F} \frac{d\mathbf{k}'}{(2\pi)^3} \frac{1}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D^2}. \tag{3}$$

We can transform variables by

$$N(\epsilon)\,d\epsilon = \frac{d\mathbf{k}}{(2\pi)^2}$$

from (2.10) in the lecture notes. Then Eq. (3) becomes

$$g^2 \int_{-k_F}^{k_F} \frac{d\mathbf{k}'}{(2\pi)^3} \frac{1}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D^2} = g^2 \int_{-\epsilon_F}^{\epsilon_F} d\epsilon' \, \frac{N(\epsilon')}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega_D^2}.$$

In the approximation $\epsilon_F \approx \mu$, we can replace the limits of integration by $(-\mu, \mu)$. For the lower bound, we assume that large negative do not contribute much: that is, $-\mu \to -\infty$. Then

$$\int \frac{d\mathbf{k}'}{(2\pi)^3} \frac{|g_{\mathbf{k}-\mathbf{k}'}|^2 n_{\mathbf{k}'}}{(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}'})^2 - \omega(\mathbf{k}-\mathbf{k}')^2} \approx g^2 \int_{-\infty}^{\mu} d\epsilon' \, \frac{N(\epsilon')}{(\epsilon' - \epsilon_{\mathbf{k}})^2 - \omega_D^2} \tag{4}$$

as desired. □

2(a) Since the dominant part of the integral comes from energies near the Fermi energy, we can usually replace $N(\epsilon)$ by $N(\mu)$. Making this approximation, show that for energies $|\epsilon_{\mathbf{k}} - \mu| \ll \omega_D$

$$\epsilon_{\mathbf{k}} - \mu = \frac{\epsilon^0_{\mathbf{k}} - \mu}{1 + \lambda}$$

where

$$\lambda = \frac{g^2 N(\mu)}{\omega_D^2}.$$

Solution. Applying Eq. (4) and this approximation, Eq. (1) becomes

$$\epsilon_{\mathbf{k}} - \mu = \epsilon^0_{\mathbf{k}} - \mu - g^2 N(\mu) \int_{-\infty}^{\mu} \frac{d\epsilon'}{(\epsilon' - \epsilon_{\mathbf{k}})^2 - \omega_D^2}. \tag{5}$$

Since we assume that only energies very close to the Fermi energy $\epsilon_F \approx \mu$ contribute, we approximate the integral by the indefinite integral at $\epsilon' = \mu$. Thus (using Mathematica)

$$\int_{-\infty}^{\mu} \frac{d\epsilon'}{(\epsilon' - \epsilon_{\mathbf{k}})^2 - \omega_D^2} \approx \int \frac{d\mu}{(\mu - \epsilon_{\mathbf{k}})^2 - \omega_D^2} = -\frac{1}{\omega_D} \tanh^{-1}\left(\frac{\mu - \epsilon_{\mathbf{k}}}{\omega_D}\right).$$

Since $|\epsilon_{\mathbf{k}} - \mu| \ll \omega_D$, we can perform a Taylor expansion (again using Mathematica):

$$\tanh^{-1}\left(\frac{\mu - \epsilon_{\mathbf{k}}}{\omega_D}\right) \approx \frac{\mu - \epsilon_{\mathbf{k}}}{\omega_D}.$$

So we have for Eq. (5)

$$\epsilon_{\mathbf{k}} - \mu = \epsilon^0_{\mathbf{k}} - \mu + g^2 N(\mu) \frac{\mu - \epsilon_{\mathbf{k}}}{\omega_D^2} = \epsilon^0_{\mathbf{k}} - \mu - \lambda(\epsilon_{\mathbf{k}} - \mu).$$

We assume that replacing $\epsilon_{\mathbf{k}}$ by $\epsilon^0_{\mathbf{k}}$ on the right side, thereby ignoring the ionic correction to the screening in that term [2, p. 520], is a valid approximation in this regime. Then we have

$$\epsilon_{\mathbf{k}} - \mu = (\epsilon^0_{\mathbf{k}} - \mu)(1 - \lambda) \approx \frac{\epsilon^0_{\mathbf{k}} - \mu}{1 + \lambda},$$

where we have simply used the Taylor expansion $1/(1+x) \approx 1 - x$ for small x. In doing so we have assumed λ, and therefore the mass enhancement, is small. Nevertheless, we have achieved the desired result. □

2(b) Making the approximation $N(\epsilon) \approx N(\mu)$, show that for energies $|\epsilon_{\mathbf{k}} - \mu|$ several times ω_D the correction to $\epsilon_{\mathbf{k}}$ is of order

$$\lambda \frac{\omega_D^2}{(\epsilon_{\mathbf{k}} - \mu)^2}(\epsilon_{\mathbf{k}} - \mu).$$

Solution. In this limit, we again approximate the integral in Eq. (5) as an antiderivative at $\epsilon' = \mu$. In this regime we also approximate the denominator of the integrand by $(\epsilon' - \epsilon_{\mathbf{k}})^2$, since $(\epsilon' - \epsilon_{\mathbf{k}})^2 \gg \omega_D^2$. Then

$$\int_{-\infty}^{\mu} \frac{d\epsilon'}{(\epsilon' - \epsilon_{\mathbf{k}})^2 - \omega_D^2} \approx \int \frac{d\mu}{(\mu - \epsilon_{\mathbf{k}})^2} = -\frac{1}{\mu - \epsilon_{\mathbf{k}}},$$

so the correction term in Eq. (5) becomes

$$\frac{g^2 N(\mu)}{\mu - \epsilon_{\mathbf{k}}} = -\frac{\lambda \omega_D^2}{\epsilon_{\mathbf{k}} - \mu} \propto \lambda \frac{\omega_D^2}{(\epsilon_{\mathbf{k}} - \mu)^2}(\epsilon_{\mathbf{k}} - \mu)$$

as we wanted to show. □

Problem 3. Cooper's problem The wavefunction of a Cooper pair of electrons added to the Fermi sea is

$$|\psi_C\rangle = \sum_{\mathbf{k} > k_F} g_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\mathrm{FS}\rangle,$$

where only terms in the sum for $k > k_F$ are allowed. We can now test out the pair wavefunction with the Hamiltonian

$$H = \sum_p \epsilon_p c^\dagger_p c_p + \frac{1}{2} \sum_{p,p',q} V_q c^\dagger_p c^\dagger_{p'} c_{p'-q} c_{p+q} \tag{6}$$

applied to the two electrons in question, but leaving the fermi sea inert. V_q is here taken to be an *attractive* interaction.

3(a) Show that the first term in Eq. (6) operating on $|\psi_C\rangle$ is

$$H_0 |\psi_C\rangle = \sum_{p,k,\sigma} \epsilon_p g_k c^\dagger_{p\sigma} c_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\mathrm{FS}\rangle = \sum_k 2\epsilon_{\mathbf{k}} g_k c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\mathrm{FS}\rangle. \tag{7}$$

The two terms in the last equation come because we must have either $p = k$, $\sigma = \uparrow$, or $p = -k$, $\sigma = \downarrow$ and $\epsilon_{-p} = \epsilon_p$.

Solution. For fermions, the anticommutators of the creation and annihilation operators are [3]

$$\{c_i, c_j^\dagger\} = c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}, \qquad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0.$$

Using these relations, note that

$$\begin{aligned}
c^\dagger_{p\sigma} c_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} &= c^\dagger_{p\sigma}(\delta_{pk}\delta_{\sigma\uparrow} - c^\dagger_{\mathbf{k}\uparrow} c_{p\sigma}) c^\dagger_{-\mathbf{k}\downarrow} \\
&= \delta_{pk}\delta_{\sigma\uparrow} c^\dagger_{p\sigma} c^\dagger_{-\mathbf{k}\downarrow} - c^\dagger_{p\sigma} c^\dagger_{\mathbf{k}\uparrow}(\delta_{p,-k}\delta_{\sigma\downarrow} - c^\dagger_{-\mathbf{k}\downarrow} c_{p\sigma}) \\
&= \delta_{pk}\delta_{\sigma\uparrow} c^\dagger_{p\sigma} c^\dagger_{-\mathbf{k}\downarrow} - \delta_{p,-k}\delta_{\sigma\downarrow} c^\dagger_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} + c^\dagger_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{p\sigma}.
\end{aligned}$$

Feeding this into Eq. (7),

$$\begin{aligned}
H_0 |\psi_C\rangle &= \sum_{p,k,\sigma} \epsilon_p g_k c^\dagger_{p\sigma} c_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\text{FS}\rangle \\
&= \sum_{p,k,\sigma} \epsilon_p g_k (\delta_{pk}\delta_{\sigma\uparrow} c^\dagger_{p\sigma} c^\dagger_{-\mathbf{k}\downarrow} - \delta_{p,-k}\delta_{\sigma\downarrow} c^\dagger_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} + c^\dagger_{p\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{p\sigma}) |\text{FS}\rangle \\
&= \sum_{p,k,\sigma} (\epsilon_p g_k \delta_{pk}\delta_{\sigma\uparrow} c^\dagger_{p\sigma} c^\dagger_{-\mathbf{k}\downarrow} - \epsilon_{-p} g_k \delta_{p,-k}\delta_{\sigma\downarrow} c^\dagger_{p\sigma} c^\dagger_{\mathbf{k}\uparrow}) |\text{FS}\rangle \\
&= \sum_{k} (\epsilon_{\mathbf{k}} g_k c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \epsilon_{\mathbf{k}} g_{-k} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow}) |\text{FS}\rangle \\
&= \sum_{k} \epsilon_{\mathbf{k}} g_k (c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \delta_{k,-k}\delta_{\uparrow\downarrow} + c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) |\text{FS}\rangle \\
&= 2\sum_{k} \epsilon_{\mathbf{k}} g_k c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\text{FS}\rangle \,,
\end{aligned}$$

where in the second term we have used $\epsilon_p = \epsilon_{-p}$. □

3(b) Similarly, show that the operation of the second term in Eq. (6) gives

$$H_{\text{int}} = \sum_{k,p,p',q,\sigma,\sigma'} V_q g_k c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} \delta_{p+q,k}\delta_{\sigma\uparrow}\delta_{p'-q,-k}\delta_{\sigma'\downarrow} |\text{FS}\rangle = \sum_{k,k'<k_F} V_{k-k'} g_{k'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} |\text{FS}\rangle \,. \tag{8}$$

Getting to the final equation involves a little crafty relabeling of the momenta in the sum. This gets us to the two-particle Schrödinger equation Eq. (7.19).

Solution. Note that

$$\begin{aligned}
c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c_{p+q,\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} &= c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} (\delta_{p+q,k}\delta_{\sigma\uparrow} - c^\dagger_{\mathbf{k}\uparrow} c_{p+q,\sigma}) c^\dagger_{-\mathbf{k}\downarrow} \\
&= \delta_{p+q,k}\delta_{\sigma\uparrow} c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c^\dagger_{-\mathbf{k}\downarrow} - c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c^\dagger_{\mathbf{k}\uparrow} c_{p+q,\sigma} c^\dagger_{-\mathbf{k}\downarrow}.
\end{aligned} \tag{9}$$

For the first term of Eq. (9),

$$c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c^\dagger_{-\mathbf{k}\downarrow} = c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} (\delta_{p'-q,-k}\delta_{\sigma'\downarrow} - c^\dagger_{-\mathbf{k}\downarrow} c_{p'-q,\sigma'}) = \delta_{p'-q,-k}\delta_{\sigma'\downarrow} c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} - c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c^\dagger_{-\mathbf{k}\downarrow} c_{p'-q,\sigma'}.$$

For the second term of Eq. (9),

$$\begin{aligned}
c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c^\dagger_{\mathbf{k}\uparrow} c_{p+q,\sigma} c^\dagger_{-\mathbf{k}\downarrow} &= c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} (\delta_{p'-q,k}\delta_{\sigma'\uparrow} - c^\dagger_{\mathbf{k}\uparrow} c_{p'-q,\sigma'})(\delta_{p+q,-k}\delta_{\sigma\downarrow} - c^\dagger_{-\mathbf{k}\downarrow} c_{p+q,\sigma}) \\
&= c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} (\delta_{p'-q,k}\delta_{\sigma'\uparrow}\delta_{p+q,-k}\delta_{\sigma\downarrow} - \delta_{p'-q,k}\delta_{\sigma'\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{p+q,\sigma} \\
&\qquad - \delta_{p+q,-k}\delta_{\sigma\downarrow} c^\dagger_{\mathbf{k}\uparrow} c_{p'-q,\sigma'} + c^\dagger_{\mathbf{k}\uparrow} c_{p'-q,\sigma'} c^\dagger_{-\mathbf{k}\downarrow} c_{p+q,\sigma}).
\end{aligned}$$

For the last term here,

$$c^\dagger_{\mathbf{k}\uparrow} c_{p'-q,\sigma'} c^\dagger_{-\mathbf{k}\downarrow} c_{p+q,\sigma} = c^\dagger_{\mathbf{k}\uparrow} (\delta_{p'-q,-k}\delta_{\sigma'\downarrow} - c^\dagger_{-\mathbf{k}\downarrow} c_{p'-q,\sigma'}) c_{p+q,\sigma} = \delta_{p'-q,-k}\delta_{\sigma'\downarrow} c^\dagger_{\mathbf{k}\uparrow} c_{p+q,\sigma} - c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{p'-q,\sigma'} c_{p+q,\sigma}.$$

We neglect the terms that end in an annihilation operator, since they will disappear when they act on $|\text{FS}\rangle$. So Eq. (9) becomes (loosely speaking)

$$\begin{aligned}
c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c_{p+q,\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} &= \delta_{p+q,k}\delta_{\sigma\uparrow}\delta_{p'-q,-k}\delta_{\sigma'\downarrow} c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} - \delta_{p'-q,k}\delta_{\sigma'\uparrow}\delta_{p+q,-k}\delta_{\sigma\downarrow} c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} \\
&= \delta_{p+q,k}\delta_{\sigma\uparrow}\delta_{p'-q,-k}\delta_{\sigma'\downarrow} c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} - \delta_{p'-q,k}\delta_{\sigma'\uparrow}\delta_{p+q,-k}\delta_{\sigma\downarrow} (\delta_{pp'}\delta_{\sigma\sigma'} - c^\dagger_{p'\sigma'} c^\dagger_{p\sigma}) \\
&= 2\delta_{p+q,k}\delta_{\sigma\uparrow}\delta_{p'-q,-k}\delta_{\sigma'\downarrow} c^\dagger_{p\sigma} c^\dagger_{p'\sigma'}.
\end{aligned}$$

Thus the operation of the second term in Eq. (6) is

$$H_{\text{int}} = \frac{1}{2}\sum_{k,p,p',q,\sigma,\sigma'} V_q g_k c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} c_{p'-q,\sigma'} c_{p+q,\sigma} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \left|\text{FS}\right\rangle = \sum_{k,p,p',q,\sigma,\sigma'} V_q g_k c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} \delta_{p+q,k}\delta_{\sigma\uparrow}\delta_{p'-q,-k}\delta_{\sigma'\downarrow} \left|\text{FS}\right\rangle$$

as in Eq. (8). Rewriting the Kronecker deltas, we have

$$H_{\text{int}} = \sum_{k,p,p',q,\sigma,\sigma'} V_q g_k c^\dagger_{p\sigma} c^\dagger_{p'\sigma'} \delta_{p,k-q}\delta_{\sigma\uparrow}\delta_{p',q-k}\delta_{\sigma'\downarrow} \left|\text{FS}\right\rangle = \sum_{k,q} V_q g_k c^\dagger_{k-q,\uparrow} c^\dagger_{q-k,\downarrow} \left|\text{FS}\right\rangle .$$

Let $q = k - k'$. Then

$$H_{\text{int}} = \sum_{k,k'<k_F} V_{k-k'} g_k c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \left|\text{FS}\right\rangle .$$

Swapping the labels k and k', and assuming $V_{k-k'} = V_{k'-k}$, we have

$$H_{\text{int}} = \sum_{k,k'<k_F} V_{k-k'} g_{k'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \left|\text{FS}\right\rangle$$

as we wanted to show. □

References

[1] C. Kittel, "Introduction to Solid State Physics". Wiley, 8th edition, 2005.

[2] N. W. Ashcroft and N. D. Mermin, "Solid State Physics". Harcourt College Publishers, 1976.

[3] Wikipedia contributors, "Creation and annihilation operators." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Creation_and_annihilation_operators`.