Problem 1. Exotic contributions to g − 2 (Peskin & Schroeder 6.3) Any particles that couples to the electron can produce a correction to the electron-photon form factors and, in particular, a correction to $g-2$. Because the electron $g-2$ agrees with QED to high accuracy, these corrections allow us to constrain the properties of hypothetical new particles.

1(a) The unified theory of weak and electromagnetic interactions contains a scalar particle h called the *Higgs boson*, which couples to the electron according to

$$H_{\text{int}} = \int d^3x \, \frac{\lambda}{\sqrt{2}} h\bar{\psi}\psi.$$

Compute the contribution of a virtual Higgs boson to the electron $(g-2)$, in terms of λ and the mass m_h of the Higgs boson.

Solution. The Higgs field is a scalar Yukawa field, so we can use the form of the Yukawa interaction Hamiltonian of Peskin & Schroeder (4.112) and the appropriate Feynman rules to write [1, p. 118]

$$(\text{vertex}) = -i\frac{\lambda}{\sqrt{2}}, \qquad\qquad (\text{propagator}) = \frac{i}{q^2 - m_h^2 + i\epsilon}.$$

We are interested in the diagram (draw it)

This is similar to the one on p. 189 in Peskin & Schroeder. We can then adapt (6.38) to write

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2 - m_h^2 + i\epsilon} \bar{u}(p') \left(-i\frac{\lambda}{\sqrt{2}}\right) \frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon} \left(-i\frac{\lambda}{\sqrt{2}}\right) u(p) \\
&= i\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)}{[(k-p)^2 - m_h^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} u(p). \qquad (1)
\end{aligned}$$

To evaluate the integral, we use Peskin & Schroeder (6.41) with $n = 3$:

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^n}.$$

Applying this to the denominator of the integrand of Eq. (1) gives us

$$\frac{1}{[(k-p)^2 - m_h^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{D^3}, \qquad (2)$$

where [1, pp. 190–191]

$$\begin{aligned}
D &= x(k^2 - m_e^2) + y(k'^2 - m_e^2) + z[(k-p)^2 - m_h^2] + (x+y+z)i\epsilon \\
&= x(k^2 - m_e^2) + y(k^2 + 2kq + q^2 - m_e^2) + z(k^2 - 2kp + p^2 - m_h^2) + i\epsilon \\
&= (x+y+z)k^2 - (x+y)m_e^2 + 2k(qy - pz) + z(p^2 - m_h^2) + i\epsilon \\
&= k^2 + 2k(qy - pz) + z(p^2 - m_h^2) - (1-z)m_e^2 + i\epsilon.
\end{aligned}$$

Here we have used $x + y + z = 1$ and $k' = k + q$. Let $\ell \equiv k + yq - zp$ [1, p. 191]. Then

$$D = \ell^2 + xyq^2 - (1-z)^2 m_e^2 - m_h^2 z + i\epsilon \equiv \ell^2 - \Delta + i\epsilon, \qquad (3)$$

where we have defined $\Delta \equiv -xyq^2 + (1-z)^2 m_e^2 + z m_h^2$ [1, p. 191].

For the numerator of Eq. (1), let $N \equiv \bar{u}(p')(\not{k}'+m_e)\gamma^\mu(\not{k}+m_e)u(p)$. Then using $k' = k+q$ and $\ell \equiv k+yq-zp$ [1, p. 191],

$$N = \bar{u}(p')(\not{k}+\not{q}+m_e)\gamma^\mu(\not{k}+m_e)u(p) = \bar{u}(p')[\not{\ell}+(1-y)\not{q}+z\not{p}+m_e]\gamma^\mu(\not{\ell}-y\not{q}+z\not{p}+m_e)u(p). \tag{4}$$

We should be able to write this as an expression of the form given in (6.31) of Peskin & Schroeder [1, p. 191],

$$\Gamma^\mu = \gamma^\mu \cdot A + (p^{\mu\prime}+p^\mu)\cdot B + (p^{\mu\prime}-p^\mu)\cdot C = \gamma^\mu \cdot A + (p^{\mu\prime}+p^\mu)\cdot B + q^\mu \cdot C. \tag{5}$$

But from (6.45),

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu}{D^3} = 0.$$

This means we can discard terms of $\mathcal{O}(\ell)$. We also know from (6.33) that

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_e}F_2(q^2). \tag{6}$$

Since the correction to $g-2$ is given by $F_2(q^2=0) = 0 + \delta F_2(q^2=0)$ (since $F_2 = 0$ to lowest order), we can discard terms of $\mathcal{O}(\gamma^\mu)$ in Eq. (4) [1, pp. 186, 196]. So Eq. (4) becomes

$$\begin{aligned} N &= \bar{u}(p')[\not{\ell}+(1-y)\not{q}+z\not{p}+m_e]\gamma^\mu(\not{\ell}-y\not{q}+z\not{p}+m_e)u(p) \\ &= \bar{u}(p')[\not{\ell}\gamma^\mu\not{\ell} - y(1-y)\not{q}\gamma^\mu\not{q} + z(1-y)\not{q}\gamma^\mu\not{p} + m_e(1-y)\not{q}\gamma^\mu - yz\not{p}\gamma^\mu\not{q} \\ &\qquad + z^2\not{p}\gamma^\mu\not{p} + m_e z\not{p}\gamma^\mu - m_e y\gamma^\mu\not{q} + m_e z\gamma^\mu\not{p}]u(p). \end{aligned} \tag{7}$$

To simplify these terms, we use Peskin & Schroeder (6.46):

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu\ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4}\frac{g^{\mu\nu}\ell^2}{4D^3},$$

as well as [1, pp. 191–192]

$$\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}, \qquad \not{p}u(p) = m_e u(p), \qquad \bar{u}(p')\not{p}' = \bar{u}(p')m_e$$

and [2?]

$$\not{a}\not{b} = a\cdot b, \qquad \not{a}\not{b}+\not{b}\not{a} = 2a\cdot b.$$

We find

$$\begin{aligned} \not{\ell}\gamma^\mu\not{\ell} &= (2\ell^\mu - \gamma^\mu\not{\ell})\not{\ell} = 2\ell^\mu\ell^\nu\gamma_\nu - \gamma^\mu\not{\ell}\not{\ell} \to \frac{\ell^2 g^{\mu\nu}\gamma_\nu}{2} - \gamma^\mu\ell^2 = -\frac{\ell^2\gamma^\mu}{2} \\ &\to 0, \\ \not{q}\gamma^\mu\not{q} &= (2q^\mu - \gamma^\mu\not{q})\not{q} \to -\gamma^\mu\not{q}\not{q} = -q^2\gamma^\mu \\ &\to 0, \\ \not{q}\gamma^\mu\not{p} &\to \not{q}\gamma^\mu m_e = (\not{p}'-\not{p})\gamma^\mu m_e \to (m_e - \not{p})\gamma^\mu m_e = m_e^2\gamma^\mu - 2m_e p^\mu + m_e\gamma^\mu\not{p} \\ &\to -2m_e p^\mu, \\ \not{q}\gamma^\mu &= (\not{p}'-\not{p})\gamma^\mu \to m_e\gamma^\mu - \not{p}\gamma^\mu \to -2p^\mu + \gamma^\mu\not{p} \to -2p^\mu + \gamma^\mu m_e \\ &\to -2p^\mu, \\ \not{p}\gamma^\mu\not{q} &= (2p^\mu - \gamma^\mu\not{p})\not{q} \to -\gamma^\mu\not{p}\not{q} = -2\gamma^\mu p\cdot q + \gamma^\mu\not{q}\not{p} \to -2\gamma^\mu p\cdot q + \gamma^\mu(\not{p}'-\not{p})m_e \\ &\to \gamma^\mu q^2 + m_e\gamma^\mu\not{p}' - m_e^2\gamma^\mu = \gamma^\mu q^2 + m_e(2p^{\mu\prime} - \not{p}'\gamma^\mu) - m_e^2\gamma^\mu \to \gamma^\mu q^2 + 2m_e p^{\mu\prime} - 2m_e^2\gamma^\mu \\ &\to 2m_e p^{\mu\prime}, \end{aligned}$$

where we have used

$$2p\cdot q = p\cdot q + q\cdot p = p\cdot q + p'\cdot q - q^2 = p'^2 + p'\cdot p - p'\cdot p - p^2 - q^2 \to m_e^2 - m_e^2 - q^2 = -q^2,$$

and

$$\begin{aligned}
&\not{p}\gamma^\mu\not{p} \to m_e\not{p}\gamma^\mu = m_e(2p^\mu - \gamma^\mu\not{p}) \to 2m_e p^\mu - m_e^2\gamma^\mu \to 2m_e p^\mu,\\
&\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p} = 2p^\mu - \gamma^\mu m_e \to 2p^\mu,\\
&\gamma^\mu\not{q} = \gamma^\mu(\not{p}' - \not{p}) \to \gamma^\mu\not{p}' - \gamma^\mu m_e \to 2p^{\mu\prime} - \not{p}'\gamma^\mu \to 2p^{\mu\prime} - m_e\gamma^\mu \to 2p^{\mu\prime},\\
&\gamma^\mu\not{p} \to 0.
\end{aligned}$$

Feeding these into Eq. (7), we obtain

$$\begin{aligned}
N &= \bar{u}(p')[-2m_e z(1-y)p^\mu - 2m_e(1-y)p^\mu - 2m_e yzp^{\mu\prime} + 2m_e z^2 p^\mu + 2m_e z p^\mu - 2m_e y p^{\mu\prime}]u(p)\\
&= 2m_e\bar{u}(p')\{[z^2 + z - z(1-y) - (1-y)]p^\mu - y(1+z)p^{\mu\prime}\}u(p)\\
&= 2m_e\bar{u}(p')\{[z^2 + y(1+z) - 1]p^\mu - y(1+z)p^{\mu\prime}\}u(p)\\
&= 2m_e\bar{u}(p')[(z^2-1)p^\mu + y(1+z)(p^\mu - p^{\mu\prime})]u(p)\\
&= m_e\bar{u}(p')[(z^2-1)p^\mu + (z^2-1)p^\mu + 2y(1+z)(p^\mu - p^{\mu\prime}) + (z^2-1)p^{\mu\prime} - (z^2-1)p^{\mu\prime}]u(p)\\
&= m_e\bar{u}(p')[(z^2-1)(p^\mu + p^{\mu\prime}) + (z^2-1)(p^\mu - p^{\mu\prime}) + 2y(1+z)(p^\mu - p^{\mu\prime})]u(p)\\
&= m_e\bar{u}(p')[(z^2-1)(p^\mu + p^{\mu\prime}) - (z^2 + 2y(1+z) - 1)(p^{\mu\prime} - p^\mu)]u(p),
\end{aligned} \tag{8}$$

which has the form of the second two terms of Eq. (5). According to the Ward identity, the coefficient of $q^\mu = p^\mu - p^\mu$ vanishes [1, p. 192]. Further, according to the Gordon identity given by Peskin & Schroeder (6.32),

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{p^{\mu\prime} + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p).$$

So Eq. (8) becomes

$$N = im_e\bar{u}(p')(1-z^2)\sigma^{\mu\nu}g^{\mu\nu}u(p). \tag{9}$$

Feeding Eqs. (2), (3), and (9) into Eq. (1), we have (ignoring the $\mathcal{O}(\gamma^\mu)$ term)

$$\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) \to i\frac{\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p')\frac{im_e(1-z^2)\sigma^{\mu\nu}g^{\mu\nu}}{(\ell^2-\Delta+i\epsilon)^3}u(p).$$

From Eq. (6), we can write

$$\delta F_2(q^2) = i\frac{\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2m_e^2(1-z^2)}{(\ell^2-\Delta+i\epsilon)^3}.$$

Computing the integral using Peskin & Schroeder (6.49),

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}},$$

we find

$$\delta F_2(q^2) = \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{m_e^2(1-z^2)}{\Delta}$$

so

$$\delta F_2(q^2=0) = \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{m_e^2(1-z^2)}{(1-z)^2m_e^2 + zm_h^2}$$

where we have used Eq. (3).

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] Wikipedia contributors, "Gamma matrices." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Gamma_matrices`.