

1 Reduced three-body problem

The problem of three point particles interacting gravitationally has a particularly simple limit: let the third body $m_3 \ll m_2, m_1$ so that its effect on the motions of m_1 and m_2 is negligible. Assume in addition that m_3 moves in the same orbital plane as m_1 and m_2 . For simplicity, consider only the case of m_1 and m_2 in circular orbit about their center of mass.

1.1 Switch into a reference frame rotating with angular velocity ω associated with the circular orbit for the two-body problem. Choose the center of mass of the two-body problem to be the origin. Choose the x axis to go through m_1 and m_2 . Show that the (now stationary) m_1 and m_2 are located at $-r_c\mu/m_1$ and $r_c\mu/m_2$.

Solution. Call the stationary coordinate system $\mathbf{R} = (X, Y, Z)$, and choose (X, Y) as the orbital plane. Call the rotating coordinate system $\mathbf{r} = (x, y, z)$, which is rotated about the Z axis by angle ωt . This gives us the transformation

$$x = X \cos \omega t + Y \sin \omega t, \quad (1)$$

$$y = Y \cos \omega t - X \sin \omega t, \quad (2)$$

$$z = Z. \quad (3)$$

From our choice of orbital plane, there is no motion in the z direction. Let the locations of m_1 and m_2 be given by $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in the rotating frame.

From our choice of x axis, we know that $y_1 = y_2 = 0$. From our choice of the origin as the center of mass, we have

$$m_1 x_1 + m_2 x_2 = 0. \quad (4)$$

By construction, m_1 and m_2 are stationary in the rotating frame, so $dx_1/dt = dx_2/dt = 0$. In other words, x_1 and x_2 must both be constant. Therefore, let

$$r_c = x_2 - x_1 \quad (5)$$

be the constant distance between m_1 and m_2 . Now we have the system of two equations (4) and (5), so we can solve for x_1 and x_2 . Substituting (5) as $x_2 = r_c + x_1$ into (4),

$$m_1 x_1 + m_2 (r_c + x_1) = 0 \implies x_1 = -\frac{m_2}{m_1 + m_2} r_c. \quad (6)$$

Now substituting (6) back into (4),

$$r_c = x_2 + \frac{m_2}{m_1 + m_2} r_c \implies x_2 = \frac{m_1}{m_1 + m_2} r_c. \quad (7)$$

Note that the reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$. Substituting μ into (6) and (7) yields

$$\mathbf{r}_1 = x_1 = -r_c \mu / m_1, \quad \mathbf{r}_2 = x_2 = r_c \mu / m_2 \quad (8)$$

as desired. \square

1.2 Show that the Lagrangian governing the equation of motion of m_3 at location $(x(t), y(t))$ is

$$L_3 = \frac{m_3}{2} [(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2] - U_{13} - U_{23}, \quad (9)$$

where $U_{13}(x, y)$ is the gravitational interaction of m_3 with m_1 , while $U_{23}(x, y)$ is associated with m_3 and m_2 .

Solution. In general, the Lagrangian for m_3 is given by

$$L_3 = T_3 - U_3, \quad (10)$$

where T_3 is the kinetic energy of m_3 and U_3 is its potential energy.

Beginning with U_3 , the only forces acting upon m_3 are the gravitational interactions with m_1 and m_2 . We know from the problem statement that these interactions are independent of each other; m_3 has a negligible effect on the motions of each m_1 and m_2 , so it cannot couple them in any way. Thus, we can write

$$U_3 = -G \frac{m_1 m_3}{r_{13}} - G \frac{m_2 m_3}{r_{23}} \equiv U_{13} + U_{23}, \quad (11)$$

where r_{13} (r_{23}) is the separation between m_3 and m_1 (m_2), and we have defined U_{13} and U_{23} .

Now we will find an expression for T_3 . Let $\mathbf{R}_3 = (X(t), Y(t))$ be the position of m_3 in the stationary coordinate system. Then

$$T_3 = \frac{m_3}{2} \dot{\mathbf{R}}_3^2 = \frac{m_3}{2} (\dot{X} + \dot{Y})^2. \quad (12)$$

We want to find an expression for T_3 in the rotating coordinate system. We can define an inverse transformation back to the stationary coordinate system by simply rotating the (x, y) plane about the z axis in the opposite direction; that is, by angle $-\omega t$. This inverse transformation is

$$X = x \cos \omega t - y \sin \omega t, \quad (13)$$

$$Y = x \sin \omega t + y \cos \omega t, \quad (14)$$

$$Z = z. \quad (15)$$

It follows from (13) and (14) that

$$\dot{X} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x} \frac{dx}{dt} + \frac{\partial X}{\partial y} \frac{dy}{dt} = -\omega x \sin \omega t - \omega y \cos \omega t + \dot{x} \cos \omega t - \dot{y} \sin \omega t \quad (16)$$

$$= (\dot{x} - \omega y) \cos \omega t - (\dot{y} + \omega x) \sin \omega t, \quad (17)$$

$$\dot{Y} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x} \frac{dx}{dt} + \frac{\partial Y}{\partial y} \frac{dy}{dt} = \omega x \cos \omega t - \omega y \sin \omega t + \dot{x} \sin \omega t + \dot{y} \cos \omega t \quad (18)$$

$$= (\dot{y} + \omega x) \cos \omega t + (\dot{x} - \omega y) \sin \omega t. \quad (19)$$

Now using the forms of (16) and (18),

$$\dot{X}^2 = (\dot{x} - \omega y)^2 \cos^2 \omega t - 2(\dot{x} - \omega y)(\dot{y} + \omega x) \cos \omega t \sin \omega t + (\dot{y} + \omega x)^2 \sin^2 \omega t, \quad (20)$$

$$\dot{Y}^2 = (\dot{y} + \omega x)^2 \cos^2 \omega t + 2(\dot{x} - \omega y)(\dot{y} + \omega x) \cos \omega t \sin \omega t + (\dot{x} - \omega y)^2 \sin^2 \omega t, \quad (21)$$

which implies

$$\dot{X}^2 + \dot{Y}^2 = (\dot{x} - \omega y)^2 (\cos^2 \omega t + \sin^2 \omega t) + (\dot{y} + \omega x)^2 (\cos^2 \omega t + \sin^2 \omega t) \quad (22)$$

$$= (\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2. \quad (23)$$

Substituting (23) into (12), we have

$$T_3 = \frac{m_3}{2} [(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2]. \quad (24)$$

Finally, substituting (11) and (24) into (10) yields (9) as desired. \square

1.3 Show that the mechanical system described by L_3 has five locations in mechanical equilibrium. These are known as *Lagrange points*. (Hint: the graphical method is perfectly good for demonstrating a real root exists in a particular instance.)

Solution. The Euler-Lagrange equations for m_3 are given by

$$0 = \frac{\partial L_3}{\partial x} - \frac{d}{dt} \frac{\partial L_3}{\partial \dot{x}}, \quad (25)$$

$$0 = \frac{\partial L_3}{\partial y} - \frac{d}{dt} \frac{\partial L_3}{\partial \dot{y}}. \quad (26)$$

We will attack each term of the Lagrangian in (10) separately. Beginning with T_3 , note that

$$\frac{\partial T_3}{\partial x} = m_3(\omega \dot{y} + \omega^2 x), \quad \frac{\partial T_3}{\partial y} = m_3(-\omega \dot{x} + \omega^2 y), \quad (27)$$

$$\frac{\partial T_3}{\partial \dot{x}} = m_3(\dot{x} - \omega y), \quad \frac{\partial T_3}{\partial \dot{y}} = m_3(\dot{y} + \omega x). \quad (28)$$

In turn, (28) implies

$$\frac{d}{dt} \frac{\partial T_3}{\partial \dot{x}} = m_3(\ddot{x} - \omega \dot{y}), \quad \frac{d}{dt} \frac{\partial T_3}{\partial \dot{y}} = m_3(\ddot{y} + \omega \dot{x}). \quad (29)$$

Now for U_3 , we can find explicitly the r_{13} and r_{23} appearing in (11) using the positions of m_1 and m_2 on the x axis given by (8). These are

$$r_{13} = \sqrt{\left(x + \frac{r_c \mu}{m_1}\right)^2 + y^2}, \quad r_{23} = \sqrt{\left(x - \frac{r_c \mu}{m_2}\right)^2 + y^2}. \quad (30)$$

It follows from (30) that

$$\frac{\partial r_{13}}{\partial x} = \frac{1}{r_{13}} \left(x + \frac{r_c \mu}{m_1}\right), \quad \frac{\partial r_{23}}{\partial x} = \frac{1}{r_{23}} \left(x - \frac{r_c \mu}{m_2}\right), \quad (31)$$

$$\frac{\partial r_{13}}{\partial y} = \frac{y}{r_{13}}, \quad \frac{\partial r_{23}}{\partial y} = \frac{y}{r_{23}}, \quad (32)$$

$$\frac{\partial r_{13}}{\partial \dot{x}} = \frac{\partial r_{23}}{\partial \dot{x}} = \frac{\partial r_{13}}{\partial \dot{y}} = \frac{\partial r_{23}}{\partial \dot{y}} = 0. \quad (33)$$

Then

$$\frac{\partial U_{13}}{\partial x} = \frac{\partial U_{13}}{\partial r_{13}} \frac{\partial r_{13}}{\partial x} = G \frac{m_1 m_3}{r_{13}^3} \left(x + \frac{r_c \mu}{m_1} \right), \quad \frac{\partial U_{23}}{\partial x} = G \frac{m_2 m_3}{r_{23}^3} \left(x - \frac{r_c \mu}{m_2} \right), \quad (34)$$

$$\frac{\partial U_{13}}{\partial y} = \frac{\partial U_{13}}{\partial r_{13}} \frac{\partial r_{13}}{\partial y} = G \frac{m_1 m_3}{r_{13}^3} y, \quad \frac{\partial U_{23}}{\partial y} = G \frac{m_2 m_3}{r_{23}^3} y, \quad (35)$$

$$\frac{\partial U_{13}}{\partial \dot{x}} = \frac{\partial U_{23}}{\partial \dot{x}} = \frac{\partial U_{13}}{\partial \dot{y}} = \frac{\partial U_{23}}{\partial \dot{y}} = 0. \quad (36)$$

Making the appropriate substitutions, (25) becomes

$$0 = m_3(\omega \dot{y} + \omega^2 x) - G \frac{m_1 m_3}{r_{13}^3} \left(x + \frac{r_c \mu}{m_1} \right) - G \frac{m_2 m_3}{r_{23}^3} \left(x - \frac{r_c \mu}{m_2} \right) - m_3(\ddot{x} - \omega \dot{y}) \quad (37)$$

$$\implies \ddot{x} = 2\omega \dot{y} + \omega^2 x - G \frac{m_1}{r_{13}^3} \left(x + \frac{r_c \mu}{m_1} \right) - G \frac{m_2}{r_{23}^3} \left(x - \frac{r_c \mu}{m_2} \right), \quad (38)$$

and (26) becomes

$$0 = m_3(-\omega \dot{x} + \omega^2 y) - G \frac{m_1 m_3}{r_{13}^3} y - G \frac{m_2 m_3}{r_{23}^3} y - m_3(\ddot{y} + \omega \dot{x}) \quad (39)$$

$$\implies \ddot{y} = -2\omega \dot{x} + \omega^2 y - G y \left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3} \right). \quad (40)$$

The system is in mechanical equilibrium at points where $\dot{x} = \dot{y} = 0$. The equilibrium behavior persists over time, implying $\ddot{x} = \ddot{y} = 0$. With these restrictions, (38) and (40) become

$$x = \frac{G}{\omega^2} \frac{m_1}{r_{13}^3} \left(x + \frac{r_c \mu}{m_1} \right) + \frac{G}{\omega^2} \frac{m_2}{r_{23}^3} \left(x - \frac{r_c \mu}{m_2} \right), \quad (41)$$

$$y = \frac{G}{\omega^2} y \left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3} \right). \quad (42)$$

The real roots of (41) and (42) are the Lagrange points.

Inspection of (42) indicates that there is at least one solution where $y = 0$. In this case (42) is eliminated. Additionally, (30) becomes

$$r_{13} = \left| x + \frac{r_c \mu}{m_1} \right|, \quad r_{23} = \left| x - \frac{r_c \mu}{m_2} \right|, \quad (43)$$

and thus (41) reduces to

$$x = \frac{G}{\omega^2} \frac{m_1}{|x + r_c \mu / m_1|^3} \left(x + \frac{r_c \mu}{m_1} \right) + \frac{G}{\omega^2} \frac{m_2}{|x - r_c \mu / m_2|^3} \left(x - \frac{r_c \mu}{m_2} \right) \equiv f(x), \quad (44)$$

where we have defined $f(x)$ as the right-hand side of the equation. Note the following observations about $f(x)$:

- $f(x)$ has singularities at $x = -r_c \mu / m_1$ and $x = r_c \mu / m_2$;

- $f(x) < 0$ in the regime $x < -r_c\mu/m_1$;
- $f(x) > 0$ in the regime $x > r_c\mu/m_2$;
- $f(x)$ crosses the x axis somewhere in the regime $-r_c\mu/m_1 < x < r_c\mu/m_2$;
- $df/dx < 0$ for all defined values of x because it is dominated by negative powers of x .

Based on these observations, we can sketch $f(x)$ and x as shown in Fig. 1. The three intersection points indicate that there are three real roots of (44). These are the first three Lagrange points.

In the case $y \neq 0$, (42) may be written

$$\frac{\omega^2}{G} = \frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}. \quad (45)$$

Substituting (45) into (41),

$$\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right)x = \frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right) \implies \frac{m_1}{r_{13}^3} \frac{r_c\mu}{m_1} = \frac{m_2}{r_{23}^3} \frac{r_c\mu}{m_2} \implies r_{13} = r_{23}. \quad (46)$$

Applying this condition to (30),

$$\left(x + \frac{r_c\mu}{m_1}\right)^2 + y^2 = \left(x - \frac{r_c\mu}{m_2}\right)^2 + y^2 \implies \quad (47)$$

But why does this imply they equal r ? Geometrically, this is only possible at two locations in the (x, y) plane as shown in Fig. 1. These are the final two Lagrange points, for a total of five as desired. \square

2 Spherical pendulum

A point mass m in three spatial dimensions is connected by a light inextensible string of length ℓ to a fixed pivot and experiences a uniform gravitational field. Use spherical polar coordinates (ρ, θ, ϕ) , where ρ is the radial distance, θ the relative inclination with respect to the downward vertical, and ϕ the azimuthal angle.

2.1 Is this mechanical system integrable? In other words, does this problem have as many independent conserved quantities as there are unknown dynamical variables?

Solution. The inextensible string fixes $\rho = \ell$, so the point mass's motion is constrained to a sphere of radius ℓ . Thus the system has two unknown dynamical variables, θ and ϕ . Therefore, we will need to find two independent conserved quantities in order for the system to be integrable. We will search for these quantities by analyzing the Lagrangian of the system.

We will begin with T . In Cartesian coordinates, kinetic energy in three dimensions is given by

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \quad (48)$$

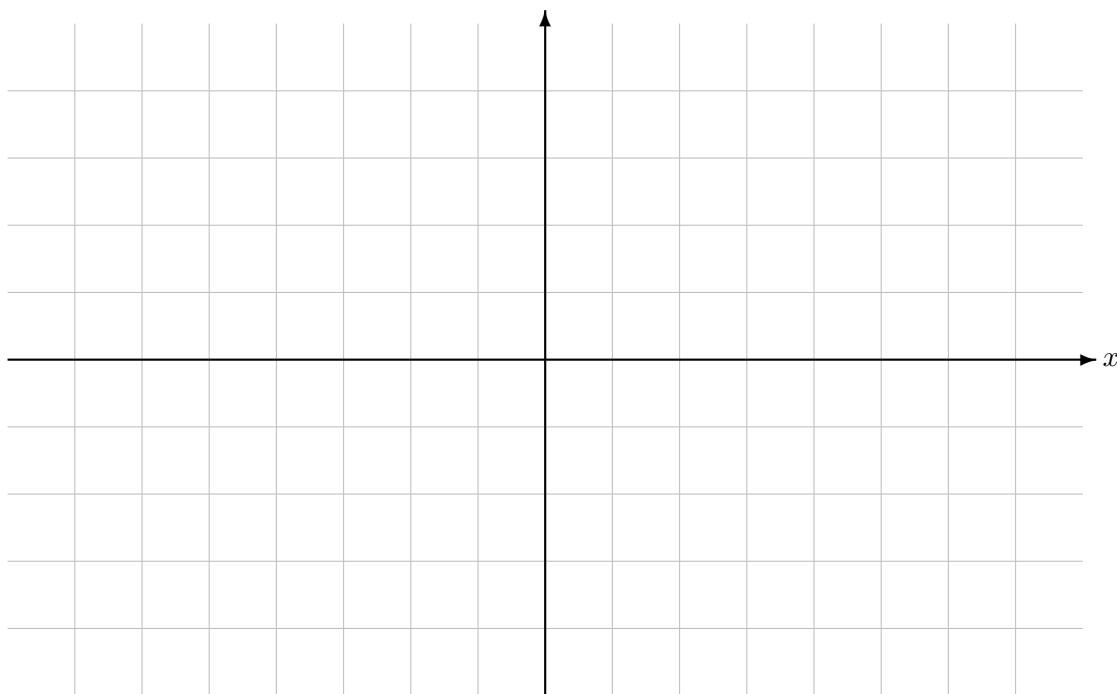


Figure 1: Three Lagrange points, indicated by roots of (44).

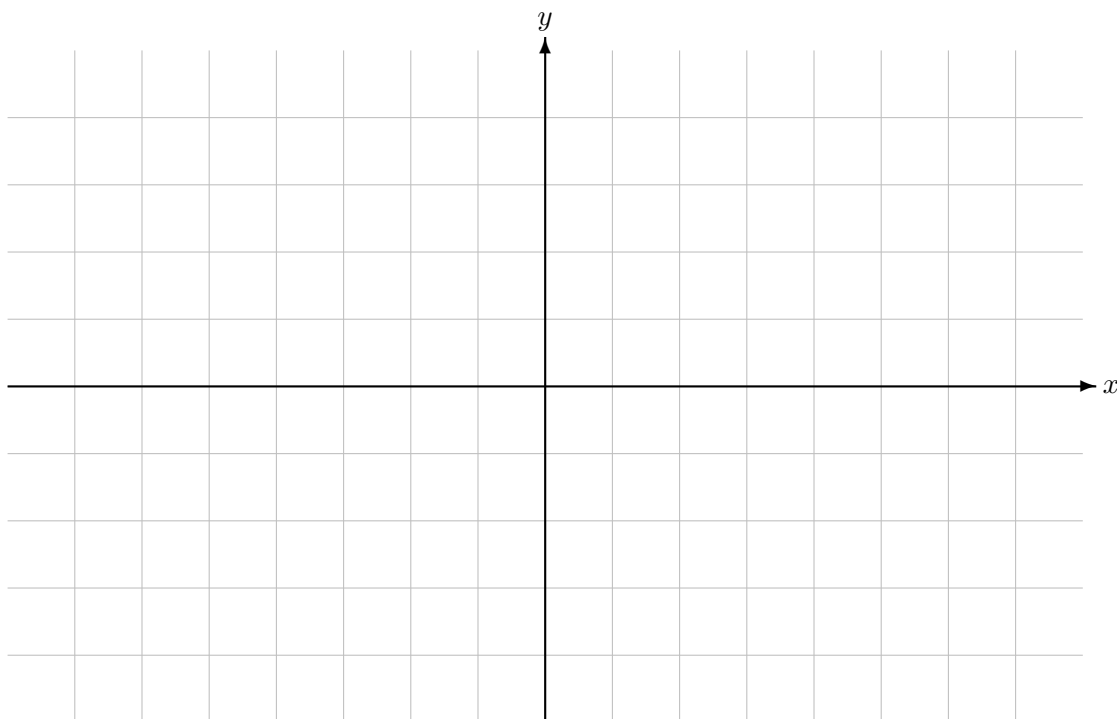


Figure 2: Two more Lagrange points, found by the geometrical argument implied by (47).

The transformation to Cartesian coordinates from the spherical coordinate system defined in the problem statement is given by

$$x = \ell \cos \phi \sin \theta, \quad (49)$$

$$y = \ell \sin \phi \sin \theta, \quad (50)$$

$$z = -\ell \cos \theta. \quad (51)$$

The time derivatives corresponding to (49)–(51) are then

$$\dot{x} = \frac{dx}{d\phi} \frac{d\phi}{dt} + \frac{dx}{d\theta} \frac{d\theta}{dt} = -\ell \sin \phi \sin \theta \dot{\phi} + \ell \cos \phi \cos \theta \dot{\theta}, \quad (52)$$

$$\dot{y} = \frac{dy}{d\phi} \frac{d\phi}{dt} + \frac{dy}{d\theta} \frac{d\theta}{dt} = \ell \cos \phi \sin \theta \dot{\phi} + \ell \sin \phi \cos \theta \dot{\theta}, \quad (53)$$

$$\dot{z} = \frac{dz}{d\phi} \frac{d\phi}{dt} + \frac{dz}{d\theta} \frac{d\theta}{dt} = \ell \sin \theta \dot{\theta}. \quad (54)$$

From (52)–(54), note that

$$\dot{x}^2 = \ell^2 (\sin^2 \phi \sin^2 \theta \dot{\phi}^2 - 2 \cos \phi \sin \phi \cos \theta \sin \theta \dot{\phi} \dot{\theta} + \cos^2 \phi \cos^2 \theta \dot{\theta}^2), \quad (55)$$

$$\dot{y}^2 = \ell^2 (\cos^2 \phi \sin^2 \theta \dot{\phi}^2 + 2 \cos \phi \sin \phi \cos \theta \sin \theta \dot{\phi} \dot{\theta} + \sin^2 \phi \cos^2 \theta \dot{\theta}^2), \quad (56)$$

$$\dot{z}^2 = \ell^2 \sin^2 \theta \dot{\theta}^2. \quad (57)$$

From (55) and (56),

$$\dot{x}^2 + \dot{y}^2 = \ell^2 \left[(\sin^2 \phi \sin^2 \theta + \cos^2 \phi \sin^2 \theta) \dot{\phi}^2 + (\cos^2 \phi \cos^2 \theta + \sin^2 \phi \cos^2 \theta) \dot{\theta}^2 \right] \quad (58)$$

$$= \ell^2 (\sin^2 \theta \dot{\phi}^2 + \cos^2 \theta \dot{\theta}^2). \quad (59)$$

Now adding (57),

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \ell^2 (\sin^2 \theta \dot{\phi}^2 + (\cos^2 \theta + \sin^2 \theta) \dot{\theta}^2) = \ell^2 (\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2). \quad (60)$$

Substituting (23), we can write (48) in polar coordinates:

$$T = \frac{m\ell^2}{2} (\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2). \quad (61)$$

Now addressing U , we know that the only external force acting upon the particle is gravity in the $-z$ direction. This is

$$U = mgz = -mg\ell \cos \theta, \quad (62)$$

where g is the acceleration of gravity and we have made the substitution (51).

Combining (61) and (62) yields the Lagrangian

$$L = \frac{m\ell^2}{2} (\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2) + mg\ell \cos \theta. \quad (63)$$

Note firstly that the Lagrangian (63) has no explicit time dependence. Thus, Noether's theorem implies that the total energy H of the system is conserved, where

$$H = \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} + \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L = \frac{m\ell^2}{2} (\sin^2 \theta \dot{\phi}^2 + \dot{\theta}^2) - mg\ell \cos \theta. \quad (64)$$

Note also that (63) has no explicit ϕ dependence. From Noether's theorem, this implies a second conserved quantity, given by

$$\frac{\partial L}{\partial \dot{\phi}} = m\ell^2 \sin^2 \theta \dot{\phi} \equiv J \quad (65)$$

where we have defined J , which is the angular momentum of the system. The two conserved quantities H and J are independent; therefore, the mechanical system is indeed integrable.

2.2 When appropriately simplified, the motion of the spherical pendulum reduces to one-dimensional motion of a point mass in an effective potential. Find the effective potential.

Solution. Fix $\phi \implies$ normal pendulum?

3 Spherical pendulum, continued

3.1 Write down the Hamiltonian describing the one-dimensional motion in problem 2.2. Sketch some time-evolution trajectories in phase space. Make sure you include all qualitatively different features and indicate the direction of time evolution. If there are fixed points corresponding to states in mechanical equilibrium, identify them. If there is a separatrix, a trajectory separating qualitatively different motion, write down the equation describing its shape and specify its energy. (You do not need to solve the equation.)

Solution. If this is the normal pendulum, it's exactly what we did in class?

3.2 Using the results derived earlier, give a simple qualitative description of the spherical pendulum motion in three-dimensional space.

Solution. Sinusoidal on the surface of a sphere

4 Double-well potential

A particle with mass m is confined to one dimension and placed in the potential

$$U(q) = U_0 - \frac{q^2}{2} + \frac{q^4}{4}. \quad (66)$$

4.1 Write down the Lagrangian and the Hamiltonian.

Solution. The Lagrangian is

$$L = T - U = m \frac{\dot{q}^2}{2} - U_0 + \frac{q^2}{2} - \frac{q^4}{4}, \quad (67)$$

and the Hamiltonian is

$$H = T + U = m \frac{\dot{q}^2}{2} + U_0 - \frac{q^2}{2} + \frac{q^4}{4}. \quad (68)$$

4.2 Sketch some time evolution trajectories in phase space. Make sure you include all qualitatively different features and indicate the direction of time evolution. If there are fixed points corresponding to states in mechanical equilibrium, identify them. If there is a separatrix, a trajectory separating qualitatively different motion, write down the equation describing its shape and specify its energy.

Solution. Using the Lagrangian of (67), the generalized momentum corresponding to the Hamiltonian in (68) is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}. \quad (69)$$

Making this substitution in (68) gives

$$H = \frac{p^2}{2m} + U_0 - \frac{q^2}{2} + \frac{q^4}{4}. \quad (70)$$

Then Hamilton's equations for the system are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \implies p = m\dot{q}, \quad (71)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = q - q^3. \quad (72)$$

We are interested in the phase space (q, p) . Fixed points (q^*, p^*) occur where $\dot{p} = \dot{q} = 0$. Making the appropriate equality from (72) and (71), the fixed points are the roots of the equation

$$q^* - (q^*)^3 = p^*. \quad (73)$$

By inspection of (73), the fixed points are located at

$$(q^*, p^*) = (0, 0), (\pm 1, 0). \quad (74)$$

These points are drawn in green in Fig. 4.

In order to classify the stability of the fixed points, note that (71) gives a relationship for the time evolution in the q direction. That is, time evolution flows in the $+q$ ($-q$) direction when $p > 0$ ($p < 0$), which corresponds to the top (bottom) half of the phase plane. By this argument,

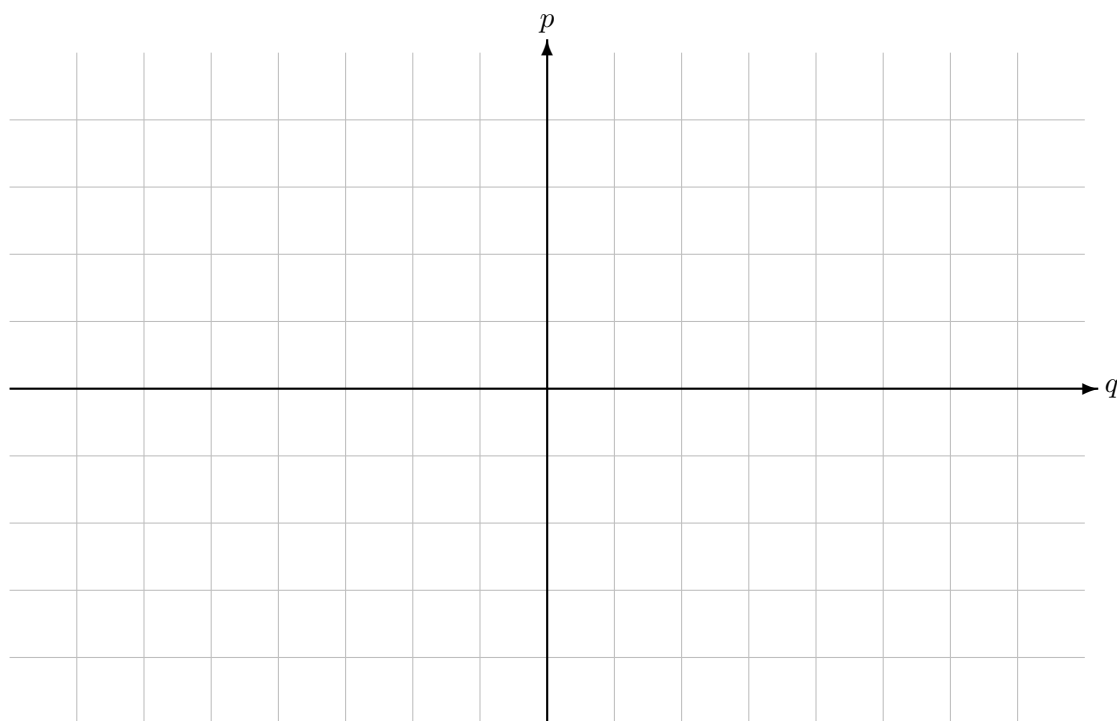


Figure 3: Time evolution trajectories in phase space for the system of Prob. 4. Fixed points are shown in green. The separatrix is shown in red.

we can figure out the behavior near the fixed points. These partial trajectories are drawn in purple in Fig. 4. We can conclude from this time evolution behavior that the fixed point at $(0,0)$ is unstable (a saddle point). The points at $(\pm 1,0)$ are stable, but not attracting; they are centers of orbital motion.

Drawing through the trajectories intersecting $(0,0)$, we identify the separatrix, drawn in red in Fig. 4. **Equation for separatrix**

In writing these solutions, I consulted David Tong's lecture notes and Steven Strogatz's *Nonlinear Dynamics and Chaos*.