Problem 1. Supersymmetry (Peskin & Schroeder 3.5) It is possible to write field theories with continuous symmetries linking fermions and bosons; such transformations are called *supersymmetries*.

1(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, written in the form
$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma} \cdot \partial \chi + F^* F.$$
Here F is an auxiliary complex scalar field whose field equation is $F = 0$. Show that this Lagrangian is invariant (up to a total divergence) under the infinitesimal transformation
$$\delta \phi = -i\epsilon^T \sigma^2 \chi, \qquad\qquad \delta \chi = \epsilon F + \sigma \cdot \partial \phi \sigma^2 \epsilon^*, \qquad\qquad \delta F = -i\epsilon^\dagger \bar{\sigma} \cdot \partial \chi, \qquad (1)$$
where the parameter ϵ_a is a 2-component spinor of Grassmann numbers.

Solution. Using the supplied transformations and dropping terms of $\mathcal{O}(\delta^2)$, we have
$$
\begin{aligned}
\mathcal{L} &\to \partial_\mu(\phi^* + \delta\phi^*)\partial^\mu(\phi + \delta\phi) + (\chi^\dagger + \delta\chi^\dagger)i\bar{\sigma}\cdot\partial(\chi + \delta\chi) + (F^* \delta F^*)(F + \delta F) \\
&\approx \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \chi^\dagger\bar{\sigma}\cdot\partial\delta\chi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F + F^*\delta F + \delta F^*F \\
&= \mathcal{L} + \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi + \chi^\dagger\bar{\sigma}\cdot\partial\delta\chi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F + \delta F^*F. \qquad (2)
\end{aligned}
$$
Note that Grassmann numbers satisfy $\alpha\beta = -\beta\alpha$ and $(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*$ for any α, β [1, p. 73]. Then
$$
\begin{aligned}
\delta\phi^* &= i(\epsilon^T\sigma^2\chi)^* = i\epsilon^\dagger\sigma^{2*}\chi^* = -i\epsilon^\dagger\sigma^2\chi^* = i\chi^\dagger\sigma^2\epsilon^*, \\
\delta\chi^\dagger &= (\epsilon F)^\dagger + (\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*)^\dagger = F^*\epsilon^\dagger + \epsilon^T\sigma^{2\dagger}\partial_\mu\phi^*\sigma^{\mu\dagger} = F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu, \\
\delta F^* &= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi = i(\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi)^* = -i\epsilon^T\bar{\sigma}^{\mu*}\partial_\mu\chi^* = i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon,
\end{aligned}
$$
where we have transposed as needed to obtain $\chi^\dagger$ or χ^*. So the $\mathcal{O}(\delta)$ terms in Eq. (2) are
$$
\begin{aligned}
\partial_\mu\phi^*\partial^\mu\delta\phi &= -i\partial_\mu\phi^*\partial^\mu(\epsilon^T\sigma^2\chi), & \partial_\mu\delta\phi^*\partial^\mu\phi &= i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi, \\
\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\epsilon F + \sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*), & \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi &= i(F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)\bar{\sigma}^\nu\partial_\nu\chi, \qquad (3) \\
F^*\delta F &= -iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi, & \delta F^*F &= i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon F.
\end{aligned}
$$

Adding the fourth and fifth terms above,
$$\delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F = iF^*\epsilon^\dagger\bar{\sigma}^\nu\partial_\nu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi - iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi = i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi.$$
Adding this to the first term of Eq. (3),
$$\partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F = -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi.$$
Note that
$$\sigma^\mu\bar{\sigma}^\nu = \frac{\sigma^\mu\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\mu + \sigma^\mu\bar{\sigma}^\nu - \bar{\sigma}^\nu\sigma^\mu}{2} = \frac{\{\sigma^\mu, \bar{\sigma}^\nu\}}{2} + \frac{[\sigma^\mu, \bar{\sigma}^\nu]}{2} = g^{\mu\nu} + \frac{[\sigma^\mu, \bar{\sigma}^\nu]}{2}$$
where we have used $\{\sigma^\mu, \bar{\sigma}^\nu\} = 2g^{\mu\nu}$ since $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ [2, p. 165]. Then
$$
\begin{aligned}
\partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F &= -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*g^{\mu\nu}\partial_\nu\chi + \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu, \bar{\sigma}^\nu] \\
&= -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\partial^\mu\chi + \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu, \bar{\sigma}^\nu] \\
&= \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu, \bar{\sigma}^\nu] \\
&= \partial_\mu\left(\frac{i}{2}\epsilon^T\sigma^2\phi^*\partial_\nu\chi[\sigma^\mu, \bar{\sigma}^\nu]\right). \qquad (4)
\end{aligned}
$$

Adding the third and sixth terms of Eq. (3),

$$\chi^\dagger i\bar\sigma^\mu \partial_\mu \delta\chi + \delta F^* F = i\chi^\dagger \bar\sigma^\mu \partial_\mu(\epsilon F) + i\chi^\dagger \bar\sigma^\mu \partial_\mu(\sigma^\nu \partial_\nu \phi \sigma^2 \epsilon^*) + i\partial_\mu \chi^\dagger \bar\sigma^{\mu\dagger}\epsilon F$$
$$= i\chi^\dagger \bar\sigma^\mu \partial_\mu(\sigma^\nu \partial_\nu \phi \sigma^2 \epsilon^*) + i\bar\sigma^\mu \partial_\mu(\chi^\dagger \epsilon F)$$
$$= i\chi^\dagger \bar\sigma^\mu \partial_\mu(\sigma^\nu \partial_\nu \phi \sigma^2 \epsilon^*) + \partial_\mu(i\bar\sigma^\mu \chi^\dagger \epsilon F)$$

Adding this to the second term of Eq. (3),

$$\chi^\dagger i\bar\sigma^\mu \partial_\mu \delta\chi + \delta F^* F + \partial_\mu \delta\phi^* \partial^\mu \phi = i\chi^\dagger \bar\sigma^\mu \sigma^\nu \partial_\mu(\partial_\nu \phi \sigma^2 \epsilon^*) + i\partial_\mu(\chi^\dagger \sigma^2 \epsilon^*)\partial^\mu \phi + \partial_\mu(i\bar\sigma^\mu \chi^\dagger \epsilon F).$$

Similar to before,

$$\bar\sigma^\mu \sigma^\nu = \frac{\bar\sigma^\mu \sigma^\nu + \sigma^\nu \bar\sigma^\mu + \bar\sigma^\mu \sigma^\nu - \sigma^\nu \bar\sigma^\mu}{2} = \frac{\{\bar\sigma^\mu, \sigma^\nu\}}{2} + \frac{[\bar\sigma^\mu, \sigma^\nu]}{2} = g^{\mu\nu} + \frac{[\bar\sigma^\mu, \sigma^\nu]}{2},$$

so

$$\chi^\dagger i\bar\sigma^\mu \partial_\mu \delta\chi + \delta F^* F + \partial_\mu \delta\phi^* \partial^\mu \phi = i\chi^\dagger g^{\mu\nu}\partial_\mu(\partial_\nu \phi \sigma^2 \epsilon^*) + \frac{i}{2}\chi^\dagger[\bar\sigma^\mu, \sigma^\nu]\partial_\mu(\partial_\nu \phi \sigma^2 \epsilon^*) + i\partial_\mu(\chi^\dagger \sigma^2 \epsilon^*)\partial^\mu \phi + \partial_\mu(i\bar\sigma^\mu \chi^\dagger \epsilon F).$$

Note that

$$\chi^\dagger[\bar\sigma^\mu, \sigma^\nu]\partial_\mu(\partial_\nu \phi \sigma^2 \epsilon^*) = \chi^\dagger[\bar\sigma^\nu, \sigma^\mu]\partial_\nu(\partial_\mu \phi \sigma^2 \epsilon^*) = -\chi^\dagger[\bar\sigma^\mu, \sigma^\nu]\partial_\mu(\partial_\nu \phi \sigma^2 \epsilon^*) = 0,$$

where we have used $[\bar\sigma^\mu, \sigma^\nu] = -[\bar\sigma^\nu, \sigma^\mu]$, since $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ [2, p. 165]. Then

$$\chi^\dagger i\bar\sigma^\mu \partial_\mu \delta\chi + \delta F^* F + \partial_\mu \delta\phi^* \partial^\mu \phi = i\chi^\dagger \partial_\mu(\partial^\mu \phi \sigma^2 \epsilon^*) + i\partial_\mu(\chi^\dagger \sigma^2 \epsilon^*)\partial^\mu \phi + \partial_\mu(i\bar\sigma^\mu \chi^\dagger \epsilon F)$$
$$= \partial_\mu(i\chi^\dagger \sigma^2 \epsilon^* \partial^\mu \phi + i\bar\sigma^\mu \chi^\dagger \epsilon F). \tag{5}$$

Finally, substituting Eqs. (4) and (5) into Eq. (2),

$$\mathcal{L} \to \mathcal{L} + \partial_\mu \left(\frac{i}{2}\epsilon^T \sigma^2 \phi^* \partial_\nu \chi[\sigma^\mu, \bar\sigma^\nu] + i\chi^\dagger \sigma^2 \epsilon^* \partial^\mu \phi + i\bar\sigma^\mu \chi^\dagger \epsilon F \right),$$

which is the same up to a total divergence. $\qquad\square$

1(b) Show that the term

$$\Delta\mathcal{L} = \left(m\phi F + \frac{i}{2}m\chi^T \sigma^2 \chi \right) + (\text{complex conjugate})$$

is also left invariant by the transformation given in 1(a). Eliminate F from the complete Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its field equation, and show that the fermion and boson fields ϕ and χ are given the same mass.

Solution. Transforming $\Delta\mathcal{L}$ and dropping terms of $\mathcal{O}(\delta^2)$ yields

$$\Delta\mathcal{L} \to m(\phi + \delta\phi)(F + \delta F) + \frac{i}{2}m(\chi^T + \delta\chi^T)\sigma^2(\chi + \delta\chi) + \text{c.c.}$$
$$\approx m\phi F + m\phi\delta F + m\delta\phi F + \frac{i}{2}m\chi^T \sigma^2 \chi + \frac{i}{2}m\chi^T \sigma^2 \delta\chi + \frac{i}{2}m\delta\chi^T \sigma^2 \chi + \text{c.c.}$$
$$= \Delta\mathcal{L} + \left(m\phi\delta F + m\delta\phi F + \frac{i}{2}m\chi^T \sigma^2 \delta\chi + \frac{i}{2}m\delta\chi^T \sigma^2 \chi + \text{c.c.} \right).$$

Applying Eqs. (1) to each term, we have

$$m\phi\delta F = -im\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi, \qquad\qquad m\delta\phi F = -im\epsilon^T\sigma^2\chi F, \qquad\qquad (6)$$

$$\frac{i}{2}m\chi^T\sigma^2\delta\chi = \frac{i}{2}m\chi^T\sigma^2(\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*), \qquad \frac{i}{2}m\delta\chi^T\sigma^2\chi = \frac{i}{2}m(F\epsilon^T - \epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T})\sigma^2\chi,,$$

where we have used

$$\delta\chi^T = F\epsilon^T - \epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T}.$$

Since $\chi^T\sigma^2\epsilon = \epsilon^T\sigma^2\chi$, adding the second, third, and fourth terms of Eq. (6) gives us

$$m\delta\phi F + \frac{i}{2}m\chi^T\sigma^2\delta\chi + \frac{i}{2}m\delta\chi^T\sigma^2\chi = -im\epsilon^T\sigma^2\chi F + \frac{i}{2}m\chi^T\sigma^2(\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*) + \frac{i}{2}m(F\epsilon^T - \epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T})\sigma^2\chi$$

$$= \frac{i}{2}m\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* - \frac{i}{2}m\epsilon^\dagger\sigma^2\partial_\mu\phi\sigma^{\mu T}\sigma^2\chi$$

$$= im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*.$$

Then adding the first term of Eq. (6) yields

$$\Delta\mathcal{L} \to \Delta\mathcal{L} + \left(-im\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi + im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* + \text{c.c.}\right)$$

$$= \Delta\mathcal{L} + \left(-im\bar{\sigma}^\mu\phi\epsilon^\dagger\partial_\mu\chi - im\bar{\sigma}^\mu\partial_\mu\phi\epsilon^\dagger\chi + \text{c.c.}\right)$$

$$= \Delta\mathcal{L} + \left(\partial_\mu(-im\bar{\sigma}^\mu\phi\epsilon^\dagger\chi) + \text{c.c.}\right)$$

where we have used $\sigma^2\sigma^\mu\sigma^2 = \bar{\sigma}^{\mu*}$ from Homework 2's 3(a). This is a total divergence and its complex conjugate, so we have shown that $\Delta\mathcal{L}$ is invariant under the supersymmetry transformations. $\square$

The complete Lagrangian is

$$\mathcal{L} + \Delta\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F + \left(m\phi F + \frac{i}{2}m\chi^T\sigma^2\chi + \text{c.c.}\right).$$

We can solve the field equation for F using the Euler-Lagrange equations, given by Peskin & Schroeder (2.3):

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Evaluating for $\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}$ and $\phi \to F$, we find

$$0 = \partial_\mu\left(\frac{\partial(\mathcal{L} + \Delta\mathcal{L})}{\partial(\partial_\mu F)}\right) - \frac{\partial(\mathcal{L} + \Delta\mathcal{L})}{\partial F} = -F^* - m\phi,$$

which implies

$$F^* = -m\phi, \qquad\qquad F = -m\phi^*.$$

Feeding these into the complete Lagrangian gives us

$$\mathcal{L} + \Delta\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + m^2|\phi|^2 - m^2|\phi|^2 - m^2|\phi|^2 + \frac{i}{2}m\chi^T\sigma^2\chi - \frac{i}{2}m\chi^\dagger\sigma^2\chi^*$$

$$= \left[\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\right] + \left[i\chi^\dagger\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right].$$

The first set of brackets is the Klein-Gordon Lagrangian describing a particle of mass m [1, p. 33], and the second set of brackets is the Majorana Lagrangian for a particle of mass m [1, p. 73]. So we have shown that the fields ϕ and χ are given the same mass. $\square$

1(c) It is possible to write supersymmetric nonlinear field equations by adding cubic and higher-order terms to the Lagrangian. Show that the following rather general field theory, containing the field (ϕ_i, χ_i), $i = 1, \ldots, n$, is supersymmetric:

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar\sigma \cdot \partial \chi_i + F_i^* F_i + \left(F_i \frac{\partial W[\phi]}{\partial \phi_i} + \frac{i}{2} \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right),$$

where $W[\phi]$ is an arbitrary function of the ϕ_i, called the *superpotential*. For the simple case $n = 1$ and $W = g\phi^3/3$, write out the field equations for ϕ and χ (after elimination of F).

Solution. We already know that the terms outside of the brackets are supersymmetric because that part is equivalent to the Lagrangian from 1(a) (but for the indices; at any rate, it will transform the same way). Then we can say

$$\mathcal{L} \to \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar\sigma \cdot \partial \chi_i + F_i^* F_i + \left((F_i + \delta F_i) \frac{\partial W[\phi + \delta\phi]}{\partial \phi_i} + \frac{i}{2} \frac{\partial^2 W[\phi + \delta\phi]}{\partial \phi_i \partial \phi_j} (\chi_i^T + \delta\chi_i^T) \sigma^2 (\chi_j + \delta\chi_j) + \text{c.c.} \right)$$

$$\approx \partial_\mu \phi_i^* \partial^\mu \phi_i + \chi_i^\dagger i\bar\sigma \cdot \partial \chi_i + F_i^* F_i + \left[(F_i + \delta F_i) \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j \right.$$

$$\left. + \frac{i}{2} \left(\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} + \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \right) (\chi_i^T \sigma^2 \chi_j + \chi_i^T \sigma^2 \delta\chi_j + \delta\chi_i^T \sigma^2 \chi_j) + \text{c.c.} \right]$$

$$= \mathcal{L} + \left[\delta F_i \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j + \frac{i}{2} \left(\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} (\chi_i^T \sigma^2 \delta\chi_j + \delta\chi_i^T \sigma^2 \chi_j) + \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \chi_i^T \sigma^2 \chi_j \right) + \text{c.c.} \right]$$

$$= \mathcal{L} + \left(\delta F_i \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \delta\chi_j + \frac{i}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \chi_i^T \sigma^2 \chi_j + \text{c.c.} \right), \qquad (7)$$

where we have used

$$\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\chi_i^T \sigma^2 \chi_j = \frac{\partial^2 W[\phi]}{\partial \phi_j \partial \phi_i} \delta\chi_j^T \sigma^2 \chi_i = \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \delta\chi_j.$$

Applying Eq. (1), we have the terms

$$\delta F_i \frac{\partial W[\phi]}{\partial \phi_i} = -i\epsilon^\dagger \bar\sigma \cdot \partial \chi_i \frac{\partial W[\phi]}{\partial \phi_i},$$

$$F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j = -iF_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \epsilon^T \sigma^2 \chi_j, \qquad (8)$$

$$i\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \delta\chi_j = i\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 (\epsilon F_j + \sigma \cdot \partial \phi_j \sigma^2 \epsilon^*),$$

$$\frac{i}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \delta\phi_k \chi_i^T \sigma^2 \chi_j = \frac{1}{2} \frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j.$$

The final term is 0:

$$\frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j = \frac{\partial^3 W[\phi]}{\partial \phi_j \partial \phi_i \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_j^T \sigma^2 \chi_i = -\frac{\partial^3 W[\phi]}{\partial \phi_i \partial \phi_j \partial \phi_k} \epsilon^T \sigma^2 \chi_k \chi_i^T \sigma^2 \chi_j = 0.$$

Adding the second and third terms of Eq. (8), we have

$$F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta\phi_j + i\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \delta\chi_j = i\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \phi_j \epsilon^* = i\frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \bar\sigma^{\mu*} \partial_\mu \phi_j \epsilon^*$$

since $\sigma^2 \sigma^\mu \sigma^2 = \sigma^{\mu *}$ and

$$F_j \frac{\partial^2 W[\phi]}{\partial \phi_j \partial \phi_i} \chi_i^T \sigma^2 \epsilon = F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \epsilon^T \sigma^2 \chi_j.$$

Adding in the first term of Eq. (8) yields a total divergence:

$$\begin{aligned}
\delta F_i \frac{\partial W[\phi]}{\partial \phi_i} + F_i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \delta \phi_j + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \delta \chi_j &= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} + i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \chi_i^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \phi_j \epsilon^* \\
&= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} - i \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \epsilon^\dagger \bar{\sigma}^\mu \chi_i \partial_\mu \phi_j \\
&= \partial_\mu \left(-i\epsilon^\dagger \bar{\sigma}^\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} \right),
\end{aligned}$$

where we have used the chain rule:

$$\partial_\mu \frac{\partial W[\phi]}{\partial \phi_i} = \frac{\partial}{\partial \phi_j} \left(\frac{\partial W[\phi]}{\partial \phi_i} \right) \frac{\partial \phi_j}{\partial x^\mu} = \frac{\partial^2 W[\phi]}{\partial \phi_i \partial \phi_j} \partial_\mu \phi_j.$$

Now using these results in Eq. (7), we have

$$\mathcal{L} \to \mathcal{L} + \partial_\mu \left(-i\epsilon^\dagger \bar{\sigma}^\mu \chi_i \frac{\partial W[\phi]}{\partial \phi_i} + \text{c.c.} \right),$$

so the field theory is indeed supersymmetric. $\qquad \square$

For $n = 1$ and $W = g\phi^3/3$, note firstly that

$$\frac{\partial W}{\partial \phi} = g\phi^2, \qquad\qquad \frac{\partial^2 W[\phi]}{\partial \phi^2} = 2g\phi.$$

Then

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma} \cdot \partial \chi + F^* F + \left(F g\phi^2 + ig\phi \chi^T \sigma^2 \chi + \text{c.c.} \right).$$

We first solve the Euler-Lagrange equations for F in order to eliminate it:

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu F)} \right) - \frac{\partial \mathcal{L}}{\partial F} = -F^* - g\phi^2,$$

so

$$F^* = -g\phi^2, \qquad\qquad F = -g^* \phi^{*2}.$$

The Lagrangian is then

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma} \cdot \partial \chi - |g|^2 |\phi|^4 + ig\phi \chi^T \sigma^2 \chi - ig^* \phi^* \chi^\dagger \sigma^2 \chi^*.$$

The field equations for ϕ are found by

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = \partial_\mu \partial^\mu \phi + 2|g|^2 |\phi|^2 \phi + ig^* \chi^\dagger \sigma^2 \chi^*,$$

and those for χ are found by

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \chi^\dagger)} \right) - \frac{\partial \mathcal{L}}{\partial \chi^\dagger} = -i\bar{\sigma}^\mu \partial_\mu \chi + ig\phi^* \sigma^2 \chi^*,$$

so the equations of motion are

$$\partial_\mu \partial^\mu \phi = -2|g|^2 |\phi|^2 \phi - ig^* \chi^\dagger \sigma^2 \chi^*, \qquad\qquad \bar{\sigma}^\mu \partial_\mu \chi = g\phi^* \sigma^2 \chi^*.$$

Problem 2.   **(Peskin & Schroeder 4.1)**   Let us return to the problem of the creation of Klein-Gordon particles by a classical source. Recall from Chapter 2 that this process can be described by the Hamiltonian

$$H = H_0 + \int d^3x \left[-j(t, \mathbf{x})\phi(x) \right],$$

where H_0 is the free Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon field, and $j(x)$ is a c-number scalar function. We found that, if the system is in the vacuum state before the source is turned on, the source will create a mean number of particles

$$\langle N \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left| \tilde{j}(p) \right|^2. \tag{9}$$

In this problem we will verify that statement, and extract more detailed information, by using a perturbation expansion in the strength of the source.

2(a)   Show that the probability that the source creates *no* particles is given by

$$P(0) = \left| \langle 0 | T \left\{ \exp\left(i \int d^4x \, j(x)\phi_I(x) \right) \right\} |0\rangle \right|^2.$$

Solution.   Both the initial and the final state are the vacuum state. The probability is

$$P(0) = | \langle 0 | U(t, t_0) |0\rangle |^2,$$

where

$$U(t, t_0) = T \left\{ \exp\left(-i \int_{t_0}^t dt' \, H_I(t') \right) \right\}$$

from Eq. (4.22). A general expression for the interaction Hamiltonian in the interaction picture is given by Peskin & Schroeder (4.19):

$$H_I(t) = e^{iH_0(t,t_0)} (H_{\text{int}}) e^{-iH_0(t,t_0)}.$$

For the given Hamiltonian $H = H_0 + H_{\text{int}}$, we have

$$H_I(t) = \int d^3x \left[-j(t, \mathbf{x})\phi_I(t, \mathbf{x}) \right],$$

where we have used (4.14),

$$\phi_I(t, \mathbf{x}) = e^{iH_0(t,t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t,t_0)}.$$

Then we have

$$U(t, t_0) = T \left\{ \exp\left(-i \int_{t_0}^t dt' \int d^3x \left[-j(t, \mathbf{x})\phi_I \right] \right) \right\} = T \left\{ \exp\left(i \int d^4x \, j(x)\phi_I(x) \right) \right\},$$

so the probability of the source's creating no particles is

$$P(0) = \left| \langle 0 | T \left\{ \exp\left(i \int d^4x \, j(x)\phi_I(x) \right) \right\} |0\rangle \right|^2,$$

as desired.   □

2(b) Evaluate the term in $P(0)$ of order j^2, and show that $P(0) = 1 - \lambda + \mathcal{O}(j^4)$, where λ equals the expression given above for $\langle N \rangle$.

Solution. The first few terms of the Taylor series expansion for e^z are [3]

$$e^z \approx 1 + z + \frac{z^2}{2}. \tag{10}$$

Then

$$\exp\left(i \int d^4x\, j(x)\phi_I(x) \right) \approx 1 + i \int d^4x\, j(x)\phi_I(x) - \frac{1}{2} \iint d^4x\, d^4y\, j(x)\phi_I(x)j(y)\phi_I(y).$$

Then the probability can be written

$$P(0) = \left| 1 + i \langle 0 | \int d^4x\, j(x)\phi_I(x) \,|0\rangle - \frac{1}{2} \langle 0 | T \left\{ \iint d^4x\, d^4y\, j(x)\phi_I(x)j(y)\phi_I(y) \right\} |0\rangle \right|^2$$

$$= \left| 1 - \frac{1}{2} \iint d^4x\, d^4y\, j(x)j(y) \langle 0|T\phi_I(x)\phi_I(y)|0\rangle \right|^2, \tag{11}$$

since $\langle 0|\phi_I|0\rangle = 0$ (and if we had an $\mathcal{O}(j^3)$ term, it would likewise vanish since there would be an uncontracted operator remaining [1, p. 89]). Applying Peskin & Schroder (4.11),

$$\langle 0|T\phi_I(x)\phi_I(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon},$$

we have [1, p. 30]

$$\iint d^4x\, d^4y\, j(x)j(y) \langle 0|T\phi_I(x)\phi_I(y)|0\rangle = \iint d^4x\, d^4y \int \frac{d^4p}{(2\pi)^4} j(x)j(y) \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

$$= i \int \frac{d^4p}{(2\pi)^4} \int d^4x\, e^{-ip\cdot x} j(x) \int d^4y\, e^{ip\cdot y} j(y) \frac{1}{p^2 - m^2 + i\epsilon}$$

$$= i \int \frac{d^4p}{(2\pi)^4} \frac{\left|\tilde{j}(p)\right|^2}{p^2 - m^2 + i\epsilon}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i\left|\tilde{j}(p)\right|^2}{p^{02} - E_{\mathbf{p}}^2 + i\epsilon},$$

where we have used [1, p. 32]

$$\tilde{j}(p) = \int d^4y\, e^{ip\cdot y} j(y), \qquad\qquad \tilde{j}^*(p) = \int d^4y\, e^{-ip\cdot y} j(y). \tag{12}$$

Then we can perform a contour integral. Letting $\epsilon = 2E_{\mathbf{p}}\epsilon'$ and neglecting terms of $\mathcal{O}(\epsilon^2)$ [4],

$$\iint d^4x\, d^4y\, j(x)j(y) \langle 0|T\phi_I(x)\phi_I(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi} \frac{i\left|\tilde{j}(p)\right|^2}{(p^0 - E_{\mathbf{p}} + i\epsilon')(p^0 + E_{\mathbf{p}} - i\epsilon')},$$

In general the poles are at $p^0 = \pm(E_{\mathbf{p}} - i\epsilon')$ [1, p. 31]. When we close the contour in the upper half plane, we enclose only the pole at $p^0 = -E_{\mathbf{p}} + i\epsilon'$. Then, applying the residue theorem [5],

$$\iint d^4x\, d^4y\, j(x)j(y) \langle 0|T\phi_I(x)\phi_I(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} 2\pi i \operatorname{Res}_{p^0 = -E_{\mathbf{p}} + i\epsilon'} \left(\frac{i\left|\tilde{j}(p)\right|^2}{(p^0 - E_{\mathbf{p}} + i\epsilon')(p^0 + E_{\mathbf{p}} - i\epsilon')} \right)$$

$$= - \int \frac{d^3p}{(2\pi)^3} \frac{\left|\tilde{j}(p)\right|^2}{-2E_{\mathbf{p}} + 2i\epsilon'}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\left|\tilde{j}(p)\right|^2}{2E_{\mathbf{p}}},$$

where in the final step we have neglected terms of $\mathcal{O}(\epsilon)$. This is identical to the given expression for $\lambda = \langle N \rangle$ in Eq. (9).
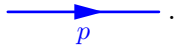
Making this substitution in Eq. (11),

$$P(0) = \left| 1 - \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{|\tilde{j}(p)|^2}{2E_{\mathbf{p}}} \right|^2 = 1 - \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{|\tilde{j}(p)|^2}{2E_{\mathbf{p}}} + \mathcal{O}(j^4) = 1 - \lambda + \mathcal{O}(j^4)$$

as we sought to show.

2(c) Represent the term computed in 2(b) as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly: $P(0) = e^{-\lambda}$.
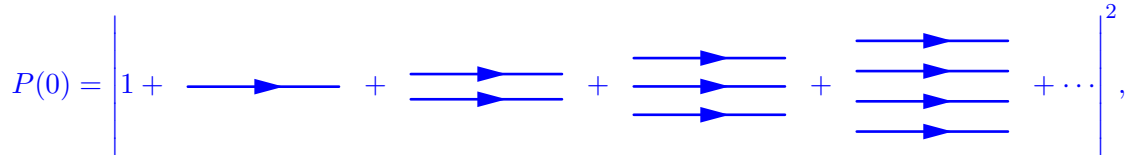
Solution. From 2(b), the term is

$$-\lambda = -\int \frac{d^4p}{(2\pi)^4} |\tilde{j}(p)|^2 \frac{i}{p^2 - m^2 + i\epsilon}.$$

According to the momentum space Feynman rules [1, p. 95], this term is represented by .

We know that we will only have terms in $P(0)$ of even powers of ϕ. Then for integer n, the term of order j^{2n} is proportional to

$$-\lambda^n = -\int d^4x_1 \cdots d^4x_n \, j(x_1) \cdots j(x_n) \, \langle 0|T\phi_1 \cdots \phi_n|0 \rangle = -\int \frac{d^4p}{(2\pi)^4} |\tilde{j}(p)|^{2n} \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^n.$$

So the whole perturbation series can be written as

where each propagator represents one factor of λ. For the symmetry factor, there are $2^{2n/2} = 2^n$ ways the $2n$ vertices can be chosen to be initial or final vertices, and a further $n!$ ways the n initial vertices can be paired with the n final vertices. This gives us the symmetry factor $2^n n!$. Then, using the power series [6]

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{13}$$

we can write

$$P(0) = \left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} \right)^2 = \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{2} \right)^n \right]^2 = (e^{-\lambda/2})^2 = e^{-\lambda}$$

as desired. $\square$

2(d) Compute the probability that the source creates one particle of momentum k. Perform this computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of 2(c) to sum the series.

Solution. The initial state is $|0\rangle$ and the final state is $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}} a_{\mathbf{k}}^{\dagger} |0\rangle$ from Peskin & Schroeder (2.35). The probability is

$$P(k) = |\langle \mathbf{k}|U(t, t_0)|0\rangle|^2 = \left| |\mathbf{k}\rangle \, T \left\{ \exp\left(i \int d^4x \, j(x)\phi_I(x) \right) \right\} |0\rangle + \mathcal{O}(j^2) \right|^2$$

from the result of 2(a). To $\mathcal{O}(j)$, this is

$$P(k) = \left| i \, |\mathbf{k}\rangle \int d^4x \, j(x)\phi_I(x) \, |0\rangle \right|^2 + \mathcal{O}(j^3) = \left| i\sqrt{2E_{\mathbf{k}}} \int d^4x \, \langle 0|a_{\mathbf{k}}j(x)\phi_I(x)|0\rangle \right|^2 + \mathcal{O}(j^3)$$

since $\langle 0|a_{\mathbf{k}}|0\rangle = 0$. At this point we need Peskin & Schroeder (2.25) [1, p. 83],

$$\phi(t_0, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{2E_{\mathbf{k}}}}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

and (2.29),

$$[a_{\mathbf{p}}, a_{\mathbf{p}}'^{\dagger}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}').$$

Then

$$P(k) = \left| i \int d^4x \, \langle 0|a_{\mathbf{k}}j(x) \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{2E_{\mathbf{k}}}}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{-ip\cdot x} \right)|0\rangle \right|^2 + \mathcal{O}(j^3)$$

$$= \left| i \int d^4x \, \langle 0|j(x) \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}} \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}} + [a_{\mathbf{k}}, a_{\mathbf{p}}^{\dagger}] \right) e^{-ip\cdot x}|0\rangle \right|^2 + \mathcal{O}(j^3)$$

$$= \left| i \int d^4x \, j(x) \int d^3p \sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}} e^{-ip\cdot x} \delta^3(\mathbf{k} - \mathbf{p}) \right|^2 + \mathcal{O}(j^3)$$

$$= \left| i \int d^4x \, j(x) e^{-ik\cdot x} \right|^2 + \mathcal{O}(j^3)$$

$$= |i\tilde{j}(k)|^2 + \mathcal{O}(j^3)$$

$$= |\tilde{j}(k)|^2 + \mathcal{O}(j^3),$$

since $\langle 0|a_{\mathbf{k}}a_{\mathbf{p}}|0\rangle = \langle 0|a_{\mathbf{p}}^{\dagger} a_{\mathbf{k}}|0\rangle = 0$, and where we have used Eq. (12).

In order to use the trick of 2(c), we consider the creation of one particle of any momentum. We integrate $P(k)$ over all possible k, inserting a prefactor of $1/2E_{\mathbf{k}}$ in order to keep Lorentz invariance [1, pp. 106–107]:

$$P(1) = \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{j}(k)|^2}{2E_{\mathbf{k}}} + \mathcal{O}(j^3) = \lambda + \mathcal{O}(j^3), \tag{14}$$

where we have applied Eq. (9). This implies

$$P(1) = \left| i\sqrt{\lambda} + \mathcal{O}(j^2) \right|^2;$$

in other words, $i\sqrt{\lambda}$ is the amplitude corresponding to the creation of a particle at lowest order. But the higher-order vacuum fluctuations we saw in 2(c) need to be taken into account as well. The lowest term of the

amplitude in 2(c) was 1, so the complete perturbation series for the amplitude of $P(1)$ is simply the amplitude for $P(0)$ multiplied by the amplitude for creating one particle:

$$P(1) = \left| i\sqrt{\lambda} \left(1 + \longrightarrow + \Longrightarrow + \Longrightarrow + \Longrightarrow + \cdots \right) \right|^2,$$

or

$$P(1) = \left| i\sqrt{\lambda} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} \right|^2 = \lambda \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{2} \right)^n \right]^2 = \lambda (e^{-\lambda/2})^2 = \lambda e^{-\lambda}.$$

(The same symmetry factor is the same as in 2(c), since there is only one particle with $1! = 1$ arrangements.)

But this is the probability to create one particle of any momentum. The probability to create one particle of momentum k can be retrieved by replacing λ by $\left| \tilde{j}(k) \right|^2$ via Eq. (14):

$$P(k) = \left| \tilde{j}(k) \right|^2 e^{-\left| \tilde{j}(k) \right|^2}.$$

2(e) Show that the probability of producing n particles is given by

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}.$$

This is a *Poisson* distribution.

Solution. In 2(d) we found that the amplitude to create one particle is $i\sqrt{\lambda}$. It follows that the amplitude to create n particles is $(i\sqrt{\lambda})^n$. The vacuum fluctuations are the same as before, but there is an additional symmetry factor of $n!$ in this case to account for the number of ways to arrange n particles. Then we have

$$P(n) = \frac{1}{n!} \left| (i\sqrt{\lambda})^n \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} \right|^2 = \frac{\lambda^n}{n!} \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{2} \right)^n \right]^2 = \frac{\lambda^n}{n!} (e^{-\lambda/2})^2 = \frac{\lambda^n e^{-\lambda}}{n!}$$

as we wanted to show. $\square$

2(f) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1, \qquad \qquad \langle N \rangle = \sum_{n=0}^{\infty} n P(n) = \lambda.$$

The first identity says that the $P(n)$s are properly normalized probabilities, while the second confirms our proposal for $\langle N \rangle$. Compute the mean square fluctuation $\left\langle (N - \langle N \rangle)^2 \right\rangle$.

Solution. Applying the result of 2(e) and Eq. (13),

$$\sum_{n=0}^{\infty} P(n) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} e^{\lambda} = 1,$$

and

$$\langle N \rangle = \sum_{n=0}^{\infty} nP(n) = \sum_{n=0}^{\infty} n\frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{(n-1)!} = \lambda e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

as desired. □

Note that $\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$, and that

$$\begin{aligned}
\langle N^2 \rangle &= \sum_{n=0}^{\infty} n^2 P(n) \\
&= \sum_{n=0}^{\infty} n^2 \frac{\lambda^n e^{-\lambda}}{n!} \\
&= \lambda e^{-\lambda}\sum_{n=1}^{\infty} n\frac{\lambda^{n-1}}{(n-1)!} \\
&= \lambda e^{-\lambda}\left(\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty}(n-1)\frac{\lambda^{n-1}}{(n-1)!}\right) \\
&= \lambda e^{-\lambda}\left(e^{\lambda} + \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-2)!}\right) \\
&= \lambda e^{-\lambda}\left(e^{\lambda} + \lambda\sum_{n=1}^{\infty}\frac{\lambda^{n-2}}{(n-2)!}\right) \\
&= \lambda e^{-\lambda}\left(e^{\lambda} + \lambda e^{\lambda}\right) \\
&= \lambda + \lambda^2.
\end{aligned}$$

Then

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \lambda + \lambda^2 - \lambda^2 = \lambda.$$

Problem 3. Decay of a scalar particle (Peskin & Schroeder 4.2) Consider the following Lagrangian, involving two real scalar fields Φ and ϕ:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi.$$

The last term is an interaction that allows a Φ particle to decay into two ϕs, provided that $M > 2m$. Assuming that this condition is met, calculate the lifetime of the Φ to lowest order in μ.

Solution. The lifetime of the Φ is the reciprocal of its decay rate into two ϕs, since this is the only allowed decay mode [1, p. 101]. Peskin & Schroeder (4.86) gives the decay rate formula

$$d\Gamma = \frac{1}{2m_{\mathcal{A}}}\left(\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}(m_{\mathcal{A}} \to \{p_f\})|^2(2\pi)^4\delta^4\left(p_{\mathcal{A}} - \sum p_f\right),$$

where $\mathcal{M}$ can be found by Peskin & Schroeder (4.73):

$$\langle \mathbf{p}_1\mathbf{p}_2\cdots|iT|\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}\rangle = (2\pi)^4\delta^4\left(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\right)\cdot i\mathcal{M}(k_{\mathcal{A}}, k_{\mathcal{B}} \to p_f). \tag{15}$$

In turn, (4.90) gives

$$\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | iT | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle = \lim_{T \to \infty(1-i\epsilon)} \left({}_0\langle \mathbf{p}_1 \mathbf{p}_2 \cdots | T \left\{ \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 \right)_{\text{connected, amputated}}.$$

Here we have one incoming Φ with mass $m_\mathcal{A} = M$ and momentum $p_\mathcal{A} = p_\Phi$, and two outgoing ϕs with momenta $p_f = p_1, p_2$ and energies $E_f = E_1, E_2$. So we have

$$d\Gamma = \frac{1}{16 M E_1 E_2} \frac{d^3 p_1\, d^3 p_2}{(2\pi)^6} |\mathcal{M}(M \to p_1 p_2)|^2 (2\pi)^4 \delta^4(p_\Phi - p_1 - p_2), \tag{16}$$

where the initial factor of $1/n! = 1/2$ is needed because we have two identical particles [1, p. 108]. We need to find $\mathcal{M}(M \to p_1 p_2)$, which we can do by evaluating (4.90). Since $H_{\text{int}} = -L_{\text{int}}$ [1, p. 77], $H_I(t) = \mu \Phi \phi \phi$ and

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{p}_\Phi \rangle = \lim_{T \to \infty(1-i\epsilon)} \left({}_0\langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ \exp\left[-i \int_{-T}^{T} dt\, \mu \Phi \phi \phi \right] \right\} | \mathbf{p}_\Phi \rangle_0 \right)_{\text{connected, amputated}}.$$

We will use the series expansion of the exponential function in Eq. (10) to first order. The zeroth order term does not contribute to $\mathcal{M}$, so we consider only the first order term [1, p. 110]:

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{p}_\Phi \rangle \approx {}_0\langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ -i\mu \int d^4 x\, \Phi \phi \phi \right\} | \mathbf{p}_\Phi \rangle_0 = {}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i\mu \int d^4 x\, \Phi \phi \phi + \text{contractions} \right\} | \mathbf{p}_\Phi \rangle_0.$$

We ignore the terms with contracted operators since only fully connected diagrams contribute to the T-matrix [1, p. 111]. From there, there are two ways to contract $\phi \phi$ with $\langle \mathbf{p}_1 \mathbf{p}_2 |$ and one way to contract Φ with $|\mathbf{p}_\Phi \rangle$. Applying Peskin & Schroeder (4.94),

$$\overbrace{\phi_I(x)|\mathbf{p}\rangle}^{} = e^{-ip\cdot x}, \qquad\qquad\qquad \overbrace{\langle \mathbf{p}|\phi_I(x)}^{} = e^{ip\cdot x},$$

we can write [1, p. 112]

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{p}_\Phi \rangle = -2i\mu \int d^4 x\, e^{ip_1 \cdot x} e^{ip_2 \cdot x} e^{-ip_\Phi \cdot x} = -2i\mu \int d^4 x\, e^{i(p_1 + p_2 - p_\Phi)\cdot x} = -2i\mu(2\pi)^4 \delta^4(p_1 + p_2 - p_\Phi),$$

where the factor of 2 comes from the two sets of contractions. Inspecting Eq. (15), we have

$$\mathcal{M}(M \to p_1 p_2) = -2\mu.$$

Feeding this into Eq. (16),

$$d\Gamma = \frac{\mu^2}{4 M E_1 E_2} \frac{d^3 p_1\, d^3 p_2}{(2\pi)^6} (2\pi)^4 \delta^4(p_\Phi - p_1 - p_2).$$

We can use any reference frame to perform the computation [1, p. 100], so we choose the rest frame of the Φ. In this frame, $p_\Phi = (M, \mathbf{0})$ and $\mathbf{p}_1 = -\mathbf{p}_2$. Let $p = |\mathbf{p}_1| = |\mathbf{p}_2|$. Then

$$E_1 = \sqrt{m^2 + p_1^2} = \sqrt{m^2 + p^2} = \sqrt{m^2 + p_2^2} = E_2,$$

so

$$d\Gamma = \frac{\mu^2}{4 M E_1 E_2} \frac{d^3 p_1\, d^3 p_2}{(2\pi)^6} (2\pi)^4 \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \delta(E_1 + E_2 - M).$$

Integrating this expression over momentum space yields

$$\begin{aligned}
\Gamma &= \frac{\mu^2}{4M} \int \frac{d^3 p_1\, d^3 p_2}{(2\pi)^6} \frac{1}{E_1 E_2} (2\pi)^4 \delta^3(\mathbf{p}_1 + \mathbf{p}_2) \delta(E_1 + E_2 - M) \\
&= \frac{\mu^2}{4M} \int \frac{d^3 p_1}{(2\pi)^2} \frac{1}{E_1^2} \delta(2E_1 - M) \\
&= \frac{\mu^2}{4M} \int \frac{dp\, d\theta\, d\varphi}{(2\pi)^2} \frac{p^2 \sin\theta}{p^2 + m^2} \delta(2\sqrt{p^2 + m^2} - M),
\end{aligned}$$

where we have let $p_1 = p$. Now we use [7]

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

where x_i are the roots of $g(x)$. We have $g(p) = 2\sqrt{p^2 + m^2} - M$ with roots $p = \pm\sqrt{M^2/4 - m^2}$. Note also that

$$g'(p) = \frac{4p}{\sqrt{m^2 + p^2}} \quad \Longrightarrow \quad g'(p_i) = \pm\frac{4\sqrt{M^2/4 - m^2}}{M/2} = \pm 4\sqrt{1 - \frac{4m^2}{M^2}},$$

which means

$$\delta(2\sqrt{p^2 + m^2} - M) = \frac{\delta(p + \sqrt{M^2/4 - m^2}) + \delta(p - \sqrt{M^2/4 - m^2})}{2\sqrt{1 - 4m^2/M^2}}.$$

Then

$$\Gamma = \frac{\mu^2}{8M\sqrt{1 - 4m^2/M^2}} \int \frac{dp\, d\theta\, d\varphi}{(2\pi)^2} \frac{p^2 \sin\theta}{p^2 + m^2} \left(\delta(p + \sqrt{M^2/4 - m^2}) + \delta(p - \sqrt{M^2/4 - m^2})\right)$$

$$= \frac{\mu^2}{16\pi M\sqrt{1 - 4m^2/M^2}} \frac{2(M^2/4 - m^2)}{M^2/4}$$

$$= \frac{\mu^2}{8\pi M} \sqrt{1 - \frac{4m^2}{M^2}}$$

so the lifetime is

$$\tau = \frac{1}{\Gamma} = \frac{8\pi M}{\mu^2} \sqrt{1 - \frac{4m^2}{M^2}}^{-1}.$$

Problem 4.   Aspects of scaling behavior in QFT

4(a) As discussed in class, massless free scalar field theory is invariant under scale transformations if we assign the appropriate scale dimension to the field ϕ. The action of dilations on coordinates is implemented by the operator $D = ix^\mu \partial_\mu$. Find the commutation relations of this operator with the operators that implement the Poincaré algebra of translations $P_\mu = i\partial_\mu$ and Lorentz transformations $J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$. Find the canonical Noether current and Noether charge for this symmetry in the free field theory, and find the commutation relations of the charge with the field ϕ to show that it implements the infinitesimal form of the symmetry on the fields. Check also that this current has the right commutation relations with the field theory Hamiltonian and momentum generators.

Solution.   Firstly, the commutators of the operators are

$$[D, P_\mu] = -x^\nu \partial_\nu \partial_\mu + \partial_\mu x^\nu \partial_\nu = -x^\nu \partial_\nu \partial_\mu + \partial_\mu(x_\nu)\partial^\nu + x^\nu \partial_\nu \partial_\mu = \delta_{\mu\nu}\partial^\nu = \partial_\mu = -iP_\mu,$$

and

$$[D, J_{\mu\nu}] = -x^\alpha \partial_\alpha (x_\mu \partial_\nu - x_\nu \partial_\mu) + (x_\mu \partial_\nu - x_\nu \partial_\mu)x^\alpha \partial_\alpha$$

$$= -x^\alpha \partial_\alpha(x_\mu \partial_\nu) + x^\alpha \partial_\alpha(x_\nu \partial_\mu) + x_\mu \partial_\nu(x^\alpha \partial_\alpha) - x_\nu \partial_\mu(x^\alpha \partial_\alpha)$$

$$= -x^\alpha \partial_\alpha(x_\mu)\partial_\nu - x^\alpha x_\mu \partial_\alpha \partial_\nu + x^\alpha \partial_\alpha(x_\nu)\partial_\mu + x^\alpha x_\nu \partial_\alpha \partial_\mu + x_\mu \partial_\nu(x_\alpha)\partial^\alpha + x_\mu x^\alpha \partial_\nu \partial_\alpha - x_\nu \partial_\mu(x_\alpha)\partial^\alpha$$

$$\quad - x_\nu x^\alpha \partial_\mu \partial_\alpha$$

$$= -x^\alpha \delta_{\mu\alpha}\partial_\nu - x^\alpha x_\mu \partial_\alpha \partial_\nu + x^\alpha \delta_{\nu\alpha}\partial_\mu + x^\alpha x_\nu \partial_\alpha \partial_\mu + x_\mu \delta_{\nu\alpha}\partial^\alpha + x_\mu x^\alpha \partial_\nu \partial_\alpha - x_\nu \delta_{\mu\alpha}\partial^\alpha - x_\nu x^\alpha \partial_\mu \partial_\alpha$$

$$= -x_\mu \partial_\nu - x^\alpha x_\mu \partial_\alpha \partial_\nu + x_\nu \partial_\mu + x^\alpha x_\nu \partial_\alpha \partial_\mu + x_\mu \partial_\nu + x_\mu x^\alpha \partial_\nu \partial_\alpha - x_\nu \partial_\mu - x_\nu x^\alpha \partial_\mu \partial_\alpha$$

$$= 0.$$

The conserved charge is given in general by Peskin & Schroeder (2.12) and (2.13),

$$Q \equiv \int_{\text{all space}} j^0 \, d^3x \,, \qquad \text{where } j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu, \qquad (17)$$

where $j^\mu(x)$ is a Noether current and J^μ is a 4-divergence that arises when transforming the Lagrangian as in Peskin & Schroeder (2.10):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x).$$

The Lagrangian of the massless free scalar field theory can be written from Peskin & Schroeder (2.14),

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi.$$

A dilation can be written $\phi \to e^{i\alpha D}\phi$ [1, p. 18]. Then

$$\mathcal{L} \to \frac{1}{2}\partial_\mu e^{-i\alpha D}\phi^* \partial^\mu e^{i\alpha D}\phi = \frac{1}{2}\partial_\mu e^{-i\alpha D}\phi^*(e^{i\alpha D}\partial_\mu + i\partial_\mu)\phi$$

The infinitesimal transformations, by analogy with Peskin & Schroeder (2.15), are

$$\alpha\Delta\phi = i\alpha D\phi, \qquad\qquad \alpha\Delta\phi^* = -i\alpha D\phi^*.$$

Then the Noether current is

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\Delta\phi^* = i\partial^\mu(\phi^*)D\phi - i\partial_\mu(\phi)D\phi^*,$$

and the Noether charge is

$$Q = i\int d^3x \left[\partial^0(\phi^*)D\phi - \partial_0(\dot\phi)D\phi^*\right] = i\int d^3x \, (\dot\phi^* D\phi - \dot\phi D\phi^*).$$

From 2(a) of Homework 1, we know the momenta conjugate to ϕ and ϕ^*,

$$\pi = \dot\phi^*, \qquad\qquad \pi^* = \dot\phi,$$

and the commutation relations

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\,\delta^3(\mathbf{x} - \mathbf{y}),$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = 0,$$
$$[\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0,$$
$$[\phi(\mathbf{x}), \pi^*(\mathbf{y})] = [\phi(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi^*(\mathbf{y})] = 0.$$

The commutation relation between the charge and the field is

$$[\phi(y), Q(x)] = i \left[\phi(y), \int d^3x \left[\pi(x)D\phi(x) - \pi^*(x)D\phi^*(x)\right]\right]$$

$$= i \int d^3x \left\{[\phi(y), \pi(x)]D\phi(x) - [\phi(y), \pi^*(x)D\phi^*(x)]\right\}$$

$$= i \int d^3x \left\{[\phi(y), \pi(x)]D\phi(x) - i\pi^*(x)[\phi(y), x^\mu\partial_\mu\phi^*(x)]\right\}$$

$$= \int d^3x \left\{i[\phi(y), \pi(x)]D\phi(x) + \pi^*(x)[\phi(y), (t\pi(x) + x^i\partial_i\phi^*(x))]\right\}$$

$$= \int d^3x \left\{i[\phi(y), \pi(x)]D\phi(x) + t\pi^*(x)[\phi(y), \pi(x)]\right\}$$

$$= i \int d^3x \left[iD\phi(x) + t\pi^*(x)\right]\delta^3(\mathbf{y} - \mathbf{x})$$

$$= it\pi^*(y) - D\phi(y)$$

$$= it\pi^*(y) - ix^\mu\partial_\mu\phi(y)$$

$$= i[t\pi^*(y) - t\pi^*(y) - x^i\partial_i\phi(y)]$$

$$= i[\mathbf{y} \cdot \boldsymbol{\nabla}\phi(y)]$$

where $x = (t, \mathbf{x})$ and $y = (t, \mathbf{y})$. This is the infinitesimal form of a spatial dilation on the field.

4(b) The sound waves in a solid obey a wave equation, which at low enough wavelengths/energies is described by the action (for simplicity, consider one spatial dimension)

$$S_0 = \int dt\, dx \left(\partial_t\phi\partial_t\phi - c_s^2\partial_x\phi\partial_x\phi\right)$$

where c_s is the speed of sound in the material. The field ϕ describes the distortion of the lattice away from equilibrium. Model this situation by a lattice of atoms (mass m, lattice spacing a) interacting through harmonic nearest-neighbor forces (spring constant k). Find the leading term in the action which corrects the above kinetic energy in a power series in $a \times$ (derivatives) (this can be done for instance by expanding the exact dispersion relation for the chain of masses and springs). Show that this correction term is an irrelevant perturbation of the action, so that all traces of the lattice structure disappear in the continuum limit $a \to 0$. Estimate the momentum scale at which the irrelevant corrections amount to 10% of the total energy of a phonon.

Solution.   The dispersion relation for a monatomic lattice is

$$\omega = \sqrt{\frac{4k}{m}}\left|\sin\left(\frac{qa}{2}\right)\right|,$$

where q is the wavenumber. It has solutions of the form

$$\phi = Ae^{iqx - \omega t} \tag{18}$$

where A is some constant [8, p. 432].

We expand the dispersion relation using the Taylor series [3]

$$\sin x \approx x - \frac{1}{6}x^3$$

to find

$$\omega \approx \sqrt{\frac{4k}{m}} \left| \frac{qa}{2} - \frac{1}{6}\left(\frac{qa}{2}\right)^3 \right| = \sqrt{\frac{4k}{m}} \left| \frac{qa}{2} - \frac{q^3 a^3}{48} \right|.$$

Note that

$$\omega^2 = \frac{4k}{m} \left| \frac{qa}{2} - \frac{q^3 a^3}{48} \right|^2 = \frac{4k}{m}\left(\frac{q^2 a^2}{4} - \frac{q^4 a^4}{48} + \frac{q^6 a^6}{48^2} \right),$$

where the lowest-order term has $\omega^2 \propto q^2$. This is the relation we would get from feeding Eq. (18) into the equation of motion, which is the wave equation [9]:

$$\frac{\partial^2 \phi}{\partial t^2} = c_s^2 \frac{\partial^2 \phi}{\partial x^2} \quad \Longrightarrow \quad -\omega^2 A e^{i(qx-\omega t)} = -c_s^2 q^2 A e^{i(qx-\omega t)} \quad \Longrightarrow \quad \omega^2 = c_s^2 q^2.$$

To account for the next-highest-order term in ω^2, we need a term with higher-order spatial derivatives. This gives us the equation of motion

$$0 = \frac{\partial^2 \phi}{\partial t^2} - \frac{ka^2}{m}\frac{\partial^2 \phi}{\partial x^2} + \frac{ka^4}{12m}\frac{\partial^4 \phi}{\partial x^4} = \frac{\partial^2 \phi}{\partial t^2} - c_s^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{a^2}{12}c_s^2 \frac{\partial^4 \phi}{\partial x^4}.$$

The Lagrangian that gives us this equation of motion can be found in general from the Euler–Lagrange equations for a single function of two variables with higher derivatives, $\mathcal{L}[x_1, x_2, f, f_1, f_2, f_{11}, f_{12}, f_{22}, \ldots]$ where $f_i = \partial f / \partial x_i$ [10]:

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial x_1}\left(\frac{\partial \mathcal{L}}{\partial f_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \mathcal{L}}{\partial f_2}\right) + \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial \mathcal{L}}{\partial f_{11}}\right) + \frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\partial \mathcal{L}}{\partial f_{12}}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial \mathcal{L}}{\partial f_{22}}\right) + \cdots = 0.$$

So the Lagrangian including the higher-order term will be

$$\mathcal{L} = \partial_t \phi \partial_t \phi - c_s^2 \partial_x \phi \partial_x \phi - \frac{a^2}{12}c_s^2 \partial_x^2 \phi \partial_x^2 \phi$$

The leading term in the action that corrects the kinetic energy is highlighted below:

$$S_q = \int dt\, dx \left(\partial_t \phi \partial_t \phi - c_s^2 \partial_x \phi \partial_x \phi - \frac{a^2}{12}c_s^2 \partial_x^2 \phi \partial_x^2 \phi \right).$$

In terms of mass dimension, $\partial_t = \partial_x = [1]$. We need the kinetic term of the Lagrangian to have mass dimension 0. Since $\partial_t = \partial_x = [1]$ and $dt = dx = [-1]$, it must be that $\phi = [0]$. Then

$$\int dt\, dx\, \partial_x^2 \phi \partial_x^2 \phi = [-1 - 1 + 2 + 0 + 2 + 0] = [2].$$

This means that this term, neglecting the coupling constant a^2, has dimension proportional to $1/(\text{length})^2$. In order for this term to contribute at larger length scales, a would need to increase proportionally to the length scale. However, this spoils the continuum limit $a \to 0$. Thus, the term is irrelevant.

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