Problem 1. Verify that the functional

$$J[u] = \int_R \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy \tag{1}$$

is invariant under a rotation of the coordinate axis:

$$\tilde{x} = x\cos\epsilon + y\sin\epsilon, \qquad \tilde{y} = -x\sin\epsilon + y\cos\epsilon. \tag{2}$$

Solution. The functional is invariant if $J[u(x,y)] = J[\tilde{u}(\tilde{x},\tilde{y})]$. By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = \cos\epsilon \frac{\partial}{\partial \tilde{x}} - \sin\epsilon\frac{\partial}{\partial \tilde{y}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = \sin\epsilon \frac{\partial}{\partial \tilde{x}} + \cos\epsilon\frac{\partial}{\partial \tilde{y}}.$$

The Jacobian for (2) is

$$A = \begin{bmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial y \\ \partial\tilde{y}/\partial x & \partial\tilde{y}/\partial y \end{bmatrix} = \begin{bmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{bmatrix} \implies \det(A) = \cos^2\epsilon + \sin^2\epsilon = 1,$$

and therefore

$$\int_R f(u)\, dx\, dy \mapsto \int_{\tilde{R}} f(\tilde{u})\, d\tilde{x}\, d\tilde{y}\,,$$

for an arbitrary function f. Making these substitutions into (1), we have

$$\begin{aligned} J[u(x,y)] &= \int_R \left[\left(\cos\epsilon\frac{\partial u}{\partial \tilde{x}} - \sin\epsilon\frac{\partial u}{\partial \tilde{y}} \right)^2 + \left(\sin\epsilon\frac{\partial u}{\partial \tilde{x}} + \cos\epsilon\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy \\ &= \int_R \left(\cos^2\epsilon\frac{\partial^2 u}{\partial \tilde{x}^2} - 2\cos\epsilon\sin\epsilon\frac{\partial^2 u}{\partial\tilde{x}\partial\tilde{y}} + \sin^2\epsilon\frac{\partial^2 u}{\partial\tilde{y}^2} + \sin^2\epsilon\frac{\partial^2 u}{\partial\tilde{x}^2} + 2\cos\epsilon\sin\epsilon\frac{\partial^2 u}{\partial\tilde{x}\partial\tilde{y}} + \cos^2\epsilon\frac{\partial^2 u}{\partial\tilde{y}^2} \right) dx\, dy \\ &= \int_R \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy = \int_{\tilde{R}} \left[\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial \tilde{u}}{\partial \tilde{y}} \right)^2 \right] d\tilde{x}\, d\tilde{y} \\ &= J[\tilde{u}(\tilde{x},\tilde{y})], \end{aligned}$$

as desired. □

Problem 2. Consider the real-valued Lagrangian density $\mathcal{L}$ depending on a complex-valued function $\phi(t,x,y)$:

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt}\phi \right) - \nabla\phi^* \cdot \nabla\phi - m^2\phi^*\phi, \tag{3}$$

where * is complex conjugation, and $\nabla\phi = (\,\partial\phi/\partial x\,,\,\partial\phi/\partial y\,)$. Treating ϕ and ϕ^* as independent objects, derive the Euler-Lagrange equations.

Solution. We will have two Euler-Lagrange equations; one for ϕ and one for ϕ^*. In general, they are given by

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi_t} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi_x} - \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi_y}, \qquad 0 = \frac{\partial\mathcal{L}}{\partial\phi^*} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi^*_t} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi^*_x} - \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi^*_y}.$$

Expanding out $\nabla\phi^* \cdot \nabla\phi$, (3) becomes

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt}\phi\right) - \frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x} - \frac{\partial\phi^*}{\partial y}\frac{\partial\phi}{\partial y} - m^2\phi^*\phi.$$

Then

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^*, \qquad \frac{\partial\mathcal{L}}{\partial\phi_t} = \frac{i}{2}\phi^*, \qquad \frac{\partial\mathcal{L}}{\partial\phi_x} = -\frac{\partial\phi^*}{\partial x}, \qquad \frac{\partial\mathcal{L}}{\partial\phi_y} = -\frac{\partial\phi^*}{\partial y},$$
$$\frac{\partial\mathcal{L}}{\partial\phi^*} = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi_t^*} = -\frac{i}{2}\phi, \qquad \frac{\partial\mathcal{L}}{\partial\phi_x^*} = -\frac{\partial\phi}{\partial x}, \qquad \frac{\partial\mathcal{L}}{\partial\phi_y^*} = -\frac{\partial\phi}{\partial y},$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi_t} = \frac{i}{2}\frac{d\phi^*}{dt}, \qquad \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi_x} = -\frac{\partial^2\phi^*}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi_y} = -\frac{\partial^2\phi^*}{\partial y^2},$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi_t^*} = -\frac{i}{2}\frac{d\phi}{dt}, \qquad \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi_x^*} = -\frac{\partial^2\phi}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi_y^*} = -\frac{\partial^2\phi}{\partial y^2}.$$

Then the Euler-Lagrange equations are

$$0 = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^* - \frac{i}{2}\frac{d\phi^*}{dt} + \frac{\partial^2\phi^*}{\partial x^2} + \frac{\partial^2\phi^*}{\partial y^2}, \qquad 0 = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi + \frac{i}{2}\frac{d\phi}{dt} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2},$$

which simplify to

$$0 = i\frac{d\phi^*}{dt} - \nabla^2\phi^* + m^2\phi^*, \qquad 0 = i\frac{d\phi}{dt} + \nabla^2\phi - m^2\phi.$$

Problem 3. The nondimensionalized, multidimensional Sine-Gordon equation,

$$\theta_{xx} + \theta_{yy} - \theta_{tt} = \sin\theta$$

for $\theta(x, y, t)$, is the Euler-Lagrange equation for the action integral

$$S[\theta] = \int_R \left\{\frac{1}{2}\left[\theta_t^2 - (\nabla\theta)^2\right] - \sin\theta\right\} dx\, dy\, dt$$

with $\nabla\theta = (\partial\theta/\partial x\,,\, \partial\theta/\partial y)$. The functional $S[\theta]$ is invariant under translation of x, y, and t. Find the associated energy-momentum tensor and energy-momentum vector.

Solution. Expanding out $(\nabla\theta)^2$, the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) - \sin\theta. \tag{4}$$

The energy-momentum tensor is defined by

$$T_{ij} = \frac{\partial\mathcal{L}}{\partial\theta_{x_i}}\frac{\partial\theta}{\partial x_j} - \mathcal{L}\,\delta_{ij},$$

where $x_i \in \{x_0, x_1, x_2\} = \{t, x, y\}$. The diagonal elements of T are then

$$T_{00} = \frac{\partial\mathcal{L}}{\partial\theta_t}\frac{\partial\theta}{\partial t} - \mathcal{L} = \theta_t^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = \frac{1}{2}(\theta_t^2 + \theta_x^2 + \theta_y^2) + \sin\theta,$$
$$T_{11} = \frac{\partial\mathcal{L}}{\partial\theta_x}\frac{\partial\theta}{\partial x} - \mathcal{L} = -\theta_x^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 + \theta_x^2 - \theta_y^2) + \sin\theta,$$
$$T_{22} = \frac{\partial\mathcal{L}}{\partial\theta_y}\frac{\partial\theta}{\partial y} - \mathcal{L} = -\theta_y^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 - \theta_x^2 + \theta_y^2) + \sin\theta,$$

and the nondiagonal elements are

$$T_{01} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial x} = \theta_t\theta_x, \qquad T_{02} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial y} = \theta_t\theta_y, \qquad T_{12} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial y} = -\theta_x\theta_y,$$
$$T_{10} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial t} = -\theta_t\theta_x, \qquad T_{20} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial t} = -\theta_t\theta_y, \qquad T_{21} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial x} = -\theta_x\theta_y.$$

In matrix form, we have

$$T = \begin{bmatrix} (\theta_t^2+\theta_x^2+\theta_y^2)/2+\sin\theta & \theta_t\theta_x & \theta_t\theta_y \\ -\theta_t\theta_x & -(\theta_t^2+\theta_x^2-\theta_y^2)/2+\sin\theta & -\theta_x\theta_y \\ -\theta_t\theta_y & -\theta_x\theta_y & -(\theta_t^2-\theta_x^2+\theta_y^2)/2+\sin\theta \end{bmatrix}.$$

The energy-momentum vector is defined by

$$P_j = \int T_{0j}\, dx_1\, dx_2\,.$$

Its components are then

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2+\theta_x^2+\theta_y^2)+\sin\theta\right] dx\, dy\,, \qquad P_1 = \int \theta_t\theta_x\, dx\, dy\,, \qquad P_2 = \int \theta_t\theta_y\, dx\, dy\,.$$

Problem 4. Extra credit

4.a Verify that the nondimensionalized, one-dimensional Sine-Gordon equation,

$$\theta_{xx} - \theta_{tt} = \sin\theta, \tag{5}$$

is also invariant under a Lorentz transformation on $(x_0 = t, x_1 = x)$. The transformation is given by

$$\begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix},$$

where $\gamma = 1/\sqrt{1-\nu^2}$.

Solution. Define $(\tilde{t}, \tilde{x})$ as the transformed coordinates. (5) is invariant if it has the same form under the substitution $\theta(t,x) \mapsto \theta(\tilde{t},\tilde{x})$. The new coordinates are given by

$$\begin{bmatrix} \tilde{t} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix}\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma(t-\nu x) \\ \gamma(x-\nu t) \end{bmatrix},$$

or

$$\tilde{t} = \gamma(t-\nu x), \qquad \tilde{x} = \gamma(x-\nu t).$$

Proceeding similarly to problem 1, the chain rule gives us

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} + \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial t} = \gamma\left(\frac{\partial}{\partial \tilde{t}} - \nu\frac{\partial}{\partial \tilde{x}}\right), \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial x} + \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} = \gamma\left(\frac{\partial}{\partial \tilde{x}} - \nu\frac{\partial}{\partial \tilde{t}}\right).$$

For the second derivatives,

$$\frac{\partial^2}{\partial t^2} = \gamma^2 \left(\frac{\partial}{\partial \tilde{t}} - \nu \frac{\partial}{\partial \tilde{x}} \right)^2 = \gamma^2 \left(\frac{\partial^2}{\partial \tilde{t}^2} - 2\nu \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2}{\partial \tilde{x}^2} \right),$$
$$\frac{\partial^2}{\partial x^2} = \gamma^2 \left(\frac{\partial}{\partial \tilde{x}} - \nu \frac{\partial}{\partial \tilde{t}} \right)^2 = \gamma^2 \left(\frac{\partial^2}{\partial \tilde{x}^2} - 2\nu \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2}{\partial \tilde{t}^2} \right).$$

Making these substitutions, (5) becomes

$$\begin{aligned}
\sin\theta &= \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial t^2} \\
&= \gamma^2 \left(\frac{\partial^2 \theta}{\partial \tilde{x}^2} - 2\nu \frac{\partial^2 \theta}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2 \theta}{\partial \tilde{t}^2} \right) - \gamma^2 \left(\frac{\partial^2 \theta}{\partial \tilde{t}^2} - 2\nu \frac{\partial^2 \theta}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2 \theta}{\partial \tilde{x}^2} \right) \\
&= \gamma^2 \left[(1-\nu^2) \frac{\partial^2 \theta}{\partial \tilde{x}^2} - (1-\nu^2) \frac{\partial^2 \theta}{\partial \tilde{t}^2} \right] \\
&= \frac{\partial^2 \theta}{\partial \tilde{x}^2} - \frac{\partial^2 \theta}{\partial \tilde{t}^2},
\end{aligned}$$

because $\gamma^2 = 1/(1-\nu^2)$. Thus, we have demonstrated the invariance of (5). □

4.b Find the associated conserved quantity. Is it analogous to a common conserved quantity in classical mechanics?

Solution. By analogy to problem 3, the Lagrangian for this system is given by

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2) - \sin\theta$$

which is like (4), but with only one spatial dimension. Continuing the analogy, the components of the energy-momentum vector are

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2 + \theta_x^2) + \sin\theta \right] dx\,, \qquad\qquad P_1 = \int \theta_t \theta_x \, dx\,.$$

These are the conserved quantitites, or "currents." The component P_0 is analagous to the classical Hamiltonian, or the total energy of the system. This corresponds to $\mathcal{L}$'s having no explicit t dependence. The component P_1 is like the momentum conjugate to x, since it corresponds to $\mathcal{L}$'s having no explicit x dependence. Since we are concerned with only one spatial dimension, P_1 is analogous to the classical total (linear) momentum of the system.

Problem 5. Interacting line vortices

A system of n vortices moving on a two-dimensional plane has the Hamiltonian

$$H = \sum_{j=1}^{n} \sum_{i=1}^{j-1} -\gamma^{(i)} \gamma^{(j)} \ln |\mathbf{r}_i - \mathbf{r}_j|, \tag{6}$$

where $\gamma^{(i)}$ is the strength of the ith line vortex, and $\mathbf{r}_i = (x_i, y_i)$ its position in the plane. Using the Poisson bracket structure

$$[f, g] = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right), \tag{7}$$

Hamilton's equations for the vortex system simplify to

$$\dot{x}_i = [x_i, H], \qquad \dot{y}_i = [y_i, H].$$

Consider two vortices. Show that the equations of motion can be solved explicitly. Most importantly, show that the solution tells us the two vortices orbit each other with a frequency that is inversely proportional to the square of their separation.

Solution. For two vortices, (6) reduces to

$$H = -\gamma^{(1)}\gamma^{(2)} \ln |\mathbf{r}_1 - \mathbf{r}_2|.$$

For $i, j \in \{1, 2\}$, note that

$$\frac{\partial x_i}{\partial x_j} = \frac{\partial y_i}{\partial y_j} = \delta_{ij}, \qquad \frac{\partial x_i}{\partial y_j} = \frac{\partial y_i}{\partial x_j} = 0.$$

Note also that

$$|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2},$$

and define $R \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ as the separation of the vortices. Define also

$$u \equiv \ln R, \qquad v \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 = R^2.$$

Then

$$\frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial R}\frac{\partial R}{\partial v}\frac{\partial v}{\partial x_1} = -\gamma^{(1)}\gamma^{(2)}\frac{1}{R}\frac{1}{2R}(2x_1 - 2x_2) = -\gamma^{(1)}\gamma^{(2)}\frac{x_1 - x_2}{(x_1 - x_2)^2 + (y_1 - y_2)^2} = -\gamma^{(1)}\gamma^{(2)}\frac{x_1 - x_2}{R^2},$$

and, likewise,

$$\frac{\partial H}{\partial x_2} = \gamma^{(1)}\gamma^{(2)}\frac{x_1 - x_2}{R^2}, \qquad \frac{\partial H}{\partial y_1} = -\gamma^{(1)}\gamma^{(2)}\frac{y_1 - y_2}{R^2}, \qquad \frac{\partial H}{\partial y_2} = \gamma^{(1)}\gamma^{(2)}\frac{y_1 - y_2}{R^2}.$$

Combining the above, we use (7) to find

$$\dot{x}_1 = \frac{1}{\gamma^{(1)}}\left(\frac{\partial x_1}{\partial x_1}\frac{\partial H}{\partial y_1} - \frac{\partial x_1}{\partial y_1}\frac{\partial H}{\partial x_1}\right) + \frac{1}{\gamma^{(2)}}\left(\frac{\partial x_1}{\partial x_2}\frac{\partial H}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial H}{\partial x_2}\right) = \frac{1}{\gamma^{(1)}}\frac{\partial H}{\partial y_1} = -\gamma^{(2)}\frac{y_1 - y_2}{R^2}, \tag{8}$$

$$\dot{x}_2 = \frac{1}{\gamma^{(1)}}\left(\frac{\partial x_2}{\partial x_1}\frac{\partial H}{\partial y_1} - \frac{\partial x_2}{\partial y_1}\frac{\partial H}{\partial x_1}\right) + \frac{1}{\gamma^{(2)}}\left(\frac{\partial x_2}{\partial x_2}\frac{\partial H}{\partial y_2} - \frac{\partial x_2}{\partial y_2}\frac{\partial H}{\partial x_2}\right) = \frac{1}{\gamma^{(2)}}\frac{\partial H}{\partial y_2} = \gamma^{(1)}\frac{y_1 - y_2}{R^2}, \tag{9}$$

$$\dot{y}_1 = \frac{1}{\gamma^{(1)}}\left(\frac{\partial y_1}{\partial x_1}\frac{\partial H}{\partial y_1} - \frac{\partial y_1}{\partial y_1}\frac{\partial H}{\partial x_1}\right) + \frac{1}{\gamma^{(2)}}\left(\frac{\partial y_1}{\partial x_2}\frac{\partial H}{\partial y_2} - \frac{\partial y_2}{\partial y_2}\frac{\partial H}{\partial x_2}\right) = -\frac{1}{\gamma^{(1)}}\frac{\partial H}{\partial x_1} = \gamma^{(2)}\frac{x_1 - x_2}{R^2}, \tag{10}$$

$$\dot{y}_2 = -\frac{1}{\gamma^{(1)}}\left(\frac{\partial y_2}{\partial x_1}\frac{\partial H}{\partial y_1} - \frac{\partial y_2}{\partial y_1}\frac{\partial H}{\partial x_1}\right) + \frac{1}{\gamma^{(2)}}\left(\frac{\partial y_2}{\partial x_2}\frac{\partial H}{\partial y_2} - \frac{\partial y_2}{\partial y_2}\frac{\partial H}{\partial x_2}\right) = -\frac{1}{\gamma^{(2)}}\frac{\partial H}{\partial x_2} = -\gamma^{(2)}\frac{x_1 - x_2}{R^2}. \tag{11}$$

Note that

$$\frac{\partial R}{\partial x_i} = \frac{\partial R}{\partial v}\frac{\partial v}{\partial x_i} = \frac{x_i - x_j}{R^2}, \qquad \frac{\partial R}{\partial y_i} = \frac{y_i - y_j}{R^2}$$

so

$$[H, R] = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(\frac{\partial H}{\partial x_i}\frac{\partial v}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial v}{\partial x_i}\right) = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(-\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{R^2}\frac{y_i - y_j}{R^2} + \gamma^{(i)}\gamma^{(j)}\frac{y_i - y_j}{R^2}\frac{x_i - x_j}{R^2}\right) = 0.$$

This means R is a conserved quantity and therefore constant.

Define $\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2 = (X, Y)$, where $|\mathbf{R}| = R$. Now we have two generalized coordinates X and Y, where $X = x_1 - x_2$ and $Y = y_1 - y_2$. This gives us the two equations of motion

$$\dot{X} = \dot{x}_1 - \dot{x}_2 = -(\gamma^{(1)} + \gamma^{(2)})\frac{y_1 - y_2}{R^2} = -\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}Y, \tag{12}$$

$$\dot{Y} = \dot{y}_1 - \dot{y}_2 = (\gamma^{(1)} + \gamma^{(2)})\frac{x_1 - x_2}{R^2} = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}X. \tag{13}$$

We can differentiate these to obtain two uncoupled second-order equations:

$$\ddot{X} = -\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}\dot{Y} = -\frac{(\gamma^{(1)} + \gamma^{(2)})^2}{R^4}X, \qquad \ddot{Y} = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}\dot{X} = -\frac{(\gamma^{(1)} + \gamma^{(2)})^2}{R^4}Y.$$

These equations have the solutions

$$X(t) = C_1 \cos\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right) + C_2 \sin\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right), \quad Y(t) = D_1 \cos\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right) + D_2 \sin\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right),$$

where C_1, C_2, D_1, D_2 are constants. Define

$$\omega \equiv \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2} \tag{14}$$

as the angular frequency. To solve for the constants, we apply (12) and (13):

$$\dot{X}(t) = -C_1\omega \sin(\omega t) + C_2\omega \cos(\omega t) = -\omega Y, \qquad \dot{Y}(t) = -D_1\omega \sin(\omega t) + D_2\omega \cos(\omega t) = \omega X.$$

This implies $C_1 = D_2$ and $C_2 = -D_1$. We can fix $C_2 = D_1 = 0$ without loss of generality, which implies $C_1 = D_1 = R$. We now have

$$X(t) = R\cos(\omega t) = x_1(t) - x_2(t), \qquad Y(t) = R\sin(\omega t) = y_1(t) - y_2(t). \tag{15}$$

We can now find the solutions to the original four equations by integrating (8)–(11) with respect to t:

$$x_1(t) = -\frac{\gamma^{(2)}}{R^2}\int Y(t)\,dt = \frac{\gamma^{(2)}}{\omega R}\cos(\omega t) = \frac{\gamma^{(2)}}{\gamma^{(1)} + \gamma^{(2)}}R\cos(\omega t),$$

$$x_2(t) = \frac{\gamma^{(1)}}{R^2}\int Y(t)\,dt = -\frac{\gamma^{(1)}}{\gamma^{(1)} + \gamma^{(2)}}R\cos(\omega t),$$

$$y_1(t) = \frac{\gamma^{(2)}}{R^2}\int X(t)\,dt = \frac{\gamma^{(2)}}{\gamma^{(1)} + \gamma^{(2)}}R\sin(\omega t),$$

$$y_2(t) = -\frac{\gamma^{(1)}}{R^2}\int X(t)\,dt = \frac{\gamma^{(1)}}{\gamma^{(1)} + \gamma^{(2)}}R\sin(\omega t),$$

where we have taken the constants of integration to be zero without loss of generality.

Thus, we have shown that the equations of motion can be solved explicitly. Since $\mathbf{R} = (X, Y)$ is the vector separating the vortices, and (15) show that it rotates in a circle, we have also shown that the vortices orbit each other. The orbital frequency ω given by (14) is clearly inversely proportional to R^2, where $R = |\mathbf{R}|$ is the magnitude of the vortices' separation.

Problem 6. Conserved quantities for a system of line vortices

Now consider the general case of n vortices.

6.a Verify that the "total linear momentum along x" and the "total linear momentum along y,"

$$P_x = \sum_{i=1}^{n} \gamma^{(i)} y_i, \qquad P_y = \sum_{i=1}^{n} -\gamma^{(i)} x_i,$$

are conserved.

Solution. In order to show that P_x and P_y are conserved, we will show that $[H, P_x] = [H, P_y] = 0$, where H is given by (6). For all i, j,

$$\frac{\partial P_x}{\partial x_i} = 0, \qquad \frac{\partial P_x}{\partial y_i} = \gamma^{(i)}, \qquad \frac{\partial P_y}{\partial x_i} = -\gamma^{(i)}, \qquad \frac{\partial P_y}{\partial y_i} = 0,$$

and

$$\frac{\partial H}{\partial x_i} = \sum_{j \neq i} -\gamma^{(i)}\gamma^{(j)} \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \qquad \frac{\partial H}{\partial y_i} = \sum_{j \neq i} -\gamma^{(i)}\gamma^{(j)} \frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}.$$

Applying (7),

$$[H, P_x] = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \left(\frac{\partial H}{\partial x_i}\frac{\partial P_x}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial P_x}{\partial x_i} \right) = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \frac{\partial H}{\partial x_i}\frac{\partial P_x}{\partial y_i} = \sum_{i=1}^{n}\sum_{j \neq i} -\gamma^{(i)}\gamma^{(j)} \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} = 0,$$

$$[H, P_y] = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \left(\frac{\partial H}{\partial x_i}\frac{\partial P_y}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial P_y}{\partial x_i} \right) = \sum_{i=1}^{n} -\frac{1}{\gamma^{(i)}} \frac{\partial H}{\partial y_i}\frac{\partial P_y}{\partial x_i} = \sum_{i=1}^{n}\sum_{j \neq i} \gamma^{(i)}\gamma^{(j)} \frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} = 0,$$

as desired. $\square$

6.b Verify that $[P_x, P_y]$ gives a conserved quantity.

Solution. As in 6.a, we will show that $[H, [P_x, P_y]] = 0$. Firstly, note that

$$[P_x, P_y] = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \left(\frac{\partial P_x}{\partial x_i}\frac{\partial P_y}{\partial y_i} - \frac{\partial P_x}{\partial y_i}\frac{\partial P_y}{\partial x_i} \right) = \sum_{i=1}^{n} -\frac{1}{\gamma^{(i)}} \frac{\partial P_x}{\partial y_i}\frac{\partial P_y}{\partial x_i} = \sum_{i=1}^{n} \gamma^{(i)}.$$

It is obvious that

$$\frac{\partial [P_x, P_y]}{\partial x_i} = \frac{\partial [P_x, P_y]}{\partial y_i} = 0,$$

for all i, j, so it follows trivially that

$$[H, [P_x, P_y]] = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \left(\frac{\partial H}{\partial x_i}\frac{\partial [P_x, P_y]}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial [P_x, P_y]}{\partial x_i} \right) = 0$$

as desired. $\square$

Problem 7. Charged particle in a magnetic field

Suppose a charged particle moves in a two-dimensional plane while experiencing a magnetic field $\mathbf{B} = (0, 0, B)$. Use the vector potential $\mathbf{A} = (-By, 0, 0)$. The Hamiltonian for the particle is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{p_y^2}{2m}.$$

7.a Write down Hamilton's equations. Verify that by appropriate manipulation we have

$$p_y + \frac{eB}{c}x = a, \qquad p_x = m\dot{x} - \frac{eB}{c}y = b,$$

where a and b are constants.

Solution. Note that

$$H = \frac{1}{2m}\left(p_x^2 + 2\frac{eB}{c}p_x y + \frac{e^2B^2}{c^2}y^2\right) + \frac{p_y^2}{2m} = \frac{p_x^2}{2m} + \frac{eB}{c}\frac{p_x y}{m} + \frac{e^2B^2}{c^2}\frac{y^2}{2m} + \frac{p_y^2}{2m}$$

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{eB}{c}y\right), \tag{16}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \tag{17}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \tag{18}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{eB}{c}\frac{1}{m}\left(p_x + \frac{eB}{c}y\right). \tag{19}$$

Substituting (16) into (19),

$$\dot{p}_y = -\frac{eB}{c}\dot{x}.$$

By integrating with respect to t, we obtain

$$p_y = -\frac{eB}{c}\int \dot{x}\,dt = -\frac{eB}{c}x + a,$$

where a is some constant. Therefore, we have

$$p_y + \frac{eB}{c}x = a, \tag{20}$$

as desired.

From (16),

$$m\dot{x} = p_x + \frac{eB}{c}y \iff p_x = m\dot{x} - \frac{eB}{c}y,$$

and from (17),

$$p_x = \int 0\,dt = b,$$

where b is some constant. Combining these, we have

$$p_x = m\dot{x} - \frac{eB}{c}y = b \tag{21}$$

as desired. □

7.b Using the relations above and the equations of motion, verify that the charged particle moves in a circle in the (x, y) plane and that the circling frequency ω is given by

$$\omega = \frac{eB}{mc}.$$

This is called the *Larmor frequency*.

Solution. Substituting (21) into (16) yields

$$\dot{x} = \frac{1}{m}\left(b + \frac{eB}{c}y\right) = \frac{eB}{mc}\left(\frac{c}{eB}b + y\right). \tag{22}$$

Similarly, solving (20) for p_y and substituting into (18) gives us

$$\dot{y} = \frac{1}{m}\left(a - \frac{eB}{c}x\right) = \frac{eB}{mc}\left(\frac{c}{eB}a - x\right). \tag{23}$$

Differentiating (22) and (23) by t, we obtain two uncoupled second-order equations:

$$\ddot{x} = \frac{eB}{mc}\dot{y} = -\frac{e^2B^2}{m^2c^2}\left(x - \frac{c}{eB}a\right), \qquad \ddot{y} = -\frac{eB}{mc}\dot{x} = -\frac{e^2B^2}{m^2c^2}\left(y + \frac{c}{eB}b\right). \tag{24}$$

Let $\tilde{x}$ and $\tilde{y}$ be new coordinates such that

$$\tilde{x} \equiv x - \frac{c}{eB}a, \qquad \tilde{y} \equiv y + \frac{c}{eB}b.$$

Then

$$\frac{d\tilde{x}}{dt} = \dot{x}, \qquad \frac{d^2\tilde{x}}{dt^2} = \ddot{x}, \qquad \frac{d\tilde{y}}{dt} = \dot{y}, \qquad \frac{d^2\tilde{y}}{dt^2} = \ddot{y},$$

and the equations (24) can be rewritten in terms of $\tilde{x}$ and $\tilde{y}$:

$$\frac{d^2\tilde{x}}{dt^2} = -\frac{e^2B^2}{m^2c^2}\tilde{x}, \qquad \frac{d^2\tilde{y}}{dt^2} = -\frac{e^2B^2}{m^2c^2}\tilde{y}.$$

These equations have the solutions

$$\tilde{x}(t) = C_1\cos\left(\frac{eB}{mc}t\right) + C_2\sin\left(\frac{eB}{mc}t\right) \equiv C_1\cos(\omega t) + C_2\sin(\omega t),$$
$$\tilde{y}(t) = D_1\cos\left(\frac{eB}{mc}t\right) + D_2\sin\left(\frac{eB}{mc}t\right) \equiv D_1\cos(\omega t) + D_2\sin(\omega t),$$

where C_1, C_2, D_1, D_2 are constants, and we have defined

$$\omega \equiv \frac{eB}{mc}. \tag{25}$$

Applying (22) and (23), we have

$$\frac{d\tilde{x}}{dt} = -C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t) = \omega\tilde{y}, \qquad \frac{d\tilde{y}}{dt} = -D_1\omega\sin(\omega t) + D_2\omega\cos(\omega t) = -\omega\tilde{x}.$$

This implies $C_1 = -D_2$ and $C_2 = D_1$. We may fix $C_1 = D_2 = 0$ and $C_2 = D_1 = R$ without loss of generality, where R is some constant. Transforming back to the original coordinates, we have

$$x(t) = R\sin(\omega t) + \frac{c}{eB}a, \qquad y(t) = R\cos(\omega t) - \frac{c}{eB}b.$$

These solutions show that the particle moves in a circle with angular frequency ω defined by (25), as desired. □

7.c Now consider the limit where the B field can be made arbitrarily strong. Compare the Poisson bracket $[x, p_x]$ for the charged particle with the Poisson bracket relation

$$[x_i, y_j] = \frac{\delta_{ij}}{\gamma^{(i)}} \tag{26}$$

for the system of line vortices described in problems 5 and 6.

Solution. In general, the Poisson bracket for the charged particle is given by

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x} - \frac{\partial f}{\partial p_x}\frac{\partial g}{\partial x}.$$

Note that

$$\frac{\partial x}{\partial x} = \frac{\partial p_x}{\partial p_x} = 1, \qquad \frac{\partial x}{\partial p_x} = \frac{\partial p_x}{\partial x} = 0,$$

so

$$[x, p_x] = \frac{\partial x}{\partial x}\frac{\partial p_x}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial p_x}{\partial x} = 1.$$

In the limit $B \to \infty$, (21) becomes

$$p_x \to -\frac{eB}{c}y.$$

It follows that

$$\frac{\partial y}{\partial p_x} = -\frac{c}{eB}, \qquad \frac{\partial y}{\partial x} = 0.$$

Thus,

$$[x, y] = \frac{\partial x}{\partial x}\frac{\partial y}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial y}{\partial x} = -\frac{c}{eB}. \tag{27}$$

Consider a particular vortex i from the line vortex problem. Its position is (x_i, y_i), and as we discussed in class, y_i may also be treated as its generalized momentum. From (26), the Poisson bracket relating the position coordinates of the vortex in the two-dimensional plane is inversely proportional to the strength of the vortex itself. This is similar to the charged particle Poisson bracket in (27), which relates the position coordinates of the particle in the plane perpendicular to a strong magnetic field. This bracket is inversely proportional to the strength of the field. So, in both cases, we see an inverse proportionality between position and "strength," which suggests a similarity between the two systems.

While writing up these solutions, I consulted Gelfand and Fomin's *Calculus of Variations*, Goldstein's *Classical Mechanics*, and Tong's *Classical Dynamics*.