Problem 1. (Peskin & Schroeder 2.1) Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

1(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying

$$E^i = -F^{0i}; \qquad \epsilon^{ijk}B^k = -F^{ij}. \tag{2}$$

Solution. We want to extremize the action,

$$S[A_\mu] = \int d^4x\, \mathcal{L}(A_\mu, \partial_\mu A_\mu)$$

Let δA_μ denote some arbitrary variation that vanishes at the boundaries of spacetime. The action for $A_\mu + \delta A_\mu$ is

$$S[A_\mu + \delta A_\mu] = \int d^4x\, \mathcal{L}(A_\mu + \delta A_\mu, \partial_\nu A_\mu + \partial_\nu \delta A_\mu).$$

Then, to first order in δA_μ, the variation of the action is

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu],$$

which we want to vanish for all δA_μ. Let $\delta F_{\mu\nu} = \partial_\mu\, \delta A_\nu - \partial_\nu\, \delta A_\mu$. Then, applying the definition of $F_{\mu\nu}$ given in Eq. (1),

$$\begin{aligned}
\delta S &= \int d^4x \left(-\frac{1}{4}(F_{\mu\nu} + \delta F_{\mu\nu})(F^{\mu\nu} + \delta F^{\mu\nu}) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\
&\approx \int d^4x \left(-\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\, \delta F^{\mu\nu} + \delta F_{\mu\nu}\, F^{\mu\nu}) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\
&= \int d^4x \left(-\frac{1}{4}(F_{\mu\nu}\, \delta F^{\mu\nu} + \delta F_{\mu\nu}\, F^{\mu\nu})\right) \\
&= \int d^4x \left(-\frac{1}{2}\delta F_{\mu\nu}\, F^{\mu\nu}\right),
\end{aligned} \tag{3}$$

where we have discarded terms of $\mathcal{O}\big((\delta A^\mu)^2\big)$ and swapped covariant and contravariant indices in going to the final equality.

Note that

$$\begin{aligned}
\delta F_{\mu\nu}\, F^{\mu\nu} &= (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \delta A_\nu\, \partial^\mu A^\nu - \partial_\mu \delta A_\nu\, \partial^\nu A^\mu - \partial_\nu \delta A_\mu\, \partial^\mu A^\nu + \partial_\nu \delta A_\mu\, \partial^\nu A^\mu.
\end{aligned} \tag{4}$$

Integrating the first term of Eq. (4) by parts, we have

$$\int d^4x\, \frac{\partial\, \delta A_\nu}{\partial x^\mu}\frac{\partial A^\nu}{\partial x_\mu} = \left[\delta A_\nu \frac{\partial A^\nu}{\partial x_\mu}\right]_{-\infty}^{\infty} - \int d^4x\, \delta A_\nu \frac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} = -\int d^4x\, \delta A_\nu\, \partial_\mu \partial^\mu A^\nu,$$

because δA^ν vanishes at $\pm\infty$. The other terms follow similarly. Then we find

$$\begin{aligned}\int d^4x\,\delta F_{\mu\nu}\,F^{\mu\nu} &= -\int d^4x\,(\delta A_\nu\,\partial_\mu\partial^\mu A^\nu - \delta A_\nu\,\partial_\mu\partial^\nu A^\mu - \delta A_\mu\,\partial_\nu\partial^\mu A^\nu + \delta A_\mu\,\partial_\nu\partial^\nu A^\mu)\\ &= -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\mu\,\partial_\nu F^{\nu\mu}) = -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\nu\,\partial_\mu F^{\mu\nu})\\ &= -2\int d^4x\,\delta A_\nu\,\partial_\mu F^{\mu\nu},\end{aligned}$$

where in going to the second-to-last equality we have simply swapped the indices.

Making this substitution in Eq. (3), we obtain

$$\delta S = \delta A_\nu \int d^4x\,\partial_\mu F^{\mu\nu}.$$

In order for the action to be at a local extremum, we need $\delta S = 0$ for any δA_ν. This implies that the integrand is 0. Thus, we obtain

$$\partial_\mu F^{\mu\nu} = 0, \tag{5}$$

which is the covariant form of the inhomogeneous Maxwell equations in a source-free region [**?** , p. 557], as we sought to derive. □

From Eq. (2) and the knowledge that $F^{\mu\nu}$ is antisymmetric [**?** , p. 556], we can write

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}.$$

The first equation of Eq. (2) is equivalent to $E^i = F^{i0}$. Then the zeroth component of Eq. (5) can be written

$$\partial_\mu F^{\mu 0} = \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = \boldsymbol{\nabla}\cdot\mathbf{E} = 0,$$

which is the differential form of Gauss's law.

For the remaining components of Eq. (5), we apply the second equation of Eq. (5) to find

$$\partial_\mu F^{\mu i} = -\frac{\partial E^i}{\partial t} + \epsilon^{ijk}\frac{\partial B^k}{\partial x^j} = 0.$$

In vector form, this is

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{0},$$

the differential form of Ampère's law.