Problem 1. Let T be a rank one tensor which takes as argument vectors in a finite-dimensional vector space equipped with an inner product (either Euclidean with all spacelike directions or Lorenztian with one timelike direction). Prove that there exists a vector V such that for all vectors W,

$$T(W) = V \cdot W. \tag{1}$$

Solution. We can expand W in terms of the basis vectors e_i:

$$W = W^i e_i.$$

By the linearity of tensors [1, p. 11],

$$T(W^i e_i) = W^i T(e_i).$$

We can also write the inner product in index notation:

$$V \cdot W = V_i W^i.$$

Then Eq. (1) holds so long as

$$V_i = T(e_i). \tag{2}$$

This definition gives us

$$T(W) = T(W^i e_i) = W^i T(e_i) = W^i V_i = V \cdot W,$$

for all W. Thus we have proven that such a V exists and has the form of Eq. (2). □

Problem 2. Relativistic gravitational force law (MCP 2.4) In Newtonian theory, the gravitational potential Φ exerts a force $\mathbf{F} = d\mathbf{p}/dt = -m\boldsymbol{\nabla}\Phi$ on a particle with mass m and momentum $\mathbf{p}$. Before Einstein formulated general relativity, some physicists constructed relativistic theories of gravity in which a Newtonian-like scalar gravitational field Φ exerted a 4-force $\vec{F} = d\vec{p}/d\tau$ on any particle with rest mass m, 4-velocity $\vec{u}$, and 4-momentum $\vec{p} = m\vec{u}$. What must that force law have been for it to (i) obey the Principle of Relativity, (ii) reduce to Newton's law in the nonrelativistic limit, and (iii) preserve the particle's rest mass as time passes?

Solution. We follow a process similar to that in Sec. 2.4.2 of MCP [1, pp. 52–53]. In order to satisfy condition (i), we need to be able to write the law in geometric, frame-independent object [1, p. 46]. To satisfy (ii), we need the 4-force $\vec{F} = d\vec{p}/d\tau$ to be proportional to m and linear in $\boldsymbol{\nabla}\Phi$, as in Newton's law in the nonrelativistic limit. A tensor is a linear function of n vectors [1, p. 11], so we need to include a second-rank tensor that takes $\boldsymbol{\nabla}\Phi$ in one of its slots. We will call it the gravitational field tensor, F. It is related to $\vec{F}$ via

$$\vec{F} = mF(\_\_, \boldsymbol{\nabla}\Phi).$$

For condition (iii), we can apply MCP (2.17), which states that the 4-force must be orthogonal to the 4-momentum for the rest mass to be preserved as time passes:

$$0 = \frac{dm^2}{d\tau} = -\frac{d\vec{p}^2}{d\tau} = -2\vec{p} \cdot \frac{d\vec{p}}{d\tau} = -2\vec{p} \cdot \vec{F}.$$

In terms of our problem, we need $\vec{F} \cdot \vec{u} = \vec{F}(\vec{u}) = 0$ where $\vec{u}$ is a timelike unit-length vector. That is, we require $F(\vec{u}, \boldsymbol{\nabla}\Phi) = 0$. So our force law is

$$\frac{d\vec{p}}{d\tau} = mF(\_\_, \boldsymbol{\nabla}\Phi), \quad \text{where} \quad F(\vec{u}, \boldsymbol{\nabla}\Phi) = 0$$

for any timelike unit-length vector $\vec{u}$.

Problem 3. Index gymnastics (MCP 2.8)

3(a) Simplify the following expression so the metric does not appear in it:

$$A^{\alpha\beta\gamma} g_{\beta\rho} S_{\gamma\lambda} g^{\rho\delta} g^{\lambda}{}_{\alpha}.$$

Solution. Equation (2.23d) in MCP states that

$$A_\alpha = g_{\alpha\beta} A^\beta, \qquad A^\alpha = g^{\alpha\beta} A_\beta, \qquad T^{\alpha}{}_{\mu\nu} \equiv g_{\mu\beta} g_{\nu\gamma} T^{\alpha\beta\gamma}, \qquad T^{\alpha\beta\gamma} \equiv g^{\beta\mu} g^{\gamma\nu} T^{\alpha}{}_{\mu\nu}.$$

We are free to reorder the factors in the expression. We find

$$A^{\alpha\beta\gamma} g_{\beta\rho} S_{\gamma\lambda} g^{\rho\delta} g^{\lambda}{}_{\alpha} = A^{\alpha}{}_{\rho}{}^{\gamma} g^{\rho\delta} S_{\gamma\lambda} g^{\lambda}{}_{\alpha} = A^{\alpha\delta\gamma} S_{\gamma\alpha}.$$

3(b) The quantity $g_{\alpha\beta} g^{\alpha\beta}$ is a scalar since it has no free indices. What is its numerical value?

Solution. From (2.23c) in MCP, $g_{\alpha\beta} = g^{\alpha\beta} = \eta_{\alpha\beta}$. Further, (2.22) states that

$$\eta_{00} \equiv -1, \qquad \eta_{11} \equiv \eta_{22} \equiv \eta_{33} \equiv 1, \qquad \eta_{\alpha\beta} \equiv 0 \text{ if } \alpha \neq \beta.$$

Thus

$$\begin{aligned} g_{\alpha\beta} g^{\alpha\beta} &= \eta_{\alpha\beta} \eta^{\alpha\beta} \\ &= \eta_{\alpha 0} \eta^{\alpha 0} + \eta_{\alpha 1} \eta^{\alpha 1} + \eta_{\alpha 2} \eta^{\alpha 2} + \eta_{\alpha 3} \eta^{\alpha 3} \\ &= \eta_{00} \eta^{00} + \eta_{11} \eta^{11} + \eta_{22} \eta^{22} + \eta_{33} \eta^{33} \\ &= 1 + 1 + 1 + 1 \\ &= 4. \end{aligned}$$

3(c) What is wrong with the following expression and equation?

$$A_\alpha{}^{\beta\gamma} S_{\alpha\gamma}; \qquad A_\alpha{}^{\beta\gamma} S_\beta T_\gamma = R_{\alpha\beta\gamma} S^\beta.$$

Solution. The expression $A_\alpha{}^{\beta\gamma} S_{\alpha\gamma}$ is nonsensical because it contains two like indices that are not contracted. If the α indices are intended to be summed, one must be an upper index and the other a lower index.

The equation $A_\alpha{}^{\beta\gamma} S_\beta T_\gamma = R_{\alpha\beta\gamma} S^\beta$ is invalid because there is a free γ index on the right-hand side, but not on the left-hand side. When contracted, the left-hand side will have only a lower α index remaining. However, the right-hand side has a free lower α index and a free lower γ index. It is impossible for the two to be equivalent.

Problem 4. Spacetime diagrams (MCP 2.14) Use spacetime diagrams to prove the following.

4(a) Two events that are simultaneous in one inertial frame are not necessarily simultaneous in another. More specifically, if frame $\overline{\mathcal{F}}$ moves with velocity $\vec{v} = \beta\vec{e}_x$ as seen in frame $\mathcal{F}$, where $\beta > 0$, then of two events that are simultaneous in $\overline{\mathcal{F}}$ the one farther "back" (with the more negative value of $\bar{x}$) will occur in $\mathcal{F}$ before the one farther "forward."

Solution. Figure 1 (a) shows a spacetime diagram for frames $\mathcal{F}$ and $\overline{\mathcal{F}}$. Two events, $\mathcal{P}_1$ and $\mathcal{P}_2$, are located on the $\bar{x}$ axis. In $\overline{\mathcal{F}}$, they occur simultaneously at $\bar{t} = 0$. The dotted lines extending from the events to the t axis show when the events occur in $\mathcal{F}$. Event $\mathcal{P}_1$ is farther back and occurs earlier in $\mathcal{F}$ than event $\mathcal{P}_2$. Thus two events that are simultaneous in $\overline{\mathcal{F}}$ are not simultaneous in $\mathcal{F}$. □

4(b) Two events that occur at the same spatial location in one inertial frame do not necessarily occur at the same spatial location in another.

Solution. Figure 1 (b) shows a spacetime diagram in which events $\mathcal{P}_1$ and $\mathcal{P}_2$ both occur at position $\bar{x} = 0$ in $\overline{\mathcal{F}}$. The dotted lines extending from the events to the x axis show that, in $\mathcal{F}$, event $\mathcal{P}_1$ occurs at $x < 0$ and event $\mathcal{P}_2$ at $x > 0$. Hence, the events do not occur at the same position in $\mathcal{F}$. □

4(c) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two events with a timelike separation, then there exists an inertial reference frame in which they occur at the same spatial location, and in that frame the time lapse between them is equal to the square root of the negative of their invariant interval, $\Delta t = \Delta\tau \equiv \sqrt{-(\Delta s)^2}$.

Solution. Figure 1 (c) shows a spacetime diagram in which axes are given arbitrary units. Events $\mathcal{P}_1$ and $\mathcal{P}_2$ occur at the same location in $\mathcal{F}$, $x = 2$. The dotted lines show the $(\bar{t}, \bar{x})$ coordinates of $\mathcal{P}_1$ and $\mathcal{P}_2$. The interval is clearly timelike as viewed from $\overline{\mathcal{F}}$ since $\Delta\bar{t} > \Delta\bar{x}$. Since the interval is invariant of reference frame, it can be calculated in one spatial and one temporal dimension by adapting (2.2a) in MCP:

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2. \tag{3}$$

In $\mathcal{F}$, $\Delta x = 0$ and $\Delta t = 1$. Then

$$(\Delta s)^2 = -1^2 + 0^2 = -1 < 0,$$

so the interval is indeed timelike [1, p. 45]. Since the events are at the same spatial location in $\mathcal{F}$, we calculate the time lapse between them in $\mathcal{F}$ as

$$\sqrt{-(\Delta s)^2} = \sqrt{1} = 1,$$

which is exactly as shown in Fig. 4 (c). □

4(d) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two events with a spacelike separation, then there exists an inertial reference frame in which they are simultaneous, and in that frame the spatial distance between them is equal to the square root of their invariant interval, $\sqrt{g_{ij}\Delta x^i \Delta x^j} = \Delta s \equiv \sqrt{(\Delta s)^2}$.

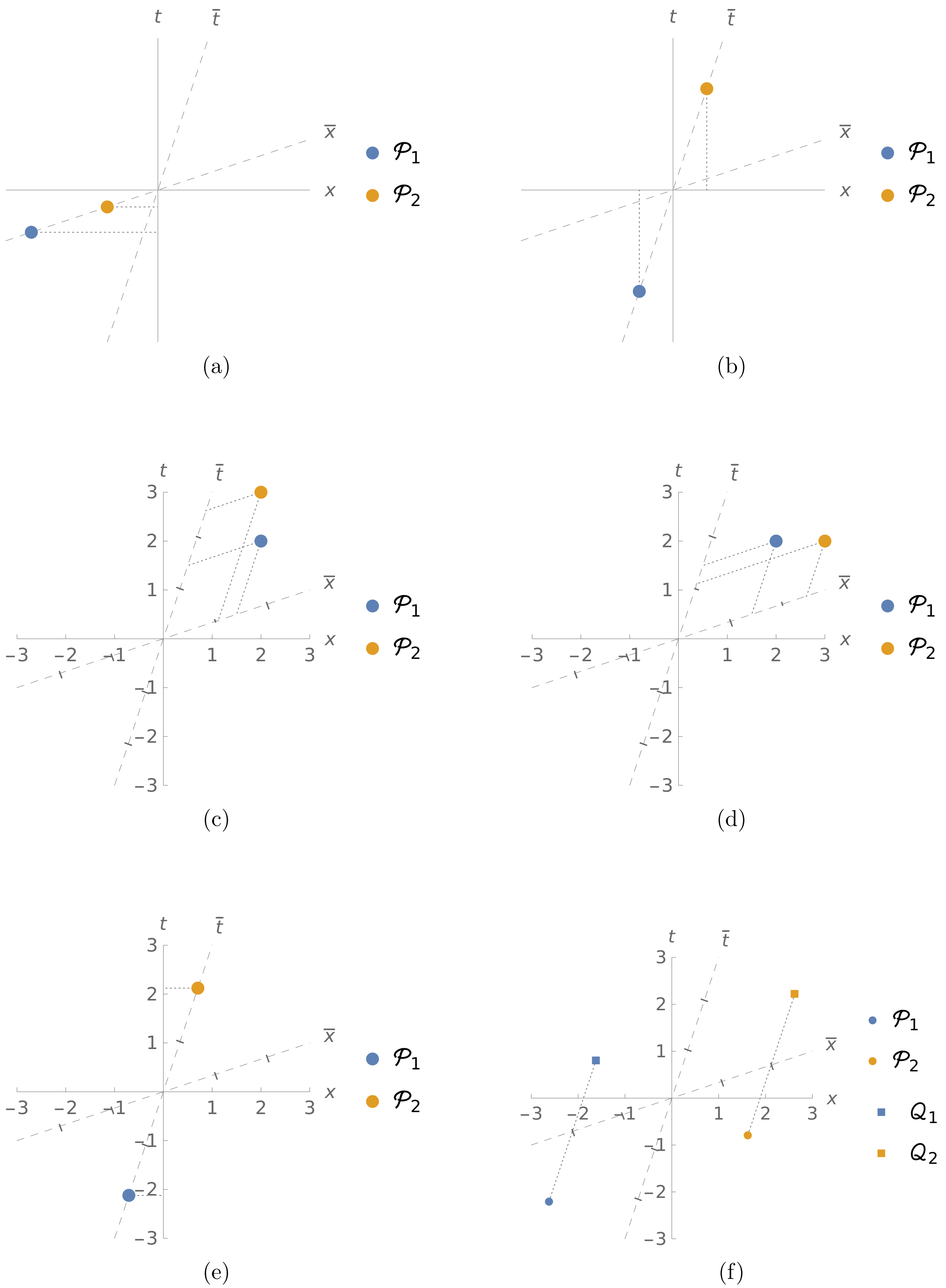

Figure 1: Spacetime diagrams for problems 4.

Solution. Figure 1 (d) shows a spacetime diagram in which events $\mathcal{P}_1$ and $\mathcal{P}_2$ have a spacelike separation. The dotted lines show the $(\bar{t}, \bar{x})$ coordinates of $\mathcal{P}_1$ and $\mathcal{P}_2$. The interval is obviously spacelike as viewed from $\overline{\mathcal{F}}$ because $\Delta\bar{t} < \Delta\bar{x}$. In $\mathcal{F}$, the events occur simultaneously at $t = 2$, and $\Delta x = 1$, $\Delta t = 0$. Using Eq. (3) to compute the interval, we find

$$(\Delta s)^2 = -0^2 + 1^2 = 1 > 0,$$

so the interval is indeed spacelike [1, p. 45]. Since the events are simultaneous in $\mathcal{F}$, we calculate the spatial distance between them as

$$\sqrt{(\Delta s)^2} = \sqrt{1} = 1,$$

which is exactly as shown in Fig. 4 (d). □

4(e) If the inertial frame $\overline{\mathcal{F}}$ moves with speed β relative to the frame $\mathcal{F}$, then a clock at rest in $\overline{\mathcal{F}}$ ticks more slowly as viewed from $\mathcal{F}$ than as viewed from $\overline{\mathcal{F}}$—more slowly by a factor $\gamma^{-1} = \sqrt{1-\beta^2}$. This is called *relativistic time dilation*. As one consequence, the lifetimes of unstable particles moving with a speed β are increased by the Lorentz factor γ.

Solution. Figure 1 (e) shows a spacetime diagram in which $\mathcal{P}_1$ represents a tick of a clock and $\mathcal{P}_2$ the clock's subsequent tick. Both events occur at $\bar{x} = 0$, so the clock is stationary in $\overline{\mathcal{F}}$. $\mathcal{P}_1$ occurs at $\bar{t} = -2$ and $\mathcal{P}_2$ occurs at $\bar{t} = 2$. In $\overline{\mathcal{F}}$, the time between ticks is $\Delta\bar{t} = 4$; in $\mathcal{F}$, the dotted lines indicate that $\Delta t > 4$. In this figure

$$\beta = \frac{1}{3}, \qquad \gamma = \frac{3}{2\sqrt{2}} \approx 1.06. \tag{4}$$

So in $\mathcal{F}$, the time between ticks is $\Delta t = 4\gamma \approx 4.24$. The dotted lines indeed intersect the t axis at $t = \pm 2\gamma \approx \pm 2.12$, which confirms time dilation. □

4(f) If the inertial frame $\overline{\mathcal{F}}$ moves with velocity $\vec{v} = \beta\vec{e}_x$ relative to the frame $\mathcal{F}$, then an object at rest in $\overline{\mathcal{F}}$ as studied in $\mathcal{F}$ appears shortened by a factor $\gamma^{-1} = \sqrt{1-\beta^2}$ along the x direction, but its length along the y and z directions is unchanged. This is called *Lorentz contraction*. As one consequence, heavy ions moving at high speeds in a particle accelerator appear to act like pancakes, squashed along their directions of motion.

Solution. Figure 1 (f) shows a spacetime diagram in which a rod at rest in $\overline{\mathcal{F}}$ has endpoints located at $\mathcal{P}_1, \mathcal{P}_2$. The dotted lines represent the worldlines of the ends of the rod as time passes and the rod moves in the x direction with velocity $\vec{v} = \vec{e}_x/3$. After some length of time has passed, the ends of the rod are located at $\mathcal{Q}_1, \mathcal{Q}_2$, but its position in $\overline{\mathcal{F}}$ has not changed.

The length of the rod in $\overline{\mathcal{F}}$ is $\Delta\bar{x} = 4$. The length of the rod in $\mathcal{F}$, Δx, is the distance between the points at which the worldlines intersect the x axis. Clearly $\Delta x < 4$. Applying Eq. (4) once more, we see that $\Delta x = 4/\gamma \approx 3.77$. The worldlines intersect the x axis at $x = \pm 1.89$, which confirms length contraction [2, p. 194].

An observer in $\mathcal{F}$ does not see the rod move in the y or z directions. Therefore special relativity is not relevant when considering these directions, so the length of the rod in the y and z directions is the same in $\mathcal{F}$ as it is in $\overline{\mathcal{F}}$. □

Problem 5. Twins paradox (MCP 2.16)

5(a) The 4-acceleration of a particle or other object is defined by $\vec{a} \equiv d\vec{u}/d\tau$, where $\vec{u}$ is its 4-velocity and τ is proper time along its world line. Show that, if an observer carries an accelerometer, the magnitude $|\mathbf{a}|$ of the 3-dimensional acceleration $\mathbf{a}$ measured by the accelerometer will always be equal to the magnitude of the observer's 4-acceleration, $|\mathbf{a}| = |\vec{a}| \equiv \sqrt{\vec{a} \cdot \vec{a}}$.

Solution. The velocity and acceleration of the observer and of the accelerometer are the same. From MCP (2.25c), their 4-velocity can be written

$$\vec{u} = (\gamma, \gamma\mathbf{v}) = \gamma(1, \mathbf{v}),$$

where $\mathbf{v} = d\mathbf{x}/dt$ and $\mathbf{x}$ is the position in 3-space. Then the acceleration is

$$\vec{a} = \frac{d\vec{u}}{d\tau} = \gamma\frac{d\vec{u}}{dt} = \gamma\left(\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\frac{d\mathbf{v}}{dt}\right),$$

since $d\tau = dt/\gamma$ [2, p. 201]. But in the rest frame of the accelerometer and observer, $\gamma = 1$ and $d\gamma/dt = 0$. So

$$\vec{a} = \left(0, \frac{d\mathbf{v}}{dt}\right) = (0, \mathbf{a}).$$

Then

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{\mathbf{a} \cdot \mathbf{a}} = |\mathbf{a}|,$$

as we wanted to show. □

5(b) In the twins paradox of Fig. 2.8a, suppose that Florence begins at rest beside Methuselah, then accelerates in Methuselah's x-direction with an acceleration a equal to one Earth gravity, g, for a time $T_{\text{Florence}}/4$ as measured by her, then accelerates in the $-x$-direction at g for a time $T_{\text{Florence}}/2$, thereby reversing her motion; then she accelerates in the $+x$-direction at g for a time $T_{\text{Florence}}/4$, thereby returning to rest beside Methuselah. (This is the type of motion shown in the figure.) Show that the total time lapse as measured by Methuselah is

$$T_{\text{Methuselah}} = \frac{4}{g}\sinh\left(\frac{gT_{\text{Florence}}}{4}\right). \tag{5}$$

Solution. Let a be Florence's acceleration as measured in Methuselah's rest frame $\mathcal{F}$. For the first leg of her journey, Florence's proper acceleration is $\bar{a} = g$. The expression for the Lorentz transformation relating the proper acceleration is [3]

$$\bar{a} = \frac{a}{(1-\beta^2)^{3/2}}.$$

Rearranging and substituting yields (for $c = 1$)

$$a = g(1-\beta^2)^{3/2} = g(1-v^2)^{3/2},$$

where v is Florence's velocity as observed by Methuselah. Substituting $a = dv/dt$ gives us a differential equation for v:

$$\frac{dv}{dt} = g(1-v)^{3/2} \quad\Longrightarrow\quad \int_0^v \frac{dv'}{(1-v'^2)^{3/2}} = g\int_0^t dt' \quad\Longrightarrow\quad \frac{v}{\sqrt{1-v^2}} = gt,$$

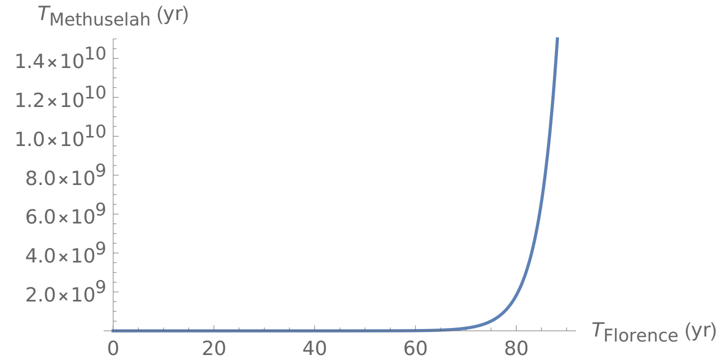

Figure 2: Plot of Eq. (5) indicating T_{Florence} when $T_{\text{Methuselah}} \approx 14 \times 10^9$ yr.

where we start counting at $t = 0$ for Methuselah, and Florence's initial velocity is 0. (This integral and all following computations in this problem are performed using Mathematica.) Solving for v yields

$$v = \frac{gt}{\sqrt{1 + (gt)^2}}.$$

Let x be Florence's position as measured by Methuselah. It may be found by substituting $v = dx/dt$ and solving the resulting differential equation:

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1 + (gt)^2}} \quad \Longrightarrow \quad \int_0^x dx' = \int_0^t dt' \, \frac{gt'}{\sqrt{1 + (gt')^2}} \quad \Longrightarrow \quad x = \frac{\sqrt{(gt)^2 + 1} - 1}{g},$$

where we have placed Florence's initial position at $x = 0$.

Then somehow we use $d\tau = dt\,/\gamma$?

5(c) Show that in the geometrized units used here, Florence's acceleration (equal to acceleration of gravity at the surface of Earth) is $g = 1.033\,\text{yr}^{-1}$. Plot $T_{\text{Methuselah}}$ as a function of T_{Florence}, and from your plot estimate T_{Florence} if $T_{\text{Methuselah}}$ is the age of the Universe, 14 billion years.

Solution. Note that

$$c = 1 \quad \Longrightarrow \quad 1\,\text{m} \approx \frac{1}{2.998 \times 10^8}\,\text{s} \approx 3.336 \times 10^{-9}\,\text{s},$$

Using this relation to convert g from SI units, as well as the fact that $1\,\text{yr} \approx 3.154 \times 10^7\,\text{s}$,

$$g = 9.807\,\text{m}\,\text{s}^{-2} = (9.807\,\text{s}^{-2})(3.336 \times 10^{-9}\,\text{s}) = (3.271 \times 10^{-8}\,\text{s}^{-1})(3.154 \times 10^7\,\text{s}\,\text{yr}^{-1}) \approx 1.033\,\text{yr}^{-1},$$

as we wanted to show. □

Figure 2 shows a plot of Eq. (5). When $T_{\text{Methuselah}} \approx 14 \times 10^9$ yr, $T_{\text{Florence}} \approx 88$ yr.

References

[1] K. S. Thorne and R. D. Blandford, "Modern Classical Physics". Princeton University Press, 2017.

[2] R. Resnick, "Introduction to Special Relativity". John Wiley & Sons, Inc., 1968.

[3] Wikipedia contributors, "Acceleration (special relativity)." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Acceleration_(special_relativity)`.