Problem 1. Linear sigma model (Peskin & Schroeder 4.3) The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Psi^i(x)$, $i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

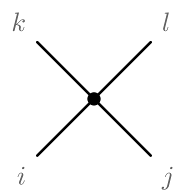
$$V(\Phi^2) = \frac{1}{2}m^2(\Phi^i)^2 + \frac{\lambda}{4}\left((\Phi^i)^2\right)^2 \tag{1}$$

is a function symmetric under rotations of $\mathbf{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

1(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$\overbracket{\Phi^i(x)\Phi^j}(y) = \delta^{ij} D_F(x-y),$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}). \tag{2}$$

Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

Solution. The Klein-Gordon Hamiltonian is given by Peskin & Schroeder (2.8),

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \tag{3}$$

For $\lambda = 0$, the linear sigma model has the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\Phi^i)^2 + \frac{1}{2}m^2(\Phi^i)^2 \right)$$

which is clearly N copies of the Klein-Gordon Hamiltonian, one for each i.

From (4.36) we know that the Feynman propagator is the contraction of two fields:

$$\overbracket{\phi(x)\phi}(y) = D_F(x-y).$$

No terms $\Phi^i\Phi^j$ for $i \neq j$ appear in the Hamiltonian, so fields with $i \neq j$ cannot be contracted. Moreover, each field is governed by its own independent Klein-Gordon Hamiltonian to zeroth order. So the propagator must be

$$\overbracket{\Phi^i(x)\Phi^j}(y) = \delta^{ij} D_F(x-y)$$

where $D_F(x-y)$ is the Klein-Gordon propagator. □

In order to determine the Feynman rules, we use Peskin & Schroeder (4.90),

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_n | iT | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p}_1 \cdots \mathbf{p}_n | T\left\{ \exp\left(-i \int_{-T}^{T} dt\, H_I(t) \right) \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

Our interaction Hamiltonian is

$$H_I = \int d^3x\, \frac{\lambda}{4} \left((\Phi^i)^2 \right)^2 = \int d^3x\, \frac{\lambda}{4} (\mathbf{\Phi} \cdot \mathbf{\Phi})^2 = \frac{\lambda}{4} \int d^3x \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right),$$

W have two final momenta, $\mathbf{p}_1$ and $\mathbf{p}_2$. Now we have

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle = \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p}_1 \mathbf{p}_2 | T\left\{ \exp\left[-i \int d^4x\, \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right) \right] \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

The first term that contributes to leading order is, by analogy to (4.92),

$$\begin{aligned} &{}_0\langle \mathbf{p}_1 \mathbf{p}_2 | T\left\{ -i \int d^4x\, \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right) \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 \\ &\qquad = {}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N\left\{ -i \int d^4x\, \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right) + \text{contractions} \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 , \end{aligned}$$

but only the terms in which none of the fields are contracted with each other will contribute [1, p. 111].

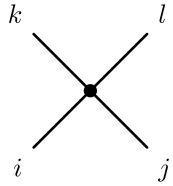
The first term represents the process $\Phi^i \Phi^i \to \Phi^i \Phi^i$:

$${}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N\left\{ -i \int d^4x\, \frac{\lambda}{4} \sum_i \Phi^i \Phi^i \Phi^i \Phi^i + \text{contractions} \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

The fields are all the same, so there are 4! ways of contracting the fields with the momenta, and we will obtain a diagram similar to (4.98). Adapting that expression, we find

$$-4!i \int \frac{\lambda}{4}\, d^4x\, e^{-i(p_\mathcal{A} + p_\mathcal{B} - p_1 - p_2)\cdot x} = -6i\lambda (4\pi)^4 \delta^4(p_\mathcal{A} + p_\mathcal{B} - p_1 - p_2).$$

The diagram in Eq. (2) is $\Phi^i \Phi^j \to \Phi^k \Phi^l$. Since $i = j = k = l$ for this term, we have



$$= -6i\lambda = -2i\lambda(1+1+1) = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

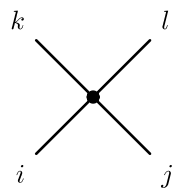
The second term can represent the processes $\Phi^i \Phi^i \to \Phi^j \Phi^j$ or $\Phi^i \Phi^j \to \Phi^i \Phi^j$ (where the indices and the order of the fields on either side is interchangeable):

$${}_0\langle \mathbf{p}_1 \mathbf{p}_2 | N\left\{ -i \int d^4x\, \frac{\lambda}{4} 2 \sum_{i \neq j} \Phi^i \Phi^i \Phi^j \Phi^j + \text{contractions} \right\} | \mathbf{p}_\mathcal{A} \mathbf{p}_\mathcal{B} \rangle_0 .$$

Now there are only $2! \times 2! = 4$ ways to contract the fields with the momenta. We have

$$-4i \int \frac{\lambda}{2}\, d^4x\, e^{-i(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)\cdot x} = -2i\lambda(4\pi)^4\delta^4(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2).$$

Here, either $i=j$ and $k=l$, $i=l$ and $j=k$, or $i=k$ and $j=l$. We have



$$= -2i\lambda = -2i\lambda(1+0+0) = -2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}).$$

Both of the terms can therefore be represented by Eq. (2) as we wanted to show. □

When all four of the particles in the interaction have the same mass, the differential cross section in the center-of-mass frame is given by Peskin & Schroeder (4.85)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2},$$

where E_{cm} is the center-of-mass energy and $\mathcal{M}$ is the invariant matrix element. We know that the diagrams we calculated before have the form $i\mathcal{M}(2\pi)^4\delta^4(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)$ [1, p. 112]. Then

$$\mathcal{M} = -6\lambda \quad \text{for} \quad \Phi^1\Phi^1 \to \Phi^1\Phi^1, \qquad\qquad \mathcal{M} = -2\lambda \quad \text{for} \quad \Phi^1\Phi^2 \to \Phi^1\Phi^2 \text{ and } \Phi^1\Phi^1 \to \Phi^2\Phi^2.$$

So the differential cross sections are, to leading order in λ,

$$(\Phi^1\Phi^2 \to \Phi^1\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-2\lambda|^2}{64\pi^2E_{\text{cm}}^2} = \frac{\lambda^2}{16\pi^2E_{\text{cm}}^2},$$
$$(\Phi^1\Phi^2 \to \Phi^1\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-6\lambda|^2}{64\pi^2E_{\text{cm}}^2} = \frac{9\lambda^2}{16\pi^2E_{\text{cm}}^2},$$
$$(\Phi^1\Phi^1 \to \Phi^2\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-6\lambda|^2}{64\pi^2E_{\text{cm}}^2} = \frac{9\lambda^2}{16\pi^2E_{\text{cm}}^2}.$$

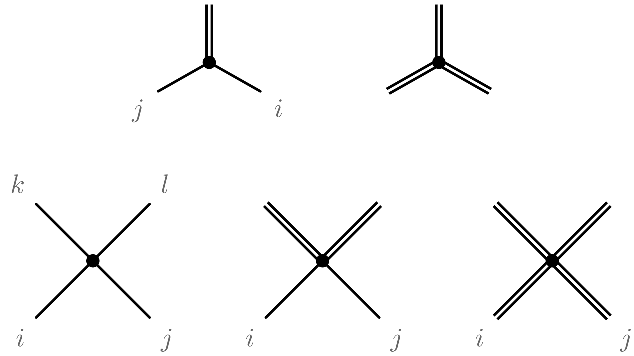
1(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \qquad i = 1,\dots,N-1, \qquad\qquad \Phi^N(x) = v + \sigma(x)$$

where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ *massless* pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:

$\contraction{}{\sigma}{}{\sigma}\sigma\sigma =$ 

$\contraction{}{\pi^i}{}{\pi^j}\pi^i\pi^j = i$  j



Solution. With the negative mass, Eq. (1) becomes

$$V(\Phi^2) = -\frac{1}{2}\mu^2(\Phi^i)^2 + \frac{\lambda}{4}\left((\Phi^i)^2\right)^2.$$

To find the minimum of V, we differentiate with respect to Φ^N. We stipulate

$$0 = \frac{\partial V}{\partial \Phi^N} = -\mu^2\Phi^N + \lambda(\mathbf{\Phi}\cdot\mathbf{\Phi})\Phi^N = -\mu^2\Phi^N + \lambda(\Phi^N)^3,$$

where we have used the chain rule to evaluate the second term, and the fact that V is minimal for all $\Phi^i = 0$ with $i \neq N$. This implies $\Phi^N = 0$ or

$$(\Phi^N)^2 = \frac{\mu^2}{\lambda}$$

when V is minimal. Thus

$$v = \frac{\mu}{\sqrt{\lambda}}.$$

In order to determine the form of the theory, we need to rewrite $V(\Phi^2)$ in the new coordinates. Note that $\mathbf{\Phi} = (\boldsymbol{\pi}, v+\sigma)$. Then

$$\begin{aligned}
V(\Phi^2) &= -\frac{1}{2}\mu^2\left[\boldsymbol{\pi}^2 + (v+\sigma)^2\right] + \frac{\lambda}{4}\left[\boldsymbol{\pi}^2 + (v+\sigma)^2\right]^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2\right) + \frac{\lambda}{4}\left(\boldsymbol{\pi}^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2\right)^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2\right) \\
&\qquad + \frac{\lambda}{4}\left((\boldsymbol{\pi}^2)^2 + 2\frac{\boldsymbol{\pi}^2\mu^2}{\lambda} + \frac{\mu^4}{\lambda^2} + 4\frac{\boldsymbol{\pi}^2\mu\sigma}{\sqrt{\lambda}} + 4\frac{\mu^3\sigma}{\lambda^{3/2}} + 2\boldsymbol{\pi}^2\sigma^2 + 6\frac{\mu^2\sigma^2}{\lambda} + 4\frac{\mu\sigma^3}{\sqrt{\lambda}} + \sigma^4\right) \\
&= -\frac{\boldsymbol{\pi}^2\mu^2}{2} - \frac{\mu^4}{2\lambda} - \frac{\mu^3\sigma}{\sqrt{\lambda}} - \frac{\mu^2\sigma^2}{2} + \frac{(\boldsymbol{\pi}^2)^2\lambda}{4} + \frac{\boldsymbol{\pi}^2\mu^2}{2} + \frac{\mu^4}{4\lambda} \\
&\qquad + \boldsymbol{\pi}^2\mu\sigma\sqrt{\lambda} + \frac{\mu^3\sigma}{\sqrt{\lambda}} + \frac{\boldsymbol{\pi}^2\sigma^2\lambda}{2} + \frac{3\mu^2\sigma^2}{2} + \mu\sigma^3\sqrt{\lambda} + \frac{\sigma^4\lambda}{4} \\
&= -\frac{\mu^4}{4\lambda} + \mu^2\sigma^2 + \frac{(\boldsymbol{\pi}^2)^2\lambda}{4} + \boldsymbol{\pi}^2\mu\sigma\sqrt{\lambda} + \frac{\boldsymbol{\pi}^2\sigma^2\lambda}{2} + \mu\sigma^3\sqrt{\lambda} + \frac{\sigma^4\lambda}{4}.
\end{aligned}$$

This expression includes a $\mu^2\sigma^2$ term, which indicates a massive sigma field. Comparing with Eq. (3), the pion mass is $\sqrt{2}\mu$. However, there is no $\mu^2\boldsymbol{\pi}^2$ term, which indicates that the pion field is massless. The terms of

$\mathcal{O}\left(\sqrt{\lambda}\right)$ and $\mathcal{O}(\lambda)$ have factors of $\boldsymbol{\pi}^4$, $\boldsymbol{\pi}^2\sigma$, $\boldsymbol{\pi}^2\sigma^2$, σ^3, and σ^4; these are all cubic and quartic factors. Since they are of $\mathcal{O}\left(\sqrt{\lambda}\right)$ and $\mathcal{O}(\lambda)$, they become small as $\lambda \to 0$. This is what we wanted to show. $\square$

For the propagators, we can use (4.46) of Peskin & Schroeder:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}.$$

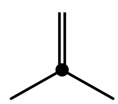
Then we can write

 $= \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip\cdot(x-y)}}{p^2 - 2\mu^2 + i\epsilon},$ i  $j = \delta^{ij} \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip\cdot(x-y)}}{p^2 - 2\mu^2 + i\epsilon}.$

We can associate each of the vertices with a term in $V(\Psi^2)$. The symmetry factors for each of the terms are

$$\boldsymbol{\pi}^2\mu\sigma\sqrt{\lambda} : 2! = 2, \qquad \mu\sigma^3\sqrt{\lambda} : 3! = 6, \qquad \frac{(\boldsymbol{\pi}^2)^2\lambda}{4} : 4! = 24, \qquad \frac{\boldsymbol{\pi}^2\sigma^2\lambda}{2} : 2!2! = 4, \qquad \frac{\sigma^4\lambda}{4} : 4! = 24.$$

Then the vertices are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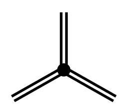
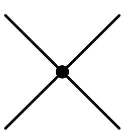
$= -2i\mu\sqrt{\lambda}\delta^{ij},$



$= -6i\mu\sqrt{\lambda},$

k l

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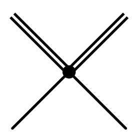
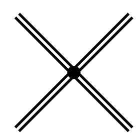
$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}),$


i j

$= -2i\lambda\delta^{ij},$

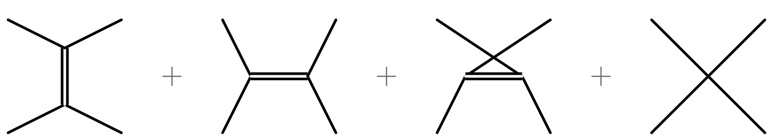

i j

$= -6i\lambda.$

1(c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. Show that, in the special case $N = 2$ (1 species of pion), the term $\mathcal{O}(p^2)$ also cancels.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.