Problem 1. Alternative regulators in QED (Peskin & Schroeder 7.2) In Section. 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Paul-Villars regularization.

1(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

Solution. From (7.47) in Peskin & Schroeder,

$$\Gamma^\mu(q=0) = \frac{1}{Z_1}\gamma^\mu, \tag{1}$$

we can find an expression for δZ_1, similar to how (7.31) is obtained from (7.26). Roughly,

$$\frac{1}{Z_1 + \delta Z_1}\gamma^\mu \approx Z_1(1-\delta Z_1)\gamma^\mu = \Gamma^\mu(q=0) + \delta\Gamma^\mu(q=0) \quad \Longrightarrow \quad \delta\Gamma^\mu(q=0) = -\delta Z_1\gamma^\mu. \tag{2}$$

According to (6.33),

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

We note that $\Gamma^\mu = \gamma^\mu$, $F_1 = 1$, and $F_2 = 0$ to lowest order [1, pp. 185–186]. Then we can write

$$\delta\Gamma^\mu(q=0) = \gamma^\mu \delta F_1(0) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\delta F_2(0). \tag{3}$$

Using this equation and the identity $\gamma^\mu\gamma_\mu = 4$ [2], Eq. (2) can be written

$$\delta Z_1 = -\frac{1}{4}\gamma_\mu\delta\Gamma^\mu(q=0) = -\delta F_1(0) - \gamma_\mu\frac{i\sigma^{\mu\nu}q_\nu}{8m}\delta F_2(0). \tag{4}$$

In order to find $\delta\Gamma^\mu$ we use (6.47):

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3}& \\ \times\,\bar{u}(p')\left\{\gamma^\mu\left[-\frac{\ell^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right]\right.& \\ \left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m}[2m^2z(1-z)]\right\}u(p),&\end{aligned} \tag{5}$$

where $\Delta \equiv -xyq^2 + (1-z)^2m^2$ by (6.44), $\ell \equiv k + yq - zp$, and $D = \ell^2 - \Delta + i\epsilon$ [1, p. 191]. The momenta k and p are assigned in the Feynman diagram on p. 189 of Peskin & Schroeder, and x, y are Feynman parameters [1, p. 190].

The forms of the two integrals we need to compute are given by Peskin & Schroeder (6.49) and (6.50). Equation (6.49) is

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}. \tag{6}$$

Here $m = 3$ because we have D^{-3} in Eq. (5). We can evaluate the left-hand side using the Euclidian 4-momentum defined in (6.48),

$$\ell^0 \equiv i\ell^0_E, \qquad\qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E. \tag{7}$$

Following the steps on p. 193, we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta)^3} = \frac{i}{(-1)^3} \frac{1}{(2\pi)^4} \int \frac{d^4\ell_E}{(\ell_E^2+\Delta)^3} = \frac{i(-1)^3}{(2\pi)^4} \int d\Omega_4 \int_0^\Lambda d\ell_E \, \frac{\ell_E^3}{(\ell_E^2+\Delta)^3},$$

where we have replaced the upper (infinite) bound of integration by a finite number Λ. Evaluating this integral using Mathematica and using $\int d\Omega_4 = 2\pi^2$ [1, p. 193], we find

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta)^3} &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{\Lambda^4}{4\Delta(\Delta+\Lambda^2)^2} \\ &= -\frac{i}{32\pi^2}\frac{\Lambda^4}{\Delta(\Delta+\Lambda^2)^2} \\ &\approx -\frac{i}{32\pi^2}\frac{1}{\Delta} \equiv \alpha, \end{aligned} \tag{8}$$

where we have taken the limit $\Lambda \gg \Delta$ [1, p. 218] and defined α. Equation (6.50) in Peskin & Schroeder is

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)} \frac{1}{\Delta^{m-3}}.$$

Following similar steps as for Eq. (5), the left-hand side is

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta)^3} &= \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int d^4\ell_E \, \frac{\ell_E^2}{(\ell_E^2+\Delta)^3} \\ &= \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E \, \frac{\ell_E^5}{(\ell_E^2+\Delta)^3} \\ &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{1}{4}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right] \\ &= -\frac{i}{32\pi^2}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right] \\ &\approx -\frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - \frac{3}{2}\right] \equiv \beta, \end{aligned} \tag{9}$$

where we have defined β and ignored terms of $\mathcal{O}\big(\Lambda^{-2}\big)$ [1, p. 218].

We now set $q^2 = 0$, and define $\Delta_0 = (1-z)^2 m^2$. Then $\Delta \to \Delta_0$ in our expression and $\alpha \to \alpha_0, \beta \to \beta_0$ (which are functions of Δ_0). Feeding in Eqs. (8) and (9), Eq. (5) can be written

$$\bar{u}(p')\delta\Gamma^\mu(q=0)u(p) = 2ie^2 \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p') \int \left\{\gamma^\mu\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right]\right\} u(p).$$

Then

$$\begin{aligned}\delta F_1(0) &= 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right] \\ &= 2ie^2\int_0^1 dz\,(1-z)\left[-\beta_0 + 2m^2(1-4z+z^2)\alpha_0\right],\end{aligned}$$

$$\begin{aligned}\delta F_2(0) &= 8ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)m^2 z(1-z)\alpha_0 \\ &= 8ie^2\int_0^1 dz\,m^2 z(1-z)^2\alpha_0.\end{aligned}$$

We ignore $\delta F_2(0)$ since it is not affected by the divergence [1, p. 196]. In order to avoid issues coming from the divergence in $\delta F_1(0)$, we add a $z\mu^2$ term to Δ_0 [1, p. 195]. So, feeding these results into Eq. (4), we obtain

$$\begin{aligned}\delta Z_1 &= -2ie^2 \int_0^1 dz\,(1-z)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right] \\ &= -2ie^2 \int_0^1 dz\,(1-z)\left\{\frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Lambda^2}{\Delta_0}\right) - \frac{3}{2}\right] - 2(1-4z+z^2)m^2\frac{i}{32\pi^2}\frac{1}{\Delta_0}\right\} \\ &= \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\ln\!\left(\frac{\Lambda^2}{\Delta_0}\right) - \frac{3}{2} - \frac{m^2(1-4z+z^2)}{\Delta_0}\right], \end{aligned} \tag{10}$$

where

$$\Delta_0 = (1-z)^2 m^2 + z\mu^2. \tag{11}$$

For δZ_2, we can use the first part of Peskin & Schroeder (7.31),

$$\delta Z_2 = \left.\frac{d\Sigma_2}{d\not{p}}\right|_{\not{p}=m}, \tag{12}$$

where Σ_2 is given by (7.17),

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\not{p} + 4m_0}{(\ell^2 - \Delta + i\epsilon)^2}, \tag{13}$$

where $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. We may once again follow the steps on p. 193 to evaluate the integral, now with $m = 2$. Changing the upper bound of integration to Λ once more, we have

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} &= \frac{i}{(-1)^2}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{1}{(\ell_E^2+\Delta)^2} \\ &= \frac{i(-1)^2}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^2} \\ &= \frac{i}{(2\pi)^4}(2\pi^2)\frac{1}{2}\left[\frac{\Delta}{\Delta+\Lambda^2} + \ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] \\ &= \frac{i}{16\pi^2}\left[\frac{\Delta}{\Delta+\Lambda^2} + \ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] \\ &\approx \frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right], \end{aligned}$$

where we have evaluated the integral using Mathematica and dropped terms of $\mathcal{O}(\Lambda^{-2})$. Substituting back into Eq. (13), we find

$$\Sigma_2(p) = -ie^2\int_0^1 dx\,(-2x\not{p} + 4m_0)\frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right].$$

Note that

$$\begin{aligned}\frac{d\Sigma_2}{d\not{p}} &= \frac{e^2}{16\pi^2}\frac{d}{d\not{p}}\left\{\int_0^1 dx\,(-2x\not{p}+4m_0)\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right]\right\} \\ &= \frac{e^2}{16\pi^2}\int_0^1 dx\left\{\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right]\frac{d}{d\not{p}}(-2x\not{p}+4m_0) + (-2x\not{p}+4m_0)\frac{d}{d\not{p}}\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right]\right\} \\ &= \frac{e^2}{16\pi^2}\int_0^1 dx\left\{\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right]\frac{d}{d\not{p}}(-2x\not{p}+4m_0) + (-2x\not{p}+4m_0)\frac{d}{d\Delta}\left[\ln\!\left(\frac{\Lambda^2}{\Delta}\right) - 1\right]\frac{d\Delta}{d\not{p}}\right\}. \end{aligned} \tag{14}$$

Using $p^2 = \not{p}^2$ [1, p. 220], note that

$$\frac{d\Delta}{d\not{p}} = \frac{d}{d\not{p}}[-x(1-x)\not{p}^2 + x\mu^2 + (1-x)m_0^2] = -2x(1-x)\not{p}. \tag{15}$$

Also, ignoring terms of $\mathcal{O}(\Lambda^{-2})$,

$$\frac{d}{d\not{p}}(-2x\not{p} + 4m_0) = -2x, \tag{16}$$

$$\frac{d}{d\Delta}\left[\ln\left(\frac{\Lambda^2}{\Delta}\right) - 1\right] = \frac{d}{d\Delta}\left[\ln(\Lambda^2) - \ln(\Delta) - 1\right] = -\frac{1}{\Delta}. \tag{17}$$

Making these substitutions in Eq. (14),

$$\frac{d\Sigma_2}{d\not{p}} = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2x\left[\ln\left(\frac{\Lambda^2}{\Delta}\right) - 1\right] - \frac{(2x\not{p} - 4m_0)[2x(1-x)\not{p}]}{\Delta}\right].$$

We now define

$$\Delta_m \equiv -x(1-x)m^2 + x\mu^2 + (1-x)m_0^2 \approx (1-x)^2m^2 + x\mu^2, \tag{18}$$

since $m \approx m_0$. Then Eq. (12) becomes

$$\begin{aligned}\delta Z_2 &= \frac{e^2}{16\pi^2}\int_0^1 dx\left[-x\left[\ln\left(\frac{\Lambda^2}{\Delta_m}\right) - 1\right] + \frac{(2xm - 4m)[2x(1-x)m]}{\Delta_m}\right] \\ &= -\frac{e^2}{8\pi^2}\int_0^1 dx\left[x\ln\left(\frac{\Lambda^2}{\Delta_m}\right) - x - \frac{2xm^2(2-x)(1-x)}{\Delta_m}\right]\end{aligned} \tag{19}$$

Now we rename $x \to z$ in δZ_2. This means $\Delta_0 = \Delta_m$ from Eqs. (11) and (18). Naming $\Delta \equiv \Delta_0 = \Delta_m$ in Eqs. (10) and (19), we have

$$\delta Z_1 = \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\ln\left(\frac{\Lambda^2}{\Delta}\right) - \frac{3}{2} - \frac{m^2(1-4z+z^2)}{\Delta}\right],$$

$$\delta Z_2 = -\frac{e^2}{8\pi^2}\int_0^1 dx\left[x\ln\left(\frac{\Lambda^2}{\Delta}\right) - x - \frac{2xm^2(2-x)(1-x)}{\Delta}\right],$$

where $\Delta = (1-z)^2m^2 + z\mu^2$. It appears that $\delta Z_1 \neq \delta Z_2$, as we wanted to show, but I am not sure how to evaluate these integrals.

1(b) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be 4×4 as usual, but note that, in d dimensions,

$$g_{\mu\nu}\gamma^\mu\gamma^\nu = d. \tag{20}$$

Show that, with this definition, $\delta Z_1 = \delta Z_2$.

Solution. We begin by finding δZ_2. We need to fix Peskin & Schroeder (7.17) so it has arbitrary d instead of $d = 4$. We begin from (7.16), changing $4 \to d$:

$$-i\Sigma_2(p) = (-ie)^2\int\frac{d^dk}{(2\pi)^d}\gamma^\mu\frac{i(\not{k} + m_0)}{k^2 + m_0^2 + i\epsilon}\gamma_\mu\frac{i}{(p-k)^2 - \mu^2 + i\epsilon}.$$

Following the procedure on pp. 217–218, we introduce the Feynman parameter x to combine the denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon}\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx\, \frac{1}{[k^2 - 2xk\cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}. \tag{21}$$

Let $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. Then

$$\begin{aligned}
-i\Sigma_2(p) &= (-ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\gamma^\mu \frac{i^2(\not{k} + m_0)}{[\ell^2 - \Delta + i\epsilon]^2}\gamma_\mu \\
&= -(-ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\gamma^\mu \frac{\not{\ell} + x\not{p} + m_0}{[\ell^2 - \Delta + i\epsilon]^2}\gamma_\mu \\
&= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\gamma^\mu \frac{2\ell^\mu - \gamma_\mu\not{\ell} + x(2p_\mu - \gamma_\mu\not{p}) + m_0\gamma_\mu}{[\ell^2 - \Delta + i\epsilon]^2} \\
&= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)x\not{p} + dm_0}{[\ell^2 - \Delta + i\epsilon]^2},
\end{aligned} \tag{22}$$

where we have applied Eq. (20) and $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}$ [1, p. 191], and we have dropped terms linear in ℓ by (6.45),

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu}{D^3} = 0.$$

To evaluate the integral, we can write it in terms of the Euclidean 4-momentum defined in Eq. (7), as on p. 193 in Peskin & Schroeder:

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(-1)^2}\frac{1}{(2\pi)^d}\int d^d\ell_E\, \frac{1}{(\ell_E^2 + \Delta)^2} = i\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2 + \Delta)^2}.$$

Then we can apply (7.84), which takes the limit as $d \to 4$:

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2 + \Delta)^2} \to \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi}{\Delta}\right)\right),$$

where $\epsilon = 4 - d$ [1, p. 250]. Making these substitutions into Eq. (22), we find

$$\Sigma_2(p) = \frac{e^2}{16\pi^2}\int_0^1 dx\, [(2-d)x\not{p} + dm_0]\left(\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi}{\Delta}\right)\right).$$

Then

$$\begin{aligned}
\frac{d\Sigma_2}{d\not{p}} &= \frac{e^2}{16\pi^2}\int_0^1 dx \left[\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi}{\Delta}\right)\right]\frac{d}{d\not{p}}[(2-d)x\not{p} + dm_0] \\
&\qquad + [(2-d)x\not{p} + dm_0]\frac{d}{d\not{p}}\left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi)\right) \\
&= \frac{e^2}{16\pi^2}\int_0^1 dx \left\{\left[\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi}{\Delta}\right)\right](2-d)x + [(2-d)x\not{p} + dm_0]\frac{2x(1-x)\not{p}}{\Delta}\right\}
\end{aligned}$$

where

$$\frac{d}{d\not{p}}(\ln\Delta) = \frac{d\Delta}{d\not{p}}\frac{d}{d\Delta}(\ln\Delta) = -\frac{2x(1-x)\not{p}}{\Delta}$$

from Eqs. (15) and (16). Then applying $m \approx m_0$ and discarding terms of $\mathcal{O}(\epsilon)$,

$$\begin{aligned}
\delta Z_2 &= \frac{e^2}{16\pi^2}\int_0^1 dx \left\{(2-d)x\left[\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi}{\Delta_m}\right)\right] + \frac{2x(1-x)[(2-d)x + d]m^2}{\Delta_m}\right\} \\
&= \frac{e^2}{8\pi^2}\int_0^1 dx \left\{x\left[-\frac{2}{\epsilon} + \gamma - \ln\!\left(\frac{4\pi}{\Delta_m}\right)\right] + \frac{2m^2x(x-1)(x-2)}{\Delta_m}\right\}.
\end{aligned} \tag{23}$$

To find δZ_1, we need to start at Peskin & Schroeder (6.38) for arbitrary dimension:

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p)\\ &= -ie^2\int\frac{d^dk}{(2\pi)^d}\bar{u}(p')\frac{\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{[(k-p)^2+i\epsilon](k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}u(p).\end{aligned} \tag{24}$$

Using an expression analogous to Eq. (21), we introduce the Feynman parameter x and define $\ell = k - xp$. We are implicitly using $q = k' - k = 0$. We also define $\Delta = x\mu^2 + (1-x)^2m^2$, where we have introduced a photon mass and let $m \approx m_0$. The denominator of Eq. (24) is $D^3 = (\ell^2 + \Delta + i\epsilon)^3$, and the numerator is

$$\begin{aligned}N &= \bar{u}(p')\gamma^\nu(\not{\ell}+x\not{p}+m)\gamma^\mu(\not{\ell}+x\not{p}+m)\gamma_\nu u(p)\\ &= \bar{u}(p')\gamma^\nu(\not{\ell}\gamma^\mu\not{\ell}+x\not{\ell}\gamma^\mu\not{p}+m\not{\ell}\gamma^\mu+x\not{p}\gamma^\mu\not{\ell}+x^2\not{p}\gamma^\mu\not{p}+mx\not{p}\gamma^\mu+m\gamma^\mu\not{\ell}+mx\gamma^\mu\not{p}+m^2\gamma^\mu)\gamma_\nu u(p)\\ &\to \bar{u}(p')\gamma^\nu(\not{\ell}\gamma^\mu\not{\ell}+x^2\not{p}\gamma^\mu\not{p}+2mx\gamma^\mu\not{p}+m^2\gamma^\mu)\gamma_\nu u(p).\end{aligned} \tag{25}$$

To simplify each of these terms we apply (7.87),

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{D^3} = \int\frac{d^d\ell}{(2\pi)^d}\frac{g^{\mu\nu}\ell^2}{dD^3}, \tag{26}$$

(7.89),

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(2-\epsilon)\gamma^\nu = -(d-2)\gamma^\nu, \tag{27}$$

as well as [1, pp. 191–192]

$$\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}, \qquad \not{p}u(p) = m_e u(p), \qquad \bar{u}(p')\not{p}' = \bar{u}(p')m_e,$$

and the Gordon identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p^{\mu\prime}+p^\mu}{2m}+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \quad\Longrightarrow\quad p^\mu \to m\gamma^\mu. \tag{28}$$

Note that

$$\begin{aligned}\gamma^\nu\not{\ell}\gamma^\mu\not{\ell}\gamma_\nu &= \gamma^\nu(2\ell^\mu-\gamma^\mu\not{\ell})\not{\ell}\gamma_\nu = \gamma^\nu(2\ell^\mu\ell^\rho\gamma_\rho-\gamma^\mu\not{\ell}\not{\ell})\gamma_\nu \to \gamma^\nu\left(\frac{2}{d}\ell^2g^{\mu\rho}\gamma_\rho-\gamma^\mu\ell^2\right)\gamma_\nu\\ &\to \gamma^\nu\left(\frac{2-d}{d}\ell^2\gamma^\mu-2m^2\gamma^\mu\right)\gamma_\nu = \frac{(2-d)^2}{d}\ell^2\gamma^\mu+2(d-2)m^2\gamma^\mu,\end{aligned}$$

$$\begin{aligned}\gamma^\nu\not{p}\gamma^\mu\not{p}\gamma_\nu &= \gamma^\nu(2p^\mu-\gamma^\mu\not{p})(2p_\nu-\gamma_\nu\not{p}) = \gamma^\nu(4p^\mu p_\nu-2p^\mu\gamma_\nu\not{p}-2\gamma^\mu\not{p}p_\nu+\gamma^\mu\not{p}\gamma_\nu\not{p})\\ &\to \gamma^\nu(4m^2\gamma^\mu\gamma_\nu-2m^2\gamma^\mu\gamma_\nu-2m\gamma^\mu\not{p}\gamma_\nu+m\gamma^\mu\not{p}\gamma_\nu) = 2m^2\gamma^\nu\gamma^\mu\gamma_\nu-m\gamma^\nu\gamma^\mu\not{p}\gamma_\nu\\ &= -2m^2(d-2)\gamma^\mu-m\gamma^\nu\gamma^\mu(2p_\nu-\gamma_\nu\not{p}) \to -2m^2(d-2)\gamma^\mu-m\gamma^\nu\gamma^\mu(2m\gamma_\nu-m\gamma_\nu)\\ &= -2m^2(d-2)\gamma^\mu+m^2(d-2)\gamma^\mu = -m^2(d-2)\gamma^\mu,\end{aligned}$$

$$\gamma^\nu\gamma^\mu\not{p}\gamma_\nu = \gamma^\nu\gamma^\mu(2p_\nu-\gamma_\nu\not{p}) \to \gamma^\nu\gamma^\mu(2m\gamma_\nu-m\gamma_\nu) = -m(d-2)\gamma^\mu.$$

Then Eq. (25) becomes

$$\begin{aligned}N &= \bar{u}(p')\left[\frac{(2-d)^2}{d}\ell^2+2(d-2)m^2-x^2m^2(d-2)-2m^2x(d-2)-m^2(d-2)\right]\gamma^\mu u(p)\\ &= \bar{u}(p')\left[\frac{(d-2)^2}{d}\ell^2+m^2(d-2)(1-2x-x^2)\right]\gamma^\mu u(p).\end{aligned}$$

Then

$$\delta Z_1 = -\delta F_1 = ie^2 \int_0^1 dx\,(1-x) \int \frac{d^d\ell}{(2\pi)^d}(d-2)\frac{(d-2)\ell^2/d + m^2(1-2x-x^2)}{(\ell^2+\Delta+i\epsilon)^3}. \tag{29}$$

Using (7.85) and (7.86) and substituting $\epsilon = 4-d$, we can write

$$\begin{aligned}
\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3} &= \frac{i}{(-1)^3}\frac{1}{(2\pi)^d}\int \frac{d^d\ell_E}{(\ell_E^2+\Delta)^3} \\
&= -i\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^3} \\
&= -\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(3-d/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-d/2} \\
&= -\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{1+\epsilon/2},
\end{aligned} \tag{30}$$

$$\begin{aligned}
\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3} &= -\frac{i}{(-1)^3}\frac{1}{(2\pi)^d}\int d^d\ell_E\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3} \\
&= i\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^3} \\
&= \frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-d/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-d/2} \\
&= \frac{i}{(4\pi)^{2-\epsilon/2}}\frac{4-\epsilon}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\epsilon/2}.
\end{aligned} \tag{31}$$

We can now write Eq. (29) as

$$\begin{aligned}
\delta Z_1 = ie^2\int_0^1 dx\,(1-x)(2-\epsilon)&\Bigg[\frac{2-\epsilon}{2}\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{4-\epsilon}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\epsilon/2} \\
&\qquad -m^2(1-2x-x^2)\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{1+\epsilon/2}\Bigg] \\
= -\frac{e^2}{32\pi^2}\int_0^1 dx\,(1-x)&\left[\frac{(2-\epsilon)^2(4-\epsilon)}{4}\Gamma(\epsilon/2)\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2} - \frac{m^2(1-2x-x^2)}{\Delta}(2-\epsilon)\Gamma(1+\epsilon/2)\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2}\right] \\
\approx -\frac{e^2}{32\pi^2}\int_0^1 dx\,(1-x)&\left\{\frac{8}{\epsilon} - 2\left[5+2\gamma-2\ln\left(\frac{4\pi}{\Delta}\right)\right] - 2\frac{m^2(1-2x-x^2)}{\Delta}\right\} \\
= -\frac{e^2}{16\pi^2}\int_0^1 dx\,(1-x)&\left[\frac{4}{\epsilon} - 5 - 2\gamma + 2\ln\left(\frac{4\pi}{\Delta}\right) - \frac{m^2(1-2x-x^2)}{\Delta}\right],
\end{aligned}$$

where we have expanded about $\epsilon = 0$ using Mathematica. Comparing this with Eq. (23), we have

$$\begin{aligned}
\delta Z_1 &= -\frac{e^2}{16\pi^2}\int_0^1 dx\,(1-x)\left[\frac{4}{\epsilon} - 5 - 2\gamma + 2\ln\left(\frac{4\pi}{\Delta}\right) - \frac{m^2(1-2x-x^2)}{\Delta}\right], \\
\delta Z_2 &= -\frac{e^2}{8\pi^2}\int_0^1 dx\,x\left[\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi}{\Delta}\right) + \frac{2m^2(x-1)(x-2)}{\Delta}\right]
\end{aligned}$$

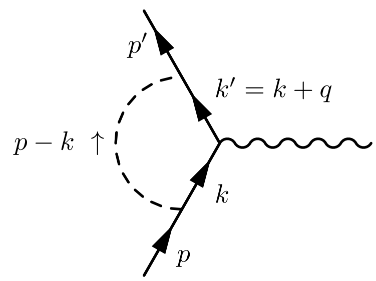
with $\Delta = x\mu^2 + (1-x)^2m^2$. Again, I do not know how to evaluate these integrals. It seems possible, although not obvious, that $\delta Z_1 = \delta Z_2$.

Problem 2. (Peskin & Schroeder 7.3) Consider a theory of elementary fermions that couple both to QED and to a Yukawa field ϕ:

$$H_{\text{int}} = \int d^3x \, \frac{\lambda}{\sqrt{2}} \phi \bar{\psi} \psi + \int d^3x \, e A_\mu \bar{\psi} \gamma^\mu \psi. \tag{32}$$

2(a) Verify that the contribution to Z_1 from the vertex diagram with a virtual ϕ equals the contribution to Z_2 from the diagram with a virtual ϕ. Use dimensional regularization. Is the Ward identity generally true in this theory?

Solution. We begin by finding δZ_1. We are interested in the diagram



(33)

We considered a similar diagram in Homework 6 of Physics 443, and we build on that work here. We adapted Peskin & Schroeder (6.38) using the scalar field Feynman rules to write [1, p. 123]

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p,p')u(p) &= \int \frac{d^dk}{(2\pi)^d} \frac{i}{(k-p)^2 - m_\phi^2 + i\epsilon} \bar{u}(p') \left(-i\frac{\lambda}{\sqrt{2}}\right) \frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon} \left(-i\frac{\lambda}{\sqrt{2}}\right) u(p) \\
&= \frac{i\lambda^2}{2} \int \frac{d^dk}{(2\pi)^d} \bar{u}(p') \frac{(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)}{[(k-p)^2 - m_\phi^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} u(p).
\end{aligned} \tag{34}$$

We use Peskin & Schroeder (6.41) to write

$$\frac{1}{[(k-p)^2 - m_\phi^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{D^3}, \tag{35}$$

where [1, pp. 190–191]

$$D = k^2 + 2k(qy - pz) + z(p^2 - m_\phi^2) - (1-z)m_e^2 + i\epsilon \equiv \ell^2 - \Delta + i\epsilon, \tag{36}$$

where we use $x + y + z = 1$ and $k' = k + q$, and we have defined $\ell \equiv k + yq - zp$ [1, p. 191], and

$$\Delta \equiv -xyq^2 + (1-z)^2 m_e^2 + z m_\phi^2.$$

For the numerator of Eq. (34), we use $k' = k + q$ and $\ell \equiv k + yq - zp$ [1, p. 191], and define

$$N \equiv \bar{u}(p')[\not{\ell} + (1-y)\not{q} + z\not{p} + m_e]\gamma^\mu(\not{\ell} - y\not{q} + z\not{p} + m_e)u(p) \tag{37}$$

Again discarding terms linear in ℓ, Eq. (37) simplifies to

$$\begin{aligned}
N = \bar{u}(p')[&\not{\ell}\gamma^\mu\not{\ell} - y(1-y)\not{q}\gamma^\mu\not{q} + z(1-y)\not{q}\gamma^\mu\not{p} + m_e(1-y)\not{q}\gamma^\mu - yz\not{p}\gamma^\mu\not{q} + z^2\not{p}\gamma^\mu\not{p} + m_e z\not{p}\gamma^\mu - m_e y\gamma^\mu\not{q} \\
&+ m_e z\gamma^\mu\not{p} + m_e^2\gamma^\mu]u(p).
\end{aligned} \tag{38}$$

Using Eqs. (26)–(28), the terms above reduce to

$$
\begin{aligned}
\not{\ell}\gamma^\mu\not{\ell} &= (2\ell^\mu - \gamma^\mu\not{\ell})\not{\ell} = 2\ell^\mu\ell^\nu\gamma_\nu - \gamma^\mu\not{\ell}\not{\ell} \to \frac{2}{d}\ell^2 g^{\mu\nu}\gamma_\nu - \gamma^\mu\ell^2 \to \frac{2-d}{d}\ell^2\gamma^\mu - 2m_e^2\gamma^\mu = 0,\\
\not{q}\gamma^\mu\not{q} &\to 0,\\
\not{q} &\to 0,\\
\not{q}\gamma^\mu &\to 0,\\
\not{p}\gamma^\mu\not{q} &\to 0,\\
\not{p}\gamma^\mu\not{p} &\to m_e\not{p}\gamma^\mu = m_e(2p^\mu - \gamma^\mu\not{p}) \to 2m_e p^\mu - m_e^2\gamma^\mu \to 2m_e^2\gamma^\mu - m_e^2\gamma^\mu = m_e^2\gamma^\mu,\\
\not{p}\gamma^\mu &= 2p^\mu - \gamma^\mu\not{p} = 2p^\mu - \gamma^\mu m_e \to 2m_e\gamma^\mu - \gamma^\mu m_e = m_e\gamma^\mu,\\
\gamma^\mu\not{q} &\to 0,\\
\gamma^\mu\not{p} &\to m_e\gamma^\mu.
\end{aligned}
$$

Feeding these back into Eq. (38), we obtain

$$
N \to \bar{u}(p')\left[\frac{2-d}{d}\ell^2 + z^2m_e^2 + m_e^2 z + m_e^2 z + m_e^2\right]\gamma^\mu u(p) = \bar{u}(p')\left[\frac{2-d}{d}\ell^2 + (1+z)^2 m_e^2\right]\gamma^\mu u(p).
$$

Then, letting $\Delta_0 = (1-z)^2m_e^2 + zm_\phi^2$,

$$
\begin{aligned}
\delta Z_1 &= -\delta F_1\\
&= -i\lambda^2\int\frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{(2-d)\ell^2/d + (1+z)^2m_e^2}{(\ell^2-\Delta_0)^3}\\
&= -i\lambda^2\int\frac{d^d\ell}{(2\pi)^d}\int_0^1 dz\,(1-z)\frac{(2-d)\ell^2/d + (1+z)^2m_e^2}{(\ell^2-\Delta_0)^3}.
\end{aligned}
\tag{39}
$$

Applying Eqs. (30) and (31), Eq. (39) becomes

$$
\begin{aligned}
\delta Z_1 &= -i\lambda^2\int_0^1 dz\,(1-z)\left[\frac{2-d}{d}\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{d}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta_0}\right)^{\epsilon/2} - (1+z)^2m_e^2\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta_0}\right)^{1+\epsilon/2}\right]\\
&= \frac{1}{\Gamma(3)}\frac{\lambda^2}{(4\pi)^{2-\epsilon/2}}\int_0^1 dz\,(1-z)\left[\frac{\epsilon-2}{2}\Gamma(\epsilon/2)\left(\frac{1}{\Delta_0}\right)^{\epsilon/2} - (1+z)^2m_e^2\Gamma(1+\epsilon/2)\left(\frac{1}{\Delta_0}\right)^{1+\epsilon/2}\right]\\
&= \frac{1}{\Gamma(3)}\frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\,(1-z)\left[\frac{\epsilon-2}{2}\Gamma(\epsilon/2)\left(\frac{4\pi}{\Delta_0}\right)^{\epsilon/2} - \frac{(1+z)^2m_e^2}{\Delta}\Gamma(1+\epsilon/2)\left(\frac{4\pi}{\Delta_0}\right)^{\epsilon/2}\right]\\
&\approx -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\,(1-z)\left[\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi}{\Delta}\right) + \frac{(1+z)^2m_e^2}{\Delta}\right],
\end{aligned}
\tag{40}
$$

where we have performed the expansion with Mathematica once more.

Now we must find δZ_2. We begin with a diagram that looks like that in (7.15) of Peskin & Schroeder, but with a scalar instead of a photon. We need to adapt their (7.16) for a scalar field. Using the Feynman rules for a scalar field [1, p. 118], we write

$$
-i\Sigma_2(p) = \left(-\frac{i\lambda}{\sqrt{2}}\right)^2\int\frac{d^dk}{(2\pi)^d}\frac{i(\not{k}+m_e)}{k^2-m_e^2+i\epsilon}\frac{i}{(p-k)^2-m_\phi^2+i\epsilon}.
$$

Let $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + xm_\phi^2 + (1-x)m_e^2$. Then from Eq. (22),

$$
\Sigma_2(p) = \frac{i\lambda^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\not{\ell}+x\not{p}+m_e}{(\ell^2-\Delta+i\epsilon)^2} \to \frac{i\lambda^2}{2}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{x\not{p}+m_e}{(\ell^2-\Delta+i\epsilon)^2},
\tag{41}
$$

where we have again dropped terms linear in ℓ. Then from (7.85),

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2} &= \frac{i}{(-1)^2}\frac{1}{(2\pi)^d}\int \frac{d^d\ell_E}{(\ell_E^2+\Delta)^2} \\ &= i\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} \\ &= \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-d/2} \\ &= \frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(\epsilon/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{\epsilon/2}.\end{aligned}$$

Feeding this into Eq. (**??**) and Taylor expanding as before,

$$\begin{aligned}\Sigma_2(p) &= \frac{i\lambda^2}{2}\int_0^1 dx\,(x\not p+m_e)\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(\epsilon/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{\epsilon/2} \\ &= -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\,(x\not p+m_e)\Gamma(\epsilon/2)\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2} \\ &\approx -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\,(x\not p+m_e)\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)\right].\end{aligned}$$

Note that

$$\frac{d\Sigma_2(p)}{d\not p} = -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\left\{\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)\right]\frac{d}{d\not p}(x\not p+m_e)+(x\not p+m_e)\frac{d}{d\Delta}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)\right]\frac{d\Delta}{d\not p}\right\},$$

where

$$\frac{d}{d\not p}(x\not p+m_e)=x, \qquad \frac{d}{d\Delta}\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)\right]=-\frac{1}{\Delta}, \qquad \frac{d\Delta}{d\not p}=-2x(1-x)\not p.$$

Then

$$\frac{d\Sigma_2(p)}{d\not p} = -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\left\{x\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)\right]+(x\not p+m_e)\frac{2x(1-x)\not p}{\Delta}\right\},$$

so from Eq. (12),

$$\delta Z_2 = -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\,x\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta_m}\right)+\frac{2m_e^2(1+x)(1-x)}{\Delta_m}\right]. \tag{42}$$

Comparing with Eq. (40), we have found

$$\begin{aligned}\delta Z_1 &= -\frac{\lambda^2}{32\pi^2}\int_0^1 dz\,(1-z)\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)+\frac{(1+z)^2m_e^2}{\Delta}\right], \\ \delta Z_2 &= -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\,x\left[\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)+\frac{2m_e^2(1+x)(1-x)}{\Delta}\right],\end{aligned}$$

where $\Delta=(1-x)^2m_e^2+xm_\phi^2$. If the Ward identity is generally true in this theory as we expect, we would have $\delta Z_1=\delta Z_2$, but it looks like there is a sign error here, at the very least.

2(b) Now consider the renormalization of the $\phi\bar\psi\psi$ vertex. Show that the rescaling of this vertex at $q^2=0$ is *not* canceled by the correction to Z_2. (It suffices to compute the ultraviolet-divergent parts of the diagrams.) In this theory, the vertex and field-strength rescaling give additional shifts of the observable coupling constant relative to its bare value.

Solution. In order to account for both terms of the Hamiltonian, we need to sum the diagrams with scalar propagators and with photon propagators. Both of the diagrams have scalar external legs. This means we need to replace Eq. (4) with a relation that makes sense for a scalar vertex. We adapt (7.47) as

$$\Gamma(q=0) = Z_1^{-1} \quad \Longrightarrow \quad \delta\Gamma(q=0) = -\delta Z_1. \tag{43}$$

To compute δZ_1, we must sum two vertex diagrams similar to the one on p. 189 of Peskin & Schroeder. The first has the external leg replaced with a scalar:

$$\begin{aligned}\bar{u}(p')\delta\Gamma_1(p,p')u(p) &= \int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m_e)}{k'^2-m_e^2+i\epsilon}\frac{i(\not{k}+m_e)}{k^2-m_e^2+i\epsilon}(-ie\gamma^\rho)u(p)\\ &= -ie^2\int\frac{d^dk}{(2\pi)^d}\bar{u}(p')\gamma^\nu\frac{(\not{k}'+m_e)(\not{k}+m_e)}{[(k-p)^2+i\epsilon][k'^2-m_e^2+i\epsilon][k^2-m_e^2+i\epsilon]}\gamma_\nu u(p)\end{aligned} \tag{44}$$

We can simplify the calculations at this step (contrary to what we did in 2(a)) by setting $q = k'-k = 0$ now. We introduce Feynman parameters analogously to Eq. (35). Let $\ell = k - xp$. We also introduce the photon mass now, and so define $\Delta_1 = (1-z)^2m^2 + z\mu^2$ as in Eq. (11). Then the denominator of the integrand is $D_1 = \ell^2 + \Delta_1 + i\epsilon$, and the numerator is

$$\begin{aligned}N_1 &= \bar{u}(p')\gamma^\nu(\not{k}+m_e)(\not{k}+m_e)\gamma_\nu u(p)\\ &= \bar{u}(p')\gamma^\nu(\not{\ell}+z\not{p}+m_e)(\not{\ell}+z\not{p}+m_e)\gamma_\nu u(p)\\ &= \bar{u}(p')\gamma^\nu(\not{\ell}\not{\ell}+z\not{\ell}\not{p}+m_e\not{\ell}+z^2\not{p}\not{\ell}+z\not{p}\not{p}+zm_e\not{p}+m_e\not{\ell}+zm_e\not{p}+m_e^2)\gamma_\nu u(p)\\ &\to \bar{u}(p')\gamma^\nu(\ell^2+z^2m_e^2+2zm_e\not{p}+m_e^2)\gamma_\nu u(p)\\ &= \bar{u}(p')[d\ell^2+dz^2m_e^2+2zm_e\gamma^\nu(2p_\nu-\gamma_\nu\not{p})+dm_e^2]u(p)\\ &= \bar{u}(p')[d\ell^2+dm_e^2(z^2+1)+2zm_e^2(2-d)]u(p),\end{aligned}$$

where we have dropped terms linear in ℓ. Then applying Eqs. (43) and (44), we can write an expression for the contribution to δZ_1, which we call δZ_1^1. We also make use of Eqs. (30) and (31). We find

$$\begin{aligned}\delta Z_1^1 &= ie^2\int_0^1 dz\,(1-z)\int\frac{d^d\ell}{(2\pi)^d}\frac{d\ell^2+dm_e^2(z^2+1)+2zm_e^2(2-d)}{(\ell^2+\Delta_1+i\epsilon)^3}\\ &= ie^2\int_0^1 dz\,(1-z)\Bigg[\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{(4-\epsilon)^2}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta_1}\right)^{\epsilon/2}\\ &\qquad +m_e^2[(4-\epsilon)(z^2+1)+2z(\epsilon-2)]\left(-\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta_1}\right)^{1+\epsilon/2}\right)\Bigg]\\ &= -\frac{e^2}{32\pi^2}\int_0^1 dz\,(1-z)\Bigg[\frac{(4-\epsilon)^2}{2}\Gamma(\epsilon/2)\left(\frac{4\pi}{\Delta_1}\right)^{\epsilon/2}-\frac{m_e^2(z^2+1)}{\Delta_1}(4-\epsilon)\Gamma(1+\epsilon/2)\left(\frac{4\pi}{\Delta_1}\right)^{\epsilon/2}\\ &\qquad -\frac{2m_e^2z}{\Delta_1}(\epsilon-2)\Gamma(1+\epsilon/2)\left(\frac{4\pi}{\Delta_1}\right)^{\epsilon/2}\Bigg]\\ &\approx -\frac{e^2}{32\pi^2}\int_0^1 dz\,(1-z)\left[\frac{16}{\epsilon}-8-8\gamma+8\ln\!\left(\frac{4\pi}{\Delta_1}\right)-4\frac{m_e^2(z^2+1)}{\Delta_1}+4\frac{m_e^2z}{\Delta_1}\right]\\ &= -\frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\frac{4}{\epsilon}-2-2\gamma+2\ln\!\left(\frac{4\pi}{\Delta_1}\right)-\frac{m_e^2(z^2-z+1)}{\Delta_1}\right].\end{aligned} \tag{45}$$

The second Feynman diagram looks like Eq. (33), but with a scalar loop instead of a photon loop:

$$\bar{u}(p')\delta\Gamma_2(p,p')u(p) = \int \frac{d^dk}{(2\pi)^d}\frac{i}{(k-p)^2 - m_\phi^2 + i\epsilon}\bar{u}(p')\left(-\frac{i\lambda}{\sqrt{2}}\right)\frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon}\frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon}\left(-\frac{i\lambda}{\sqrt{2}}\right)u(p)$$
$$= \frac{i\lambda^2}{2}\int \frac{d^dk}{(2\pi)^d}\bar{u}(p')\frac{(\not{k}' + m_e)(\not{k} + m_e)}{[(k-p)^2 - m_\phi^2 + i\epsilon][k'^2 - m_e^2 + i\epsilon][k^2 - m_e^2 + i\epsilon]}u(p) \tag{46}$$

Again setting $q = 0$, we again apply the appropriate analogue to Eq. (35) and let $\Delta_2 = (1-z)^2 m_e^2 + z m_\phi^2$. Then the denominator of the integrand is $D_2 = \ell^2 - \Delta_2 + i\epsilon$, and the numerator is (again dropping terms linear in ℓ)

$$\begin{aligned}
N_2 &= \bar{u}(p')(\not{k} + m_e)(\not{k} + m_e)\bar{u} \\
&= \bar{u}(p')(\not{\ell}\not{\ell} + z\not{\ell}\not{p} + m_e\not{\ell} + z\not{p}\not{\ell} + z\not{p}\not{p} + zm_e\not{p} + m_e\not{\ell} + zm_e\not{p} + m_e^2)\bar{u} \\
&= \bar{u}(p')(\ell^2 + 2z\not{\ell}m_e + 2m_e\not{\ell} + z^2m_e^2 + 2zm_e^2 + m_e^2)\bar{u} \\
&\to \bar{u}(p')[\ell^2 + m_e^2(1+z)^2]\bar{u}.
\end{aligned}$$

We call the contribution to δZ_1 from this diagram δZ_1^2. Again using Eqs. (30) and (31) yields

$$\begin{aligned}
\delta Z_1^2 &= -\frac{i\lambda^2}{2}\int_0^1 dz\,(1-z)\int\frac{d^dk}{(2\pi)^d}\frac{\ell^2 + m_e^2(1+x)^2}{(\ell^2 - \Delta_2 + i\epsilon)^3} \\
&= -\frac{i\lambda^2}{2}\int_0^1 dz\,(1-z)\left[\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{4-\epsilon}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta_2}\right)^{\epsilon/2} - m_e^2(1+z)^2\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta_2}\right)^{1+\epsilon/2}\right] \\
&= \frac{\lambda^2}{64\pi^2}\int_0^1 dz\,(1-z)\left[\frac{4-\epsilon}{2}\Gamma(\epsilon/2)\left(\frac{4\pi}{\Delta_2}\right)^{\epsilon/2} - \frac{m_e^2(1+z)^2}{\Delta_2}\Gamma(1+\epsilon/2)\left(\frac{4\pi}{\Delta_2}\right)^{1+\epsilon/2}\right] \\
&\approx \frac{\lambda^2}{64\pi^2}\int_0^1 dz\,(1-z)\left[\frac{4}{\epsilon} - 1 - 2\gamma + 2\ln\left(\frac{4\pi}{\Delta_2}\right) - \frac{m_e^2(1+z)^2}{\Delta_2}\right].
\end{aligned}$$

With Eq. (45), we find

$$\delta Z_1 = \int_0^1 dz\,(1-z)\left\{\frac{\lambda^2}{64\pi^2}\left[\frac{4}{\epsilon} - 1 - 2\gamma + 2\ln\left(\frac{4\pi}{\Delta_2}\right) - \frac{m_e^2(1+z)^2}{\Delta_2}\right] - \frac{e^2}{8\pi^2}\left[\frac{4}{\epsilon} - 2 - 2\gamma + 2\ln\left(\frac{4\pi}{\Delta_1}\right) - \frac{m_e^2(z^2 - z + 1)}{\Delta_1}\right]\right\}.$$

For δZ_2, we need to add the loop diagrams of a photon and a scalar. We calculated the former in 1(b) and the latter in 2(a). Applying Eqs. (23) and (42), we have

$$\delta Z_2 = \int_0^1 dx\left\{\frac{\lambda^2}{32\pi^2}x\left[-\frac{2}{\epsilon} + \gamma - \ln\left(\frac{4\pi}{\Delta_2}\right) - \frac{2m_e^2(1+x)(1-x)}{\Delta_2}\right] - \frac{e^2}{16\pi^2}(1-x)\left[\frac{4}{\epsilon} - 5 - 2\gamma + 2\ln\left(\frac{4\pi}{\Delta_1}\right) - \frac{m^2(1-2x-x^2)}{\Delta_1}\right]\right\}.$$

where $\Delta_1 = (1-z)^2m^2 + z\mu^2$ and $\Delta_2 = (1-z)^2m_e^2 + zm_\phi^2$. It certainly appears that $\delta Z_1 \neq \delta Z_2$ in this case, but I am not sure whether this is because I actually computed each correctly.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] Wikipedia contributors, "Gamma matrices." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Gamma_matrices`.