Problem 1.

1(a) Show by explicit computation the Lorentz invariance of the Dirac Lagrangian, by considering a Lorentz transformation of the fields.

Solution. The Dirac Lagrangian is given by Eq. (3.34) in Peskin & Schroder:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

According to their Eq. (3.33), $\bar{\psi}$ transforms as $\bar{\psi} \to \bar{\psi}\Lambda^{-1}_{\frac{1}{2}}$; also, $\psi \to \Lambda_{\frac{1}{2}}\psi$. The Lorentz transformation of the Dirac Lagrangian is then [1, p. 42]

$$\begin{aligned}\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) &\to \bar{\psi}(\Lambda^{-1}x)\Lambda^{-1}_{\frac{1}{2}}[i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m)]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)\\ &= \bar{\psi}(\Lambda^{-1}x)[i\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda^{-1}_{\frac{1}{2}}\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)\\ &= \bar{\psi}(\Lambda^{-1}x)[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\Lambda^{-1}x),\end{aligned}$$

where we have used Peskin & Schroeder (3.29), $\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. Then

$$\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \to \bar{\psi}(\Lambda^{-1}x)[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m)]\psi(\Lambda^{-1}x) = \bar{\psi}(\Lambda^{-1}x)(i\gamma^\nu\partial_\nu - m)\psi(\Lambda^{-1}x),$$

which has the same form as $\mathcal{L}_{\text{Dirac}}$. So we have shown that the Dirac Lagrangian is Lorentz invariant. □

1(b) Consider the chiral rotation of a massless Dirac field $\psi' = e^{i\alpha\gamma^5}\psi$. Find the corresponding Noether current. Show that the corresponding Noether charge measures the total helicity of a collection of massless Dirac particles, and that the addition of a mass term to the Lagrangian violates the symmetry. Find an equation that expresses the violation of current conservation by the mass.

Solution. The conserved charge is given in general by Peskin & Schroeder (2.12) and (2.13),

$$Q \equiv \int_{\text{all space}} j^0\, d^3x\,, \qquad\qquad \text{where } j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu, \tag{1}$$

where J^μ is a 4-divergence that arises when transforming the Lagrangian as in Peskin & Schroeder (2.10):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu J^\mu(x). \tag{2}$$

Under the rotation $\psi \to e^{i\alpha\gamma^5}\psi$, $\psi^\dagger \to \psi^\dagger e^{-i\alpha\gamma^5}$. Then, using $\bar{\psi} = \psi^\dagger\gamma^0$ as defined in Peskin & Schroeder (3.32),

$$\bar{\psi} \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 = -\psi^\dagger\gamma^0 e^{-i\alpha\gamma^5} = -\bar{\psi}e^{-i\alpha\gamma^5},$$

since $\{\gamma^\mu, \gamma^5\} = 0$ from Peskin & Schroeder (3.70). Then, using $m = 0$ in the Dirac Lagrangian, we have

$$\mathcal{L}_{\text{Dirac}} = i\bar{\psi}\gamma^\mu\partial_\mu\psi \to -i\bar{\psi}e^{-i\alpha\gamma^5}\gamma^\mu\partial_\mu e^{i\alpha\gamma^5}\psi = i\bar{\psi}\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu e^{i\alpha\gamma^5}\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi,$$

so the Dirac Lagrangian is indeed invariant under chiral transformations, and $J^\mu = 0$.

The infinitesimal transformations associated with $\psi \to e^{i\alpha\gamma^5}\psi$ are

$$\alpha\Delta\psi = i\alpha\gamma^5\psi, \alpha\Delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5.$$

Then we have the Noether current [1, p. 50]

$$j^\mu = -\left[\frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu \psi)}\Delta\psi + \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu \bar{\psi})}\Delta\bar{\psi}\right] = \bar{\psi}\gamma^\mu\gamma^5\psi,$$

where we have multiplied by an arbitrary constant [1, p. 18].

Peskin & Schroeder (3.76) defines

$$j^\mu_L = \bar{\psi}\gamma^\mu\frac{1-\gamma^5}{2}\psi, \qquad j^\mu_R = \bar{\psi}\gamma^\mu\frac{1+\gamma^5}{2}\psi,$$

as the electric current densities of left- and right-handed particles. Note that $j^\mu = j^\mu_R - j^\mu_L$. Then we have the conserved charge

$$Q = \int d^3x\, \bar{\psi}\gamma^0\gamma^5\psi = \int d^3x\,(j^0_R - j^0_L),$$

which tells us the total helicity of a collection of massless Dirac particles. □

If $m \neq 0$ in the Dirac Lagrangian, then it transforms as

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \to -\bar{\psi}e^{-i\alpha\gamma^5}(i\gamma^\mu\partial_\mu - m)e^{i\alpha\gamma^5}\psi &= \bar{\psi}(\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu + e^{-i\alpha\gamma^5}m)e^{i\alpha\gamma^5}\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu + m)\psi,\end{aligned}$$

which is not of the same form. So the symmetry is violated for nonzero m. □

In order for the current to be conserved, we need the divergence $\partial_\mu j^\mu = 0$. Note that

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi.$$

Since ψ satisfies the Dirac equation, we can make use of the Dirac equation and its conjugate, given by Eqs. (3.31) and (3.35) in Peskin & Schroeder:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \qquad -i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0.$$

So the divergence can be written [1, p. 51]

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi = im\bar{\psi}\gamma^5\psi + \bar{\psi}\gamma^5 im\psi = 2im\bar{\psi}\gamma^5\psi,$$

which is zero only if m is zero.

1(c) Find the Noether current related to charge conservation by considering a phase rotation of a Dirac field (of arbitrary mass) $\psi' = e^{i\alpha}\psi$.

Solution. We will once again use Eqs. (1) and (2). Under the rotation $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$. Then the Dirac Lagrangian transforms as

$$\mathcal{L}_{\text{Dirac}} \to \bar{\psi}e^{i\alpha}(i\gamma^\mu\partial_\mu - m)e^{-i\alpha}\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

so again $J^\mu = 0$.

The infinitesimal translations are

$$\alpha\Delta\psi = i\alpha\psi, \alpha\Delta\bar{\psi} = -i\alpha\bar{\psi},$$

and the Noether current is [1, p. 50]

$$j^\mu = -\left[\frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu \psi)}\Delta\psi + \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu \bar{\psi})}\Delta\bar{\psi}\right] = \bar{\psi}\gamma^\mu\psi.$$

Problem 2. Lorentz group (Peskin & Schroeder 3.1) Recall from Eq. (3.17) the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

2(a) Define the generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i},$$

where $i, j, k = 1, 2, 3$. An infinitesimal Lorentz transformation can then be written

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K})\Phi. \tag{3}$$

Write the commutation relations of these vector operators explicitly. (For example, $[L^i, L^j] = i\epsilon^{ijk}L^k$.) Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}), \qquad \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

commute with one another and separately satisfy the commutation relations of angular momentum.

Solution. Firstly, using Eq. (3.18),

$$\begin{aligned}
[L^i, L^j] &= \left[\frac{1}{2}\epsilon^{i\mu\nu}J^{\mu\nu}, \frac{1}{2}\epsilon^{j\rho\sigma}J^{\rho\sigma}\right] = \frac{1}{4}\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}[J^{\mu\nu}, J^{\rho\sigma}] = \frac{i}{4}\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\mu\rho}J^{\nu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\sigma}J^{\mu\rho} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\mu\sigma}J^{\nu\rho}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\nu\mu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\sigma\rho}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\nu\mu}\epsilon^{j\sigma\rho}g^{\nu\rho}J^{\mu\sigma}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma}) \\
&= i\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma},
\end{aligned}$$

where we have simply relabeled indices. Then, using $g^{ij} = -\delta^{ij}$ and $\epsilon^{ijk}\epsilon^{pqk} = \delta^{ip}\delta^{jq} - \delta^{iq}\delta^{jp}$ [3],

$$\begin{aligned}
[L^i, L^j] &= -i\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}\delta^{\nu\rho}J^{\mu\sigma} = -i\epsilon^{i\mu\nu}\epsilon^{j\nu\sigma}J^{\mu\sigma} = i\epsilon^{i\mu\nu}\epsilon^{j\sigma\nu}J^{\mu\sigma} = i(\delta^{ij}\delta^{\mu\sigma} - \delta^{i\sigma}\delta^{\mu j})J^{\mu\sigma} = i(\delta^{ij}J^{\mu\mu} - \delta^{i\sigma}J^{j\sigma}) \\
&= -iJ^{ji} = iJ^{ij},
\end{aligned}$$

where we have used the antisymmetry of J^{ij}. From $L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$, we can write

$$\epsilon^{i\rho\sigma}L^i = \frac{1}{2}\epsilon^{ijk}\epsilon^{i\rho\sigma}J^{jk} = \frac{1}{2}(\delta^{j\rho}\delta^{k\sigma} - \delta^{j\sigma}\delta^{k\rho})J^{jk} = \frac{1}{2}(\delta^{j\rho}J^{j\sigma} - \delta^{j\sigma}J^{j\rho}) = \frac{1}{2}(J^{\rho\sigma} - J^{\sigma\rho}) = J^{\rho\sigma}.$$

Then we see that

$$[L^i, L^j] = i\epsilon^{ijk}L^k.$$

Secondly,

$$[K^i, K^j] = [J^{0i}, J^{0j}] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) = -iJ^{ij} = -i\epsilon^{ijk}L^k,$$

and thirdly,

$$\begin{aligned}
[K^i, L^j] &= \left[J^{0i}, \frac{1}{2}\epsilon^{j\rho\sigma}J^{\rho\sigma}\right] = \frac{1}{2}\epsilon^{j\rho\sigma}[J^{0i}, J^{\rho\sigma}] = \frac{i}{2}\epsilon^{j\rho\sigma}(g^{i\rho}J^{0\sigma} - g^{0\rho}J^{i\sigma} - g^{i\sigma}J^{0\rho} + g^{0\sigma}J^{i\rho}) \\
&= \frac{i}{2}(\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} - \epsilon^{j\rho\sigma}g^{i\sigma}J^{0\rho}) = \frac{i}{2}(\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} - \epsilon^{j\sigma\rho}g^{i\rho}J^{0\sigma}) = i\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} = -i\epsilon^{j\rho\sigma}\delta^{i\rho}J^{0\sigma} = i\epsilon^{ij\sigma}J^{0\sigma} \\
&= i\epsilon^{ijk}K^k.
\end{aligned}$$

Next we want to show that $[\mathbf{J}_+, \mathbf{J}_-] = 0$. Note that

$$\begin{aligned}[J_+^i, J_-^j] &= \left[\frac{1}{2}(L^i + iK^i), \frac{1}{2}(L^j - iK^j)\right] = \frac{1}{4}\left([L^i, L^j] - i[L^i, K^j] + i[K^i, L^j] + [K^i, K^j]\right)\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{jik}K^k - \epsilon^{ijk}K^k - i\epsilon^{ijk}L^k\right)\\ &= 0,\end{aligned}$$

so $[\mathbf{J}_+, \mathbf{J}_-] = 0$. $\square$

The angular momentum commutation relations are given by Peskin & Schroeder Eq. (3.12): $[J^i, J^j] = i\epsilon^{ijk}J^k$. We have

$$\begin{aligned}[J_\pm^i, J_\pm^j] &= \left[\frac{1}{2}(L^i \pm iK^i), \frac{1}{2}(L^j \pm iK^j)\right] = \frac{1}{4}\left([L^i, L^j] \pm i[L^i, K^j] \pm i[K^i, L^j] - [K^i, K^j]\right)\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k \pm \epsilon^{jik}K^k \mp \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right) = \frac{1}{2}\left(i\epsilon^{ijk}L^k \mp \epsilon^{ijk}K^k\right) = \frac{1}{2}i\epsilon^{ijk}\left(L^k \pm iK^k\right)\\ &= i\epsilon^{ijk}J_\pm^k,\end{aligned}$$

as desired. $\square$

2(b) The finite-dimensional representations of the rotation group correspond precisely to the allowed values for angular momentum: integers or half-integers. The result of 2(a) implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half integers, (j_+, j_-), corresponding to pairs of representations of the rotation group. Using the fact that $\mathbf{J} = \boldsymbol{\sigma}/2$ in the spin-1/2 representation of angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. Show that these correspond precisely to the transformations of ψ_L and ψ_R given in (3.37).

Solution. Equation (3.37) is

$$\psi_L \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_L, \qquad \psi_R \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_R,$$

where ψ_L and ψ_R are the left- and right-handed Weyl spinors, respectively.

We can rewrite Eq. (3) in terms of $\mathbf{J}_+$ and $\mathbf{J}_-$:

$$\Phi \to [1 - i\boldsymbol{\theta}\cdot(\mathbf{J}_+ + \mathbf{J}_-) - \boldsymbol{\beta}\cdot(\mathbf{J}_+ - \mathbf{J}_-)]\Phi = [1 - (i\boldsymbol{\theta} + \boldsymbol{\beta})\cdot\mathbf{J}_+ + (i\boldsymbol{\theta} - \boldsymbol{\beta})\cdot\mathbf{J}_-)]\Phi.$$

From the final expression, we associate $\mathbf{J}_+$ and $\mathbf{J}_-$ with $\boldsymbol{\sigma}/2$ in turn, with $\mathbf{J}_+ = \boldsymbol{\sigma}/2$ corresponding to the $(\frac{1}{2}, 0)$ representation and $\mathbf{J}_- = \boldsymbol{\sigma}/2$ corresponding to the $(0, \frac{1}{2})$ representation. The transformation laws are

$$\Phi \to \begin{cases}\left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\Phi & (\frac{1}{2}, 0) \text{ representation,}\\ \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\Phi & (0, \frac{1}{2}) \text{ representation.}\end{cases}$$

Comparing to Eq. (3.37), we see that Φ transforms as ψ_L under the $(\frac{1}{2}, 0)$ representation and as ψ_R under the $(0, \frac{1}{2})$ representation. $\square$

2(c) The identity $\boldsymbol{\sigma}^T = -\sigma^2\boldsymbol{\sigma}\sigma^2$ allows us to rewrite the ψ_L transformations in the unitarily equivalent form

$$\psi' \to \psi'\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right),$$

where $\psi' = \psi_L^T\sigma^2$. Using this law, we can represent the object that transforms as $(\frac{1}{2},\frac{1}{2})$ as a 2×2 matrix that has the ψ_R transformations law on the left and, simultaneously, the transposed ψ_L transformation on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^3 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}.$$

Show that the object V^μ transforms as a 4-vector.

Solution. Peskin & Schroeder (3.19) shows an infinitesimal Lorentz transformation:

$$V^\alpha \to \left[\delta^\alpha{}_\beta - \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta\right]V^\beta,$$

where V is a 4-vector, $\omega_{\mu\nu}$ is an antisymmetric tensor that gives the infinitesimal angles, and $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha)$ from Peskin & Schroeder (3.18). Using this definition, the transformation is

$$\begin{aligned}
V^\alpha &\to \left[\delta^\alpha{}_\beta + \frac{1}{2}\omega_{\mu\nu}(\delta^{\mu\alpha}\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^{\nu\alpha})\right]V^\beta = \left[\delta^\alpha{}_\beta + \frac{1}{2}\omega_{\mu\nu}g_{\beta\gamma}(\delta^{\mu\alpha}\delta^{\nu\gamma} - \delta^{\mu\gamma}\delta^{\nu\alpha})\right]V^\beta \\
&= \left[\delta^\alpha{}_\beta + \frac{1}{2}g_{\beta\gamma}(\omega^{\alpha\gamma} - \omega^{\gamma\alpha})\right]V^\beta = (\delta^\alpha{}_\beta + g_{\beta\gamma}\omega^{\alpha\gamma})V^\beta \\
&= (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta, \qquad (4)
\end{aligned}$$

where we have used the antisymmetry of $\omega^{\mu\nu}$.

For the problem at hand, note that

$$V_\mu\sigma^\mu = \begin{pmatrix} V^0 & 0 \\ 0 & V^0 \end{pmatrix} - \begin{pmatrix} 0 & V^1 \\ V^1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -iV^2 \\ iV^2 & 0 \end{pmatrix} - \begin{pmatrix} V^3 & 0 \\ 0 & -V^3 \end{pmatrix} = \begin{pmatrix} V^0 + V^3 & V^1 - iV^3 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}.$$

Then the transformation is

$$\begin{aligned}
V_\mu\sigma^\mu &\to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)V_\mu\sigma^\mu\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right) = \left(1 + (\boldsymbol{\beta} - i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2}\right)V_\mu\sigma^\mu\left(1 + (\boldsymbol{\beta} + i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2}\right) \\
&= V_\mu\sigma^\mu + V_\mu\sigma^\mu(\boldsymbol{\beta} + i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2} + (\boldsymbol{\beta} - i\boldsymbol{\theta})\cdot\frac{\boldsymbol{\sigma}}{2}V_\mu\sigma^\mu,
\end{aligned}$$

where we note that θ and β are infinitesimal angles and drop terms of $\mathcal{O}(\theta^2) = \mathcal{O}(\beta^2) = \mathcal{O}(\theta\beta)$. Then

$$\begin{aligned}
V_\mu\sigma^\mu &\to V_\mu\sigma^\mu + \frac{1}{2}V_\mu\sigma^\mu(\beta_i\sigma^i + i\theta_j\sigma^j) + \frac{1}{2}V_\mu(\beta_k\sigma^k - i\theta_l\sigma^l)\sigma^\mu \\
&= V_\mu\sigma^\mu + \frac{1}{2}V_\mu\left(\beta_i\sigma^\mu\sigma^i + i\theta_j\sigma^\mu\sigma^j + \beta_k\sigma^k\sigma^\mu - i\theta_l\sigma^l\sigma^\mu\right) \\
&= V_\mu\sigma^\mu + \frac{1}{2}V_\mu\left[\beta_i(\sigma^\mu\sigma^i + \sigma^i\sigma^\mu) + i\theta_j(\sigma^\mu\sigma^j - \sigma^j\sigma^\mu)\right] = V_\mu\sigma^\mu + \frac{1}{2}V_\mu\left(\beta_i\{\sigma^\mu,\sigma^i\} + i\theta_j[\sigma^\mu,\sigma^j]\right) \\
&= V_\mu\sigma^\mu + \frac{1}{2}V_0\left(\beta_i\{\sigma^0,\sigma^i\} + i\theta_j[\sigma^0,\sigma^j]\right) + \frac{1}{2}V_k\left(\beta_i\{\sigma^k,\sigma^i\} + i\theta_j[\sigma^k,\sigma^j]\right) \\
&= V_\mu\sigma^\mu + \beta_iV_0\sigma^i + V_k\left(\beta_i\delta^{ik} - \theta_j\epsilon^{kji}\sigma^i\right) = V_\mu\sigma^\mu + \beta_iV_0\sigma^i + V_i\beta_i + V_k\theta_j\epsilon^{ijk}\sigma^i \\
&= V_\mu\sigma^\mu + V_0\beta^i\sigma_i - V_i\beta^i - V_j\epsilon^{ijk}\theta^i\sigma^k
\end{aligned}$$

where we have used $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ and $[\sigma^i,\sigma^j] = 2i\epsilon^{ijk}\sigma^k$ [4, p. 165], as well as $\{\sigma^0,\sigma^i\} = 2\sigma^i$ and $[\sigma^0,\sigma^i] = 0$.

Referring to Eq. (3.19), we define

$$\omega^{0j} = \beta^j, \qquad \omega^{ij} = \epsilon^{ijk}\theta^k.$$

Then we have

$$\begin{aligned} V_\mu\sigma^\mu \to V_\mu\sigma^\mu + V_0\omega^{0i}\sigma_i - V_i\omega^{0i} - V_j\omega^{ij}\sigma^j &= V_\mu\sigma^\mu + V^0\omega_{0i}\sigma^i - V^i\omega_{0i}\sigma^0 + V^j\omega_{ij}\sigma^j \\ = V_\mu\sigma^\mu + V^0\omega_{0\mu}\sigma^\mu + V^i\omega_{i0}\sigma^0 + V^j\omega_{ij}\sigma^j &= V_\mu\sigma^\mu + V^\nu\omega_{\nu\mu}\sigma^\mu = (\delta^\nu{}_\mu + \omega^\nu{}_\mu)V_\nu\sigma^\mu \end{aligned}$$

or

$$V^\alpha\sigma_\alpha \to (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta\sigma_\alpha \qquad \Longrightarrow \qquad V^\alpha \to (\delta^\alpha{}_\beta + \omega^\alpha{}_\beta)V^\beta,$$

since σ^μ forms a complete basis. This is identical to Eq. (4), so we have shown that V^μ transforms as a 4-vector. □

Problem 3. Majorana fermions (Peskin & Schroeder 3.4) Recall from Eq. (3.40) that one can write a relativistic equation for a massless 2-component fermion field that transforms as the upper two components of a Dirac spinor (ψ_L). Call such a 2-component field $\chi_a(x)$, $a = 1, 2$.

3(a) Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0. \tag{5}$$

That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

Solution. The expression

$$\left(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \left[1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right]^\mu{}_\nu \gamma^\nu$$

is the infinitesimal form of Peskin & Schroeder (3.29), $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$ [1, p. 42]. We proved a similar expression in 2(c):

$$\left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)V_\mu\sigma^\mu\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right) = \left[1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right]^\mu{}_\nu V^\nu\sigma_\mu. \tag{6}$$

From Peskin & Schroeder (3.41),

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}), \qquad \bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}),$$

which means we can write

$$g_{\mu\nu}\sigma^\nu = \bar{\sigma}^\mu, \qquad g_{\mu\nu}\bar{\sigma}^\nu = \sigma^\mu.$$

Then Eq. (6) can be rewritten as

$$\left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)V_\mu g_{\mu\alpha}\bar{\sigma}^\alpha\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right) = \left[1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right]^\mu{}_\nu V^\nu g_{\mu\alpha}\bar{\sigma}^\alpha,$$

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