Problem 1. Consider the following probabalistic game: There are four doors (Q, R, S, T). Behind each door is a device which displays ± 1 randomly according to the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$. Alice and Bob are on the same team. Alice has to choose either Q and R, and then Bob has to choose either S and T. When the numbers match, they get $+1$ point; when the numbers do not match, they get -1 point. However, when they open Q and T, it's an exception. When the numbers (do not) match, they get -1 $(+1)$.

1.1 Let's assume Alice and Bob open the doors completely randomly. When all numbers are $+1$ with probability 1, what is the expectation value of the point they get?

Solution. Let $\mathbf{E}$ be the expectation value of the number of points. In this case, the numbers behind the two doors will always match. So

$$\mathbf{E} = \frac{QS + RS + RT - QT}{4} = \frac{1 + 1 + 1 - 1}{4} = \frac{1}{2}.$$

1.2 As it turns out, irrespective of how hard you fine tune the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$, the expectation value of the point Alice and Bob get cannot exceed a certain value Max:

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} \leq \text{Max}.$$

Here, $\mathbf{E}(QS)$, etc. is the expectation value of the point when Alice opens Q and Bob opens S. This is a Bell inequality. Determine Max.

Hint: For a given realization of the numbers $Q = \pm 1$, $R = \pm 1$, $S = \pm 1$, $T = \pm 1$, which occurs with probability $P(Q, R, S, T)$, note that $QS + RS + RT - QT = (Q + R)S + (R - Q)T$, where one of $\{(R + Q), (R - Q)\}$ is 2 and the other 0.

Solution. In addition to the information provided in the hint, both S and T must be ± 1. This means the only possibilities for the number of points earned are

$$\frac{(Q + R)S + (R - Q)T}{4} = \begin{cases} \dfrac{(0)(-1) + (2)(1)}{4} = \dfrac{1}{2}, \\ \dfrac{(0)(1) + (2)(-1)}{4} = -\dfrac{1}{2}. \end{cases}$$

Thus,

$$\text{Max} = \frac{1}{2}.$$

1.3 Frustrated by the upper bound set by the Bell inequality, Bob decides to cheat. He now changes the value of T after Alice chooses Q or R. Assume Q, R, S are set to be $+1$ with probability 1. To make the expectation value of the point they get equal to $+1$, what values should Bob set after Alice chooses Q or R?

Solution. If Alice chooses R, Bob should set $T = 1$. If Alice chooses Q, Bob should set $T = -1$. This way,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1 + 1 + 1 + 1}{4} = 1.$$

1.4 Now consider a quantum mechanical version of the game. There are quantum states of two spin-1/2 degrees of freedom shared by Alice and Bob. Alice can measure the z component or x components of the first spin $\mathbf{S}^A$. (This corresponds to $Q = \pm 1$ or $R = \pm 1$.) Bob can measure the $-(z+x)$ component or the $(z-x)$ component of the second spin $\mathbf{S}^B$. (This corresponds to $S = \pm 1$ or $T = \pm 1$.)

More specifically, Alice and Bob share the quantum state

$$|\psi\rangle = \frac{|{\uparrow_z}\rangle \otimes |{\downarrow_z}\rangle - |{\downarrow_z}\rangle \otimes |{\uparrow_z}\rangle}{\sqrt{2}}.$$

The operators to be measured are

$$Q = S_z^A, \qquad R = S_x^A, \qquad S = -\frac{S_z^B + S_x^B}{\sqrt{2}}, \qquad T = \frac{S_z^B - S_x^B}{\sqrt{2}}.$$

Let us consider the case when Alice measures Q and Bob measures T. Calculate the probability $P(Q,T)$ for Alice and Bob getting the measurement outcomes $(Q,T) = (\pm 1, \pm 1)$.

Solution. All of the spin operators have eigenvalues $\pm\hbar/2$, but in order to maintain consistency with the classical example, we will let $S_z \to \sigma_1$ and $S_z \to \sigma_3$ so the eigenvalues are instead ± 1.

The Pauli matrices are given by Eq. (3.2.32) in Sakurai,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then in the S_z^A basis, Q and its eigenvectors are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad |Q_+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |Q_-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In the S_z^B basis, T can be written

$$T = \frac{\hbar}{2\sqrt{2}} \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{\hbar}{2\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

The eigenvectors of T corresponding to eigenvalues $\pm\hbar/2$ can be found by

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \pm\sqrt{2} \begin{bmatrix} u \\ v \end{bmatrix},$$

which is satisfied when

$$v = (1 \mp \sqrt{2})u, \qquad u = -(1 \pm \sqrt{2})v.$$

We will fix $u = 1$. Then the normalization constants $A_\pm$ are found by

$$1 = |\langle T_\pm | T_\pm \rangle|^2 = A_\pm^2 \begin{bmatrix} 1 & 1 \mp \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \mp \sqrt{2} \end{bmatrix} = A_\pm^2 (4 \mp 2\sqrt{2}) \implies A_\pm = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}}. \tag{1}$$

The normalized eigenvectors are

$$|T_+\rangle = \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{bmatrix} 1 \\ 1 - \sqrt{2} \end{bmatrix}, \qquad |T_-\rangle = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \begin{bmatrix} 1 \\ 1 + \sqrt{2} \end{bmatrix}. \tag{2}$$

The probability that Alice and Bob obtain $(Q,T)=(+1,+1)$ is

$$\begin{aligned}
P(+1,+1) = |\langle Q_+, T_+|\psi\rangle|^2 &= \left(\begin{bmatrix}1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix}1 & 1-\sqrt{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right)^2 \\
&= \left(\frac{1}{\sqrt{2}} \otimes -\frac{1}{\sqrt{4-2\sqrt{2}}}\right)^2 = \frac{1}{8-4\sqrt{2}},
\end{aligned}$$

and the probability that they obtain $(Q,T)=(-1,-1)$ is

$$\begin{aligned}
P(-1,-1) = |\langle Q_-, T_-|\psi\rangle|^2 &= \left(\begin{bmatrix}0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1 & 1+\sqrt{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right)^2 \\
&= \left(-\frac{1}{\sqrt{2}} \otimes \frac{1}{\sqrt{4+2\sqrt{2}}}\right)^2 = \frac{1}{8+4\sqrt{2}}.
\end{aligned}$$

1.5 Similarly, consider the case when Alice measures R and Bob measures T. Calculate the probability $P(R,T)$ for Alice and Bob getting the measurement outcomes $(R,T)=(\pm1,\pm1)$.

Solution. In the S_z^A basis, R and its eigenvectors are

$$R = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad |R_+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad |R_-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}.$$

Using (2), the probability for Alice and Bob obtain $(R,T)=(+1,+1)$ is

$$P(+1,+1) = |\langle R_+, T_+|\psi\rangle|^2 = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix}1 & 1-\sqrt{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right)^2 = 0,$$

and the probability that they obtain $(R,T)=(-1,-1)$ is

$$\begin{aligned}
P(-1,-1) = |\langle R_-, T_-|\psi\rangle|^2 &= \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1 & 1+\sqrt{2}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right)^2 \\
&= \left(1 \otimes \frac{1}{\sqrt{4+2\sqrt{2}}}\right)^2 = \frac{1}{4+2\sqrt{2}}.
\end{aligned}$$

1.6 Compute the expectation values $\mathbf{E}(QS)$, $\mathbf{E}(RS)$, $\mathbf{E}(QT)$, and $\mathbf{E}(RT)$. Compute

$$\frac{\mathbf{E}(QS)+\mathbf{E}(RS)+\mathbf{E}(RT)-\mathbf{E}(QT)}{4}.$$

Solution. Firstly, in the S_z^B basis, S can be written

$$S = -\frac{1}{\sqrt{2}}\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & -1\\-1 & 1\end{bmatrix}.$$

Then the expectation values are

$$\begin{aligned}
\mathbf{E}(QS) &= \langle\psi|QS|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}\otimes\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\otimes\frac{1}{2\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix} = 0,
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(RS) &= \langle\psi|RS|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}\otimes\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}\otimes\frac{1}{2\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix} = -1\otimes\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}},
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(QT) &= \langle\psi|QT|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}\otimes\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}\otimes\frac{1}{2\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-2\\ 0\end{bmatrix} = 0,
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(RT) &= \langle\psi|RT|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}\otimes\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix}\\
&= \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix}\otimes\frac{1}{2\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}-2\\ 0\end{bmatrix} = -1\otimes\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.
\end{aligned}$$

Then

$$\frac{\mathbf{E}(QS)+\mathbf{E}(RS)+\mathbf{E}(RT)-\mathbf{E}(QT)}{4} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4},$$

which is not greater than the classical Max $= 1/2$, thus not violating Bell's inequality.

Problem 2. Consider a quantum particle with mass m moving in the presence of the square well potential

$$V(|r|) = \begin{cases} -V_0 & r \le a,\\ 0 & r > a. \end{cases}$$

2.1 Writing the wave function in polar coordinates as $\psi(\mathbf{r}) = R_l(r)\,Y_{lm}(\theta,\phi)$, write down the Schrödinger equation obeyed by R_l.

2.2 When V_0 is a certain value, there is one bound state for the s wave ($l = 0$). The bound state energy ε is small ($0 < |\varepsilon| \ll V_0$). Obtain the range of the depth of the well V_0 ($? \le V_0 < ?$). Also, calculate for the bound state the probability for the particle to exist outside of the well.

2.3 Consider the scattering problem by the well. For each l, for large enough r, when $R_l(r)$ is given by $R_l(r) \sim A_l \sin(kr - l\pi/2 + \delta_l)/r$, δ_l is called the scattering phase shift. For the value of V_0 within the range you obtained in the above problem, when the energy of the incident wave is is $E = 9V_0/16$, calculate $\tan\delta_0$ (where δ_0 is the scattering phase shift for the s wave).

2.4 Now consider the S matrix, $S \equiv \exp(2i\delta_0) = \exp(i\delta_0)/\exp(-i\delta_0)$. Compare the condition on s wave bound state energies and the zero of the denominator of S. Explain their relation.

Problem 3. Consider a three dimensional potential

$$V(|r|) = \frac{\hbar^2\gamma}{2m}\delta(|r| - a).$$

The s wave Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\chi_0(r)}{dr^2} + \frac{\hbar^2\gamma}{2m}\delta(r-a)\,\chi_0(r) = E\,\chi_0(r).$$

The s wave function must be regular (zero) at $r = 0$. At $r = a$, it is continuous, but its derivative can jump.

3.1 Calculate the s wave scattering phase shift (k), where k is related to E as $E = \hbar^2k^2/2m$.

3.2 When $\gamma \gg k$, $1/a$ and when $\sin ka$ is not small, discuss the behavior of the scattering phase shift.

3.3 Obtain the condition to have resonant states and calculate the energy of the resonant states.

3.4 Calculate the width Γ of the resonance. Discuss its behavior when γ is big.

3.5 When the velocity of the incident wave is small, obtain the scattering cross section.

I consulted Sakurai's *Modern Quantum Mechanics*, Shankar's *Principles of Quantum Mechanics*, and Wolfram MathWorld while writing up these solutions.