Problem 1. Stress-energy tensor and energy-momentum conservation for a perfect fluid (MCP 2.26)

1(a) Derive the frame-independent expression (2.74b) for the perfect fluid stress-energy tensor from its rest-frame components (2.74a).

Solution. From MCP (2.74a), the nonzero rest-frame components of the tensor are

$$T^{00} = \rho, \qquad\qquad T^{jk} = P\delta^{jk}.$$

We know from MCP (2.23c) that $g^{\alpha\beta} = \eta^{\alpha\beta}$, and from (2.22) that $\eta_{00} = -1$, $\eta_{11} = \eta_{22} = \eta_{33} = 1$. Then, using the form $\mathsf{T} \equiv T^{\alpha\beta}\vec{e}_\alpha \otimes \vec{e}_\beta$ of (2.23a), we can write

$$\mathsf{T} = (\rho + P)\vec{e}_0 \otimes \vec{e}_0 + P\boldsymbol{g}.$$

Note that in the local rest frame, the fluid is stationary so its 4-velocity is $(1, 0, 0, 0)$. That is, $\vec{u} \otimes \vec{u}$ simplifies to $\vec{e}_0 \otimes \vec{e}_0$ in the local rest frame. So the frame-independent expression is

$$\mathsf{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\boldsymbol{g},$$

which is identical to (2.74b). □

1(b) Explain why the projection of $\vec{\nabla} \cdot \mathsf{T} = 0$ along the fluid 4-velocity, $\vec{u} \cdot (\vec{\nabla} \cdot \mathsf{T}) = 0$, should represent energy conservation as viewed by the fluid itself. Show that this equation reduces to

$$\frac{d\rho}{d\tau} = -(\rho + P)\vec{\nabla} \cdot \vec{u}.$$

With the aid of Eq. (2.65), bring this into the form

$$\frac{d(\rho V)}{d\tau} = -P\frac{dV}{d\tau},$$

where V is the 3-volume of some small fluid element as measured in the fluid's local rest frame. What are the physical interpretations of the left- and right-hand sides of this equation, and how is it related to the first law of thermodynamics?

Solution. We know that $\vec{\nabla} \cdot \mathsf{T}$ represents the energy-momentum (i.e., 4-momentum) flow of the system, and that $\vec{\nabla} \cdot \mathsf{T} = 0$ tells us 4-momentum is conserved [1, pp. 83–85]. So $\vec{u} \cdot (\vec{\nabla} \cdot \mathsf{T})$ tells us how the energy-momentum flow looks in the local rest frame of the fluid, and $\vec{u} \cdot (\vec{\nabla} \cdot \mathsf{T}) = 0$ thus indicates that energy is conserved in that frame.

Applying (2.74b) and the product rule, note that

$$\begin{aligned}
\vec{\nabla} \cdot \mathsf{T} &= \partial_\alpha T^{\alpha\beta} \\
&= \partial_\alpha[(\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}] \\
&= u^\alpha u^\beta \partial_\alpha(\rho + P) + (\rho + P)(u^\beta \partial_\alpha u^\alpha + u^\alpha \partial_\alpha u^\beta) + g^{\alpha\beta}\partial_\alpha P \\
&= u^\alpha u^\beta \partial_\alpha(\rho + P) + (\rho + P)(u^\beta \partial_\alpha u^\alpha + u^\alpha \partial_\alpha u^\beta) + \partial^\beta P.
\end{aligned} \tag{1}$$

Then

$$\begin{aligned}\vec{u}\cdot(\vec{\nabla}\cdot T) &= u_\beta \partial_\alpha T^{\alpha\beta} \\ &= u_\beta u^\alpha u^\beta \partial_\alpha(\rho+P) + u_\beta(\rho+P)(u^\beta\partial_\alpha u^\alpha + u^\alpha \partial_\alpha u^\beta) + u_\beta \partial^\beta P \\ &= -u^\alpha\partial_\alpha(\rho+P) + (\rho+P)(-\partial_\alpha u^\alpha + u_\beta u^\alpha \partial_\alpha u^\beta) + u_\beta\partial^\beta P \\ &= -u^\alpha\partial_\alpha(\rho+P) - (\rho+P)\partial_\alpha u^\alpha + u_\beta \partial^\beta P, \end{aligned} \tag{2}$$

where we have used MCP (2.9), $\vec{u}^2=-1$, and that as a consequence [2, p. 36],

$$0 = \partial_\alpha(u_\beta u^\beta) = u_\beta\partial_\alpha u^\beta + u^\beta \partial_\alpha u_\beta = 2u_\beta \partial_\alpha u^\beta. \tag{3}$$

Picking back up at Eq. (2),

$$\vec{u}\cdot(\vec{\nabla}\cdot T) = -u^\alpha\partial_\alpha\rho - u^\alpha\partial_\alpha P - (\rho+P)\partial_\alpha u^\alpha + u_\beta\partial^\beta P = -u^\alpha\partial_\alpha \rho - (\rho+P)\partial_\alpha u^\alpha. \tag{4}$$

We can evaluate $u^\alpha\partial_\alpha\rho$ in the fluid's local rest frame where $\vec{u}=(1,0,0,0)$. Its value will be the same in every rest frame because the inner product is Lorentz invariant [3, p. 541]. Thus

$$u^\alpha\partial_\alpha\rho = \frac{d\rho}{d\tau} + u^i\frac{d\rho}{dx^i} = \frac{d\rho}{d\tau}.$$

Applying this and $\vec{u}\cdot(\vec{\nabla}\cdot T)=0$ to Eq. (4), we find

$$\frac{d\rho}{d\tau} = -(\rho+P)\partial_\alpha u^\alpha \quad\Longrightarrow\quad \frac{d\rho}{d\tau} = -(\rho+P)\vec{\nabla}\cdot\vec{u} \quad\Longrightarrow\quad \frac{d\rho}{d\tau} = -(\rho+P)\vec{\nabla}\cdot\vec{u}, \tag{5}$$

as we wanted to show. □

MCP (2.65) states

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{V}\frac{dV}{d\tau}.$$

Feeding this into Eq. (5) and applying the product rule yields

$$\frac{d\rho}{d\tau} = -(\rho+P)\frac{1}{V}\frac{dV}{d\tau} \quad\Longrightarrow\quad V\frac{d\rho}{d\tau} = -(\rho+P)\frac{dV}{d\tau} \quad\Longrightarrow\quad V\frac{d\rho}{d\tau} + \rho\frac{dV}{d\tau} = \frac{d(\rho V)}{d\tau} = -P\frac{dV}{d\tau}, \tag{6}$$

as we wanted to show. □

The left-hand side of Eq. (6) represents the rate of change of energy ($=\rho V$). The right-hand side represents the product of pressure and the rate of change of volume. Essentially, the equation is relating the change in energy of the fluid to the change in its volume under constant pressure. The first law of thermodynamics is given by MCP (5.7):

$$d\mathcal{E} = T\,dS + \tilde{\mu}\,dN - P\,dV\,,$$

where $\mathcal{E}$ is energy, T is temperature, S is entropy, $\tilde{\mu}$ is chemical potential, and N is number of particles. Eq. (6) is the first law of thermodynamics for a perfect fluid in the case of constant entropy and constant number of particles.

1(c) Read the discussion in Ex. 2.10 about the tensor $P = g + \vec{u}\otimes\vec{u}$ that projects into the 3-space of the fluid's rest frame. Explain why $P_{\alpha\beta}T^{\alpha\beta}{}_{;\beta}=0$ should represent the law of force balance (momentum conservation) as seen by the fluid. Show that this equation reduces to

$$(\rho+P)\vec{a} = -P\cdot\boldsymbol{\nabla}P,$$

where $\vec{a} = d\vec{u}/d\tau$ is the fluid's 4-acceleration. This equation is a relativistic version of Newton's $\mathbf{F}=m\mathbf{a}$. Explain the physical meanings of the left- and right-hand sides. Infer that $\rho+P$ must be the fluid's inertial mass per unit volume.

Solution. Projecting $\vec{\nabla}\cdot T$ into the 3-space of the fluid's rest frame leaves us with only the flow of 3-momentum (as opposed to 4-momentum, as we saw in 1(b)). So $P_{\alpha\beta}T^{\alpha\beta}{}_{;\beta} = 0$ means that 3-momentum must be conserved in the fluid's rest frame, which is equivalent to force balance.

We write

$$P_{\alpha\beta}T^{\alpha\beta}{}_{;\beta} = P_{\alpha\beta}\partial_\gamma T^{\alpha\gamma}$$

and apply Eq. (1) to $\partial_\gamma T^{\alpha\gamma}$. Now we invoke MCP (2.31a), which we can write as $P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$:

$$\begin{aligned}
P_{\alpha\beta}T^{\alpha\beta}{}_{;\beta} &= (g_{\alpha\beta} + u_\alpha u_\beta)[u^\gamma u^\beta \partial_\gamma(\rho+P) + (\rho+P)(u^\alpha\partial_\gamma u^\gamma + u^\gamma\partial_\gamma u^\alpha) + \partial^\alpha P] \\
&= u^\gamma u_\alpha \partial_\gamma(\rho+P) - u_\alpha u^\gamma \partial_\gamma(\rho+P) + (g_{\alpha\beta} + u_\alpha u_\beta)[(\rho+P)(u^\alpha\partial_\gamma u^\gamma + u^\gamma\partial_\gamma u^\alpha) + \partial^\alpha P] \\
&= (\rho+P)(u_\beta\partial_\gamma u^\gamma + u^\gamma\partial_\gamma u_\beta - u_\beta\partial_\gamma u^\gamma + u_\beta u^\gamma u_\alpha\partial_\gamma u^\alpha) + (g_{\alpha\beta} + u_\alpha u_\beta)\partial^\alpha P \\
&= (\rho+P)u^\gamma\partial_\gamma u_\beta + P_{\alpha\beta}\partial^\alpha P, \qquad (7)
\end{aligned}$$

where we have again used $\vec{u}^2 = -1$ and Eq. (3). As in 1(b), we take advantage of the Lorentz invariance of the dot product to evaluate $u^\gamma\partial_\gamma u_\beta$ in the fluid's local rest frame:

$$u^\gamma\partial_\gamma u_\beta = \partial_t.$$

Applying this result and $P_{\alpha\beta}T^{\alpha\beta}{}_{;\beta} = 0$ to Eq. (7), we have

$$-P_{\alpha\beta}\partial^\alpha P = (\rho+P)\partial_t u_\beta = (\rho+P)a_\beta \quad \implies \quad (\rho+P)\vec{a} = -\mathsf{P}\cdot\boldsymbol{\nabla}P, \qquad (8)$$

as we wanted to show. □

The left-hand side of Eq. (8) represents the fluid's inertial force per unit volume, since $\rho+P$ is its inertial mass per unit volume. The right-hand side is the 3-space projection of the pressure gradient, so it represents the spatial distribution of the pressure that the fluid sees.

Problem 2. Inertial mass per unit volume (MCP 2.27) Suppose that some medium has a rest frame (unprimed frame) in which its energy flux and momentum density vanish, $T^{0j} = T^{j0} = 0$. Suppose that the medium moves in the x direction with speed very small compared to light, $v \ll 1$, as seen in a (primed) laboratory frame, and ignore factors of order v^2. The ratio of the medium's momentum density $G_{j'} = T^{j'0'}$ (as measured in the laboratory frame) to its velocity $v_i = \delta_{ix}$ is called its total *inertial mass per unit volume* and is denoted ρ^{inert}_{ji}:

$$T^{j'0'} = \rho^{\text{inert}}_{ji} v_i. \qquad (9)$$

In other words, ρ^{inert}_{ji} is the 3-dimensional tensor that gives the momentum density $G_{j'}$ when the medium's small velocity is put into its second slot.

2(a) Using a Lorentz transformation from the medium's (unprimed) rest frame to the (primed) laboratory frame, show that

$$\rho^{\text{inert}}_{ji} = T^{00}\delta_{ji} + T_{ji}. \qquad (10)$$

Solution. In the rest frame of the medium,

$$[T^{\mu\nu}] = \begin{bmatrix} T^{00} & 0 & 0 & 0 \\ 0 & T_{11} & T_{12} & T_{13} \\ 0 & T_{12} & T_{22} & T_{23} \\ 0 & T_{13} & T_{23} & T_{33} \end{bmatrix}.$$

In the limit $v \ll 1$, $\gamma = 1/\sqrt{1-v^2} \approx 1$. Since we are boosting in the x direction, the Lorentz transformation matrix we need is given by MCP (2.37a):

$$\left[L^{\alpha}{}_{\bar{\mu}}\right] = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & v & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To perform a boost on a tensor like $T^{\mu\nu}$, we invoke MCP (2.36a):

$$T^{\bar{\mu}\bar{\nu}\bar{\rho}} = L^{\bar{\mu}}{}_{\alpha} L^{\bar{\nu}}{}_{\beta} L^{\bar{\rho}}{}_{\gamma} T^{\alpha\beta\gamma}.$$

We need to perform the operation

$$T^{\mu'\nu'} = L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} T^{\mu\nu}.$$

We can write this as a matrix equation, and solve it using Mathematica:

$$\left[T^{\mu'\nu'}\right] = \begin{bmatrix} 1 & v & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T^{00} & 0 & 0 & 0 \\ 0 & T_{11} & T_{12} & T_{13} \\ 0 & T_{12} & T_{22} & T_{23} \\ 0 & T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} 1 & v & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} T^{00} & v(T^{00}+T_{11}) & vT_{12} & vT_{13} \\ v(T^{00}+T_{11}) & T_{11} & T_{12} & T_{13} \\ vT_{12} & T_{12} & T_{22} & T_{23} \\ vT_{13} & T_{13} & T_{23} & T_{33} \end{bmatrix},$$

where we have neglected terms of $\mathcal{O}(v^2)$. Then we have

$$\left[T^{j'0'}\right] = v \begin{bmatrix} T^{00} + T_{11} \\ T_{12} \\ T_{13} \end{bmatrix}. \tag{11}$$

Now we feed Eq. (10) into Eq. (9), and find

$$T^{j'0'} = (T^{00}\delta_{ji} + T_{ji})v_i = v(T^{00}\delta_{ji} + T_{ji})\delta_{i1} = v(T^{00}\delta_{j1} + T_{j1}),$$

or

$$\left[T^{j'0'}\right] = v \begin{bmatrix} T^{00} + T_{11} \\ T_{12} \\ T_{13} \end{bmatrix}.$$

This is identical to Eq. (11), the result we found from boosting $T^{\mu\nu}$ from the rest frame to the lab frame. So we have found that $\rho^{\text{inert}}_{ji} = T^{00}\delta_{ji} + T_{ji}$, as we wanted to show. □

2(b) Give a physical explanation of the contribution T_{ji} to the momentum density.

Solution. The momentum flux (stress) that we observe in the medium rest frame contributes to the momentum density in the lab frame. We can think of the momentum flux as seen from the rest frame as moving at velocity v with the fluid in the lab frame. Therefore it contributes to the momentum density of the medium.

2(c) Show that for a perfect fluid [Eq. (2.74b)] the inertial mass per unit volume is isotropic and has magnitude $\rho + P$, where ρ is the mass-energy density, and P is the pressure measured in the fluid's rest frame:

$$\rho^{\text{inert}}_{ji} = (\rho + P)\delta_{ji}.$$

Solution. MCP (2.74b) states that, for a perfect fluid,

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}.$$

Feeding this into Eq. (10), we find

$$\rho_{ji}^{\text{inert}} = [(\rho + P)u^0 u^0 + Pg^{00}]\delta_{ji} + (\rho + P)u_j u_i + Pg_{ji}. \tag{12}$$

Note that $u^0 = \gamma \approx 1$, and also that $g^{00} = \eta^{00} = -1$ and $g_{ji} = \delta_{ji}$. So Eq. (12) becomes

$$\rho_{ji}^{\text{inert}} = \rho\delta_{ji} + (\rho + P)u_j u_i + P\delta_{ji} = (\rho + P)(\delta_{ji} + u_j u_i).$$

Since velocity v is very small, $u_j u_i \approx v^2 \approx 0$. This gives us

$$\rho_{ji}^{\text{inert}} = (\rho + P)\delta_{ji},$$

as we wanted to show. □

Problem 3. Index-manipulation rules from duality (MCP 24.4) For an arbitrary basis $\{\vec{e}_\alpha\}$ and its dual basis $\{\vec{e}^\mu\}$, use (i) the duality relation (24.8), (ii) the definition (24.9) of components of a tensor, and (iii) the relation $\vec{A} \cdot \vec{B} = \mathsf{g}(\vec{A}, \vec{B})$ between the metric and the inner product to deduce the following results.

3(a) The relations

$$\vec{e}^\mu = \mathsf{g}^{\mu\alpha}\vec{e}_\alpha, \qquad \vec{e}_\alpha = \mathsf{g}_{\alpha\mu}\vec{e}^\mu.$$

Solution. MCP (24.8) is

$$\vec{e}^\mu \cdot \vec{e}_\beta = \mathsf{g}(\vec{e}^\mu, \vec{e}_\beta) = \delta^\mu{}_\beta, \tag{13}$$

and MCP (24.9) is

$$F^{\mu\nu} = \mathsf{F}(\vec{e}^\mu, \vec{e}^\nu), \qquad F_{\alpha\beta} = \mathsf{F}(\vec{e}_\alpha, \vec{e}_\beta), \qquad F^\mu{}_\beta = \mathsf{F}(\vec{e}^\mu, \vec{e}_\beta). \tag{14}$$

For the first relation, we take the dot product of both sides with $\vec{e}^\beta$. Beginning with the right-hand side,

$$(\mathsf{g}^{\mu\alpha}\vec{e}_\alpha) \cdot \vec{e}^\beta = \mathsf{g}^{\mu\alpha}(\vec{e}^\beta \cdot \vec{e}_\alpha) = \mathsf{g}^{\mu\alpha}\mathsf{g}(\vec{e}^\beta, \vec{e}_\alpha) = \mathsf{g}^{\mu\alpha}\delta^\beta{}_\alpha = \mathsf{g}^{\mu\beta} = \mathsf{g}(\vec{e}^\mu, \vec{e}_\beta) = \vec{e}^\mu \cdot \vec{e}^\beta$$

which proves that $\vec{e}^\mu = \mathsf{g}^{\mu\alpha}\vec{e}_\alpha$. □

For the second relation, the proof follows the same path:

$$(\mathsf{g}_{\alpha\mu}\vec{e}^\mu) \cdot \vec{e}_\beta = \mathsf{g}_{\alpha\mu}(\vec{e}_\beta \cdot \vec{e}^\mu) = \mathsf{g}_{\alpha\mu}\mathsf{g}(\vec{e}_\beta, \vec{e}^\mu) = \mathsf{g}_{\alpha\mu}\delta_\beta{}^\mu = \mathsf{g}_{\alpha\beta} = \mathsf{g}(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta,$$

which proves that $\vec{e}_\alpha = \mathsf{g}_{\alpha\mu}\vec{e}^\mu$. □

3(b) The fact that indices on the components of tensors can be raised and lowered using the components of the metric:

$$F^{\mu\nu} = \mathsf{g}^{\mu\alpha}F_\alpha{}^\nu, \qquad p_\alpha = \mathsf{g}_{\alpha\beta}p^\beta.$$

Solution. We once again apply Eqs. (13) and (14). For the first relation, we also use the linearity of tensors [1, p. 11] and the result of 3(a). Beginning with the right-hand side,

$$g^{\mu\alpha}F_{\alpha}{}^{\nu} = g^{\mu\alpha}F(\vec{e}_\alpha, \vec{e}^{\nu}) = F(g^{\mu\alpha}\vec{e}_\alpha, \vec{e}^{\nu}) = F(\vec{e}^{\mu}, \vec{e}^{\nu}) = F^{\mu\nu}$$

as we wanted to show. □

For the second relation, the proof is similar:

$$g_{\alpha\beta}p^{\beta} = g_{\alpha\beta}p(\vec{e}^{\beta}) = p(g_{\alpha\beta}\vec{e}^{\beta}) = p(\vec{e}_\alpha) = p_\alpha,$$

as we wanted to show. □

3(c) The fact that a tensor can be reconstructed from its components in the manner of Eq. (24.11).

Solution. MCP (24.11) is

$$F = F^{\mu\nu}\vec{e}_\mu \otimes \vec{e}_\nu = F_{\alpha\beta}\vec{e}^{\alpha} \otimes \vec{e}^{\beta} = F^{\mu}{}_{\beta}\vec{e}_\mu \otimes \vec{e}^{\beta}.$$

In addition to Eqs. (13) and (14), we invoke the definition of the tensor product given by MCP (1.5a):

$$\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}(\mathbf{E}, \mathbf{F}, \mathbf{G}) = \mathbf{A}(\mathbf{E})\mathbf{B}(\mathbf{F})\mathbf{C}(\mathbf{G}) = (\mathbf{A} \cdot \mathbf{E})(\mathbf{B} \cdot \mathbf{F})(\mathbf{C} \cdot \mathbf{G}).$$

Then [1, p. 55]

$$F(\vec{e}^{\mu}, \vec{e}^{\nu}) = F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} = F_{\alpha\beta}(\vec{e}^{\alpha} \cdot \vec{e}^{\mu})(\vec{e}^{\beta} \cdot \vec{e}^{\nu}) = F_{\alpha\beta}\vec{e}^{\alpha} \otimes \vec{e}^{\beta}(\vec{e}^{\mu}, \vec{e}^{\nu}).$$

Comparing the first and last expressions, we have shown that $F = F_{\alpha\beta}\vec{e}^{\alpha} \otimes \vec{e}^{\beta}$ as desired. The proofs for the other two expressions are equivalent. □

Problem 4. Transformation matrices for circular polar bases (MCP 24.5) Consider the circular polar coordinate system $\{\varpi, \phi\}$ and its coordinate bases and orthonormal bases as shown in Fig. 24.3 and discussed in the associated text. These coordinates are related to Cartesian coordinates $\{x, y\}$ by the usual relations: $x = \varpi\cos\phi$, $y = \varpi\sin\phi$.

4(a) Evaluate the components ($L^{x}{}_{\varpi}$, etc.) of the transformation matrix that links the two coordinate bases $\{\vec{e}_x, \vec{e}_y\}$ and $\{\vec{e}_\varpi, \vec{e}_\phi\}$. Also evaluate the components ($L^{\varpi}{}_{x}$, etc.) of the inverse transformation matrix.

Solution. MCP (24.17) states that

$$\vec{e}_\alpha = \vec{e}_{\bar{\mu}}L^{\bar{\mu}}{}_{\alpha}, \qquad \vec{e}_{\bar{\mu}} = \vec{e}_\alpha L^{\alpha}{}_{\bar{\mu}}, \tag{15}$$

and MCP (24.20) states that

$$L^{\bar{\mu}}{}_{\alpha} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\alpha}}, \qquad L^{\alpha}{}_{\bar{\mu}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\mu}}}. \tag{16}$$

Then

$$L^{x}{}_{\varpi} = \frac{\partial x}{\partial \varpi} = \cos\phi, \qquad L^{y}{}_{\varpi} = \frac{\partial y}{\partial \varpi} = \sin\phi, \qquad L^{x}{}_{\phi} = \frac{\partial x}{\partial \phi} = -\varpi\sin\phi, \qquad L^{y}{}_{\phi} = \frac{\partial y}{\partial \phi} = \varpi\cos\phi.$$

We know the inverse transformation matrix must be the inverse of the matrix we just found [1, p. 1164]. We can write

$$\begin{bmatrix}\vec{e}_\varpi \\ \vec{e}_\phi\end{bmatrix} = \begin{bmatrix}\cos\phi & -\varpi\sin\phi \\ \sin\phi & \varpi\cos\phi\end{bmatrix}\begin{bmatrix}\vec{e}_x \\ \vec{e}_y\end{bmatrix} = \begin{bmatrix}L^x{}_\varpi & L^x{}_\phi \\ L^y{}_\varpi & L^y{}_\phi\end{bmatrix}\begin{bmatrix}\vec{e}_x \\ \vec{e}_y\end{bmatrix} \equiv \mathsf{L}\begin{bmatrix}\vec{e}_x \\ \vec{e}_y\end{bmatrix}, \tag{17}$$

where we have defined the matrix L. The inverse of 2×2 matrix can be found using the general expression [4]

$$\mathsf{A}^{-1} = \frac{1}{|\mathsf{A}|}\begin{bmatrix}A_{22} & -A_{12} \\ -A_{21} & A_{11}\end{bmatrix}. \tag{18}$$

Our determinant is

$$|\mathsf{L}| = L^x{}_\varpi L^y{}_\phi - L^x{}_\phi L^y{}_\varpi = \varpi\cos^2\phi + \varpi\sin^2\psi = \varpi,$$

so

$$\mathsf{L}^{-1} = \frac{1}{\varpi}\begin{bmatrix}L^y{}_\phi & -L^x{}_\phi \\ -L^y{}_\varpi & L^x{}_\varpi\end{bmatrix} = \frac{1}{\varpi}\begin{bmatrix}\varpi\cos\phi & \varpi\sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi \\ -\sin(\phi)/\varpi & \cos(\phi)/\varpi\end{bmatrix}.$$

In other words,

$$L^\varpi{}_x = \cos\phi, \qquad L^\varpi{}_y = \sin\phi, \qquad L^\phi{}_x = -\frac{\sin\phi}{\varpi}, \qquad L^\phi{}_y = \frac{\cos\phi}{\varpi}. \tag{19}$$

4(b) Similarly, evaluate the components of the transformation matrix and its inverse linking the bases $\{\vec{e}_x, \vec{e}_y\}$ and $\{\vec{e}_{\hat\varpi}, \vec{e}_{\hat\phi}\}$.

Solution. We know that $\vec{e}_{\hat\phi} = (1/\varpi)\vec{e}_\phi$ and $\vec{e}_{\hat\varpi} = \vec{e}_{\hat\varpi}$ [1, p. 1163]. Applying Eq. (16) and the chain rule, we find

$$L^x{}_{\hat\varpi} = \frac{\partial x}{\partial\hat\varpi} = \frac{\partial\varpi}{\partial\hat\varpi}\frac{\partial x}{\partial\varpi} = \cos\phi, \qquad L^y{}_{\hat\varpi} = \frac{\partial y}{\partial\hat\varpi} = \frac{\partial\varpi}{\partial\hat\varpi}\frac{\partial y}{\partial\varpi} = \sin\phi,$$
$$L^x{}_{\hat\phi} = \frac{\partial x}{\partial\hat\phi} = \frac{\partial\phi}{\partial\hat\phi}\frac{\partial x}{\partial\varpi} = -\sin\phi, \qquad L^y{}_{\hat\phi} = \frac{\partial y}{\partial\hat\phi} = \frac{\partial\phi}{\partial\hat\phi}\frac{\partial y}{\partial\varpi} = \cos\phi.$$

For the inverse, we may apply Eq. (18) once more. Our determinant is

$$|\hat{\mathsf{L}}| = L^x{}_{\hat\varpi}L^y{}_{\hat\phi} - L^x{}_{\hat\phi}L^y{}_{\hat\varpi} = \cos^2\phi + \sin^2\phi = 1,$$

so

$$\hat{\mathsf{L}}^{-1} = \begin{bmatrix}L^y{}_{\hat\phi} & -L^x{}_{\hat\phi} \\ -L^y{}_{\hat\varpi} & L^x{}_{\hat\varpi}\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos(\phi)/\varpi\end{bmatrix}.$$

In other words,

$$L^{\hat\varpi}{}_x = \cos\phi, \qquad L^{\hat\varpi}{}_y = \sin\phi, \qquad L^{\hat\phi}{}_x = -\sin\phi, \qquad L^{\hat\phi}{}_y = \cos\phi. \tag{20}$$

4(c) Consider the vector $\vec{A} = \vec{e}_x + 2\vec{e}_y$. What are its components in the other two bases?

Solution. Applying Eqs. (15) and (19),

$$\vec{e}_x = \vec{e}_\varpi L^\varpi{}_x + \vec{e}_\phi L^\phi{}_x = \cos\phi\vec{e}_\varpi - \frac{\sin\phi}{\varpi}\vec{e}_\phi, \qquad \vec{e}_y = \vec{e}_\varpi L^\varpi{}_y + \vec{e}_\phi L^\phi{}_y = \sin\phi\vec{e}_\varpi + \frac{\cos\phi}{\varpi}\vec{e}_\phi,$$

so

$$\vec{A} = (\cos\phi\vec{e}_\varpi - \frac{\sin\phi}{\varpi}\vec{e}_\phi) + 2(\sin\phi\vec{e}_\varpi + \frac{\cos\phi}{\varpi}\vec{e}_\phi) = (\cos\phi + 2\sin\phi)\vec{e}_\varpi + \frac{1}{\varpi}(2\cos\phi - \sin\phi)\vec{e}_\phi.$$

Now applying Eqs. (15) and (20),

$$\vec{e}_x = \vec{e}_{\hat{\varpi}} L^{\hat{\varpi}}{}_x + \vec{e}_{\hat{\phi}} L^{\hat{\phi}}{}_x = \cos\phi \vec{e}_{\hat{\varpi}} - \sin\phi \vec{e}_{\hat{\phi}}, \qquad \vec{e}_y = \vec{e}_{\hat{\varpi}} L^{\hat{\varpi}}{}_y + \vec{e}_{\hat{\phi}} L^{\hat{\phi}}{}_y = \sin\phi \vec{e}_{\hat{\varpi}} + \cos\phi \vec{e}_{\hat{\phi}},$$

so

$$\vec{A} = (\cos\phi \vec{e}_{\hat{\varpi}} - \sin\phi \vec{e}_{\hat{\phi}}) + 2(\sin\phi \vec{e}_{\hat{\varpi}} + \cos\phi \vec{e}_{\hat{\phi}}) = (\cos\phi + 2\sin\phi)\vec{e}_{\hat{\varpi}} + (2\cos\phi - \sin\phi)\vec{e}_{\hat{\phi}}.$$

Problem 5. Gauss's theorem (MCP 24.11) In 3-dimensional Euclidean space Maxwell's equation $\nabla \cdot \mathbf{E} = \rho_e/\epsilon_0$ can be combined with Gauss's theorem to show that the electric flux through the surface $\partial\mathcal{V}$ of a sphere is equal to the charge in the sphere's interior $\mathcal{V}$ divided by ϵ_0:

$$\int_{\partial\mathcal{V}} \mathbf{E} \cdot d\mathbf{\Sigma} = \int_{\mathcal{V}} \frac{\rho_e}{\epsilon_0}\, dV\,. \tag{21}$$

Introduce spherical polar coordinates so the sphere's surface is at some radius $r = R$. Consider a surface element on the sphere's surface with vectorial legs $d\phi\, \partial/\partial\phi$ and $d\theta\, \partial/\partial\theta$. Evaluate the components $d\Sigma_j$ of the surface integration element $d\mathbf{\Sigma} = \boldsymbol{\epsilon}(\ldots, d\theta\, \partial/\partial\theta\,, d\phi\, \partial/\partial\phi)$. (Here $\boldsymbol{\epsilon}$ is the Levi-Civita tensor.) Similarly, evaluate dV in terms of vectorial legs in the sphere's interior. Then use these results for $d\Sigma_j$ and dV to convert Eq. (21) into an explicit form in terms of integrals over r, θ, and ϕ. The final answer should be obvious, but the above steps in deriving it are informative.

Solution. We begin by evaluating dV. From MCP (1.20), a parallelepiped whose edges are the n vectors $\mathbf{A}, \mathbf{B}, \ldots, \mathbf{F}$ has volume given by

$$\text{volume} = \boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \ldots, \mathbf{F}).$$

Then [1, p. 1175]

$$dV = \boldsymbol{\epsilon}\left(dr\,\frac{\partial}{\partial r}, d\theta\,\frac{\partial}{\partial\theta}, d\phi\,\frac{\partial}{\partial\phi}\right) = \boldsymbol{\epsilon}(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)\, dr\, d\theta\, d\phi = \epsilon_{r\theta\phi}\, dr\, d\theta\, d\phi\,, \tag{22}$$

where we have used MCP (1.9g), $\mathcal{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{ijk} A_i B_j C_k$. Now we apply MCP (24.44),

$$\epsilon_{\alpha\beta\ldots\nu} = \sqrt{|g|}[\alpha\beta\ldots\nu],$$

where $|g|$ is the determinant of the metric tensor. In spherical coordinates g is diagonal, and [1, p. 1175]

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2\sin^2\theta,$$

so

$$\sqrt{|g|} = \sqrt{r^4\sin^2\theta} = r^2\sin\theta.$$

Then Eq. (22) becomes

$$dV = r^2\sin\theta\, dr\, d\theta\, d\phi\,. \tag{23}$$

For $d\Sigma_j$,

$$d\mathbf{\Sigma} = \boldsymbol{\epsilon}\left(\_\_, d\theta\,\frac{\partial}{\partial\theta}, d\phi\,\frac{\partial}{\partial\phi}\right) = \boldsymbol{\epsilon}(\_\_, \vec{e}_\theta, \vec{e}_\phi)\, d\theta\, d\phi$$

so

$$d\Sigma_j = \epsilon_{j\theta\phi}\, d\theta\, d\phi = \sqrt{|g|}[j\theta\phi]\, d\theta\, d\phi = R^2\sin^2\theta[j\theta\phi]\, d\theta\, d\phi\,. \tag{24}$$

Here we are on the surface of the sphere, where $r = R$ and so

$$g_{rr} = 1, \qquad g_{\theta\theta} = R^2, \qquad g_{\phi\phi} = R^2 \sin^2\theta.$$

By the definition of the Levi-Civita tensor, the only nonzero $d\Sigma_j$ in Eq. (24) is $d\Sigma_r$. Thus

$$d\mathbf{\Sigma} = d\Sigma_r\,\hat{\mathbf{r}} = R^2 \sin^2\theta\, d\theta\, d\phi\,\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector in the r direction. Feeding this result and Eq. (23) into Eq. (21) yields

$$\int_0^{2\pi}\int_0^{\pi}(\mathbf{E}\cdot\hat{\mathbf{r}})R^2\sin^2\theta\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{\rho_e}{\epsilon_0}r^2\sin\theta\,dr\,d\theta\,d\phi\,,$$

as we expect. □

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