## Problem 1. Stress-energy tensor and energy-momentum conservation for a perfect fluid (MCP 2.26)

**1(a)** Derive the frame-independent expression (2.74b) for the perfect fluid stress-energy tensor from its rest-frame components (2.74a).

**1(b)** Explain why the projection of $\vec{\nabla} \cdot \mathsf{T} = 0$ along the fluid 4-velocity, $\vec{u} \cdot (\vec{\nabla} \cdot \mathsf{T}) = 0$, should represent energy conservation as viewed by the fluid itself. Show that this equation reduces to

$$\frac{d\rho}{d\tau} = -(\rho + P)\vec{\nabla} \cdot \vec{u}.$$

With the aid of Eq. (2.65), bring this into the form

$$\frac{d(\rho V)}{d\tau} = -P\frac{dV}{d\tau},$$

where $V$ is the 3-volume of some small fluid element as measured in the fluid's local rest frame. What are the physical interpretations of the left- and right-hand sides of this equation, and how is it related to the first law of thermodynamics?

**1(c)** Read the discussion in Ex. 2.10 about the tensor $\mathsf{P} = \mathsf{g} + \vec{u} \otimes \vec{u}$ that projects into the 3-space of the fluid's rest frame. Explain why $P_{\alpha\beta}T^{\alpha\beta}{}_{;\beta} = 0$ should represent the law of force balance (momentum conservation) as seen by the fluid. Show that this equation reduces to

$$(\rho + P)\vec{a} = -\mathsf{P} \cdot \boldsymbol{\nabla} P,$$

where $\vec{a} = d\vec{u}/d\tau$ is the fluid's 4-acceleration. This equation is a relativistic version of Newton's $\mathbf{F} = m\mathbf{a}$. Explain the physical meanings of the left- and right-hand sides. Infer that $\rho + P$ must be the fluid's inertial mass per unit volume. It is also the enthalpy per unit volume, including the contribution of rest mass; see Ex. 5.5 and Box 13.2.

## Problem 2. Inertial mass per unit volume (MCP 2.27)

Suppose that some medium has a rest frame (unprimed frame) in which its energy flux and momentum density vanish, $T^{0j} = T^{j0} = 0$. Suppose that the medium moves in the $x$ direction with speed very small compared to light, $v \ll 1$, as seen in a (primed) laboratory frame, and ignore factors of order $v^2$. The ratio of the medium's momentum density $G_{j'} = T^{j'0'}$ (as measured in the laboratory frame) to its velocity $v_i = \delta_{ix}$ is called its total *inertial mass per unit volume* and is denoted $\rho_{ji}^{\text{inert}}$:

$$T^{j'0'} = \rho_{ji}^{\text{inert}} v_i.$$

In other words, $\rho_{ji}^{\text{inert}}$ is the 3-dimensional tensor that gives the momentum density $G_{j'}$ when the medium's small velocity is put into its second slot.

**2(a)** Using a Lorentz transformation from the medium's (unprimed) rest frame to the (primed) laboratory frame, show that

$$\rho_{ji}^{\text{inert}} = T^{00}\delta_{ji} + T_{ji}.$$

**2(b)** Give a physical explanation of the contribution $T_{ji}$ to the momentum density.

**2(c)** Show that for a perfect fluid [Eq. (2.74b)] the inertial mass per unit volume is isotropic and has magnitude $\rho + P$, where $\rho$ is the mass-energy density, and $P$ is the pressure measured in the fluid's rest frame:

$$\rho_{ji}^{\text{inert}} = (\rho + P)\delta_{ji}.$$

See Ex. 2.26 for this inertial-mass role of $\rho + P$ in the law of force balance.

**Problem 3. Index-manipulation rules from duality (MCP 24.4)** For an arbitrary basis $\{\vec{e}_\alpha\}$ and its dual basis $\{\vec{e}^\mu\}$, use (i) the duality relation (24.8), (ii) the definition (24.9) of components of a tensor, and (iii) the relation $\vec{A} \cdot \vec{B} = \boldsymbol{g}(\vec{A}, \vec{B})$ between the metric and the inner product to deduce the following results.

**3(a)** The relations

$$\vec{e}^\mu = g^{\mu\alpha}\vec{e}_\alpha, \qquad \vec{e}_\alpha = g_{\alpha\mu}\vec{e}^\mu.$$

**3(b)** The fact that indices on the components of tensors can be raised and lowered using the components of the metric:

$$F^{\mu\nu} = g^{\mu\alpha}F^\alpha{}_\nu, \qquad p_\alpha = g_{\alpha\beta}p^\beta.$$

**3(c)** The fact that a tensor can be reconstructed from its components in the manner of Eq. (24.11).

**Problem 4. Transformation matrices for circular polar bases (MCP 24.5)** Consider the circular polar coordinate system $\{\varpi, \phi\}$ and its coordinate bases and orthonormal bases as shown in Fig. 24.3 and discussed in the associated text. These coordinates are related to Cartesian coordinates $\{x, y\}$ by the usual relations: $x = \varpi\cos\phi$, $y = \varpi\sin\phi$.

**4(a)** Evaluate the components ($L^x{}_\varpi$, etc.) of the transformation matrix that links the two coordinate bases $\{\vec{e}_x, \vec{e}_y\}$. Also evaluate the components ($L^\varpi{}_x$, etc.) of the inverse transformation matrix.

**4(b)** Similarly, evaluate the components of the transformation matrix and its inverse linking the bases $\{\vec{e}_x, \vec{e}_y\}$ and $\{\vec{e}_{\hat{\varpi}}, \vec{e}_{\hat{\phi}}\}$.

**4(c)** Consider the vector $\vec{A} = \vec{e}_x + 2\vec{e}_y$. What are its components in the other two bases?

**Problem 5. Gauss's theorem (MCP 24.11)** In 3-dimensional Euclidean space Maxwell's equation $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho_e/\epsilon_0$ can be combined with Gauss's theorem to show that the electric flux through the surface $\partial\mathcal{V}$ of a sphere is equal to the charge in the sphere's interior $\mathcal{V}$ divided by $\epsilon_0$:

$$\int_{\partial\mathcal{V}} \mathbf{E} \cdot d\boldsymbol{\Sigma} = \int_{\mathcal{V}} \frac{\rho_e}{\epsilon_0}\, dV\,.$$

Introduce spherical polar coordinates so the sphere's surface is at some radius $r = R$. Consider a surface element on the sphere's surface with vectorial legs $d\phi\, \partial/\partial\phi$ and $d\theta\, \partial/\partial\theta$. Evaluate the components $d\Sigma_j$ of the surface integration element $d\boldsymbol{\Sigma} = \boldsymbol{\epsilon}(\ldots, d\theta\, \partial/\partial\theta\,, d\phi\, \partial/\partial\phi)$. (Here $\boldsymbol{\epsilon}$ is the Levi-Civita tensor.) Similarly, evaluate $dV$ in terms of vectorial legs in the sphere's interior. Then use these results for $d\Sigma_j$ and $dV$ to convert Eq. (24.47) into an explicit form in terms of integrals over $r$, $\theta$, and $\phi$. The final answer should be obvious, but the above steps in deriving it are informative.

## References

[1] K. S. Thorne and R. D. Blandford, “Modern Classical Physics”. Princeton University Press, 2017.

[2] R. Resnick, “Introduction to Special Relativity”. John Wiley & Sons, Inc., 1968.

[3] Wikipedia contributors, “Acceleration (special relativity).” From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Acceleration_(special_relativity)`.

[4] T. Müller, A. King, and D. Adis, “A trip to the end of the universe and the twin “paradox””, *Am. J. Phys.* **76** (2008) 360–373, `doi:10.1119/1.2830528`.