Problem 1. Show that for an arbitrary spatially bound charge-current source, the electric dipole moment $\mathbf{p}$ satisfies

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{J}\, d^3x\,.$$

Solution. The electric dipole moment $\mathbf{p}$ is defined by Eq. (2.36),

$$\mathbf{p} = \int \mathbf{x}\,\rho(x)\, d^3x\,. \tag{1}$$

Differentiating both sides with respect to t, we find

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int \mathbf{x}\rho\, d^3x = \int \frac{d}{dt}(\mathbf{x}\rho)\, d^3x = \int \mathbf{x}\frac{\partial\rho}{\partial t}\, d^3x\,, \tag{2}$$

because $\mathbf{x}$ is simply the point at which we are evaluating the potential, and is therefore independent of time.

The charge-current conservation law is given by Eq. (5.8),

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0. \tag{3}$$

Multiplying by $\mathbf{x}$ on both sides and integrating over all space, we obtain

$$\int \mathbf{x}\frac{\partial\rho}{\partial t}\, d^3x + \int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = 0.$$

Applying (2), we have

$$\frac{d\mathbf{p}}{dt} = -\int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x\,. \tag{4}$$

It remains to be shown that the right side is equal to the integral of $\mathbf{J}$ over all space.

Vector identity (5) in Griffiths is

$$\boldsymbol{\nabla}\cdot(f\mathbf{a}) = f(\boldsymbol{\nabla}\cdot\mathbf{a}) + \mathbf{a}\cdot(\boldsymbol{\nabla}f).$$

Writing the right side of (4) in component notation and applying the identity gives us

$$-\int x_i(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = \int \mathbf{J}\cdot(\boldsymbol{\nabla}x_i)\, d^3x - \int \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x\,. \tag{5}$$

Gauss's theorem is given by Eq. (2.6),

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{v}\, d^3x = \int_S \mathbf{v}\cdot\hat{\mathbf{n}}\, dS\,,$$

Here, let $\mathcal{V}$ be a ball of radius R, with R large enough that the entire charge-current source is enclosed. Then S is a sphere of radius R, and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Applying Gauss's theorem to the second integral on the right side of (5), we have

$$\int \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x = \lim_{R\to\infty}\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x = \lim_{R\to\infty}\int_S x_i\mathbf{J}\cdot\hat{\mathbf{r}}\, dS = 0,$$

since $\mathbf{J}$ is bounded, meaning that $\mathbf{J}$ evaluated on S reaches zero well before x_i becomes very large.

Returning to (5), we now have

$$-\int x_i(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = \int \mathbf{J}\cdot(\boldsymbol{\nabla}x_i)\, d^3x = \sum_j \int J_j\partial_j x_i\, d^3x = \sum_j \int J_j\delta_{ij}\, d^3x = \int J_i\, d^3x\,,$$

where we have followed the proof in Eq. (4.24) of the course notes. Finally, (4) becomes

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{J}\, d^3x$$

as desired. □

Problem 2. A particle of charge q_1 moves with velocity v in a circular orbit of radius R about the origin in the xy plane, such that its φ coordinate varies as $\varphi = \omega t$, with $\omega = v/R$. Assume that $v \ll c$. Another particle of charge q_2 is at rest at point $\mathbf{x}$, where $|\mathbf{x}| \gg R$. To order $1/|\mathbf{x}|$, find the force $\mathbf{F}$ on the particle of charge q_2 at time t.

Solution. The Lorentz force equation, Eq. (1.25), is written

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{6}$$

where $\mathbf{v}$ is the velocity of the charge q on which the force is exerted, and $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields. For this problem, we are interested in the force acting on a stationary point charge q_2, so $\mathbf{v}_2 = 0$. Additionally, we do not have to consider the self-field contribution to $\mathbf{E}$, since static charge distributions do not experience any self force. Thus we need only find the electric field due to q_1, $\mathbf{E}_1$. The multipole expansion of the electric field in electrodynamics is given by Eq. (5.70),

$$\mathbf{E}(t, \mathbf{x}) = \frac{1}{c^2|\mathbf{x}|}\left[\left(\hat{\mathbf{x}} \cdot \frac{d^2\mathbf{p}}{dt^2}\right)\hat{\mathbf{x}} - \frac{d^2\mathbf{p}}{dt^2}\right]_{\text{ret}} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right), \tag{7}$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ is the unit vector in the direction of the point at which we are evaluating the field, and $\mathbf{p}$ is the dipole moment defined by (1). In addition, (7) relies upon the assumption that the velocity of q_1, v, satisfies $v \ll c$.

The position of q_1 at time t can be expressed as

$$\mathbf{x}_1(t) = R\cos(\omega t)\,\hat{\mathbf{x}} + R\sin(\omega t)\,\hat{\mathbf{y}},$$

so the charge density for q_1 everywhere is

$$\rho_1(t, \mathbf{x}) = q_1\,\delta\big(\mathbf{x} - \mathbf{x}_1(t)\big).$$

Then the dipole moment $\mathbf{p}_1(t, \mathbf{x})$ is

$$\mathbf{p}_1(t, \mathbf{x}) = \int \mathbf{x}\,\rho_1(t, \mathbf{x})\, d^3x = q_1 \int \mathbf{x}\,\delta\big(\mathbf{x} - \mathbf{x}_1(t)\big)\, d^3x = q_1\mathbf{x}_1(t) = q_1 R\cos(\omega t)\,\hat{\mathbf{x}} + q_1 R\sin(\omega t)\,\hat{\mathbf{y}},$$

and so its second time derivative is

$$\frac{d^2\mathbf{p}_1(t)}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{p}_1}{dt}\right) = \frac{d}{dt}\Big(-q_1 R\omega\sin(\omega t)\,\hat{\mathbf{x}} + q_1 R\omega\cos(\omega t)\,\hat{\mathbf{y}}\Big) = -q_1 R\omega^2\cos(\omega t)\,\hat{\mathbf{x}} - q_1 R\omega^2\sin(\omega t)\,\hat{\mathbf{y}}.$$

To this order, the retarded time t' is defined

$$t' = t - \frac{|\mathbf{x}|}{c}. \tag{8}$$

In (7), let $\mathbf{x} \to \mathbf{r}$. Then $\hat{\mathbf{x}} \to \hat{\mathbf{r}}$, which is the radial unit vector, and $|\mathbf{x}| \to r$. To first order in $1/|\mathbf{x}|$, we get

$$\begin{aligned}
\mathbf{E}_1(t,\mathbf{x}) &= \frac{1}{c^2 r}\left[-\frac{q_1 R\omega^2}{r}\left[\cos\big(\omega t'\big)(\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}) + \sin\big(\omega t'\big)(\hat{\mathbf{r}}\cdot\hat{\mathbf{y}})\right]\mathbf{r} + q_1 R\omega^2\big[\cos\big(\omega t'\big)\,\hat{\mathbf{x}} + \sin\big(\omega t'\big)\,\hat{\mathbf{y}}\big]\right]_{\text{ret}} \\
&= q_1\frac{R\omega^2}{c^2 r}\left[\cos\big(\omega t'\big)\big[\hat{\mathbf{x}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{x}})\hat{\mathbf{r}}\big] + \sin\big(\omega t'\big)\big[\hat{\mathbf{y}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{y}})\hat{\mathbf{r}}\big]\right]_{\text{ret}} \\
&= q_1\frac{R\omega^2}{c^2 r}\left\{\cos\Big(\omega t - \frac{\omega r}{c}\Big)\big[\hat{\mathbf{x}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{x}})\hat{\mathbf{r}}\big] + \sin\Big(\omega t - \frac{\omega r}{c}\Big)\big[\hat{\mathbf{y}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{y}})\hat{\mathbf{r}}\big]\right\}.
\end{aligned}$$

Note that

$$\hat{\mathbf{x}} = \sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}, \qquad \hat{\mathbf{y}} = \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}},$$

so

$$\begin{aligned}
\hat{\mathbf{x}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{x}})\hat{\mathbf{r}} &= \hat{\mathbf{x}} - \sin\theta\cos\varphi\,\hat{\mathbf{r}} = \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}, \\
\hat{\mathbf{y}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{y}})\hat{\mathbf{r}} &= \hat{\mathbf{y}} - \sin\theta\sin\varphi\,\hat{\mathbf{r}} = \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}},
\end{aligned}$$

and then

$$\mathbf{E}_1(t,\mathbf{x}) = q_1\frac{R\omega^2}{c^2 r}\left[\cos\Big(\omega t - \frac{\omega r}{c}\Big)(\cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}) + \sin\Big(\omega t - \frac{\omega r}{c}\Big)(\cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}})\right].$$

Applying (6) with $\mathbf{v}_2 = 0$, we have

$$\begin{aligned}
\mathbf{F}(t,\mathbf{x}) &= q_2\mathbf{E}_1 \\
&= q_1 q_2\frac{R\omega^2}{c^2 r}\left[\cos\Big(\omega t - \frac{\omega r}{c}\Big)(\cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}) + \sin\Big(\omega t - \frac{\omega r}{c}\Big)(\cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}})\right] \\
&= q_1 q_2\frac{R\omega^2}{c^2 r}\left\{\left[\cos\Big(\omega t - \frac{\omega r}{c}\Big)\cos\varphi + \sin\Big(\omega t - \frac{\omega r}{c}\Big)\sin\varphi\right]\cos\theta\,\hat{\boldsymbol{\theta}}\right. \\
&\qquad\qquad \left. + \left[\sin\Big(\omega t - \frac{\omega r}{c}\Big)\cos\varphi - \cos\Big(\omega t - \frac{\omega r}{c}\Big)\sin\varphi\right]\hat{\boldsymbol{\varphi}}\right\}.
\end{aligned}$$

Problem 3. An "antenna" is a segment of conducting wire in which a current flows (driven by an external power supply). Suppose an antenna of length L is placed on the z axis between $z = 0$ and $z = L$, and suppose that the current in the antenna is

$$\mathbf{J}(t,z) = I_0\sin\Big(\frac{\pi z}{L}\Big)\cos(\omega t)\,\delta(x)\,\delta(y)\,\hat{\mathbf{z}}. \tag{9}$$

3.a Find the charge density $\rho(t,z)$ in the antenna.

Solution. From the charge-current conservation law (3), we have

$$\rho(t,z) = -\int \boldsymbol{\nabla}\cdot\mathbf{J}\,dt\,.$$

For $\mathbf{J}$ given by (9),

$$\boldsymbol{\nabla}\cdot\mathbf{J} = \frac{\partial J_z}{\partial z} = \frac{\pi}{L}I_0\cos\Big(\frac{\pi z}{L}\Big)\cos(\omega t)\,\delta(x)\,\delta(y),$$

and so, discarding the constant of integration,

$$\rho(t,z) = -\frac{\pi}{L}I_0\cos\Big(\frac{\pi z}{L}\Big)\,\delta(x)\,\delta(y)\int\cos(\omega t)\,dt = -\frac{\pi}{L}\frac{I_0}{\omega}\cos\Big(\frac{\pi z}{L}\Big)\sin(\omega t)\,\delta(x)\,\delta(y)$$

for $0 \le z \le L$.

3.b Assume that $\omega L \ll c$. Find the electric and magnetic fields, $\mathbf{E}(t, z)$ and $\mathbf{B}(t, z)$, at large distances from the antenna (valid to order $1/|\mathbf{x}|$).

Solution. We will use (7) to find $\mathbf{E}(t, z)$. From Eq. (5.68), we know

$$\int \mathbf{J}(\mathbf{x})\, d^3x = \frac{d\mathbf{p}}{dt},$$

so from (9) we have

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= I_0 \cos(\omega t)\, \hat{\mathbf{z}} \int_0^L \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin\Big(\frac{\pi z}{L}\Big)\, \delta(x)\, \delta(y)\, dx\, dy\, dz = I_0 \cos(\omega t)\, \hat{\mathbf{z}} \int_0^L \sin\Big(\frac{\pi z}{L}\Big)\, dz \\ &= I_0 \cos(\omega t)\, \hat{\mathbf{z}} \left[-\frac{L}{\pi} \cos\Big(\frac{\pi z}{L}\Big)\right]_0^L = \frac{2L}{\pi} I_0 \cos(\omega t)\, \hat{\mathbf{z}}.\end{aligned}$$

Then

$$\frac{d^2\mathbf{p}}{dt^2} = -\frac{2L}{\pi} I_0 \omega \sin(\omega t)\, \hat{\mathbf{z}}.$$

Using the retarded time (8), to first order in $1/|\mathbf{x}|$ we obtain

$$\begin{aligned}\mathbf{E}(t, \mathbf{x}) &= \frac{1}{c^2 r} \left[\Big(-\hat{\mathbf{r}} \cdot \frac{2L}{\pi} I_0 \omega \sin(\omega t)\, \hat{\mathbf{z}}\Big)\, \hat{\mathbf{r}} + \frac{2L}{\pi} I_0 \omega \sin(\omega t)\, \hat{\mathbf{z}}\right]_{\text{ret}} = \frac{2L}{\pi c^3} \frac{I_0 \omega}{r} \Big[\sin(\omega t)\big[\hat{\mathbf{z}} - (\mathbf{r} \cdot \hat{\mathbf{z}})\hat{\mathbf{r}}\big]\Big]_{\text{ret}} \\ &= \frac{2L}{\pi c^2} \frac{I_0 \omega}{r} \sin\Big(\omega t - \frac{\omega r}{c}\Big) \big[\hat{\mathbf{z}} - (\mathbf{r} \cdot \hat{\mathbf{z}})\hat{\mathbf{r}}\big].\end{aligned}$$

Note that $\hat{\mathbf{z}} = \cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}$, so $\hat{\mathbf{z}} - (\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})\hat{\mathbf{r}} = \hat{\mathbf{z}} - \cos\theta\, \hat{\mathbf{r}} = -\sin\theta\, \hat{\boldsymbol{\theta}}$, and then

$$\mathbf{E}(t, \mathbf{x}) = -\frac{2L}{\pi c^2} \frac{I_0 \omega}{r} \sin\Big(\omega t - \frac{\omega r}{c}\Big) \sin\theta\, \hat{\boldsymbol{\theta}}.$$

The multipole expansion of the magnetic field in electrodynamics is given by Eq. (5.73),

$$\mathbf{B}(t, \mathbf{x}) = -\frac{1}{c^2|\mathbf{x}|} \hat{\mathbf{x}} \times \left[\frac{d^2\mathbf{p}}{dt^2}\right]_{\text{ret}} + \mathcal{O}\Big(\frac{1}{|\mathbf{x}|^2}\Big). \tag{10}$$

To first order in $1/|\mathbf{x}|$, we obtain

$$\mathbf{B}(t, \mathbf{x}) = \frac{1}{c^2 r} \hat{\mathbf{r}} \times \left[\frac{2L}{\pi} I_0 \omega \sin(\omega t)\, \hat{\mathbf{z}}\right]_{\text{ret}} = \frac{2L}{\pi c^2} \frac{I_0 \omega}{r} \hat{\mathbf{r}} \times \Big[\sin(\omega t)\, \hat{\mathbf{z}}\Big]_{\text{ret}} = \frac{2L}{\pi c^2} \frac{I_0 \omega}{r} \sin\Big(\omega t - \frac{\omega r}{c}\Big) (\hat{\mathbf{r}} \times \hat{\mathbf{z}}).$$

Again using spherical coordinates, $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\sin\theta\, \hat{\boldsymbol{\varphi}}$, and so

$$\mathbf{B}(t, \mathbf{x}) = -\frac{2L}{\pi c^2} \frac{I_0 \omega}{r} \sin\Big(\omega t - \frac{\omega r}{c}\Big) \sin\theta\, \hat{\boldsymbol{\varphi}}.$$

In addition to the course lecture notes, I consulted Griffiths's *Introduction to Electrodynamics* while writing up these solutions.