Problem 1. Connection coefficients for spherical polar coordinates (MCP 24.9)

1(a) Consider spherical polar coordinates in 3-dimensional space, and verify that the nonzero connection coefficients, assuming an orthonormal basis, are given by Eq. (11.71).

Solution. We follow the procedure on pp. 1171–1172 of MCP for computing the connection coefficients. We first evaluate the commutation coefficients ${c_{\alpha\beta}}^{\rho}$ using MCP (24.38a),

$$ {c_{\alpha\beta}}^{\rho} = \vec{e}^{\rho} \cdot [\vec{e}_\alpha, \vec{e}_\beta], \tag{1} $$

We lower the last index using (24.38b),

$$ c_{\alpha\beta\gamma} = {c_{\alpha\beta}}^{\rho} g_{\rho\gamma}. $$

Then we use (24.38c) to compute

$$ \Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}), \tag{2} $$

and raise the first index using (24.38d),

$$ {\Gamma^{\mu}}_{\beta\gamma} = g^{\mu\alpha}\Gamma_{\alpha\beta\gamma}. \tag{3} $$

From (24.40), the commutator is given by

$$ [\vec{A}, \vec{B}] = \nabla_{\vec{A}}\vec{B} - \nabla_{\vec{B}}\vec{A}. \tag{4} $$

We also note that $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$ [1, p. 1161].

For an orthonormal basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$, g is the Kronecker delta [1, p. 614]. In spherical coordinates, the gradient is

$$ \boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\phi}, $$

and its components are [2]

$$ \begin{aligned} &\nabla_r\hat{\mathbf{r}} = \mathbf{0}, && \nabla_\theta\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\theta}}, && \nabla_\phi\hat{\mathbf{r}} = \frac{1}{r}\hat{\boldsymbol{\phi}}, \\ &\nabla_r\hat{\boldsymbol{\theta}} = \mathbf{0}, && \nabla_\theta\hat{\boldsymbol{\theta}} = -\frac{1}{r}\hat{\mathbf{r}}, && \nabla_\phi\hat{\boldsymbol{\theta}} = \frac{1}{r\tan\theta}\hat{\boldsymbol{\phi}}, \\ &\nabla_r\hat{\boldsymbol{\phi}} = \mathbf{0}, && \nabla_\theta\hat{\boldsymbol{\phi}} = \mathbf{0}, && \nabla_\phi\hat{\boldsymbol{\phi}} = -\frac{1}{r\tan\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r}\hat{\mathbf{r}}. \end{aligned} $$

Applying Eq. (4) and the above, we find

$$ \begin{aligned} &[\hat{\mathbf{r}}, \hat{\mathbf{r}}] = \nabla_r\hat{\mathbf{r}} - \nabla_r\hat{\mathbf{r}} = \mathbf{0}, && [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = \nabla_r\hat{\boldsymbol{\theta}} - \nabla_\theta\hat{\mathbf{r}} = -\frac{1}{r}\hat{\boldsymbol{\theta}}, && [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = \nabla_r\hat{\boldsymbol{\phi}} - \nabla_\phi\hat{\mathbf{r}} = -\frac{1}{r}\hat{\boldsymbol{\phi}}, \\ &[\hat{\boldsymbol{\theta}}, \hat{\mathbf{r}}] = -[\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = \frac{1}{r}\hat{\boldsymbol{\theta}}, && [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] = \nabla_\theta\hat{\boldsymbol{\theta}} - \nabla_\theta\hat{\boldsymbol{\theta}} = \mathbf{0}, && [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = \nabla_\theta\hat{\boldsymbol{\phi}} - \nabla_\phi\hat{\boldsymbol{\theta}} = -\frac{1}{r\tan\theta}\hat{\boldsymbol{\phi}}, \\ &[\hat{\boldsymbol{\phi}}, \hat{\mathbf{r}}] = -[\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] = \frac{1}{r}\hat{\boldsymbol{\phi}}, && [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}] = -[\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] = \frac{1}{r\tan\theta}\hat{\boldsymbol{\phi}}, && [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] = \nabla_\phi\hat{\boldsymbol{\phi}} - \nabla_\phi\hat{\boldsymbol{\phi}} = \mathbf{0}. \end{aligned} $$

Since g is the Kronecker delta, we can immediately write from Eq. (1)

$$ \begin{aligned} &c_{rrr} = [\hat{\mathbf{r}}, \hat{\mathbf{r}}] \cdot \hat{\mathbf{r}} = 0, && c_{r\theta r} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\mathbf{r}} = 0, && c_{r\phi r} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\mathbf{r}} = 0, \\ &c_{\theta rr} = -c_{r\theta r} = 0, && c_{\theta\theta r} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\mathbf{r}} = 0, && c_{\theta\phi r} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\mathbf{r}} = 0, \\ &c_{\phi rr} = -c_{r\phi r} = 0, && c_{\phi\theta r} = -c_{\theta\phi r} = 0, && c_{\phi\phi r} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\mathbf{r}} = 0, \end{aligned} $$

$$c_{rr\theta} = [\hat{\mathbf{r}}, \hat{\mathbf{r}}] \cdot \hat{\boldsymbol{\theta}} = 0, \qquad c_{r\theta\theta} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\theta}} = -\frac{1}{r}, \qquad c_{r\phi\theta} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\theta}} = 0,$$
$$c_{\theta r\theta} = -c_{r\theta\theta} = \frac{1}{r}, \qquad c_{\theta\theta\theta} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\theta}} = 0, \qquad c_{\theta\phi\theta} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\theta}} = 0,$$
$$c_{\phi r\theta} = -c_{r\phi\theta} = 0, \qquad c_{\phi\theta\theta} = -c_{\theta\phi\theta} = 0, \qquad c_{\phi\phi\theta} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\theta}} = 0,$$

$$c_{rr\phi} = [\hat{\mathbf{r}}, \hat{\mathbf{r}}] \cdot \hat{\boldsymbol{\phi}} = 0, \qquad c_{r\theta\phi} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\phi}} = 0, \qquad c_{r\phi\phi} = [\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\phi}} = -\frac{1}{r},$$
$$c_{\theta r\phi} = -c_{r\theta\phi} = 0, \qquad c_{\theta\theta\phi} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] \cdot \hat{\boldsymbol{\phi}} = 0, \qquad c_{\theta\phi\phi} = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\phi}} = -\frac{1}{r\tan\theta},$$
$$c_{\phi r\phi} = -c_{r\phi\phi} = \frac{1}{r}, \qquad c_{\phi\theta\phi} = -c_{\theta\phi\phi} = \frac{1}{r\tan\theta}, \qquad c_{\phi\phi\phi} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\phi}}] \cdot \hat{\boldsymbol{\phi}} = 0.$$

From Eq. (2) we again use the fact that g is the identity to write

$$\Gamma_{rrr} = \frac{c_{rrr} + c_{rrr} - c_{rrr}}{2} = 0, \qquad \Gamma_{rr\theta} = \frac{c_{rr\theta} + c_{r\theta r} - c_{r\theta r}}{2} = 0, \qquad \Gamma_{rr\phi} = \frac{c_{rr\phi} + c_{r\phi r} - c_{r\phi r}}{2} = 0,$$
$$\Gamma_{r\theta r} = \frac{c_{r\theta r} + c_{rr\theta} - c_{\theta rr}}{2} = 0, \qquad \Gamma_{r\theta\theta} = \frac{c_{r\theta\theta} + c_{r\theta\theta} - c_{\theta\theta r}}{2} = -\frac{1}{r}, \qquad \Gamma_{r\theta\phi} = \frac{c_{r\theta\phi} + c_{r\phi\theta} - c_{\theta\phi r}}{2} = 0,$$
$$\Gamma_{r\phi r} = \frac{c_{r\phi r} + c_{rr\phi} - c_{\phi rr}}{2} = 0, \qquad \Gamma_{r\phi\theta} = \frac{c_{r\phi\theta} + c_{r\theta\phi} - c_{\phi\theta r}}{2} = 0, \qquad \Gamma_{r\phi\phi} = \frac{c_{r\phi\phi} + c_{r\phi\phi} - c_{\phi\phi r}}{2} = -\frac{1}{r},$$

$$\Gamma_{\theta rr} = \frac{c_{\theta rr} + c_{\theta rr} - c_{rr\theta}}{2} = 0, \qquad \Gamma_{\theta r\theta} = \frac{c_{\theta r\theta} + c_{\theta\theta r} - c_{r\theta\theta}}{2} = \frac{1}{r}, \qquad \Gamma_{\theta r\phi} = \frac{c_{\theta r\phi} + c_{\theta\phi r} - c_{r\phi\theta}}{2} = 0,$$
$$\Gamma_{\theta\theta r} = \frac{c_{\theta\theta r} + c_{\theta r\theta} - c_{\theta r\theta}}{2} = 0, \qquad \Gamma_{\theta\theta\theta} = \frac{c_{\theta\theta\theta} + c_{\theta\theta\theta} - c_{\theta\theta\theta}}{2} = 0, \qquad \Gamma_{\theta\theta\phi} = \frac{c_{\theta\theta\phi} + c_{\theta\phi\theta} - c_{\theta\phi\theta}}{2} = 0,$$
$$\Gamma_{\theta\phi r} = \frac{c_{\theta\phi r} + c_{\theta r\phi} - c_{\phi r\theta}}{2} = 0, \qquad \Gamma_{\theta\phi\theta} = \frac{c_{\theta\phi\theta} + c_{\theta\theta\phi} - c_{\phi\theta\theta}}{2} = 0, \qquad \Gamma_{\theta\phi\phi} = \frac{c_{\theta\phi\phi} + c_{\theta\phi\phi} - c_{\phi\phi\theta}}{2} = -\frac{1}{r\tan\theta},$$

$$\Gamma_{\phi rr} = \frac{c_{\phi rr} + c_{\phi rr} - c_{rr\phi}}{2} = 0, \qquad \Gamma_{\phi r\theta} = \frac{c_{\phi r\theta} + c_{\phi\theta r} - c_{r\theta\phi}}{2} = 0, \qquad \Gamma_{\phi r\phi} = \frac{c_{\phi r\phi} + c_{\phi\phi r} - c_{r\phi\phi}}{2} = \frac{1}{r},$$
$$\Gamma_{\phi\theta r} = \frac{c_{\phi\theta r} + c_{\phi r\theta} - c_{\theta r\phi}}{2} = 0, \qquad \Gamma_{\phi\theta\theta} = \frac{c_{\phi\theta\theta} + c_{\phi\theta\theta} - c_{\theta\theta\phi}}{2} = 0, \qquad \Gamma_{\phi\theta\phi} = \frac{c_{\phi\theta\phi} + c_{\phi\phi\theta} - c_{\theta\phi\phi}}{2} = \frac{1}{r\tan\theta},$$
$$\Gamma_{\phi\phi r} = \frac{c_{\phi\phi r} + c_{\phi r\phi} - c_{\phi r\phi}}{2} = 0, \qquad \Gamma_{\phi\phi\theta} = \frac{c_{\phi\phi\theta} + c_{\phi\theta\phi} - c_{\phi\theta\phi}}{2} = 0, \qquad \Gamma_{\phi\phi\phi} = \frac{c_{\phi\phi\phi} + c_{\phi\phi\phi} - c_{\phi\phi\phi}}{2} = 0.$$

In summary, we have the nonzero connection coefficients

$$\Gamma_{r\theta\theta} = \Gamma_{r\phi\phi} = -\frac{1}{r}, \qquad \Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = \frac{1}{r}, \qquad \Gamma_{\theta\phi\phi} = -\frac{1}{r\tan\theta}, \qquad \Gamma_{\phi\theta\phi} = \frac{1}{r\tan\theta}.$$

This is in agreement with MCP (11.71), which gives the nonzero connection coefficients as

$$\Gamma_{\theta r\theta} = \Gamma_{\phi r\phi} = -\Gamma_{r\theta\theta} = -\Gamma_{r\phi\phi} = \frac{1}{r}, \qquad \Gamma_{\phi\theta\phi} = -\Gamma_{\theta\phi\phi} = \frac{\cot\theta}{r}. \quad \square$$

1(b) Repeat the exercise in 1(a) assuming a coordinate basis with

$$\mathbf{e}_r \equiv \frac{\partial}{\partial r}, \qquad \mathbf{e}_\theta \equiv \frac{\partial}{\partial \theta}, \qquad \mathbf{e}_\phi \equiv \frac{\partial}{\partial \phi}.$$

Solution. In a coordinate basis, it is always true that $[\vec{e}_\alpha, \vec{e}_\beta] = 0$ [1, p. 1168]. In this case, the nonzero elements of g are [2]

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2 \sin^2\theta, \tag{5}$$

which implies

$$g^{rr} = 1, \qquad g^{\theta\theta} = \frac{1}{r^2}, \qquad g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta},$$

since the matrix of contravariant components of the metric is inverse to that of the covariant components [1, p. 1162]. The only nonzero derivatives are

$$g_{\theta\theta,r} = 2r, \qquad g_{\phi\phi,r} = 2r\sin^2\theta, \qquad g_{\phi\phi,\theta} = 2r^2\sin\theta\cos\theta.$$

From Eq. (2), the $\Gamma_{\alpha\beta\gamma}$ are

$$\begin{aligned}
\Gamma_{rrr} &= \frac{g_{rr,r} + g_{rr,r} - g_{rr,r}}{2} = 0, & \Gamma_{rr\theta} &= \frac{g_{rr,\theta} + g_{r\theta,r} - g_{r\theta,r}}{2} = 0,\\
\Gamma_{rr\phi} &= \frac{g_{rr,\phi} + g_{r\phi,r} - g_{r\phi,r}}{2} = 0, & \Gamma_{r\theta r} &= \frac{g_{r\theta,r} + g_{rr,\theta} - g_{rr,\theta}}{2} = 0,\\
\Gamma_{r\theta\theta} &= \frac{g_{r\theta,\theta} + g_{r\theta,\theta} - g_{\theta\theta,r}}{2} = -r, & \Gamma_{r\theta\phi} &= \frac{g_{r\theta,\phi} + g_{r\phi,\theta} - g_{\theta\phi,r}}{2} = 0,\\
\Gamma_{r\phi r} &= \frac{g_{r\phi,r} + g_{rr,\phi} - g_{\phi r,r}}{2} = 0, & \Gamma_{r\phi\theta} &= \frac{g_{r\phi,\theta} + g_{r\theta,\phi} - g_{\phi\theta,r}}{2} = 0,\\
\Gamma_{r\phi\phi} &= \frac{g_{r\phi,\phi} + g_{r\phi,\phi} - g_{\phi\phi,r}}{2} = -r\sin^2\theta, & &
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\theta rr} &= \frac{g_{\theta r,r} + g_{\theta r,r} - g_{rr,\theta}}{2} = 0, & \Gamma_{\theta r\theta} &= \frac{g_{\theta r,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}}{2} = r,\\
\Gamma_{\theta r\phi} &= \frac{g_{\theta r,\phi} + g_{\theta\phi,r} - g_{r\phi,\theta}}{2} = 0, & \Gamma_{\theta\theta r} &= \frac{g_{\theta\theta,r} + g_{\theta r,\theta} - g_{\theta r,\theta}}{2} = r,\\
\Gamma_{\theta\theta\theta} &= \frac{g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}}{2} = 0, & \Gamma_{\theta\theta\phi} &= \frac{g_{\theta\theta,\phi} + g_{\theta\phi,\theta} - g_{\theta\phi,\theta}}{2} = 0,\\
\Gamma_{\theta\phi r} &= \frac{g_{\theta\phi,r} + g_{\theta r,\phi} - g_{\phi r,\theta}}{2} = 0, & \Gamma_{\theta\phi\theta} &= \frac{g_{\theta\phi,\theta} + g_{\theta\theta,\phi} - g_{\phi\theta,\theta}}{2} = 0,\\
\Gamma_{\theta\phi\phi} &= \frac{g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta}}{2} = -r^2\sin\theta\cos\theta, & &
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\phi rr} &= \frac{g_{\phi r,r} + g_{\phi r,r} - g_{rr,\phi}}{2} = 0, & \Gamma_{\phi r\theta} &= \frac{g_{\phi r,\theta} + g_{\phi\theta,r} - g_{r\theta,\phi}}{2} = 0,\\
\Gamma_{\phi r\phi} &= \frac{g_{\phi r,\phi} + g_{\phi\phi,r} - g_{r\phi,\phi}}{2} = r\sin^2\theta, & \Gamma_{\phi\theta r} &= \frac{g_{\phi\theta,r} + g_{\phi r,\theta} - g_{\theta r,\phi}}{2} = 0,\\
\Gamma_{\phi\theta\theta} &= \frac{g_{\phi\theta,\theta} + g_{\phi\theta,\theta} - g_{\theta\theta,\phi}}{2} = 0, & \Gamma_{\phi\theta\phi} &= \frac{g_{\phi\theta,\phi} + g_{\phi\phi,\theta} - g_{\theta\phi,\phi}}{2} = r^2\sin\theta\cos\theta,\\
\Gamma_{\phi\phi r} &= \frac{g_{\phi\phi,r} + g_{\phi r,\phi} - g_{\phi r,\phi}}{2} = r\sin^2\theta, & \Gamma_{\phi\phi\theta} &= \frac{g_{\phi\phi,\theta} + g_{\phi\theta,\phi} - g_{\phi\theta,\phi}}{2} = r^2\sin\theta\cos\theta,\\
\Gamma_{\phi\phi\phi} &= \frac{g_{\phi\phi,\phi} + g_{\phi\phi,\phi} - g_{\phi\phi,\phi}}{2} = 0. & &
\end{aligned}$$

Now applying Eq. (3),

$$\begin{aligned}
\Gamma^r{}_{rr} &= g^{rr}\Gamma_{rrr} = 0, & \Gamma^r{}_{r\theta} &= g^{rr}\Gamma_{rr\theta} = 0, & \Gamma^r{}_{r\phi} &= g^{rr}\Gamma_{rr\phi} = 0,\\
\Gamma^r{}_{\theta r} &= g^{rr}\Gamma_{r\theta r} = 0, & \Gamma^r{}_{\theta\theta} &= g^{rr}\Gamma_{r\theta\theta} = -r, & \Gamma^r{}_{\theta\phi} &= g^{rr}\Gamma_{r\theta\phi} = 0,\\
\Gamma^r{}_{\phi r} &= g^{rr}\Gamma_{r\phi r} = 0, & \Gamma^r{}_{\phi\theta} &= g^{rr}\Gamma_{r\phi r} = 0, & \Gamma^r{}_{\phi\phi} &= g^{rr}\Gamma_{r\phi\phi} = -r\sin^2\theta,
\end{aligned}$$

$$\begin{aligned}
\Gamma^\theta{}_{rr} &= g^{\theta\theta}\Gamma_{\theta rr} = 0, & \Gamma^\theta{}_{r\theta} &= g^{\theta\theta}\Gamma_{\theta r\theta} = \frac{1}{r}, & \Gamma^\theta{}_{r\phi} &= g^{\theta\theta}\Gamma_{\theta r\phi} = 0,\\
\Gamma^\theta{}_{\theta r} &= g^{\theta\theta}\Gamma_{\theta\theta r} = \frac{1}{r}, & \Gamma^\theta{}_{\theta\theta} &= g^{\theta\theta}\Gamma_{\theta\theta\theta} = 0, & \Gamma^\theta{}_{\theta\phi} &= g^{\theta\theta}\Gamma_{\theta\theta\phi} = 0,\\
\Gamma^\theta{}_{\phi r} &= g^{\theta\theta}\Gamma_{\theta\phi r} = 0, & \Gamma^\theta{}_{\phi\theta} &= g^{\theta\theta}\Gamma_{\theta\phi\theta} = 0, & \Gamma^\theta{}_{\phi\phi} &= g^{\theta\theta}\Gamma_{\theta\phi\phi} = -\sin\theta\cos\theta,
\end{aligned}$$

$$\Gamma^\phi{}_{rr} = g^{\phi\phi}\Gamma_{\phi rr} = 0, \qquad \Gamma^\phi{}_{r\theta} = g^{\phi\phi}\Gamma_{\phi r\theta} = 0, \qquad \Gamma^\phi{}_{r\phi} = g^{\phi\phi}\Gamma_{\phi r\phi} = \frac{1}{r},$$
$$\Gamma^\phi{}_{\theta r} = g^{\phi\phi}\Gamma_{\phi\theta r} = 0, \qquad \Gamma^\phi{}_{\theta\theta} = g^{\phi\phi}\Gamma_{\phi\theta\theta} = 0, \qquad \Gamma^\phi{}_{\theta\phi} = g^{\phi\phi}\Gamma_{\phi\theta\phi} = \frac{1}{\tan\theta},$$
$$\Gamma^\phi{}_{\phi r} = g^{\phi\phi}\Gamma_{\phi\phi r} = \frac{1}{r}, \qquad \Gamma^\phi{}_{\phi\theta} = g^{\phi\phi}\Gamma_{\phi\phi\theta} = \frac{1}{\tan\theta}, \qquad \Gamma^\phi{}_{\phi\phi} = g^{\phi\phi}\Gamma_{\phi\phi\phi} = 0.$$

Thus we have found that the nonzero connection coeffients are

$$\Gamma^r{}_{\theta\theta} = -r, \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta, \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \frac{1}{\tan\theta},$$
$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}.$$

1(c) Repeat both computations in 1(a) and 1(b) using symbolic manipulation software on a computer.

Solution. For 1(b), we use the Mathematica notebook from Ref. [3]:

```
In[∘]:= n = 3
Out[∘]= 3

In[∘]:= coord = {r, θ, ϕ}
Out[∘]= {r, θ, ϕ}

In[∘]:= metric = {{1, 0, 0}, {0, r^2, 0}, {0, 0, r^2 Sin[θ]^2}}
Out[∘]= {{1, 0, 0}, {0, r^2, 0}, {0, 0, r^2 Sin[θ]^2}}

In[∘]:= inversemetric = Simplify[Inverse[metric]]
Out[∘]= {{1, 0, 0}, {0, 1/r^2, 0}, {0, 0, Csc[θ]^2/r^2}}

In[∘]:= affine := affine = Simplify[Table[(1/2)*Sum[(inversemetric[[i, s]])*
            (D[metric[[s, j]], coord[[k]]] +
              D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]), {s, 1, n}],
        {i, 1, n}, {j, 1, n}, {k, 1, n}]]

In[∘]:= listaffine :=
      Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[i, j, k]], affine[[i, j, k]]}] ,
       {i, 1, n}, {j, 1, n}, {k, 1, j}]

In[∘]:= TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
      TableSpacing → {2, 2}]

Out[∘]//TableForm=
      Γ[1, 2, 2]   -r
      Γ[1, 3, 3]   -r Sin[θ]^2
      Γ[2, 2, 1]   1/r
      Γ[2, 3, 3]   -Cos[θ] Sin[θ]
      Γ[3, 3, 1]   1/r
      Γ[3, 3, 2]   Cot[θ]
```

Here $r \to 1$, $\theta \to 2$, and $\phi \to 3$. Taking into account that in a coordinate basis $\Gamma_{\alpha\beta\gamma}$ is symmetric in its last two indices [1, p. 1172], these match our result from 1(b).

For 1(a), I wrote a Mathematica notebook of my own, taking some inspiration from Ref. [3] (and I later loaned this notebook to Morgan Lynn):

```
In[∘]:= vr = (Sin[θ] Cos[ϕ]
              Sin[θ] Sin[ϕ]
              Cos[θ]); vθ = (Cos[θ] Cos[ϕ]
                             Cos[θ] Sin[ϕ]
                             -Sin[θ]); vϕ = (-Sin[ϕ]
                                             Cos[ϕ]
                                             0);

In[∘]:= coords = {r, θ, ϕ}; vecs = {vr, vθ, vϕ}; grad = {1, 1/r, 1/(r Sin[θ])};

In[∘]:= comm[i_, j_] := grad[[i]] * D[vecs[[j]], coords[[i]]] - grad[[j]] * D[vecs[[i]], coords[[j]]]

In[∘]:= commcoeff[i_, j_, k_] := Simplify[Transpose[comm[i, j]].vecs[[k]]]

In[∘]:= conncoeff[i_, j_, k_] :=
          First[First[Simplify[1/2 (commcoeff[i, j, k] + commcoeff[i, k, j] - commcoeff[j, k, i])]]]

In[∘]:= table :=
          Table[If[UnsameQ[conncoeff[i, j, k], 0], {ToString[Γ[i, j, k]], conncoeff[i, j, k]}],
           {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]

In[∘]:= TableForm[Partition[DeleteCases[Flatten[table], Null], 2], TableSpacing → {2, 2}]
```

Out[∘]//TableForm=

Γ[1, 2, 2]	$-\frac{1}{r}$
Γ[1, 3, 3]	$-\frac{1}{r}$
Γ[2, 1, 2]	$\frac{1}{r}$
Γ[2, 3, 3]	$-\frac{\mathrm{Cot}[\theta]}{r}$
Γ[3, 1, 3]	$\frac{1}{r}$
Γ[3, 2, 3]	$\frac{\mathrm{Cot}[\theta]}{r}$

For the result, we again have $r \to 1$, $\theta \to 2$, and $\phi \to 3$. These match our result from 1(a).

Problem 2. Let V be a vector field. Prove the covariant divergence formula valid in a coordinate basis

$$\nabla_\alpha V^\alpha = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} V^\alpha),$$

where g is the determinant of the metric.

Solution. From Lecture 7, the covariant derivative can be written

$$\nabla_\beta V^\beta = \partial_\beta V^\beta + \Gamma^\gamma{}_{\beta\gamma} V^\beta. \tag{6}$$

Applying Eqs. (3) and (2),

$$\Gamma^\gamma{}_{\beta\gamma} = g^{\gamma\alpha} \Gamma_{\alpha\beta\gamma} = \frac{1}{2} g^{\gamma\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + c_{\alpha\beta\gamma} + c_{\alpha\gamma\beta} - c_{\beta\gamma\alpha}) = \frac{1}{2} g^{\gamma\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}),$$

where we have used the fact that $[\vec{e}_\alpha, \vec{e}_\beta] = 0$ in a coordinate basis [1, p. 1168], rendering all of the commutation coefficients zero. Then

$$\begin{aligned}
\Gamma^\gamma{}_{\beta\gamma} &= \frac{1}{2} g^{\gamma\alpha}(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) \\
&= \frac{1}{2}(g^{\gamma\alpha}\partial_\gamma g_{\alpha\beta} + g^{\gamma\alpha}\partial_\beta g_{\alpha\gamma} - g^{\gamma\alpha}\partial_\gamma g_{\alpha\beta}) \\
&= \frac{1}{2} g^{\gamma\alpha} \partial_\beta g_{\alpha\gamma}, \qquad (7)
\end{aligned}$$

where we have used the symmetry of the metric. Since $\text{tr}(AB) = A_{ij}B_{ji}$ [4], we can write Eq. (7) as

$$\Gamma^{\gamma}{}_{\beta\gamma} = \frac{1}{2}\,\text{tr}(\boldsymbol{g}\partial_\beta \boldsymbol{g}). \tag{8}$$

We now apply the identity [5, p. 106]

$$\text{tr}\big[M^{-1}(x)\partial_\lambda M(x)\big] = \partial_\lambda[\ln\det M(x)].$$

Using also the fact that $\boldsymbol{g}^{\mu\beta}\boldsymbol{g}_{\beta\nu} = \delta^{\mu}{}_{\nu}$ by MCP (24.10), Eq. (8) becomes [5, p. 107]

$$\Gamma^{\gamma}{}_{\beta\gamma} = \frac{1}{2}\,\text{tr}(\boldsymbol{g}^{\gamma\alpha}\partial_\beta \boldsymbol{g}_{\alpha\gamma}) = \frac{1}{2}\partial_\beta(\ln\det \boldsymbol{g}_{\alpha\gamma}) = \frac{1}{2}\partial_\beta(\ln g) = \partial_\beta(\ln\sqrt{|g|}) = \frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}).$$

Feeding this into Eq. (8) and integrating by parts, we have

$$\nabla_\beta V^\beta = \partial_\beta V^\beta + V^\beta \frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}) = \frac{1}{\sqrt{|g|}}\partial_\beta(\sqrt{|g|}V^\beta)$$

as we wanted to show. □

Problem 3. In this problem you will explore the geometry of a sphere S^2 of radius R.

3(a) A vector $\vec{V} = V^\theta \vec{e}_\theta + V^\phi \vec{e}_\phi$ is defined at a point (θ, ϕ) on the sphere. It is then parallel transported around the circle of constant θ with $\phi \to \phi + 2\pi$. What are its resulting components? What is its length?

Solution. From Lecture 7, the parallel transport equation is

$$\frac{dV^\beta}{d\tau} + \Gamma^{\beta}{}_{\mu\alpha} V^\mu(\tau)\frac{\partial x^\alpha}{\partial \tau} = 0. \tag{9}$$

Let $\vec{u} = d\vec{x}/d\tau$ be the vector tangent to the curve along which we are transporting $\vec{V}$.

For our problem, let $\vec{x} = (\theta_0, \phi)$, with θ_0 constant. Then $\tau \to \phi$, and so $\vec{u} = (0, 1)$. Then Eq. (9) becomes two expressions, one for $\beta \to \theta$ and one for $\beta \to \phi$. We have

$$0 = \frac{dV^\theta}{d\phi} + \Gamma^{\theta}{}_{\mu\phi}V^\mu = \frac{dV^\theta}{d\phi} + \Gamma^{\theta}{}_{\theta\phi}V^\theta + \Gamma^{\theta}{}_{\phi\phi}V^\phi = \frac{dV^\theta}{d\phi} - \sin\theta_0\cos\theta_0 V^\phi, \tag{10}$$

$$0 = \frac{dV^\phi}{d\phi} + \Gamma^{\phi}{}_{\mu\phi}V^\mu = \frac{dV^\phi}{d\phi} + \Gamma^{\phi}{}_{\theta\phi}V^\theta + \Gamma^{\phi}{}_{\phi\phi}V^\phi = \frac{dV^\phi}{d\phi} + \frac{1}{\tan\theta_0}V^\theta, \tag{11}$$

where we have used our commutation coefficients from 1(b). This gives us a system of equations that we can use to solve for the components of $\vec{V}$. We can differentiate one of the equations and feed in the other:

$$0 = \frac{d^2V^\theta}{d\phi^2} - \sin\theta_0\cos\theta_0\frac{dV^\phi}{d\phi} = \frac{d^2V^\theta}{d\phi^2} + \cos^2\theta_0 V^\theta,$$

$$0 = \frac{d^2V^\phi}{d\phi^2} + \frac{1}{\tan\theta_0}\frac{dV^\theta}{d\phi} = \frac{d^2V^\phi}{d\phi^2} + \cos^2\theta_0 V^\phi.$$

The solutions to these equations are [6, p. 207]

$$V^\theta = C_1\cos(\omega\phi) + C_2\sin(\omega\phi), \qquad V^\phi = D_1\cos(\omega\phi) + D_2\sin(\omega\phi), \tag{12}$$

where $\omega = \cos\theta_0$. Let the coordinates of our vector at $\phi = 0$ be given by $\vec{V}(0) = (V_0^\theta, V_0^\phi)$. Applying this condition to Eq. (12) gives us

$$V_0^\theta = C_1, \qquad V_0^\phi = D_1.$$

Involving Eqs. (10) and (11) as well gives us

$$\left.\frac{dV^\theta}{d\phi}\right|_0 = \sin\theta_0\cos\theta_0 V^\phi = \sin\theta_0\cos\theta_0[V_0^\phi\cos(\omega\phi) + D_2\sin(\omega\phi)] = -V_0^\theta\omega\sin(\omega\phi) + C_2\omega\cos(\omega\phi),$$
$$\left.\frac{dV^\phi}{d\phi}\right|_0 = -\frac{1}{\tan\theta_0}V^\theta = -\frac{1}{\tan\theta_0}[V_0^\theta\cos(\omega\phi) + C_2\sin(\omega\phi)] = -V_0^\phi\omega\sin(\omega\phi) + D_2\omega\cos(\omega\phi),$$

and solving these two equations for C_2 and D_2 yields

$$C_2 = V_0^\phi\sin\theta_0, \qquad D_2 = -\frac{V_0^\theta}{\sin\theta_0}.$$

So referring once more to Eq. (12), we may finally write the components of $\vec{V}$:

$$\vec{V} = \left[V_0^\theta\cos(\omega\phi) + V_0^\phi\sin\theta_0\sin(\omega\phi)\right]\vec{e}_\theta + \left[V_0^\phi\cos(\omega\phi) - \frac{V_0^\theta}{\sin\theta_0}\sin(\omega\phi)\right]\vec{e}_\phi$$

with $\omega = \cos\theta_0$.

We can find the length V using the dot product:

$$\begin{aligned}
V^2 &= \vec{V}\cdot\vec{V} \\
&= g_{\alpha\beta}V^\alpha V^\beta \\
&= g_{\theta\theta}V^\theta V^\theta + g_{\phi\phi}V^\phi V^\phi \\
&= R^2\left[V_0^\theta\cos(\omega\phi) + V_0^\phi\sin\theta_0\sin(\omega\phi)\right]^2 + R^2\sin^2\theta_0\left[V_0^\phi\cos(\omega\phi) - \frac{V_0^\theta}{\sin\theta_0}\sin(\omega\phi)\right]^2 \\
&= R^2\left[V_0^\theta\cos(\omega\phi) + V_0^\phi\sin\theta_0\sin(\omega\phi)\right]^2 + R^2\left[V_0^\phi\sin\theta_0\cos(\omega\phi) - V_0^\theta\sin(\omega\phi)\right]^2 \\
&= R^2\Big[V_0^{\theta\,2}\cos^2(\omega\phi) + 2V_0^\theta V_0^\phi\sin\theta_0\sin(\omega\phi)\cos(\omega\phi) + V_0^{\phi\,2}\sin^2\theta_0\sin^2(\omega\phi) + V_0^{\phi\,2}\sin^2\theta_0\cos^2(\omega\phi) \\
&\qquad -2V_0^\phi V_0^\theta\sin\theta_0\sin(\omega\phi)\cos(\omega\phi) + V_0^{\theta\,2}\sin^2(\omega\phi)\Big] \\
&= R^2\left[V_0^{\theta\,2} + V_0^{\phi\,2}\sin^2(\omega\phi)\right] \\
&= R^2V_0^{\theta\,2} + R^2\sin^2(\omega\phi)V_0^{\phi\,2},
\end{aligned}$$

where we have used Eq. (5). Thus the length of $\vec{V}$ is

$$V = \sqrt{R^2V_0^{\theta\,2} + R^2\sin^2(\omega\phi)V_0^{\phi\,2}},$$

which is unchanged as we would expect.

3(b) Write the geodesic equation in (θ, ϕ) angular coordinates. Show that the solutions are *great circles*, i.e. circles on the sphere of largest diameter.

Solution. From Lecture 7, the geodesic equation is

$$\frac{du^\alpha}{d\tau} + u^\mu u^\nu {\Gamma^\alpha}_{\mu\nu} = 0.$$

We have two equations, one with $\alpha \to \theta$ and one with $\alpha \to \phi$:

$$0 = \frac{du^\theta}{d\tau} + u^\mu u^\nu {\Gamma^\theta}_{\mu\nu} = \frac{du^\theta}{d\tau} + u^\phi u^\phi {\Gamma^\theta}_{\phi\phi} = \frac{du^\theta}{d\tau} - \sin\theta\cos\theta {u^\phi}^2, \tag{13}$$

$$0 = \frac{du^\phi}{d\tau} + u^\mu u^\nu {\Gamma^\phi}_{\mu\nu} = \frac{du^\phi}{d\tau} + u^\theta u^\phi {\Gamma^\theta}_{\theta\phi} + u^\phi u^\theta {\Gamma^\phi}_{\phi\theta} = \frac{du^\phi}{d\tau} + 2\frac{1}{\tan\theta} u^\theta u^\phi. \tag{14}$$

In order to solve these equations, we choose the initial conditions

$$\theta_0 = \frac{\pi}{2}, \qquad \phi_0 = 0, \qquad u_0^\phi = 1, \qquad u_0^\theta = 0.$$

This set of initial conditions is completely general because we can always rotate our coordinate system. Applying these conditions to Eqs. (13) and (14),

$$\left.\frac{du^\theta}{d\tau}\right|_0 = \sin(\pi/2)\cos(\pi/2) = 0, \qquad \left.\frac{du^\phi}{d\tau}\right|_0 = 0.$$

This means both components of $\vec{u}$ are constant, and we have

$$\vec{u} = (0, 1) \quad \Longrightarrow \quad \frac{d\theta}{d\tau} = 0, \qquad \frac{d\phi}{d\tau} = 1,$$

which implies $\theta = \pi/2$ is constant and $\phi = \tau$. So our parameterized curve traces out a circle of radius R at $\theta = \pi/2$. This is a great circle because its radius is equal to the radius of S^2, and we could rotate our coordinate system to transform this to any other great circle. □

3(c) Consider a disk of radius ϵ on the sphere. Working in the limit of small ϵ, compute the area of the disk to order ϵ^4. Compare your results to $\mathbb{R}^2$ with the flat metric.

Solution. We place the disk at the "top" of the sphere so its center is at $\theta = 0$. The area element for a sphere of radius R is $R^2 \sin\theta \, d\theta \, d\phi$ [2]. We need to integrate over the surface of the sphere from $\theta = 0$ to $\theta = \epsilon/R$:

$$dA = \int_0^{2\pi}\int_0^{\epsilon/R} R^2 \sin\theta \, d\theta \, d\phi = R^2 \int_0^{2\pi} d\phi \int_0^{\epsilon/R} \sin\theta \, d\theta = 2\pi R^2 \Big[-\cos\theta\Big]_0^{\epsilon/R} = 2\pi R^2 \left[1 - \cos\!\left(\frac{\epsilon}{R}\right)\right].$$

The Maclaurin series expansion for $\cos x$ is [7]

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4,$$

so our area is

$$dA = 2\pi R^2 \left[\frac{1}{2}\frac{\epsilon^2}{R^2} - \frac{1}{24}\frac{\epsilon^4}{R^4}\right] = \pi\epsilon^2 - \frac{\pi}{12}\frac{\epsilon^4}{R^2}.$$

In $\mathbb{R}^2$ with the flat metric, $dA = \pi\epsilon^2$. So the area on the sphere is smaller.

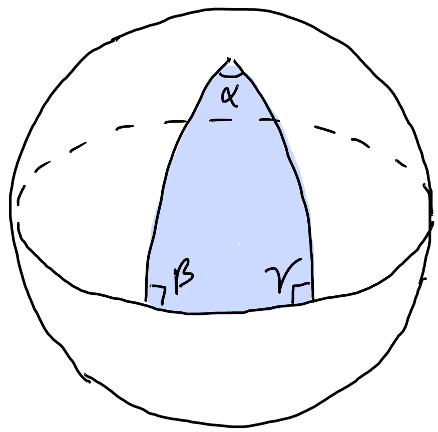


Figure 1: A triangle on the sphere with $\alpha > 0$ and $\beta = \gamma = \pi/2$.

3(d) A spherical triangle is made from three points on the sphere pairwise connected by geodesics. Let the angles on the triangle be α, β, and γ. By drawing pictures, show that $\alpha + \beta + \gamma$ can be larger than π.

Solution. We can draw a triangle on the sphere where $\alpha > 0$ and $\beta = \gamma = \pi/2$, so $\alpha + \beta + \gamma > \pi$. This is shown in Fig. 2.

3(e) Define the excess angle E of a spherical triangle by $E = \alpha + \beta + \gamma - \pi$. Prove that the area of the triangle is $R^2 E$.

Solution. The vertices of any triangle on the great sphere can be defined by the intersection points of three great circles. In fact, two triangles of identical area are defined this way. This is illustrated in Fig. **??**(a), with both of the triangles shaded. Call the area of one of these triangles $A_{\alpha\beta\gamma}$.

We can also define a region on the surface of the sphere by the intersection of two great circles at some angle θ. Two regions identical in area can be defined this way, as illustrated by the shaded areas in Fig. **??**(b)–(d). Call the area of one of these regions A_θ. We can find its area using

$$\frac{4\pi R^2}{\pi} = \frac{2A_\theta}{\theta} \quad \Longrightarrow \quad A_\theta = 2\theta R^2, \tag{15}$$

since drawing two great circles separated by angle π covers the entire sphere in this manner.

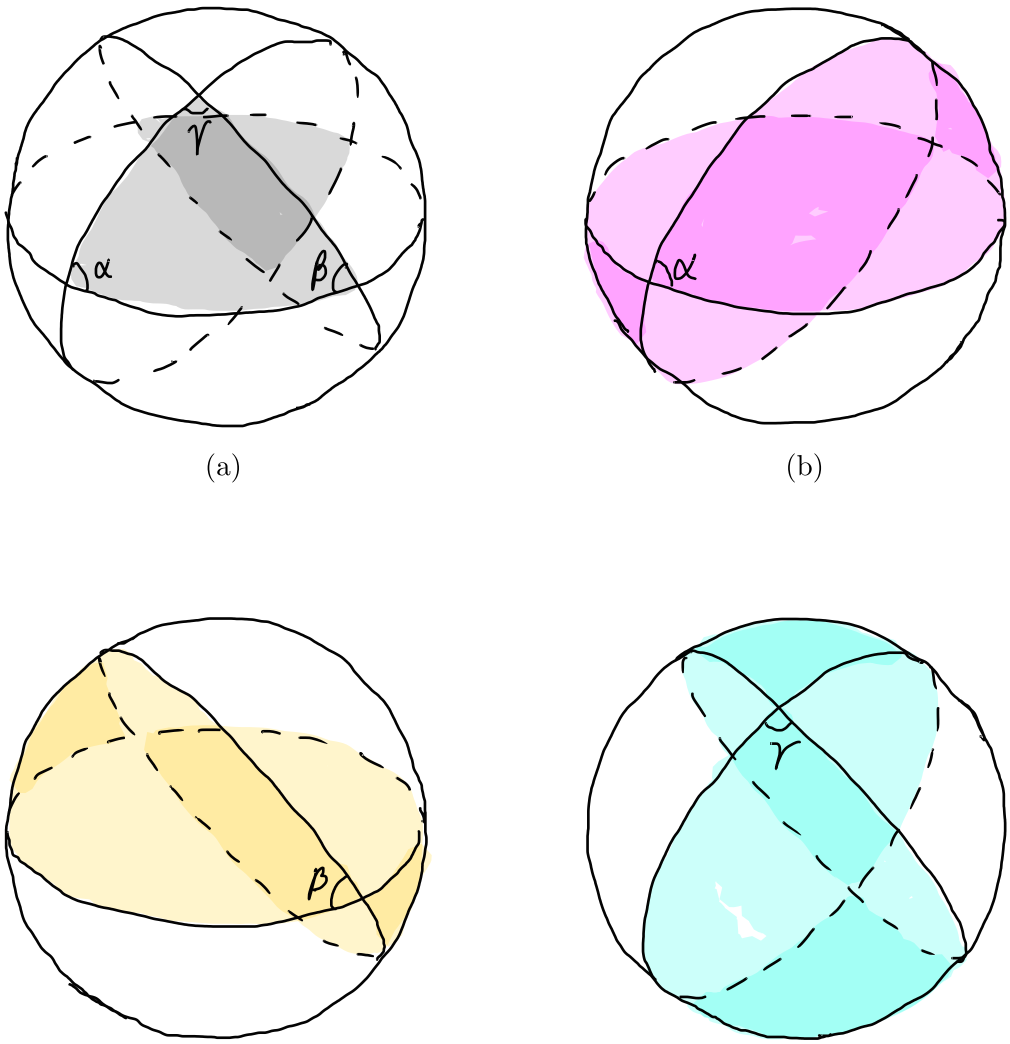
If we add up all of the shaded areas in Fig. **??**(b)–(d), we have covered each of the triangles three times (that is, we have covered $6A_{\alpha\beta\gamma}$) and the rest of the sphere's area once. The area we have covered can be expressed

$$4\pi R^2 + 4A_{\alpha\beta\gamma} = 2(A_\alpha + A_\beta + A_\gamma) = 4R^2(\alpha + \beta + \gamma),$$

using Eq. (15), which implies

$$A_{\alpha\beta\gamma} = 4R^2(\alpha + \beta + \gamma - \pi) = 4R^2 E$$

as we wanted to show. $\square$



(a) (b) (c) (d)

Figure 2: The areas (a) $2A_{\alpha\beta\gamma}$, (b) $2A_{\alpha}$, (c) $2A_{\beta}$, and (d) $2A_{\gamma}$.

Problem 4. In this problem you will explore the geometry on the space of possible inertial velocities.

4(a) Suppose two inertial frames move with 3-velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to a fixed inertial frame. Show that their relative velocity $\mathbf{v}$ has magnitude v given by

$$v^2 = \frac{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2}. \tag{16}$$

Solution. We can associate each 3-velocity with the 4-momentum of some particle:

$$\vec{p}_1 = m_1\gamma_1(1, \mathbf{v}_1), \qquad \vec{p}_2 = m_2\gamma_2(1, \mathbf{v}_2),$$

where m_1, m_2 are the masses of the two particles and γ_1, γ_2 are their Lorentz factors. Here we have consulted MCP (2.25c), (2.25d), (2.26a), and (2.26d) to write $\vec{p} = m\gamma(1, \mathbf{v})$. Let $\gamma = 1/\sqrt{1-v^2}$ be the relative Lorentz factor between them. In the rest frame of particle 2, then, we can write [**?** , p. 35]

$$\vec{p}_1 \cdot \vec{p}_2 = m_1\gamma(1, \mathbf{v}_1') \cdot m_2(1, \mathbf{0}) = \gamma m_1 m_2 = \frac{m_1 m_2}{\sqrt{1-v^2}}.$$

This implies

$$v = \sqrt{1 - \frac{(m_1 m_2)^2}{(\vec{p}_1 \cdot \vec{p}_2)^2}} \quad \Longrightarrow \quad v^2 = 1 - \frac{(m_1 m_2)^2}{(\vec{p}_1 \cdot \vec{p}_2)^2}. \tag{17}$$

We can make use of the Lorentz-invariant dot product $\vec{p}_1 \cdot \vec{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_2 - \mathcal{E}_1\mathcal{E}_2$. By (2.26a), $\mathcal{E} = \gamma m$. So we have

$$\vec{p}_1 \cdot \vec{p}_2 = (m_1\gamma_1\mathbf{v}_1) \cdot (m_2\gamma_2\mathbf{v}_2) - \gamma_1 m_1 \gamma_2 m_2 = m_1 m_2 \frac{\mathbf{v}_1 \cdot \mathbf{v}_2 - 1}{\sqrt{(1-v_1^2)(1-v_2^2)}},$$

which implies

$$(\vec{p}_1 \cdot \vec{p}_2)^2 = (m_1 m_2)^2 \frac{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2}{(1-v_1^2)(1-v_2^2)}$$

Feeding this into Eq. (17), we find [8, p. 35]

$$\begin{aligned}
v^2 &= 1 - \frac{(1-v_1^2)(1-v_2^2)}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2} \\
&= \frac{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2 - (1-v_1^2)(1-v_2^2)}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2} \\
&= \frac{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2 - (v_1^2-1)(v_2^2-1)}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2} \\
&= \frac{1 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - v_1^2 v_2^2 + v_1^2 + v_2^2 - 1}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2} \\
&= \frac{-2(\mathbf{v}_1 \cdot \mathbf{v}_2) + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - (\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) + \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2} \\
&= \frac{\mathbf{v}_1 \cdot \mathbf{v}_1 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \mathbf{v}_2 \cdot \mathbf{v}_2 - (\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2} \\
&= \frac{(\mathbf{v}_1 - \mathbf{v}_2)^2 - (\mathbf{v}_1 \times \mathbf{v}_2)^2}{(1 - \mathbf{v}_1 \cdot \mathbf{v}_2)^2},
\end{aligned}$$

as we wanted to show. In performing the last step we have used the vector quadruple product [9],

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}),$$

to write $(\mathbf{v}_1 \times \mathbf{v}_2)^2 = (\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2$. □

4(b) We define a metric on the space of all possible 3-velocities by defining the distance between two nearby velocities to be their relative velocity. Using the result from 4(a), show that this metric is

$$ds^2 = d\chi^2 + \sinh^2(\chi)[d\theta^2 + \sin^2(\theta)\, d\phi^2],$$

where χ is the rapidity $v = \tanh(\chi)$, and θ, ϕ are polar and azimuthal angles defined relative to $\mathbf{v}$.

Solution. Let $d\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1$. The line element ds is the relative velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ [8, p. 35]. Feeding this into Eq. (16), we can write

$$ds^2 = \frac{(d\mathbf{v})^2 - (\mathbf{v} \times d\mathbf{v})^2}{(1-v^2)^2} = \frac{d\mathbf{v}^2 - \mathbf{v}^2\, d\mathbf{v}^2 + (\mathbf{v} \cdot d\mathbf{v})^2}{(1-v^2)^2} = \frac{d\mathbf{v}^2\,(1-v^2) + (\mathbf{v} \cdot d\mathbf{v})^2}{(1-v^2)^2}, \tag{18}$$

where he have again used the vector quadruple product. Using the line element in spherical coordinates [2], we can express $d\mathbf{v}$ in terms of unit vectors. We have

$$d\mathbf{v} = dv\,\hat{\mathbf{v}} + v\,d\theta\,\hat{\boldsymbol{\theta}} + v\sin\theta\hat{\boldsymbol{\phi}}, \qquad \mathbf{v} = v\hat{\mathbf{v}},$$

which implies

$$d\mathbf{v}^2 = dv^2 + v^2\,d\theta^2 + v^2\sin^2\theta\,d\phi^2\,, \qquad (\mathbf{v} \cdot d\mathbf{v})^2 = v^2\,dv^2\,.$$

So Eq. (18) becomes [8, p. 35]

$$\begin{aligned} ds^2 &= \frac{(1-v^2)(dv^2 + v^2\,d\theta^2 + v^2\sin^2\theta\,d\phi^2) + v^2\,dv^2}{(1-v^2)^2} \\ &= \frac{dv^2 + v^2\,d\theta^2 + v^2\sin^2\theta - v^4\,d\theta^2 - v^4\sin^2\theta\,d\phi^2}{(1-v^2)^2} \\ &= \frac{dv^2 + v^2(1-v^2)(d\theta^2 + \sin^2\theta\,d\phi^2)}{(1-v^2)^2} \\ &= \frac{dv^2}{(1-v^2)^2} + \frac{v^2(d\theta^2 + \sin^2\theta\,d\phi^2)}{1-v^2}. \end{aligned}$$

Since $v = \tanh\chi$, $dv = \operatorname{sech}^2\chi\,d\chi = d\chi/\cosh^2\chi$. Using also $\tanh x = \sinh x/\cosh x$ and $\cosh^2\chi - \sinh^2 x = 1$ [10], we have

$$\begin{aligned} ds^2 &= \frac{\operatorname{sech}^2\chi\,d\chi^2}{(1-\tanh^2\chi)^2} + \frac{\tanh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)}{1-\tanh^2\chi} \\ &= \frac{d\chi^2}{\cosh^4\chi(1-\tanh^2\chi)^2} + \frac{\sinh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)}{\cosh^2(1-\tanh^2\chi)} \\ &= \frac{d\chi^2}{(\cosh^2\chi - \sinh^2\chi)^2} + \frac{\sinh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)}{\cosh^2\chi - \sinh^2\chi} \\ &= d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2), \end{aligned}$$

as we wanted to show. □

4(c) Show that the geodesics of this metric are paths of minimum fuel use for a rocket ship changing its velocity.

Solution. A geodesic is the shortest possible path between two points in a manifold [5, p. 77]. This means that a geodesic of the metric in our inertial velocity space is the shortest way to change from one velocity to another. Taking the shortest route uses as little fuel as possible, so we can conclude that the geodesics of this metric are paths of minimum fuel use. $\square$

4(d) A rocket ship in interstellar travel with velocity $\mathbf{v}_1$ relative to earth changes to a new velocity $\mathbf{v}_2$ in a manner that uses the least amount of fuel. What is the ship's smallest velocity relative to earth during the change?

Solution. I could not figure out how to solve this one. I think we want to find some kind of minimum along the geodesic between $\mathbf{v}_1$ and $\mathbf{v}_2$, but I am not sure how to do this.

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