1 Reduced three-body problem

The problem of three point particles interacting gravitationally has a particularly simple limit: let the third body $m_3 \ll m_2, m_1$ so that its effect on the motions of m_1 and m_2 is negligible. Assume in addition that m_3 moves in the same orbital plane as m_1 and m_2. For simplicity, consider only the case of m_1 and m_2 in circular orbit about their center of mass.

1.a Switch into a reference frame rotating with angular velocity ω associated with the circular orbit for the two-body problem. Choose the center of mass of the two-body problem to be the origin. Choose the x axis to go through m_1 and m_2. Show that the (now stationary) m_1 and m_2 are located at $-r_c\mu/m_1$ and $r_c\mu/m_2$.

Solution. Call the stationary coordinate system (X, Y, Z), and choose (X, Y) as the orbital plane. Call the rotating coordinate system (x, y, z), which is rotated about the Z axis by angle ωt. This gives us the transformation

$$x = X\cos\omega t + Y\sin\omega t, \tag{1}$$
$$y = Y\cos\omega t - X\sin\omega t, \tag{2}$$
$$z = Z. \tag{3}$$

From our choice of orbital plane, there is no motion in the z direction. Let the locations of m_1 and m_2 be given by $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ in the rotating frame.

From our choice of x axis, we know that $y_1 = y_2 = 0$. From our choice of the origin as the center of mass, we have

$$m_1x_1 + m_2x_2 = 0. \tag{4}$$

By construction, m_1 and m_2 are stationary in the rotating frame, so $\mathrm{d}x_1/\mathrm{d}t = \mathrm{d}x_2/\mathrm{d}t = 0$. In other words, x_1 and x_2 must both be constant. Therefore, let

$$r_c = x_2 - x_1 \tag{5}$$

be the constant distance between m_1 and m_2. Now we have the system of two equations (4) and (5), so we can solve for x_1 and x_2. Substituting (5) into (4) yields

$$m_1x_1 + m_2(r_c + x_1) = 0 \implies x_1 = -\frac{m_2}{m_1 + m_2}r_c. \tag{6}$$

Now substituting (6) back into (4),

$$r_c = x_2 + \frac{m_2}{m_1 + m_2}r_c \implies x_2 = \frac{m_1}{m_1 + m_2}r_c. \tag{7}$$

Note that the reduced mass $\mu = m_1m_2/(m_1 + m_2)$. Substituting μ into (6) and (7) yields

$$\mathbf{r}_1 = x_1 = -r_c\mu/m_1, \qquad \mathbf{r}_2 = x_2 = r_c\mu/m_2 \tag{8}$$

as desired. □

1.b Show that the Lagrangian governing the equation of motion of m_3 at location $(x(t), y(t))$ is

$$L_3 = \frac{m_3}{2}\left[(\dot{x}-\omega y)^2 + (\dot{y}+\omega x)^2\right] - U_{13} - U_{23}, \tag{9}$$

where $U_{13}(x,y)$ is the gravitational interaction of m_3 with m_1, while $U_{23}(x,y)$ is associated with m_3 and m_2.

Solution. In general, the Lagrangian for m_3 is given by

$$L_3 = T_3 - U_3, \tag{10}$$

where T_3 is the kinetic energy of m_3 and U_3 is its potential energy.

Beginning with U_3, the only forces acting upon m_3 are the gravitational interactions with m_1 and m_2. We know from the problem statement that these interactions are independent of each other; m_3 has a negligible effect on the motions of each m_1 and m_2, so it cannot couple them in any way. Thus, we can write

$$U_3(x,y) = -G\frac{m_1 m_3}{r_{13}} - G\frac{m_2 m_3}{r_{23}} \equiv U_{13} + U_{23}, \tag{11}$$

where r_{13} (r_{23}) is the separation between m_3 and m_1 (m_2) in the rotating frame. Using the positions of m_1 and m_2 on the x axis given by (8), they are given by

$$r_{13} = \sqrt{\left(x + \frac{r_c\mu}{m_1}\right)^2 + y^2}, \qquad r_{23} = \sqrt{\left(x - \frac{r_c\mu}{m_2}\right)^2 + y^2}. \tag{12}$$

So in (11) we have defined $U_{13}(x,y)$ and $U_{23}(x,y)$.

Now we will find an expression for T_3. Let $\mathbf{R}_3 = (X(t), Y(t))$ be the position of m_3 in the stationary frame. Then

$$T_3 = \frac{m_3}{2}\dot{\mathbf{R}}_3^2 = \frac{m_3}{2}(\dot{X}+\dot{Y})^2. \tag{13}$$

We want to find an expression for T_3 in the rotating coordinate system. We can define an inverse transformation back to the stationary frame by simply rotating the (x,y) plane about the z axis in the opposite direction; that is, by angle $-\omega t$. This inverse transformation is

$$X = x\cos\omega t - y\sin\omega t, \tag{14}$$
$$Y = x\sin\omega t + y\cos\omega t, \tag{15}$$
$$Z = z. \tag{16}$$

It follows from (14) and (15) that

$$\dot{X} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial X}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = -\omega x\sin\omega t - \omega y\cos\omega t + \dot{x}\cos\omega t - \dot{y}\sin\omega t \tag{17}$$
$$= (\dot{x}-\omega y)\cos\omega t - (\dot{y}+\omega x)\sin\omega t, \tag{18}$$
$$\dot{Y} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial Y}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = \omega x\cos\omega t - \omega y\sin\omega t + \dot{x}\sin\omega t + \dot{y}\cos\omega t \tag{19}$$
$$= (\dot{y}+\omega x)\cos\omega t + (\dot{x}-\omega y)\sin\omega t. \tag{20}$$

Now using the forms of (17) and (19),

$$\dot{X}^2 = (\dot{x} - \omega y)^2 \cos^2 \omega t - 2(\dot{x} - \omega y)(\dot{y} + \omega x) \cos \omega t \sin \omega t + (\dot{y} + \omega x)^2 \sin^2 \omega t, \tag{21}$$
$$\dot{Y}^2 = (\dot{y} + \omega x)^2 \cos^2 \omega t + 2(\dot{x} - \omega y)(\dot{y} + \omega x) \cos \omega t \sin \omega t + (\dot{x} - \omega y)^2 \sin^2 \omega t, \tag{22}$$

which implies

$$\begin{aligned} \dot{X}^2 + \dot{Y}^2 &= (\dot{x} - \omega y)^2 (\cos^2 \omega t + \sin^2 \omega t) + (\dot{y} + \omega x)^2 (\cos^2 \omega t + \sin^2 \omega t) && (23) \\ &= (\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2. && (24) \end{aligned}$$

Substituting (24) into (13), we have

$$T_3 = \frac{m_3}{2} \left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 \right]. \tag{25}$$

Finally, substituting (11) and (25) into (10) yields (9) as desired. □

1.c Show that the mechanical system described by L_3 has five locations in mechanical equilibrium. These are known as *Lagrange points*. (Hint: the graphical method is perfectly good for demonstrating a real root exists in a particular instance.)

Solution. The Euler-Lagrange equations for m_3 are given by

$$0 = \frac{\partial L_3}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_3}{\partial \dot{x}}, \tag{26}$$
$$0 = \frac{\partial L_3}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_3}{\partial \dot{y}}. \tag{27}$$

We will attack each term of the Lagrangian in (10) separately. Beginning with T_3, note that

$$\frac{\partial T_3}{\partial x} = m_3(\omega \dot{y} + \omega^2 x), \qquad \frac{\partial T_3}{\partial y} = m_3(-\omega \dot{x} + \omega^2 y), \tag{28}$$
$$\frac{\partial T_3}{\partial \dot{x}} = m_3(\dot{x} - \omega y), \qquad \frac{\partial T_3}{\partial \dot{y}} = m_3(\dot{y} + \omega x). \tag{29}$$

In turn, (29) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T_3}{\partial \dot{x}} = m_3(\ddot{x} - \omega \dot{y}), \qquad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T_3}{\partial \dot{y}} = m_3(\ddot{y} + \omega \dot{x}). \tag{30}$$

Now for U_3, it follows from (12) that

$$\frac{\partial r_{13}}{\partial x} = \frac{1}{r_{13}} \left(x + \frac{r_c \mu}{m_1} \right), \qquad \frac{\partial r_{23}}{\partial x} = \frac{1}{r_{23}} \left(x - \frac{r_c \mu}{m_2} \right), \tag{31}$$
$$\frac{\partial r_{13}}{\partial y} = \frac{y}{r_{13}}, \qquad \frac{\partial r_{23}}{\partial y} = \frac{y}{r_{23}}, \tag{32}$$
$$\frac{\partial r_{13}}{\partial \dot{x}} = \frac{\partial r_{23}}{\partial \dot{x}} = \frac{\partial r_{13}}{\partial \dot{y}} = \frac{\partial r_{23}}{\partial \dot{y}} = 0. \tag{33}$$

Then

$$\frac{\partial U_{13}}{\partial x} = \frac{\partial U_{13}}{\partial r_{13}}\frac{\partial r_{13}}{\partial x} = G\frac{m_1 m_3}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right), \qquad \frac{\partial U_{23}}{\partial x} = G\frac{m_2 m_3}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right), \tag{34}$$

$$\frac{\partial U_{13}}{\partial y} = \frac{\partial U_{13}}{\partial r_{13}}\frac{\partial r_{13}}{\partial y} = G\frac{m_1 m_3}{r_{13}^3}y, \qquad \frac{\partial U_{23}}{\partial y} = G\frac{m_2 m_3}{r_{23}^3}y, \tag{35}$$

$$\frac{\partial U_{13}}{\partial \dot{x}} = \frac{\partial U_{23}}{\partial \dot{x}} = \frac{\partial U_{13}}{\partial \dot{y}} = \frac{\partial U_{23}}{\partial \dot{y}} = 0. \tag{36}$$

Making the appropriate substitutions, (26) becomes

$$0 = m_3(\omega\dot{y} + \omega^2 x) - G\frac{m_1 m_3}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) - G\frac{m_2 m_3}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right) - m_3(\ddot{x} - \omega\dot{y}) \tag{37}$$

$$\implies \ddot{x} = 2\omega\dot{y} + \omega^2 x - G\frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) - G\frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right), \tag{38}$$

and (27) becomes

$$0 = m_3(-\omega\dot{x} + \omega^2 y) - G\frac{m_1 m_3}{r_{13}^3}y - G\frac{m_2 m_3}{r_{23}^3}y - m_3(\ddot{y} + \omega\dot{x}) \tag{39}$$

$$\implies \ddot{y} = -2\omega\dot{x} + \omega^2 y - Gy\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right). \tag{40}$$

The system is in mechanical equilibrium at points where $\dot{x} = \dot{y} = 0$. The equilibrium behavior persists over time, implying $\ddot{x} = \ddot{y} = 0$. With these restrictions, the equations of motion (38) and (40) become

$$x = \frac{G}{\omega^2}\frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{G}{\omega^2}\frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right), \tag{41}$$

$$y = \frac{G}{\omega^2}y\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right). \tag{42}$$

The real roots of (41) and (42) are the Lagrange points.

Inspection of (42) indicates that there is at least one solution where $y = 0$. In this case (42) is eliminated. Additionally, (12) becomes

$$r_{13} = \left|x + \frac{r_c\mu}{m_1}\right|, \qquad r_{23} = \left|x - \frac{r_c\mu}{m_2}\right|, \tag{43}$$

and thus (41) reduces to

$$x = \frac{G}{\omega^2}\frac{m_1}{|x + r_c\mu/m_1|^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{G}{\omega^2}\frac{m_2}{|x - r_c\mu/m_2|^3}\left(x - \frac{r_c\mu}{m_2}\right) \equiv f(x), \tag{44}$$

where we have defined $f(x)$ as the right-hand side of the equation. We make the following observations about $f(x)$:

- $f(x)$ has singularities at $x = -r_c\mu/m_1$ and $x = r_c\mu/m_2$;

- $f(x) < 0$ in the regime $x < -r_c\mu/m_1$;
- $f(x) > 0$ in the regime $x > r_c\mu/m_2$;
- $f(x)$ crosses the x axis somewhere in the regime $-r_c\mu/m_1 < x < r_c\mu/m_2$;
- $\mathrm{d}f/\mathrm{d}x < 0$ for all defined values of x because it is dominated by negative powers of x.

Based on these observations, we can sketch $f(x)$ and x as shown in figure 1. The three intersection points indicate that there are three real roots of (44). These are the first three Lagrange points.

In the case $y \neq 0$, (42) may be written

$$\frac{\omega^2}{G} = \frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}. \tag{45}$$

Substituting (45) into (41),

$$\left(\frac{m_1}{r_{13}^3} + \frac{m_2}{r_{23}^3}\right) x = \frac{m_1}{r_{13}^3}\left(x + \frac{r_c\mu}{m_1}\right) + \frac{m_2}{r_{23}^3}\left(x - \frac{r_c\mu}{m_2}\right) \implies \frac{m_1}{r_{13}^3}\frac{r_c\mu}{m_1} = \frac{m_2}{r_{23}^3}\frac{r_c\mu}{m_2} \implies r_{13} = r_{23}. \tag{46}$$

Geometrically, this is only possible at two locations in the (x, y) plane as shown in figure 2. These are the final two Lagrange points, for a total of five as desired. $\square$

2 Spherical pendulum

A point mass m in three spatial dimensions is connected by a light inextensible string of length ℓ to a fixed pivot and experiences a uniform gravitational field. Use spherical polar coordinates (ρ, θ, ϕ), where ρ is the radial distance, θ the relative inclination with respect to the downward vertical, and ϕ the azimuthal angle.

2.a Is this mechanical system integrable? In other words, does this problem have as many independent conserved quantities as there are unknown dynamical variables?

Solution. The inextensible string fixes $\rho = \ell$, so the point mass's motion is constrained to a sphere of radius ℓ. Thus the system has two unknown dynamical variables, θ and ϕ. Therefore, we will need to find two independent conserved quantities in order for the system to be integrable. We will search for these quantities by analyzing the Lagrangian of the system.

We will begin with T. In Cartesian coordinates, kinetic energy in three dimensions is given by

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{47}$$

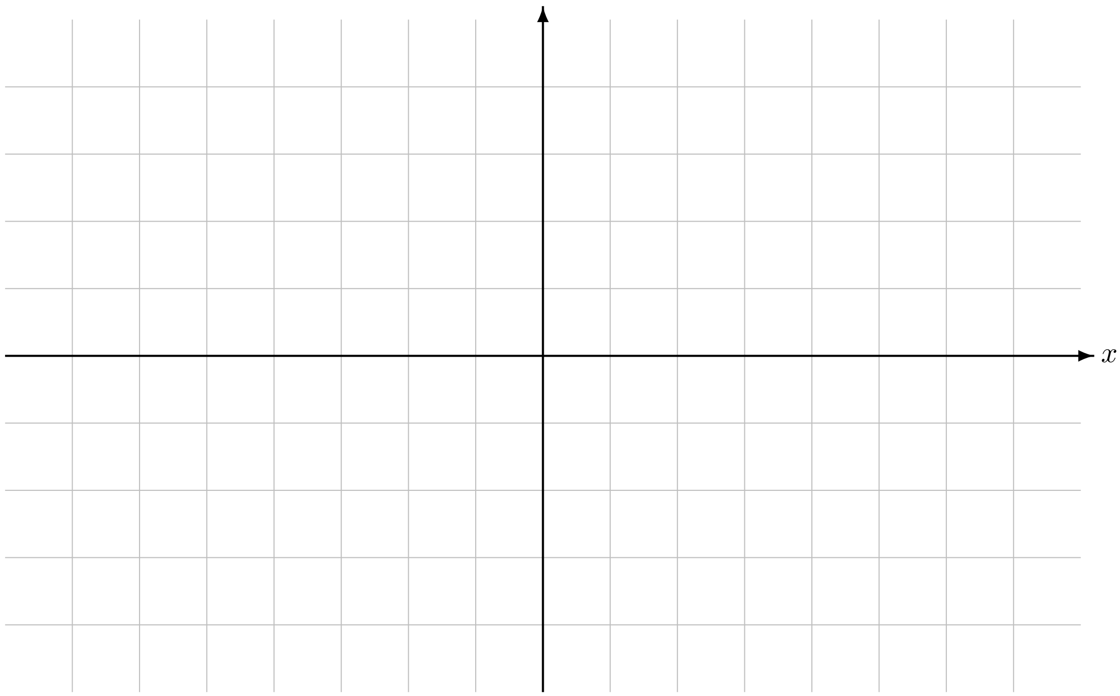

Figure 1: Three Lagrange points, shown as red dots, indicated by roots of (44). The blue curve is $f(x)$ and the purple line is x.

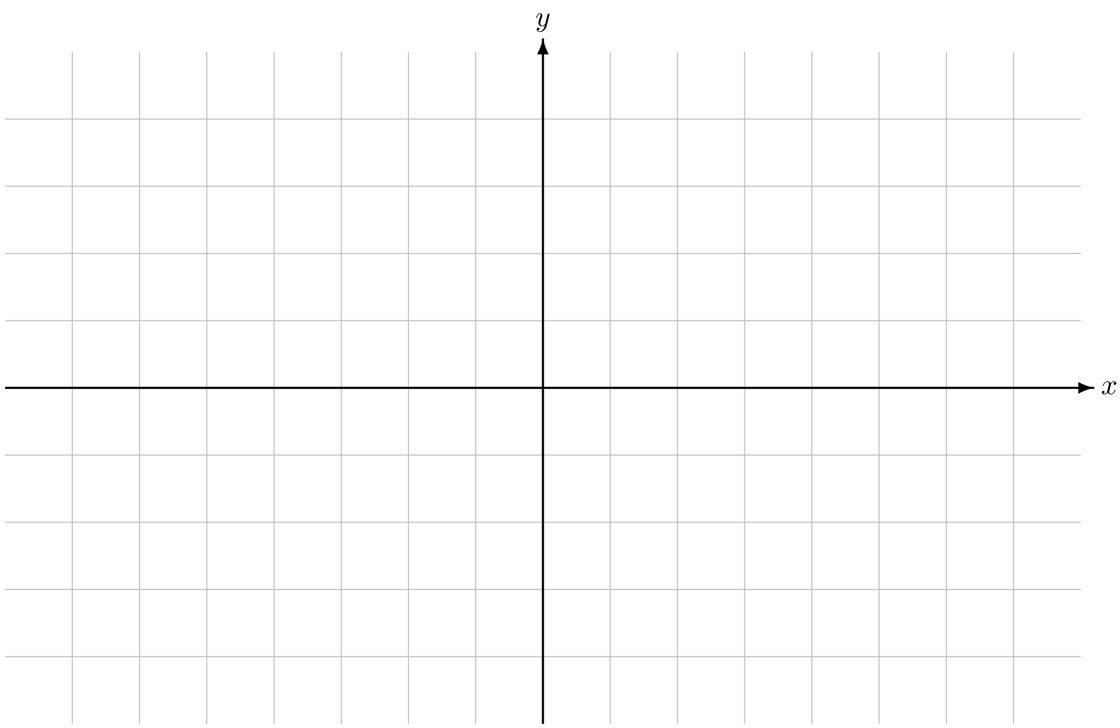

Figure 2: Two more Lagrange points, shown as red dots, found by the geometrical argument implied by (46). At these points, $r_{13} = r_{23} = r^*$.

The transformation to Cartesian coordinates from the spherical coordinate system defined in the problem statement is given by

$$x = \ell \cos\phi \sin\theta, \tag{48}$$
$$y = \ell \sin\phi \sin\theta, \tag{49}$$
$$z = -\ell \cos\theta. \tag{50}$$

Note that these transformations are defined only for the intervals $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi)$. The time derivatives corresponding to (48)–(50) are

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\ell \sin\phi \sin\theta\,\dot{\phi} + \ell\cos\phi\cos\theta\,\dot{\theta}, \tag{51}$$
$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \ell \cos\phi \sin\theta\,\dot{\phi} + \ell\sin\phi\cos\theta\,\dot{\theta}, \tag{52}$$
$$\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \ell \sin\theta\,\dot{\theta}. \tag{53}$$

From (51)–(53), note that

$$\dot{x}^2 = \ell^2(\sin^2\phi\sin^2\theta\,\dot{\phi}^2 - 2\cos\phi\sin\phi\cos\theta\sin\theta\,\dot{\phi}\,\dot{\theta} + \cos^2\phi\cos^2\theta\,\dot{\theta}^2), \tag{54}$$
$$\dot{y}^2 = \ell^2(\cos^2\phi\sin^2\theta\,\dot{\phi}^2 + 2\cos\phi\sin\phi\cos\theta\sin\theta\,\dot{\phi}\,\dot{\theta} + \sin^2\phi\cos^2\theta\,\dot{\theta}^2), \tag{55}$$
$$\dot{z}^2 = \ell^2\sin^2\theta\,\dot{\theta}^2. \tag{56}$$

From (54) and (55),

$$\dot{x}^2 + \dot{y}^2 = \ell^2\left[(\sin^2\phi\sin^2\theta + \cos^2\phi\sin^2\theta)\,\dot{\phi}^2 + (\cos^2\phi\cos^2\theta + \sin^2\phi\cos^2\theta)\,\dot{\theta}^2\right] \tag{57}$$
$$= \ell^2(\sin^2\theta\,\dot{\phi}^2 + \cos^2\theta\,\dot{\theta}^2). \tag{58}$$

Now adding (56),

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \ell^2(\sin^2\theta\,\dot{\phi}^2 + (\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2) = \ell^2(\sin^2\theta\,\dot{\phi}^2 + \dot{\theta}^2). \tag{59}$$

Substituting (24), we can write (47) in polar coordinates:

$$T = \frac{m\ell^2}{2}(\sin^2\theta\,\dot{\phi}^2 + \dot{\theta}^2). \tag{60}$$

Now addressing U, we know that the only external force acting upon the particle is gravity in the $-z$ direction. This implies

$$U = mgz = -mg\ell\cos\theta, \tag{61}$$

where g is the acceleration of gravity and we have made use of (50).

Combining (60) and (61) yields the Lagrangian

$$L = T - U = \frac{m\ell^2}{2}(\sin^2\theta\,\dot{\phi}^2 + \dot{\theta}^2) + mg\ell\cos\theta. \tag{62}$$

Note firstly that the Lagrangian (62) has no explicit time dependence. Thus, the total energy H of the system is conserved, where

$$H = T + U = \frac{m\ell^2}{2}(\sin^2\theta\,\dot{\phi}^2 + \dot{\theta}^2) - mg\ell\cos\theta. \tag{63}$$

Note also that (62) has no explicit ϕ dependence. From Noether's theorem, this implies a second conserved quantity, given by

$$\frac{\partial L}{\partial\dot{\phi}} = m\ell^2\sin^2\theta\,\dot{\phi} \equiv J \tag{64}$$

where we have defined J, which is the angular momentum of the system. The two conserved quantities H and J are independent; therefore, the mechanical system is indeed integrable.

2.b When appropriately simplified, the motion of the spherical pendulum reduces to one-dimensional motion of a point mass in an effective potential. Find the effective potential.

Solution. The Euler-Lagrange equations for the Lagrangian (62) are

$$0 = \frac{\partial L}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\phi}} = -\frac{\mathrm{d}}{\mathrm{d}t}J = m\ell^2\sin^2\theta\ddot{\phi}, \tag{65}$$

$$0 = \frac{\partial L}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} = -mg\ell\sin\theta + m\ell^2\cos\theta\sin\theta\,\dot{\phi}^2 - m\ell^2\,\ddot{\theta}. \tag{66}$$

Note that (65) is simply a restatement of the conservation of angular momentum that was shown in 2.a. Only (66) is relevant to the motion of the system; therfore, we are concerned with one-dimensional motion. We can rewrite (66) as

$$\ddot{\theta} = \cos\theta\sin\theta\,\dot{\phi}^2 - \frac{g}{\ell}\sin\theta \equiv -\frac{\partial U_{\text{eff}}}{\partial\theta} \tag{67}$$

where we are defining the effective potential U_{eff} by

$$U_{\text{eff}} = \frac{\sin^2\theta}{2}\,\dot{\phi}^2 - \frac{g}{\ell}\cos\theta = \frac{J^2}{2m^2\ell^4}\frac{1}{\sin^2\theta} - \frac{g}{\ell}\cos\theta, \tag{68}$$

using J as defined by (64).

3 Spherical pendulum, continued

3.a Write down the Hamiltonian describing the one-dimensional motion in problem 2.b. Sketch some time-evolution trajectories in phase space. Make sure you include all qualitatively different features and indicate the direction of time evolution. If there are fixed points corresponding to states in mechanical equilibrium, identify them. If there is a separatrix, a trajectory separating qualitatively different motion, write down the equation describing its shape and specify its energy.

Solution. The Hamiltonian for the one-dimensional system is given by

$$H = T_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}\dot{\theta}^2 + U_{\text{eff}}(\theta), \tag{69}$$

with the definition of U_{eff} given by (68). The mass does not show up in the kinetic energy term T_{eff} because it was eliminated from (67), which we used to find U_{eff}.

We know from the form of (69) that fixed points of this system will exist at θ^* such that $U_{\text{eff}}(\theta^*)$ is an extremum. The condition for this to occur is

$$\left.\frac{\partial U_{\text{eff}}}{\partial \theta}\right|_{\theta^*} = 0. \tag{70}$$

Substituting J as defined in (64) into (67), we find

$$\frac{\partial U_{\text{eff}}}{\partial \theta} = \frac{g}{\ell}\sin\theta - \frac{J^2}{m^2\ell^4}\frac{\cos\theta}{\sin^3\theta} \equiv f(\theta) - g(\theta), \tag{71}$$

where we have defined $f(\theta)$ and $g(\theta)$. We will make a geometrical argument similar to that for the first three Lagrange points of problem 1.c to find the roots θ^* of $\partial U_{\text{eff}}/\partial\theta = 0$, which exist where $f(\theta)$ and $g(\theta)$ intersect. Recall when we defined the coordinate transformation in (48)–(50) that we are only concerned with the interval $\theta \in [0, \pi)$; in this range, $\sin\theta \geq 0$. We make the following observations about $g(\theta)$:

- $g(\theta)$ has singularities at $\theta = 0$ and $\theta = \pi$;
- $g(\pi/2) = 0$;
- $\mathrm{d}g/\mathrm{d}\theta < 0$ for $\theta \in [0, \pi)$ because it is dominated by negative powers of $\sin\theta$.

Based on these observations, we can sketch $f(x)$ and $g(x)$ qualitatively as shown in figure 3. Their intersection indicates that there exists exactly one $\theta^* \in [0, \pi/2)$.

Using figure 3 as a reference, we can geometrically perform the subtraction $f(\theta) - g(\theta)$ to obtain a sketch of $\frac{\partial U_{\text{eff}}}{\partial \theta}$ as a function of θ. This is shown in figure 4, where the positive slope at θ^* indicates $\partial^2 U_{\text{eff}}/\partial\theta^2\,|_{\theta^*} > 0$. This means θ^* is a minimum of U_{eff}; in fact, it is the global minimum in the interval $\theta \in [0, \pi)$. Therefore, θ^* is a stable fixed point.

Now we want to draw trajectories in the phase plane (θ, p_θ), where p_θ is the generalized momentum corresponding to θ. Substituting (64) into (62) gives the Lagrangian

$$L = \frac{m\ell^2}{2}\dot{\theta}^2 + \frac{J^2}{2m\ell^2\sin^2\theta} + mg\ell\cos\theta. \tag{72}$$

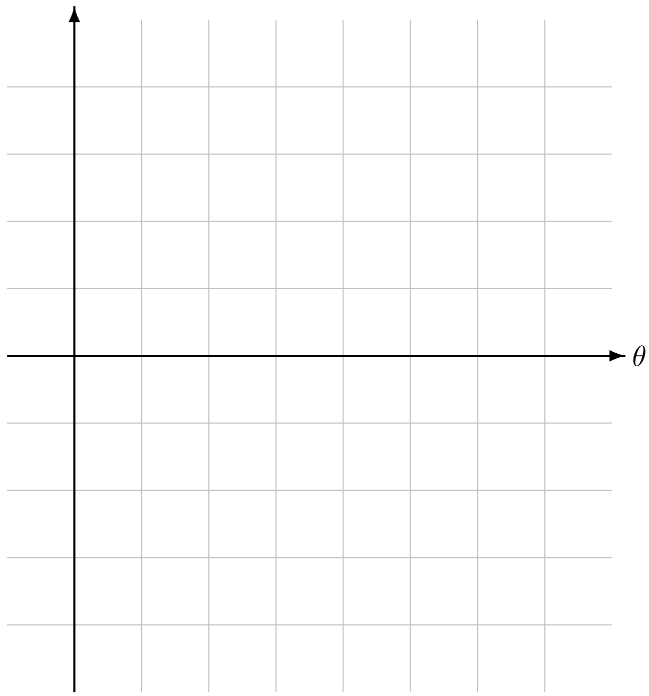

Figure 3: Intersection of $f(\theta)$ (purple line) and $g(\theta)$ (blue line) as defined in (71), showing the existence of a fixed point at $0 < \theta^* < \pi/2$ (red dot).

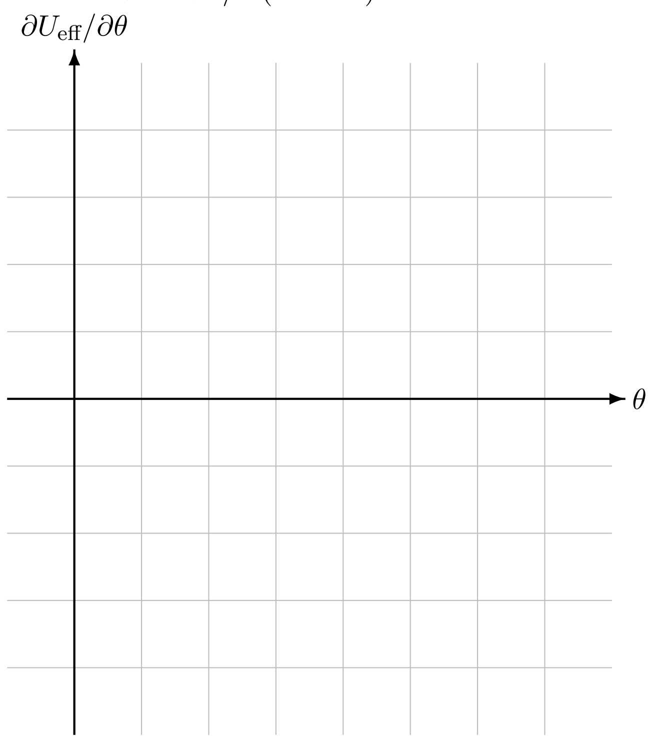

Figure 4: Qualitative sketch of $\partial U_{\text{eff}}/\partial\theta$, whose slope indicates a local minimum at fixed point θ^* (red dot).

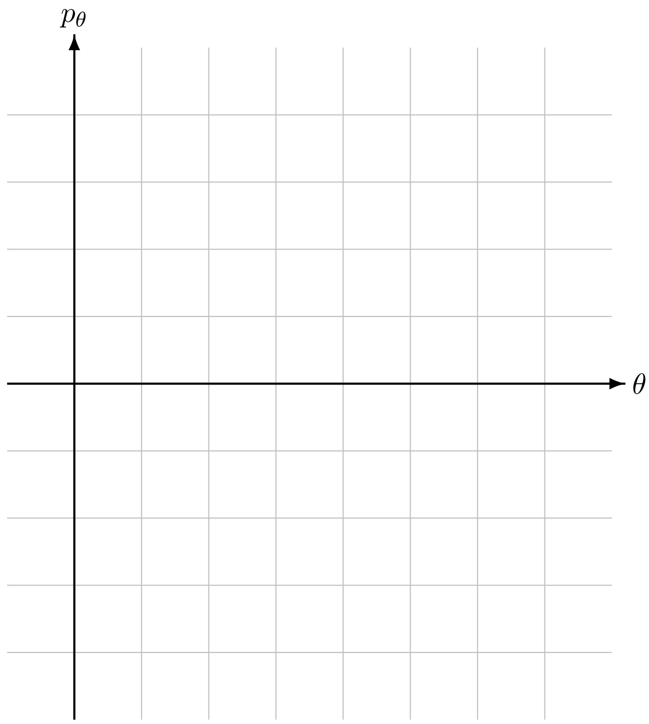

Figure 5: Time evolution trajectories (blue) in phase space for the system of problem 3.a. The fixed point $(\theta^*, 0)$ is shown in red.

Using the Lagrangian (72), the generalized momentum corresponding to θ is given by

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}. \tag{73}$$

In general, fixed points (θ^*, p_θ^*) occur where $\dot{\theta} = \dot{p}_\theta = 0$. It follows immediately from (73) that $p_\theta^* = 0$. The fixed point $(\theta^*, 0)$ is drawn in red in figure 3. In order to characterize the trajectories near this fixed point, note that (73) gives a relationship for the time evolution in the θ direction. That is, time evolution flows in the $+\theta$ $(-\theta)$ direction when $p_\theta > 0$ $(p_\theta < 0)$, which corresponds to the top (bottom) half of the phase plane. By this argument, and our knowledge of the stability of the fixed point, we can draw the trajectories shown in blue in figure 3. We see that the fixed point, while stable, is not an attractor; it is a center of orbital motion.

Because there is only this one fixed point in the region of interest, all motion is orbital, and so there is no separatrix in figure 3.

3.b Using the results derived earlier, give a simple qualitative description of the spherical pendulum motion in three-dimensional space.

Solution. The mass moves on the surface of a sphere of radius ℓ, as noted in 2.a. It rotates with constant angular velocity $\dot{\phi} = \omega$ about the z axis. The motion in the θ direction depends on the initial condition $\theta_0 = \theta(t = 0)$. If $\theta_0 = \theta^*$, it mass's inclination does not change, so it rotates in a circular orbit. If $\theta_0 \neq \theta_*$, the motion is oscillatory in θ. If $\theta_0 < \theta_*$ ($\theta_0 > \theta_*$), θ_0 is the minimum (maximum) angle of inclination.

4 Double-well potential

A particle with mass m is confined to one dimension and placed in the potential

$$U(q) = U_0 - \frac{q^2}{2} + \frac{q^4}{4}. \tag{74}$$

4.a Write down the Lagrangian and the Hamiltonian.

Solution. The Lagrangian is

$$L = T - U = m\frac{\dot{q}^2}{2} - U_0 + \frac{q^2}{2} - \frac{q^4}{4}, \tag{75}$$

and the Hamiltonian is

$$H = T + U = m\frac{\dot{q}^2}{2} + U_0 - \frac{q^2}{2} + \frac{q^4}{4}. \tag{76}$$

4.b Sketch some time evolution trajectories in phase space. Make sure you include all qualitatively different features and indicate the direction of time evolution. If there are fixed points corresponding to states in mechanical equilibrium, identify them. If there is a separatrix, a trajectory separating qualitatively different motion, write down the equation describing its shape and specify its energy. (You do not need to solve the equation.)

Solution. Using the Lagrangian of (75), the generalized momentum corresponding to the Hamiltonian in (76) is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}. \tag{77}$$

Making this substitution in (76) gives

$$H = \frac{p^2}{2m} + U_0 - \frac{q^2}{2} + \frac{q^4}{4}. \tag{78}$$

Then Hamilton's equations for the system are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \implies p = m\dot{q}, \tag{79}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = q - q^3. \tag{80}$$

We are interested in the phase space (q, p). Fixed points (q^*, p^*) occur where $\dot{q} = \dot{p} = 0$. Making the appropriate equality from (80) and (79), the fixed points are the roots of the equation

$$q^* - (q^*)^3 = p^*. \tag{81}$$

By inspection of (81), the fixed points are located at

$$(q^*, p^*) = (0, 0),\ (\pm 1, 0). \tag{82}$$

These points are drawn in red in figure 4.

In order to classify the stability of the fixed points, we can examine the curvature of $U(q)$ at each. By definition, $\partial U/\partial q\,|_{q^*} = 0$, meaning that each fixed point corresponds to an extremum of U. From the definition (74), we have

$$\frac{\partial^2 U}{\partial q^2} = -2 + 3q^2. \tag{83}$$

If $\partial^2 U/\partial q^2\,|_{q^*} > 0$, q^* is a local minimum of U and therefore a stable fixed point. On the contrary, if $\partial^2 U/\partial q^2\,|_{q^*} < 0$, q^* is a local maximum and thus unstable. Evaluating (83) at each of the fixed points, we have

$$\left.\frac{\partial^2 U}{\partial q^2}\right|_{q^*=0} = -2 < 0, \qquad \left.\frac{\partial^2 U}{\partial q^2}\right|_{q^*=1} = 1 > 0, \qquad \left.\frac{\partial^2 U}{\partial q^2}\right|_{q^*=-1} = 1 > 0. \tag{84}$$

Therefore, $(0, 0)$ is an unstable fixed point, and $(\pm 1, 0)$ are both stable.

In order to determine the behavior near the fixed points, we use the same reasoning as in problem 3.a to draw the trajectories. That is, (79) implies that time evolution flows in the $+q$ $(-q)$ direction in the top (bottom) half of the phase plane. Using our knowledge of the stability of each point, we can draw the partial trajectories shown in purple in figure 4. We see that the fixed point at $(0, 0)$ is a saddle point. The points at $(\pm 1, 0)$ are stable, but not attracting; they are centers of orbital motion.

Drawing through the trajectories intersecting $(0, 0)$, we identify the separatrix, drawn in green in figure 4. To find the equation for the separatrix, note that the Hamiltonian in (76) is conserved because the Lagrangian in (75) is time independent. We know that the separatrix intersects $(q, p) = (0, 0)$. Substituting this condition into (78) gives us

$$H = U_0 \tag{85}$$

as the energy of the separatrix. This energy is conserved, so $H = U_0$ at any point along the separatrix. Substituting back into (78) gives us

$$\frac{q^4}{2} = q^2 - \frac{p^2}{m} \tag{86}$$

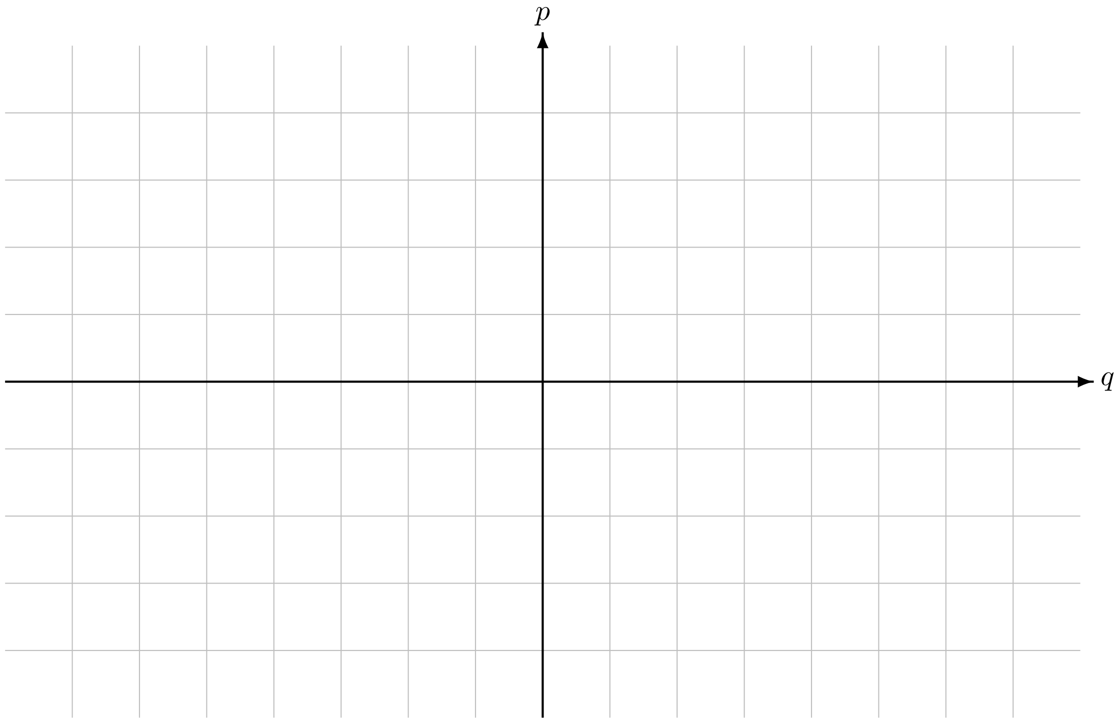

Figure 6: Time evolution trajectories in phase space for the system of problem 4. Fixed points are shown in red. The separatrix is shown in green.

as the equation describing the shape of the separatrix.

In writing these solutions, I consulted David Tong's lecture notes and Steven Strogatz'z *Nonlinear Dynamics and Chaos*. I also got some advice from Morgan Lynn.