Problem 1. Consider a dielectric ball of radius R with dielectric constant ϵ. Obtain a multipole expansion for the field, $\phi(\mathbf{x})$, of a point charge q placed at a point $\mathbf{x}'$ with $|\mathbf{x}'| = d > R$ (so the charge is outside of the dielectric ball).

Hint: Follow the procedure we used in class to find the multipole expansion of a point charge without the dielectric, but now consider the three regions $r \leq R$, $R \leq r \leq d$, and $r \geq d$. Obtain the form of the solution in these regions and match suitably.

Solution. The multipole expansion in spherical harmonics is given by Eq. (2.79) in the course notes,

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi), \tag{1}$$

where the spherical multipole moments q_{lm} are defined in Eq. (2.80),

$$q_{lm} \equiv \int \rho(\mathbf{x}')\, r'^l\, Y^*_{lm}(\theta', \phi')\, d^3x' \,.$$

Note that (1) is valid only for $|\mathbf{x}| \geq R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only within $|\mathbf{x}'| \leq R$, which is outside the dielectric.

The spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi},$$

and the associated Legendre polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l.$$

Problem 2. A dielectric ball of radius R and dielectric constant ϵ is placed in the external electrostatic potential $\phi_0 = \alpha(2z^2 - x^2 - y^2)$ where α is a constant, with the center of the ball at $\mathbf{x} = 0$.

2.a Find the total electrostatic potential ϕ everywhere.

Hint: It is useful to note that the external potential is proportional to $r^2\, Y_{20}(\theta, \phi)$. This should allow you to determine/guess the form of the total potential inside and outside the dielectric up to unknown constants, which can then be determined by matching.

Solution. Poisson's equation inside a dielectric is given by Eq. (3.22) in the course notes,

$$\nabla^2 \langle \phi \rangle = -\frac{4\pi}{\epsilon} \langle \rho_f \rangle \,.$$

Here, $\langle \rho_f \rangle = 0$ since there are no free charges within the dielectric, so this reduces to Laplace's equation. The general solution to Laplace's equation is given by Eq. (3.61) in Jackson,

$$\langle \phi \rangle (r, \theta, \varphi) = \sum_{l,m} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \phi), \tag{2}$$

where A_{lm} and B_{lm} are constant coefficients.

In the region $r < R$, we must have $B_{lm} = 0$ because $1/r^{l+1}$ is undefined at the origin. In the region $r > R$, we may invoke the boundary condition at infinity:

$$\phi(r > R, \theta, \varphi) \to \alpha r^2\, Y_{20}(\theta, \phi),$$

where we note that $\langle\phi\rangle = \phi$ for $r > R$. This implies that the only nonzero A_{lm} here is $A_{20} = \alpha$. Thus we have

$$\langle\phi\rangle(r, \theta, \varphi) = \begin{cases} \sum_{l,m} A_{lm} r^l\, Y_{lm}(\theta, \phi) & \text{if } r \le R, \\ \alpha r^2\, Y_{20}(\theta, \phi) + \sum_{l,m} \dfrac{B_{lm}}{r^{l+1}}\, Y_{lm}(\theta, \phi) & \text{if } r \ge R. \end{cases}$$

To solve for the remaining coefficients, we invoke the boundary conditions at $r = R$. Firstly, $\langle\phi\rangle$ must be continuous at the boundary. This gives us

$$\langle\phi\rangle(R, \theta, \varphi) = \sum_{l,m} A_{lm} R^l\, Y_{lm}(\theta, \phi) = \alpha R^2\, Y_{20}(\theta, \phi) + \sum_{l,m} \frac{B_{lm}}{R^{l+1}}\, Y_{lm}(\theta, \phi),$$

so

$$A_{20} = \alpha + \frac{B_{20}}{R^5}, \qquad A_{lm} = \frac{B_{lm}}{R^{l+3}} \quad \text{for } (l, m) \ne (2, 0). \tag{3}$$

Secondly, we require that $\hat{\mathbf{n}} \cdot \langle\mathbf{D}\rangle$ is also continuous at the boundary, where

$$\langle\mathbf{D}\rangle = \epsilon\, \langle\mathbf{E}\rangle$$

inside a dielectric, from Eq. (3.20) in the course notes. (In vacuum, $\mathbf{D} = \mathbf{E}$.) Here we are only concerned with the r component of $\langle\mathbf{E}\rangle$. Applying $\langle\mathbf{E}\rangle = -\boldsymbol{\nabla}\langle\phi\rangle$, we have

$$\langle E_r\rangle(r, \theta, \phi) = \begin{cases} \sum_{l,m} A_{lm} l r^{l-1}\, Y_{lm}(\theta, \phi) & \text{if } r \le R, \\ 2\alpha r\, Y_{20}(\theta, \phi) - \sum_{l,m} (l+1) \dfrac{B_{lm}}{r^{l+2}}\, Y_{lm}(\theta, \phi) & \text{if } r \ge R. \end{cases}$$

Then we need to satisfy

$$\langle\mathbf{D}\rangle(R, \theta, \varphi) = \epsilon \sum_{l,m} A_{lm} l R^{l-1}\, Y_{lm}(\theta, \phi) = 2\alpha R\, Y_{20}(\theta, \phi) - \sum_{l,m} (l+1) \frac{B_{lm}}{R^{l+2}}\, Y_{lm}(\theta, \phi),$$

which stipulates

$$A_{20} = \frac{1}{\epsilon}\left(\alpha - \frac{3}{2}\frac{B_{20}}{R^5}\right), \qquad A_{lm} = -\frac{1}{\epsilon}\frac{(l+1)}{l}\frac{B_{lm}}{R^{2l+1}} \quad \text{for } (l, m) \ne (2, 0). \tag{4}$$

Eliminating B_{lm} from (3) and (4), we obtain

$$A_{20} = \frac{5\alpha}{2\epsilon + 3}, \qquad A_{lm} = 0 \quad \text{for } (l, m) \ne (2, 0),$$

and substituting back into (3) yields

$$B_{20} = 2\alpha R^5 \frac{1 - \epsilon}{2\epsilon + 3}, \qquad B_{lm} = 0 \quad \text{for } (l, m) \ne (2, 0).$$

Finally, the total electrostatic potential everywhere is

$$\langle\phi\rangle(r, \theta, \varphi) = \begin{cases} \dfrac{5\alpha}{2\epsilon + 3} r^2\, Y_{20}(\theta, \phi) & \text{if } r \le R, \\ \alpha r^2\, Y_{20}(\theta, \phi) + 2\alpha \dfrac{1 - \epsilon}{2\epsilon + 3} \dfrac{R^5}{r^{l+1}}\, Y_{20}(\theta, \phi) & \text{if } r \ge R. \end{cases}$$

2.b Calculate the interaction energy between the field produced by the dielectric and the external field. Assume that the potential arises from "distant charges" so that the formula for $\mathcal{E}_{\text{int}}$ given in class and the notes can be used.

2.c Calculate the total force needed to hold the dielectric ball in place.

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics* while writing up these solutions.