Problem 1. Let $\mathcal{V}$ be a bounded region of space and let ϕ be an electrostatic potential that is source free in this region, so that $\nabla^2\phi = 0$ throughout $\mathcal{V}$. Suppose that for all $\mathbf{x}$ lying on the boundary $S = \partial\mathcal{V}$, we have

$$\phi(\mathbf{x}) = -f(\mathbf{x})\hat{\mathbf{n}} \cdot \nabla\psi(\mathbf{x}), \tag{1}$$

where f is a positive function ($f(\mathbf{x}) \geq 0$) and $\hat{\mathbf{n}}$ is the outward pointing normal. Show that $\phi = 0$ throughout $\mathcal{V}$.

Solution. Clearly $\phi = 0$ trivially satisfies the Laplace equation $\nabla^2\phi = 0$ and the boundary condition (1). We will show that this solution is unique.

Suppose to the contrary that there exists another solution ψ. This means $\nabla^2\psi = 0$ snd that ψ satisfies the boundary condition

$$\psi(\mathbf{x}) = -f(\mathbf{x})\hat{\mathbf{n}} \cdot \nabla\psi(\mathbf{x}). \tag{2}$$

Multiplying both sides of $\nabla^2\psi = 0$ by ψ and integrating over $\mathcal{V}$,

$$\begin{aligned} 0 &= \int_{\mathcal{V}} \psi\nabla^2\psi \, d^3x = \int_{\mathcal{V}} \left(\nabla \cdot (\psi\nabla\psi) - |\nabla\psi|^2\right) d^3x \\ &= \int_{\mathcal{V}} \nabla \cdot (\psi\nabla\psi) \, d^3x - \int_{\mathcal{V}} |\nabla\psi|^2 \, d^3x \,, \end{aligned} \tag{3}$$

where we have factored out the divergence. Applying Gauss's theorem, Eq. (2.6) in the course notes, to the first integral on the right-hand side of (3), we have

$$\int_{\mathcal{V}} \nabla \cdot (\psi\nabla\psi) \, d^3x = \int_S \psi\hat{\mathbf{n}} \cdot \nabla\psi \, dS = -\int_S f(\mathbf{x})(\hat{\mathbf{n}} \cdot \nabla\psi)^2 \, dS \,,$$

where we have made the substitution (2). Substituting this back into (3) yields

$$0 = \int_S f(\mathbf{x})(\hat{\mathbf{n}} \cdot \nabla\psi)^2 \, dS + \int_{\mathcal{V}} |\nabla\psi|^2 \, d^3x \,.$$

Each of the integrands is positive, because $f(\mathbf{x})$ is known to be positive and the other quantities are squared. Therefore, in order to satisfy the equality, we must have $\nabla\psi = 0$ and $\hat{\mathbf{n}} \cdot \nabla\psi = 0$. Feeding the latter into (2), we conclude that $\psi = 0$. Therefore, we have shown that $\phi = 0$ is indeed the only solution to the Laplace equation subject to the boundary condition (1). □

Problem 2. Consider the following classical model of a hydrogen atom. The proton is taken to be a uniformly charged ball of radius $R = 10 \times 10^{-13}\,\text{cm}$ and total charge $e = 4.8 \times 10^{-10}\,\text{esu}$. The electron is taken to have charge density $\rho = -e|\psi|^2$, where ψ is the ground state wave function, $\psi(r) = e^{-r/a}/\sqrt{\pi a^3}$, with $a = \hbar/me^2 = 5.3 \times 10^{-9}\,\text{cm}$.

2.a What is the electromagnetic energy of the proton? Compare this with the mass of the proton.

Solution. From elementary electrostatics, the electric field due to the proton is given by

$$\mathbf{E}(\mathbf{x}) = \begin{cases} \dfrac{er}{R^3}\,\hat{\mathbf{r}} & r < R, \\ \dfrac{e}{r^2}\,\hat{\mathbf{r}} & r > R, \end{cases}$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction and r is the radial spherical coordinate. Then the electrostatic potential is given by

$$\phi(\mathbf{x}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r}' = \begin{cases} \dfrac{e}{2R^3}(3R^2 - r^2) & r < R, \\ \dfrac{e}{r} & r > R. \end{cases}$$

Let ρ_p be the charge density, which is given by

$$\rho(\mathbf{x}) = \begin{cases} \dfrac{3e}{4\pi R^3} & r < R, \\ 0 & r > R. \end{cases}$$

The total electrostatic energy is then given by Eq. (2.25) in the lecture notes. For the proton,

$$\begin{aligned} \mathscr{E} &= \frac{1}{2}\int \phi(\mathbf{x})\, \rho_p(\mathbf{x})\, d^3x \\ &= \frac{1}{2}\int_0^R \frac{3e}{4\pi R^3}\frac{e}{2R^3}(3R^2 - r^2)4\pi r^2\, dr = \frac{3e^2}{4R^6}\int_0^R (3R^2r^2 - r^4)\, dr = \frac{3e^2}{4R^6}\left[R^2r^3 - \frac{1}{5}r^5\right]_0^R = \frac{3e^2}{4R^6}\left(R^5 - \frac{R^5}{5}\right) \\ &= \frac{3e^2}{5R}. \end{aligned}$$

Plugging in numbers, we have

$$\mathscr{E} = \frac{3(4.8 \times 10^{-10}\,\mathrm{esu})^2}{5(10 \times 10^{-13}\,\mathrm{cm})} = 1.4 \times 10^{-7}\,\mathrm{erg}.$$

The mass of the proton is $m_p = 1.673 \times 10^{-24}$ g, so the fraction of the proton mass due to electromagnetic energy is

$$\frac{\mathscr{E}}{m_pc^2} = \frac{1.4 \times 10^{-7}\,\mathrm{erg}}{(1.673 \times 10^{-24}\,\mathrm{g})(3.00 \times 10^{10}\,\mathrm{cm\,s^{-1}})^2} = 9 \times 10^{-5}.$$

This makes sense, because the proton's mass is primarily due to the strong interactions that bind its constituent quarks.

2.b What is the electromagnetic interaction energy of the proton and electron? (Since $R \ll a$, you may treat the proton as a point charge located at the origin.) Compare this with the ground state energy of hydrogen.

Solution. For this model, the electrostatic potential due to the proton is simply

$$\phi_p(\mathbf{x}) = \frac{e}{r}.$$

Let ρ_e denote the charge density of the electron. The electromagnetic interaction energy between two charged bodies is given by Eq. (2.30) in the course notes. For the proton and electron, this is

$$\begin{aligned} \mathscr{E}_{\text{int}} &= \int \rho_e \phi_p\, d^3x \\ &= \int_0^\infty -e\left|\frac{e^{-r/a}}{\sqrt{\pi a^3}}\right|^2 \frac{e}{r}4\pi r^2\, dr = -\frac{4e^2}{a^3}\int_0^\infty re^{-2r/a}\, dr = -\frac{4e^2}{a^3}\frac{a^2}{4}\left[e^{-2r/a}\right]_0^\infty \\ &= -\frac{e^2}{a}. \end{aligned}$$

Plugging in numbers, we have

$$\mathscr{E}_{\text{int}} = -\frac{(4.8\times10^{-10}\,\text{esu})^2}{5.3\times10^{-9}\,\text{cm}} = -4.35\times10^{-11}\,\text{erg}.$$

The ground state energy of hydrogen is $E = 13.6\,\text{eV}$, which is equivalent to 2.18×10^{-11} erg. Then the fraction of the ground state energy due to the electromagnetic interaction is

$$\frac{\mathscr{E}_{\text{int}}}{E} = \frac{-4.35\times10^{-11}\,\text{erg}}{2.18\times10^{-11}\,\text{erg}} = -2,$$

so we see that the interaction energy has twice the magnitude of the ground state energy. This makes sense when we consider the virial theorem $T = -U/2$, where T is the electron's kinetic energy and U its potential energy. If we consider $\mathscr{E}_{\text{int}}$ as the electron's potential energy, then its kinetic energy is equal in magnitude to the ground state energy of hydrogen.

Problem 3. The potential of an electrostatic dipole of dipole moment $\mathbf{p}$ located at $\mathbf{x}'$ is given by

$$\phi(\mathbf{x}) = \frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{4}$$

Suppose a dipole $\mathbf{p}_1$ is located at $\mathbf{x}_1$ and another dipole $\mathbf{p}_2$ is located at $\mathbf{x}_2$.

3.a What is the electrostatic force on the second dipole?

Solution. The total force on a charged body in electrostatics is given by Eq. (2.42) in the text,

$$\mathbf{F} = \int \rho(\mathbf{x})\mathbf{E}_0(\mathbf{x})\,d^3x\,,$$

where ρ is the charge density of the body and $\mathbf{E}_0$ is the external field. For the force on $\mathbf{p}_2$, this becomes

$$\mathbf{F} = \int \rho_2(\mathbf{x})\mathbf{E}_0(\mathbf{x})\,d^3x\,.$$

Since $\mathbf{p}_2$ consists of two point charges,

$$\rho_2(\mathbf{x}) = Q_2[\delta(\mathbf{x}_2) - \delta(\mathbf{x}_2-\mathbf{d}_2)], \tag{5}$$

where we have defined the displacement vector of $\mathbf{p}_2$ as $\mathbf{d}_2$. Let ϕ_1 be the potential of $\mathbf{p}_1$; that is,

$$\phi_1(\mathbf{x}) = \frac{\mathbf{p}_1\cdot(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}. \tag{6}$$

Let $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$, and assume $d_2 \ll r$. The force on $\mathbf{p}_2$ is given by

$$\begin{aligned}
\mathbf{F} &= -\int Q_2[\delta(\mathbf{x}_2) - \delta(\mathbf{x}_2-\mathbf{d}_2)]\nabla\phi_1\,d^3x \\
&= Q_2\left[\nabla\left(\frac{\mathbf{p}_1\cdot(\mathbf{x}_2-\mathbf{d}_2-\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{d}_2-\mathbf{x}_1|^3}\right) - \nabla\left(\frac{\mathbf{p}_1\cdot(\mathbf{x}_2-\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{x}_1|^3}\right)\right] = Q_2\nabla\left(\frac{\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{d}_2)}{|\mathbf{r}-\mathbf{d}_2|^3} - \frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3}\right) \\
&\approx Q_2\nabla\left[\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3} - \mathbf{p}_1\cdot(\mathbf{r}-\mathbf{d}_2)\left(\frac{1}{r^3} + \frac{3\mathbf{d}_2\cdot\mathbf{r}}{r^5}\right)\right] = Q_2\nabla\left(\frac{\mathbf{p}_1\cdot\mathbf{r}}{r^3} - \frac{\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{d}_2)}{r^3} + \frac{3\mathbf{p}_1\cdot(\mathbf{r}-\mathbf{d}_2)(\mathbf{d}_2\cdot\mathbf{r})}{r^5}\right) \\
&\approx Q_2\nabla\left(\frac{\mathbf{p}_1\cdot\mathbf{d}_2}{r^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{d}_2\cdot\mathbf{r})}{r^5}\right) = \nabla\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})}{r^5}\right) \\
&= \nabla\left(\frac{3[\mathbf{p}_1\cdot(\mathbf{x}_2-\mathbf{x}_1)][\mathbf{p}_2\cdot(\mathbf{x}_2-\mathbf{x}_1)]}{|\mathbf{x}_2-\mathbf{x}_1|^5} - \frac{\mathbf{p}_1\cdot\mathbf{p}_2}{|\mathbf{x}_2-\mathbf{x}_1|^3}\right),
\end{aligned}$$

where we have used the Taylor series expansion

$$\frac{1}{|\mathbf{r}-\mathbf{d}_2|^3} = \frac{1}{r^3} + \frac{3\mathbf{d}_2 \cdot \mathbf{r}}{r^5} + \cdots$$

and neglected terms of $\mathcal{O}\big(d^2\big)$.

3.b What is the electrostatic interaction energy of the two dipoles?

Solution. Equation (2.30) in the course notes gives the interaction energy of two charge distributions:

$$\mathscr{E}_{\text{int}} = \int \rho_2 \phi_1 \, d^3x \,, \tag{7}$$

where ρ_2 is the charge density of $\mathbf{p}_2$ and ϕ_1 is given by (6). Applying (5), (7) becomes

$$\begin{aligned}\mathscr{E}_{\text{int}} &= Q_2 \int [\delta(\mathbf{x}_2) - \delta(\mathbf{x}_2 - \mathbf{d}_2)]\phi_1 \, d^3x = Q_2 \left(\frac{\mathbf{p}_1 \cdot \mathbf{r}}{r^3} - \frac{\mathbf{p}_1 \cdot (\mathbf{r} - \mathbf{d}_2)}{|\mathbf{r} - \mathbf{d}_2|^3} \right) \\ &= Q_2 \left(\frac{3[\mathbf{p}_1 \cdot (\mathbf{x}_2 - \mathbf{x}_1)][\mathbf{p}_2 \cdot (\mathbf{x}_2 - \mathbf{x}_1)]}{|\mathbf{x}_2 - \mathbf{x}_1|^5} - \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{x}_2 - \mathbf{x}_1|^3} \right),\end{aligned}$$

where we have repeated the calculations of 3.a. So we see that the force on the second dipole may be found by $F = -\nabla \mathscr{E}_{\text{int}}$.

In addition to the course lecture notes, I consulted Griffiths's *Introduction to Electrodynamics* and David Tong's lecture notes on electrodynamics while writing up these solutions.