Problem 1. (Peskin & Schroeder 2.1) Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

1(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

1(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\nu}A^\nu,$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{E^2 + B^2}{2}; \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

Problem 2. The complex scalar field (Peskin & Schroeder 2.2) Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x \, (\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi).$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

2(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \, (\pi^*\pi + \boldsymbol{\nabla}\phi^* \cdot \boldsymbol{\nabla}\phi + m^2\phi^*\phi).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

2(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

2(c) Rewrite the conserved charge

$$Q = \int d^3x \, \frac{i}{2}(\phi^*\pi^* - \pi\phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

2(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part 2(c), and the other three given by

$$Q^i = \int d^3x \, \frac{i}{2}(\phi_a^* \sigma^i{}_{ab} \pi_b^* - \pi_a \sigma^i{}_{ab} \phi_b),$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

Problem 3. (Peskin & Schroeder 2.3) Evaluate the function

$$\langle 0|\phi(x)\,\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{ip\,(x-y)},$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicitly in terms of Bessel functions.

Problem 4. The classical limit of a harmonic oscillator can be described in terms of *coherent states*,

$$|\alpha\rangle = \exp\left(\alpha a^\dagger - \frac{1}{2}|\alpha|^2\right)|0\rangle\,.$$

When α is large, the oscillator state is semiclassical. Proceeding similarly for the Fourier modes of the quantum Klein-Gordon field,

$$|f\rangle = N_f \exp\left(i \int \frac{d^3p}{(2\pi)^3} f(\mathbf{p})\, a_{\mathbf{p}}^\dagger\right)|0\rangle\,, \qquad N_f = \exp\left(-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}|f(\mathbf{p})|^2\right).$$

4(a) Evaluate the expectation value of the field operator $\langle f|\phi(x)|f\rangle$ and show that it satisfies the Klein-Gordon equation.

4(b) Evaluate the relative mean square fluctuation of the occupation number of the mode with momentum $\mathbf{p}$ and the relative mean square fluctuation in the total energy:

$$\frac{\langle \hat{n}_{\mathbf{p}}^2\rangle - \langle \hat{n}_{\mathbf{p}}\rangle^2}{\langle \hat{n}_{\mathbf{p}}\rangle^2}, \qquad \frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2}.$$

Is either of these a good measure of the degree to which the field is classical? Justify your answer.

4(c) Take $\Delta(x-y) = \langle 0|\phi(\mathbf{x})\,\phi(\mathbf{y})|0\rangle$ (equal times) as a measure of the fluctuations or correlations of the field amplitude. Use your result from problem 3 to evaluate this quantity. What is the meaning of the divergence as $\mathbf{x} \to \mathbf{y}$?