Problem 1. Consider the following probabalistic game: There are four doors (Q, R, S, T). Behind each door is a device which displays ± 1 randomly according to the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$. Alice and Bob are on the same team. Alice has to choose either Q and R, and then Bob has to choose either S and T. When the numbers match, they get $+1$ point; when the numbers do not match, they get -1 point. However, when they open Q and T, it's an exception. When the numbers (do not) match, they get -1 $(+1)$.

1.1 Let's assume Alice and Bob open the doors completely randomly. When all numbers are $+1$ with probability 1, what is the expectation value of the point they get?

Solution. Let $\mathbf{E}$ be the expectation value of the number of points. In this case, the numbers behind the two doors will always match. So

$$\mathbf{E} = \frac{QS + RS + RT - QT}{4} = \frac{1 + 1 + 1 - 1}{4} = \frac{1}{2}.$$

1.2 As it turns out, irrespective of how hard you fine tune the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$, the expectation value of the point Alice and Bob get cannot exceed a certain value Max:

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} \le \text{Max}.$$

Here, $\mathbf{E}(QS)$, etc. is the expectation value of the point when Alice opens Q and Bob opens S. This is a Bell inequality. Determine Max.

Hint: For a given realization of the numbers $Q = \pm 1$, $R = \pm 1$, $S = \pm 1$, $T = \pm 1$, which occurs with probability $P(Q, R, S, T)$, note that $QS + RS + RT - QT = (Q + R)S + (R - Q)T$, where one of $\{(R + Q), (R - Q)\}$ is 2 and the other 0.

Solution. In addition to the information provided in the hint, both S and T must be ± 1. This means the only possibilities for the number of points earned are

$$\frac{(Q + R)S + (R - Q)T}{4} = \begin{cases} \dfrac{(0)(-1) + (2)(1)}{4} = \dfrac{1}{2}, \\ \dfrac{(0)(1) + (2)(-1)}{4} = -\dfrac{1}{2}. \end{cases}$$

Thus,

$$\text{Max} = \frac{1}{2}.$$

1.3 Frustrated by the upper bound set by the Bell inequality, Bob decides to cheat. He now changes the value of T after Alice chooses Q or R. Assume Q, R, S are set to be $+1$ with probability 1. To make the expectation value of the point they get equal to $+1$, what values should Bob set after Alice chooses Q or R?

Solution. If Alice chooses R, Bob should set $T = 1$. If Alice chooses Q, Bob should set $T = -1$. This way,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1 + 1 + 1 + 1}{4} = 1.$$

1.4 Now consider a quantum mechanical version of the game. There are quantum states of two spin-1/2 degrees of freedom shared by Alice and Bob. Alice can measure the z component or x components of the first spin $\mathbf{S}^A$. (This corresponds to $Q = \pm 1$ or $R = \pm 1$.) Bob can measure the $-(z + x)$ component or the $(z - x)$ component of the second spin $\mathbf{S}^B$. (This corresponds to $S = \pm 1$ or $T = \pm 1$.)

More specifically, Alice and Bob share the quantum state

$$|\psi\rangle = \frac{|\uparrow_z\rangle \otimes |\downarrow_z\rangle - |\downarrow_z\rangle \otimes |\uparrow_z\rangle}{\sqrt{2}}.$$

The operators to be measured are

$$Q = S_z^A, \qquad R = S_x^A, \qquad S = -\frac{S_z^B + S_x^B}{\sqrt{2}}, \qquad T = \frac{S_z^B - S_x^B}{\sqrt{2}}.$$

Let us consider the case when Alice measures Q and Bob measures T. Calculate the probability $P(Q, T)$ for Alice and Bob getting the measurement outcomes $(Q, T) = (\pm 1, \pm 1)$.

Solution. From Sakurai (3.9.11), the probability of measuring $\mathbf{S} \cdot \hat{\mathbf{a}}$ and $\mathbf{S} \cdot \hat{\mathbf{b}}$ to both be positive is

$$P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+) = \frac{1}{2} \sin^2\left(\frac{\theta_{ab}}{2}\right),$$

where the 1/3 comes from the probability of measuring θ_{ab} is the angle between the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ directions. For the other combinations, we may generalize this expression using Fig. (3.9) in Sakurai: This gives us

$$P(\hat{\mathbf{a}}-; \hat{\mathbf{b}}-) = \frac{1}{2} \sin^2\left(\frac{\theta_{ab}}{2}\right) = P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}+), \quad P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}-) = \frac{1}{2} \sin^2\left(\frac{\theta_{ab} + \pi/2}{2}\right) = \frac{1}{2} \cos^2\left(\frac{\theta_{ab}}{2}\right) = P(\hat{\mathbf{a}}+; \hat{\mathbf{b}}-). \tag{1}$$

For Q and T, $\theta_{ab} = \pi/4$. So we have

$$P(Q = \pm 1, T = \pm 1) = \frac{1}{2} \sin^2\left(\frac{\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2 - \sqrt{2}}}{2}\right)^2 = \frac{1}{2} \frac{2 - \sqrt{2}}{4} = \frac{2 - \sqrt{2}}{8} \approx 0.073,$$
$$P(Q = \pm 1, T = \mp 1) = \frac{1}{2} \cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2 + \sqrt{2}}}{2}\right)^2 = \frac{1}{2} \frac{2 + \sqrt{2}}{4} = \frac{2 + \sqrt{2}}{8} \approx 0.427.$$

1.5 Similarly, consider the case when Alice measures R and Bob measures T. Calculate the probability $P(R, T)$ for Alice and Bob getting the measurement outcomes $(R, T) = (\pm 1, \pm 1)$.

Solution. Again applying (1), for R and T, $\theta_{ab} = 3\pi/4$. So we have

$$P(R = \pm 1, T = \pm 1) = \frac{1}{2} \sin^2\left(\frac{3\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2 + \sqrt{2}}}{2}\right)^2 = \frac{1}{2} \frac{2 + \sqrt{2}}{4} = \frac{2 + \sqrt{2}}{8} \approx 0.427,$$
$$P(R = \pm 1, T = \mp 1) = \frac{1}{2} \cos^2\left(\frac{3\pi}{8}\right) = \frac{1}{2} \left(\frac{\sqrt{2 - \sqrt{2}}}{2}\right)^2 = \frac{1}{2} \frac{2 - \sqrt{2}}{4} = \frac{2 - \sqrt{2}}{8} \approx 0.073.$$

1.6 Compute the expectation values $\mathbf{E}(QS)$, $\mathbf{E}(RS)$, $\mathbf{E}(QT)$, and $\mathbf{E}(RT)$. Compute

$$\frac{\mathbf{E}(QS)+\mathbf{E}(RS)+\mathbf{E}(RT)-\mathbf{E}(QT)}{4}.$$

Solution. We need to find the probabilities of obtaining $(Q,S)=(\pm1,\pm1)$ and $(R,S)=(\pm1,\pm1)$. For Q and S, $\theta_{ab}=3\pi/4$, so

$$P(Q=\pm1,S=\pm1)=P(R=\pm1,T=\pm1), \qquad P(Q=\pm1,S=\mp1)=P(R=\pm1,T=\mp1).$$

For R and S, $\theta_{ab}=5\pi/4$, so

$$P(R=\pm1,S=\pm1)=\frac{1}{2}\sin^2\left(\frac{5\pi}{8}\right)=\frac{1}{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^2=\frac{1}{2}\frac{2+\sqrt{2}}{4}=\frac{2+\sqrt{2}}{8}\approx 0.427,$$
$$P(R=\pm1,S=\mp1)=\frac{1}{2}\cos^2\left(\frac{5\pi}{8}\right)=\frac{1}{2}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)^2=\frac{1}{2}\frac{2-\sqrt{2}}{4}=\frac{2-\sqrt{2}}{8}\approx 0.073.$$

The expectation value of a random variable X is defined

$$E(X)=\sum_i p_i x_i,$$

where x_i are all of the possible values of X, and p_i the probabilities associated with each. Then

$$\mathbf{E}(QS)=2P(Q=\pm1,S=\pm1)-2P(Q=\pm1,S=\mp1)=\frac{2+\sqrt{2}}{4}-\frac{2-\sqrt{2}}{4}=\frac{\sqrt{2}}{2},$$
$$\mathbf{E}(RS)=2P(R=\pm1,S=\pm1)-2P(R=\pm1,S=\mp1)=\frac{2+\sqrt{2}}{4}-\frac{2-\sqrt{2}}{4}=\frac{\sqrt{2}}{2},$$
$$\mathbf{E}(RT)=2P(R=\pm1,T=\pm1)-2P(R=\pm1,T=\mp1)=\frac{2+\sqrt{2}}{4}-\frac{2-\sqrt{2}}{4}=\frac{\sqrt{2}}{2},$$
$$\mathbf{E}(QT)=2P(Q=\pm1,T=\pm1)-2P(Q=\pm1,T=\mp1)=\frac{2-\sqrt{2}}{4}-\frac{2+\sqrt{2}}{4}=-\frac{\sqrt{2}}{2}.$$

Finally,

$$\frac{\mathbf{E}(QS)+\mathbf{E}(RS)+\mathbf{E}(RT)-\mathbf{E}(QT)}{4}=\frac{1}{4}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2},$$

which is greater than Max, thereby violating Bell's inequality.

Problem 2. Consider a quantum particle with mass m moving in the presence of the square well potential

$$V(r) = \begin{cases} -V_0 & r \leq a, \\ 0 & r > a. \end{cases}$$

2.1 Writing the wave function in polar coordinates as $\psi(\mathbf{r}) = R_l(r)\, Y_{lm}(\theta, \phi)$, write down the Schrödinger equation obeyed by R_l.

Solution. From (A.5.1) in Sakurai, the full Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi_E}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \psi_E}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi_E}{\partial \phi^2}\right] + V(r)\,\psi_E = E\psi_E,$$

where the angular part of ψ_E satisfies (A.5.4),

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm} = l(l+1)Y_{lm}.$$

Then the equivalent one-dimensional Schrödinger equation is the equation immediately following (A.5.8),

$$-\frac{\hbar^2}{2m}\frac{d^2 u_E}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right] u_E = E u_E, \tag{2}$$

where $u_E(r) = rR(r)$. In terms of R_l,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}(rR_l) + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right] rR_l = ErR_l.$$

or

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r) + \frac{l(l+1)}{r^2}\right] R_l(r) = E_l\, R_l(r).$$

From (7.7.1), the effective potential at low energies for the lth partial wave is

$$V_{\text{eff}} = V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2},$$

so the Schrödinger equation can be rewritten as

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V_{\text{eff}}\right] R_l(r) = E_l\, R_l(r).$$

2.2 When V_0 is a certain value, there is one bound state for the s wave ($l = 0$). The bound state energy ε is small ($0 < |\varepsilon| \ll V_0$). Obtain the range of the depth of the well V_0 ($? \leq V_0 < ?$). Also, calculate for the bound state the probability for the particle to exist outside of the well.

Solution. Inside the well, R_l are given by (A.5.16),

$$R_l(r) = \text{constant } j_l(\alpha r),$$

where α is defined in Eq. (A.5.17),

$$\alpha = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}, \quad r < a,$$

and the spherical Bessel functions j_l is given by (A.5.12),

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+1/2}(\rho).$$

For the s wave, the relevant Bessel function is given by (A.5.12),

$$j_0(\rho) = \frac{\sin\rho}{\rho}.$$

But for $l - 0$, V_{eff} reduces to $V(r)$, so (2) reduces to the one-dimensional problem for u_E,

$$-\frac{\hbar^2}{2m}\frac{d^2u_E}{dr^2} + V(r)u_E = Eu_E.$$

The bound-state solutions are given by (A.2.6),

$$u_E \sim \begin{cases} e^{-\kappa r} & & \text{for } r > a, \\ \cos kr & \text{(even parity)} & \text{for } r < a, \\ \sin kr & \text{(even parity)} & \text{for } r > a, \end{cases}$$

where k and κ are defined by (A.2.7),

$$k = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}, \qquad \kappa = \sqrt{\frac{2m|E|}{\hbar^2}}.$$

So we see that $\alpha = k$, and thus we are interested in the odd-parity solutions to the one-dimensional problem.

For the one-dimensional problem, the allowed values of bound-state energy

$$E = -\frac{\hbar^2\kappa^2}{2m}$$

can be found by solving (A.2.8),

$$ka\tan ka = \kappa a \quad \text{(even parity)}, \qquad ka\cot ka = -\kappa a \quad \text{(odd parity)}.$$

We are interested in the odd parity solutions, so

$$\frac{\pi^2\hbar^2}{8ma^2} < V_0 < \infty.$$

For some reason, $l = 0$ has odd parity.

2.3 Consider the scattering problem by the well. For each l, for large enough r, when $R_l(r)$ is given by $R_l(r) \sim A_l \sin(kr - l\pi/2 + \delta_l)/r$, δ_l is called the scattering phase shift. For the value of V_0 within the range you obtained in the above problem, when the energy of the incident wave is is $E = 9V_0/16$, calculate $\tan\delta_0$ (where δ_0 is the scattering phase shift for the s wave).

2.4 Now consider the S matrix, $S \equiv \exp(2i\delta_0) = \exp(i\delta_0)/\exp(-i\delta_0)$. Compare the condition on s wave bound state energies and the zero of the denominator of S. Explain their relation.

Problem 3. Consider a three dimensional potential

$$V(|r|) = \frac{\hbar^2\gamma}{2m}\delta(|r| - a).$$

The s wave Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\chi_0(r)}{dr^2} + \frac{\hbar^2\gamma}{2m}\delta(r-a)\,\chi_0(r) = E\,\chi_0(r).$$

The s wave function must be regular (zero) at $r = 0$. At $r = a$, it is continuous, but its derivative can jump.

3.1 Calculate the s wave scattering phase shift (k), where k is related to E as $E = \hbar^2k^2/2m$.

3.2 When $\gamma \gg k$, $1/a$ and when $\sin ka$ is not small, discuss the behavior of the scattering phase shift.

3.3 Obtain the condition to have resonant states and calculate the energy of the resonant states.

3.4 Calculate the width Γ of the resonance. Discuss its behavior when γ is big.

3.5 When the velocity of the incident wave is small, obtain the scattering cross section.

I consulted Sakurai's *Modern Quantum Mechanics*, Shankar's *Principles of Quantum Mechanics*, and the Wikipedia article on a particle in a spherically symmetric potential while writing up these solutions.