

Problem 1. (Jackson 14.1) Verify by explicit calculation that the Liénard-Wiechert expressions for *all* components of \mathbf{E} and \mathbf{B} for a particle moving with constant velocity agree with the ones obtained in the text by means of a Lorentz transformation. Follow the general method at the end of Section 14.1.

Solution. The Liénard-Wiechert expressions for the fields are given by Jackson (14.13–14):

$$\mathbf{B} = [\hat{\mathbf{n}} \times \mathbf{E}]_{\text{ret}}, \quad \mathbf{E}(\mathbf{x}, t) = e \left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \right]_{\text{ret}} + \frac{e}{c} \left[\frac{\hat{\mathbf{n}} \times \{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R} \right]_{\text{ret}}, \quad (1)$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ with \mathbf{v} being the particle's velocity, R is the distance from the observation point to the particle's position, and $\hat{\mathbf{n}}$ is a unit vector defined by $\mathbf{x} - \mathbf{r}(\tau) = R \hat{\mathbf{n}}$. Here, $\mathbf{r}(\tau)$ is the particle's present position and τ the proper time.

The expressions for the components of \mathbf{E} and \mathbf{B} obtained by a Lorentz transformation are given by Jackson (11.152):

$$E_1 = -\frac{e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \quad E_2 = \frac{e\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \quad E_3 = B_1 = B_2 = 0, \quad B_3 = \beta E_2, \quad (2)$$

where the particle is moving in the x_1 direction at impact parameter b on the x_2 axis, as shown in Fig. (1).

For a particle moving with constant velocity in the x_1 direction with velocity v as shown in Fig. (1), $\boldsymbol{\beta} = \beta \hat{\mathbf{x}}_1$ and $\dot{\boldsymbol{\beta}} = 0$. From Jackson (14.16), note that

$$(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2 R^2 = b^2 + v^2 t^2 - \beta^2 b^2 = \frac{b^2 + \gamma^2 v^2 t^2}{\gamma^2} \implies (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2 = \frac{(b^2 + \gamma^2 v^2 t^2)^{3/2}}{R\gamma^3}.$$

This calculation comes from Fig. (2), where O is the observation point, P is the present position of the particle, and P' its retarded position. Also from Fig. 2,

$$\hat{\mathbf{n}} = \cos \theta \hat{\mathbf{x}}_1 + \sin \theta \hat{\mathbf{x}}_2 = \frac{\beta R - vt}{R} \hat{\mathbf{x}}_1 + \frac{b}{R} \hat{\mathbf{x}}_2.$$

Making these substitutions in the expression for $\mathbf{E}(\mathbf{x}, t)$ in Eq. (1),

$$\mathbf{E}(\mathbf{x}, t) = e \left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \right]_{\text{ret}} = e \left[\frac{(\beta - vt/R - \beta) \hat{\mathbf{x}}_1 + (b/R) \hat{\mathbf{x}}_2}{\gamma^2(b^2 + \gamma^2 v^2 t^2)^{3/2}} R\gamma^3 \right]_{\text{ret}} = e\gamma \frac{-vt \hat{\mathbf{x}}_1 + b \hat{\mathbf{x}}_2}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \quad (3)$$

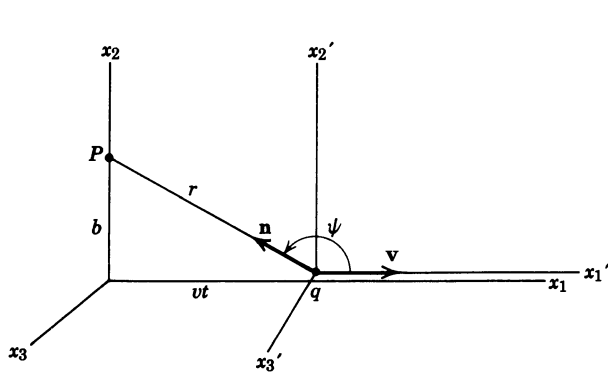


Figure 1: (Jackson Fig. 11.8) Particle of charge q moving at constant velocity \mathbf{v} passes an observation point P at impact parameter b .

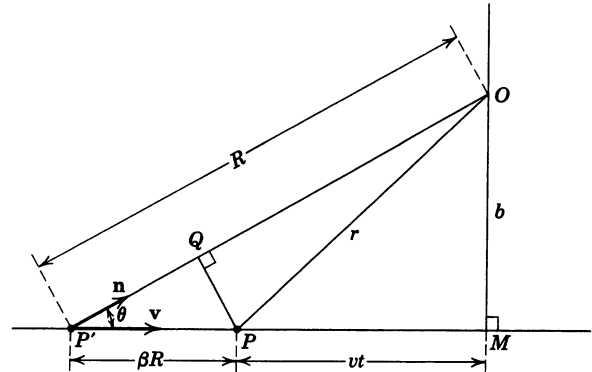


Figure 2: (Jackson Fig. 14.2) Present and retarded positions of a charge in uniform motion.

For $\mathbf{B}(\mathbf{x}, t)$, note that

$$\hat{\mathbf{n}} \times \mathbf{E} \propto \left(\frac{\beta R - vt}{R} \hat{\mathbf{x}}_1 + \frac{b}{R} \hat{\mathbf{x}}_2 \right) \times (-vt \hat{\mathbf{x}}_1 + b \hat{\mathbf{x}}_2) = \left(b \frac{\beta R - vt}{R} + \frac{bvt}{R} \right) \hat{\mathbf{x}}_3 = \beta b,$$

so

$$\mathbf{B}(\mathbf{x}, t) = e\gamma \frac{\beta b \hat{\mathbf{x}}_3}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \quad (4)$$

Writing Eqs. (3) and (4) in component notation, we find

$$\begin{aligned} E_1 &= -\frac{e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, & E_2 &= \frac{e\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, & E_3 &= 0, \\ B_1 &= 0, & B_2 &= 0, & B_3 &= \frac{e\gamma \beta b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \beta E_2, \end{aligned}$$

which are identical to Eq. (2) as was to be shown. \square

Problem 2. (Jackson 14.3) The Heaviside-Feynman expression for the electric field of a particle of charge e in arbitrary motion, an alternative to the Liénard-Wiechert expression in Eq. (1), is

$$\mathbf{E} = e \left[\frac{\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} + e \left[\frac{R}{c} \right]_{\text{ret}} \frac{d}{dt} \left[\frac{\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} + \frac{e}{c^2} \frac{d^2}{dt^2} [\hat{\mathbf{n}}]_{\text{ret}}, \quad (5)$$

where the time derivatives are with respect to the time at the observation point. Using the fact that the retarded time is $t' = t - R(t')/c$ and that, as a result,

$$\frac{dt}{dt'} = 1 - \beta(t') \cdot \hat{\mathbf{n}}(t'),$$

show that the form above yields the expression for \mathbf{E} in Eq. (1) when the time differentiations are performed.

Solution. Since $R(t') = c(t - t')$, note that

$$\left[\frac{dR}{dt} \right]_{\text{ret}} = \frac{dR(t')}{dt'} = c \left(\frac{dt}{dt'} - 1 \right) = c(1 - \beta(t') \cdot \hat{\mathbf{n}}(t') - 1) = -[\hat{\mathbf{n}} \cdot \mathbf{v}]_{\text{ret}}.$$

The definition of $\hat{\mathbf{n}}$ is given on p. 663 of Jackson as $\hat{\mathbf{n}} = [\mathbf{x} - \mathbf{r}(\tau)]/R$. From Jackson (11.26), $d\tau = dt/\gamma$. Then

$$\frac{d\mathbf{r}(\tau)}{dt} = \frac{1}{\gamma} \frac{d\mathbf{r}(\tau)}{d\tau} = \frac{\mathbf{v}}{\gamma}, \quad \frac{d\mathbf{v}(\tau)}{dt} = \frac{\dot{\mathbf{v}}}{\gamma},$$

and

$$\begin{aligned} \frac{d^2 \mathbf{r}(\tau)}{dt^2} &= \frac{d}{dt} \left(\frac{\mathbf{v}(\tau)}{\gamma} \right) = \frac{1}{\gamma} \frac{d}{d\tau} \left(\mathbf{v}(\tau) \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \right) = \frac{\dot{\mathbf{v}}}{\gamma} \left(\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - \frac{\mathbf{v}^2}{c^2 \sqrt{1 - \mathbf{v}^2/c^2}} \right) = \frac{\dot{\mathbf{v}}}{\gamma^2} - \beta^2 \dot{\mathbf{v}} \\ &= \frac{\dot{\mathbf{v}}}{\gamma^2} - \left(1 - \frac{1}{\gamma^2} \right) \dot{\mathbf{v}} = 2 \frac{\dot{\mathbf{v}}}{\gamma^2} - \dot{\mathbf{v}}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d}{dt'} [\hat{\mathbf{n}}]_{\text{ret}} &= \left[\frac{d\hat{\mathbf{n}}}{dt} \right]_{\text{ret}} = \left[\frac{1}{R^2} \left(R \frac{d}{dt} [\mathbf{x} - \mathbf{r}(\tau)] - [\mathbf{x} - \mathbf{r}(\tau)] \frac{dR}{dt} \right) \right]_{\text{ret}} = \left[\frac{1}{R^2} \left(-R \frac{\mathbf{v}}{\gamma} + R(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}} \right) \right]_{\text{ret}} \\ &= \left[\frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{R} - \frac{\mathbf{v}}{\gamma R} \right]_{\text{ret}}, \end{aligned}$$

$$\begin{aligned}\frac{d}{dt'} \left[\frac{\hat{\mathbf{n}}}{R} \right]_{\text{ret}} &= \left[\frac{d}{dt} \left(\frac{\hat{\mathbf{n}}}{R} \right) \right]_{\text{ret}} = \left[\frac{1}{R^2} \left(R \frac{d\hat{\mathbf{n}}}{dt} - \hat{\mathbf{n}} \frac{dR}{dt} \right) \right]_{\text{ret}} = \left[\frac{1}{R^2} \left\{ R \left(\frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R} - \frac{\mathbf{v}}{\gamma R} \right) + (\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}} \right\} \right]_{\text{ret}} \\ &= \left[\frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}}{\gamma R^2} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} = \left[2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}}{\gamma R^2} \right]_{\text{ret}},\end{aligned}$$

$$\begin{aligned}\frac{d}{dt'} \left[\frac{\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} &= \left[\frac{d}{dt} \left(\frac{\hat{\mathbf{n}}}{R^2} \right) \right]_{\text{ret}} = \left[\frac{1}{R^2} \left\{ R \frac{d}{dt} \left(\frac{\hat{\mathbf{n}}}{R} \right) - \frac{\hat{\mathbf{n}}}{R} \frac{dR}{dt} \right\} \right]_{\text{ret}} = \left[\frac{1}{R^2} \left\{ R \left(2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}}{\gamma R^2} \right) + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R} \right\} \right]_{\text{ret}} \\ &= \left[2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^3} - \frac{\mathbf{v}}{\gamma R^3} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^3} \right]_{\text{ret}} = \left[3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^3} - \frac{\mathbf{v}}{\gamma R^3} \right]_{\text{ret}}.\end{aligned}$$

For the second derivative of $\hat{\mathbf{n}}$, note that

$$\begin{aligned}\frac{d}{dt'} [\hat{\mathbf{n}} \cdot \mathbf{v}]_{\text{ret}} &= \left[\frac{d}{dt} (\hat{\mathbf{n}} \cdot \mathbf{v}) \right]_{\text{ret}} = \left[\hat{\mathbf{n}} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\hat{\mathbf{n}}}{dt} \cdot \mathbf{v} \right]_{\text{ret}} = \left[\hat{\mathbf{n}} \cdot \frac{\dot{\mathbf{v}}}{\gamma} + \left(\frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R} - \frac{\mathbf{v}}{\gamma R} \right) \cdot \mathbf{v} \right]_{\text{ret}} \\ &= \left[\frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{\gamma} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{R} - \frac{\mathbf{v}^2}{\gamma R} \right]_{\text{ret}},\end{aligned}$$

$$\begin{aligned}\frac{d}{dt'} \left[\frac{\mathbf{v}}{\gamma R} \right]_{\text{ret}} &= \left[\frac{d}{dt} \left(\frac{\mathbf{v}}{\gamma R} \right) \right]_{\text{ret}} = \left[\frac{1}{R^2} \left(R \frac{d}{dt} \left(\frac{\mathbf{v}}{\gamma} \right) - \frac{\mathbf{v}}{\gamma} \frac{dR}{dt} \right) \right]_{\text{ret}} = \left[\frac{1}{R^2} \left\{ R \left(2 \frac{\dot{\mathbf{v}}}{\gamma^2} - \dot{\mathbf{v}} \right) + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{v}}{\gamma} \right\} \right]_{\text{ret}} \\ &= \left[2 \frac{\dot{\mathbf{v}}}{\gamma^2 R} - \frac{\dot{\mathbf{v}}}{R} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{v}}{\gamma R^2} \right]_{\text{ret}},\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dt'^2} [\hat{\mathbf{n}}]_{\text{ret}} &= \left[\frac{d^2 \hat{\mathbf{n}}}{dt^2} \right]_{\text{ret}} = \left[\frac{d}{dt} \left(\frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R} - \frac{\mathbf{v}}{\gamma R} \right) \right]_{\text{ret}} = \left[(\hat{\mathbf{n}} \cdot \mathbf{v}) \frac{d}{dt} \left(\frac{\hat{\mathbf{n}}}{R} \right) + \frac{\hat{\mathbf{n}}}{R} \frac{d}{dt} (\hat{\mathbf{n}} \cdot \mathbf{v}) - \frac{d}{dt} \left(\frac{\mathbf{v}}{\gamma R} \right) \right]_{\text{ret}} \\ &= \left[(\hat{\mathbf{n}} \cdot \mathbf{v}) \left(2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}}{\gamma R^2} \right) + \frac{\hat{\mathbf{n}}}{R} \left(\frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{\gamma} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{R} - \frac{\mathbf{v}^2}{\gamma R} \right) - \left(2 \frac{\dot{\mathbf{v}}}{\gamma^2 R} - \frac{\dot{\mathbf{v}}}{R} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{v}}{\gamma R^2} \right) \right]_{\text{ret}} \\ &= \left[2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}}}{R^2} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{v}}{\gamma R^2} + \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{\gamma R} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}^2 \hat{\mathbf{n}}}{\gamma R^2} - 2 \frac{\dot{\mathbf{v}}}{\gamma^2 R} + \frac{\dot{\mathbf{v}}}{R} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{v}}{\gamma R^2} \right]_{\text{ret}} \\ &= \left[3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}}}{R^2} - 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{v}}{\gamma R^2} + \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{\gamma R} - \frac{\mathbf{v}^2 \hat{\mathbf{n}}}{\gamma R^2} - 2 \frac{\dot{\mathbf{v}}}{\gamma^2 R} + \frac{\dot{\mathbf{v}}}{R} \right]_{\text{ret}}\end{aligned}$$

By the chain rule,

$$\frac{d}{dt} = \frac{dt'}{dt} \frac{d}{dt'} = \frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \frac{d}{dt'} = \left[\frac{1}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} \right]_{\text{ret}} \frac{d}{dt'}.$$

For the second derivative, note that

$$\begin{aligned}\frac{d}{dt'} \left[\frac{1}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} \right]_{\text{ret}} &= \left[\frac{d}{dt} \left(\frac{1}{1 - \mathbf{v} \cdot \hat{\mathbf{n}}/c} \right) \right]_{\text{ret}} = \left[\frac{1}{(1 - \mathbf{v} \cdot \hat{\mathbf{n}}/c)^2} \frac{d}{dt} \left(1 - \frac{\hat{\mathbf{n}} \cdot \mathbf{v}}{c} \right) \right]_{\text{ret}} \\ &= \frac{1}{c} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2} \left(\frac{\mathbf{v}^2}{\gamma R} - \frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{\gamma} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{R} \right) \right]_{\text{ret}},\end{aligned}$$

so

$$\begin{aligned}\frac{d^2}{dt'^2} &= \frac{d}{dt} \left(\frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \frac{d}{dt'} \right) = \frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \frac{d}{dt'} \left(\frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \frac{d}{dt'} \right) \\ &= \frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \left\{ \frac{d}{dt'} \left(\frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \right) \frac{d}{dt'} + \frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')} \frac{d^2}{dt'^2} \right\} \\ &= \frac{1}{c} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3} \left(\frac{\mathbf{v}^2}{\gamma R} - \frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{\gamma} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{R} \right) \right]_{\text{ret}} \frac{d}{dt'} + \left[\frac{1}{[1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}]^2} \right]_{\text{ret}} \frac{d^2}{dt'^2}.\end{aligned}$$

The first term of Eq. (5) can be written

$$e \left[\frac{\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} = e \left[\frac{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 \hat{\mathbf{n}}}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \right]_{\text{ret}} = e \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(\hat{\mathbf{n}} - 3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{c} + 3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}}}{c^2} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^3 \hat{\mathbf{n}}}{c^3} \right) \right]_{\text{ret}}.$$

The second term of Eq. (5) becomes

$$\begin{aligned} e \left[\frac{R}{c} \right]_{\text{ret}} \frac{d}{dt} \left[\frac{\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} &= \frac{e}{c} \left[\frac{R}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} \right]_{\text{ret}} \frac{d}{dt'} \left[\frac{\hat{\mathbf{n}}}{R^2} \right]_{\text{ret}} = \frac{e}{c} \left[\frac{R}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} \left(3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{R^3} - \frac{\mathbf{v}}{\gamma R^3} \right) \right]_{\text{ret}} \\ &= \frac{e}{c} \left[\frac{1}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} \left(3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}}{\gamma R^2} \right) \right]_{\text{ret}} = \frac{e}{c} \left[\frac{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(3(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}} - \frac{\mathbf{v}}{\gamma} \right) \right]_{\text{ret}} \\ &= \frac{e}{c} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(1 - 2 \frac{\hat{\mathbf{n}} \cdot \mathbf{v}}{c} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{c^2} \right) \left(3(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}} - \frac{\mathbf{v}}{\gamma} \right) \right]_{\text{ret}} \\ &= \frac{e}{c} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(3(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}} - \frac{\mathbf{v}}{\gamma} - 6 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}}}{c} + 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \mathbf{v}}{c \gamma} + 3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^3 \hat{\mathbf{n}}}{c^2} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \mathbf{v}}{c^2 \gamma} \right) \right]_{\text{ret}}. \end{aligned}$$

The third term of Eq. (5) becomes

$$\frac{e}{c^2} \frac{d^2}{dt^2} [\hat{\mathbf{n}}]_{\text{ret}} = \frac{e}{c^3} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3} \left(\frac{\mathbf{v}^2}{\gamma R} - \frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{\gamma} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{R} \right) \right]_{\text{ret}} \frac{d}{dt'} [\hat{\mathbf{n}}]_{\text{ret}} + \frac{e}{c^2} \left[\frac{1}{[1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}]^2} \right]_{\text{ret}} \frac{d^2}{dt'^2} [\hat{\mathbf{n}}]_{\text{ret}}. \quad (6)$$

The first term of Eq. (6) is

$$\begin{aligned} &\frac{e}{c^3} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3} \left(\frac{\mathbf{v}^2}{\gamma R} - \frac{\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}}{\gamma} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2}{R} \right) \left(\frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{R} - \frac{\mathbf{v}}{\gamma R} \right) \right]_{\text{ret}} \\ &= \frac{e}{c^3} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3} \left(\frac{\mathbf{v}^2 (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma R^2} - \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma R} - \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^3 \hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}^2 \mathbf{v}}{\gamma^2 R^2} + \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \mathbf{v}}{\gamma^2 R} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \mathbf{v}}{\gamma R^2} \right) \right]_{\text{ret}} \\ &= \frac{e}{c^3} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(\frac{\mathbf{v}^2 (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma} - \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma} - (\hat{\mathbf{n}} \cdot \mathbf{v})^3 \hat{\mathbf{n}} - \frac{\mathbf{v}^2 \mathbf{v}}{\gamma^2} + \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \mathbf{v}}{\gamma^2} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \mathbf{v}}{\gamma} \right) \right]_{\text{ret}} \end{aligned}$$

and the second term of Eq. (6) is

$$\begin{aligned} &\frac{e}{c^2} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2} \left(3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}}}{R^2} - 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \mathbf{v}}{\gamma R^2} + \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \hat{\mathbf{n}}}{\gamma R} - \frac{\mathbf{v}^2 \hat{\mathbf{n}}}{\gamma R^2} - 2 \frac{\dot{\mathbf{v}}}{\gamma^2 R} + \frac{\dot{\mathbf{v}}}{R} \right) \right]_{\text{ret}} \\ &= \frac{e}{c^2} \left[\frac{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(3(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}} - 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \mathbf{v}}{\gamma} + \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \hat{\mathbf{n}}}{\gamma} - \frac{\mathbf{v}^2 \hat{\mathbf{n}}}{\gamma} - 2 \frac{R \dot{\mathbf{v}}}{\gamma^2} + R \dot{\mathbf{v}} \right) \right]_{\text{ret}} \\ &= \frac{e}{c^2} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(3(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}} - 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \mathbf{v}}{\gamma} + \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \hat{\mathbf{n}}}{\gamma} - \frac{\mathbf{v}^2 \hat{\mathbf{n}}}{\gamma} - 2 \frac{R \dot{\mathbf{v}}}{\gamma^2} + R \dot{\mathbf{v}} \right) \right]_{\text{ret}} \\ &\quad + \frac{e}{c^3} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(-3(\hat{\mathbf{n}} \cdot \mathbf{v})^3 \hat{\mathbf{n}} + 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \mathbf{v}}{\gamma} - \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma} + \frac{\mathbf{v}^2 (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma} + 2 \frac{R (\hat{\mathbf{n}} \cdot \mathbf{v}) \dot{\mathbf{v}}}{\gamma^2} - R (\hat{\mathbf{n}} \cdot \mathbf{v}) \dot{\mathbf{v}} \right) \right]_{\text{ret}} \end{aligned}$$

Summing the two terms, we find

$$\begin{aligned} \frac{e}{c^2} \frac{d^2}{dt^2} [\hat{\mathbf{n}}]_{\text{ret}} &= \frac{e}{c^2} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(3(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \hat{\mathbf{n}} - 2 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v}) \mathbf{v}}{\gamma} + \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \hat{\mathbf{n}}}{\gamma} - \frac{\mathbf{v}^2 \hat{\mathbf{n}}}{\gamma} - 2 \frac{R \dot{\mathbf{v}}}{\gamma^2} + R \dot{\mathbf{v}} \right) \right]_{\text{ret}} \\ &\quad + \frac{e}{c^3} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(2 \frac{\mathbf{v}^2 (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma} - 2 \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) (\hat{\mathbf{n}} \cdot \mathbf{v}) \hat{\mathbf{n}}}{\gamma} - 4(\hat{\mathbf{n}} \cdot \mathbf{v})^3 \hat{\mathbf{n}} - \frac{\mathbf{v}^2 \mathbf{v}}{\gamma^2} + \frac{R (\hat{\mathbf{n}} \cdot \dot{\mathbf{v}}) \mathbf{v}}{\gamma^2} + 3 \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})^2 \mathbf{v}}{\gamma} + 2 \right) \right]_{\text{ret}} \end{aligned}$$

One of the vector identities on the inside cover of Jackson is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

From this,

$$\begin{aligned} \hat{\mathbf{n}} \times [(\hat{\mathbf{n}} \times \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] &= (\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})(\hat{\mathbf{n}} - \boldsymbol{\beta}) - [\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} - \boldsymbol{\beta})]\dot{\boldsymbol{\beta}} = (\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})\boldsymbol{\beta} - \dot{\boldsymbol{\beta}} + (\hat{\mathbf{n}} \cdot \boldsymbol{\beta})\dot{\boldsymbol{\beta}} \\ &= \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{c} - \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v}}{c^2} - \frac{\dot{\mathbf{v}}}{c} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}}{c^2}. \end{aligned}$$

Eq. (1) can be written

$$\begin{aligned} \mathbf{E} &= e \left[\frac{1}{\gamma^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(\hat{\mathbf{n}} - \frac{\mathbf{v}}{c} \right) \right]_{\text{ret}} + \frac{e}{c} \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R} \left(\frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{c} - \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v}}{c^2} - \frac{\dot{\mathbf{v}}}{c} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}}{c^2} \right) \right]_{\text{ret}} \\ &= e \left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2} \left(\frac{\hat{\mathbf{n}}}{\gamma^2} - \frac{\mathbf{v}}{c\gamma^2} + \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{c^2} - \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v}}{c^3} - \frac{\dot{\mathbf{v}}}{c^2} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}}{c^3} \right) \right]_{\text{ret}} \end{aligned}$$

Problem 3. (Jackson 14.4) Using the Liénard-Wiechart fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a particle with charge e , moving as described below. Sketch the angular distribution of the radiation and determine the total power radiated in each case.

3(a) The particle is moving along the z axis with instantaneous position $z(t) = \alpha \cos \omega_0 t$.

Solution. For a nonrelativistic particle, the power radiated per unit solid angle is given by Jackson (14.21),

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2} |\dot{\mathbf{v}}|^2 \sin^2 \Theta, \quad (7)$$

where Θ is the angle between $\dot{\mathbf{v}}$ and $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing toward the observer. The total instantaneous power radiated is given by Jackson (14.22):

$$P = \frac{2}{3} \frac{e^2}{c^3} |\dot{\mathbf{v}}|^2. \quad (8)$$

In this case, we have

$$\mathbf{x}(t) = \alpha \cos \omega_0 t \hat{\mathbf{x}}_3, \quad \mathbf{v}(t) = -\alpha \omega_0 \sin \omega_0 t \hat{\mathbf{x}}_3, \quad \dot{\mathbf{v}}(t) = -\alpha \omega_0^2 \cos \omega_0 t \hat{\mathbf{x}}_3.$$

The system is azimuthally symmetric since $\dot{\mathbf{v}}$ always points along the z axis. Thus, $\Theta = \theta$ where θ is the polar angle in spherical coordinates. Equation (7) becomes

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2} |-\alpha \omega_0^2 \cos \omega_0 t \hat{\mathbf{x}}_3|^2 \sin^2 \theta = \frac{e^2 \alpha^2 \omega_0^4}{4\pi c^2} \cos^2 \omega_0 t \sin^2 \theta,$$

so the time-averaged power radiated per unit solid angle is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^2} \alpha^2 \omega_0^4 \langle \cos^2 \omega_0 t \rangle \sin^2 \theta = \frac{e^2 \alpha^2 \omega_0^4}{8\pi c^2} \sin^2 \theta. \quad (9)$$

A plot of the angular distribution of the radiation is shown in Fig. 3 in the xz plane, and in three dimensions in Fig. 5.

Equation (8) becomes

$$P = \frac{2}{3} \frac{e^2}{c^3} |-\alpha \omega_0^2 \cos \omega_0 t \hat{\mathbf{x}}_3|^2 = \frac{2}{3} \frac{e^2 \alpha^2 \omega_0^4}{c^3} \cos^2 \omega_0 t,$$

so the time-averaged total power radiated is

$$\langle P \rangle = \frac{2}{3} \frac{e^2 \alpha^2 \omega_0^4}{c^3} \langle \cos^2 \omega_0 t \rangle = \frac{e^2 \alpha^2 \omega_0^4}{3c^3}.$$

3(b) The particle is moving in a circle of radius R in the xy plane with constant angular frequency ω_0 .

Solution. For a charge moving counter-clockwise,

$$\mathbf{x}(t) = R \cos \omega_0 t \hat{\mathbf{x}}_1 - R \sin \omega_0 t \hat{\mathbf{x}}_2,$$

$$\mathbf{v}(t) = -R\omega_0 \sin \omega_0 t \hat{\mathbf{x}}_1 - R\omega_0 \cos \omega_0 t \hat{\mathbf{x}}_2,$$

$$\dot{\mathbf{v}}(t) = -R\omega_0^2 \cos \omega_0 t \hat{\mathbf{x}}_1 + R\omega_0^2 \sin \omega_0 t \hat{\mathbf{x}}_2.$$

This system is also azimuthally symmetric, so it is sufficient to restrict the position of the observer to the yz plane. In polar coordinates, $\hat{\mathbf{n}} = \sin \theta \hat{\mathbf{x}}_2 + \cos \theta \hat{\mathbf{x}}_3$. Then $\sin^2 \Theta$ can be found by

$$\sin^2 \Theta = 1 - \cos^2 \Theta = 1 - \frac{(\dot{\mathbf{v}} \cdot \hat{\mathbf{n}})^2}{\dot{v}^2} = 1 - \sin^2 \theta \sin^2 \omega_0 t.$$

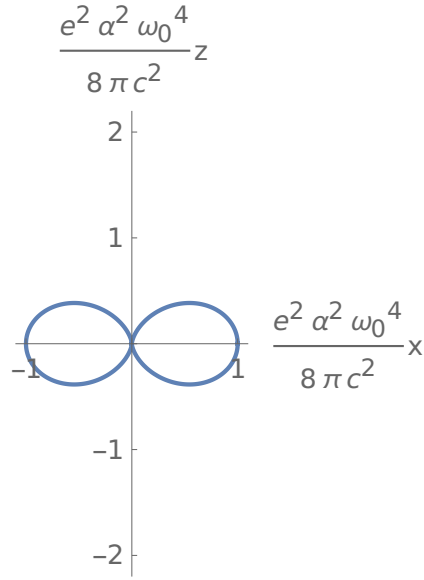


Figure 3: Plot of Eq. (9) in the xz plane.

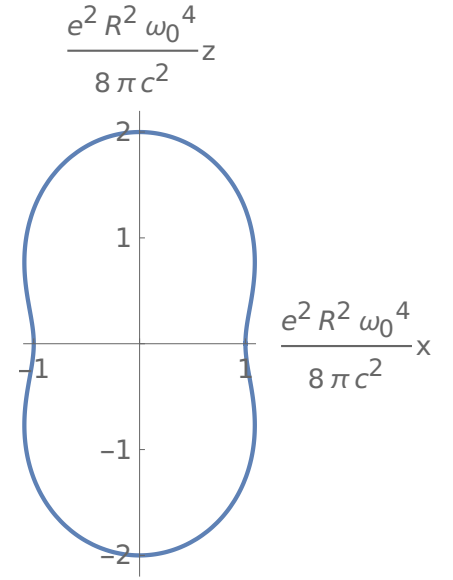


Figure 4: Plot of Eq. (10) in the xz plane.

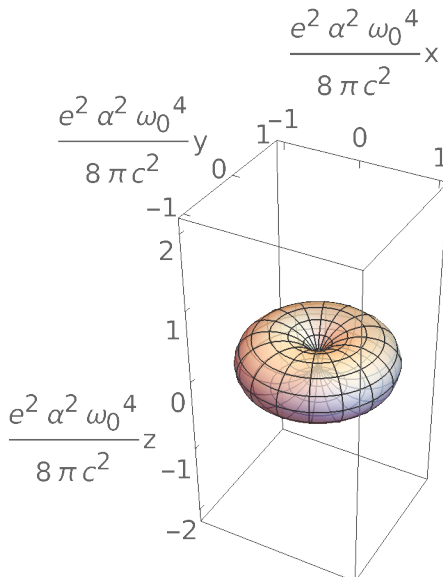


Figure 5: Three-dimensional plot of Eq. (9).

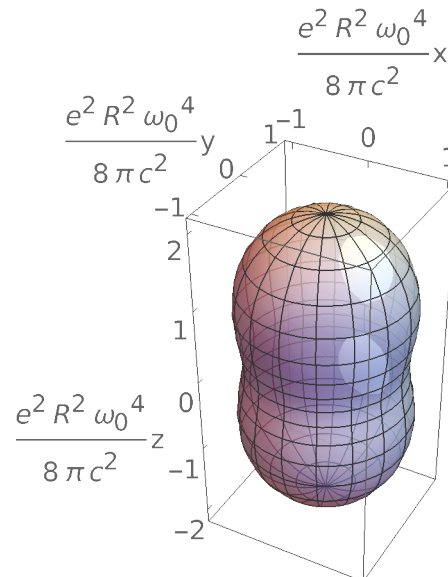


Figure 6: Three-dimensional plot of Eq. (10).

With these substitutions, Eq. (7) becomes

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{e^2}{4\pi c^2} \left| -R\omega_0^2 \cos \omega_0 t \hat{\mathbf{x}}_1 + R\omega_0^2 \sin \omega_0 t \hat{\mathbf{x}}_2 \right|^2 (1 - \sin^2 \theta \sin^2 \omega_0 t) \\ &= \frac{e^2 R^2 \omega_0^4}{4\pi c^2} (\cos^2 \omega_0 t + \sin^2 \omega_0 t) (1 - \sin^2 \theta \sin^2 \omega_0 t) = \frac{e^2 R^2 \omega_0^4}{4\pi c^2} (1 - \sin^2 \theta \sin^2 \omega_0 t),\end{aligned}$$

giving us the time-averaged power radiated per unit solid angle:

$$\begin{aligned}\left\langle \frac{dP}{d\Omega} \right\rangle &= \frac{e^2 R^2 \omega_0^4}{4\pi c^2} (1 - \sin^2 \theta \langle \sin^2 \omega_0 t \rangle) = \frac{e^2 R^2 \omega_0^4}{4\pi c^2} \left(1 - \frac{\sin^2 \theta}{2} \right) = \frac{e^2 R^2 \omega_0^4}{4\pi c^2} \left(1 - \frac{1 - \cos^2 \theta}{2} \right) \\ &= \frac{e^2 R^2 \omega_0^4}{8\pi c^2} (1 + \cos^2 \theta).\end{aligned}\tag{10}$$

A plot of the angular distribution of the radiation is shown in Fig. 4, in the xz plane, and in three dimensions in Fig. 6.

From Eq. (8), we have

$$P = \frac{2}{3} \frac{e^2}{c^3} \left| -R\omega_0^2 \cos \omega_0 t \hat{\mathbf{x}}_1 + R\omega_0^2 \sin \omega_0 t \hat{\mathbf{x}}_2 \right|^2 = \frac{2}{3} \frac{e^2 R^2 \omega_0^4}{c^3} (\cos^2 \omega_0 t + \sin^2 \omega_0 t) = \frac{2}{3} \frac{e^2 R^2 \omega_0^4}{c^3} = \langle P \rangle.$$