Problem 1 (Jackson 11.3). Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \tag{1}$$

This is an alternative way to derive the parallel-velocity addition law.

Solution. The general expression for a Lorentz transformation in the x_1 direction is given by Jackson (11.16),

$$ct' = \gamma(ct - \beta x), \qquad x' = \gamma(x - \beta ct), \qquad y' = y, \qquad z' = z. \tag{2}$$

Define β, β_1, and β_2 by

$$\beta = \frac{v}{c}, \qquad \beta_1 = \frac{v_1}{c}, \qquad \beta_2 = \frac{v_2}{c},$$

and define γ, γ_1, and γ_2 correspondingly.

First applying the boost corresponding to β_1 yields

$$ct' = \gamma_1(ct - \beta_1 x), \qquad x' = \gamma_1(x - \beta_1 ct). \tag{3}$$

Successively applying the boost corresponding to β_2 yields

$$ct'' = \gamma_2(ct' - \beta_2 x'), \qquad x'' = \gamma_2(x' - \beta_2 ct'). \tag{4}$$

Substituting (3) into (4) gives us

$$\begin{aligned} ct'' &= \gamma_1\gamma_2[(ct - \beta_1 x) - \beta_2(x - \beta_1 ct)] = \gamma_1\gamma_2[(1 + \beta_1\beta_2)ct - (\beta_1 + \beta_2)x] \\ x'' &= \gamma_1\gamma_2[(x - \beta_1 ct) - \beta_2(ct - \beta_1 x)] = \gamma_1\gamma_2[(1 + \beta_1\beta_2)x - (\beta_1 + \beta_2)ct], \end{aligned}$$

or

$$ct'' = (1 + \beta_1\beta_2)\gamma_1\gamma_2\left(ct - \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}x\right), \qquad x'' = (1 + \beta_1\beta_2)\gamma_1\gamma_2\left(x - \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}ct\right). \tag{5}$$

Note that

$$\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{1}{c}\frac{v_1 + v_2}{1 + v_1 v_2/c^2} = \frac{v}{c} = \beta,$$

where v is defined by (1), and that

$$\begin{aligned} (1 + \beta_1\beta_2)\gamma_1\gamma_2 &= \frac{1 + \beta_1\beta_2}{\sqrt{(1 - \beta_1^2)(1 - \beta_2^2)}} = \sqrt{\frac{(1 + \beta_1\beta_2)^2}{(1 - \beta_1^2)(1 - \beta_2^2)}} = \sqrt{\frac{(1 + \beta_1\beta_2)^2}{1 - \beta_1^2 - \beta_2^2 + \beta_1^2\beta_2^2}} \\ &= \sqrt{\frac{(1 + \beta_1\beta_2)^2}{1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2 - (\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2)}} = \sqrt{\frac{(1 + \beta_1\beta_2)^2}{(1 + \beta_1\beta_2)^2 - (\beta_1 + \beta_2)^2}} \\ &= \sqrt{1 - \left(\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\right)^2}^{\,-1} = \sqrt{1 - \beta^2}^{\,-1} = \gamma. \end{aligned}$$

Then (5) becomes

$$x_0'' = \gamma\left(x_0 - \beta x_1\right), \qquad x_1'' = \gamma\left(x_1 - \beta x_0\right),$$

which is a single Lorentz transformation with velocity v, as we wanted to show. $\square$

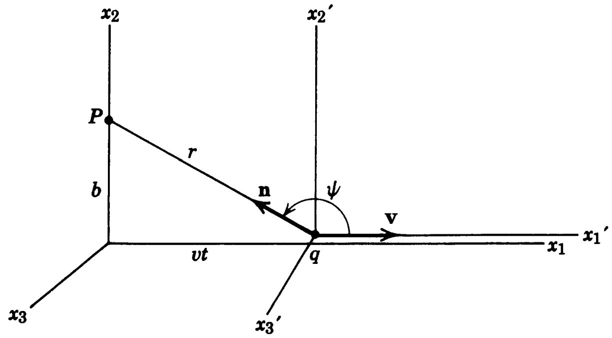

Figure 1: (Jackson Fig. 11.8) Particle of charge q moving at constant velocity $\mathbf{v}$ passes an observation point P at impact parameter b.

Problem 2 (Jackson 11.17). The electric and magnetic fields (7) of a charge in uniform motion can be obtained from Coulomb's law in the charge's rest frame and the fact that the field strength $F^{\alpha\beta}$ is an antisymmetric tensor of rank 2 without considering *explicitly* the Lorentz transformation. The idea is the following. For a charge in uniform motion the only relevant variables are the charge's 4-velocity U^α and the relative coordinate $X^\alpha = x_p^\alpha - x_q^\alpha$, where x_p^α and x_q^α are the 4-vector coordinates of the observation point and the charge, respectively. The only antisymmetric tensor that can be formed is $(X^\alpha U^\beta - X^\beta U^\alpha)$. Thus the electromagnetic field $F^{\alpha\beta}$ must be this tensor multiplied by some scalar function of the possible scalar products, $X_\alpha X^\alpha$, $X_\alpha U^\alpha$, $U_\alpha U^\alpha$.

2.a For the geometry of Fig. 1 the coordinates of P and q at a common time in K can be written $x_p^\alpha = (ct, \mathbf{b})$, $x_q^\alpha = (ct, \mathbf{v}t)$, with $\mathbf{b}\cdot\mathbf{v} = 0$. By considering the general form of $F^{\alpha\beta}$ in the rest frame of the charge, show that

$$F^{\alpha\beta} = \frac{q}{c}\frac{X^\alpha U^\beta - X^\beta U^\alpha}{[(U_\alpha X^\alpha/c)^2 - X_\alpha X^\alpha]^{3/2}}. \tag{6}$$

Verify that this yields the expressions

$$E_1 = E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2 = \gamma E_2' = \frac{\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad B_3 = \gamma\beta E_2' = \beta E_2, \tag{7}$$

with all other components vanishing, in the inertial frame K.

Solution. From Jackson (11.137),

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{8}$$

and from the equation immediately preceding Jackson (11.151),

$$E_1' = -\frac{qvt'}{r'^3}, \qquad E_2' = \frac{qb}{r'^3}, \qquad E_3' = 0, \qquad B_1' = 0, \qquad B_2' = 0, \qquad B_3' = 0,$$

in the rest frame of the charge for the geometry in Fig. **??**. Here, $r' = \sqrt{b^2+v^2t'^2}$. Then, in K',

$$F'^{\alpha\beta} = \frac{q}{(b^2+v^2t'^2)^{3/2}}\begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

Now we will boost into the frame K. From Jackson (11.147), $F' = \Lambda F \tilde{\Lambda}$, although we need $F = \Lambda F' \tilde{\Lambda}$, where we boost in the direction opposite the particle's motion. According to Jackson (11.113), the Lorentz boost in the $-x'$ direction is

$$\Lambda = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

Then

$$\begin{aligned} F^{\alpha\beta} &= \frac{q}{(b^2 + v^2 t'^2)^{3/2}} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \frac{q}{(b^2 + v^2 t'^2)^{3/2}} \begin{bmatrix} -\gamma\beta vt' & \gamma vt' & -\gamma b & 0 \\ -\gamma vt' & \gamma\beta vt' & -\gamma\beta b & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{q}{(b^2 + v^2 t'^2)^{3/2}} \begin{bmatrix} 0 & vt' & -\gamma b & 0 \\ -vt' & 0 & -\gamma\beta b & 0 \\ \gamma b & \gamma\beta b & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

From (2), $t' = \gamma t$ since $x = 0$. Finally,

$$F^{\alpha\beta} = \frac{\gamma q}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

Now we will begin from (6) and find $F^{\alpha\beta}$ directly in K. In accordance with Fig. 1,

$$X^\alpha = (0, \mathbf{b} - \mathbf{v}t) = (0, -vt, b, 0), \qquad U^\alpha = \gamma(c, \mathbf{v}) = \gamma(c, v, 0, 0),$$

and so

$$X^\alpha U^\beta - X^\beta U^\alpha = \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ -cvt & -v^2 t & 0 & 0 \\ cb & vb & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \gamma \begin{bmatrix} 0 & -cvt & cb & 0 \\ 0 & -v^2 t & vb & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \gamma \begin{bmatrix} 0 & cvt & -cb & 0 \\ -cvt & 0 & -vb & 0 \\ cb & vb & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Additionally,

$$U_\alpha X^\alpha = \gamma \begin{bmatrix} c & -v & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -vt \\ b \\ 0 \end{bmatrix} = \gamma v^2 t, \qquad X_\alpha X^\alpha = \begin{bmatrix} 0 & vt & -b & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -vt \\ b \\ 0 \end{bmatrix} = -v^2 t^2 - b^2.$$

Then, applying (6),

$$F^{\alpha\beta} = \frac{\gamma q}{(\gamma^2 v^4 t^2 / c^2 + v^2 t^2 + b^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that

$$v^2 t^2 + \frac{\gamma^2 v^4 t^2}{c^2} = v^2 t^2 \left(1 + \gamma^2 \frac{v^2}{c^2}\right) = v^2 t^2 \left(1 + \frac{\beta^2}{1 - \beta^2}\right) = v^2 t^2 \frac{1 - \beta^2 + \beta^2}{1 - \beta^2} = \gamma^2 v^2 t^2,$$

so we have again arrived at (11). Thus, we have proven (6).

In addition, comparing (11) with (8), we see that

$$E_1 = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2 = \frac{\gamma qb}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad B_3 = \frac{\gamma\beta qb}{(b^2+\gamma^2v^2t^2)^{3/2}} = \beta E_2.$$

Comparing (9) with (8) as well, and making the substitution $t' = \gamma t$, yields

$$E_1' = -\frac{q\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2' = \frac{qb}{(b^2+\gamma^2v^2t^2)^{3/2}},$$

so we have also verified (7). $\square$

2.b Repeat the calculation, using as the starting point the common-time coordinates in the rest frame, $x'^{\alpha}_{p} = (ct', \mathbf{b} - \mathbf{v}t')$ and $x'^{\alpha}_{q} = (ct', 0)$. Show that

$$F^{\alpha\beta} = \frac{q}{c}\frac{Y^{\alpha}U^{\beta} - Y^{\beta}U^{\alpha}}{(-Y_{\alpha}Y^{\alpha})^{3/2}}, \tag{12}$$

where $Y'^{\alpha} = x'^{\alpha}_{p} - x'^{\alpha}_{q}$. Verify that the fields are the same as in 2.a. Note that to obtain the results of (7) it is necessary to use the time t of the observation point P in K as the time parameter.

Solution. Firstly, note that

$$Y'^{\alpha} = (0, \mathbf{b} - \mathbf{v}t') = (0, -vt', b, 0), \qquad U'^{\alpha} = (c, \mathbf{0}) = (c, 0, 0, 0),$$

Then

$$Y'^{\alpha}U'^{\beta} - Y'^{\beta}U'^{\alpha} = c\begin{bmatrix} 0 & 0 & 0 & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - c\begin{bmatrix} 0 & -vt' & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = c\begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Y'_{\alpha}Y'^{\alpha} = \begin{bmatrix} 0 & vt' & -b & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -vt' \\ b \\ 0 \end{bmatrix} = -v^2t'^2 - b^2,$$

so, from (12), in K' we have

$$F'^{\alpha\beta} = \frac{q}{(b^2+v^2t'^2)^{3/2}}\begin{bmatrix} 0 & vt' & -b & 0 \\ -vt' & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is identical to (9). We know that boosting into K yields (11).

Now we will find $F^{\alpha\beta}$ directly in K by boosting Y'^{α} and U'^{α}. From Jackson (11.84), $x' = \Lambda x$ (where x represents x^{α}), and we once again use Λ given by (10) to perform $x = \Lambda x'$. We obtain

$$Y = \Lambda Y' = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -vt' \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} -\gamma\beta vt' \\ -\gamma vt' \\ b \\ 0 \end{bmatrix}, \qquad U = \Lambda U' = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \gamma c\begin{bmatrix} 1 \\ \beta \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$Y^\alpha U^\beta - Y^\beta U^\alpha = \gamma c \begin{bmatrix} -\gamma\beta v t' & -\gamma\beta^2 v t' & 0 & 0 \\ -\gamma v t' & -\gamma\beta v t' & 0 & 0 \\ b & \beta b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \gamma c \begin{bmatrix} -\gamma\beta v t' & -\gamma v t' & b & 0 \\ -\gamma\beta^2 v t' & -\gamma\beta v t' & \beta b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = c \begin{bmatrix} 0 & v t' & -\gamma b & 0 \\ -v t' & 0 & -\gamma\beta b & 0 \\ \gamma b & \gamma\beta b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$Y_\alpha Y^\alpha = \begin{bmatrix} -\gamma\beta v t' & \gamma v t' & -b & 0 \end{bmatrix} \begin{bmatrix} -\gamma\beta v t' \\ -\gamma v t' \\ b \\ 0 \end{bmatrix} = \gamma^2\beta^2 v^2 t'^2 - \gamma^2 v^2 t'^2 - b^2 = -v^2 t'^2 - b^2.$$

Making these substitutions into (12), and using $t' = \gamma t$,

$$F^{\alpha\beta} = \frac{q}{(b^2 + v^2 t'^2)^{3/2}} \begin{bmatrix} 0 & v t' & -\gamma b & 0 \\ -v t' & 0 & -\gamma v b/c & 0 \\ \gamma b & \gamma v b/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{\gamma q}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \begin{bmatrix} 0 & v t & -b & 0 \\ -v t & 0 & -v b/c & 0 \\ b & v b/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is identical to (11), and therefore gives the fields from (7) as in 2.a. Thus, we have proven (12). $\square$

2.c Finally, consider the coordinate $x_p^\alpha = (ct, \mathbf{b})$ and the "retarded-time" coordinate $x_q^\alpha = [ct - R, \boldsymbol{\beta}(ct - R)]$ where R is the distance between P and q at the retarded time. Define the difference as $Z^\alpha = [R, \mathbf{b} - \boldsymbol{\beta}(ct - R)]$. Show that in terms of Z^α and U^α the field is

$$F^{\alpha\beta} = \frac{q}{c} \frac{Z^\alpha U^\beta - Z^\beta U^\alpha}{(U_\alpha Z^\alpha / c)^3}. \tag{13}$$

Solution. Referring to Fig 1,

$$Z^\alpha = (R, \mathbf{b} - \mathbf{v}t + \mathbf{v}R/c) = [R, -v(t - R/c), b, 0], \qquad U^\alpha = \gamma(c, \mathbf{v}) = \gamma(c, v, 0, 0).$$

Then

$$\begin{aligned} Z^\alpha U^\beta - Z^\beta U^\alpha &= \gamma \begin{bmatrix} cR & Rv & 0 & 0 \\ -v(ct - R) & -v^2(t - R/c) & 0 & 0 \\ cb & bv & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \gamma \begin{bmatrix} cR & -v(ct - R) & cb & 0 \\ Rv & -v^2(t - R/c) & bv & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \gamma c \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$U_\alpha Z^\alpha = \gamma \begin{bmatrix} c & -v & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ -v(t - R/c) \\ b \\ 0 \end{bmatrix} = \gamma c R + \gamma v^2 (t - R/c),$$

so

$$\frac{U_\alpha Z^\alpha}{c} = \gamma R + \gamma\beta^2 ct - \gamma\beta^2 R = (1 - \beta^2)\gamma R + \gamma\beta^2 ct = \frac{R}{\gamma} + \gamma\beta^2 ct.$$

Note that x_p^α and x_q^α, as they are defined here, have lightlike separation since R/c is, by definition, the time it takes light to travel from x_q^α to x_p^α. Then

$$0 = Z_\alpha Z^\alpha = \begin{bmatrix} R & v(t - R/c) & -b & 0 \end{bmatrix} \begin{bmatrix} R \\ -v(t - R/c) \\ b \\ 0 \end{bmatrix} = R^2 - v^2(t - R/c)^2 - b^2,$$

which implies

$$R^2 = b^2 + v^2(t - R/c)^2.$$

This is corroborated by the geometry of Fig. 1, since $t - R/c$ is the retarded time. Then, referring to the denominator of (11), we find

$$\begin{aligned} b^2 + \gamma^2 v^2 t^2 &= R^2 - v^2(t - R/c)^2 + \gamma^2 v^2 t^2 = R^2 - \beta^2(c^2t^2 - 2Rct + R^2) + \gamma^2\beta^2 c^2 t^2 \\ &= (1 - \beta^2)R^2 + 2R\beta^2 ct + (\gamma^2 - 1)\beta^2 c^2 t^2 = \frac{R^2}{\gamma^2} + 2R\beta^2 ct + \gamma^2\beta^4 c^2 t^2 = \left(\frac{R}{\gamma} + \gamma\beta^2 ct\right)^2 \\ &= \left(\frac{U_\alpha Z^\alpha}{c}\right)^2. \end{aligned}$$

In summary, we have found

$$F^{\alpha\beta} = \frac{\gamma q}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \begin{bmatrix} 0 & vt & -b & 0 \\ -vt & 0 & -vb/c & 0 \\ b & vb/c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is identical to (11). Thus, we have proven (13). □

Problem 3 (Jackson 11.18). The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by (7), become more and more concentrated as $\beta \to 1$. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\mathbf{r}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

3.a Show that the fields can be written as

$$\mathbf{E} = 2q\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct - z), \qquad \mathbf{B} = 2q\frac{\hat{\mathbf{v}} \times \mathbf{r}_\perp}{r_\perp^2}\delta(ct - z), \tag{14}$$

where $\hat{\mathbf{v}}$ is a unit vector in the direction of the particle's velocity.

Solution. Let K' denote the rest frame of this charge. In this frame,

$$\mathbf{E}' = \frac{q}{r'^3}\mathbf{r}', \qquad \mathbf{B}' = \mathbf{0},$$

where $\mathbf{r}'$ is the observation point in K'.

First we will boost into the lab frame. Jackson (11.148) gives the transformations of the electric field for a boost in the x direction. Adapting these for the z direction, we have

$$E_1 = \gamma(E_1' - \beta B_3') \qquad E_2 = \gamma(E_2' + \beta B_2') \qquad E_3 = E_3',$$

where $\beta < 0$ indicates a boost in the $-z$ direction. Note also that

$$ct' = \gamma(ct + \beta z), \qquad x' = x, \qquad y' = y, \qquad z' = \gamma(z + \beta ct).$$

Let $\beta = v/c$, so $\beta < 0$. Then

$$\begin{gathered} r'^2 = x'^2 + y'^2 + z'^2 = x^2 + y^2 + \gamma^2(z - \beta ct)^2 = r_\perp^2 + \gamma^2(z - vt)^2, \\ \mathbf{r}' = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}' + z'\,\hat{\mathbf{z}}' = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + \gamma(z - \beta ct)\,\hat{\mathbf{z}} = \mathbf{r}_\perp + \gamma(z - vt)\,\hat{\mathbf{z}}, \end{gathered}$$

and in the lab frame, the perpendicular component of the electric field is

$$\mathbf{E}_\perp = \gamma q \frac{\mathbf{r}_\perp}{(r_\perp^2 + \gamma^2(z - vt)^2)^{3/2}}.$$

Taking the limit of this expression as $\beta \to 1$ is identical to taking the limit as $\gamma \to \infty$. Doing so, we find

$$\lim_{\gamma\to\infty} \mathbf{E}_\perp = q\mathbf{r}_\perp \lim_{\gamma\to\infty} \frac{\gamma}{(r_\perp^2 + \gamma^2(z - vt)^2)^{3/2}} = \begin{cases} \infty & \text{if } z = vt, \\ 0 & \text{otherwise,} \end{cases}$$

so we can conclude that

$$\lim_{\gamma\to\infty} \mathbf{E}_\perp = kq\,\delta(vt - z)\,\mathbf{r}_\perp = kq\,\delta(ct - z)\,\mathbf{r}_\perp$$

where k is some constant, and we have made the replacement $v \to c$ as $\beta \to 1$. To find k, we use the fact that the integral of $\delta(ct - z)$ from $z = -\infty$ to $z = \infty$ must be 1. Let $u = \gamma(z - vt)$. Then $dz = du\,/\gamma$, and

$$k = \int_{-\infty}^{\infty} \frac{\gamma}{(r_\perp^2 + \gamma^2(z - vt)^2)^{3/2}}\,dz = \int_{-\infty}^{\infty} \frac{du}{(r_\perp^2 + u^2)^{3/2}} = \left[\frac{u}{r_\perp^2\sqrt{r_\perp^2 + u^2}}\right]_{-\infty}^{\infty} = \frac{2}{r_\perp^2}.$$

Finally, we have shown

$$\mathbf{E}_\perp = 2q\frac{\mathbf{r}_\perp}{r_\perp^2}\,\delta(ct - z)$$

as desired.

For the magnetic field, Jackson (11.150) states that $\mathbf{B} = \boldsymbol{\beta} \times \mathbf{E}$. In the limit $\beta \to 1$, $\boldsymbol{\beta} \to \hat{\mathbf{z}} = \hat{\mathbf{v}}$. Then we have

$$\mathbf{B} = 2q\frac{\hat{\mathbf{v}} \times \mathbf{r}_\perp}{r_\perp^2}\delta(ct - z),$$

as desired. This completes our proof of (14). □

3.b Show that by substitution into the Maxwell equations that these fields are consistent with a 4-vector source density,

$$J^\alpha = qcv^\alpha \delta^2(\mathbf{r}_\perp)\,\delta(ct - z),$$

where the 4-vector $v^\alpha = (1, \hat{\mathbf{v}})$.

Solution. From Jackson (11.128), $J^\alpha = (c\rho, \mathbf{J})$. From Wald (5.4) and (5.5), the inhomogeneous Maxwell equations are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}. \tag{15}$$

For the first equation,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 2q\,\delta(ct-z)\left(\boldsymbol{\nabla}_\perp\cdot\frac{\mathbf{r}_\perp}{r_\perp^2}\right).$$

Gauss's theorem is given by Wald (2.6),

$$\int_{\mathcal{V}}\boldsymbol{\nabla}\cdot\mathbf{v}\,d^3r = \int_{\partial\mathcal{V}}\mathbf{v}\cdot\hat{\mathbf{n}}\,d(\partial\mathcal{V}),$$

where $\mathcal{V}$ is a three-dimensional bounded region with surface $\partial\mathcal{V}$, $\mathbf{v}$ is an arbitrary vector field, and $\hat{\mathbf{n}}$ is an outward pointing unit vector. In two dimensions, this becomes

$$\int_S\boldsymbol{\nabla}_\perp\cdot\mathbf{v}\,d^2r = \int_{\partial S}\mathbf{v}\cdot\hat{\mathbf{n}}\,d(\partial S),$$

where S is a two-dimensional bounded region with boundary ∂S, and $\boldsymbol{\nabla}_\perp$ is the two-dimensional gradient. Taking the surface as a circle of radius $r_\perp$ in the xy plane, $d(\partial S) = r_\perp\,d\theta$ in plane polar coordinates. Then

$$\int_S\boldsymbol{\nabla}_\perp\cdot\frac{\mathbf{r}_\perp}{r_\perp^2}\,d^2r = \int_{\partial S}\frac{\mathbf{r}_\perp}{r_\perp^2}\cdot\frac{\mathbf{r}_\perp}{r_\perp}\,d(\partial S) = \int_{\partial S}\frac{d(\partial S)}{r_\perp} = \int_0^{2\pi}\frac{r_\perp}{r_\perp}\,d\theta = \int_0^{2\pi}d\theta = 2\pi,$$

which implies

$$\boldsymbol{\nabla}_\perp\cdot\frac{\mathbf{r}_\perp}{r_\perp^2} = 2\pi\,\delta^2(\mathbf{r}_\perp).$$

Substituting into (15), we have

$$4\pi q\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z) = 4\pi\rho \quad\Longrightarrow\quad \rho = q\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z) \quad\Longrightarrow\quad J^0 = cq\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z). \tag{16}$$

For the second equation of (15), note that $\partial\mathbf{E}/\partial t = 0$, and

$$\boldsymbol{\nabla}\times\mathbf{B} = 2q\,\delta(ct-z)\left[\boldsymbol{\nabla}\times\left(\hat{\mathbf{v}}\times\frac{\mathbf{r}_\perp}{r_\perp^2}\right)\right].$$

From the inside cover of Jackson,

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b}.$$

Note that

$$\boldsymbol{\nabla}\cdot\frac{\mathbf{r}_\perp}{r_\perp^2} = 2\pi\,\delta^2(\mathbf{r}_\perp), \qquad \boldsymbol{\nabla}\cdot\hat{\mathbf{v}} = 0, \qquad \left(\frac{\mathbf{r}_\perp}{r_\perp^2}\cdot\boldsymbol{\nabla}\right)\mathbf{v} = 0, \qquad \mathbf{v}\cdot\boldsymbol{\nabla} = 0,$$

so

$$\boldsymbol{\nabla}\times\left(\hat{\mathbf{v}}\times\frac{\mathbf{r}_\perp}{r_\perp^2}\right) = 2\pi\hat{\mathbf{v}}\,\delta^2(\mathbf{r}_\perp).$$

Substituting into (15), we find

$$4\pi q\hat{\mathbf{v}}\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z) = \frac{4\pi}{c}\mathbf{J} \quad\Longrightarrow\quad \mathbf{J} = qc\hat{\mathbf{v}}\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z).$$

Combining this result with (16), we have shown

$$J^\alpha = (cq\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z), qc\hat{\mathbf{v}}\,\delta^2(\mathbf{r}_\perp)\,\delta(ct-z)) = qcv^\alpha\delta^2(\mathbf{r}_\perp)\,\delta(ct-z),$$

as desired. $\square$

3.c Show that the fields of (3.a) are derivable from either of the following 4-vector potentials,

$$A^0 = A^z = -2q\,\delta(ct-z)\ln(\lambda r_\perp), \qquad \mathbf{A}_\perp = \mathbf{0}, \tag{17}$$

or

$$A^0 = 0 = A^z, \qquad \mathbf{A}_\perp = -2q\,\Theta(ct-z)\,\boldsymbol{\nabla}_\perp \ln(\lambda r_\perp), \tag{18}$$

where λ is an irrelevant parameter setting the scale of the logarithm.

Show that the two potentials differ by a gauge transformation and find the gauge function, χ.

Solution. From Jackson (11.132), $A^\alpha = (\phi, \mathbf{A})$. From Wald (5.2) and (5.3), the fields are derived from the potentials as

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{19}$$

For the potentials of (17), note that

$$\begin{aligned}
\boldsymbol{\nabla} A^0 &= -2q\,\delta(ct-z)\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\ln\left(\lambda\sqrt{x^2+y^2}\right) = -2q\,\delta(ct-z)\left(\frac{x}{x^2+y^2}\hat{\mathbf{x}} + \frac{y}{x^2+y^2}\hat{\mathbf{y}}\right)\\
&= -2q\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct-z),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{A} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\right)\times A^z\,\hat{\mathbf{z}} = \left(\frac{\partial}{\partial y}\hat{\mathbf{x}} - \frac{\partial}{\partial x}\hat{\mathbf{y}}\right)A^z = -2q\,\delta(ct-z)\left(\frac{\partial}{\partial y}\hat{\mathbf{x}} - \frac{\partial}{\partial x}\hat{\mathbf{y}}\right)\ln\left(\lambda\sqrt{x^2+y^2}\right)\\
&= -2q\,\delta(ct-z)\left(\frac{y}{x^2+y^2}\hat{\mathbf{x}} - \frac{x}{x^2+y^2}\hat{\mathbf{y}}\right) = -2q\frac{\mathbf{r}_\perp\times\hat{\mathbf{z}}}{r_\perp^2}\,\delta(ct-z) = 2q\frac{\hat{\mathbf{v}}\times\mathbf{r}_\perp}{r_\perp^2}\,\delta(ct-z),
\end{aligned}$$

while $\partial\mathbf{A}_\perp/\partial t = \mathbf{0}$. Substitution into (19) yields

$$\mathbf{E} = 2q\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct-z), \qquad \mathbf{B} = 2q\frac{\hat{\mathbf{v}}\times\mathbf{r}_\perp}{r_\perp^2}\,\delta(ct-z),$$

which are identical to (14), as desired.

For the potentials of (18), we assume that $\Theta(x)$ denotes the Heaviside step function. Then, according to Wolfram Mathworld,

$$\frac{d\Theta(x)}{dx} = \delta(x).$$

Note also that

$$\mathbf{A}_\perp = -2q\,\Theta(ct-z)\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}\right)\ln\left(\lambda\sqrt{x^2+y^2}\right) = -2q\frac{\mathbf{r}_\perp}{r_\perp^2}\,\Theta(ct-z),$$

so

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{A} &= -2q\left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\times\Theta(ct-z)\left(\frac{x}{x^2+y^2}\hat{\mathbf{x}} + \frac{y}{x^2+y^2}\hat{\mathbf{y}}\right)\\
&= -2q\left[\Theta(ct-z)\left(\frac{\partial}{\partial x}\frac{y}{x^2+y^2} - \frac{\partial}{\partial y}\frac{x}{x^2+y^2}\right)\hat{\mathbf{z}} + \left(\frac{x}{x^2+y^2}\hat{\mathbf{y}} - \frac{y}{x^2+y^2}\hat{\mathbf{x}}\right)\frac{\partial}{\partial z}\Theta(ct-z)\right]\\
&= -2q\left[\Theta(ct-z)\left(-\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right)\hat{\mathbf{z}} + \left(\frac{y}{x^2+y^2}\hat{\mathbf{x}} - \frac{x}{x^2+y^2}\hat{\mathbf{y}}\right)\delta(ct-z)\right]\\
&= -2q\frac{\mathbf{r}_\perp\times\hat{\mathbf{z}}}{r_\perp^2}\,\delta(ct-z) = 2q\frac{\hat{\mathbf{v}}\times\mathbf{r}_\perp}{r_\perp^2}\,\delta(ct-z),
\end{aligned}$$

and

$$\frac{\partial \mathbf{A}_\perp}{\partial t} = -2qc\frac{\mathbf{r}_\perp}{r_\perp^2}\delta(ct-z),$$

while $\boldsymbol{\nabla}_\perp A^0 = \mathbf{0}$. Substituting into (19), we once again recover (14).

According to Wald (1.13), the general gauge transformations for A^α are given by

$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}, \qquad\qquad \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi,$$

where $\chi = \chi(t, \mathbf{r})$ is a gauge function. Inspecting (17) and (18), we make the ansatz

$$\chi = -2q\,\Theta(ct-z)\ln(\lambda r_\perp), \tag{20}$$

which we will now prove. Denote (17) by A^α and (18) by A'^α. Then we have

$$\begin{aligned} A^0 - \frac{1}{c}\frac{\partial \chi}{\partial t} &= -2q\,\delta(ct-z)\ln(\lambda r_\perp) + \frac{2q}{c}\ln(\lambda r_\perp)\frac{\partial}{\partial t}\Theta(ct-z) = -2q\,\delta(ct-z)\ln(\lambda r_\perp) + 2q\,\delta(ct-z)\ln(\lambda r_\perp) \\ &= 0 = A'^0, \end{aligned}$$

$$\mathbf{A}_\perp + \boldsymbol{\nabla}_\perp\chi = -2q\,\Theta(ct-z)\,\boldsymbol{\nabla}_\perp\ln(\lambda r_\perp) = \mathbf{A}'_\perp,$$

$$\begin{aligned} A^z + \frac{\partial \chi}{\partial z} &= -2q\,\delta(ct-z)\ln(\lambda r_\perp) - 2q\ln(\lambda r_\perp)\frac{\partial}{\partial z}\Theta(ct-z) = -2q\,\delta(ct-z)\ln(\lambda r_\perp) + 2q\,\delta(ct-z)\ln(\lambda r_\perp) \\ &= 0 = A'^z, \end{aligned}$$

so we have shown that (17) and (18) differ by the gauge transformation in (20). □

Problem 4 (Jackson 11.20). The lambda particle (Λ) is a neutral baryon of mass $M = 1115\,\text{MeV}$ that decays with a lifetime of $\tau = 2.9\times10^{-10}\,\text{s}$ into a nucleon of mass $m_1 \approx 939\,\text{MeV}$ and a pi-meson of mass $m_2 \approx 140\,\text{MeV}$. It was first observed in flight by its charged decay mode $\Lambda \to p + \pi^-$ in cloud chambers. The charged tracks originate from a single point and have the appearance of an inverted vee or lambda. The particles' identities and momenta can be inferred from their ranges and curvature in the magnetic field of the chamber.

4.a Using conservation of momentum and energy and the invariance of scalar products of 4-vectors show that, if the opening angle θ between the two tracks is measured, the mass of the decaying particle can be found from the formula

$$M^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta,$$

where here p_1 and p_2 are the magnitudes of the 3-momenta.

Solution. The general momentum 4-vector for a particle is $P^\mu = (E/c, \mathbf{p})$, where E is the energy of the particle and $\mathbf{p}$ its three-dimensional momentum. Let ${P_1}^\mu$ and ${P_2}^\mu$ be the momentum 4-vectors for the two particles, and define

$$P^\mu = {P_1}^\mu + {P_2}^\mu.$$

Firstly, note that

$$P^\mu P_\mu = \frac{E^2}{c^2} - p^2.$$

According to Jackson (11.55),

$$E = \sqrt{c^2p^2 + m^2c^4}, \tag{21}$$

so we have

$$P^\mu P_\mu = \frac{c^2p^2 + M^2c^4}{c^2} - p^2 = M^2c^2. \tag{22}$$

Note also that

$$P^\mu P_\mu = ({P_1}^\mu + {P_2}^\mu)(P_{1\mu} + P_{2\mu}) = \frac{(E_1 + E_2)^2}{c^2} - (\mathbf{p}_1 + \mathbf{p}_2)^2 = \frac{E_1^2 + 2E_1E_2 + E_2^2}{c^2} - p_1^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 - p_2^2,$$

and once again making use of (21),

$$P^\mu P_\mu = \frac{c^2p_1^2 + m_1^2c^4 + 2E_1E_2 + c^2p_2^2 + m_2^2c^4}{c^2} - p_1^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 - p_2^2 = m_1^2c^2 + m_2^2c^2 + 2E_1E_2/c^2 - 2p_1p_2\cos\theta. \tag{23}$$

Equating (22) and (25), we have

$$M^2c^2 = m_1^2c^2 + m_2^2c^2 + 2E_1E_2/c^2 - 2p_1p_2\cos\theta.$$

Taking $c = 1$, this becomes

$$M^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\theta$$

as desired. □

4.b A lambda particle is created with a total energy of 10 GeV in a collision in the top plate of a cloud chamber. How far on the average will it travel in the chamber before decaying? What range of opening angles will occur for a 10 GeV lambda if the decay is more or less isotropic in the lambda's rest frame?

Solution. According to Jackson (11.51), $E = \gamma mc^2$. For the Λ, this gives us $\gamma = E/M$ in natural units. Substituting this into the expression for time dilation, we find

$$\Delta t = \gamma\tau = \frac{E}{M}\tau$$

as the average lifetime in the lab frame. We also find

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{M^2}{E^2}}.$$

Then the average distance the Λ travels before decaying is

$$d = v\,\Delta t = c\tau\sqrt{1 - \frac{M^2}{E^2}}\frac{E}{M} = (3.00 \times 10^8\,\mathrm{m\,s^{-1}})\sqrt{1 - \frac{(1115\,\mathrm{MeV})^2}{(10\,\mathrm{GeV})^2}}\frac{10\,\mathrm{GeV}}{1115\,\mathrm{MeV}}(2.9 \times 10^{-10}\,\mathrm{s}) = 77\,\mathrm{cm}.$$

Let K' denote the rest frame of the Λ. Using the result of 4.a and the fact that M, m_1, and m_2 are Lorentz invariant, we can write

$$E_1'E_2' - p_1'p_2'\cos\theta' = E_1E_2 - p_1p_2\cos\theta, \tag{24}$$

where θ' is the angle between the daughter particles in K'. We know from conservation of momentum that they must be "back to back" in this frame, meaning $\theta' = \pi$. Taking this into account and rearranging, we find

$$\cos\theta = \frac{E_1E_2 - E_1'E_2' - p_1'p_2'}{p_1p_2}. \tag{25}$$

Say that the Λ is moving in the z direction in the lab frame K. The opening angle θ is minimized when the particles are emitted along the $\pm z'$ axis in K', because the boost of the Λ in K is much greater than that of either daughter particle in K'. In this scenario, both of the daughter particles travel along the z axis in the $+z$ direction in K, so the minimum possible opening angle $\min\theta = 0$.

By a similar argument, θ is maximized when the daughter particles are emitted transverse to the z' axis in K'. For this scenario, and using natural units, in the K' frame we have

$$P'^{\mu} = (M, \mathbf{0}), \qquad P_1'^{\mu} = (E_1', \mathbf{p}_1'), \qquad P_2'^{\mu} = (E_2', \mathbf{p}_2').$$

Conservation of momentum and energy stipulates that

$$M = E_1' + E_1' = \gamma_1' m_1 + \gamma_2' m_2, \qquad \mathbf{0} = \mathbf{p}_1' + \mathbf{p}_2' = \gamma_1'\mathbf{p}_1' + \gamma_2'\mathbf{p}_2' = \gamma_1' m_1 \mathbf{v}_1' + \gamma_2' m_2 \mathbf{v}_2',$$

where γ_1' and γ_2' are associated with boosting to the rest frames of m_1 and m_2, respectively, from K'. Note that

$$\gamma_1' m_1 \mathbf{v}_1' = -\gamma_2' m_2 \mathbf{v}_2' \quad\Longrightarrow\quad {\gamma_1'}^2 m_1^2 {\mathbf{v}_1'}^2 = {\gamma_2'}^2 m_2^2 {\mathbf{v}_2'}^2,$$

and that, in natural units,

$$\gamma^2 v^2 = \gamma^2\beta^2 = \gamma^2\left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 - 1.$$

Then

$$({\gamma_1'}^2 - 1)m_1^2 = ({\gamma_2'}^2 - 1)m_2^2 \quad\Longrightarrow\quad {E_1'}^2 - m_1^2 = {E_2'}^2 - m_2^2 \quad\Longrightarrow\quad m_1^2 - m_2^2 = {E_1'}^2 - {E_2'}^2,$$

which implies

$$m_1^2 - m_2^2 = {E_1'}^2 - (M - E_1')^2 = -M^2 - 2ME_1' \quad\Longrightarrow\quad E_1' = \frac{M^2 + m_1^2 - m_2^2}{2M}.$$

Similarly,

$$E_2' = \frac{M^2 - m_1^2 + m_2^2}{2M}.$$

Let m_1 denote the nucleon and m_2 the pion. Then we have

$$\begin{aligned}
E_1' &= \frac{1}{2}\frac{(1115\,\mathrm{MeV})^2 + (939\,\mathrm{MeV})^2 - (140\,\mathrm{MeV})^2}{1115\,\mathrm{MeV}} \approx 944\,\mathrm{MeV},\\
E_2' &= \frac{1}{2}\frac{(1115\,\mathrm{MeV})^2 - (939\,\mathrm{MeV})^2 + (140\,\mathrm{MeV})^2}{1115\,\mathrm{MeV}} \approx 171\,\mathrm{MeV}.
\end{aligned}$$

For the momenta, note that

$$p_1' = m_1\sqrt{{\gamma_1'}^2 - 1} = m_2\sqrt{{\gamma_2'}^2 - 1} = p_2',$$

where

$$\gamma_1' = \frac{E_1'}{m_1} = \frac{944\,\mathrm{MeV}}{939\,\mathrm{MeV}} \approx 1.005, \qquad \gamma_2' = \frac{E_2'}{m_2} = \frac{171\,\mathrm{MeV}}{140\,\mathrm{MeV}} \approx 1.221.$$

Then

$$p_1' = p_2' = (140\,\mathrm{MeV})\sqrt{1.221^2 - 1} \approx 98\,\mathrm{MeV}.$$

For the left side of (24), note that

$$\gamma = \frac{E}{M} = \frac{10\,\mathrm{GeV}}{1115\,\mathrm{MeV}} \approx 8.969 \quad\Longrightarrow\quad \beta = \sqrt{1 - \frac{M^2}{E^2}} \approx 0.994,$$

whereas

$$\beta_1' = \sqrt{1 - \frac{1}{{\gamma_1'}^2}} \approx 0.104, \qquad \beta_2' = \sqrt{1 - \frac{1}{{\gamma_2'}^2}} \approx 0.573.$$

Since these are not nearly as relativistic as β, and we have stipulated that $\boldsymbol{\beta}_1'$ and $\boldsymbol{\beta}_2'$ are transverse to the z' axis, it may be sufficient to approximate $\gamma_1, \gamma_2 \approx \gamma$ in K. Then we have

$$\begin{aligned} E_1 &\approx \gamma m_1 \approx 8.969(939\,\text{MeV}) \approx 8422\,\text{MeV}, \qquad & p_1 &\approx m_1\sqrt{\gamma^2-1} \approx (939\,\text{MeV})\sqrt{8.969^2-1} \approx 8369\,\text{MeV}, \\ E_2 &\approx \gamma m_2 \approx 8.969(140\,\text{MeV}) \approx 1256\,\text{MeV}, \qquad & p_2 &\approx m_2\sqrt{\gamma^2-1} \approx (140\,\text{MeV})\sqrt{8.969^2-1} \approx 1248\,\text{MeV}. \end{aligned}$$

Making these substitutions into (25), we have

$$\cos\theta \approx \frac{(8422\,\text{MeV})(1256\,\text{MeV}) - (944\,\text{MeV})(171\,\text{MeV}) - (98\,\text{MeV})^2}{(8369\,\text{MeV})(1248\,\text{MeV})} \approx 0.996$$

which implies $\max\theta \approx \cos^{-1}(0.996) \approx 5.1^\circ$. This gives the possible range of θ as

$$0 \leq \theta \lessapprox 5.1^\circ.$$

Problem 5. Show that $\mathbf{E}\cdot\mathbf{B}$ is Lorentz invariant. You can do this either by using the Lorentz transformation laws for $\mathbf{E}$ and $\mathbf{B}$ derived in class, or by writing $\mathbf{E}\cdot\mathbf{B}$ in a manifestly Lorentz invariant (and gauge invariant) form.

Solution. From Jackson (11.140),

$$\tilde{F}^{\alpha\beta} = \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix}.$$

Then, applying (8) as well,

$$\begin{aligned} F_{\alpha\beta}\tilde{F}^{\alpha\beta} &= \sum_{\alpha,\beta} \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{bmatrix} = \sum_{\alpha,\beta} \begin{bmatrix} \mathbf{E}\cdot\mathbf{B} & 0 & 0 & 0 \\ 0 & \mathbf{E}\cdot\mathbf{B} & 0 & 0 \\ 0 & 0 & \mathbf{E}\cdot\mathbf{B} & 0 \\ 0 & 0 & 0 & \mathbf{E}\cdot\mathbf{B} \end{bmatrix} \\ &= 4\mathbf{E}\cdot\mathbf{B}. \end{aligned}$$

Here we have shown that $\mathbf{E}\cdot\mathbf{B}$ is directly proportional to the inner product of two 4-tensors. Thus, it is Lorentz invariant. □