Problem 1. LCR circuit An electrical circuit consists of an inductance L, resistance R and capacitance C in series, driven by a voltage source $V(t) = V_0 \cos(\omega t)$.

1(a) Show that the charge $q(t)$ on the capacitor satisfies the equation

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V(t), \tag{1}$$

and use it to define the complex susceptibility from

$$q(\omega) = \chi(\omega)V(\omega). \tag{2}$$

Solution. An example LCR circuit is shown in Fig. 1. We can use Kirchoff's loop rule to obtain the differential equation for this circuit. Beginning from the bottom left corner of the circuit and moving clockwise, we have [1, pp. 849, 1007]

$$0 = V(t) - IR - L\frac{dI}{dt} - \frac{q}{C},$$

where we have applied Ohm's law $V_{ab} = IR$, the potential difference across an inductor $V_{ab} = L\, dI/dt$, and the definition of capacitance $C = q/V_{ab}$ [1, pp. 782, 999]. The current $I(t) = dq(t)/dt$ and charge $q(t)$ are identical at all points in a series circuit. Feeding in $I = dq(t)/dt$, this relation becomes

$$V(t) = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

as we wanted to show. $\square$

For the complex susceptibility, we recall that differentiating in the time domain is equivalent to multiplying by $i\omega$ in the frequency domain [3]:

$$\mathcal{F}_x[f^{(n)}(x)](\omega) = (i\omega)^n \mathcal{F}[f(x)](\omega).$$

We Fourier transform both sides of Eq. (1):

$$V(\omega) = L(i\omega)^2 q(\omega) + R(i\omega)q(\omega) + \frac{q(\omega)}{C} = \left(i\omega R - \omega^2 L^2 + \frac{1}{C}\right) q(\omega) \quad \Longrightarrow \quad q(\omega) = \frac{V(\omega)}{i\omega R - \omega^2 L^2 + 1/C}.$$

Applying Eq. (2), we find

$$\chi(\omega) = \frac{1}{i\omega R - \omega^2 L^2 + 1/C}. \tag{3}$$

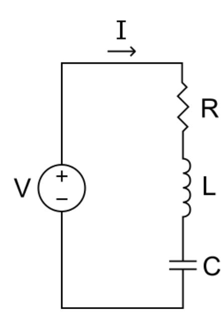

Figure 1: An LCR series circuit [2].

1(b) Show that the forced solution of this equation is

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}},$$

where

$$\tan(\phi) = \frac{\omega R}{\omega^2 L - 1/C}.$$

Solution. Equation (1) is an ODE representing forced damped motion of a mass-spring system. Its solution can be written as the sum of the homogeneous solution, which dies out with time, and a particular solution [4, pp. 38, 40, 50–51]. Rewriting Eq. (1) as

$$\frac{V_0}{L}\cos(\omega t) = \ddot{q} + 2p\dot{q} + \omega_0^2 q$$

where $p = R/2L$ and $\omega_0^2 = 1/LC$, the ansatz for the particular solution is

$$q(t) = A_c \cos(\omega t) + A_s \sin(\omega t).$$

Feeding this into the ODE and collecting terms yields

$$-A_c\omega^2 + 2pA_s\omega + \omega_0^2 A_c = \frac{V_0}{L}, \qquad -A_s\omega^2 - 2pA_c\omega + \omega_0^2 A_s = 0.$$

This system has the solutions [4, p. 51]

$$A_s = \frac{2p\omega V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}, \qquad A_c = \frac{(\omega^2 - \omega_0^2)V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}. \tag{4}$$

If we define

$$A = \sqrt{A_c^2 + A_s^2} \tag{5}$$

and write

$$q(t) = A\left(\frac{A_c}{A}\cos(\omega t) + \frac{A_s}{A}\sin(\omega t),\right)$$

there exists an angle ϕ such that $\cos(\phi) = A_c/A$, $\sin(\phi) = A_s/A$, and $\tan(\phi) = A_s/A_c$. Thus

$$q(t) = A[\cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega).$$

Using the identity

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta),$$

it follows that [4, pp. 39, 51]

$$q(t) = A\cos(\omega t - \phi).$$

The amplitude A is given by [4, p. 51]

$$A = \frac{CV_0}{\sqrt{(\nu^2 - 1)^2 + 4c^2\nu^2}}, \qquad \text{where} \quad c = \frac{R}{2\sqrt{L/C}}, \quad \nu = \frac{\omega}{\omega_0}.$$

Substituting back to the original quantities, this is

$$A = \frac{CV_0}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + (R^2C/L)(\omega^2/\omega_0^2)}} = \frac{CV_0}{\sqrt{(LC\omega^2 - 1)^2 + C^2 sR^2\omega^2}} = \frac{V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Additionally, from $\tan(\phi) = A_s/A_c$,

$$\tan(\phi) = \frac{2p\omega}{\omega^2 - \omega0^2} = \frac{R\omega/L}{\omega^2 - 1/LC} = \frac{\omega R}{\omega^2 L - 1/C}.$$

Hence we have shown

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}$$

as desired. □

1(c) Show that the mean rate of power dissipation is

$$W = \frac{1}{2}\frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Solution. The average power into a general AC circuit is [1, p. 1032]

$$P_{\text{av}} = \frac{1}{2} V I \sin(\phi),$$

where I is the current amplitude, V is the voltage amplitude, and ϕ is the phase angle determined in 1 [1, pp. 1028, 1032]. Assuming the circuit is perfectly efficient, the average power into the circuit is equal to the average power it dissipates, so $W = P_{\text{av}}$. Clearly $V = V_0$. For I,

$$I(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\left(\frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}\right) = -\frac{\omega V_0 \sin(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}, \tag{6}$$

so

$$I = \frac{\omega V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}. \tag{7}$$

Thus

$$W = \frac{1}{2} V I \sin(\phi) = \frac{1}{2}\frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}} \tag{8}$$

as we wanted to show. □

1(d) Sketch the real and imaginary parts of χ as a function of frequency, for the cases $Q \ll 1$, $Q \approx 1$, and $Q \gg 1$, where $Q = \sqrt{L/C}/R$ is the "quality factor."

Where are the poles of χ in the complex ω plane?

Solution. From Eq. (3), we can write $\chi(\omega)$ in terms of Q as

$$\chi(\omega) = \frac{1}{i\omega Q\sqrt{C/L} + Q^2/R^2 L - \omega^2 Q^4 C^2/R^4}.$$

In the case $Q \ll 1$, the Taylor series expansion to $\mathcal{O}(Q)$ is

$$\chi(\omega) \approx \frac{L^2}{CR^2\omega^2} + i\left(\frac{QL^3}{\omega^3 R^4 C\sqrt{C/L}} - \frac{1}{\omega Q\sqrt{C/L}}\right) = \frac{L^2}{\omega^2 CR^2} + i\left(\frac{L^4}{\omega^3 R^5 C^2} - \frac{LR}{\omega C}\right),$$

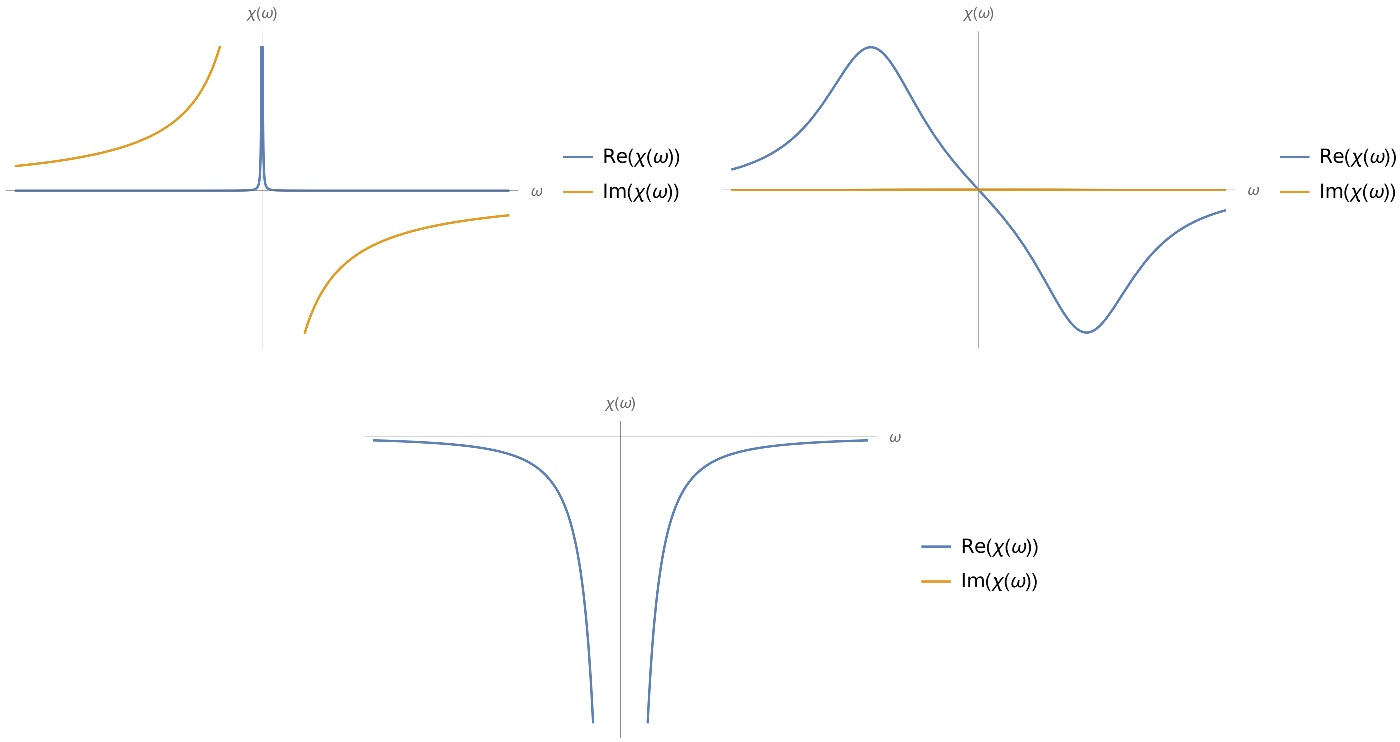

Figure 2: Real (blue line) and imaginary (gold line) parts of $\chi(\omega)$ in the cases $Q \ll 1$ (top left), $Q \approx 1$ (top right), and $Q \gg 1$ (bottom). These cases are given by Eqs. (6), (7), and (8), respectively.

which we have evaluated using Mathematica. In the case $Q \approx 1$,

$$\chi(\omega) \sim \frac{1}{i\omega R - \omega^2 L^2 + 1/C}$$

so

$$\text{Re}[\chi(\omega)] = \frac{1/C - \omega^2 L^2}{(1/C - \omega^2 L^2)^2 + \omega^2 R^2}, \qquad \text{Im}[\chi(\omega)] = -\frac{\omega R}{(1/C - \omega^2 L^2)^2 + \omega^2 R^2}.$$

In the case $Q \gg 1$, the Taylor series expansion to $\mathcal{O}(Q^4)$ is

$$\chi(\omega) \approx -\frac{R^4}{\omega^2 Q^2 C^2} = -\frac{R^2}{\omega^2 LC},$$

which is real.

Figure 2 shows plots of the real (blue line) and imaginary (gold line) parts of $\chi(\omega)$ in the cases $Q \ll 1$ (top left), $Q \approx 1$ (top right), and $Q \gg 1$ (bottom).

The function has poles at $\bar{\omega}$ such that $1/\chi(\bar{\omega}) = 0$ [5]:

$$0 = i\bar{\omega}R - \bar{\omega}^2 L^2 + \frac{1}{C} \qquad \Longrightarrow \qquad \bar{\omega} = \frac{iR \pm \sqrt{4L^2/C - R^2}}{2L^2}.$$

Problem 2. Landau theory of phase transitions A ferroelectric crystal is one that supports a macroscopic polarization P, which usually arises because the underlying crystal structure does not have inversion symmetry. However, as temperature or pressure is changed, the crystal may recover the inversion symmetry. This can be modeled by Landau's theory of second order phase transitions, where we postulate a form for the free energy density (per unit volume)

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4 + \frac{c}{6}P^6 + \cdots, \tag{9}$$

where the coefficient $a = a_0(T - T_c)$ is temperature dependent and all the other coefficients are constant. Although the polarization P is of course a vector, we assume that it can point only in a symmetry direction of the crystal, and so it is replaced by a scalar.

2(a) Assume that $b > 0$ and $c = 0$. Use Eq. (9) to determine the form for the equilibrium $P(T)$.

Solution. When $b > 0$ and $c = 0$, Eq. (9) becomes

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4.$$

The equilibrium $P(T)$ occurs at the minima of $\mathcal{F}$, where $d\mathcal{F}/dP = 0$ [6]:

$$\frac{d\mathcal{F}}{dP} = aP + bP^3 = 0.$$

This implies

$$P = 0, \qquad\qquad P = \pm\sqrt{-\frac{a}{b}}.$$

Note, however, that $P = 0$ is a local maximum of $\mathcal{F}$:

$$\left.\frac{d^2\mathcal{F}}{dP^2}\right|_{P=0} = \left[a + 2bP^2\right]_{P=0} = a_0(T - T_c) < 0 \quad \text{when } T < T_c,$$

which is the regime we are interested in for a ferroelectric [7, p. 556][6]. Thus the equiliibrium $P(T)$ is given by

$$P(T) = \pm\sqrt{\frac{a_0}{b}(T_c - T)}.$$

2(b) Including in $\mathcal{F}$ the energy of the polarization coupled to an external electric field E, determine the dielectric susceptibility $\chi = dP/dE$ both above and below the critical temperature.

2(c) Sketch curves for $P(T)$, $\chi^{-1}(T)$, and $\chi(T)$.

2(d) In a different material, the free energy is described by a similar form to Eq. (9), but with $b < 0$ and $c > 0$. By sketching $\mathcal{F}$ at different temperatures, discuss the behavior of the equilibrium polarization and the linear susceptibility, contrasting the results with those found in 2(b).

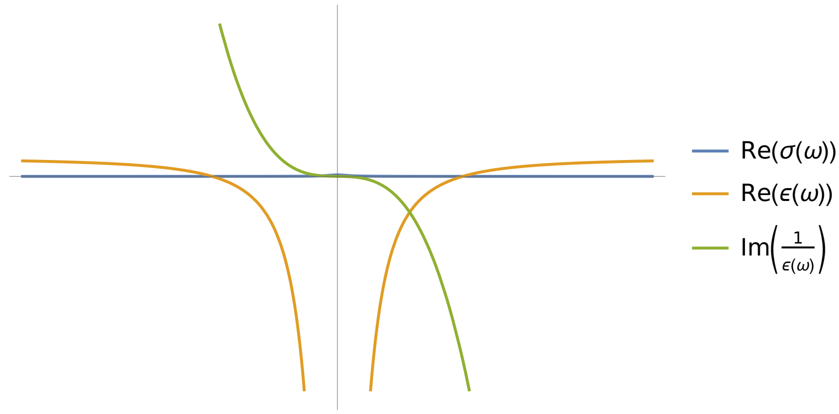

Figure 3: Plots of $\mathrm{Re}[\sigma(\omega)]$ (blue), $\mathrm{Re}[\epsilon(\omega)]$ (gold), and $\mathrm{Im}[1/\epsilon(\omega)]$ (green). These expressions are given by Eqs. (11), (12), and (14), respectively.

Problem 3. Reflectivity of metals The phase velocity of light in a conducting medium is the speed of light divided by the complex dielectric constant $N(\omega) = \sqrt{\epsilon(\omega)}$ where we may use for ϵ the Drude result

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega/\tau}. \tag{10}$$

In a good Drude metal, we have $1/\tau \ll \omega_p$.

3(a) Sketch curves of

(i) $\mathrm{Re}[\sigma(\omega)]$,

(ii) $\mathrm{Re}[\epsilon(\omega)]$,

(iii) $\mathrm{Im}[1/\epsilon(\omega)]$.

Solution. The conductivity is defined in (5.25) of the lecture notes:

$$\sigma(\omega) = \frac{\omega_p^2}{4\pi(1/\tau - i\omega)}.$$

Thus

$$\mathrm{Re}[\sigma(\omega)] = \frac{\omega_p^2}{4\pi\tau}\frac{1}{1/\tau^2 + \omega^2}. \tag{11}$$

Note also that

$$\mathrm{Re}[\epsilon(\omega)] = 1 - \frac{\omega_p^2}{1/\tau^2 + \omega^2}. \tag{12}$$

and that

$$\mathrm{Im}[\epsilon(\omega)] = -\frac{\omega_p^2}{\tau\omega}\frac{1}{1/\tau^2 + \omega^2}, \tag{13}$$

so

$$\mathrm{Im}[1/\epsilon(\omega)] = -\frac{\tau\omega}{\omega_p^2}\left(\frac{1}{\tau^2} + \omega^2\right). \tag{14}$$

Figure 3 shows plots of $\mathrm{Re}[\sigma(\omega)]$ (blue), $\mathrm{Re}[\epsilon(\omega)]$ (gold), and $\mathrm{Im}[1/\epsilon(\omega)]$ (green).

3(b) Consider a light wave with the electric field polarized in the x direction at normal incidence from the vacuum on a good Drude metal occupying the region $z > 0$. In the vacuum ($z < 0$), the incident E_1 and reflected E_2 waves give rise to a field

$$E_x = E_1 e^{i\omega(z/c-t)} + E_2 e^{-i\omega(z/c+t)},$$

whereas in the medium, the electric field is

$$E_x = E_0 e^{i\omega[N(\omega)z/c-t]}.$$

Matching the electric and magnetic fields on the boundary, show that

$$E_0 = E_1 + E_2, \qquad NE_0 = E_1 - E_2,$$

and hence show that the reflection coefficient satisfies

$$R = \left|\frac{E_2}{E_1}\right|^2 = \left|\frac{1-N}{1+N}\right|^2. \tag{15}$$

Solution. By (5.15) of the lecture notes, the boundary condition is

$$\epsilon_{\|} E_{\|} - D_{\|},$$

where $\epsilon_{\|}$ is given by Eq. (10). In this problem the surface of the metal is the xy plane. We require

$$E_1 e^{i\omega(z/c-t)} + E_2 e^{-i\omega(z/c+t)} = \epsilon(\omega) E_0 e^{i\omega[N(\omega)z/c-t]}.$$

First making the ansatz $E_0 = E_1 + E_2$,

$$\begin{aligned}
E_1 e^{i\omega(z/c-t)} + E_2 e^{-i\omega(z/c+t)} &= \epsilon(\omega)(E_1 + E_2) e^{i\omega[N(\omega)z/c-t]} \\
E_1[e^{i\omega(z/c-t)} - \epsilon(\omega) e^{i\omega[N(\omega)z/c-t]}] &= -E_2[e^{-i\omega(z/c+t)} - \epsilon(\omega) e^{i\omega[N(\omega)z/c-t]}] \\
E_1[e^{-2i\omega z/c} - \epsilon(\omega) e^{i\omega z[N(\omega)+1]/c}] &= -E_2[1 - \epsilon(\omega) e^{i\omega z[N(\omega)+1]/c}]
\end{aligned}$$

??????

3(c) Using the Drude formula above, show that

$$R \approx \begin{cases} 1 - 2\sqrt{\dfrac{\omega}{2\pi\sigma(0)}} & \omega \ll 1/\tau, \\ 1 - \dfrac{2}{\omega_p \tau} & 1/\tau \ll \omega \ll \omega_p, \\ 0 & \omega_p \ll \omega, \end{cases}$$

and sketch the reflectivity $R(\omega)$.

Solution. From Eq. (15), we can write

$$\begin{aligned}
R &= \left|\frac{1-N}{1+N}\right|^2 \\
&= \frac{(1-\mathrm{Re}[N])^2+\mathrm{Im}[N]^2}{(1+\mathrm{Re}[N])^2+\mathrm{Im}[N]^2} \\
&= \frac{(1-\sqrt{\mathrm{Re}[\epsilon(\omega)]})^2+\sqrt{\mathrm{Im}[\epsilon(\omega)]}^2}{(1+\sqrt{\mathrm{Re}[\epsilon(\omega)]})^2+\sqrt{\mathrm{Im}[\epsilon(\omega)]}^2} \\
&= \frac{1-2\sqrt{\mathrm{Re}[\epsilon(\omega)]}+\mathrm{Re}[\epsilon(\omega)]+\mathrm{Im}[\epsilon(\omega)]}{1+2\sqrt{\mathrm{Re}[\epsilon(\omega)]}+\mathrm{Re}[\epsilon(\omega)]+\mathrm{Im}[\epsilon(\omega)]},
\end{aligned}$$

where we have used Ashcroft & Mermin (K.6). Feeding in Eqs. (12) and (13),

$$R = \left(2-2\sqrt{1-\frac{\omega_p^2}{1/\tau^2+\omega^2}}-\omega_p^2\frac{1+1/\tau\omega}{1/\tau^2+\omega^2}\right)\left(2+2\sqrt{1-\frac{\omega_p^2}{1/\tau^2+\omega^2}}-\omega_p^2\frac{1+1/\tau\omega}{1/\tau^2+\omega^2}\right)^{-1}.$$

When $\omega \ll 1/\tau$, $\tau\omega \ll 1$. Then

$$\begin{aligned}
R &= \left(2-2\sqrt{1-\frac{\omega_p^2/\omega^2}{1/\tau^2\omega^2+1}}-\frac{\omega_p^2}{\omega}\frac{1/\omega+1/\tau\omega}{1/\tau^2\omega^2+1}\right)\left(2+2\sqrt{1-\frac{\omega_p^2/\omega^2}{1/\tau^2\omega^2+1}}-\frac{\omega_p^2}{\omega}\frac{1/\omega+1/\tau\omega}{1/\tau^2\omega^2+1}\right)^{-1} \\
&\approx \frac{2-2\sqrt{1-(\tau\omega)^2\omega_p^2/\omega^2}-(\tau\omega)^2\omega_p^2(1+1/\tau)/\omega^2}{2+2\sqrt{1-(\tau\omega)^2\omega_p^2/\omega^2}-(\tau\omega)^2\omega_p^2(1+1/\tau)/\omega^2} \\
&\approx \frac{[\omega_p^2/\omega^2-\omega_p^2(1+1/\tau)/\omega^2](\tau\omega)^2}{4-[\omega_p^2/\omega^2+\omega_p^2(1+1/\tau)/\omega^2](\tau\omega)^2} \\
&= \frac{(\omega_p^2/\tau\omega^2)(\tau\omega)^2}{4-[\omega_p^2/\omega^2+\omega_p^2(1+1/\tau)/\omega^2](\tau\omega)^2}
\end{aligned}$$

??? From (5.27) in the lecture notes,

$$\sigma(0) = \frac{\omega_p^2\tau}{4\pi}.$$

$$1-2\sqrt{\frac{\omega}{2\pi\sigma(0)}} = 1-2\sqrt{\frac{2\omega}{\omega_p^2\tau}}$$

When $\omega \ll \omega_p$ and $1/\tau \ll \omega$, $\omega_p/\omega \gg 1$ and $\tau\omega \gg 1$:

$$\begin{aligned}
R &= \left(2-2\sqrt{1-\frac{\omega_p^2/\omega^2}{1/\tau^2\omega^2+1}}-\frac{\omega_p^2}{\omega^2}\frac{1+1/\tau}{1/\tau^2\omega^2+1}\right)\left(2+2\sqrt{1-\frac{\omega_p^2/\omega^2}{1/\tau^2\omega^2+1}}-\frac{\omega_p^2}{\omega^2}\frac{1+1/\tau}{1/\tau^2\omega^2+1}\right)^{-1} \\
&\approx \frac{2-2\omega_p/\omega-\omega_p^2/\omega^2}{2+2\omega_p/\omega-\omega_p^2/\omega^2}
\end{aligned}$$

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When $\omega_p \ll \omega$, $\omega_p/\omega \ll 1$:

$$\begin{aligned}
R &= \left(2 - 2\sqrt{1 - \frac{\omega_p^2/\omega^2}{1/\tau^2\omega^2 + 1}} - \frac{\omega_p^2}{\omega^2}\frac{\omega^2 + \omega/\tau}{1/\tau^2 + \omega^2}\right)\left(2 + 2\sqrt{1 - \frac{\omega_p^2/\omega^2}{1/\tau^2\omega^2 + 1}} - \frac{\omega_p^2}{\omega^2}\frac{\omega^2 + \omega/\tau}{1/\tau^2 + \omega^2}\right)^{-1} \\
&\approx \frac{2 - 2\sqrt{1}}{2 + 2\sqrt{1}} \\
&= 0,
\end{aligned}$$

as desired. □

Problem 4. Phonons From Eq. (5.8) construct Im[χ] in the limit that $\gamma \to 0$. Use the Kramers–Krönig relation to then reconstruct Re[χ] from Im[χ] in the same limit.

Solution. Equation (5.8) in the lecture notes is

$$\chi(\mathbf{q}, \omega) = \frac{1}{-\rho\omega^2 + i\gamma\omega + Kq^2}.$$

Then

$$\mathrm{Im}[\chi] = -\frac{\gamma\omega}{(Kq^2 - \rho\omega)^2 + \gamma^2\omega^2}.$$

In the limit $\gamma \to 0$,

$$\mathrm{Im}[\chi] = -\frac{\gamma\omega}{(Kq^2 - \rho\omega^2)^2}.$$

The relevant Kramers–Krönig relation is given by (5.39),

$$\mathrm{Re}[\kappa(\omega)] = \mathrm{PV}\int \frac{d\omega'}{\pi} \frac{\mathrm{Im}[\kappa(\omega')]}{\omega' - \omega}.$$

Then

$$\mathrm{Re}[\kappa(\omega)] = -\mathrm{PV}\int \frac{d\omega'}{\pi} \frac{1}{\omega' - \omega} \frac{\gamma\omega'}{(Kq^2 - \rho\omega'^2)^2} =$$

partial fraction decomposition???

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