

# 1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \quad (1)$$

1.1 Verify the following:

- a.  $i\hbar\partial_t \langle \Psi(t)|x \rangle = -\langle \Psi(t)|H|x \rangle,$
- b.  $i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle,$
- c.  $i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = -\frac{\hbar^2}{2m} [\langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle],$
- d.  $\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle = \frac{\hbar}{i} [\langle \Phi(t)|x \rangle \partial_x \langle x|\Psi(t) \rangle - (\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle]$
- e.  $\frac{\hbar}{i} \partial_x [\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle] = \langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - m \langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle$

**Solution.**

- a. We will begin with the Schrödinger equation,

$$i\hbar\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle. \quad (2)$$

Since the Hamiltonian given by (1) is time independent, the system evolves in time under the time-evolution operator  $U(t) = \exp(-iHt/\hbar)$ . Denote the eigenkets of  $H$  by  $|E_i\rangle$  and the corresponding eigenvalues by  $E_i$ . Assuming  $V(x)$  is a real-valued function,  $H$  is Hermitian, and so  $|E_i\rangle$  form a complete orthonormal basis. Then we may rewrite  $|\Psi(t)\rangle$  in terms of  $U(t)$  and expand it in  $|E_i\rangle$ :

$$|\Psi(t)\rangle = U(t) |\Psi\rangle = e^{iHt/\hbar} \sum_i |E_i\rangle \langle E_i|\Psi\rangle = \sum_i e^{iE_i t/\hbar} |E_i\rangle \langle E_i|\Psi\rangle. \quad (3)$$

Substituting (3) into (2) and evaluating the time derivative,

$$-\sum_i E_i e^{iE_i t/\hbar} |E_i\rangle \langle E_i|\Psi\rangle = H \sum_i e^{iE_i t/\hbar} |E_i\rangle \langle E_i|\Psi\rangle. \quad (4)$$

Taking the adjoint of (4) yields

$$-\sum_i E_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar} = H \sum_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar}. \quad (5)$$

From the adjoint of (3), note that

$$i\hbar\partial_t \langle \Psi(t)| = i\hbar\partial_t \sum_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar} = \sum_i E_i \langle \Psi|E_i\rangle \langle E_i| e^{-iE_i t/\hbar}. \quad (6)$$

Making these substitutions into (5), and multiplying by  $|x\rangle$  on the right, we have

$$-i\hbar\partial_t \langle \Psi(t)| = H \langle \Psi(t)| \implies i\hbar\partial_t \langle \Psi(t)|x \rangle = -\langle \Psi(t)|H|x \rangle \quad (7)$$

as we sought to prove. □

- b. Rewriting what was proven in (a) with  $\Psi \mapsto \Phi$  and then multiplying by  $\Psi(x, t)$  on the right,

$$i\hbar\partial_t \langle \Phi(t)|x \rangle = -\langle \Phi(t)|H|x \rangle \quad (8)$$

$$i\hbar(\partial_t \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle = -\langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle. \quad (9)$$

Multiplying (2) by  $\langle \Phi(t)|x \rangle \langle x|$  on the left,

$$\langle \Phi(t)|x \rangle i\hbar\partial_t \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle. \quad (10)$$

Adding (10) and (9) yields

$$\langle \Phi(t)|x \rangle i\hbar\partial_t \langle x|\Psi(t) \rangle + i\hbar(\partial_t \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle \quad (11)$$

$$i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle, \quad (12)$$

where in going to (12) we have used the product rule of differentiation on the left-hand side. (12) is what we sought to prove.  $\square$

- c. Using (1), note that:

$$\langle x|H|\Psi(t) \rangle = \langle x| \left[ \frac{p^2}{2m} + V(x) \right] |\Psi(t) \rangle = \frac{1}{2m} \langle x|p^2|\Psi(t) \rangle + \langle x|V(x)|\Psi(t) \rangle \quad (13)$$

$$= -\frac{\hbar^2}{2m} \partial_x^2 \langle x|\Psi(t) \rangle + V(x) \langle x|\Psi(t) \rangle, \quad (14)$$

where in going to (14) we have (twice) used the fact that

$$\langle x|p|\Psi(t) \rangle = -i\hbar\partial_x \langle x|\Psi(t) \rangle. \quad (15)$$

Similarly, note that

$$\langle \Phi(t)|H|x \rangle = -\frac{\hbar^2}{2m} \partial_x^2 \langle \Phi(t)|x \rangle + V(x) \langle \Phi(t)|x \rangle \quad (16)$$

where we have (twice) used the adjoint of (15) with  $\Psi \mapsto \Phi$ ,

$$\langle \Phi(t)|p|x \rangle = i\hbar\partial_x \langle \Phi(t)|x \rangle. \quad (17)$$

This follows because  $p$  is Hermitian. Making the substitutions (14) and (16) into what was proven in (b),

$$\begin{aligned} i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle \\ = \langle \Phi(t)|x \rangle \left[ -\frac{\hbar^2}{2m} \partial_x^2 \langle x|\Psi(t) \rangle + V(x) \langle x|\Psi(t) \rangle \right] - \left[ -\frac{\hbar^2}{2m} \partial_x^2 \langle \Phi(t)|x \rangle + V(x) \langle \Phi(t)|x \rangle \right] \langle x|\Psi(t) \rangle \end{aligned} \quad (18)$$

$$= -\frac{\hbar^2}{2m} \left[ \langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle \right] + [V(x) - V(x)] \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle \quad (19)$$

$$= -\frac{\hbar^2}{2m} \left[ \langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle \right], \quad (20)$$

as we sought to prove.  $\square$

- d. Applying (15) and (17) to the left-hand side of (d),

$$\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle (-i\hbar\partial_x \langle x|\Psi(t) \rangle) + (i\hbar\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle \quad (21)$$

$$= \frac{\hbar}{i} \left[ \langle \Phi(t)|x \rangle \partial_x \langle x|\Psi(t) \rangle - (\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle \right] \quad (22)$$

as we sought to prove.  $\square$

- e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle) = (\partial_x\langle\Phi(t)|x\rangle)\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\partial_x\langle x|p|\Psi(t)\rangle \quad (23)$$

Multiplying through by  $\hbar/i$ ,

$$\frac{\hbar}{i}\partial_x(\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle) = (-i\hbar\partial_x\langle\Phi(t)|x\rangle)\langle x|p|\Psi(t)\rangle - \langle\Phi(t)|x\rangle i\hbar\partial_x\langle x|p|\Psi(t)\rangle \quad (24)$$

$$= -\langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle, \quad (25)$$

where in going to (25) we have used (15) and (17). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i}\partial_x(\langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle) = (-i\hbar\partial_x\langle\Phi(t)|p|x\rangle)\langle x|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle i\hbar\partial_x\langle x|\Psi(t)\rangle \quad (26)$$

$$= -\langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle. \quad (27)$$

Adding the results of (25) and (27),

$$\begin{aligned} \frac{\hbar}{i}\partial_x[\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle] \\ = \langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|p|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle \end{aligned} \quad (28)$$

$$= \langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle \quad (29)$$

as we sought to prove.  $\square$

## 1.2 Define

$$\rho(x, t) = \langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle, \quad (30)$$

$$J_x(x, t) = \frac{1}{2m} [\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle]. \quad (31)$$

Show that  $\rho(x, t) + \partial_x J_x(x, t) = 0$ .

**Solution.** From (30),

$$\partial_t \rho(x, t) = \partial_t(\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle), \quad (32)$$

and from what was proven in 1(c),

$$\partial_t(\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle) = -\frac{1}{i\hbar} [\langle\Phi(t)|x\rangle\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle] \quad (33)$$

$$= -\frac{1}{2m}\frac{i}{\hbar} [\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle], \quad (34)$$

where we have applied (15) and (17) in going to (34). Equating (32) and (34),

$$\partial_t \rho(x, t) = -\frac{1}{2m}\frac{i}{\hbar} [\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle]. \quad (35)$$

Beginning from (31),

$$\partial_x J_x(x, t) = \frac{1}{2m}\partial_x [\langle\Phi(t)|x\rangle\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle] \quad (36)$$

$$= \frac{1}{2m}\frac{i}{\hbar} [\langle\Phi(t)|x\rangle\langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle\langle x|\Psi(t)\rangle], \quad (37)$$

where in going to (37) we have used what was proven in 1(e). Summing (35) and (37), we have

$$\partial_t \rho(x, t) + \partial_x J_x(x, t) = \left( -\frac{1}{2m} \frac{i}{\hbar} + \frac{1}{2m} \frac{i}{\hbar} \right) [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle] = 0 \quad (38)$$

as we sought to prove. This is the continuity equation for probability.  $\square$

## 2 Problem 4

Consider a particle moving in three dimensions. Consider an operator

$$U(\phi) = \exp\left(-\frac{i}{\hbar} L_3 \phi\right), \quad L_3 = L_z = X P_y - Y P_x, \quad (39)$$

where  $X, Y$  and  $P_x, P_y$  are position and momentum operators, respectively. Define new operators

$$X(\phi) = U^\dagger(\phi) X U(\phi), \quad Y(\phi) = U^\dagger(\phi) Y U(\phi). \quad (40)$$

Note that  $X(0) = Y(0) = 0$ .

**2.1** Derive the equation

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi) [L_3, X] U(\phi) = -Y(\phi), \quad (41)$$

and a similar equation for  $dY(\phi)/d\phi$ .

**Solution.** Using the definition of  $X(\phi)$  in (40) and applying the product rule of differentiation,

$$\frac{dX(\phi)}{d\phi} = \frac{d}{d\phi} (U^\dagger X U) = \frac{dU^\dagger}{d\phi} X U + U^\dagger \frac{d}{d\phi} (X U) \quad (42)$$

$$= \frac{dU^\dagger}{d\phi} X U + U^\dagger \frac{dX}{d\phi} U + U^\dagger X \frac{dU}{d\phi}. \quad (43)$$

We know immediately that  $dX/d\phi = 0$  because  $\phi$  is not a parameter of the position operator  $X$ . From the definition of  $U(\phi)$  in (39), we know that  $[L_3, U(\phi)] = 0$ . Thus

$$\frac{dU}{d\phi} = -\frac{i}{\hbar} L_3 U = -\frac{i}{\hbar} L_3 \exp\left(-\frac{i}{\hbar} L_3 \phi\right) = -\frac{i}{\hbar} U L_3, \quad (44)$$

and likewise

$$U^\dagger = \exp\left(\frac{i}{\hbar} L_3 \phi\right) \implies \frac{dU^\dagger}{d\phi} = \frac{i}{\hbar} L_3 \exp\left(\frac{i}{\hbar} L_3 \phi\right) = \frac{i}{\hbar} L_3 U^\dagger = \frac{i}{\hbar} U^\dagger L_3 \quad (45)$$

because  $[L_3, U^\dagger] = 0$  as well. Then (43) becomes

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger L_3 X U - \frac{i}{\hbar} U^\dagger X L_3 U = \frac{i}{\hbar} U^\dagger (L_3 X - X L_3) U = \frac{i}{\hbar} U^\dagger(\phi) [L_3, X] U(\phi), \quad (46)$$

which is the first equality of what we wanted to show in (41).

From the definition of  $L_3$  in (39),

$$[L_3, X] = L_3X - XL_3 = (XP_y - YP_x)X - X(XP_y - YP_x) \quad (47)$$

$$= XP_yX - YP_xX - XXP_y + XY P_x = YXP_x - YP_xX \quad (48)$$

$$= Y[X, P_x] = i\hbar Y \quad (49)$$

where in (48) we have used  $[X, P_y] = [X, Y] = 0$ , and in (49) we have used  $[X, P_x] = i\hbar$ . Making the substitution (49) into (46), we have

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)(i\hbar Y)U(\phi) = -U^\dagger(\phi)YU(\phi) = -Y(\phi), \quad (50)$$

where the last equality is from the definition of  $Y(\phi)$  in (40). This is the second equality of what we wanted to show in (41), which completes the proof.

For  $dY(\phi)/d\phi$ , we can make the substitutions  $X(\phi) \mapsto Y(\phi)$ ,  $X \mapsto Y$  in (43) and (46) to obtain

$$\frac{dY(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)[L_3, Y]U(\phi). \quad (51)$$

Then making similar use of commutators  $[Y, P_x] = [X, Y] = 0$  and  $[Y, P_y] = i\hbar$  as for (48) and (49),

$$[L_3, Y] = L_3Y - YL_3 = (XP_y - YP_x)Y - Y(XP_y - YP_x) \quad (52)$$

$$= XP_yY - YP_xY - YXP_y + YY P_x = XP_yY - XY P_y \quad (53)$$

$$= X[P_y, Y] = -X[Y, P_y] = -i\hbar X. \quad (54)$$

Substituting (54) into (51),

$$\frac{dY(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)(-i\hbar X)U(\phi) = X(\phi), \quad (55)$$

and so we have derived

$$\frac{dY(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi)[L_3, Y]U(\phi) = X(\phi). \quad (56)$$

and (41) as desired.  $\square$

**2.2** Define  $X_\pm(\phi) = X(\phi) \pm iY(\phi)$ . From the results of previous parts, show  $X_+(\phi) = e^{i\phi}X_+$  where  $X_+ = X_+(0)$ . Derive the similar expression for  $X_-(\phi)$ .

**Solution.** Differentiating  $X_\pm(\phi)$  and making use of (41) and (56),

$$\frac{dX_\pm(\phi)}{d\phi} = \frac{dX(\phi)}{d\phi} \pm i \frac{dY(\phi)}{d\phi} = -Y(\phi) \pm iX(\phi) = \pm i[X(\phi) \pm iY(\phi)] \quad (57)$$

$$= \pm iX_\pm(\phi). \quad (58)$$

The differential equation (58) has solutions given by exponential functions of  $\pm i\phi$ . We will make the ansatz

$$X_\pm(\phi) = e^{\pm i\phi}C_\pm, \quad (59)$$

where  $C_\pm$  is an operator “constant” in  $\phi$  (that is, independent of it) and is fixed by an initial condition. Inspecting (59), clearly  $X_\pm(0) = C_\pm$  where it is defined  $X_\pm(0) \equiv X_\pm$ . All that remains is to show that (59) obeys the relation (58), as follows:

$$\frac{dX_\pm(\phi)}{d\phi} = \frac{d}{d\phi} \left( e^{\pm i\phi} \right) C_\pm = \pm i e^{\pm i\phi} C_\pm = \pm iX_\pm(\phi). \quad (60)$$

Thus, we have derived

$$X_+(\phi) = e^{i\phi}X_+, \quad X_-(\phi) = e^{-i\phi}X_- \quad (61)$$

as desired.  $\square$

**2.3** Show that  $[L_3, X_+] = \hbar X_+$ . Derive the similar expression for  $[L_3, X_-]$ .

**Solution.** Firstly, note that

$$X_{\pm} = X_{\pm}(0) = X(0) \pm iY(0) = U^\dagger(0)XU(0) \pm iU^\dagger(0)YU(0) = X \pm iY \quad (62)$$

because  $U(0) = U^\dagger(0) = I$ . Also applying the definition of  $L_3$  in (39), we have

$$[L_3, X_{\pm}] = [XP_y - YP_x, X \pm iY] = (XP_y - YP_x)(X \pm iY) - (X \pm iY)(XP_y - YP_x) \quad (63)$$

$$= XP_yX \pm iXP_yY - YP_xX \mp iYP_xY - XXP_y + XYP_x \mp iYXP_y \pm iYYP_x \quad (64)$$

$$= \pm iXP_yY - YP_xX + XYP_x \mp iYXP_y = \pm iX[P_y, Y] + Y[X, P_x] \quad (65)$$

$$= \pm \hbar X + i\hbar Y = \pm \hbar[X \pm iY] = \pm \hbar X_{\pm}. \quad (66)$$

Thus, we have shown

$$[L_3, X_+] = \hbar X_+, \quad [L_3, X_-] = -\hbar X_- \quad (67)$$

as desired.  $\square$

### 3 Problem 1

Consider a particle with coordinate  $x \in (-\infty, \infty)$ , and momentum  $p \in (-\infty, \infty)$ , along with corresponding operators  $X$  and  $P$ . We have

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}. \quad (68)$$

**3.1** Consider  $\langle p|X|\Psi\rangle$ . Express it in terms of  $\langle p|\Psi\rangle$ .

**Solution.** In the momentum space, the action of  $X$  is given by

$$\langle p|X|\Psi\rangle = i\hbar\partial_p \langle p|\Psi\rangle. \quad (69)$$

**3.2** Define a state  $|\Psi'\rangle$  from  $|\Psi\rangle$  by  $\langle p - p_0|\Psi\rangle = \langle p|\Psi'\rangle$ . Construct the unitary operator  $V(p_0)$  such that  $|\Psi'\rangle = V(p_0)|\Psi\rangle$ .

**Solution.** For an infinitesimal  $p_0$ ,

$$V^\dagger(p_0)|p\rangle = |p - p_0\rangle = e^{-p_0\partial_p}|p\rangle \quad (70)$$

and since  $\partial_p^\dagger = -\partial_p$  in the momentum basis,

$$V(p_0) = e^{p_0\partial_p} = e^{ip_0X/\hbar} \quad (71)$$

because  $X = -i\hbar\partial_p$  when acting on the  $|p\rangle$  basis, as given by the adjoint of (69). Then

$$\langle p|V(p_0)|\Psi\rangle = \langle p - p_0|\Psi\rangle = \langle p|\Psi'\rangle \quad (72)$$

as desired.

$V(p_0)$  has the following properties that were also required of  $U(a)$ :

1. In the limit  $p_0 \rightarrow 0$ ,  $V(p_0) \rightarrow I$ :

$$\lim_{p_0 \rightarrow 0} V(p_0) = \lim_{p_0 \rightarrow 0} e^{ip_0 X/\hbar} = e^0 = I. \quad (73)$$

2. Successive applications are equivalent to a single application:

$$V(p_1)V(p_2) = e^{ip_1 X/\hbar} e^{ip_2 X/\hbar} = e^{i(p_1+p_2)X/\hbar} = V(p_1 + p_2). \quad (74)$$

3. Unitarity:

$$V(p_0)V^\dagger(p_0) = e^{ip_0 X/\hbar} e^{-ip_0 X/\hbar} = I, \quad V^\dagger(p_0)V(p_0) = e^{-ip_0 X/\hbar} e^{ip_0 X/\hbar} = I. \quad (75)$$

**3.3** Consider  $|\Psi''\rangle = U(a)V(p_0)|\Psi\rangle$  where  $U(a)$  is the spatial translation operator. Express  $\langle x|\Psi''\rangle$  as

$$\langle x|\Psi''\rangle = \exp(i\Phi(x, a, p_0)) \langle x''|\Psi\rangle \quad (76)$$

where the phase  $\Phi$  and  $x''$  are to be determined as part of the problem.

**Solution.** Using the definition of  $|\Psi''\rangle$ ,

$$\langle x|\Psi''\rangle = \langle x|U(a)V(p_0)|\Psi\rangle = \langle x-a|V(p_0)|\Psi\rangle = \langle x-a|e^{ip_0 X/\hbar}|\Psi\rangle = e^{ip_0(x-a)/\hbar} \langle x-a|\Psi\rangle \quad (77)$$

which is equivalent to (76) with

$$\Phi = \frac{p_0(x-a)}{\hbar}, \quad x'' = x - a. \quad (78)$$

**3.4** Defining  $\langle X \rangle = \langle \Psi|X|\Psi \rangle$  and  $\langle P \rangle = \langle \Psi|P|\Psi \rangle$ , define formulas which express  $\langle \Psi''|X|\Psi'' \rangle$  and  $\langle \Psi''|P|\Psi'' \rangle$  in terms of  $\langle X \rangle$ ,  $\langle P \rangle$ , and constants.

**Solution.** Beginning with  $\langle \Psi''|V|\Psi'' \rangle$ , we may insert the identity operator:

$$\langle \Psi''|X|\Psi'' \rangle = \iint \langle \Psi''|x \rangle \langle x|X|x' \rangle \langle x'|\Psi'' \rangle dx dx' \quad (79)$$

$$= \iint \langle \Psi|x-a \rangle e^{-ip_0(x-a)/\hbar} x' \delta(x-x') e^{ip_0(x'-a)/\hbar} \langle x'-a|\Psi \rangle dx dx', \quad (80)$$

$$= \int \langle \Psi|x-a \rangle e^{-ip_0(x-a)/\hbar} x e^{ip_0(x-a)/\hbar} \langle x-a|\Psi \rangle dx \quad (81)$$

$$= \int \langle \Psi|x-a \rangle x \langle x-a|\Psi \rangle dx, \quad (82)$$

where in going to (80) we have substituted (77) and its adjoint. Now making the change of variable  $x-a \mapsto x$ , (82) becomes

$$\langle \Psi''|X|\Psi'' \rangle = \int \langle \Psi|x \rangle (x+a) \langle x|\Psi \rangle dx = \int \langle \Psi|x \rangle x \langle x|\Psi \rangle dx + a \int \langle \Psi|x \rangle \langle x|\Psi \rangle dx = \langle X \rangle + a. \quad (83)$$

Now proceeding similarly for  $\langle \Psi'' | P | \Psi'' \rangle$ ,

$$\langle \Psi'' | P | \Psi'' \rangle = \iint \langle \Psi'' | x \rangle \langle x | P | x' \rangle \langle x' | \Psi'' \rangle dx dx' \quad (84)$$

$$= \iint \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} \left( i\hbar \delta(x - x') \frac{\partial}{\partial x'} e^{ip_0(x'-a)/\hbar} \langle x' - a | \Psi \rangle \right) dx dx', \quad (85)$$

$$= i\hbar \int \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} \left( \frac{\partial}{\partial x} e^{ip_0(x-a)/\hbar} \langle x - a | \Psi \rangle \right) dx, \quad (86)$$

$$= i\hbar \int \langle \Psi | x - a \rangle e^{-ip_0(x-a)/\hbar} \left( \frac{\partial}{\partial x} e^{ip_0(x-a)/\hbar} \right) \langle x - a | \Psi \rangle dx + i\hbar \int \langle \Psi | x - a \rangle \left( \frac{\partial}{\partial x} \langle x - a | \Psi \rangle \right) dx, \quad (87)$$

$$= i\hbar \frac{ip_0}{\hbar} \int \langle \Psi | x - a \rangle \langle x - a | \Psi \rangle dx + i\hbar \int \langle \Psi | x - a \rangle \left( \frac{\partial}{\partial x} \langle x - a | \Psi \rangle \right) dx. \quad (88)$$

Again making the change of variable  $x - a \mapsto x$ , (88) becomes

$$\langle \Psi'' | P | \Psi'' \rangle = i\hbar \int \langle \Psi | x \rangle \left( \frac{\partial}{\partial x} \langle x | \Psi \rangle \right) dx - p_0 \int \langle \Psi | x \rangle \langle x | \Psi \rangle dx = \langle P \rangle - p_0. \quad (89)$$

In summary, we have found  $\langle \Psi'' | X | \Psi'' \rangle = \langle X \rangle + a$  and  $\langle \Psi'' | P | \Psi'' \rangle = \langle P \rangle - p_0$ .

## 4 Problem 2

Suppose we have a particle moving in one dimension ( $-\infty < x < \infty$ ), with quantum Hamiltonian given by

$$H(t) = H_0 - XF(t) \quad (90)$$

where

$$H_0 = \frac{P^2}{2m} + V(X) \quad (91)$$

where  $V(X)$  is the potential and  $F(t)$  is a c-number function. Consider a state ket  $|\Psi(t)\rangle$  which evolves in time according to  $|\Psi(t)\rangle = U(t, t') |\Psi(t')\rangle$ , where the unitary time-evolution operator satisfies

$$i\hbar \frac{\partial}{\partial t} U(t, t') = H(t) U(t, t'). \quad (92)$$

Define the expectation values

$$\langle X \rangle(t) = \langle \Psi(t) | X | \Psi(t) \rangle, \quad \langle P \rangle(t) = \langle \Psi(t) | P | \Psi(t) \rangle, \quad \langle H_0 \rangle(t) = \langle \Psi(t) | H_0 | \Psi(t) \rangle. \quad (93)$$

**4.1** Derive the formulas for  $\partial \langle X \rangle(t) / \partial t$  and  $\partial \langle P \rangle(t) / \partial t$ . Your results should include other expectation values. Show that your answer reduces to a classical expression if expectation values are replaced by classical values.

**Solution.** Beginning with  $X$ , the product rule of differentiation yields

$$\frac{\partial}{\partial t} \langle X \rangle(t) = \frac{\partial}{\partial t} \langle \Psi(t) | X | \Psi(t) \rangle = \langle \dot{\Psi}(t) | X | \Psi(t) \rangle + \langle \Psi(t) | \dot{X} | \Psi(t) \rangle + \langle \Psi(t) | X | \dot{\Psi}(t) \rangle, \quad (94)$$



where the dots indicate  $\partial/\partial t$ . Obviously  $\partial X/\partial t = 0$ . We can find the other two terms from the Schrödinger equation (2) and its adjoint, which was found in 1.1(a):

$$i\hbar\partial_t |\Psi(t)\rangle = H(t) |\Psi(t)\rangle \implies |\dot{\Psi}(t)\rangle = -\frac{i}{\hbar} H(t) |\Psi(t)\rangle, \quad (95)$$

$$i\hbar\partial_t \langle\Psi(t)| = -\langle\Psi(t)| H(t) \implies \langle\dot{\Psi}(t)| = \frac{i}{\hbar} \langle\Psi(t)| H(t). \quad (96)$$

Now (94) can be written

$$\frac{\partial}{\partial t} \langle X \rangle(t) = -\frac{i}{\hbar} \langle\Psi(t)| X H(t) |\Psi(t)\rangle + \frac{i}{\hbar} \langle\Psi(t)| H(t) X |\Psi(t)\rangle \quad (97)$$

$$= -\frac{i}{\hbar} \langle\Psi(t)| [X, H(t)] |\Psi(t)\rangle, \quad (98)$$

which is Ehrenfest's theorem. For the commutator,

$$[X, H(t)] = [X, P^2/(2m)] = \frac{[X, P^2]}{2m} = \frac{P[X, P] + [X, P]P}{2m} = \frac{i\hbar}{m} P, \quad (99)$$

so we find

$$\frac{\partial}{\partial t} \langle X \rangle(t) = -\frac{i}{\hbar} \frac{i\hbar}{m} \langle\Psi(t)| P |\Psi(t)\rangle = \frac{1}{m} \langle P \rangle(t). \quad (100)$$

Now for  $P$ , we have the commutator

$$[P, H(t)] = [P, V(X) - XF(t)] = P(V(X) - XF(t)) - (V(X) - XF(t))P \quad (101)$$

$$= PV(X) - PXF(t) - V(X)P + XF(t)P = [P, V(X)] + [X, P]F(t). \quad (102)$$

Note that

$$\langle x|[P, V(X)]|\Psi(t)\rangle = -i\hbar \frac{\partial V(x)}{\partial x} \langle x|\Psi(t)\rangle \implies [P, V(X)] = -i\hbar \frac{\partial V(X)}{\partial X} \quad (103)$$

so (98) with  $X \mapsto P$  yields

$$\frac{\partial}{\partial t} \langle P \rangle(t) = -\frac{i}{\hbar} \langle\Psi(t)| [P, H(t)] |\Psi(t)\rangle = -\frac{i}{\hbar} \langle\Psi(t)| \left( -i\hbar \frac{\partial V(X)}{\partial X} + i\hbar F(t) \right) |\Psi(t)\rangle \quad (104)$$

$$= -\langle\Psi(t)| \frac{\partial V(X)}{\partial X} |\Psi(t)\rangle + \langle\Psi(t)| F(t) |\Psi(t)\rangle = F(t) - \left\langle \frac{\partial V(X)}{\partial X} \right\rangle. \quad (105)$$

However, since

$$\frac{P}{m} = \frac{\partial H_0}{\partial P} = \frac{\partial H(t)}{\partial P}, \quad F(t) - \frac{\partial V(0)}{\partial X} = F(t) - \frac{\partial H_0}{\partial X} = -\frac{\partial H(t)}{\partial X}, \quad (106)$$

we can also write

$$\frac{\partial}{\partial t} \langle X \rangle(t) = \left\langle \frac{\partial H(t)}{\partial P} \right\rangle, \quad \frac{\partial}{\partial t} \langle P \rangle(t) = -\left\langle \frac{\partial H(t)}{\partial X} \right\rangle, \quad (107)$$

which appear similar to Hamilton's equations.

Now we will show that (107) reduce to classical expressions when expectation values are replaced by classical values. Let  $\langle X \rangle \mapsto x$ ,  $\langle P \rangle \mapsto p$ , and so on. Then (107) become

$$\frac{\partial}{\partial t} x(t) = \frac{\partial H(t)}{\partial p} = \frac{p}{m}, \quad (108)$$

$$\frac{\partial}{\partial t} p(t) = -\frac{\partial H(t)}{\partial x} = F(t) - \frac{\partial V(x)}{\partial x}, \quad (109)$$

where (108) is a classical expression for velocity, and (109) is a classical expression for force.  $\square$

**4.2** Derive a formula for  $\partial \langle H_0 \rangle / \partial t$  which involves only expectation values.

**Solution.**  $H_0$  is time independent, so we may again apply (98) with  $X \mapsto H_0$ . For the commutator,

$$[H_0, H(t)] = [P^2/(2m) + V(X), -XF(t)] = -F(t) \left( \frac{1}{2m} [P^2, X] + [V(X), X] \right) = F(t) \frac{i\hbar}{m} P, \quad (110)$$

so

$$\frac{\partial}{\partial t} \langle H_0 \rangle(t) = -\frac{i}{\hbar} \langle \Psi(t) | [H_0, H(t)] | \Psi(t) \rangle = -\frac{i}{\hbar} F(t) \frac{i\hbar}{m} \langle \Psi(t) | P | \Psi(t) \rangle = \frac{F(t)}{m} \langle P \rangle(t). \quad (111)$$

**4.3** Assume that  $F(t)$  vanishes for  $|t| \rightarrow \infty$ . In this case, it is useful to take  $t' \rightarrow -\infty$ . Derive a formula for the total energy put into the system by  $F(t)$  over the time interval  $(-\infty, \infty)$  for  $t$ . Your result will again involve expectation values. Here, the energy is defined in terms of the Hamiltonian without the external time-dependent force.

**Solution.** The total energy put into the system by  $F(t)$  is

$$\Delta E = \langle H_0 \rangle(t = \infty) - \langle H_0 \rangle(t = -\infty). \quad (112)$$

???

## 5 Problem 3

Consider the harmonic oscillator described by the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2}. \quad (113)$$

**5.1** Consider the Heisenberg operators  $X(t)$  and  $P(t)$ . Derive the Heisenberg equation of motion for  $X(t)$  and  $P(t)$ .

**Solution.** In general, the Heisenberg equations of motion are given by

$$\frac{dX(t)}{dt} = -\frac{i}{\hbar} [X(t), H], \quad \frac{dP(t)}{dt} = -\frac{i}{\hbar} [P(t), H]. \quad (114)$$

Using Sakurai's partial derivative formulation for evaluating commutators,

$$[X(t), H] = i\hbar \frac{\partial H}{\partial P(t)} = i\hbar \frac{P(t)}{m}, \quad [P(t), H] = -i\hbar \frac{\partial H}{\partial X(t)} = -i\hbar m\omega^2 X(t). \quad (115)$$

Making these substitutions into (114),

$$\frac{dX(t)}{dt} = -\frac{i}{\hbar} i\hbar \frac{P(t)}{m} = \frac{P(t)}{m}, \quad \frac{dP(t)}{dt} = \frac{i}{\hbar} i\hbar m\omega^2 X(t) = -m\omega^2 X(t) \quad (116)$$

are the Heisenberg equations of motion.

We can solve (116) by making use of the annihilation and creation operators,

$$A = \sqrt{\frac{m\omega}{2\hbar}} \left( X + \frac{iP}{m\omega} \right), \quad A^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( X - \frac{iP}{m\omega} \right). \quad (117)$$

Differentiating (117) and feeding in (116), we retrieve the differential equations

$$\frac{dA(t)}{dt} = \sqrt{\frac{m\omega}{2\hbar}} \left( \frac{dX(t)}{dt} + \frac{i}{m\omega} \frac{dP(t)}{dt} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left( \frac{P(t)}{m} - i\omega X(t) \right) = -i\omega A(t), \quad (118)$$

$$\frac{dA^\dagger(t)}{dt} = \sqrt{\frac{m\omega}{2\hbar}} \left( \frac{P(t)}{m} + i\omega X(t) \right) = i\omega A^\dagger(t), \quad (119)$$

which have solutions

$$A(t) = A e^{-i\omega t}, \quad A^\dagger(t) = A^\dagger e^{i\omega t}, \quad (120)$$

where  $A$  and  $A^\dagger$  are the Schrödinger representations, which are “constants” (that is, time independent) in the Heisenberg picture. In terms of  $X(t)$  and  $P(t)$ , (120) become

$$X(t) + \frac{i}{m\omega} P(t) = \left( X + \frac{i}{m\omega} P \right) e^{-i\omega t}, \quad (121)$$

$$X(t) - \frac{i}{m\omega} P(t) = \left( X - \frac{i}{m\omega} P \right) e^{i\omega t}, \quad (122)$$

where  $X$  and  $P$  are the Schrödinger representations. Adding (121) and (122),

$$X(t) = X(e^{-i\omega t} + e^{i\omega t}) + \frac{i}{m\omega} P(e^{-i\omega t} - e^{i\omega t}) = X \cos(\omega t) + \frac{P}{m\omega} \sin(\omega t). \quad (123)$$

Now subtracting (122) from (121),

$$P(t) = -im\omega \left( X(e^{-i\omega t} - e^{i\omega t}) + \frac{i}{m\omega} P(e^{-i\omega t} + e^{i\omega t}) \right) = P \cos(\omega t) - m\omega X \sin(\omega t). \quad (124)$$

The (solved) Heisenberg equations of motion are then

$$X(t) = X \cos(\omega t) + \frac{P}{m\omega} \sin(\omega t), \quad P(t) = P \cos(\omega t) - m\omega X \sin(\omega t). \quad (125)$$

**5.2** Consider the same oscillator classically. Derive the equations for  $x(t)$  and  $p(t)$  when the oscillator is released from rest at  $x = b$  at  $t = 0$ , where  $b$  is a constant.

**Solution.** Using Hamilton’s equations,

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad (126)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -m\omega^2 x \quad (127)$$

Writing (126) as  $p = m dx/dt$ , we can substitute into (127) to get a second-order equation in  $x$  only:

$$m \frac{d^2 x}{dt^2} = -m\omega^2 x \implies \frac{\partial^2 x}{\partial t^2} = -\omega^2 x \quad (128)$$

which has solutions

$$x(t) = A \cos(\omega t) + B \sin(\omega t), \quad (129)$$

$$p(t) = m\omega B \cos(\omega t) - m\omega A \sin(\omega t), \quad (130)$$

where  $A$  and  $B$  are constants. To find (130), we have applied (126). The equations are identical in form to (125).

Applying the given initial conditions, we have

$$x(0) = A = b, \quad p(0) = 0 = m\omega B \quad (131)$$

which fixes  $A$  and implies  $B = 0$ . Thus

$$x(t) = b \cos(\omega t), \quad p(t) = -m\omega b \sin(\omega t). \quad (132)$$

**5.3** Take the initial wave function to be

$$\langle x | \Psi(0) \rangle = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp\left( -\frac{m\omega(x-b)^2}{2\hbar} \right). \quad (133)$$

This is a displaced ground wave function for the oscillator. Show that  $\langle \Psi(0) | X | \Psi(0) \rangle$  and  $\langle \Psi(0) | P | \Psi(0) \rangle$  agree with the classical results you found in the previous problem.

**Solution.** Firstly,

$$\langle \Psi(0) | X | \Psi(0) \rangle = \iint \langle \Psi(0) | x \rangle \langle x | X | x' \rangle \langle x' | \Psi(0) \rangle dx dx' \quad (134)$$

$$= \sqrt{\frac{m\omega}{\pi \hbar}} \iint \exp\left( -\frac{m\omega(x-b)^2}{2\hbar} \right) x' \delta(x-x') \exp\left( -\frac{m\omega(x'-b)^2}{2\hbar} \right) dx dx' \quad (135)$$

$$= \sqrt{\frac{m\omega}{\pi \hbar}} \int x \exp\left( -\frac{m\omega(x-b)^2}{\hbar} \right) dx. \quad (136)$$

Making the change of variable

$$u = \sqrt{\frac{m\omega}{\hbar}}(x-b) \implies x = b + u\sqrt{\frac{\hbar}{m\omega}} \implies dx = \sqrt{\frac{\hbar}{m\omega}} du, \quad (137)$$

(136) becomes

$$\langle \Psi(0) | X | \Psi(0) \rangle = \frac{1}{\sqrt{\pi}} \int \left( b + u\sqrt{\frac{\hbar}{m\omega}} \right) e^{-u^2} du = \frac{b}{\sqrt{\pi}} \int e^{-u^2} du + \sqrt{\frac{\hbar}{m\pi\omega}} \int u e^{-u^2} du = b. \quad (138)$$

From the classical equation in (132),  $x(0) = b$  as well.

Secondly,

$$\langle \Psi(0) | P | \Psi(0) \rangle = \iint \langle \Psi(0) | x \rangle \langle x | P | x' \rangle \langle x' | \Psi(0) \rangle dx dx' \quad (139)$$

$$= i\hbar \sqrt{\frac{m\omega}{\pi \hbar}} \iint \exp\left( -\frac{m\omega(x-b)^2}{2\hbar} \right) \delta(x-x') \frac{\partial}{\partial x'} \exp\left( -\frac{m\omega(x'-b)^2}{2\hbar} \right) dx dx' \quad (140)$$

$$= i\hbar \sqrt{\frac{m\omega}{\pi \hbar}} \int \exp\left( -\frac{m\omega(x-b)^2}{2\hbar} \right) \frac{\partial}{\partial x} \exp\left( -\frac{m\omega(x-b)^2}{2\hbar} \right) dx. \quad (141)$$

Again making the change of variable (137), note that  $\partial/\partial x = \sqrt{m\omega/\hbar} \partial/\partial u$ . Making these substitutions in (141),

$$\langle \Psi(0)|P|\Psi(0) \rangle = i \frac{m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \frac{\partial}{\partial u} e^{-u^2/2} du = -i \frac{m\omega}{\sqrt{\pi}} \int u e^{-u^2} du = 0. \quad (142)$$

From the classical equation in (132),  $p(0) = 0$  as well. So the results agree with the classical limit for both cases, as we wanted to show.  $\square$

**5.4** Now consider uncertainties at  $t = 0$ . Define

$$\langle \Delta X^2 \rangle = \langle \Psi(0)|X^2|\Psi(0) \rangle - (\langle \Psi(0)|X|\Psi(0) \rangle)^2, \quad \langle \Delta P^2 \rangle = \langle \Psi(0)|P^2|\Psi(0) \rangle - (\langle \Psi(0)|P|\Psi(0) \rangle)^2. \quad (143)$$

Calculate  $\langle \Delta X^2 \rangle \langle \Delta P^2 \rangle$ .

**Solution.** Once again using the change of variable (137),

$$\langle \Psi(0)|X^2|\Psi(0) \rangle = \iint \langle \Psi(0)|x \rangle \langle x|X^2|x' \rangle \langle x'|\Psi(0) \rangle dx dx' \quad (144)$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) x'^2 \delta(x-x') \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) dx dx' \quad (145)$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int x^2 \exp\left(-\frac{m\omega(x-b)^2}{\hbar}\right) dx = \frac{1}{\sqrt{\pi}} \int \left(b + u\sqrt{\frac{\hbar}{m\omega}}\right)^2 e^{-u^2} du \quad (146)$$

$$= \frac{b^2}{\sqrt{\pi}} \int e^{-u^2} du + \frac{2b}{\sqrt{\pi}} \sqrt{\frac{\hbar}{m\omega}} \int u e^{-u^2} du + \frac{1}{\sqrt{\pi}} \frac{\hbar}{m\omega} \int u^2 e^{-u^2} du \quad (147)$$

$$= b^2 + \frac{\hbar}{2m\omega}, \quad (148)$$

and

$$\langle \Psi(0)|P^2|\Psi(0) \rangle = \iint \langle \Psi(0)|x \rangle \langle x|P^2|x' \rangle \langle x'|\Psi(0) \rangle dx dx' \quad (149)$$

$$= -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \delta(x-x') \frac{\partial^2}{\partial x'^2} \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) dx dx' \quad (150)$$

$$= -\hbar^2 \sqrt{\frac{m\omega}{\pi\hbar}} \int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \frac{\partial^2}{\partial x^2} \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) dx \quad (151)$$

$$= -\frac{\hbar m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \frac{\partial^2}{\partial u^2} e^{-u^2/2} du = -\frac{\hbar m\omega}{\sqrt{\pi}} \int (u^2 - 1) e^{-u^2} du \quad (152)$$

$$= -\frac{\hbar m\omega}{\sqrt{\pi}} \left( \frac{\sqrt{\pi}}{2} - \sqrt{\pi} \right) = \frac{\hbar m\omega}{2} \quad (153)$$

. Then, using the results from (138) and (142),

$$\langle \Delta X^2 \rangle = b^2 + \frac{\hbar}{2m\omega} - b^2 = \frac{\hbar}{2m\omega}, \quad \langle \Delta P^2 \rangle = \frac{\hbar m\omega}{2}, \quad \langle \Delta X^2 \rangle \langle \Delta P^2 \rangle = \frac{\hbar^2}{4}. \quad (154)$$

**5.5** Now consider

$$\langle \Delta X^2 \rangle(t) = \langle \Psi(t)|X^2|\Psi(t) \rangle - (\langle \Psi(t)|X|\Psi(t) \rangle)^2, \quad \langle \Delta P^2 \rangle(t) = \langle \Psi(t)|P^2|\Psi(t) \rangle - (\langle \Psi(t)|P|\Psi(t) \rangle)^2. \quad (155)$$

Calculate  $\langle \Delta X^2 \rangle(t) \langle \Delta P^2 \rangle(t)$ .

**Solution.** In the Heisenberg picture, (155) becomes

$$\langle \Delta X^2 \rangle(t) = \langle \Psi(0) | X(t)^2 | \Psi(0) \rangle - (\langle \Psi(0) | X(t) | \Psi(0) \rangle)^2, \quad (156)$$

$$\langle \Delta P^2 \rangle(t) = \langle \Psi(0) | P(t)^2 | \Psi(0) \rangle - (\langle \Psi(0) | P(t) | \Psi(0) \rangle)^2, \quad (157)$$

Using the expressions for  $X(t)$  and  $P(t)$  in (125),

$$\langle \Psi(0) | X(t) | \Psi(0) \rangle = \langle \Psi(0) | \left( X \cos(\omega t) + \frac{P}{m\omega} \sin(\omega t) \right) | \Psi(0) \rangle = \cos(\omega t) \langle X \rangle + \frac{\sin(\omega t)}{m\omega} \langle P \rangle = b \cos(\omega t) \quad (158)$$

where we use the notation  $\langle X \rangle = \langle \Psi(0) | X | \Psi(0) \rangle$  and so forth, as well as the results of (138) and (142). Continuing on,

$$\langle \Psi(0) | X(t)^2 | \Psi(0) \rangle = \langle \Psi(0) | \left( X \cos(\omega t) + \frac{P}{m\omega} \sin(\omega t) \right)^2 | \Psi(0) \rangle \quad (159)$$

$$= \cos^2(\omega t) \langle X^2 \rangle + \frac{\sin^2(\omega t)}{m^2 \omega^2} \langle P^2 \rangle + \frac{\cos(\omega t) \sin(\omega t)}{m\omega} \langle XP \rangle + \frac{\cos(\omega t) \sin(\omega t)}{m\omega} \langle PX \rangle. \quad (160)$$

Again using the change of variable (137), note that

$$\langle \Psi(0) | XP | \Psi(0) \rangle = \iint \langle \Psi(0) | x \rangle \langle x | XP | x' \rangle \langle x' | \Psi(0) \rangle dx dx' \quad (161)$$

$$= i\hbar \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) x' \delta(x-x') \frac{\partial}{\partial x'} \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) dx dx' \quad (162)$$

$$= i\hbar \sqrt{\frac{m\omega}{\pi\hbar}} \int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) x \frac{\partial}{\partial x} \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) dx \quad (163)$$

$$= i \frac{m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \left( b + u \sqrt{\frac{\hbar}{m\omega}} \right) \frac{\partial}{\partial u} e^{-u^2/2} du = -i \frac{m\omega}{\sqrt{\pi}} \int u e^{-u^2} \left( b + u \sqrt{\frac{\hbar}{m\omega}} \right) du \quad (164)$$

$$= -ib \frac{m\omega}{\sqrt{\pi}} \int u e^{-u^2/2} du - i \frac{m\omega}{\sqrt{\pi}} \sqrt{\frac{\hbar}{m\omega}} \int u^2 e^{-u^2} du = -i \frac{m\omega}{2} \sqrt{\frac{\hbar}{m\omega}}, \quad (165)$$

and

$$\langle \Psi(0) | PX | \Psi(0) \rangle = \iint \langle \Psi(0) | x \rangle \langle x | PX | x' \rangle \langle x' | \Psi(0) \rangle dx dx' \quad (166)$$

$$= i\hbar \sqrt{\frac{m\omega}{\pi\hbar}} \iint \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \delta(x-x') \frac{\partial}{\partial x'} x' \exp\left(-\frac{m\omega(x'-b)^2}{2\hbar}\right) dx dx' \quad (167)$$

$$= i\hbar \sqrt{\frac{m\omega}{\pi\hbar}} \int \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \left( \frac{\partial}{\partial x} x \exp\left(-\frac{m\omega(x-b)^2}{2\hbar}\right) \right) dx \quad (168)$$

$$= i \frac{m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \left[ \frac{\partial}{\partial u} \left( b + u \sqrt{\frac{\hbar}{m\omega}} \right) e^{-u^2/2} \right] du \quad (169)$$

$$= ib \frac{m\omega}{\sqrt{\pi}} \int e^{-u^2/2} \frac{\partial}{\partial u} e^{-u^2/2} du + i \frac{m\omega}{\sqrt{\pi}} \sqrt{\frac{\hbar}{m\omega}} \int e^{-u^2/2} \left( \frac{\partial}{\partial u} u e^{-u^2/2} \right) du \quad (170)$$

$$= -ib \frac{m\omega}{\sqrt{\pi}} \int u e^{-u^2} du + i \frac{m\omega}{\sqrt{\pi}} \sqrt{\frac{\hbar}{m\omega}} \int (1-u^2) e^{-u^2} du = i \frac{m\omega}{\sqrt{\pi}} \sqrt{\frac{\hbar}{m\omega}} \left( \sqrt{\pi} - \frac{\sqrt{\pi}}{2} \right) \quad (171)$$

$$= i \frac{m\omega}{2} \sqrt{\frac{\hbar}{m\omega}}, \quad (172)$$

so  $\langle XP \rangle + \langle PX \rangle = 0$  and

$$\langle \Psi(0) | X(t)^2 | \Psi(0) \rangle = \cos^2(\omega t) \langle X^2 \rangle + \frac{\sin^2(\omega t)}{m^2 \omega^2} \langle P^2 \rangle = \cos^2(\omega t) \left( b^2 + \frac{\hbar}{2m\omega} \right) + \sin^2(\omega t) \frac{\hbar}{2m\omega} \quad (173)$$

$$= b^2 \cos^2(\omega t) + \frac{\hbar}{2m\omega}, \quad (174)$$

where we have substituted (148) and (153). Similarly,

$$\langle \Psi(0) | P(t) | \Psi(0) \rangle = \langle \Psi(0) | (P \cos(\omega t) - m\omega X \sin(\omega t)) | \Psi(0) \rangle = \cos(\omega t) \langle P \rangle - m\omega \sin(\omega t) \langle X \rangle \quad (175)$$

$$= -bm\omega \sin(\omega t), \quad (176)$$

and

$$\langle \Psi(0) | P(t)^2 | \Psi(0) \rangle = \langle \Psi(0) | (P \cos(\omega t) - m\omega X \sin(\omega t))^2 | \Psi(0) \rangle \quad (177)$$

$$= \cos^2(\omega t) \langle P^2 \rangle + m^2 \omega^2 \sin^2(\omega t) \langle X^2 \rangle - m\omega \sin(\omega t) \cos(\omega t) (\langle PX \rangle + \langle XP \rangle) \quad (178)$$

$$= \frac{\hbar m \omega}{2} \cos^2(\omega t) + m^2 \omega^2 \sin^2(\omega t) \left( b^2 + \frac{\hbar}{2m\omega} \right) = b^2 m^2 \omega^2 \sin^2(\omega t) + \frac{\hbar m \omega}{2}. \quad (179)$$

Using (158), (174), (176), and (179)

$$\langle \Delta X^2 \rangle(t) = b^2 \cos^2(\omega t) + \frac{\hbar}{2m\omega} - [b \cos(\omega t)]^2 = \frac{\hbar}{2m\omega}, \quad (180)$$

$$\langle \Delta P^2 \rangle(t) = b^2 m^2 \omega^2 \sin^2(\omega t) + \frac{\hbar m \omega}{2} - [-bm\omega \sin(\omega t)]^2 = \frac{\hbar m \omega}{2}. \quad (181)$$

Finally,

$$\langle \Delta X^2 \rangle(t) \langle \Delta P^2 \rangle(t) = \frac{\hbar}{4} \quad (182)$$

which is unchanged from the initial state of the system considered in 5.4.