Problem 1. (Peskin & Schroeder 2.1) Classical electromagnetism (with no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

1(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying

$$E^i = -F^{0i}; \qquad \epsilon^{ijk} B^k = -F^{ij}. \tag{2}$$

Solution. We want to extremize the action,

$$S[A_\mu] = \int d^4x \, \mathcal{L}(A_\mu, \partial_\mu A_\mu),$$

where $\mathcal{L}$ is the integrand of Eq. (1). Let δA_μ denote some arbitrary variation that vanishes at the boundaries of spacetime. The action for $A_\mu + \delta A_\mu$ is

$$S[A_\mu + \delta A_\mu] = \int d^4x \, \mathcal{L}(A_\mu + \delta A_\mu, \partial_\nu A_\mu + \partial_\nu \delta A_\mu).$$

Then, to first order in δA_μ, the variation of the action is

$$\delta S = S[A_\mu + \delta A_\mu] - S[A_\mu],$$

which we want to vanish for all δA_μ. Let $\delta F_{\mu\nu} = \partial_\mu \, \delta A_\nu - \partial_\nu \, \delta A_\mu$. Then, applying the definition of $F_{\mu\nu}$ given in Eq. (1),

$$\begin{aligned}
\delta S &= \int d^4x \left(-\frac{1}{4} (F_{\mu\nu} + \delta F_{\mu\nu})(F^{\mu\nu} + \delta F^{\mu\nu}) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\
&\approx \int d^4x \left(-\frac{1}{4} (F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \, \delta F^{\mu\nu} + \delta F_{\mu\nu} \, F^{\mu\nu}) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \\
&= \int d^4x \left(-\frac{1}{4} (F_{\mu\nu} \, \delta F^{\mu\nu} + \delta F_{\mu\nu} \, F^{\mu\nu}) \right) \\
&= \int d^4x \left(-\frac{1}{2} \delta F_{\mu\nu} \, F^{\mu\nu} \right),
\end{aligned} \tag{3}$$

where we have discarded terms of $\mathcal{O}\big((\delta A^\mu)^2\big)$ and swapped covariant and contravariant indices in going to the final equality.

Note that

$$\begin{aligned}
\delta F_{\mu\nu} \, F^{\mu\nu} &= (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \delta A_\nu \, \partial^\mu A^\nu - \partial_\mu \delta A_\nu \, \partial^\nu A^\mu - \partial_\nu \delta A_\mu \, \partial^\mu A^\nu + \partial_\nu \delta A_\mu \, \partial^\nu A^\mu.
\end{aligned} \tag{4}$$

Integrating the first term of Eq. (4) by parts, we have

$$\int d^4x \, \frac{\partial \, \delta A_\nu}{\partial x^\mu} \frac{\partial A^\nu}{\partial x_\mu} = \left[\delta A_\nu \frac{\partial A^\nu}{\partial x_\mu} \right]_{-\infty}^{\infty} - \int d^4x \, \delta A_\nu \frac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} = - \int d^4x \, \delta A_\nu \, \partial_\mu \partial^\mu A^\nu,$$

because δA^ν vanishes at $\pm\infty$. The other terms follow similarly. Then we find

$$\begin{aligned}\int d^4x\,\delta F_{\mu\nu}\,F^{\mu\nu} &= -\int d^4x\,(\delta A_\nu\,\partial_\mu\partial^\mu A^\nu - \delta A_\nu\,\partial_\mu\partial^\nu A^\mu - \delta A_\mu\,\partial_\nu\partial^\mu A^\nu + \delta A_\mu\,\partial_\nu\partial^\nu A^\mu)\\ &= -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\mu\,\partial_\nu F^{\nu\mu}) = -\int d^4x\,(\delta A_\nu\,\partial_\mu F^{\mu\nu} + \delta A_\nu\,\partial_\mu F^{\mu\nu})\\ &= -2\int d^4x\,\delta A_\nu\,\partial_\mu F^{\mu\nu},\end{aligned}$$

where in going to the second-to-last equality we have simply swapped the indices.

Making this substitution in Eq. (3), we obtain

$$\delta S = \delta A_\nu \int d^4x\,\partial_\mu F^{\mu\nu}.$$

In order for the action to be at a local extremum, we need $\delta S = 0$ for any δA_ν. This implies that the integrand is 0. Thus, we obtain

$$\partial_\mu F^{\mu\nu} = 0, \tag{5}$$

which is the covariant form of the inhomogeneous Maxwell equations in a source-free region [1, p. 557], as we sought to derive. □

From Eq. (2) and the knowledge that $F^{\mu\nu}$ is antisymmetric [1, p. 556], we can write

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}. \tag{6}$$

The first equation of Eq. (2) is equivalent to $E^i = F^{i0}$. Then the zeroth component of Eq. (5) can be written

$$\partial_\mu F^{\mu 0} = \frac{\partial E^x}{\partial x} + \frac{\partial E^y}{\partial y} + \frac{\partial E^z}{\partial z} = \boldsymbol{\nabla}\cdot\mathbf{E} = 0,$$

which is the differential form of Gauss's law.

For the remaining components of Eq. (5), we apply the second equation of Eq. (5) to find

$$\partial_\mu F^{\mu i} = -\frac{\partial E^i}{\partial t} + \epsilon^{ijk}\frac{\partial B^k}{\partial x^j} = 0.$$

In vector form, this is

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0},$$

the differential form of Ampère's law.

1(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{7}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu, \tag{8}$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{E^2 + B^2}{2}; \qquad\qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

Solution. We want to evaluate Eq. (2.17) of Peskin & Schroeder,

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L}\delta^{\mu}{}_{\nu} \quad \Longrightarrow \quad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - \mathcal{L} g^{\mu\nu}, \tag{9}$$

where we have associated the field ϕ with A^λ. In order to evaluate the derivatives, we can use the variational method to calculate $\partial\mathcal{L}/\partial(\partial_\alpha A_\beta)$ by letting $\partial_\alpha A_\beta \to \partial_\alpha A_\beta + \delta\partial_\alpha A_\beta$ [2, p. 81]. Let

$$\delta\mathcal{L} = \mathcal{L}(\partial_\alpha A_\beta) - \mathcal{L}(\partial_\alpha A_\beta + \delta\partial_\alpha A_\beta).$$

Note that

$$\mathcal{L}(\partial_\alpha A_\beta + \delta\partial_\alpha A_\beta) = -\frac{1}{4}(F_{\alpha\beta} + \delta F_{\alpha\beta})(F^{\alpha\beta} + \delta F^{\alpha\beta}) \approx -\frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta} + F_{\alpha\beta}\,\delta F^{\alpha\beta} + \delta F_{\alpha\beta}\,F^{\alpha\beta}),$$

so

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{4}(F_{\alpha\beta}\,\delta F^{\alpha\beta} + \delta F_{\alpha\beta}\,F^{\alpha\beta}) = -\frac{1}{2}\delta F_{\alpha\beta}\,F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha\,\delta A_\beta - \partial_\beta\,\delta A_\alpha)F^{\alpha\beta} = -\frac{1}{2}(\partial_\alpha\,\delta A_\beta + \partial_\alpha\,\delta A_\beta)F^{\alpha\beta} \\ &= -\partial_\alpha\,\delta A_\beta\,F^{\alpha\beta}, \end{aligned}$$

where we have used the antisymmetry of $F^{\alpha\beta}$. This gives us

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta} \quad \Longrightarrow \quad \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A^\beta)} = -F^{\alpha}{}_{\beta},$$

and then we find

$$T^{\mu\nu} = -F^{\mu}{}_{\lambda}\,\partial^\nu A^\lambda + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu}{}_{\lambda}\,\partial^\nu A^\lambda. \tag{10}$$

Adding $K^{\lambda\mu\nu}$ as defined in Eq. (8), Eq. (7) becomes

$$\hat{T}^{\mu\nu} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu}{}_{\lambda}\,\partial^\nu A^\lambda + \partial_\lambda(F^{\mu\lambda}A^\nu). \tag{11}$$

Applying the product rule to the third term, we find

$$\partial_\lambda(F^{\mu\lambda}A^\nu) = A^\nu\partial_\lambda F^{\mu\lambda} + F^{\mu\lambda}\partial_\lambda A^\nu = -A^\nu\partial_\lambda F^{\lambda\mu} + F^{\mu\lambda}\partial_\lambda A^\nu = F^{\mu\lambda}\partial_\lambda A^\nu,$$

where we have applied the antisymmetry of $F^{\mu\nu}$ and Eq. (5). Making this substitution in Eq. (11),

$$\begin{aligned} \hat{T}^{\mu\nu} &= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu}{}_{\lambda}\,\partial^\nu A^\lambda + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu}{}_{\lambda}\partial^\lambda A^\nu - F^{\mu}{}_{\lambda}\,\partial^\nu A^\lambda = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F^{\mu}{}_{\lambda}F^{\lambda\nu} \\ &= \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^{\nu}{}_{\lambda}. \end{aligned} \tag{12}$$

To show that $\hat{T}^{\mu\nu}$ is symmetric, note that

$$\hat{T}^{\nu\mu} = \frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta} - F^{\nu\lambda}F^{\mu}{}_{\lambda} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu}{}_{\lambda}F^{\nu\lambda} = \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\lambda}F^{\nu}{}_{\lambda} = \hat{T}^{\mu\nu}$$

as desired. □

For the energy and momentum densities, from Eq. (12) we have

$$\hat{T}^{00} = \frac{1}{4}g^{00}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^{0}{}_{\lambda} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + F^{0\lambda}F_{\lambda}{}^{0}, \tag{13}$$

$$\hat{T}^{0i} = \frac{1}{4}g^{0i}F_{\alpha\beta}F^{\alpha\beta} - F^{0\lambda}F^{i}{}_{\lambda} + = F^{0\lambda}F_{\lambda}{}^{i}. \tag{14}$$

Using Eq. (6),

$$F_{\mu\nu}F^{\mu\nu} = -E^{x2} - E^{y2} - E^{z2} - E^{x2} + B^{z2} + B^{y2} - E^{y2} + B^{z2} + B^{x2} - E^{z2} + B^{y2} + B^{x2} = 2(\mathbf{B}^2 - \mathbf{E}^2).$$

Note also from Eq. (6) that

$$F_{\lambda}{}^{\nu} = g_{\lambda\mu}F^{\mu\nu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{bmatrix},$$

so

$$F^{0\lambda}F_{\lambda}{}^{0} = (-\mathbf{E}) \cdot (-\mathbf{E}) = \mathbf{E}^2, \qquad F^{0\lambda}F_{\lambda}{}^{i} = B_jE_k - E_kB_j = (\mathbf{E} \times \mathbf{B})_i.$$

Equations (13–14) are then

$$\hat{T}^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \mathcal{E}, \qquad \hat{T}^{0i} = \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})_i = \mathbf{S},$$

as we sought to show. □

Problem 2. The complex scalar field (Peskin & Schroeder 2.2) Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x\,(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi). \tag{15}$$

It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

2(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x\,(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi). \tag{16}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

Solution. The momentum density conjugate to $\phi(x)$ is defined in Peskin & Schroeder (2.4):

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}$$

Here, $\mathcal{L}$ is the integrand of Eq. (15). Expanding its first term yields

$$\mathcal{L} = \dot{\phi}\dot{\phi}^* - \nabla\phi \cdot \nabla\phi^*, \tag{17}$$

so then

$$\pi(x) = \dot{\phi}^*, \qquad \pi^*(x) = \dot{\phi}, \tag{18}$$

where $\pi^*(x)$ is the momentum conjugate to $\phi^*(x)$. The canonical commutation relations follow from Peskin & Schroeder (2.20):

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\,\delta^3(\mathbf{x}-\mathbf{y}), \tag{19}$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\phi^*(\mathbf{x}), \phi^*(\mathbf{y})] = 0, \tag{20}$$
$$[\pi(\mathbf{x}), \pi(\mathbf{y})] = [\pi^*(\mathbf{x}), \pi^*(\mathbf{y})] = 0, \tag{21}$$
$$[\phi(\mathbf{x}), \pi^*(\mathbf{y})] = [\phi(\mathbf{x}), \phi^*(\mathbf{y})] = [\pi(\mathbf{x}), \pi^*(\mathbf{y})] = 0. \tag{22}$$

The Hamiltonian is given in general for a single field by Peskin & Schroeder (2.5),

$$H = \int d^3x \left(\pi(x)\,\dot{\phi}(x) - \mathcal{L}\right).$$

For the two fields $\phi(x)$ and $\phi^*(x)$, this becomes

$$\begin{aligned}
H &= \int d^3x \left(\pi(x)\,\dot{\phi}(x) + \pi^*(x)\,\dot{\phi}^*(x) - \mathcal{L}\right) \\
&= \int d^3x \left(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \dot{\phi}\dot{\phi}^* + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi^* + m^2\phi^*\phi\right) \\
&= \int d^3x \left(\pi\pi^* + \dot{\phi}\dot{\phi}^* - \dot{\phi}\dot{\phi}^* + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi^* + m^2\phi^*\phi\right) \\
&= \int d^3x \left(\pi^*\pi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi^* + m^2\phi^*\phi\right),
\end{aligned}$$

where we have used Eqs. (17) and (18) as well as the commutation relations. So we have proven Eq. (16). □

The Heisenberg equation of motion is Peskin & Schroeder (2.44),

$$i\frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H],$$

where $\mathcal{O}$ is an arbitrary operator. Then

$$\begin{aligned}
i\frac{\partial\phi(x)}{\partial t} &= [\phi(x), H] \\
&= \left[\phi(\mathbf{x},t), \int d^3x'\,\pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] + \left[\phi(\mathbf{x},t), \int d^3x'\,\boldsymbol{\nabla}'\phi(\mathbf{x}',t)\cdot\boldsymbol{\nabla}'\phi^*(\mathbf{x}',t)\right] \\
&\qquad + m^2\left[\phi(\mathbf{x},t), \int d^3x'\,\phi^*(\mathbf{x}',t)\,\phi(\mathbf{x}',t)\right] \\
&= \left[\phi(\mathbf{x},t), \int d^3x'\,\pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] = i\int d^3x'\,\delta^3(\mathbf{x}-\mathbf{x}')\,\pi^*(\mathbf{x}',t) = i\pi^*(x),
\end{aligned}$$

$$\begin{aligned}
i\frac{\partial\phi^*(x)}{\partial t} &= [\phi^*(x), H] \\
&= \left[\phi^*(\mathbf{x},t), \int d^3x'\,\pi^*(\mathbf{x}',t)\,\pi(\mathbf{x}',t)\right] = i\int d^3x'\,\delta^3(\mathbf{x}-\mathbf{x}')\,\pi(\mathbf{x}',t) = i\pi(x),
\end{aligned}$$

$$
\begin{aligned}
i\frac{\partial \pi(x)}{\partial t} &= [\pi(x), H] \\
&= \left[\pi(\mathbf{x},t), \int d^3x'\, \pi^*(\mathbf{x}',t)\, \pi(\mathbf{x}',t)\right] + \left[\pi(\mathbf{x},t), \int d^3x'\, \boldsymbol{\nabla}'\phi(\mathbf{x}',t) \cdot \boldsymbol{\nabla}'\phi^*(\mathbf{x}',t)\right] \\
&\qquad + m^2 \left[\pi(\mathbf{x},t), \int d^3x'\, \phi^*(\mathbf{x}',t)\, \phi(\mathbf{x}',t)\right] \\
&= -i \int d^3x' \left[\boldsymbol{\nabla}'\delta^3(\mathbf{x}-\mathbf{x}') \cdot \boldsymbol{\nabla}'\phi^*(\mathbf{x}',t) + m^2\delta^3(\mathbf{x}-\mathbf{x}')\, \phi^*(\mathbf{x}',t)\right] \\
&= -i \int d^3x' \left[-\delta^3(\mathbf{x}-\mathbf{x}') \cdot \nabla'^2\phi^*(\mathbf{x}',t) + m^2\delta^3(\mathbf{x}-\mathbf{x}')\, \phi^*(\mathbf{x}',t)\right] = -i(-\nabla^2 + m^2)\, \phi^*(x),
\end{aligned}
$$

$$
\begin{aligned}
i\frac{\partial \pi^*(x)}{\partial t} &= [\pi^*(x), H] \\
&= -i \int d^3x' \left[\boldsymbol{\nabla}'\phi(\mathbf{x}',t) \cdot \boldsymbol{\nabla}'\delta(\mathbf{x}-\mathbf{x}') + m^2\delta^3(\mathbf{x}-\mathbf{x}')\, \phi(\mathbf{x}',t)\right] \\
&= -i \int d^3x' \left[-\delta^3(\mathbf{x}-\mathbf{x}') \cdot \nabla'^2\phi(\mathbf{x}',t) + m^2\delta^3(\mathbf{x}-\mathbf{x}')\, \phi(\mathbf{x}',t)\right] = -i(-\nabla^2 + m^2)\, \phi(x).
\end{aligned}
$$

Thus we have obtained

$$
\frac{\partial \phi(x)}{\partial t} = \pi^*(x), \qquad \frac{\partial \phi^*(x)}{\partial t} = \pi(x), \qquad \frac{\partial \pi(x)}{\partial t} = (\nabla^2 - m^2)\, \phi^*(x), \qquad \frac{\partial \pi^*(x)}{\partial t} = (\nabla^2 - m^2)\, \phi(x).
$$

Combining these results yields

$$
\frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - m^2)\phi, \qquad\qquad \frac{\partial^2 \phi^*}{\partial t^2} = (\nabla^2 - m^2)\phi^*,
$$

which is the Klein-Gordon equation and its complex conjugate, as we sought to show. □

2(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

Solution. Peskin & Schroeder (2.21) gives the Klein-Gordon equation in the momentum basis,

$$
\left(\frac{\partial^2}{\partial t^2} + \mathbf{p}^2 + m^2\right) \phi(\mathbf{p},t) = 0.
$$

This is the same as the harmonic oscillator equation of motion. It has solutions [3]

$$
\phi(\mathbf{p},t) = A(\mathbf{p})\, e^{i\omega_{\mathbf{p}}t} + B(\mathbf{p})\, e^{-i\omega_{\mathbf{p}}t},
$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ as in Peskin & Schroeder Eq. (2.22), and $A(\mathbf{p})$ and $B(\mathbf{p})$ are arbitrary functions of $\mathbf{p}$. The complex conjugate of this solution is

$$
\phi^*(\mathbf{p},t) = B^*(\mathbf{p})\, e^{i\omega_{\mathbf{p}}t} + A^*(\mathbf{p})\, e^{-i\omega_{\mathbf{p}}t}.
$$

The field ϕ in the position basis can be expanded as [4, p. 20],

$$
\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\, \phi(\mathbf{p},t).
$$

so we can write, as explained in class,

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \qquad \phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(b_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

where $a_\mathbf{p}^\dagger, b_\mathbf{p}^\dagger$ ($a_\mathbf{p}, b_\mathbf{p}$) are creation (annihilation) operators. By analogy to Eq. (2.26) of Peskin & Schroeder, we can also write

$$\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(b_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} - a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \qquad \pi^*(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} - b_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right).$$

Simplifying these expressions as in their Eqs. (2.27) and (2.28), we have

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(a_\mathbf{p} + b_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \phi^*(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(b_\mathbf{p} + a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{23}$$

$$\pi(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(b_\mathbf{p} - a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p}\cdot\mathbf{x}}, \qquad \pi^*(\mathbf{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(a_\mathbf{p} - b_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{24}$$

Also generalizing their Eq. (2.24),

$$[a_\mathbf{p}, a_{\mathbf{p}'}^\dagger] = [b_\mathbf{p}, b_{\mathbf{p}'}^\dagger] = (2\pi)^3 \, \delta^3(\mathbf{p} - \mathbf{p}'), \qquad [a_\mathbf{p}, b_{\mathbf{p}'}^\dagger] = [b_\mathbf{p}, a_{\mathbf{p}'}^\dagger] = 0.$$

Feeding Eqs. (23) and (24) into Eq. (16) yields

$$H = \int d^3x \int \frac{d^3p \, d^3p'}{(2\pi)^6} \, e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \left[-\frac{\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}{2} \left(a_\mathbf{p} - b_{-\mathbf{p}}^\dagger \right) \left(b_{\mathbf{p}'} - a_{-\mathbf{p}'}^\dagger \right) + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}} \left(a_\mathbf{p} + b_{-\mathbf{p}}^\dagger \right) \left(b_{\mathbf{p}'} + a_{-\mathbf{p}'}^\dagger \right) \right].$$

Using the delta function identity [5]

$$\delta(x-a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-a)} \, dp \, ,$$

this becomes

$$\begin{aligned}
H &= \int \frac{d^3p \, d^3p'}{(2\pi)^3} \, \delta^3(\mathbf{p} + \mathbf{p}') \left[-\frac{\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}{2} \left(a_\mathbf{p} - b_{-\mathbf{p}}^\dagger \right) \left(b_{\mathbf{p}'} - a_{-\mathbf{p}'}^\dagger \right) + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}} \left(a_\mathbf{p} + b_{-\mathbf{p}}^\dagger \right) \left(b_{\mathbf{p}'} + a_{-\mathbf{p}'}^\dagger \right) \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \left[-\frac{\omega_\mathbf{p}}{2} \left(a_\mathbf{p} - b_{-\mathbf{p}}^\dagger \right) \left(b_{-\mathbf{p}} - a_\mathbf{p}^\dagger \right) + \frac{\mathbf{p}^2 + m^2}{2\omega_\mathbf{p}} \left(a_\mathbf{p} + b_{-\mathbf{p}}^\dagger \right) \left(b_{-\mathbf{p}} + a_\mathbf{p}^\dagger \right) \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_\mathbf{p}}{2} \left[a_\mathbf{p} b_{-\mathbf{p}} + a_\mathbf{p} a_\mathbf{p}^\dagger + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} + b_{-\mathbf{p}}^\dagger a_\mathbf{p}^\dagger - \left(a_\mathbf{p} b_{-\mathbf{p}} - a_\mathbf{p} a_\mathbf{p}^\dagger - b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} + b_{-\mathbf{p}}^\dagger a_\mathbf{p}^\dagger \right) \right] \\
&= \int \frac{d^3p}{(2\pi)^3} \, \omega_\mathbf{p} \left(a_\mathbf{p} a_\mathbf{p}^\dagger + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} \right) = \int \frac{d^3p}{(2\pi)^3} \, \omega_\mathbf{p} \left(a_\mathbf{p}^\dagger a_\mathbf{p} + b_\mathbf{p}^\dagger b_\mathbf{p} + [a_\mathbf{p}, a_{\mathbf{p}'}^\dagger] \right).
\end{aligned}$$

Ignoring the infinite constant term [4, p. 21], we have

$$H = \int \frac{d^3p}{(2\pi)^3} \, \omega_\mathbf{p} \left(a_\mathbf{p}^\dagger a_\mathbf{p} + b_\mathbf{p}^\dagger b_\mathbf{p} \right). \tag{25}$$

To show that the theory contains two sets of particles of mass m, we evaluate the commutators [4, p. 22]:

$$[H, a_\mathbf{p}^\dagger] = \left[\int \frac{d^3p'}{(2\pi)^3} \omega_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger a_{\mathbf{p}'}, a_\mathbf{p}^\dagger \right] = \omega_\mathbf{p} a_\mathbf{p}^\dagger, \qquad [H, a_\mathbf{p}] = \left[\int \frac{d^3p'}{(2\pi)^3} \omega_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger a_{\mathbf{p}'}, a_\mathbf{p} \right] = -\omega_\mathbf{p} a_\mathbf{p},$$

$$[H, b_\mathbf{p}^\dagger] = \left[\int \frac{d^3p'}{(2\pi)^3} \omega_{\mathbf{p}'} b_{\mathbf{p}'}^\dagger b_{\mathbf{p}'}, b_\mathbf{p}^\dagger \right] = \omega_\mathbf{p} b_\mathbf{p}^\dagger, \qquad [H, b_\mathbf{p}] = \left[\int \frac{d^3p'}{(2\pi)^3} \omega_{\mathbf{p}'} b_{\mathbf{p}'}^\dagger b_{\mathbf{p}'}, b_\mathbf{p} \right] = -\omega_\mathbf{p} b_\mathbf{p}.$$

Then we can define the eigenstates of the Hamiltonian by

$$(a^\dagger_{\mathbf{p}})^{n_a}\,(b^\dagger_{\mathbf{p}})^{n_b}\,|0,0\rangle \equiv |n_a, n_b\rangle\,,$$

which have eigenvalues $(n_a + n_b)\omega_{\mathbf{p}}$. So the expression for the Hamiltonian in Eq. (25) is diagonal in the occupation number basis $\{|n_a, n_b\rangle\}$, where n_a indicates the number of particles created with $a^\dagger_{\mathbf{p}}$ and n_b the number of antiparticles created with $b^\dagger_{\mathbf{p}}$. The ground state is $|0,0\rangle$; it has zero energy since its eigenvalue is zero. Since each operation of $a^\dagger_{\mathbf{p}}$ or $b^\dagger_{\mathbf{p}}$ imparts energy $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ to the system, and each operation of $a_{\mathbf{p}}$ or $b_{\mathbf{p}}$ removes energy $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ from the system, we can conclude that each of the two sets of operators corresponds to a set of particles of mass m. □

2(c) Rewrite the conserved charge

$$Q = \int d^3x\, \frac{i}{2}(\phi^*\pi^* - \pi\phi) \tag{26}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

Solution. Applying Eqs. (23) and (24), we find

$$\begin{aligned}
Q &= \frac{1}{4}\int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\, e^{i\mathbf{p}\cdot\mathbf{x}} \left[\left(b_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p}'} - b^\dagger_{-\mathbf{p}'}\right) - \left(b_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p}'} + b^\dagger_{-\mathbf{p}'}\right)\right] \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3} \left[\left(b_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{-\mathbf{p}} - b^\dagger_{\mathbf{p}}\right) - \left(b_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)\left(a_{-\mathbf{p}} + b^\dagger_{\mathbf{p}}\right)\right] \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3} \left[\left(b_{\mathbf{p}}a_{-\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} - a^\dagger_{-\mathbf{p}}b^\dagger_{-\mathbf{p}}\right) - \left(b_{\mathbf{p}}a_{-\mathbf{p}} + b_{\mathbf{p}}b^\dagger_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} - a^\dagger_{-\mathbf{p}}b^\dagger_{\mathbf{p}}\right)\right] \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \left(a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} - b_{\mathbf{p}}b^\dagger_{\mathbf{p}}\right) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} - [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] - b^\dagger_{\mathbf{p}}b_{\mathbf{p}} - [b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}]\right) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right),
\end{aligned}$$

where in the fifth line we have used $a = (q + ip)/\sqrt{2}$ and $a^\dagger = (q - ip)$ [6]. The particles associated with $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ must have positive charge, since $a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$ represents their number and has the same sign as the conserved charge. Similarly, the antiparticles associated with $b^\dagger_{\mathbf{p}}b_{\mathbf{p}}$ must have negative charge.

2(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part 2(c), and the other three given by

$$Q^i = \int d^3x\, \frac{i}{2}(\phi^*_a {\sigma^i}_{ab}\pi^*_b - \pi_a {\sigma^i}_{ab}\phi_b),$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

Solution. Generalizing Eq. (15), the Lagrangian for the two Klein-Gordon fields is

$$\mathcal{L} = \partial_\mu\phi^*_1\partial^\mu\phi_1 - m^2\phi^*_1\phi_1 + \partial_\mu\phi^*_2\partial^\mu\phi_2 - m^2\phi^*_2\phi_2. \tag{27}$$

The conserved charge is given in general by Peskin & Schroeder (2.12) and (2.13),

$$Q \equiv \int_{\text{all space}} j^0\, d^3x\,, \qquad\qquad \text{where } j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu,$$

where J^μ is a 4-divergence that arises when transforming the Lagrangian as in Peskin & Schroeder (2.10):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x).$$

For the first conserved charge, we note that the Lagrangian in Eq. (27) is invariant under the transformations $\phi_a \to e^{i\alpha}\phi_a$ [4, p. 18]:

$$\mathcal{L} \to \sum_a \left[\partial_\mu(e^{-i\alpha}\phi_a^*)\,\partial^\mu(e^{i\alpha}\phi_a) - m^2 e^{-i\alpha}\phi_a^*\,e^{i\alpha}\phi_a\right] = \mathcal{L},$$

so $J^\mu(x) = 0$. The relevant infinitesimal transformations are found by generalizing Peskin & Schroeder (2.15):

$$\alpha\,\Delta\phi_a = i\alpha\phi_a, \qquad\qquad \alpha\,\Delta\phi_a^* = -i\alpha\phi_a^*. \tag{28}$$

These transformations yield the conserved current

$$j^\mu = -\frac{1}{2}\sum_a \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a^*)}\Delta\phi_a^*\right) = -\frac{i}{2}\sum_a \left(\phi_a\partial^\mu\phi_a^* - \phi_a^*\partial^\mu\phi_a\right),$$

where we have arbitrarily chosen the overall constant [4, p. 18]. Then, generalizing Eq. (18), the corresponding conserved charge is

$$\begin{aligned} Q^0 &= \int d^3x\, j^0 = -\frac{i}{2}\int d^3x \sum_a \left(\phi_a\dot{\phi}_a^* - \phi_a^*\dot{\phi}_a\right) = -\frac{i}{2}\int d^3x \sum_a \left(\phi_a\pi_a - \phi_a^*\pi_a^*\right) \\ &= \int d^3x\,\frac{i}{2}\left(\phi_1^*\pi_1^* - \phi_1\pi_1 + \phi_2^*\pi_2^* - \phi_2\pi_2\right), \end{aligned}$$

which is the generalization of Eq. (26) for two fields.

From the problem statement, we make the ansatz that $\mathcal{L}$ is also invariant under rotations, $\phi \to e^{i\alpha^i\sigma^i/2}\phi$ where $\phi = (\phi_1, \phi_2)$ is a two-component spinor, from Peskin & Schroeder (15.19) and (15.20). To verify,

$$\mathcal{L} \to \sum_a \left[\partial_\mu(e^{-i\alpha^i\sigma^i/2}\phi_a^*)\,\partial^\mu(e^{i\alpha^i\sigma^i/2}\phi_a) - m^2 e^{-i\alpha^i\sigma^i/2}\phi_a^*\,e^{i\alpha^i\sigma^i/2}\phi_a\right] = \mathcal{L}.$$

Again, $J^\mu = 0$. By analogy with Eq. (28), the infinitesimal transformations are

$$\alpha^i\,\Delta\phi = \frac{i}{2}\phi\alpha^i\sigma^i, \qquad\qquad \alpha^i\,\Delta\phi_a = -\frac{i}{2}\phi_a\alpha^i\sigma^i.$$

We have the conserved currents

$$j_i^\mu = -\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*\right) = \frac{i}{2}\left(\phi^*\sigma^i\,\partial^\mu\phi - \phi\sigma^i\,\partial^\mu\phi^*\right),$$

where we have chosen a different overall constant factor. Then the corresponding conserved charges are

$$Q^i = \int d^3x\, j_i^0 = \int d^3x\,\frac{i}{2}\left(\phi^*\sigma^i\dot{\phi} - \phi\sigma^i\dot{\phi}^*\right) = \int d^3x\,\frac{i}{2}\left(\phi^*\sigma^i\pi^* - \phi\sigma^i\pi\right) = \int d^3x\,\frac{i}{2}(\phi_a^*\sigma^i{}_{ab}\pi_b^* - \pi_a\sigma^i{}_{ab}\phi_b),$$

as desired. $\square$

If Q^i have the commutation relations of angular momentum, then we want to show $[Q^i, Q^j] = i\epsilon^{ijk}Q^k$ [7, p. 158]. Generalizing Eq. (19) to two fields, the canonical commutation relations are

$$\begin{aligned} [\phi_a(\mathbf{x}), \pi_b(\mathbf{y})] &= [\phi_a^*(\mathbf{x}), \pi_b^*(\mathbf{y})] = i\,\delta_{ab}\,\delta^3(\mathbf{x}-\mathbf{y}), \\ [\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] &= [\phi_a^*(\mathbf{x}), \phi_b^*(\mathbf{y})] = 0, \\ [\pi_a(\mathbf{x}), \pi_b(\mathbf{y})] &= [\pi_a^*(\mathbf{x}), \pi_b^*(\mathbf{y})] = 0, \\ [\phi_a(\mathbf{x}), \pi_b^*(\mathbf{y})] &= [\phi_a(\mathbf{x}), \phi_b^*(\mathbf{y})] = [\pi_a(\mathbf{x}), \pi_b^*(\mathbf{y})] = 0. \end{aligned}$$

The commutation relations among the Pauli matrices are $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$ [7, p. 165].

Note that

$$
\begin{aligned}
[Q^i, Q^j] &= \left[\int d^3x\, \frac{i}{2}[\phi_a^*(x)\, \sigma^i{}_{ab}\, \pi_b^*(x) - \pi_a(x)\, \sigma^i{}_{ab}\, \phi_b(x)], \int d^3x'\, \frac{i}{2}[\phi_c^*(x')\, \sigma^j{}_{cd}\, \pi_d^*(x') - \pi_c(x')\, \sigma^j{}_{cd}\, \phi_d(x')]\right] \\
&= -\frac{1}{4}\int d^3x \int d^3x' \Big\{ \left[\phi_a^*(x)\, \sigma^i{}_{ab}\, \pi_b^*(x), \phi_c^*(x')\, \sigma^j{}_{cd}\, \pi_d^*(x')\right] - \left[\phi_a^*(x)\, \sigma^i{}_{ab}\, \pi_b^*(x), \pi_c(x')\, \sigma^j{}_{cd}\, \phi_d(x')\right] \\
&\qquad - \left[\pi_a(x)\, \sigma^i{}_{ab}\, \phi_b(x), \phi_c^*(x')\, \sigma^j{}_{cd}\, \pi_d^*(x')\right] + \left[\pi_a(x)\, \sigma^i{}_{ab}\, \phi_b(x), \pi_c(x')\, \sigma^j{}_{cd}\, \phi_d(x')\right] \Big\}. \qquad (29)
\end{aligned}
$$

For the first term in Eq. (29),

$$
\begin{aligned}
&\int d^3x \int d^3x' \left[\phi_a^*(x)\, \sigma^i{}_{ab}\, \pi_b^*(x), \phi_c^*(x')\, \sigma^j{}_{cd}\, \pi_d^*(x')\right] \\
&= \int d^3x \int d^3x' \left\{\phi_a^*(x)\, [\phi_c^*(x')\, \pi_b^*(x) - i\, \delta_{bc}\, \delta^3(x-x')]\, \pi_d^*(x')\, \sigma^i{}_{ab}\, \sigma^j{}_{cd} \right. \\
&\qquad \left. - \phi_c^*(x')\, [\phi_a^*(x)\, \pi_d^*(x') - i\, \delta_{ad}\, \delta^3(x-x')]\, \pi_b^*(x)\, \sigma^j{}_{cd}\, \sigma^i{}_{ab}\right\} \\
&= \int d^3x \int d^3x' \left[\phi_a^*(x)\, \pi_b^*(x)\, \phi_c^*(x')\, \pi_d^*(x')\, \sigma^i{}_{ab}\, \sigma^j{}_{cd} - i\, \delta_{bc}\, \delta^3(x-x')\, \phi_a^*(x)\, \pi_d^*(x')\, \sigma^i{}_{ab}\, \sigma^j{}_{cd} \right. \\
&\qquad \left. - \phi_c^*(x')\, \pi_d^*(x')\, \phi_a^*(x)\, \pi_b^*(x)\, \sigma^j{}_{cd}\, \sigma^i{}_{ab} + i\, \delta_{ad}\, \phi_c^*(x')\, \delta^3(x-x')\, \pi_b^*(x)\, \sigma^j{}_{cd}\, \sigma^i{}_{ab}\right] \\
&= \int d^3x \int d^3x'\; i\delta^3(x-x') \left[\phi_c^*(x')\, \pi_b^*(x)\, (\sigma^j\sigma^i)_{cb} - \phi_a^*(x)\, \pi_d^*(x')\, (\sigma^i\sigma^j)_{ad}\right] \\
&= \int d^3x\; i \left[\phi_a^*(x)\, \pi_b^*(x)\, (\sigma^j\sigma^i)_{ab} - \phi_a^*(x)\, \pi_b^*(x)\, (\sigma^i\sigma^j)_{ab}\right] = -\int d^3x\; i\, \phi_a^*(x)\, \pi_b^*(x)\, [\sigma^i, \sigma^j]_{ab} \\
&= 2\epsilon^{ijk} \int d^3x\; \phi_a^*(x)\, \sigma^k{}_{ab}\, \pi_b^*(x),
\end{aligned}
$$

where we have interchanged $a \leftrightarrow c$ and $b \leftrightarrow d$ in one of the terms that canceled, and we have also relabeled indices in the second to last line.

The second and third terms of Eq. (29) both vanish since they consist entirely of fields that commute. The fourth term may be found using similar arithmetic as for the first:

$$
\int d^3x \int d^3x' \left[\pi_a(x)\, \sigma^i{}_{ab}\, \phi_b(x), \pi_c(x')\, \sigma^j{}_{cd}\, \phi_d(x')\right] = -2\epsilon^{ijk} \int d^3x\; \phi_a(x)\, \sigma^k{}_{ab}\, \pi_b(x).
$$

Feeding these results into Eq. (29), we have

$$
[Q^i, Q^j] = -\frac{\epsilon^{ijk}}{2} \int d^3x \left[\phi_a^*(x)\, \sigma^k{}_{ab}\, \pi_b^*(x) - \phi_a(x)\, \sigma^k{}_{ab}\, \pi_b(x)\right] = i\epsilon^{ijk} Q^k,
$$

as we sought to show. □

Problem 3. (Peskin & Schroeder 2.3) Evaluate the function

$$
\langle 0|\phi(x)\, \phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{ip\cdot(x-y)}, \qquad (30)
$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicitly in terms of Bessel functions.

Solution. We choose our reference frame such that $x^0 - y^0 = 0$ and let $\mathbf{x} - \mathbf{y} = \mathbf{r}$. Then

$$p \cdot (x-y) = -\mathbf{p} \cdot \mathbf{r} = -pr\cos\theta,$$

where we have defined θ as the angle between $\mathbf{p}$ and $\mathbf{r}$. We choose our coordinates such that θ is also the spherical polar coordinate. Substituting this expression and $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ [4, p. 22] into Eq. (30),

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ipr\cos\theta} = \frac{1}{2(2\pi)^3} \int_0^\infty dp\, \frac{p^2}{\sqrt{p^2+m^2}} \int_{-1}^{1} d(\cos\theta)\, e^{-ipr\cos\theta} \int_0^{2\pi} d\phi \\ &= \frac{1}{2(2\pi)^3} \int_0^\infty dp\, \frac{p^2}{\sqrt{p^2+m^2}} \left[-\frac{ie^{-ipr\cos\theta}}{pr} \right]_{-1}^{1} \Big[\phi\Big]_0^{2\pi} = \frac{1}{(2\pi)^2} \int_0^\infty dp\, \frac{p\sin(pr)}{r\sqrt{p^2+m^2}}. \end{aligned} \tag{31}$$

An integral formula for the modified Bessel function of the second kind of order 0 is [8]

$$K_0(x) = \int_0^\infty dt\, \frac{\cos(xt)}{\sqrt{t^2+1}},$$

which has the derivative

$$\frac{dK_0(x)}{dx} = -\int_0^\infty dt\, \frac{t\sin(xt)}{\sqrt{t^2+1}} = -K_1(x),$$

where we have used the identity $d[x^{-p}K_p(x)]/dx = -x^{-p}K_{p+1}(x)$ [9, p. 386]. So Eq. (31) can be written

$$D(x-y) = \frac{m}{(2\pi)^2 r} K_1(mr) = \frac{m}{(2\pi)^2 |x-y|} K_1(m|x-y|). \tag{32}$$

Problem 4. The classical limit of a harmonic oscillator can be described in terms of *coherent states*,

$$|\alpha\rangle = \exp\left(\alpha a^\dagger - \frac{1}{2}|\alpha|^2\right) |0\rangle .$$

When α is large, the oscillator state is semiclassical. Proceeding similarly for the Fourier modes of the quantum Klein-Gordon field,

$$|f\rangle = N_f \exp\left(i \int \frac{d^3p}{(2\pi)^3} f(\mathbf{p})\, a_{\mathbf{p}}^\dagger\right) |0\rangle , \qquad N_f = \exp\left(-\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} |f(\mathbf{p})|^2\right).$$

4(a) Evaluate the expectation value of the field operator $\langle f|\phi(x)|f\rangle$ and show that it satisfies the Klein-Gordon equation.

Solution. The field operator is given in terms of the creation and annihilation operators by Peskin & Schroeder (2.47),

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \Big|_{p^0 = E_{\mathbf{p}}}.$$

Note that

$$\begin{aligned} a_{\mathbf{p}} |f\rangle &= a_{\mathbf{p}} N_f \exp\left(i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a_{\mathbf{p}'}^\dagger\right) |0\rangle = N_f a_{\mathbf{p}} \sum_{n=0}^{\infty} \frac{i}{n!} \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a_{\mathbf{p}'}^\dagger |0\rangle \\ &= N_f a_{\mathbf{p}} \left[1 + i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a_{\mathbf{p}'}^\dagger + \frac{1}{2} \left(i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a_{\mathbf{p}'}^\dagger \right)^2 + \cdots \right] |0\rangle , \end{aligned} \tag{33}$$

where we have used the Maclaurin series expansion for the exponential function [**?**]. The first term vanishes. For the second term,

$$i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a_{\mathbf{p}} a^\dagger_{\mathbf{p}'} |0\rangle = i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}') \left[a^\dagger_{\mathbf{p}'} a_{\mathbf{p}} + (2\pi)^3\, \delta^3(\mathbf{p} - \mathbf{p}') \right] |0\rangle$$
$$= i \left(f(\mathbf{p}) + \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}') a^\dagger_{\mathbf{p}'} a_{\mathbf{p}} \right) |0\rangle = i f(\mathbf{p})\, |0\rangle\, .$$

For the third,

$$\frac{i^2}{2} \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a_{\mathbf{p}} a^\dagger_{\mathbf{p}'} \int \frac{d^3p''}{(2\pi)^3} f(\mathbf{p}'')\, a^\dagger_{p''} |0\rangle = \frac{i^2}{2} \left(f(\mathbf{p}) + \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}') a^\dagger_{\mathbf{p}'} a_{\mathbf{p}} \right) \int \frac{d^3p''}{(2\pi)^3} f(\mathbf{p}'')\, a^\dagger_{p''} |0\rangle$$
$$= \left[\frac{i^2}{2} f(\mathbf{p}) \int \frac{d^3p''}{(2\pi)^3} f(\mathbf{p}'')\, a^\dagger_{p''} + \frac{i^2}{2} \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}') a^\dagger_{\mathbf{p}'} \left(f(\mathbf{p}) + \int \frac{d^3p''}{(2\pi)^3} f(\mathbf{p}'') a^\dagger_{p''} a_{\mathbf{p}} \right) \right] |0\rangle$$
$$= i^2 f(\mathbf{p}) \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a^\dagger_{\mathbf{p}'}\, |0\rangle\, .$$

Returning to Eq. (33), we have

$$a_{\mathbf{p}} |f\rangle = i f(\mathbf{p}) N_f \left[1 + i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a^\dagger_{\mathbf{p}'} + \frac{1}{2} \left(i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a^\dagger_{\mathbf{p}'} \right)^2 + \cdots \right] |0\rangle$$
$$= i f(\mathbf{p}) N_f \exp\!\left(i \int \frac{d^3p'}{(2\pi)^3} f(\mathbf{p}')\, a^\dagger_{\mathbf{p}'} \right) |0\rangle = i f(\mathbf{p})\, |f\rangle\, ,$$

where the Maclaurin series has shifted by one term. Likewise, $\langle f|\, a^\dagger_{\mathbf{p}} = -i f^*(\mathbf{p})$.

Then, using these results, we can evaluate the expectation value:

$$\langle f|\phi(x)|f\rangle = \langle f| \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right) |f\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(\langle f|a_{\mathbf{p}}|f\rangle\, e^{-ip\cdot x} + \langle f|a^\dagger_{\mathbf{p}}|f\rangle\, e^{ip\cdot x} \right)$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{\sqrt{2E_{\mathbf{p}}}} \left[f(\mathbf{p})\, e^{-ip\cdot x} - f^*(\mathbf{p})\, e^{ip\cdot x} \right] .$$

The Klein-Gordon equation is given by Peskin & Schroeder (2.7), $(\partial^2/\partial t^2 - \nabla^2 + m^2)\phi = 0$. Then

$$(\partial^\mu \partial_\mu + m^2)\, \langle f|\phi(x)|f\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{i}{\sqrt{2E_{\mathbf{p}}}} \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right] . \quad (34)$$

For the time derivative,

$$\frac{\partial^2}{\partial t^2} \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right] = \frac{\partial}{\partial t} \left[-iE_{\mathbf{p}}\, f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - iE_{\mathbf{p}}\, f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right]$$
$$= -E_{\mathbf{p}}^2 \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right] ,$$

and for the spatial derivative,

$$\nabla^2 \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right] = \mathbf{p}^2 \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right] .$$

Feeding these back into Eq. (34) and using $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ [4, p. 22],

$$(\partial^2/\partial t^2 - \nabla^2 + m^2)\, \langle f|\phi(x)|f\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{i}{\sqrt{2E_{\mathbf{p}}}} (\mathbf{p}^2 + m^2 - E_{\mathbf{p}}^2) \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right]$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{i}{\sqrt{2E_{\mathbf{p}}}} (E_{\mathbf{p}}^2 - E_{\mathbf{p}}^2) \left[f(\mathbf{p})\, e^{-i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} - f^*(\mathbf{p})\, e^{i(E_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} \right]$$
$$= 0,$$

so $\langle f|\phi(x)|f\rangle$ satisfies the Klein-Gordon equation. $\square$

4(b) Evaluate the relative mean square fluctuation of the occupation number of the mode with momentum $\mathbf{p}$ and the relative mean square fluctuation in the total energy:

$$\frac{\langle \hat{n}_{\mathbf{p}}^2\rangle - \langle \hat{n}_{\mathbf{p}}\rangle^2}{\langle \hat{n}_{\mathbf{p}}\rangle^2}, \qquad \frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2}.$$

Is either of these a good measure of the degree to which the field is classical? Justify your answer.

Solution. Note that $\hat{n}_{\mathbf{p}} = a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ [7, p. 90]. Then

$$\begin{aligned}
\langle \hat{n}_{\mathbf{p}}\rangle &= \langle f|a_{\mathbf{p}}^\dagger a_{\mathbf{p}}|f\rangle = f^*(\mathbf{p})f(\mathbf{p}) = |f(\mathbf{p})|^2,\\
\langle \hat{n}_{\mathbf{p}}^2\rangle &= \langle f|a_{\mathbf{p}}^\dagger a_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}|f\rangle = |f(\mathbf{p})|^2\,\langle f|a_{\mathbf{p}}a_{\mathbf{p}}^\dagger|f\rangle = |f(\mathbf{p})|^2\,\langle f|a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]|f\rangle = |f(\mathbf{p})|^4 + |f(\mathbf{p})|\,(2\pi)^3\,\delta^3(0)
\end{aligned}$$

so

$$\frac{\langle \hat{n}_{\mathbf{p}}^2\rangle - \langle \hat{n}_{\mathbf{p}}\rangle^2}{\langle \hat{n}_{\mathbf{p}}\rangle^2} = \frac{|f(\mathbf{p})|^4 + |f(\mathbf{p})|\,(2\pi)^3\,\delta^3(0) - |f(\mathbf{p})|^4}{|f(\mathbf{p})|^4} = \frac{(2\pi)^3\,\delta^3(0)}{|f(\mathbf{p})|^2},$$

which diverges.

From Peskin & Schroeder (2.31),

$$\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}},$$

so

$$\langle H\rangle = \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\,\langle f|a_{\mathbf{p}}^\dagger a_{\mathbf{p}}|f\rangle = \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}|f(\mathbf{p})|^2,$$

and

$$\begin{aligned}
\langle H^2\rangle &= \int \frac{d^3p\,d^3p'}{(2\pi)^6}E_{\mathbf{p}}E_{\mathbf{p}'}\,\langle f|a_{\mathbf{p}}^\dagger a_{\mathbf{p}}a_{\mathbf{p}'}^\dagger a_{\mathbf{p}'}|f\rangle = \int \frac{d^3p\,d^3p'}{(2\pi)^6}\,E_{\mathbf{p}}E_{\mathbf{p}'}\,f^*(\mathbf{p})\,f(\mathbf{p}')\,\langle f|a_{\mathbf{p}}a_{\mathbf{p}'}^\dagger|f\rangle\\
&= \int \frac{d^3p\,d^3p'}{(2\pi)^6}\,E_{\mathbf{p}}E_{\mathbf{p}'}\,f^*(\mathbf{p})\,f(\mathbf{p}')\,\langle f|a_{\mathbf{p}'}^\dagger a_{\mathbf{p}} + [a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger]|f\rangle\\
&= \int \frac{d^3p\,d^3p'}{(2\pi)^6}\,E_{\mathbf{p}}E_{\mathbf{p}'}\,f^*(\mathbf{p})\,f(\mathbf{p}')\,\left[f^*(\mathbf{p})f(\mathbf{p}') + (2\pi)^3\,\delta^3(\mathbf{p}-\mathbf{p}')\right]\\
&= \left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}|f(\mathbf{p})|^2\right)^2 + \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}^2|f(\mathbf{p})|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\langle H^2\rangle - \langle H\rangle^2}{\langle H\rangle^2} &= \frac{\left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\,|f(\mathbf{p})|^2\right)^2 + \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}^2\,|f(\mathbf{p})|^2 - \left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}|f(\mathbf{p})|^2\right)^2}{\left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}|f(\mathbf{p})|^2\right)^2}\\
&= \int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}^2\,|f(\mathbf{p})|^2 \bigg/ \left(\int \frac{d^3p}{(2\pi)^3}\,E_{\mathbf{p}}\,|f(\mathbf{p})|^2\right)^2.
\end{aligned}$$

Both quantities can tell us something about how classical the field is. Both quantities are zero in a classical system, since quantum fluctuations do not occur. The infinite mean square fluctuation of the occupation number indicates that, in any given mode, any number of particles may be created or destroyed at any given time. However, the changing number of particles in each state does not correspond to a large change in the total energy of the system, since the mean square fluctuation in the total energy is finite. Both mean square fluctuations tend to zero as $f(\mathbf{p}) \to \infty$. If $f(\mathbf{p})$ is analogous to α in that a large $f(\mathbf{p})$ indicates a semiclassical system, then this is the expected behavior.

4(c) Take $\Delta(x-y) = \langle 0|\phi(\mathbf{x})\,\phi(\mathbf{y})|0\rangle$ (equal times) as a measure of the fluctuations or correlations of the field amplitude. Use your result from problem 3 to evaluate this quantity. What is the meaning of the divergence as $\mathbf{x} \to \mathbf{y}$?

Solution. Since $x-y$ is described as spacelike, the solution is the same as Eq. (32):

$$\Delta(x-y) = \frac{m}{(2\pi)^2|x-y|}\,K_1(m\,|x-y|).$$

This quantity represents the amplitude for a particle to propagate from y to x [4, p. 27]. When $x^0 = y^0$ and $\mathbf{x} \to \mathbf{y}$, the divergence means that this amplitude becomes infinite. This seems to imply that, in this limit, a particle located at $\mathbf{x}$ would remain there indefinitely.

References

[1] J. D. Jackson, “Classical Electrodynamics”. Wiley, 3rd edition, 1999.

[2] L. D. Landau and E. M. Lifshitz, “The Classical Theory of Fields”, volume 2. Pergamon Press, 3rd edition, 1971.

[3] E. W. Weisstein, “Simple Harmonic Motion.” From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/SimpleHarmonicMotion.html`.

[4] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”. Perseus Books Publishing, 1995.

[5] Wikipedia contributors, “Dirac delta function.” From Wikipedia, the Free Encyclodia. `https://en.wikipedia.org/wiki/Dirac_delta_function`.

[6] Wikipedia contributors, “Creation and annihilation operators.” From Wikipedia, the Free Encyclodia. `https://en.wikipedia.org/wiki/Creation_and_annihilation_operators`.

[7] J. J. Sakurai, “Modern Quantum Mechanics”. Addison-Wesley Publishing Company, revised edition, 1994.

[8] E. W. Weisstein, “Modified Bessel Function of the Second Kind.” From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/ModifiedBesselFunctionoftheSecondKind.html`.

[9] W. E. Olmstead and V. A. Volpert, “Differential Equations in Applied Mathematics”. 2014.