Problem 1. Consider the charge density $\rho(\mathbf{x})$ given by

$$\rho(\mathbf{x}) = \begin{cases} (R-r)(1-\cos\theta)^2 & \text{for } |\mathbf{x}| \leq R, \\ 0 & \text{for } |\mathbf{x}| \geq R. \end{cases} \tag{1}$$

Find the electrostatic potential, $\phi(\mathbf{x})$, of this charge distribution at all $\mathbf{x}$ with $|\mathbf{x}| \geq R$.

Solution. The multipole expansion in spherical harmonics is given by Eq. (2.79) in the course notes,

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta,\phi), \tag{2}$$

where the spherical multipole moments q_{lm} are defined in Eq. (2.80),

$$q_{lm} \equiv \int \rho(\mathbf{x}')\, r'^l\, Y^*_{lm}(\theta',\phi')\, d^3x' \,.$$

Note that (2) is valid only for $|\mathbf{x}| \geq R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only within $|\mathbf{x}'| \leq R$, which is the regime we are interested in here.

The spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi},$$

and the Lagrange polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}}{}^{\,l} \,,$$

although in practice I am taking all spherical harmonics from the table in Jackson.

We can write the angular component of $\rho(\mathbf{x})$ as an expansion of spherical harmonics. Inspecting (1), we will only have terms of $l = 0, 1, 2$ and $m = 0$. The relevant spherical harmonics are

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}, \qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \qquad Y_{20}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).$$

Then we have

$$\begin{aligned} \rho(r,\theta,\ vph) &= (R-r)(1-2\cos\theta+\cos^2\theta) \\ &= (R-r)\left(\frac{2}{3}\sqrt{\frac{4\pi}{5}} Y_{20}(\theta,\phi) - 2\sqrt{\frac{4\pi}{3}} Y_{10}(\theta,\phi) + 4\frac{\sqrt{4\pi}}{3} Y_{00}(\theta,\phi)\right). \end{aligned}$$

The only nonzero q_{lm} are q_{00}, q_{10}, and q_{20}:

$$\begin{aligned} q_{00} &= \int_0^{2\pi}\int_{-1}^{1}\int_0^R \rho(\mathbf{x}')\, r'^0\, Y^*_{00}(\theta',\phi')\, r'^2\, dr'\, d(\cos\theta')\, d\varphi' \\ &= 4\frac{\sqrt{4\pi}}{3}\int_0^{2\pi}\int_{-1}^{1} Y^*_{00}(\theta',\phi') Y_{00}(\theta',\phi')\, d(\cos\theta')\, d\varphi' \int_0^R (R-r')r'^2\, dr' \\ &= 4\frac{\sqrt{4\pi}}{3}\left[\frac{Rr'^3}{3} - \frac{r'^4}{4}\right]_0^R = 4\frac{\sqrt{4\pi}}{3}\frac{R^4}{12} = \frac{2\sqrt{\pi}}{9}R^4, \end{aligned}$$

$$
\begin{aligned}
q_{10} &= \int_0^{2\pi}\int_{-1}^{1}\int_0^R \rho(\mathbf{x}')\, r'^1\, Y_{10}^*(\theta',\phi')\, r'^2\, dr'\, d(\cos\theta')\, d\varphi' \\
&= -2\sqrt{\frac{4\pi}{3}}\int_0^{2\pi}\int_{-1}^{1} Y_{10}^*(\theta',\phi')Y_{10}(\theta',\phi')\, d(\cos\theta')\, d\varphi' \int_0^R (R-r')r'^3\, dr' \\
&= -2\sqrt{\frac{4\pi}{3}}\left[\frac{Rr'^4}{4}-\frac{r'^5}{5}\right]_0^R = -2\sqrt{\frac{4\pi}{3}}\frac{R^5}{20} = -\frac{1}{5}\sqrt{\frac{\pi}{3}}R^5,
\end{aligned}
$$

$$
\begin{aligned}
q_{20} &= \int_0^{2\pi}\int_{-1}^{1}\int_0^R \rho(\mathbf{x}')\, r'^2\, Y_{20}^*(\theta',\phi')\, r'^2\, dr'\, d(\cos\theta')\, d\varphi' \\
&= \frac{2}{3}\sqrt{\frac{4\pi}{5}}\int_0^{2\pi}\int_{-1}^{1} Y_{20}^*(\theta',\phi')Y_{20}(\theta',\phi')\, d(\cos\theta')\, d\varphi' \int_0^R (R-r')r'^4\, dr' \\
&= \frac{2}{3}\sqrt{\frac{4\pi}{5}}\left[\frac{Rr'^5}{5}-\frac{r'^6}{6}\right]_0^R = \frac{2}{3}\sqrt{\frac{4\pi}{5}}\frac{R^6}{30} = \frac{2}{45}\sqrt{\frac{\pi}{5}}R^6.
\end{aligned}
$$

Then ϕ is given by

$$
\begin{aligned}
\phi(\mathbf{x}) &= \frac{4\pi}{1}\frac{q_{00}}{r^1}Y_{00}(\theta,\phi) + \frac{4\pi}{2+1}\frac{q_{10}}{r^2}Y_{10}(\theta,\phi) + \frac{4\pi}{5}\frac{q_{20}}{r^3}Y_{20}(\theta,\phi) \\
&= (4\pi)\frac{2\sqrt{\pi}}{9}\frac{R^4}{r}\frac{1}{\sqrt{4\pi}} - \frac{4\pi}{3}\frac{1}{5}\sqrt{\frac{\pi}{3}}\frac{R^5}{r^2}\sqrt{\frac{3}{4\pi}}\cos\theta + \frac{4\pi}{5}\frac{2}{45}\sqrt{\frac{\pi}{5}}\frac{R^6}{r^3}\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right) \\
&= \frac{4\pi}{9}\frac{R^4}{r} - \frac{2\pi}{15}\frac{R^5}{r^2}\cos\theta + \frac{2\pi}{225}\frac{R^6}{r^3}(3\cos^2\theta-1).
\end{aligned}
$$

Problem 2. Let $\mathcal{V}$ be an arbitrary bounded region of space and suppose that a total charge Q is to be distributed in $\mathcal{V}$ in an arbitrary way, with $\rho = 0$ outside of $\mathcal{V}$. Show that the total energy is minimized if the charge is distributed the way that it would be if $\mathcal{V}$ were a conductor, so that $\phi = \text{const.}$ within $\mathcal{V}$ (and thus, in particular, all of the charge lies on the boundary of $\mathcal{V}$).

Hint: Let $\phi_0(\mathbf{x})$ be the potential one would obtain if $\mathcal{V}$ were filled by a conducting body. Consider the energy of $\phi_0 + \phi'$, where the source ρ' of ϕ' vanishes outside of $\mathcal{V}$ and has no net charge within $\mathcal{V}$.

Solution. Let $S = \partial\mathcal{V}$ denote the boundary of $\mathcal{V}$. Suppose, to the contrary, that there is charge enclosed within $\mathcal{V}$. Call this source ρ'. By the superposition principle, we may write

$$\rho = \rho_0 + \rho', \qquad\qquad \phi = \phi_0 + \phi',$$

where ρ_0 is the charge of a conducting body filling $\mathcal{V}$, ϕ_0 is the electrostatic potential due to ρ_0, ρ' is the charge distribution within $\mathcal{V}$, and ϕ' is the electrostatic potential due to ρ'. Without loss of generality, we may require

$$\int_{\mathcal{V}} \rho'\, d^3x = 0. \tag{3}$$

For the entire body to have charge Q, we need

$$\int \rho_0\, d^3x = Q.$$

By definition, $\rho_0 = 0$ everywhere *but* on the boundary. It follows that $\phi_0 = \text{const.}$ everywhere.

The total energy is given by Eq. (2.25) in the course notes,

$$\mathscr{E} = \frac{1}{8\pi}\int_{\mathcal{V}} |\mathbf{E}|^2\, d^3x = \frac{1}{2}\int \phi\rho\, d^3x\,. \tag{4}$$

So

$$\begin{aligned}\mathscr{E} &= \frac{1}{2}\int (\phi_0+\phi')(\rho_0+\rho')\, d^3x = \frac{1}{2}\left(\int \phi_0(\rho_0+\rho')\, d^3x + \int \phi'(\rho_0+\rho')\, d^3x\right)\\ &= \frac{1}{2}\left(\phi_0 Q + \int_{\mathcal{V}} \phi'\rho'\, d^3x + \int_{\mathcal{V}} \phi'\rho_0\, d^3x\right). \end{aligned}\tag{5}$$

Applying (4), we can rewrite the second term:

$$\int_{\mathcal{V}} \phi'\rho'\, d^3x = \frac{1}{4\pi}\int_{\mathcal{V}} |\mathbf{E}'|^2\, d^3x \geq 0.$$

Eq. (2.30) gives the expression for interaction energy,

$$\mathscr{E}_{\text{int}} = \int \rho_1\phi_2\, d^3x = \int \rho_2\phi_1\, d^3x\,,$$

so we can rewrite the third term of (5) as follows:

$$\int_{\mathcal{V}} \phi'\rho_0\, d^3x = \int_{\mathcal{V}} \phi_0\rho'\, d^3x = \phi_0\int_{\mathcal{V}} \rho'\, d^3x = 0.$$

Now (5) becomes

$$\mathscr{E} = \frac{1}{2}\left(\phi_0 Q + \frac{1}{4\pi}\int_{\mathcal{V}} |\mathbf{E}'|^2\, d^3x\right),$$

which is minimal when

$$0 = \frac{1}{4\pi}\int_{\mathcal{V}} |\mathbf{E}'|^2\, d^3x = \int_{\mathcal{V}} \phi'\rho'\, d^3x\,.$$

This is only posisble if

$$\phi' = 0 \text{ or } \rho' = 0, \qquad \rho' = \text{const. and } \int_{\mathcal{V}} \phi'\, d^3x = 0, \qquad \phi' = \text{const. and } \int_{\mathcal{V}} \rho'\, d^3x = 0.$$

The first is trivial, and the second contradicts (3). So we are left with the third option, and thus conclude that $\phi' = \text{const}$. However, this implies that ρ' is distributed as it would be for a conductor, which contradicts our initial assumption. Thus, we have shown that the total energy is minimized for charge distributed as it is in a conductor. □

Problem 3. Charge is distributed on a (nonconducting) sphere of radius R, i.e., the charge density throughout space is of the form $\rho(\mathbf{x}) = \sigma(\theta,\phi)\,\delta(r-R)$. The surface charge distribution σ on the sphere is chosen in such a way that the electrostatic potential on the sphere is $\phi(r=R,\theta,\varphi) = \alpha\cos\theta$, where α is a constant.

3.a Find the electrostatic potential $\phi(\mathbf{x})$ at all $r \leq R$.

Solution. The electrostatic potential can be found using the Green's function for electrostatics, $G(\mathbf{x}, \mathbf{x}')$, as given by Eq. (2.23),

$$\phi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}')\, \rho(\mathbf{x}')\, d^3x' \,.$$

$G(\mathbf{x}, \mathbf{x}')$ can be expanded in spherical harmonics according to Eq. (2.78):

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \begin{cases} \sum_{l,m} \dfrac{4\pi}{2l+1} \dfrac{r^l}{r'^{l+1}} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi) & \text{if } r < r', \\ \sum_{l,m} \dfrac{4\pi}{2l+1} \dfrac{r'^l}{r^{l+1}} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi) & \text{if } r > r'. \end{cases}$$

We can also write ϕ in terms of spherical harmonics:

$$\psi(r = R, \theta, \varphi) = \alpha \sqrt{\frac{4\pi}{3}} Y_{10}(\theta, \phi) = \alpha \sqrt{\frac{4\pi}{3}} Y^*_{10}(\theta, \phi).$$

For $r \leq r'$, the potential is

$$\begin{aligned} \phi(\mathbf{x}) &= \int_0^{2\pi} \int_{-1}^1 \int_0^\infty \sigma(\theta', \phi')\, \delta(r' - R) \sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi) r'^2\, dr'\, d(\cos\theta')\, d\varphi' \\ &= \sum_{l,m} \frac{4\pi}{2l+1} r^l Y_{lm}(\theta, \phi) \int_0^\infty \delta(r' - R) \frac{1}{r'^{l-1}}\, dr' \int_0^{2\pi} \int_{-1}^1 \sigma(\theta', \phi')\, Y^*_{lm}(\theta', \phi')\, d(\cos\theta')\, d\varphi' \\ &= \sum_{l,m} \frac{4\pi}{2l+1} r^l Y_{lm}(\theta, \phi) \frac{1}{R^{l-1}} \int_0^{2\pi} \int_{-1}^1 \sigma(\theta', \phi')\, Y^*_{lm}(\theta', \phi')\, d(\cos\theta')\, d\varphi' \,. \end{aligned} \tag{6}$$

Plugging in $r = R$,

$$\alpha \cos\theta = \sum_{l,m} \frac{4\pi}{2l+1} R Y_{lm}(\theta, \phi) \int_0^{2\pi} \int_{-1}^1 \sigma(\theta', \phi')\, Y^*_{lm}(\theta', \phi')\, d(\cos\theta')\, d\varphi' \,,$$

which implies that $l = 1$ and $m = 0$ are the only Y_{lm} with nonzero coefficients. Therefore,

$$\begin{aligned} \alpha \cos\theta &= \frac{4\pi}{3} R \sqrt{\frac{3}{4\pi}} \cos\theta \int_0^{2\pi} \int_{-1}^1 \sigma(\theta', \phi')\, Y^*_{lm}(\theta', \phi')\, d(\cos\theta')\, d\varphi' \\ \implies & \int_0^{2\pi} \int_{-1}^1 \sigma(\theta', \phi')\, Y^*_{lm}(\theta', \phi')\, d(\cos\theta')\, d\varphi' = \sqrt{\frac{3}{4\pi}} \frac{\alpha}{R}, \end{aligned} \tag{7}$$

so (6) becomes

$$\phi(\mathbf{x}) = \frac{4\pi}{3} r \sqrt{\frac{3}{4\pi}} \cos\theta \frac{1}{R^{l-1}} \sqrt{\frac{3}{4\pi}} \frac{\alpha}{R} = \alpha \frac{r}{R} \cos\theta.$$

3.b Find the electrostatic potential $\phi(\mathbf{x})$ at all $r \geq R$.

Solution. For $r \geq r'$, the potential is

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} Y_{lm}(\theta, \phi) \int_0^\infty \delta(r' - R) r'^{l+2}\, dr' \int_0^{2\pi} \int_{-1}^1 \sigma(\theta', \phi') Y^*_{lm}(\theta', \phi')\, d(\cos\theta')\, d\varphi'$$

$$= \sum_{l,m} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} Y_{lm}(\theta,\phi) R^{l+2} \int_0^{2\pi} \int_{-1}^{1} \sigma(\theta',\phi') Y_{lm}^*(\theta',\phi')\, d(\cos\theta')\, d\varphi' \,.$$

By the same arguments as in 3.a, we restrict ourselves to $l = 0$ and $m = 1$ and make the substitution (7). This gives us

$$\phi(\mathbf{x}) = \frac{4\pi}{3} \frac{R^3}{r^2} \sqrt{\frac{3}{4\pi}} \cos\theta \sqrt{\frac{3}{4\pi}} \frac{\alpha}{R} = \alpha \frac{R^2}{r^2} \cos\theta.$$

3.c Find the surface charge density $\sigma(\theta,\varphi)$ that was required in order to produce this potential ϕ.

Solution. From (7) and the fact that $l = 1$ and $m = 0$, we need $\sigma(\theta,\phi) = C\, Y_{10}(\theta,\phi)$ where C is a constant. Then

$$\sqrt{\frac{3}{4\pi}} \frac{\alpha}{R} = C \int_0^{2\pi} \int_{-1}^{1} Y_{10}(\theta',\phi') Y_{10}^*(\theta',\phi')\, d(\cos\theta')\, d\varphi' = C$$

which implies

$$\sigma(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \frac{\alpha}{R} \sqrt{\frac{3}{4\pi}} \cos\theta = \frac{3}{4\pi} \frac{\alpha}{R} \cos\theta.$$

3.d Find the total electrostatic energy.

Solution. The total energy is given by (4). Since ρ is nonzero only on the boundary, we can use the given expression for ϕ on the boundary. Feeding in our result from 3.c,

$$\begin{aligned}
\mathscr{E} &= \frac{1}{2} \int \phi\rho\, d^3x = \frac{1}{2} \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} \frac{3}{4\pi} \frac{\alpha}{R} \cos\theta\, \delta(r-R)\, \alpha \cos\theta r^2\, dr\, d(\cos\theta)\, d\varphi \\
&= \frac{3}{8\pi} \frac{\alpha^2}{R} \int_0^{2\pi} d\varphi \int_{-1}^{1} \cos^2\theta\, d(\cos\theta) \int_0^{\infty} \delta(r-R)\, r^2\, dr = \frac{3}{8\pi} \frac{\alpha^2}{R} \Big[\varphi\Big]_0^{2\pi} \left[\frac{\cos^3\theta}{3}\right]_{-1}^{1} R^2 = \frac{3}{8\pi} \alpha^2 R (2\pi) \frac{2}{3} \\
&= \frac{1}{2} \alpha^2 R.
\end{aligned}$$

Problem 4. A point charge of charge q is placed at point $\mathbf{x}'$ inside a conducting spherical shell of radius R. There is no net charge on the conductor. The potential inside the sphere is thus given by $q\, G_D(\mathbf{x},\mathbf{x}')$, where the explicit formula for $G_D(\mathbf{x},\mathbf{x}')$ for a spherical cavity is given in the lecture notes.

4.a Find the surface charge density $\sigma(\theta,\varphi)$ on the conducting shell.

Solution. The Green's function for a spherical cavity is given by Eq. (2.91),

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\alpha}{|\mathbf{x}-\mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}' \frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \alpha = -\frac{R}{|\mathbf{x}'|}.$$

The surface charge density can be found from Eq. (2.86),

$$\mathbf{E} \cdot \hat{\mathbf{n}} = 4\pi\sigma, \tag{8}$$

where $\mathbf{E} = -\nabla\phi$ in electrostatics.

We will begin by finding $\mathbf{E}$. We will orient our coordinate system such that $\mathbf{x}'$ (and consequently $\mathbf{x}''$) points along the z axis. Note that

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{R}{|\mathbf{x}'|\left|\mathbf{x} - \frac{R^2}{|\mathbf{x}'|^2}\mathbf{x}'\right|} = \frac{1}{\sqrt{\mathbf{x}^2 - 2\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'^2}} - \frac{R}{|\mathbf{x}'|\sqrt{\mathbf{x}^2 - 2\frac{R^2}{\mathbf{x}'^2}\mathbf{x}\cdot\mathbf{x}' + \frac{R^4}{\mathbf{x}'^4}\mathbf{x}'^2}}.$$

In spherical coordinates, we have

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{r^2 - 2rr'\cos\theta + r'^2}} - \frac{R}{r'}\frac{1}{\sqrt{r^2 - 2R^2 r\cos\theta/r' + R^4/r'^2}},$$

where we note that θ is the angle between $\mathbf{x}$ and the z axis. The gradient in spherical coordinates is given by

$$\nabla = \frac{\partial}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\,\hat{\boldsymbol{\varphi}}.$$

The r component of the electric field inside the conductor is then

$$E_r(\mathbf{x}) = -q\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial r} = q\left(\frac{r - r'\cos\theta}{(r^2 - 2rr'\cos\theta + r'^2)^{3/2}} - \frac{R}{r'}\frac{r - R^2\cos\theta/r'}{(r^2 - 2R^2 r\cos\theta/r' + R^4/r'^2)^{3/2}}\right).$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ for the inner surface of a sphere, we are interested in only the r component of the field. On the surface of the sphere, the field is $E_r(r=R)\,\hat{\mathbf{r}}$. So we have

$$\begin{aligned}
E_r(r=R) &= q\left(\frac{R - r'\cos\theta}{(R^2 - 2Rr'\cos\theta + r'^2)^{3/2}} - \frac{R}{r'}\frac{R - R^2\cos\theta/r'}{(R^2 - 2R^3\cos\theta/r' + R^4/r'^2)^{3/2}}\right)\\
&= q\left(\frac{R - r'\cos\theta}{r'^3(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}} - \frac{R}{r'}\frac{R - R^2\cos\theta/r'}{R^3(1 - 2R\cos\theta/r' + R^2/r'^2)^{3/2}}\right)\\
&= \frac{q}{r'}\frac{R^3 - R^2 r'\cos\theta - Rr'^2 + R^2 r'\cos\theta}{R^2 r'^2(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}} = \frac{q}{Rr'^3}\frac{R^2 - r'^2}{(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}}.
\end{aligned}$$

Finally, feeding this into (8),

$$\sigma = -\frac{\mathbf{E}\cdot\hat{\mathbf{r}}}{4\pi} = \frac{q}{4\pi Rr'^3}\frac{r'^2 - R^2}{(R^2/r'^2 - 2R\cos\theta/r' + 1)^{3/2}} = \frac{q}{4\pi R|\mathbf{x}'|^3}\frac{|\mathbf{x}'|^2 - R^2}{(R^2/|\mathbf{x}'|^2 - 2R\cos\theta/|\mathbf{x}'| + 1)^{3/2}}.$$

4.b Find the force $\mathbf{F}$ that must be exerted on the point charge in order to hold it in place.

Solution. The total force on a charge distribution arises only from the external electric field $\mathbf{E}_0$, and is given by Eq. (2.42) in the lecture notes:

$$\mathbf{F} = \int \rho(\mathbf{x})\,\mathbf{E}_0(\mathbf{x})\,d^3x\,.$$

The force required to keep the point charge in place is equal and opposite to this force, so we need to insert a minus sign. We also need the θ component of the field inside the conductor, which is

$$E_\theta(\mathbf{x}) = -\frac{q}{r}\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial\theta} = -q\left(\frac{r'\sin\theta}{(r^2 - 2rr'\cos\theta + r'^2)^{3/2}} - \frac{R^3\sin\theta}{r'^2(r^2 - 2R^2 r\cos\theta/r' + R^4/r'^2)^{3/2}}\right).$$

The charge density for a point charge located at $\mathbf{x}'$ is given by $\rho(\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}')$. Evaluating the integral, we have

$$\mathbf{F} = -\int q\,\delta(\mathbf{x}-\mathbf{x}')\,\mathbf{E}(\mathbf{x})\,d^3x = -q\mathbf{E}(\mathbf{x}').$$

Recall that we chose $\mathbf{x}'$ to point along the z axis, so $\theta' = 0$. The θ component of $\mathbf{F}$ is then 0, and the r component is

$$F_r = -q^2\left(\frac{r'-r'}{(r'^2-2r'^2+r'^2)^{3/2}} - \frac{R}{r'}\frac{r'-R^2/r'}{(r'^2-2R^2+R^4/r'^2)^{3/2}}\right) = q^2Rr'^2\frac{r'-R^2/r'}{(r'^4-2R^2r'^2+R^4)^{3/2}}$$
$$= -q^2Rr'^2\frac{(r'^2-R^2)/r'}{(r'^2-R^2)^3} = -q^2\frac{Rr'}{(r'^2-R^2)^2}.$$

Since only the r component of $\mathbf{F}$ is nonzero, it points in the z direction, which we chose to be equivalent to the unit vector $\mathbf{x}'/|\mathbf{x}'|$. Therefore,

$$\mathbf{F} = -q^2\frac{R|\mathbf{x}'|}{(R^2-|\mathbf{x}'|^2)^2}\frac{\mathbf{x}'}{|\mathbf{x}'|} = -q^2\frac{R}{(R^2-|\mathbf{x}'|^2)^2}\mathbf{x}'.$$

Problem 5. The "mean value theorem" is stated as follows: For any solution ϕ to $\nabla^2\phi = 0$, the value of ϕ at $\mathbf{x}$ is equal to the average value of ϕ on a sphere of radius R (for any R) centered at $\mathbf{x}$.

5.a Prove the mean value theorem. Hint: Apply Green's theorem to ϕ and $1/|\mathbf{x}-\mathbf{x}'|$ for a suitable choice of region and a suitable choice of $\mathbf{x}'$.

Solution. Green's theorem is given by Eq. (2.96),

$$\int_S \hat{\mathbf{n}}\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1)\,dS = -4\pi\int_{\mathcal{V}}(\phi_1\rho_2 - \phi_2\rho_1)\,d^3x\,.$$

We will choose our volume as a sphere centered at $\mathbf{x}$ with radius r, so $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Suppose $\phi_1 = \phi(\mathbf{x})$ is a solution to Laplace's equation as stated. Let $\mathbf{x}'$ point radially from $\mathbf{x}$, located at the center of the sphere, to a point a distance r away; that is, $\mathbf{x}' = \mathbf{x} + r\,\hat{\mathbf{r}}$. Then

$$\phi_2 = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|-r\,\hat{\mathbf{r}}|} = \frac{1}{r}.$$

From Poisson's equation, $\nabla^2\phi = -4\pi\rho$ in general. This means $\rho_1 = 0$. For the Green's function, $\rho_2 = \delta(\mathbf{x}-\mathbf{x}') = \delta(r)$.

Applying Green's theorem,

$$\int_S \hat{\mathbf{r}}\cdot\left(\phi(\mathbf{x})\nabla\frac{1}{r} - \frac{1}{r}\nabla\phi(\mathbf{x})\right)dS = -4\pi\int_{\mathcal{V}}\phi\,\delta(r)\,d^3x \tag{9}$$

For the first term on the left side, note that

$$\hat{\mathbf{r}}\cdot\nabla\frac{1}{r} = \frac{\partial}{\partial r}\frac{1}{r}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = -\frac{1}{r^2}.$$

Gauss's theorem is given by Eq. (2.6),

$$\int_{\mathcal{V}}\nabla\cdot\mathbf{v}\,d^3x = \int_S \mathbf{v}\cdot\hat{\mathbf{n}}\,dS\,.$$

Applying this to the second term on the left side of (9),

$$-\int_S \hat{\mathbf{n}}\cdot\frac{1}{r}\nabla\phi(\mathbf{x})\,dS = -\int_{\mathcal{V}}\nabla\cdot\frac{1}{r}\nabla\phi(\mathbf{x})\,d^3x = -\frac{1}{r}\nabla^2\phi(\mathbf{x}) = 0.$$

For the right side of (9),

$$-4\pi \int_{\mathcal{V}} \phi(\mathbf{x})\, \delta(r)\, d^3x = -4\pi\phi(0).$$

Putting this together, (9) becomes

$$-\int_S \frac{\phi(\mathbf{x})}{r^2}\, dS = -4\pi\phi(0)\, dS\,.$$

We can choose $\mathbf{x} = 0$ without loss of generality and switch $\mathbf{x}$ with $\mathbf{x}'$, which gives us

$$\phi(\mathbf{x}) = \frac{1}{4\pi r^2} \int_S \phi(\mathbf{x}')\, dS'\,. \tag{10}$$

This equation demonstrates that the value of ϕ at $\mathbf{x}$ is equal to its average value on a sphere of arbitrary radius r. Thus, we have proven the mean value theorem. $\square$

5.b Use this result to show that a point charge can never be in stable equilibrium if placed in an electric field $\mathbf{E}$ that is source free in a neighborhood of the charge.

Solution. Let $\mathcal{V}$ denote the neighborhood of the point charge, which can be described as a sphere of radius r centerd at the location of the point charge. We will choose this point as the origin. Let S denote the boundary of $\mathcal{V}$.

Suppose, contrary to the problem statement, that the point charge is in stable equilibrium. This means that the electrostatic potential ϕ has a local minimum at the origin, and so $\phi(0) < \phi(\mathbf{x})$ for all other $\mathbf{x} \neq 0$ within $\mathcal{V}$. In particular, $\phi(0) < \phi|_S$ at all points on the boundary, and so

$$\phi(0) < \frac{1}{4\pi r^2} \int_S \phi(\mathbf{x})\, dS\,. \tag{11}$$

However, ϕ must satisfy $\nabla^2\phi = 0$, since $\mathcal{V}$ is source free. As proven in 5.a, ϕ therefore obeys (10), which contradicts (11) and therefore our assumption that the point charge is in stable equilibrium. So we have shown that stable equilibrium is impossible in this situation. $\square$

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics* while writing up these solutions.