Problem 1. Renormalization of Yukawa theory (P&S 10.2) Consider the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi, \tag{1}$$

where ϕ is a real scalar field and ψ is a Dirac fermion. Notice that this Lagrangian is invariant under the parity transformation $\psi(t, \mathbf{x}) \to \gamma^0\psi(zt, -\mathbf{x})$, $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$, in which the field ϕ carries odd parity.

1(a) Determine the superficially divergent amplitudes and work out the Feynman rules for renormalized perturbation theory for this Lagrangian. Include all necessary counterterm vertices. Show that the theory contains a superficially divergent 4ϕ amplitude. This means that the theory cannot be renormalized unless one includes a scalar self-interaction,

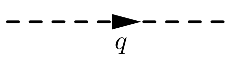
$$\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4, \tag{2}$$

and a counterterm of the same form. It is of course possible to set the renormalized value of this coupling to zero, but that is not a natural choice, since the counterterm will still be nonzero. Are any further interactions required?

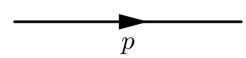
Solution. We write Eq. (1) explicitly in terms of the bare masses m_0, M_0 and the bare coupling constant g_0:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M_0)\psi - ig_0\bar{\psi}\gamma^5\psi\phi, \tag{3}$$

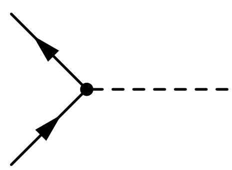
The Feynman rules for a pseudoscalar Yukawa theory are [1, pp. 24–25]



$$= \frac{i}{q^2 - m_0^2 + i\epsilon}$$



$$= \frac{i(p\!\!\!/ + M_0)}{p^2 - M_0^2 + i\epsilon}$$



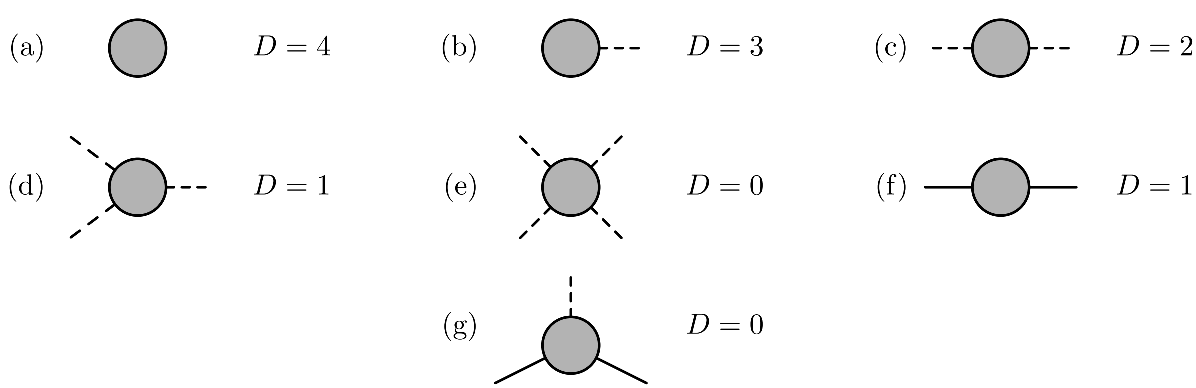
$$= g_0\gamma^5$$

These Feynman rules are similar enough to those for QED; that is, the powers of k are the same, each propagator has a momentum integral, each vertex has a delta function, and each vertex involves one ϕ line and two fermion lines [2, p. 316]. So we can adapt P&S (10.4) for the superficial degree of divergence:

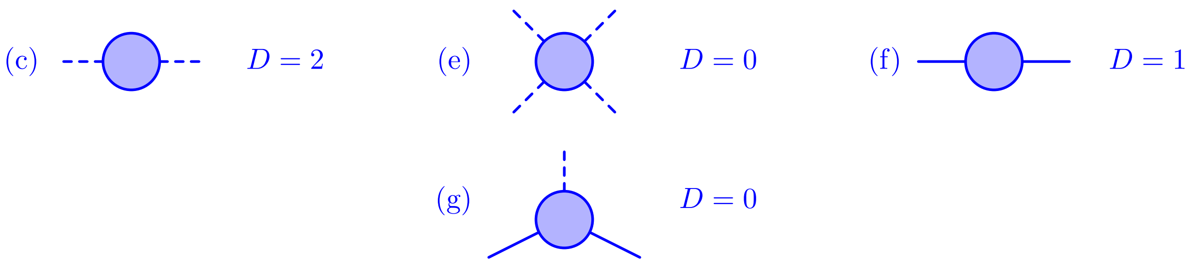
$$D = 4 - N_\phi - \frac{3}{2}N_f,$$

where N_ϕ is the number of external ϕ lines and N_f is the number of external fermion lines.

This means the superficially divergent amplitudes are a subset of those appearing in Fig. 10.2 of P&S, with the photon lines replaced by pseudoscalar lines:

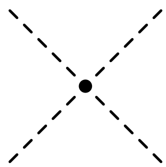


We ignore (a) since it is irrelevant to scattering processes [2, pp. 317–318]. Amplitudes (b) and (d) vanish because the theory is invariant under the parity transformation, which means all amplitudes with zero external fermion legs and an odd number of external ϕ legs vanish [2, pp. 318, 323–324]. So the superficially divergent amplitudes are



(4)

Note that amplitude (e) is a 4ϕ amplitude. Since it is superficially divergent, according to the problem statement we must introduce the scalar self-interaction given by Eq. (2). We subtract this term as in the ϕ^4 theory [2, p. 324]. The Feynman rule for this vertex is [2, p. 325]


$= -i\lambda_0.$

With the addition of this new term, our Lagrangian in Eq. (3) becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M_0)\psi - ig_0\bar{\psi}\gamma^5\psi\phi - \frac{\lambda_0}{4!}\phi^4, \tag{5}$$

where λ_0 is the bare coupling constant for the scalar self-interaction. To work out the renormalized theory, we rescale the field as in P&S (10.15):

$$\phi = Z_1^{1/2}\phi_r.$$

The rescaling for the fermion is [2, p. 330]

$$\psi = Z_2^{1/2}\psi_r.$$

Feeding these into Eq. (5), we obtain the renormalized Lagrangian [2, p. 324]

$$\mathcal{L} = \frac{1}{2}Z_1(\partial_\mu \phi)^2 - \frac{1}{2}Z_1 m_0^2\phi^2 + Z_2\bar{\psi}(i\partial\!\!\!/ - M_0)\psi - iZ_1^{1/2}Z_2 g_0\bar{\psi}\gamma^5\psi\phi - \frac{\lambda_0}{4!}Z_1^2\phi^4. \tag{6}$$

Define [2, pp. 324, 331]

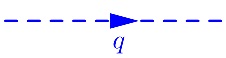
$$\begin{aligned} &\delta_{Z_1} = Z_1 - 1, && \delta_{Z_2} = Z_2 - 1, && \delta_m = m_0^2 Z_1 - m^2, \\ &\delta_M = M_0 Z_2 - M, && \delta_g = (g_0/g)Z_1^{1/2}Z_2 - 1, && \delta_\lambda = \lambda_0 Z_1^2 - \lambda \end{aligned}$$

Then Eq. (6) becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(1+\delta_{Z_1})(\partial_\mu \phi)^2 - \frac{1}{2}(m^2+\delta_m)\phi^2 + \bar{\psi}[i(\delta_{Z_2}+1)\partial\!\!\!/ - (M+\delta_M)]\psi - ig(1+\delta_g)\bar{\psi}\gamma^5\psi\phi + \frac{\lambda+\delta_\lambda}{4!}\phi^4 \\ &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \\ &\qquad + \frac{1}{2}\delta_{Z_1}(\partial_\mu \phi)^2 - \frac{1}{2}\delta_m\phi^2 + \bar{\psi}(i\delta_{Z_2}\partial\!\!\!/ - \delta_M)\psi - ig\delta_g\bar{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4. \end{aligned}$$

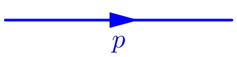
Here the first first five terms look like Eq. (5), but written in terms of the physical masses and couplings. The last five terms are the counterterms [2, p. 325].

The Feynman rules for the renormalized theory are [2, p. 325]

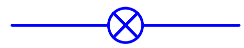


$$= \frac{i}{q^2 - m^2 + i\epsilon}$$

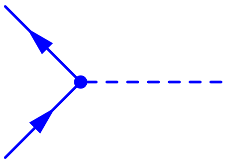
$$= i(p^2\delta_{Z_1} - \delta_m)$$



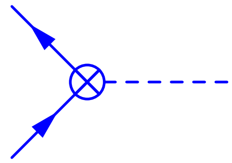
$$= \frac{i(\not{p} + M)}{p^2 - M^2 + i\epsilon}$$



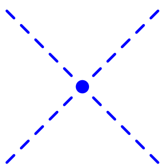
$$= i(\not{p}\delta_{Z_2} - \delta_M)$$



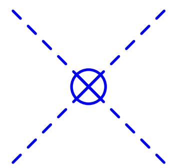
$$= g\gamma^5$$



$$= g\delta_g\gamma^5$$



$$= -i\lambda$$

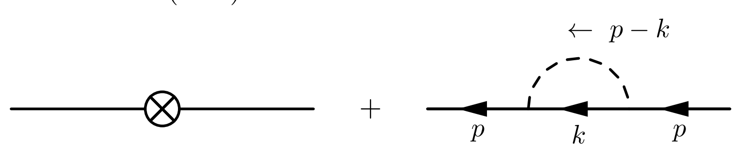


$$= -i\delta_\lambda$$

No further interactions are required because once we have added the ϕ^4 term, the Lagrangian in Eq. (6) contains terms that reflect all of the amplitudes in Eq. (4).

1(b) Compute the divergent part (the pole as $d \to 4$) of each counterterm, to the one-loop order of perturbation theory, implementing a sufficient set of renormalization conditions. You need not worry about finite parts of the counterterms. Since the divergent parts must have a fixed dependence on the external momenta, you can simplify this calculation by choosing the momenta in the simplest possible form.

Solution. To compute the divergent part of the fermion propagator counterterm to one-loop order, we include the fermion self-energy similar to P&S (7.15):



The fermion-self energy here looks similar to that in QED, so we may adapt P&S (7.16) for that term. Using our Feynman rules from 1(a), we have

$$\begin{aligned}
-iM^2(p^2) &= i(\not{p}\delta_{Z_2} - \delta_M) + g^2 \int \frac{d^dk}{(2\pi)^d}\gamma^5 \frac{i(\not{k} + M)}{p^2 - M^2 + i\epsilon}\gamma^5 \frac{i}{(p-k)^2 - m^2 + i\epsilon} \\
&= i(\not{p}\delta_{Z_2} - \delta_M) + g^2 \int \frac{d^dk}{(2\pi)^d} \frac{\not{k} - M}{(p^2 - M^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]},
\end{aligned} \tag{7}$$

where we have used P&S (3.70), $(\gamma^5)^2 = 1$, and (3.71), $\{\gamma^5, \gamma^\mu\} = 0$, which implies $\gamma^5\gamma^\mu\gamma^5 = -\gamma^\mu$. Following the procedure on pp. 217–218, we introduce the Feynman parameter x to combine the denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon}\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx\, \frac{1}{[k^2 - 2xk\cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}. \tag{8}$$

Let $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + xm^2 + (1-x)M^2$. Then Eq. (7) can be written

$$-iM^2(p^2) = i(\not{p}\delta_{Z_2} - \delta_M) + g^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{x\not{p} - M}{(\ell^2 - \Delta + i\epsilon)^2}. \tag{9}$$

To evaluate the integral, we can write it in terms of the Euclidean 4-momentum defined by [2, p. 193]

$$\ell^0 \equiv i\ell_E^0, \qquad\qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E. \tag{10}$$

Then we can write

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(-1)^2} \frac{1}{(2\pi)^d} \int d^d\ell_E \, \frac{1}{(\ell_E^2 + \Delta)^2} = i \int \frac{d^d\ell_E}{(2\pi)^2} \frac{1}{(\ell_E^2 + \Delta)^2}. \tag{11}$$

Then we can apply (7.84), which takes the limit as $d \to 4$:

$$\int \frac{d^d\ell_E}{(2\pi)^2} \frac{1}{(\ell_E^2 + \Delta)^2} \to \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi}{\Delta} \right) \right) \to \frac{1}{8\pi^2\epsilon}, \tag{12}$$

where $\epsilon = 4 - d$ [2, p. 250], and we have omitted the finite parts. Making these substitutions into Eq. (**??**), we find

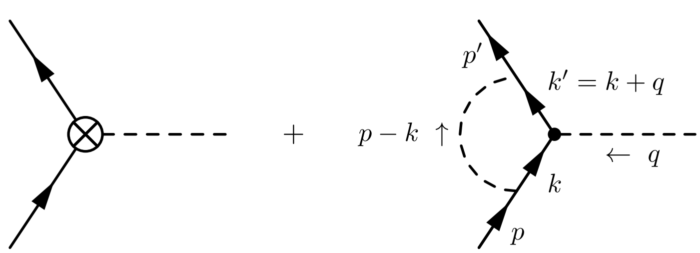
$$\begin{aligned}
-iM^2(p^2) &= i(\not{p}\delta_{Z_2} - \delta_M) + \frac{ig^2}{8\pi^2\epsilon} \int_0^1 dx \, (x\not{p} - M) \\
&= i(\not{p}\delta_{Z_2} - \delta_M) + \frac{ig^2}{8\pi^2\epsilon} \left[\frac{x^2}{2}\not{p} - Mx \right]_0^1 \\
&= i(\not{p}\delta_{Z_2} - \delta_M) + \frac{ig^2}{8\pi^2\epsilon} \left(\frac{\not{p}}{2} - M \right) \\
&= i\not{p}\left(\delta_{Z_2} + \frac{g^2}{16\pi^2\epsilon} \right) - i\left(\delta_M + \frac{g^2}{8\pi^2\epsilon} M \right).
\end{aligned}$$

This implies that

$$\delta_{Z_1} = -\frac{g^2}{16\pi^2\epsilon}, \qquad\qquad \delta_M = -\frac{g^2}{8\pi^2\epsilon} M \tag{13}$$

are the conditions to eliminate the divergence.

For the scalar-fermion vertex, we can adapt some of our work from Prob. 2(a) of Homework 1. With the one-loop diagram similar to the one on p. 189 of P&S, we have



We adapt Peskin & Schroeder (6.38) using the pseudoscalar field Feynman rules to write [2, p. 123]

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p,p')u(p) &= \bar{u}(p')g\delta_g\gamma^5 u(p) + ig^3 \int \frac{d^dk}{(2\pi)^d} \bar{u}(p') \frac{\gamma^5(\not{k}' + M)\gamma^5(\not{k} + M)\gamma^5}{[(k-p)^2 - m^2 + i\epsilon](k'^2 - M^2 + i\epsilon)(k^2 - M^2 + i\epsilon)} u(p) \\
&= \bar{u}(p')g\delta_g\gamma^5 u(p) + ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \bar{u}(p') \frac{(\not{k}' + M)(\not{k} - M)}{[(k-p)^2 - m^2 + i\epsilon](k'^2 - M^2 + i\epsilon)(k^2 - M^2 + i\epsilon)} u(p),
\end{aligned} \tag{14}$$

where we have once more used $(\gamma^5)^2 = 1$ and $\{\gamma^5, \gamma^\mu\} = 0$. We use Peskin & Schroeder (6.41) to write

$$\frac{1}{[(k-p)^2 - m^2 + i\epsilon](k'^2 - M^2 + i\epsilon)(k^2 - M^2 + i\epsilon)} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\frac{2}{D^3}, \tag{15}$$

where [2, pp. 190–191]

$$\begin{aligned} D &= k^2 + 2k(qy - pz) + z(p^2 - m^2) - (1-z)M^2 + i\epsilon \\ &= k^2 - 2kpz + z(p^2 - m^2) - (1-z)M^2 + i\epsilon \\ &= \ell^2 - \Delta + i\epsilon. \end{aligned} \tag{16}$$

Here we have used $x + y + z = 1$ and set $q = 0$ (so $k' = k$) as in Prob. 2(a) of Homework 1. We have defined $\ell \equiv k - zp$ [2, p. 191], and $\Delta \equiv (1-z)^2M^2 + zm^2$. For the numerator of Eq. (14), we use $\ell \equiv k - zp$ [2, p. 191], and define

$$\begin{aligned} N &\equiv \bar{u}(p')(\not{\ell} + z\not{p} + M)(\not{\ell} + z\not{p} - M)u(p) \\ &= \bar{u}(p')(\not{\ell}\not{\ell} + z\not{\ell}\not{p} - M\not{\ell} + z\not{p}\not{\ell} + z^2\not{p}\not{p} - zM\not{p} + M\not{\ell} + zM\not{p} - M^2)u(p). \end{aligned} \tag{17}$$

To simplify N we apply (7.87),

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{D^3} = \int \frac{d^d\ell}{(2\pi)^d}\frac{g^{\mu\nu}\ell^2}{dD^3},$$

the fact that we may drop terms linear in ℓ by (6.45),

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu}{D^3} = 0,$$

as well as [2, pp. 191–192]

$$\not{p}u(p) = Mu(p), \qquad \bar{u}(p')\not{p}' = \bar{u}(p')M,$$

The Eq. (17) becomes

$$N = \bar{u}(p')(\ell^2 + z^2M^2 - zM^2 + zM^2 - M^2)u(p) = \bar{u}(p')[\ell^2 + (z^2-1)M^2]u(p).$$

With this and Eqs. (15) and (16), we can write Eq. (14) in the form

$$\delta\Gamma^\mu(p,p') = g\delta_g\gamma^5 + 2ig^3\gamma^5\int_0^1 dz\,\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2 + (z^2-1)M^2}{(\ell^2 - \Delta + i\epsilon)^3}. \tag{18}$$

To solve the integrals over ℓ, we use some work from Prob. 1(b) of Homework 1. We substituted $\epsilon = 4 - d$ into P&S (7.85) and (7.86) and found

$$\begin{aligned} \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3} &= -\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{1+\epsilon/2}, \\ \int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3} &= \frac{i}{(4\pi)^{2-\epsilon/2}}\frac{4-\epsilon}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\epsilon/2}. \end{aligned}$$

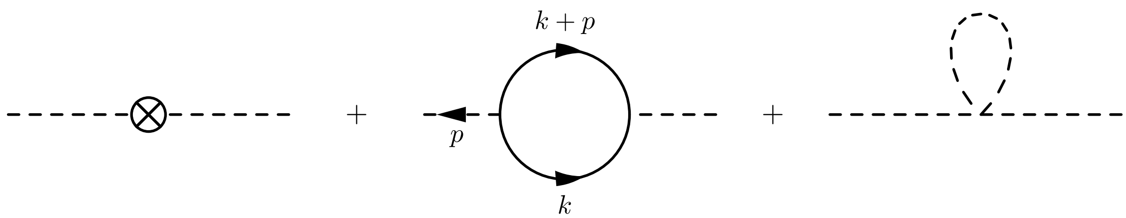
Then for Eq. (18), we have

$$\begin{aligned} \delta\Gamma^\mu(p,p') &= g\delta_g\gamma^5 - \frac{2}{(4\pi)^{2-\epsilon/2}}g^3\gamma^5\int_0^1 dz\,(1-z)\left[\frac{4-\epsilon}{2}\frac{\Gamma(\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{\epsilon/2} - (z^2-1)M^2\frac{\Gamma(1+\epsilon/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{1+\epsilon/2}\right] \\ &= g\delta_g\gamma^5 - \frac{g^3}{(4\pi)^2}\gamma^5\int_0^1 dz\,(1-z)\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2}\left[\frac{4-\epsilon}{2}\Gamma(\epsilon/2) + \frac{(1-z^2)M^2}{\Delta}\Gamma(1+\epsilon/2)\right] \\ &\to g\delta_g\gamma^5 - \frac{g^3}{(4\pi)^2}\frac{4}{\epsilon}\gamma^5\int_0^1 dz\,(1-z) \\ &= g\gamma^5\left(\delta_g - \frac{g^2}{8\pi^2}\frac{1}{\epsilon}\right)\gamma^5, \end{aligned}$$

where we have taken the $\epsilon \to 0$ limit using Mathematica and retained only divergent terms. Then in order to eliminate the divergence, we need

$$\delta_g = \frac{g^2}{8\pi^2}\frac{1}{\epsilon}. \tag{19}$$

For the divergent part of the pseudoscalar propagator, we must include the pseudoscalar self-energy. Since our Feynman rules include two different vertices involving the pseudoscalar, we can draw two different self-energy diagrams:



(20)

Adapting P&S (10.32) for the second diagram, we have

$$-g^2\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\left[\frac{i\gamma^5(\not{k}+\not{p}+M)}{(k+p)^2-M^2+i\epsilon}\frac{i\gamma^5(\not{k}+M)}{k^2-M^2+i\epsilon}\right] = -g^2\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\Big[-\frac{(\not{k}+\not{p}-M)(\not{k}+M)}{[(k+p)^2-M^2+i\epsilon][k^2-M^2+i\epsilon]}\Big].$$

We introduce the Feynman parameter x [2, pp. 217, 327]:

$$\frac{1}{(k+p)^2-M^2+i\epsilon}\frac{1}{k^2-M^2+i\epsilon} = \int_0^1 dx\,\frac{1}{[k^2+2xk\cdot p+xp^2-M^2+i\epsilon]^2} \equiv \int_0^1 dx\,\frac{1}{D^2}. \tag{21}$$

Let $\ell = k + xp$ and $\Delta = M^2 - x(1-x)p^2$ [2, pp. 327, 329]. Then $D = \ell^2 - \Delta + i\epsilon$. For the numerator, note that

$$(\not{k}+\not{p}-M)(\not{k}+M) = \not{k}\not{k} + M\not{k} + \not{p}\not{k} + M\not{p} - M\not{k} - M^2,$$

so

$$\begin{aligned} N \equiv \mathrm{tr}\big[-(\not{k}+\not{p}-M)(\not{k}+M)\big] &= 4[k\cdot(p+k) - M^2] \\ &= 4[(\ell - xp)\cdot(p+\ell-xp) - M^2] \\ &= 4[\ell^2 + (1-x)\ell\cdot p - xp\cdot\ell + x(1-x)p^2 - M^2] \\ &= 4[\ell^2 + x(1-x)p^2 - M^2] \end{aligned} \tag{22}$$

since

$$\mathrm{tr}(\mathbf{1}) = 0, \qquad \mathrm{tr}(\text{any odd number of }\gamma\text{'s}) = 0, \qquad \mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{23}$$

by (A.27). This is similar to the expression obtained in P&S (10.32). Now we can evaluate the integral using (A.45) with $n = 2$,

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^2} = \frac{(-1)i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-d/2)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{1-d/2} = -\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{4-\epsilon}{2}\Gamma(\epsilon/2-1)\left(\frac{1}{\Delta}\right)^{\epsilon/2-1} \to -\frac{\Delta}{4\pi^2\epsilon},$$

as well as Eqs. (11) and (12). Utilizing our work in Eqs. (21) and (22), then, yields

$$\begin{aligned}
-g^2\int\frac{d^dk}{(2\pi)^d}\operatorname{tr}\left[-\frac{(\not{k}+\not{p}-M)(\not{k}+M)}{[(k+p)^2-M^2+i\epsilon][k^2-M^2+i\epsilon]}\right] &= -4g^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2-x(1-x)p^2-M^2}{(\ell^2-\Delta)^2} \\
&\to -4ig^2\int_0^1 dx\left[\frac{\Delta}{4\pi^2\epsilon}-\frac{x(1-x)p^2+M^2}{8\pi^2\epsilon}\right] \\
&= -\frac{ig^2}{\pi^2\epsilon}\int_0^1 dx\left[M^2-x(1-x)p^2-\frac{1}{2}x(1-x)p^2-\frac{M^2}{2}\right] \\
&= -\frac{ig^2}{2\pi^2\epsilon}\int_0^1 dx\left[M^2-3x(1-x)p^2\right] \\
&= \frac{ig^2}{2\pi^2\epsilon}\left(\frac{p^2}{2}-M^2\right) \qquad (24)
\end{aligned}$$

for the second diagram.

For the third diagram in Eq. (20), we may adapt (10.29):

$$-i\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}=\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2+i\epsilon}.$$

Let $\ell = k$ and $\Delta = m^2$. Then we can use P&S (A.44) with $n = 1$ and substitute $\epsilon = 4 - d$:

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2-\Delta}=-\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(1-d/2)}{\Gamma(1)}\left(\frac{1}{\Delta}\right)^{1-d/2}=-\frac{i}{(4\pi)^{2-\epsilon/2}}\Gamma(\epsilon/2-1)\left(\frac{1}{\Delta}\right)^{\epsilon/2-1}.$$

This gives us

$$\frac{\lambda}{2}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2-\Delta}\to\frac{i\lambda}{16\pi^2\epsilon}m^2 \qquad (25)$$

for the third diagram.

Feeding Eqs. (24) and (25) into the sum of all three diagrams in Eq. (20), we have

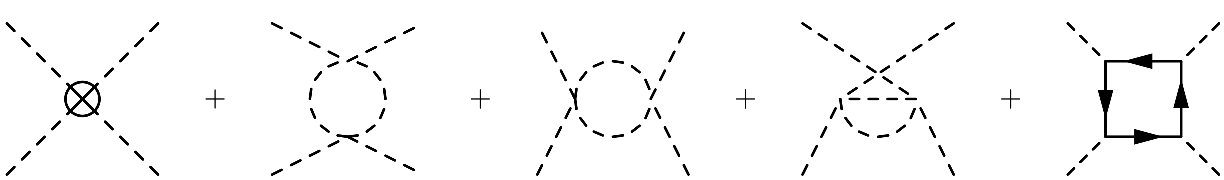
$$\begin{aligned}
-iM^2(p^2) &= i(p^2\delta_{Z_2}-\delta_m)-g^2\int\frac{d^dk}{(2\pi)^d}\operatorname{tr}\left[-\frac{(\not{k}+\not{p}-M)(\not{k}+M)}{[(k+p)^2-M^2+i\epsilon][k^2-M^2+i\epsilon]}\right]+\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2+i\epsilon} \\
&= i(p^2\delta_{Z_2}-\delta_m)+\frac{ig^2}{2\pi^2\epsilon}\left(\frac{p^2}{2}-M^2\right)+\frac{i\lambda}{16\pi^2\epsilon}m^2 \\
&= i\left\{p^2\left[\delta_{Z_2}+\frac{g^2}{4\pi^2\epsilon}\right]-\left[\delta_m+\left(\frac{g^2M^2}{2\pi^2\epsilon}-\frac{\lambda m^2}{16\pi^2\epsilon}\right)\right]\right\},
\end{aligned}$$

which implies

$$\delta_{Z_2}=-\frac{g^2}{4\pi^2\epsilon}, \qquad \delta_m=\frac{\lambda m^2}{16\pi^2\epsilon}-\frac{g^2M^2}{2\pi^2\epsilon} \qquad (26)$$

is needed.

For the 4ϕ vertex, we need to consider all of the one-loop diagrams of ϕ^4 theory [2, p. 326], as well as a diagram with a fermion loop [2, p. 121]:



(27)

The second diagram can be computed in an identical manner to the corresponding scalar diagram in P&S (10.20) and (10.23):

$$\frac{(-i\lambda)^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(k+p)^2-m^2+i\epsilon}=\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2+i\epsilon)[(k+p)^2-m^2+i\epsilon]}. \tag{28}$$

We introduce the Feynman parameter x as in (10.23):

$$\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(k+p)^2-m^2+i\epsilon}=\int_0^1 dx\,\frac{1}{[k^2+2xk\cdot p+xp^2-m^2]^2}\equiv\int_0^1\frac{1}{D^2}.$$

We note that this D is identical to the one in Eq. (21) with $M\to m$, so we may reuse our work from that calculation. Let $\Delta=m^2-x(1-x)p^2$, so $D=\ell^2-\Delta+i\epsilon$. Once more applying Eqs. (11) and (12), we have

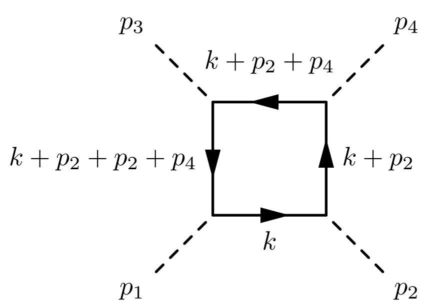
$$\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2+i\epsilon)[(k+p)^2-m^2+i\epsilon]}\to\frac{\lambda^2}{2}\frac{i}{8\pi^2\epsilon}=\frac{i\lambda^2}{16\pi^2\epsilon} \tag{29}$$

as $\epsilon\to 0$.

We note that the third and fourth diagrams in Eq. (27) also take the form of Eq. (28) since $p^2=s$, where s is a Mandelstam variable. We can evaluate the third and fourth diagrams by replacing $p^2=s$ by t and by u, respectively [2, pp. 156, 326]. However, we see in Eq. (29) that the divergent term does not depend on the Mandelstam variable, so we conclude that the sum of these three diagrams is

$$\frac{3i\lambda^2m^2}{16\pi^2\epsilon}. \tag{30}$$

For the final (fermion loop) diagram, we need to evaluate the trace of the four vertices [2, p. 120]. We label the momenta as on p. 326 of P&S:



We have

$$-g^4\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\left[\frac{\gamma^5(\not{k}+M)}{k^2-M^2+i\epsilon}\frac{\gamma^5(\not{k}+p_2+M)}{(k+p_2)^2-M^2+i\epsilon}\frac{\gamma^5(\not{k}+p_2+p_4+M)}{(k+p_2+p_4)^2-M^2+i\epsilon}\frac{\gamma^5(\not{k}+\not{p}_2+\not{p}_3+\not{p}_4+M)}{(k+p_2+p_3+\not{p}_4)^2-M^2+i\epsilon}\right]$$
$$=-g^4]int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\left[\frac{(\not{k}-M)(\not{k}+\not{p}_2+M)(\not{k}+\not{p}_2+\not{p}_4-M)(\not{k}+\not{p}_2+\not{p}_3+\not{p}_4+M)}{(k^2-M^2+i\epsilon)[(k+p_2)^2-M^2+i\epsilon][(k+\not{p}_2+\not{p}_4)^2-M^2+i\epsilon][(k+\not{p}_2+\not{p}_3+\not{p}_4)^2-M^2+i\epsilon]}\right]. \tag{31}$$

Now we introduce the Feynman parameters, using Peskin & Schroeder (6.41) with $n=4$:

$$\frac{1}{(k^2-M^2+i\epsilon)[(k+p_2)^2-M^2+i\epsilon][(k+\not{p}_2+\not{p}_4)^2-M^2+i\epsilon][(k+\not{p}_2+\not{p}_3+\not{p}_4)^2-M^2+i\epsilon]}$$
$$=\int_0^1 dw\,dx\,dy\,dz\,\frac{3!\,\delta(1-w-x-y-z)}{\{w(k^2-M^2+i\epsilon)+x[(k+p_2)^2-M^2+i\epsilon]+y[(k+\not{p}_2+\not{p}_4)^2-M^2+i\epsilon]+z[(k+\not{p}_2+\not{p}_3+\not{p}_4)^2-M^2+i\epsilon]\}^4}$$
$$\equiv 3!\int_0^1 dw\,dx\,dy\,dz\,\frac{\delta(1-w-x-y-z)}{(\ell^2-\Delta+i\epsilon)^4},$$

where we have defined ℓ and Δ, although not explicitly, in hopes that the divergent terms are independent of Δ. We do know that $\ell = k+$(other terms), so we see that the numerator in Eq. (31) has terms of $\mathcal{O}(\ell^4)$, $\mathcal{O}(\ell^2)$, and $\mathcal{O}(\ell^0)$. The terms of odd powers of ℓ vanish when we take the trace by (A.27), which tells us that

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{32}$$

along with the identities in Eq. (23). From P&S (A.44) and (A.45),

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^4} = \frac{(-1)^4 i}{(4\pi)^{d/2}}\frac{\Gamma(4-d/2)}{\Gamma(4)}\left(\frac{1}{\Delta}\right)^{4-d/2} = \frac{i}{(4\pi)^{2-\epsilon/2}}\frac{\Gamma(2+\epsilon/2)}{6}\left(\frac{1}{\Delta}\right)^{2+\epsilon/2} \to 0,$$
$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^4} = \frac{(-1)^3 i}{(4\pi)^2}\frac{d}{2}\frac{\Gamma(4-d/2-1)}{\Gamma(4)}\left(\frac{1}{\Delta}\right)^{4-d/2-1} = -\frac{i}{(4\pi)^{2-\epsilon/2}}\frac{4-\epsilon}{2}\frac{\Gamma(1+\epsilon/2)}{6}\left(\frac{1}{\Delta}\right)^{1+\epsilon/2} \to 0,$$

where in the final step we have taken the $\epsilon \to 0$ limit and kept only terms of $\mathcal{O}(\epsilon^{-1})$. So these terms do not contribute to the divergence. Finally, from (A.47),

$$\begin{aligned}\int \frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^2}{(\ell^2-\Delta)^4} &= \frac{(-1)^4 i}{(4\pi)^2}\frac{d(d+2)}{4}\frac{\Gamma(4-d/2-2)}{\Gamma(4)}\left(\frac{1}{\Delta}\right)^{4-d/2-2} \\ &= \frac{i}{(4\pi)^{2-\epsilon/2}}\frac{(4-\epsilon)(6-\epsilon)}{4}\frac{\Gamma(\epsilon/2)}{6}\left(\frac{1}{\Delta}\right)^{\epsilon/2} \\ &\to \frac{i}{8\pi^2\epsilon}.\end{aligned}$$

We note that the ℓ^4 term in the numerator of Eq. (31) picks up a factor of 4 when we take the trace by Eq. (32). So for the fourth diagram in Eq. (27), we have

$$-g^4\int \frac{d^dk}{(2\pi)^d}\operatorname{tr}\left[\cdots\right] = -(3!)(4)\int_0^1 dw\,dx\,dy\,dz\,\delta(1-w-x-y-z)\frac{ig^4}{8\pi^2\epsilon} = \frac{ig^4}{2\pi^2\epsilon}, \tag{33}$$

where we have used Mathematica to evaluate the integral.

Summing Eqs. (30) and (33) with the Feynman rule for the appropriate counterterm, the sum of the diagrams in Eq. (27) is

$$-iM^2(p^2) = \frac{3i\lambda^2m^2}{16\pi^2\epsilon} - \frac{ig^4}{2\pi^2\epsilon} - i\delta_\lambda = i\left[\left(\frac{3\lambda^2m^2}{16\pi^2\epsilon} - \frac{g^4}{2\pi^2\epsilon}\right) - \delta_\lambda\right],$$

which implies that we need

$$\delta_\lambda = \frac{3\lambda^2m^2}{16\pi^2\epsilon} - \frac{g^4}{2\pi^2\epsilon} \tag{34}$$

in order to eliminate the divergence.

Summing up our results in Eqs. (13), (19), (26), and (34), the renormalization conditions are

$$\delta_{Z_1} = -\frac{g^2}{16\pi^2\epsilon}, \qquad \delta_M = -\frac{g^2}{8\pi^2\epsilon}M, \qquad \delta_g = \frac{g^2}{8\pi^2}\frac{1}{\epsilon},$$
$$\delta_{Z_2} = -\frac{g^2}{4\pi^2\epsilon}, \qquad \delta_m = \frac{\lambda m^2}{16\pi^2\epsilon} - \frac{g^2M^2}{2\pi^2\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2m^2}{16\pi^2\epsilon} - \frac{g^4}{2\pi^2\epsilon}.$$

References

[1] C. Blair, "Quantum Field Theory—Useful Formulae and Feynman Rules", May, 2010. `https://www.maths.tcd.ie/~cblair/notes/list.pdf`.

[2] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.