**Problem 1.** **(Jackson 14.1)** Verify by explicit calculation that the Liénard-Wiechert expressions for *all* components of $\mathbf{E}$ and $\mathbf{B}$ for a particle moving with constant velocity agree with the ones obtained in the text by means of a Lorentz transformation. Follow the general method at the end of Section 14.1.

**Solution.** The Liénard-Wiechert expressions for the fields are given by Jackson (14.13–14):

$$\mathbf{B} = [\hat{\mathbf{n}} \times \mathbf{E}]_{\text{ret}}, \qquad \mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3 R^2}\right]_{\text{ret}} + \frac{e}{c}\left[\frac{\hat{\mathbf{n}} \times \{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3 R}\right]_{\text{ret}}, \tag{1}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ with $\mathbf{v}$ being the particle's velocity, $R$ is the distance from the observation point to the particle's position, and $\hat{\mathbf{n}}$ is a unit vector defined by $\mathbf{x} - \mathbf{r}(\tau) = R\,\hat{\mathbf{n}}$. Here, $\mathbf{r}(\tau)$ is the particle's present position and $\tau$ the proper time.

The expressions for the components of $\mathbf{E}$ and $\mathbf{B}$ obtained by a Lorentz transformation are given by Jackson (11.152):

$$E_1 = -\frac{e\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2 = \frac{e\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_3 = B_1 = B_2 = 0, \qquad B_3 = \beta E_2, \tag{2}$$

where the particle is moving in the $x_1$ direction at impact parameter $b$ on the $x_2$ axis, as shown in Fig. (1).

For a particle moving with constant velocity in the $x_1$ direction with velocity $v$ as shown in Fig. (1), $\boldsymbol{\beta} = \beta\,\hat{\mathbf{x}}_1$ and $\dot{\boldsymbol{\beta}} = 0$. From Jackson (14.16), note that

$$(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2R^2 = b^2 + v^2t^2 - \beta^2b^2 = \frac{b^2+\gamma^2v^2t^2}{\gamma^2} \qquad \Longrightarrow \qquad (1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2 = \frac{(b^2+\gamma^2v^2t^2)^{3/2}}{R\gamma^3}.$$

This calculation comes from Fig. (2), where $O$ is the observation point, $P$ is the present position of the particle, and $P'$ its retarded position. Also from Fig. 2,

$$\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}}_1 + \sin\theta\,\hat{\mathbf{x}}_2 = \frac{\beta R - vt}{R}\,\hat{\mathbf{x}}_1 + \frac{b}{R}\,\hat{\mathbf{x}}_2.$$

Making these substitutions in the expression for $\mathbf{E}(\mathbf{x}, t)$ in Eq. (1),

$$\mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\text{ret}} = e\left[\frac{(\beta - vt/R - \beta)\,\hat{\mathbf{x}}_1 + (b/R)\,\hat{\mathbf{x}}_2}{\gamma^2(b^2+\gamma^2v^2t^2)^{3/2}}R\gamma^3\right]_{\text{ret}} = e\gamma\frac{-vt\,\hat{\mathbf{x}}_1 + b\,\hat{\mathbf{x}}_2}{(b^2+\gamma^2v^2t^2)^{3/2}}. \tag{3}$$

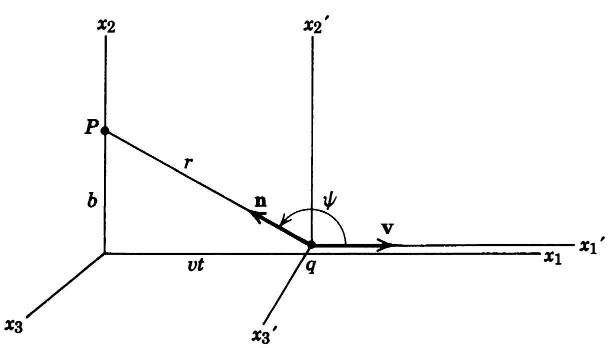


Figure 1: (Jackson Fig. 11.8) Particle of charge $q$ moving at constant velocity $\mathbf{v}$ passes an observation point $P$ at impact parameter $b$.

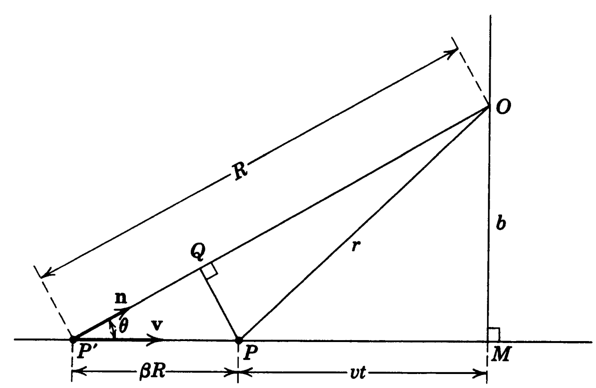


Figure 2: (Jackson Fig. 14.2) Present and retarded positions of a charge in uniform motion.

For $\mathbf{B}(\mathbf{x}, t)$, note that

$$\hat{\mathbf{n}} \times \mathbf{E} \propto \left(\frac{\beta R - vt}{R}\,\hat{\mathbf{x}}_\mathbf{1} + \frac{b}{R}\,\hat{\mathbf{x}}_\mathbf{2}\right) \times (-vt\,\hat{\mathbf{x}}_\mathbf{1} + b\,\hat{\mathbf{x}}_\mathbf{2}) = \left(b\frac{\beta R - vt}{R} + \frac{bvt}{R}\right)\hat{\mathbf{x}}_\mathbf{3} = \beta b,$$

so

$$\mathbf{B}(\mathbf{x}, t) = e\gamma\frac{\beta b\,\hat{\mathbf{x}}_\mathbf{3}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \tag{4}$$

Writing Eqs. (3) and (4) in component notation, we find

$$E_1 = -\frac{e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_2 = \frac{e\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_3 = 0,$$

$$B_1 = 0, \qquad B_2 = 0, \qquad B_3 = \frac{e\gamma\beta b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \beta E_2,$$

which are identical to Eq. (2) as was to be shown. □

**Problem 2.** **(Jackson 14.3)** The Heaviside-Feynman expression for the electric field of a particle of charge $e$ in arbitrary motion, an alternative to the Liénard-Wiechert expression in Eq. (1), is

$$\mathbf{E} = e\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + e\left[\frac{R}{c}\right]_{\text{ret}}\frac{d}{dt}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + \frac{e^2}{c^2}\frac{d^2}{dt^2}[\hat{\mathbf{n}}]_{\text{ret}},$$

where the time derivatives are with respect to the time at the observation point. Using the fact that the retarded time is $t' = t - R(t')/c$ and that, as a result,

$$\frac{dt}{dt'} = 1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t'),$$

show that the form above yields the expression for $\mathbf{E}$ in Eq. (1) when the time differentiations are performed.

**Solution.** By the chain rule,

$$\frac{d}{dt} = \frac{dt'}{dt}\frac{d}{dt'} = \frac{1}{1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')}\frac{d}{dt'}, \qquad \frac{d^2}{dt^2} = \left(\frac{dt'}{dt}\frac{d}{dt'}\right)^2 = \frac{1}{[1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t')]^2}\frac{d^2}{dt'^2}.$$

Note also that $\boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t') = dR/dt$.

**Problem 3.** **(Jackson 14.4)** Using the Liénard-Wiechart fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a particle with charge $e$, moving as described below. Sketch the angular distribution of the radiation and determine the total power radiated in each case.

**3(a)** The particle is moving along the $z$ axis with instantaneous position $z(t) = \alpha \cos \omega_0 t$.

**Solution.** For a nonrelativistic particle, the power radiated per unit solid angle is given by Jackson (14.21),

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2} |\dot{\mathbf{v}}|^2 \sin^2 \Theta, \tag{5}$$

where $\Theta$ is the angle between $\dot{\mathbf{v}}$ and $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing toward the observer. The total instantaneous power radiated is given by Jackson (14.22):

$$P = \frac{2}{3} \frac{e^2}{c^3} |\dot{\mathbf{v}}|^2. \tag{6}$$

In this case, we have

$$\mathbf{x}(t) = \alpha \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{3}, \qquad \mathbf{v}(t) = -\alpha \omega_0 \sin \omega_0 t \, \hat{\mathbf{x}}_\mathbf{3}, \qquad \dot{\mathbf{v}}(t) = -\alpha \omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{3}.$$

The system is azimuthally symmetric since $\dot{\mathbf{v}}$ always points along the $z$ axis. Thus, $\Theta = \theta$ where $\theta$ is the polar angle in spherical coordinates. Equation (5) becomes

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2} \left| -\alpha \omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{3} \right|^2 \sin^2 \theta = \frac{e^2 \alpha^2 \omega_0^4}{4\pi c^2} \cos^2 \omega_0 t \sin^2 \theta,$$

so the time-averaged power radiated per unit solid angle is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^2} \alpha^2 \omega_0^4 \left\langle \cos^2 \omega_0 t \right\rangle \sin^2 \theta = \frac{e^2 \alpha^2 \omega_0^4}{8\pi c^2} \sin^2 \theta. \tag{7}$$

A plot of the angular distribution of the radiation is shown in Fig. 3, in the $xz$ plane.

Equation (6) becomes

$$P = \frac{2}{3} \frac{e^2}{c^3} \left| -\alpha \omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{3} \right|^2 = \frac{2}{3} \frac{e^2 \alpha^2 \omega_0^4}{c^3} \cos^2 \omega_0 t,$$

so the time-averaged total power radiated is

$$\langle P \rangle = \frac{2}{3} \frac{e^2 \alpha^2 \omega_0^4}{c^3} \left\langle \cos^2 \omega_0 t \right\rangle = \frac{e^2 \alpha^2 \omega_0^4}{3c^3}.$$

**3(b)** The particle is moving in a circle of radius $R$ in the $xy$ plane with constant angular frequency $\omega_0$.

**Solution.** For a charge moving counter-clockwise,

$$\begin{aligned} \mathbf{x}(t) &= R \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{1} - R \sin \omega_0 t \, \hat{\mathbf{x}}_\mathbf{2}, \\ \mathbf{v}(t) &= -R\omega_0 \sin \omega_0 t \, \hat{\mathbf{x}}_\mathbf{1} - R\omega_0 \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{2}, \\ \dot{\mathbf{v}}(t) &= -R\omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_\mathbf{1} + R\omega_0^2 \sin \omega_0 t \, \hat{\mathbf{x}}_\mathbf{2}. \end{aligned}$$

This system is also azimuthally symmetric, so it is sufficient to restrict the position of the observer to the $yz$ plane. In polar coordinates, $\hat{\mathbf{n}} = \sin\theta\,\hat{\mathbf{x}}_\mathbf{2} + \cos\theta\,\hat{\mathbf{x}}_\mathbf{3}$. Then $\sin^2\Theta$ can be found by

$$\sin^2\Theta = 1 - \cos^2\Theta = 1 - \frac{(\dot{\mathbf{v}}\cdot\hat{\mathbf{n}})^2}{\dot{v}^2} = 1 - \sin^2\theta\sin^2\omega_0 t.$$

With these substitutions, Eq. (5) becomes

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{e^2}{4\pi c^2}\left|-R\omega_0^2\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} + R\omega_0^2\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2}\right|^2(1-\sin^2\theta\sin^2\omega_0 t)\\ &= \frac{e^2R^2\omega_0^4}{4\pi c^2}(\cos^2\omega_0 t + \sin^2\omega_0 t)(1-\sin^2\theta\sin^2\omega_0 t) = \frac{e^2R^2\omega_0^4}{4\pi c^2}(1-\sin^2\theta\sin^2\omega_0 t),\end{aligned}$$

giving us the time-averaged power radiated per unit solid angle:

$$\begin{aligned}\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{e^2R^2\omega_0^4}{4\pi c^2}(1-\sin^2\theta\left\langle\sin^2\omega_0 t\right\rangle) = \frac{e^2R^2\omega_0^4}{4\pi c^2}\left(1-\frac{\sin^2\theta}{2}\right) = \frac{e^2R^2\omega_0^4}{4\pi c^2}\left(1-\frac{1-\cos^2\theta}{2}\right)\\ &= \frac{e^2R^2\omega_0^4}{8\pi c^2}(1+\cos^2\theta).\end{aligned} \tag{8}$$

A plot of the angular distribution of the radiation is shown in Fig. 4, in the $xz$ plane.

From Eq. (6), we have

$$P = \frac{2}{3}\frac{e^2}{c^3}\left|-R\omega_0^2\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} + R\omega_0^2\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2}\right|^2 = \frac{2}{3}\frac{e^2R^2\omega_0^4}{c^3}(\cos^2\omega_0 t + \sin^2\omega_0 t) = \frac{2}{3}\frac{e^2R^2\omega_0^4}{c^3} = \langle P\rangle\,.$$

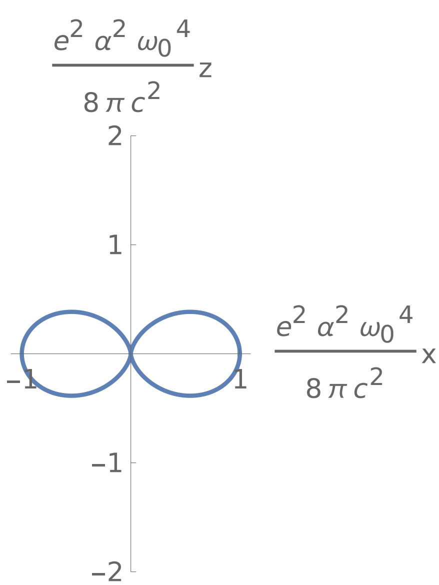


Figure 3: Plot of Eq. (7) in the $xz$ plane.

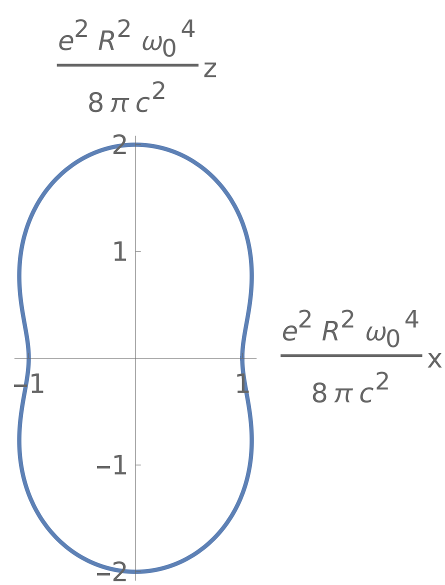


Figure 4: Plot of Eq. (8) in the $xz$ plane.