1 Elastically fastened ends

Consider an ideal stretched string in two dimensions with length ℓ, density per unit length ρ, and effective elastic modulus k. Suppose its two ends are fastened *elastically* by two springs with spring constant k_0 so that a nonzero deflection $u(x,t)$ of the end location from either $(0,0)$ or $\ell, 0)$ is penalized by a linear restrictive force $-ku$. Adapt the derivation in class for a stretched spring with two fixed ends to this situation. What are the Euler-Lagrange equations and the associated boundary conditions?

Solution. We will begin with the expression for the kinetic energy T of the string. Let $\mathrm{d}x$ denote an infinitesimal length of string. Its mass $\mathrm{d}m = \rho\,\mathrm{d}x$, so its kinetic energy $\mathrm{d}T$ is

$$\mathrm{d}T = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x \implies T = \frac{\rho}{2}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,, \tag{1}$$

where we have integrated over the length of the string to obtain T.

For the potential energy, let U_1 be the work required to stretch the string, and U_2 the work to compress and decompress the springs. (The addition of U_2 is what differs from the derivation in class.). For U_1, consider an infinitesimal length of string $\mathrm{d}x$. If this length is stretched by some amount Δu to a total length

$$\mathrm{d}x + \Delta u = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}u)^2}, \tag{2}$$

it has potential energy $\mathrm{d}U_1 = k\,\Delta u$. Performing a Taylor series expansion for a small Δu and integrating to obtain U_1,

$$\mathrm{d}U_1 = k\,\Delta u = k(\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}u)^2} - \mathrm{d}x) \approx \frac{k}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x \implies U_1 = \frac{k}{2}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,. \tag{3}$$

This approximation is sufficient because we consider only small oscillations. For U_2, the potential energy in the two springs is given by

$$U_2 = \frac{k_0}{2}u_0^2 + \frac{k_0}{2}u_\ell^2, \tag{4}$$

where $u_0 = u_0(t) = u(0,t)$ and $u_\ell = u_\ell(t) = u(\ell,t)$. The total potential energy $U = U_1 + U_2$.

Using (1), (3), and (4), we can write an expression for the action of the string:

$$\begin{aligned} S[u] = \int_{t_0}^{t_1} (T-U)\,\mathrm{d}t &= \int_{t_0}^{t_1}\left[\frac{\rho}{2}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x - \frac{k}{2}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x - \frac{k_0}{2}u_0^2 - \frac{k_0}{2}u_\ell^2\right]\mathrm{d}t & (5)\\ &= \frac{\rho}{2}\int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{k}{2}\int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{k_0}{2}\int_{t_0}^{t_1}\left(u_\ell^2 + u_0^2\right)\mathrm{d}t & (6)\\ &= \int_{t_0}^{t_1}\int_0^\ell \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t\,, & (7)\end{aligned}$$

where $\mathcal{L}$ is the Lagrangian density. Consider some variation of the action $\Delta S = S[u+\epsilon\psi] - S[u]$, where $\psi = \psi(x,t)$ is a function representing a variation and $\epsilon \ll 1$. The principle component of ΔS, δS, is given by

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x}\right)\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{t_0}^{t_1}\int_0^\ell \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\psi\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\psi\right)\right]\mathrm{d}x\,\mathrm{d}t\,, \tag{8}$$

where $u_t = \partial u/\partial t$ and $u_x = \partial u/\partial x$. Note that

$$\frac{\partial \mathcal{L}}{\partial u_t} = \rho\frac{\partial u}{\partial t}, \qquad \frac{\partial \mathcal{L}}{\partial u_x} = -k\frac{\partial u}{\partial x}, \tag{9}$$

so

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1} \int_0^{\ell} \left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} \right) \psi \, \mathrm{d}x \, \mathrm{d}t + \rho \int_{t_0}^{t_1} \int_0^{\ell} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\psi \right) \mathrm{d}x \, \mathrm{d}t - k \int_{t_0}^{t_1} \int_0^{\ell} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\psi \right) \mathrm{d}x \, \mathrm{d}t - k_0 \int_{t_0}^{t_1} (u_\ell \psi_\ell + u_0\psi_0) \, \mathrm{d}t \,, \tag{10}$$

where $\psi_0 = \psi_0(t) = \psi(0,t)$ and $\psi_\ell = \psi_\ell(t) = \psi(\ell, t)$. We stipulate that $\psi(x, t_0) = \psi(x, t_1) = 0$ for $x \in [0, \ell]$ and that $\psi(0,t) = \psi(\ell, t) = 0$ for $t \in [t_0, t_1]$. Then (10) becomes

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1} \int_0^{\ell} \left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} \right) \psi \, \mathrm{d}x \, \mathrm{d}t + k \int_{t_0}^{t_1} \left(\psi_0 \left.\frac{\partial u}{\partial x}\right|_{x=0} - \psi_\ell \left.\frac{\partial u}{\partial x}\right|_{x=\ell} \right) \mathrm{d}t - k_0 \int_{t_0}^{t_1} (u_\ell \psi_\ell + u_0\psi_0) \, \mathrm{d}t \tag{11}$$

$$= \int_{t_0}^{t_1} \int_0^{\ell} \left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} \right) \psi \, \mathrm{d}x \tag{12}$$

By the principle of least action, $\delta S = 0$ for the actual solution $u(x,t)$:

$$0 = \int_{t_0}^{t_1} \int_0^{\ell} \left(k\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} \right) \psi \, \mathrm{d}x \implies \frac{\partial^2 u}{\partial t^2} = \frac{k}{\rho}\frac{\partial^2 u}{\partial x^2} \tag{13}$$

for $x \in (0, \ell)$ and for $t \in (-\infty, \infty)$, since the time interval $[t_0, t_1]$ was arbitrary. (13) is the Euler-Lagrange equation for the system. (This is the same as we derived in class.)

In order to evaluate the boundary conditions, we remove the stipulation $\psi(0,t) = \psi(\ell, t) = 0$ for $t \in [t_0, t_1]$. Under the condition that is satisfied, (11) reduces to

$$\frac{\delta S}{\epsilon} = k \int_{t_0}^{t_1} \left(\psi_0 \left.\frac{\partial u}{\partial x}\right|_{x=0} - \psi_\ell \left.\frac{\partial u}{\partial x}\right|_{x=\ell} \right) \mathrm{d}t - k_0 \int_{t_0}^{t_1} (u_\ell \psi_\ell + u_0\psi_0) \, \mathrm{d}t \,, \tag{14}$$

and once again invoking the principle of least action,

$$\delta S = 0 \implies k \left(\psi_0 \left.\frac{\partial u}{\partial x}\right|_{x=0} - \psi_\ell \left.\frac{\partial u}{\partial x}\right|_{x=\ell} \right) = k_0(u_\ell \psi_\ell + u_0\psi_0) = 0. \tag{15}$$

Rearranging the result of (15), we find

$$0 = \frac{k_0}{k}u(0,t) - \left.\frac{\partial u}{\partial x}\right|_{x=0}, \qquad 0 = \frac{k_0}{k}u(\ell,t) + \left.\frac{\partial u}{\partial x}\right|_{x=\ell} \tag{16}$$

as the boundary conditions for (13).

2

Check that the boundary conditions you derived in problem 1 are reasonable by considering two limiting scenarios: $k_0/k \to 0$ and $k_0/k \to \infty$. Show that the boundary conditions simplify to the forms one would expect based on physical intuition.

Solution. For the case $k_0/k \to \infty$, $k/k_0 \to 0$ and the boundary conditions in reduce to

$$0 = u(0,t), \qquad 0 = u(\ell, t). \tag{17}$$

These are the boundary conditions for a string with fixed endpoints, as we discussed in class. In the limit $k_0/k \to \infty$, infinite force is required to deflect the ends of the string. Hence, the ends are effectively fixed. The vibrations of the string are standing waves in the (x, u) plane, with the ends as nodes.

For the case $k_0/k \to 0$, the boundary conditions in reduce to

$$0 = \left.\frac{\partial u}{\partial x}\right|_{x=0}, \qquad 0 = \left.\frac{\partial u}{\partial x}\right|_{x=\ell}. \tag{18}$$

In the limit $k_0/k \to 0$, the force that penalizes displacements at the ends of the string disappears. Physically, this means the ends of the string are allowed to slide up and down freely. In this situation, the vibrations in (x, u) are standing waves with the ends as antinodes. An antinode is a local maximum or minimum of a wave, so the slope of the wave at such a point must be zero, as represented in the boundary conditions.

3 Stretched membrane

Consider the transverse motion of a stretched membrane. Assume the membrane is homogeneous, with density per unit area ρ, and let $u(x, y)$ denote the displacement from the stable equilibrium position. For simplicity you may assume the boundary of the membrane is fixed. Adapt the derivation for a stretched string to a membrane and find the Euler-Lagrange equations.

Solution. Let the membrane cover a square in the (x, y) plane with corners at $(0, 0)$, $(0, \ell)$, $(\ell, 0)$, and (ℓ, ℓ). Assume the boundary around the edge of the square is fixed. Let k be the effective elastic modulus of the membrane.

Analogous to problem 1, let $\mathrm{d}x\,\mathrm{d}y$ denote an infinitesimal area of the membrane with mass. Then we can find the kinetic energy as in (1):

$$\mathrm{d}T = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y \implies T = \frac{\rho}{2}\int_0^\ell \int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y\,. \tag{19}$$

Because we are assuming a fixed boundary, we will only have one potential energy term, which is the work required to stretch the membrane. In two dimensions, (3) becomes

$$\mathrm{d}U_1 = k\,\Delta u = k(\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}u)^2} - \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} \approx \frac{k}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x + \frac{k}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}y\,. \tag{20}$$

Note that the cross terms in the expansion are all zero. Then

$$U_1 = \frac{k}{2}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + \frac{k}{2}\int \left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}y\,. \tag{21}$$

Using (19) and (21), we have as the action

$$S[u] = \frac{\rho}{2}\int_{t_0}^{t_1}\int_0^\ell\int_0^\ell \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \frac{k}{2}\int_{t_0}^{t_1}\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}t - \frac{k}{2}\int_0^\ell\int_0^\ell \left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}y\,\mathrm{d}t\,. \tag{22}$$

In general, the Euler-Lagrange equation is given by

$$0 = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y}. \tag{23}$$

Here,

$$\frac{\partial \mathcal{L}}{\partial u} = 0, \qquad \frac{\partial \mathcal{L}}{\partial u_t} = \rho \frac{\partial u}{\partial t}, \qquad \frac{\partial \mathcal{L}}{\partial u_x} = -k \frac{\partial u}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial u_y} = -k \frac{\partial u}{\partial y}, \tag{24}$$

so (23) becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{25}$$

In writing these solutions, I consulted Gelfand's *Calculus of Variations*.