Problem 1. Momentum operator Show that the state $\psi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ is an eigenstate of the momentum operator $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$ and find the eigenvalue.

Solution. A state $|a\rangle$ is an eigenstate of an operator A if $A|a\rangle = a|a\rangle$, where a is a number and eigenvalue of $|a\rangle$ [**?** , p. 12]. Note that

$$\hat{\mathbf{p}}\,\psi_k(\mathbf{r}) = -i\hbar\boldsymbol{\nabla}e^{i\mathbf{k}\cdot\mathbf{r}} = -i\hbar(i\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}} = \hbar\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}} = \hbar\mathbf{k}\,\psi_k(\mathbf{r}),$$

so we have shown that $\psi_k(\mathbf{r})$ is an eigenvector of $\hat{\mathbf{p}}$ with eigenvalue $\hbar\mathbf{k}$. □

Problem 2. Density of states for free electrons

2(a) What is the fermi wavevector and fermi energy as a function of particle density for a free electron gas in one and two dimensions (define density appropriately)?

Solution. In three dimensions, the total number of occupied states inside the fermi sphere is given by Eq. (2.8) of the lecture notes,

$$N = 2\frac{4\pi k_F^3/3}{(2\pi/L)^3}, \tag{1}$$

where k_F is the fermi wavevector, $L^3 = V$ is the volume, and 2 is the number of electron spin states. By inspection, then, the one- and two-dimensional equivalents to this equation are

$$N = 2\frac{k_F}{2\pi/L} \quad (d=1), \qquad\qquad N = 2\frac{\pi k_F^2}{(2\pi/L)^2} \quad (d=2),$$

where k_F is the length of a one-dimensional fermi line and πk_F^2 is the area of a fermi circle of radius k_F. The one-dimensional volume is L, and the two-dimensional volume is L^2. Solving for the wavevectors and writing them in terms of particle density $n = N/V$, we obtain

$$k_F = \pi n \quad (d=1), \qquad\qquad k_F = \sqrt{2\pi n} \quad (d=2).$$

The fermi momentum is $p_F = \hbar k_F$ and the fermi energy is $E_F = p_F^2/2m$ [**?** , p. 36]. Both definitions hold regardless of dimension. So the fermi energy in one and two dimensions is

$$E_F = \frac{\pi^2\hbar^2 n^2}{2m} \quad (d=1), \qquad\qquad E_F = \frac{\pi\hbar^2 n}{m} \quad (d=2).$$

2(b) Calculate the density of states in energy for free electrons in one and two dimensions.

Solution. According to Eq. (2.10) of the lecture notes, the density of states $g(E)$ can be found by

$$g(E)\,dE = 2\cdot\frac{\text{Volume of shell in } k \text{ space}}{\text{Volume of } k \text{ space per state}} = 2\frac{4\pi k^2\,dk}{(2\pi)^3/V},$$

where the final equality is for the three-dimensional case. For one and two dimensions, the equivalent expressions are

$$g(E)\,dE = 2\frac{dk}{2\pi/L} \quad (d=1), \qquad\qquad g(E)\,dE = 2\frac{2\pi k\,dk}{(2\pi)^2/L^2} \quad (d=2).$$

Noting that

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad \Longrightarrow \quad \frac{dk}{dE} = \sqrt{\frac{m}{2\hbar^2 E}},$$

which again is true regardless of dimension. So we find

$$(d=1) \quad g(E) = \frac{L}{\pi}\frac{dk}{dE} = \frac{L}{\pi}\sqrt{\frac{m}{2\hbar^2 E}},$$
$$(d=2) \quad g(E) = \frac{L^2 k}{\pi}\frac{dk}{dE} = \frac{L^2}{\pi}\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{m}{2\hbar^2 E}} = \frac{L^2 m}{\pi\hbar^2}.$$

Per unit volume, we have

$$g(E) = \frac{1}{\pi}\sqrt{\frac{m}{2\hbar^2 E}} \quad (d=1), \qquad\qquad g(E) = \frac{L^2 m}{\pi\hbar^2} \quad (d=1).$$

2(c) Show how the 3D density of states can be rewritten as

$$\frac{3}{2}\frac{n}{E_F}\sqrt{\frac{E}{E_F}} \tag{2}$$

with $n = N/V$.

Solution. The 3D density of states per unit volume is given by Eq. (2.11) in the lecture notes,

$$g(E) = \frac{V}{\pi^2}\frac{m}{\hbar^2}\sqrt{\frac{2mE}{\hbar^2}}.$$

We will work backward to reach this form from Eq. (2).

Equation (1) can be written as follows:

$$N = 2\frac{4\pi/3}{(2\pi)^3/V}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2} \quad \Longrightarrow \quad n = \frac{(2mE_F)^{3/2}}{3\pi^2\hbar^3} \quad \Longrightarrow \quad E_F^3 = \frac{(3\pi^2\hbar^3 n)^2}{(2m)^3},$$

where we have used $k = \sqrt{2mE/\hbar^2}$. Feeding the last expression into Eq. (2), we obtain

$$g(E) = \frac{3}{2}n\sqrt{\frac{E}{E_F^3}} = \frac{3}{2}n\sqrt{E\frac{(2m)^3}{(3\pi^2\hbar^3 n)^2}} = \frac{1}{\pi^2}\frac{m}{\hbar}\sqrt{\frac{2mE}{\hbar^2}}$$

as desired. □