Problem 1. Consider the following probabalistic game: There are four doors (Q, R, S, T). Behind each door is a device which displays ± 1 randomly according to the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$. Alice and Bob are on the same team. Alice has to choose either Q and R, and then Bob has to choose either S and T. When the numbers match, they get $+1$ point; when the numbers do not match, they get -1 point. However, when they open Q and T, it's an exception. When the numbers (do not) match, they get -1 $(+1)$.

1.1 Let's assume Alice and Bob open the doors completely randomly. When all numbers are $+1$ with probability 1, what is the expectation value of the point they get?

Solution. Let $\mathbf{E}$ be the expectation value of the number of points. In this case, the numbers behind the two doors will always match. So

$$\mathbf{E} = \frac{QS + RS + RT - QT}{4} = \frac{1 + 1 + 1 - 1}{4} = \frac{1}{2}.$$

1.2 As it turns out, irrespective of how hard you fine tune the probability $P(Q = \pm 1, R = \pm 1, S = \pm 1, T = \pm 1)$, the expectation value of the point Alice and Bob get cannot exceed a certain value Max:

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} \leq \text{Max}.$$

Here, $\mathbf{E}(QS)$, etc. is the expectation value of the point when Alice opens Q and Bob opens S. This is a Bell inequality. Determine Max.

Hint: For a given realization of the numbers $Q = \pm 1$, $R = \pm 1$, $S = \pm 1$, $T = \pm 1$, which occurs with probability $P(Q, R, S, T)$, note that $QS + RS + RT - QT = (Q + R)S + (R - Q)T$, where one of $\{(R + Q), (R - Q)\}$ is 2 and the other 0.

Solution. In addition to the information provided in the hint, both S and T must be ± 1. This means the only possibilities for the number of points earned are

$$\frac{(Q + R)S + (R - Q)T}{4} = \begin{cases} \dfrac{(0)(-1) + (2)(1)}{4} = \dfrac{1}{2}, \\ \dfrac{(0)(1) + (2)(-1)}{4} = -\dfrac{1}{2}. \end{cases}$$

Thus,

$$\text{Max} = \frac{1}{2}.$$

1.3 Frustrated by the upper bound set by the Bell inequality, Bob decides to cheat. He now changes the value of T after Alice chooses Q or R. Assume Q, R, S are set to be $+1$ with probability 1. To make the expectation value of the point they get equal to $+1$, what values should Bob set after Alice chooses Q or R?

Solution. If Alice chooses R, Bob should set $T = 1$. If Alice chooses Q, Bob should set $T = -1$. This way,

$$\frac{\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)}{4} = \frac{1 + 1 + 1 + 1}{4} = 1.$$

1.4 Now consider a quantum mechanical version of the game. There are quantum states of two spin-1/2 degrees of freedom shared by Alice and Bob. Alice can measure the z component or x components of the first spin $\mathbf{S}^A$. (This corresponds to $Q = \pm 1$ or $R = \pm 1$.) Bob can measure the $-(z+x)$ component or the $(z-x)$ component of the second spin $\mathbf{S}^B$. (This corresponds to $S = \pm 1$ or $T = \pm 1$.)

More specifically, Alice and Bob share the quantum state

$$|\psi\rangle = \frac{|\uparrow_z\rangle \otimes |\downarrow_z\rangle - |\downarrow_z\rangle \otimes |\uparrow_z\rangle}{\sqrt{2}}.$$

The operators to be measured are

$$Q = S_z^A, \qquad R = S_x^A, \qquad S = -\frac{S_z^B + S_x^B}{\sqrt{2}}, \qquad T = \frac{S_z^B - S_x^B}{\sqrt{2}}.$$

Let us consider the case when Alice measures Q and Bob measures T. Calculate the probability $P(Q,T)$ for Alice and Bob getting the measurement outcomes $(Q,T) = (\pm 1, \pm 1)$.

Solution. From Sakurai (3.9.11), the probability of measuring $\mathbf{S}\cdot\hat{\mathbf{a}}$ and $\mathbf{S}\cdot\hat{\mathbf{b}}$ to both be positive is

$$P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+) = \frac{1}{2}\sin^2\left(\frac{\theta_{ab}}{2}\right),$$

where the 1/3 comes from the probability of measuring θ_{ab} is the angle between the $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ directions. For the other combinations, we may generalize this expression using Fig. (3.9) in Sakurai: This gives us

$$P(\hat{\mathbf{a}}-;\hat{\mathbf{b}}-) = \frac{1}{2}\sin^2\left(\frac{\theta_{ab}}{2}\right) = P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}+), \quad P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}-) = \frac{1}{2}\sin^2\left(\frac{\theta_{ab}+\pi/2}{2}\right) = \frac{1}{2}\cos^2\left(\frac{\theta_{ab}}{2}\right) = P(\hat{\mathbf{a}}+;\hat{\mathbf{b}}-). \tag{1}$$

For Q and T, $\theta_{ab} = \pi/4$. So we have

$$P(Q=\pm 1, T=\pm 1) = \frac{1}{2}\sin^2\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)^2 = \frac{1}{2}\frac{2-\sqrt{2}}{4} = \frac{2-\sqrt{2}}{8} \approx 0.073,$$
$$P(Q=\pm 1, T=\mp 1) = \frac{1}{2}\cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^2 = \frac{1}{2}\frac{2+\sqrt{2}}{4} = \frac{2+\sqrt{2}}{8} \approx 0.427.$$

1.5 Similarly, consider the case when Alice measures R and Bob measures T. Calculate the probability $P(R,T)$ for Alice and Bob getting the measurement outcomes $(R,T) = (\pm 1, \pm 1)$.

Solution. Again applying (1), for R and T, $\theta_{ab} = 3\pi/4$. So we have

$$P(R=\pm 1, T=\pm 1) = \frac{1}{2}\sin^2\left(\frac{3\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^2 = \frac{1}{2}\frac{2+\sqrt{2}}{4} = \frac{2+\sqrt{2}}{8} \approx 0.427,$$
$$P(R=\pm 1, T=\mp 1) = \frac{1}{2}\cos^2\left(\frac{3\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)^2 = \frac{1}{2}\frac{2-\sqrt{2}}{4} = \frac{2-\sqrt{2}}{8} \approx 0.073.$$

1.6 Compute the expectation values $\mathbf{E}(QS)$, $\mathbf{E}(RS)$, $\mathbf{E}(QT)$, and $\mathbf{E}(RT)$. Compute

$$\frac{\mathbf{E}(QS)+\mathbf{E}(RS)+\mathbf{E}(RT)-\mathbf{E}(QT)}{4}.$$

Solution. We need to find the probabilities of obtaining $(Q,S)=(\pm1,\pm1)$ and $(R,S)=(\pm1,\pm1)$. For Q and S, $\theta_{ab}=3\pi/4$, so

$$P(Q=\pm1,S=\pm1)=P(R=\pm1,T=\pm1), \qquad P(Q=\pm1,S=\mp1)=P(R=\pm1,T=\mp1).$$

For R and S, $\theta_{ab}=5\pi/4$, so

$$P(R=\pm1,S=\pm1)=\frac{1}{2}\sin^2\left(\frac{5\pi}{8}\right)=\frac{1}{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)^2=\frac{1}{2}\frac{2+\sqrt{2}}{4}=\frac{2+\sqrt{2}}{8}\approx 0.427,$$
$$P(R=\pm1,S=\mp1)=\frac{1}{2}\cos^2\left(\frac{5\pi}{8}\right)=\frac{1}{2}\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)^2=\frac{1}{2}\frac{2-\sqrt{2}}{4}=\frac{2-\sqrt{2}}{8}\approx 0.073.$$

The expectation value of a random variable X is defined

$$E(X)=\sum_i p_i x_i,$$

where x_i are all of the possible values of X, and p_i the probabilities associated with each. Then

$$\mathbf{E}(QS)=2P(Q=\pm1,S=\pm1)-2P(Q=\pm1,S=\mp1)=\frac{2+\sqrt{2}}{4}-\frac{2-\sqrt{2}}{4}=\frac{\sqrt{2}}{2},$$
$$\mathbf{E}(RS)=2P(R=\pm1,S=\pm1)-2P(R=\pm1,S=\mp1)=\frac{2+\sqrt{2}}{4}-\frac{2-\sqrt{2}}{4}=\frac{\sqrt{2}}{2},$$
$$\mathbf{E}(RT)=2P(R=\pm1,T=\pm1)-2P(R=\pm1,T=\mp1)=\frac{2+\sqrt{2}}{4}-\frac{2-\sqrt{2}}{4}=\frac{\sqrt{2}}{2},$$
$$\mathbf{E}(QT)=2P(Q=\pm1,T=\pm1)-2P(Q=\pm1,T=\mp1)=\frac{2-\sqrt{2}}{4}-\frac{2+\sqrt{2}}{4}=-\frac{\sqrt{2}}{2}.$$

Finally,

$$\frac{\mathbf{E}(QS)+\mathbf{E}(RS)+\mathbf{E}(RT)-\mathbf{E}(QT)}{4}=\frac{1}{4}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2},$$

which is greater than Max, thereby violating Bell's inequality.

Problem 2. Consider a quantum particle with mass m moving in the presence of the square well potential

$$V(|r|) = \begin{cases} -V_0 & |r| \leq a, \\ 0 & |r| > a. \end{cases}$$

2.1 Writing the wave function in polar coordinates as $\psi(\mathbf{r}) = R_l(r)\, Y_{lm}(\theta, \phi)$, write down the Schrödinger equation obeyed by R_l.

Solution. In three dimensions, the Hamiltonian for the square well is

$$H = \begin{cases} \dfrac{\mathbf{p}^2}{2m} - V_0 & |r| \leq a, \\ \dfrac{\mathbf{p}^2}{2m} & |r| > a. \end{cases}$$

In the position basis in spherical coordinates, (3.6.21) in Sakurai tells us that the kinetic energy can be written

$$\frac{1}{2m} \langle \mathbf{x}' | \mathbf{p}^2 | \alpha \rangle = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} \langle \mathbf{x}' | \alpha \rangle + \frac{2}{r} \frac{\partial}{\partial r} \langle \mathbf{x}' | \alpha \rangle - \frac{1}{\hbar^2 r^2} \langle \mathbf{x}' | \mathbf{L}^2 | \alpha \rangle \right).$$

From (3.6.27) in Sakurai,

$$\mathbf{L}^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle ,$$

so the angular part $Y_{lm}(\theta, \phi)$ obeys

$$\mathbf{L}^2 Y_{lm}(\theta, \phi) = l(l+1)\hbar^2 Y_{lm}(\theta, \phi). \tag{2}$$

The time-independent Schrödinger equation is

$$H |\psi\rangle = E |\psi\rangle ,$$

where E is the energy associated with $|\psi\rangle$. Substituting in H and ψ in the range $|r| \leq a$, this becomes

$$\frac{\hbar^2}{2m} \left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\mathbf{L}^2}{\hbar^2 r^2} - V_0 \right] R_l(r)\, Y_{lm}(\theta, \phi) = E_l\, R_l(r)\, Y_{lm}(\theta, \phi).$$

The angular part acts only on Y_{lm}, so substituting in (2) gives us

$$\frac{\hbar^2}{2m} \left[-\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{l(l+1)}{r^2} - V_0 \right] R_l(r) = E_l\, R_l(r).$$

From (7.7.1), the effective potential at low energies for the lth partial wave is

$$V_{\text{eff}} = V(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2},$$

so the Schrödinger equation can be rewritten as

$$\left[-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + V_{\text{eff}} \right] R_l(r) = E_l\, R_l(r),$$

where

$$V_{\text{eff}} = \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} - V_0.$$

2.2 When V_0 is a certain value, there is one bound state for the s wave ($l = 0$). The bound state energy ε is small ($0 < |\varepsilon| \ll V_0$). Obtain the range of the depth of the well V_0 ($? \leq V_0 < ?$). Also, calculate for the bound state the probability for the particle to exist outside of the well.

Solution. The bound state must have energy $E_0 < -V_0$...

(A.1.2)

$$\mathbf{k} = \frac{\mathbf{p}}{\hbar}, \qquad \omega = \frac{E}{\hbar} = \frac{\mathbf{p}^2}{2m\hbar} = \frac{\hbar\mathbf{k}^2}{2m}$$

2.3 Consider the scattering problem by the well. For each l, for large enough r, when $R_l(r)$ is given by $R_l(r) \sim A_l \sin(kr - l\pi/2 + \delta_l)/r$, δ_l is called the scattering phase shift. For the value of V_0 within the range you obtained in the above problem, when the energy of the incident wave is is $E = 9V_0/16$, calculate $\tan\delta_0$ (where δ_0 is the scattering phase shift for the s wave).

2.4 Now consider the S matrix, $S \equiv \exp(2i\delta_0) = \exp(i\delta_0)/\exp(-i\delta_0)$. Compare the condition on s wave bound state energies and the zero of the denominator of S. Explain their relation.

Problem 3. Consider a three dimensional potential

$$V(|r|) = \frac{\hbar^2\gamma}{2m}\delta(|r| - a).$$

The s wave Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{d^2\chi_0(r)}{dr^2} + \frac{\hbar^2\gamma}{2m}\delta(r-a)\,\chi_0(r) = E\,\chi_0(r).$$

The s wave function must be regular (zero) at $r = 0$. At $r = a$, it is continuous, but its derivative can jump.

3.1 Calculate the s wave scattering phase shift (k), where k is related to E as $E = \hbar^2k^2/2m$.

3.2 When $\gamma \gg k$, $1/a$ and when $\sin ka$ is not small, discuss the behavior of the scattering phase shift.

3.3 Obtain the condition to have resonant states and calculate the energy of the resonant states.

3.4 Calculate the width Γ of the resonance. Discuss its behavior when γ is big.

3.5 When the velocity of the incident wave is small, obtain the scattering cross section.

I consulted Sakurai's *Modern Quantum Mechanics*, Shankar's *Principles of Quantum Mechanics*, and the Wikipedia article on a particle in a spherically symmetric potential while writing up these solutions.