Problem 1.

1(a) Show by explicit computation the Lorentz invariance of the Dirac Lagrangian, by considering a Lorentz transformation of the fields.

Solution. The Dirac Lagrangian is given by Eq. (3.34) in Peskin & Schroder:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

According to their Eq. (3.33), $\bar{\psi}$ transforms as $\bar{\psi} \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}$; also, $\psi \to \Lambda_{\frac{1}{2}}\psi$. The Lorentz transformation of the Dirac Lagrangian is then [1, p. 42]

$$\begin{aligned}
\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) &\to \bar{\psi}(\Lambda^{-1}x)\Lambda_{\frac{1}{2}}^{-1}[i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m)]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \bar{\psi}(\Lambda^{-1}x)[i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \bar{\psi}(\Lambda^{-1}x)[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m]\psi(\Lambda^{-1}x),
\end{aligned}$$

where we have used Peskin & Schroeder (3.29), $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. Then

$$\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \to \bar{\psi}(\Lambda^{-1}x)[i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m)]\psi(\Lambda^{-1}x) = \bar{\psi}(\Lambda^{-1}x)(i\gamma^\nu\partial_\nu - m)\psi(\Lambda^{-1}x),$$

which has the same form as $\mathcal{L}_{\text{Dirac}}$. So we have shown that the Dirac Lagrangian is Lorentz invariant. □

1(b) Consider the chiral rotation of a massless Dirac field $\psi' = e^{i\alpha\gamma^5}\psi$. Find the corresponding Noether current. Show that the corresponding Noether charge measures the total helicity of a collection of massless Dirac particles, and that the addition of a mass term to the Lagrangian violates the symmetry. Find an equation that expresses the violation of current conservation by the mass.

Solution. The conserved charge is given in general by Peskin & Schroeder (2.12) and (2.13),

$$Q \equiv \int_{\text{all space}} j^0\, d^3x\,, \qquad \text{where } j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu, \tag{1}$$

where J^μ is a 4-divergence that arises when transforming the Lagrangian as in Peskin & Schroeder (2.10):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu J^\mu(x). \tag{2}$$

Under the rotation $\psi \to e^{i\alpha\gamma^5}\psi$, $\psi^\dagger \to \psi^\dagger e^{-i\alpha\gamma^5}$. Then, using $\bar{\psi} = \psi^\dagger\gamma^0$ as defined in Peskin & Schroeder (3.32),

$$\bar{\psi} \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 = -\psi^\dagger\gamma^0 e^{-i\alpha\gamma^5} = -\bar{\psi}e^{-i\alpha\gamma^5},$$

since $\{\gamma^\mu, \gamma^5\} = 0$ from Peskin & Schroeder (3.70). Then, using $m = 0$ in the Dirac Lagrangian, we have

$$\mathcal{L}_{\text{Dirac}} = i\bar{\psi}\gamma^\mu\partial_\mu\psi \to -i\bar{\psi}e^{-i\alpha\gamma^5}\gamma^\mu\partial_\mu e^{i\alpha\gamma^5}\psi = i\bar{\psi}\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu e^{i\alpha\gamma^5}\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi,$$

so the Dirac Lagrangian is indeed invariant under chiral transformations, and $J^\mu = 0$.

The infinitesimal transformations associated with $\psi \to e^{i\alpha\gamma^5}\psi$ are

$$\alpha\Delta\psi = i\alpha\gamma^5\psi, \alpha\Delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5.$$

Then we have the Noether current [1, p. 50]

$$j^\mu = -\left[\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\bar{\psi})}\Delta\bar{\psi}\right] = \bar{\psi}\gamma^\mu\gamma^5\psi,$$

where we have multiplied by an arbitrary constant [1, p. 18].

Peskin & Schroeder (3.76) defines

$$j_L^\mu = \bar{\psi}\gamma^\mu\frac{1-\gamma^5}{2}\psi, \qquad j_R^\mu = \bar{\psi}\gamma^\mu\frac{1+\gamma^5}{2}\psi,$$

as the electric current densities of left- and right-handed particles. Note that $j^\mu = j_R^\mu - j_L^\mu$. Then we have the conserved charge

$$Q = \int d^3x\,\bar{\psi}\gamma^0\gamma^5\psi = \int d^3x\,(j_R^0 - j_L^0),$$

which tells us the total helicity of a collection of massless Dirac particles. □

If $m \neq 0$ in the Dirac Lagrangian, then it transforms as

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \to -\bar{\psi}e^{-i\alpha\gamma^5}(i\gamma^\mu\partial_\mu - m)e^{i\alpha\gamma^5}\psi &= \bar{\psi}(\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu + e^{-i\alpha\gamma^5}m)e^{i\alpha\gamma^5}\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu + m)\psi,\end{aligned}$$

which is not of the same form. So the symmetry is violated for nonzero m. □

In order for the current to be conserved, we need the divergence $\partial_\mu j^\mu = 0$. Note that

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi.$$

Since ψ satisfies the Dirac equation, we can make use of the Dirac equation and its conjugate, given by Eqs. (3.31) and (3.35) in Peskin & Schroeder:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \qquad -i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0.$$

So the divergence can be written [1, p. 51]

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi = im\bar{\psi}\gamma^5\psi + \bar{\psi}\gamma^5 im\psi = 2im\bar{\psi}\gamma^5\psi,$$

which is zero only if m is zero.

1(c) Find the Noether current related to charge conservation by considering a phase rotation of a Dirac field (of arbitrary mass) $\psi' = e^{i\alpha}\psi$.

Solution. We will once again use Eqs. (1) and (2). Under the rotation $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\alpha}$. Then the Dirac Lagrangian transforms as

$$\mathcal{L}_{\text{Dirac}} \to \bar{\psi}e^{i\alpha}(i\gamma^\mu\partial_\mu - m)e^{-i\alpha}\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

so again $J^\mu = 0$.

The infinitesimal translations are

$$\alpha\Delta\psi = i\alpha\psi, \alpha\Delta\bar{\psi} \qquad = -i\alpha\bar{\psi},$$

and the Noether current is [1, p. 50]

$$j^\mu = -\left[\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\bar{\psi})}\Delta\bar{\psi}\right] = \bar{\psi}\gamma^\mu\psi.$$

Problem 2. (Peskin & Schroeder 3.1) Lorentz group Recall from Eq. (3.17) the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

2(a) Define the generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \qquad K^i = J^{0i},$$

where $i, j, k = 1, 2, 3$. An infinitesimal Lorentz transformation can then be written

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K})\Phi.$$

Write the commutation relations of these vector operators explicitly. (For example, $[L^i, L^j] = i\epsilon^{ijk}L^k$.) Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}), \qquad \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

commute with one another and separately satisfy the commutation relations of angular momentum.

Solution. Firstly, using the Eq. (3.18)

$$\begin{aligned}
[L^i, L^j] &= \left[\frac{1}{2}\epsilon^{i\mu\nu}J^{\mu\nu}, \frac{1}{2}\epsilon^{j\rho\sigma}J^{\rho\sigma}\right] = \frac{1}{4}\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}[J^{\mu\nu}, J^{\rho\sigma}] = \frac{i}{4}\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\mu\rho}J^{\nu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\sigma}J^{\mu\rho} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\mu\sigma}J^{\nu\rho}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\nu\mu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} - \epsilon^{i\mu\nu}\epsilon^{j\sigma\rho}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\nu\mu}\epsilon^{j\sigma\rho}g^{\nu\rho}J^{\mu\sigma}) \\
&= \frac{i}{4}(\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma} + \epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma}) \\
&= i\epsilon^{i\mu\nu}\epsilon^{j\rho\sigma}g^{\nu\rho}J^{\mu\sigma},
\end{aligned}$$

where we have simply relabeled indices. Then, using $g_\alpha{}^\beta = \delta_\alpha^\beta$ [2] and $\epsilon^{ijk}\epsilon^{pqk} = \delta^{ip}\delta^{jq} - \delta^{iq}\delta^{jp}$ [3],

$$\begin{aligned}
[L^i, L^j] &= i\epsilon^{i\mu}{}_\nu\epsilon^{j\rho\sigma}\delta^\rho_\nu J^{\mu\sigma} = i\epsilon^{i\mu}{}_\nu\epsilon^{j\nu\sigma}J^{\mu\sigma} = i\epsilon^{i\mu\nu}\epsilon^{j\sigma\nu}J^{\mu\sigma} = i(\delta^{ij}\delta^{\mu\sigma} - \delta^{i\sigma}\delta^{\mu j})J^{\mu\sigma} = i(\delta^{ij}J^{\mu\mu} - \delta^{i\sigma}J^{j\sigma}) = -iJ^{ji} \\
&= iJ^{ij},
\end{aligned}$$

where we have used the antisymmetry of J^{ij}. From $L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$, we can write

$$\epsilon^{i\rho\sigma}L^i = \frac{1}{2}\epsilon^{ijk}\epsilon^{i\rho\sigma}J^{jk} = \frac{1}{2}(\delta^{j\rho}\delta^{k\sigma} - \delta^{j\sigma}\delta^{k\rho})J^{jk} = \frac{1}{2}(\delta^{j\rho}J^{j\sigma} - \delta^{j\sigma}J^{j\rho}) = \frac{1}{2}(J^{\rho\sigma} - J^{\sigma\rho}) = J^{\rho\sigma}.$$

Then we see that

$$[L^i, L^j] = i\epsilon^{ijk}L^k.$$

Secondly,

$$[K^i, K^j] = [J^{0i}, J^{0j}] = i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) = -iJ^{ij} = -i\epsilon^{ijk}L^k. =,$$

and thirdly,

$$\begin{aligned}
[K^i, L^j] &= \left[J^{0i}, \frac{1}{2}\epsilon^{j\rho\sigma}J^{\rho\sigma}\right] = \frac{1}{2}\epsilon^{j\rho\sigma}[J^{0i}, J^{\rho\sigma}] = \frac{i}{2}\epsilon^{j\rho\sigma}(g^{i\rho}J^{0\sigma} - g^{0\rho}J^{i\sigma} - g^{i\sigma}J^{0\rho} + g^{0\sigma}J^{i\rho}) \\
&= \frac{i}{2}(\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} - \epsilon^{j\rho\sigma}g^{i\sigma}J^{0\rho}) = \frac{i}{2}(\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} - \epsilon^{j\sigma\rho}g^{i\rho}J^{0\sigma}) = i\epsilon^{j\rho\sigma}g^{i\rho}J^{0\sigma} = i\epsilon^{j}{}_\rho{}^\sigma\delta^i_\rho J^{0\sigma} = -i\epsilon^{ij\sigma}J^{0\sigma} \\
&= -i\epsilon^{ijk}K^k.
\end{aligned}$$

References

[1] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”. Perseus Books Publishing, 1995.

[2] E. W. Weisstein, “Metric Tensor.” From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/MetricTensor.html`.

[3] E. W. Weisstein, “Permutation Symbol.” From MathWorld—A Wolfram Web Resource. `https://mathworld.wolfram.com/PermutationSymbol.html`.

[4] L. D. Landau and E. M. Lifshitz, “The Classical Theory of Fields”, volume 2. Pergamon Press, 3rd edition, 1971.

[5] J. D. Jackson, “Classical Electrodynamics”. Wiley, 3rd edition, 1999.

[6] J. J. Sakurai, “Modern Quantum Mechanics”. Addison-Wesley Publishing Company, revised edition, 1994.

[7] W. E. Olmstead and V. A. Volpert, “Differential Equations in Applied Mathematics”. 2014.