Problem 1. The CP^N model (P&S 13.3) The nonlinear sigma model discussed in the text can be thought of as a quantum theory of fields that are coordinates on the unit sphere. A slightly more complicated space of high symmetry is complex projective space, CP^N. This space can be defined as the space of $(N+1)$-dimensional complex vectors $(z_1, \ldots, z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1,$$

with points related by an overall phase rotation identified, that is,

$$(e^{i\alpha} z_1, \ldots, e^{i\alpha} z_{N+1},) \text{ identified with } (z_1, \ldots, z_{N+1}).$$

In this problem, we study that two-dimensional quantum field theory whose fields are coordinates on this space.

1(a) One way to represent a theory of coordinates on CP^N is to write a Lagrangian depending on fields $z_j(x)$, subject to the constraint, which also has the total symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x), \tag{1}$$

independently at each point x. Show that the following Lagrangian has this symmetry:

$$\mathcal{L} = \frac{1}{g^2}\left[|\partial_\mu z_j|^2 - |z_j^* \partial_\mu z_j|^{\,2} \right]. \tag{2}$$

To prove the invariance, you will need to use the constraint on the z_j, and its consequence

$$z_j^* \partial_\mu z_j = -(\partial_\mu z_j^*) z_j. \tag{3}$$

Show that the nonlinear sigma model for the case $N = 3$ can be converted to the CP^N model for the case $N = 1$ by the substitution

$$n^i = z^* \sigma^i z, \tag{4}$$

where σ^i are the Pauli sigma matrices.

Solution. The original Lagrangian can be written

$$\mathcal{L} = \frac{1}{g^2}\left[(\partial_\mu z_j)(\partial_\mu z_j^*) - z_j^* z_k (\partial_\mu z_j)(\partial_\mu z_k^*) \right].$$

For the transformation,

$$\begin{aligned}
\mathcal{L} &\to \frac{1}{g^2}\left[\left|\partial_\mu (e^{i\alpha} z_j)\right|^2 - \left|(e^{i\alpha} z_j)^* \partial_\mu (e^{i\alpha} z_j)\right|^2 \right] \\
&= \frac{1}{g^2}\left[\left|z_j \partial_\mu e^{i\alpha} + e^{i\alpha}\partial_\mu z_j\right|^2 - \left|e^{-i\alpha} z_j^* (z_j \partial_\mu e^{i\alpha} + e^{i\alpha}\partial_\mu z_j)\right|^2 \right] \\
&= \frac{1}{g^2}\left[\left|z_j \partial_\mu e^{i\alpha} + e^{i\alpha}\partial_\mu z_j\right|^2 - z_j^* z_k (z_j \partial_\mu e^{i\alpha} + e^{i\alpha}\partial_\mu z_j)(z_k^* \partial_\mu e^{-i\alpha} + e^{-i\alpha}\partial_\mu z_k^*) \right] \\
&= \frac{1}{g^2}\left[\left|z_j \partial_\mu e^{i\alpha} + e^{i\alpha}\partial_\mu z_j\right|^2 - (|z_j|^2 \partial_\mu e^{i\alpha} + z_j^* e^{i\alpha}\partial_\mu z_j)(|z_k|^2 \partial_\mu e^{-i\alpha} - z_k e^{-i\alpha}\partial_\mu z_k^*) \right] \\
&= \frac{1}{g^2}\left[(\partial_\mu e^{i\alpha})(\partial_\mu e^{-i\alpha}) + e^{-i\alpha} z_j (\partial_\mu e^{i\alpha})(\partial_\mu z_j^*) + e^{i\alpha} z_j^* (\partial_\mu e^{-i\alpha})(\partial_\mu z_j) + (\partial_\mu z_j)(\partial_\mu z_j^*) \right. \\
&\qquad \left. - (\partial_\mu e^{i\alpha})(\partial_\mu e^{-i\alpha}) - e^{-i\alpha} z_k (\partial_\mu e^{i\alpha})(\partial_\mu z_k^*) - e^{i\alpha} z_j^* (\partial_\mu e^{-i\alpha})(\partial_\mu z_j) - z_j^* z_k (\partial_\mu z_j)(\partial_\mu z_k^*) \right] \\
&= \frac{1}{g^2}\left[(\partial_\mu z_j)(\partial_\mu z_j^*) - z_j^* z_k (\partial_\mu z_j)(\partial_\mu z_k^*) \right],
\end{aligned}$$

where we have used $|z_j|^2 = 1$. So the Lagrangian has the symmetry Eq. (1).

The Lagrangian for the nonlinear sigma model is given by P&S (13.67),

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu \vec{n}|^2, \tag{5}$$

where $\vec{n}(x)$ is an N-component vector field constrained to satisfy P&S (13.66),

$$\sum_{i=1}^{N} \left|n^i(x)\right|^2 = 1.$$

Making the substitution Eq. (4) in Eq. (5),

$$\begin{aligned}
\mathcal{L} &\to \frac{1}{2g^2}\left|\partial_\mu (z^* \sigma^i z)\right|^2 \\
&= \frac{1}{2g^2}\left|(\partial_\mu z^*)\sigma^i z + z^* \sigma^i (\partial_\mu z)\right|^2 \\
&= \frac{1}{2g^2}\left[(\partial_\mu z^*)\sigma^i z + z^* \sigma^i (\partial_\mu z)\right]\left[(\partial_\mu z^*)\sigma^i z + z^* \sigma^i (\partial_\mu z)\right] \\
&= \frac{1}{2g^2}\left[(\partial_\mu z^*)\sigma^i z(\partial_\mu z^*)\sigma^i z + (\partial_\mu z^*)\sigma^i z z^* \sigma^i (\partial_\mu z) + z^* \sigma^i (\partial_\mu z)(\partial_\mu z^*)\sigma^i z + z^* \sigma^i (\partial_\mu z) z^* \sigma^i (\partial_\mu z)\right]
\end{aligned}$$

where we have used ${\sigma^\dagger}^i = \sigma^i$.

1(b) To write the Lagrangian in a simpler form, introduce a scalar Lagrange multiplier λ which implements the constraint and also a vector Lagrange multiplier A_μ to express the local symmetry. More specifically, show that the Lagrangian of the CP^N model is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\left(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)\right), \tag{6}$$

where $D_\mu = (\partial_\mu + iA_\mu)$, by functionally integrating over the fields λ and A_μ.

Solution. The path integral for Eq. (6) is

$$\int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}^2 z\,\exp\left[\frac{i}{g^2}\int d^2x\left(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)\right)\right].$$

Since λ is a Lagrange multiplier, we can compute this integral easily:

$$\int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}^2 z\,\exp\left[\frac{i}{g^2}\int d^2x\left(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)\right)\right] = \int \mathcal{D}A\,\mathcal{D}^2 z\,\delta^{(2)}(|z_j|^2 - 1)\exp\left[\frac{i}{g^2}\int d^2x\,|D_\mu z_j|^2\right],$$

where we have applied [1]

$$\int \frac{d^dk}{(2\pi)^d} e^{ik(x-y)} = \delta^{(d)}(x-y)$$

and ignored the overall constant. Now we have

$$\begin{aligned}
&\int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}^2 z\,\exp\left[\frac{i}{g^2}\int d^2x\left(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)\right)\right] \\
&\qquad = \int \mathcal{D}A\,\mathcal{D}^2 z\,\delta^{(2)}(|z_j|^2 - 1)\exp\left[\frac{i}{g^2}\int d^2x\,(\partial_\mu + iA_\mu)z_j(\partial^\mu - iA^{\mu *})z_j^*\right] \\
&\qquad = \int \mathcal{D}A\,\mathcal{D}^2 z\,\delta^{(2)}(|z_j|^2 - 1)\exp\left[\frac{i}{g^2}\int d^2x\left\{|\partial_\mu z_j|^2 + 2i(\partial^\mu z_j^*)z_j A_\mu + |A|^2\right\}\right],
\end{aligned} \tag{7}$$

where we have imposed $|z_j|^2 = 1$. We apply [1]

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\,\exp\left(-\frac{1}{2}ax^2 + Jx\right) &= \exp\left(\frac{J^2}{2a}\right)\int_{-\infty}^{\infty} dx\,\exp\left[-\frac{1}{a}\left(x - \frac{J}{a}\right)^2\right] \\
&= \exp\left(\frac{J^2}{2a}\right)\int_{-\infty}^{\infty} dw\,\exp\left(-\frac{1}{2}aw^2\right) \\
&= \sqrt{\frac{2\pi}{a}}\exp\left(\frac{J^2}{2a}\right).
\end{aligned}$$

In Eq. (7), $a = -2i/g^2$ and $J = 2(\partial^\mu z_j^*)z_j/g^2$. Then

$$\frac{J^2}{2a} = -\frac{4\left|(\partial^\mu z_j^*)z_j\right|^2}{g^4}\frac{g^2}{2i} = \frac{i}{g^2}\left|z_j\partial^\mu z_j^*\right|^2 = \frac{i}{g^2}\left|z_j^*\partial^\mu z_j\right|^2,$$

where we have used Eq. (3). Finally, Eq. (7) gives us

$$\begin{aligned}
&\int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}^2 z\,\exp\left[\frac{i}{g^2}\int d^2x\left(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)\right)\right] \\
&\qquad = \mathcal{D}^2 z\,\delta^{(2)}(|z_j|^2 - 1)\exp\left[\frac{i}{g^2}\int d^2x\left\{|\partial_\mu z_j|^2 - \left|z_j^*\partial^\mu z_j\right|^2\right\}\right],
\end{aligned}$$

where we have again ignored the overall constant. Thus we have shown that the Lagrangian Eq. (2) can be obtained from the Lagrangian Eq. (6) by functionally integrating over the Lagrange multipliers. □

1(c) We can solve the CP^N model in the limit $N \to \infty$ by integrating over the fields z_j. Show that this integral leads to the expression

$$Z = \int \mathcal{D}A\,\mathcal{D}\lambda \exp\left(-N \operatorname{tr}\ln\left(-D^2 - \lambda\right) + \frac{i}{g^2}\int d^2x\,\lambda\right),$$

where we have kept only the leading terms for $N \to \infty$, g^2N fixed. Using methods similar to those we used for the nonlinear sigma model, examine the conditions for minimizing the exponent with respect to λ and A_μ. Show that these conditions have a solution at $A_\mu = 0$ and $\lambda = m^2 > 0$. Show that, if g^2 is renormalized at the scale M, m can be written as

$$m = M \exp\left(-\frac{2\pi}{g^2N}\right).$$

1(d) Now expand the exponent about $A_\mu = 0$. Show that the first nontrivial term in this expansion is proportional to the vacuum polarization of massive scalar fields. Evaluate this expression using dimensional regularization, and show that it yields a standard kinetic energy term for A_μ. Thus the strange nonlinear field theory that we started with is finally transformed into a theory of $N+1$ massive scalar fields interacting with a massless photon.

References

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[2] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.