

1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \quad (1)$$

1.1 Verify the following:

- a. $i\hbar\partial_t \langle \Psi(t)|x \rangle = -\langle \Psi(t)|H|x \rangle,$
- b. $i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle,$
- c. $i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = -\frac{\hbar^2}{2m} [\langle \Phi(t)|x \rangle \partial_x^2 \langle x|\Psi(t) \rangle - (\partial_x^2 \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle],$
- d. $\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle = \frac{\hbar}{i} [\langle \Phi(t)|x \rangle \partial_x \langle x|\Psi(t) \rangle - (\partial_x \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle]$
- e. $\frac{\hbar}{i} \partial_x [\langle \Phi(t)|x \rangle \langle x|p|\Psi(t) \rangle + \langle \Phi(t)|p|x \rangle \langle x|\Psi(t) \rangle] = \langle \Phi(t)|x \rangle \langle x|p^2|\Psi(t) \rangle - m\langle \Phi(t)|p^2|x \rangle \langle x|\Psi(t) \rangle$

Solution.

- a. Beginning with Schrödinger's equation, note that

$$i\hbar\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \quad (2)$$

$$i\hbar\partial_t \langle x|\Psi(t) \rangle = \langle x|H|\Psi(t) \rangle \quad (3)$$

$$(i\hbar\partial_t \langle x|\Psi(t) \rangle)^* = (\langle x|H|\Psi(t) \rangle)^* \quad (4)$$

$$-i\hbar\partial_t \langle \Psi(t)|x \rangle = \langle \Psi(t)|H|x \rangle \quad (5)$$

$$i\hbar\partial_t \langle \Psi(t)|x \rangle = -\langle \Psi(t)|H|x \rangle, \quad (6)$$

where in going to (5) we have used the fact that H is Hermitian, and (6) is what we sought to prove. \square

- b. Beginning with what was proven in (a),

$$i\hbar\partial_t \langle \Phi(t)|x \rangle = -\langle \Phi(t)|H|x \rangle \quad (7)$$

$$i\hbar(\partial_t \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle = -\langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle. \quad (8)$$

From (3), we can write

$$\langle \Phi(t)|x \rangle i\hbar\partial_t \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle. \quad (9)$$

Adding (15) and (17) yields

$$\langle \Phi(t)|x \rangle i\hbar\partial_t \langle x|\Psi(t) \rangle + i\hbar(\partial_t \langle \Phi(t)|x \rangle) \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle \quad (10)$$

$$i\hbar\partial_t \langle \Phi(t)|x \rangle \langle x|\Psi(t) \rangle = \langle \Phi(t)|x \rangle \langle x|H|\Psi(t) \rangle - \langle \Phi(t)|H|x \rangle \langle x|\Psi(t) \rangle, \quad (11)$$

where in going to (11) we have used the product rule of differentiation. (11) is what we sought to prove. \square

c. Using (1), note that:

$$\langle x|H|\Psi(t)\rangle = \langle x|\left[\frac{p^2}{2m} + V(x)\right]|\Psi(t)\rangle \quad (12)$$

$$= \frac{1}{2m} \langle x|p^2|\Psi(t)\rangle + \langle x|V(x)|\Psi(t)\rangle \quad (13)$$

$$= \frac{(-i\hbar\partial_x)^2}{2m} \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle \quad (14)$$

$$= -\frac{\hbar^2}{2m} \partial_x^2 \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle, \quad (15)$$

where in going to (14) we have used the fact that

$$\langle x|p|\Psi(x)\rangle = -i\hbar\partial_x \langle x|\Psi(t)\rangle. \quad (16)$$

Similarly, note that

$$\langle \Phi(t)|H|x\rangle = -\frac{\hbar^2}{2m} \partial_x^2 \langle \Phi(t)|x\rangle + V(x) \langle \Phi(t)|x\rangle \quad (17)$$

where we have used

$$\langle \Phi(t)|p|x\rangle = i\hbar\partial_x \langle \Phi(t)|x\rangle. \quad (18)$$

Making the substitutions (15) and (17) into what was proven in (b),

$$\begin{aligned} i\hbar\partial_t \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle &= \langle \Phi(t)|x\rangle \left[-\frac{\hbar^2}{2m} \partial_x^2 \langle x|\Psi(t)\rangle + V(x) \langle x|\Psi(t)\rangle \right] \\ &\quad - \left[-\frac{\hbar^2}{2m} \partial_x^2 \langle \Phi(t)|x\rangle + V(x) \langle \Phi(t)|x\rangle \right] \langle x|\Psi(t)\rangle \end{aligned} \quad (19)$$

$$\begin{aligned} &= -\frac{\hbar^2}{2m} [\langle \Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle] \\ &\quad + V(x) \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle - V(x) \langle \Phi(t)|x\rangle \langle x|\Psi(t)\rangle \end{aligned} \quad (20)$$

$$= -\frac{\hbar^2}{2m} [\langle \Phi(t)|x\rangle \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle], \quad (21)$$

as we sought to prove. \square

d. Applying (16) and (18) to the left-hand side of (d),

$$\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle = \langle \Phi(t)|x\rangle (-i\hbar\partial_x \langle x|\Psi(t)\rangle) + (i\hbar\partial_x \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle \quad (22)$$

$$= \frac{\hbar}{i} [\langle \Phi(t)|x\rangle \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle \Phi(t)|x\rangle) \langle x|\Psi(t)\rangle] \quad (23)$$

as we sought to prove. \square

e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x (\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle) = (\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \partial_x \langle x|p|\Psi(t)\rangle \quad (24)$$

Multiplying through by \hbar/i ,

$$\frac{\hbar}{i} \partial_x (\langle \Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle) = (-i\hbar\partial_x \langle \Phi(t)|x\rangle) \langle x|p|\Psi(t)\rangle - \langle \Phi(t)|x\rangle i\hbar\partial_x \langle x|p|\Psi(t)\rangle \quad (25)$$

$$= -\langle \Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle + \langle \Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle, \quad (26)$$

where in going to (26) we have used (16) and (18). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i} \partial_x (\langle \Phi(t) | p | x \rangle \langle x | \Psi(t) \rangle) = (-i\hbar \partial_x \langle \Phi(t) | p | x \rangle) \langle x | \Psi(t) \rangle - \langle \Phi(t) | p | x \rangle i\hbar \partial_x \langle x | \Psi(t) \rangle \quad (27)$$

$$= -\langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle + \langle \Phi(t) | p | x \rangle \langle x | p | \Psi(t) \rangle. \quad (28)$$

Adding the results of (26) and (28),

$$\begin{aligned} \frac{\hbar}{i} \partial_x [\langle \Phi(t) | x \rangle \langle x | p | \Psi(t) \rangle + \langle \Phi(t) | p | x \rangle \langle x | \Psi(t) \rangle] &= \langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p | x \rangle \langle x | p | \Psi(t) \rangle \\ &\quad + \langle \Phi(t) | p | x \rangle \langle x | p | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle \end{aligned} \quad (29)$$

$$= \langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle \quad (30)$$

as we sought to prove. \square

1.2 Define

$$\rho(x, t) = \langle \Phi(t) | x \rangle \langle x | \Psi(t) \rangle, \quad (31)$$

$$J_x(x, t) = \frac{1}{2m} [\langle \Phi(t) | x \rangle \langle x | p | \Psi(t) \rangle + \langle \Phi(t) | p | x \rangle \langle x | \Psi(t) \rangle]. \quad (32)$$

Show that $\rho(x, t) + \partial_x J_x(x, t) = 0$.

Solution. From (31),

$$\partial_t \rho(x, t) = \partial_t (\langle \Phi(t) | x \rangle \langle x | \Psi(t) \rangle), \quad (33)$$

and from what was proven in 1(c),

$$\partial_t (\langle \Phi(t) | x \rangle \langle x | \Psi(t) \rangle) = -\frac{1}{i\hbar} [\langle \Phi(t) | x \rangle \partial_x^2 \langle x | \Psi(t) \rangle - (\partial_x^2 \langle \Phi(t) | x \rangle) \langle x | \Psi(t) \rangle] \quad (34)$$

$$= -\frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle], \quad (35)$$

where we have applied (16) and (18) in going to (35). Equating (33) and (35),

$$\partial_t \rho(x, t) = -\frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle]. \quad (36)$$

Beginning from (32),

$$\partial_x J_x(x, t) = \frac{1}{2m} \partial_x [\langle \Phi(t) | x \rangle \langle x | p | \Psi(t) \rangle + \langle \Phi(t) | p | x \rangle \langle x | \Psi(t) \rangle] \quad (37)$$

$$= \frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle], \quad (38)$$

where in going to (38) we have used what was proven in 1(e). Summing (36) and (38), we have

$$\begin{aligned} \partial_t \rho(x, t) + \partial_x J_x(x, t) &= -\frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle] \\ &\quad + \frac{1}{2m} \frac{i}{\hbar} [\langle \Phi(t) | x \rangle \langle x | p^2 | \Psi(t) \rangle - \langle \Phi(t) | p^2 | x \rangle \langle x | \Psi(t) \rangle] \end{aligned} \quad (39)$$

$$= 0 \quad (40)$$

as we sought to prove. This is the continuity equation. \square

2 Problem 4

Consider a particle moving in three dimensions. Consider an operator

$$U(\phi) = \exp\left(-\frac{i}{\hbar} L_3 \phi\right), \quad L_3 = L_z = X P_y - Y P_x, \quad (41)$$

where X, Y and P_x, P_y are position and momentum operators, respectively. Define new operators

$$X(\phi) = U^\dagger(\phi) X U(\phi), \quad (42)$$

$$Y(\phi) = U^\dagger(\phi) Y U(\phi). \quad (43)$$

Note that $X(0) = Y(0) = 0$.

2.1 Derive the equation

$$\frac{dX(\phi)}{d\phi} = \frac{i}{\hbar} U^\dagger(\phi) [L_3, X] U(\phi) = -Y(\phi), \quad (44)$$

and a similar equation for $dY(\phi)/d\phi$.

Solution. Using the definition of $X = X(\phi)$ in (42) and applying the product rule of differentiation,

$$\frac{dX}{d\phi} = \frac{d}{d\phi} (U^\dagger X U) = \frac{dU^\dagger}{d\phi} X U + U^\dagger \frac{d}{d\phi} (X U) = \frac{dU^\dagger}{d\phi} X U + U^\dagger \frac{dX}{d\phi} U + U^\dagger X \frac{dU}{d\phi} \quad (45)$$