

Problem 1. Linear sigma model (Peskin & Schroeder 4.3) The interactions of pions at low energy can be described by a phenomenological model called the *linear sigma model*. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Psi^i(x)$, $i = 1, \dots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\Phi^i)^2 + V(\Phi^2) \right),$$

where $(\Phi^i)^2 = \Phi \cdot \Phi$, and

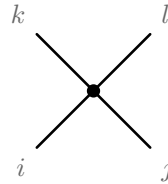
$$V(\Phi^2) = \frac{1}{2}m^2(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2 \quad (1)$$

is a function symmetric under rotations of Φ . For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H ; hence, V is the field potential energy.

1(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ . Show that the propagator is

$$\overline{\Phi^i(x)\Phi^j(y)} = \delta^{ij}D_F(x-y),$$

where D_F is the standard Klein-Gordon propagator for mass m , and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}). \quad (2)$$

Compute, to leading order in λ , the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \rightarrow \Phi^1\Phi^2,$$

$$\Phi^1\Phi^1 \rightarrow \Phi^2\Phi^2,$$

$$\Phi^1\Phi^1 \rightarrow \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

Solution. The Klein-Gordon Hamiltonian is given by Peskin & Schroeder (2.8),

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \quad (3)$$

For $\lambda = 0$, the linear sigma model has the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}(\Pi^i)^2 + \frac{1}{2}(\nabla\Phi^i)^2 + \frac{1}{2}m^2(\Phi^i)^2 \right)$$

which is clearly N copies of the Klein-Gordon Hamiltonian, one for each i .

From (4.36) we know that the Feynman propagator is the contraction of two fields:

$$\overline{\phi(x)\phi(y)} = D_F(x-y).$$

No terms $\Phi^i\Phi^j$ for $i \neq j$ appear in the Hamiltonian, so fields with $i \neq j$ cannot be contracted. Moreover, each field is governed by its own independent Klein-Gordon Hamiltonian to zeroth order. So the propagator must be

$$\overline{\Phi^i(x)\Phi^j(y)} = \delta^{ij}D_F(x-y)$$

where $D_F(x - y)$ is the Klein-Gordon propagator. \square

In order to determine the Feynman rules, we use Peskin & Schroeder (4.90),

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_n | iT | \mathbf{p}_A \mathbf{p}_B \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} {}_0 \langle \mathbf{p}_1 \cdots \mathbf{p}_n | T \left\{ \exp \left(-i \int_{-T}^T dt H_I(t) \right) \right\} | \mathbf{p}_A \mathbf{p}_B \rangle_0.$$

Our interaction Hamiltonian is

$$H_I = \int d^3x \frac{\lambda}{4} ((\Phi^i)^2)^2 = \int d^3x \frac{\lambda}{4} (\Phi \cdot \Phi)^2 = \frac{\lambda}{4} \int d^3x \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right),$$

We have two final momenta, \mathbf{p}_1 and \mathbf{p}_2 . Now we have

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{p}_A \mathbf{p}_B \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} {}_0 \langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ \exp \left[-i \int d^4x \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right) \right] \right\} | \mathbf{p}_A \mathbf{p}_B \rangle_0.$$

The first term that contributes to leading order is, by analogy to (4.92),

$$\begin{aligned} {}_0 \langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ -i \int d^4x \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right) \right\} | \mathbf{p}_A \mathbf{p}_B \rangle_0 \\ = {}_0 \langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i \int d^4x \frac{\lambda}{4} \left(\sum_i (\Phi^i)^4 + 2 \sum_{i \neq j} (\Phi^i)^2 (\Phi^j)^2 \right) + \text{contractions} \right\} | \mathbf{p}_A \mathbf{p}_B \rangle_0, \end{aligned}$$

but only the terms in which none of the fields are contracted with each other will contribute [1, p. 111].

The first term represents the process $\Phi^i \Phi^i \rightarrow \Phi^i \Phi^i$:

$${}_0 \langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i \int d^4x \frac{\lambda}{4} \sum_i \Phi^i \Phi^i \Phi^i \Phi^i + \text{contractions} \right\} | \mathbf{p}_A \mathbf{p}_B \rangle_0.$$

The fields are all the same, so there are 4! ways of contracting the fields with the momenta, and we will obtain a diagram similar to (4.98). Adapting that expression, we find

$$-4!i \int \frac{\lambda}{4} d^4x e^{-i(p_A + p_B - p_1 - p_2) \cdot x} = -6i\lambda(4\pi)^4 \delta^4(p_A + p_B - p_1 - p_2).$$

The diagram in Eq. (2) is $\Phi^i \Phi^j \rightarrow \Phi^k \Phi^l$. Since $i = j = k = l$ for this term, we have

$$\begin{aligned} \begin{array}{c} k \quad l \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ i \quad j \end{array} = -6i\lambda = -2i\lambda(1 + 1 + 1) = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}). \end{aligned}$$

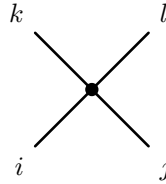
The second term can represent the processes $\Phi^i \Phi^i \rightarrow \Phi^j \Phi^j$ or $\Phi^i \Phi^j \rightarrow \Phi^i \Phi^j$ (where the indices and the order of the fields on either side is interchangeable):

$${}_0 \langle \mathbf{p}_1 \mathbf{p}_2 | N \left\{ -i \int d^4x \frac{\lambda}{4} 2 \sum_{i \neq j} \Phi^i \Phi^i \Phi^j \Phi^j + \text{contractions} \right\} | \mathbf{p}_A \mathbf{p}_B \rangle_0.$$

Now there are only $2! \times 2! = 4$ ways to contract the fields with the momenta. We have

$$-4i \int \frac{\lambda}{2} d^4x e^{-i(p_A + p_B - p_1 - p_2) \cdot x} = -2i\lambda(4\pi)^4 \delta^4(p_A + p_B - p_1 - p_2).$$

Here, either $i = j$ and $k = l$, $i = l$ and $j = k$, or $i = k$ and $j = l$. We have



$$= -2i\lambda = -2i\lambda(1 + 0 + 0) = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

Both of the terms can therefore be represented by Eq. (2) as we wanted to show. \square

When all four of the particles in the interaction have the same mass, the differential cross section in the center-of-mass frame is given by Peskin & Schroeder (4.85)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2},$$

where E_{cm} is the center-of-mass energy and \mathcal{M} is the invariant matrix element. We know that the diagrams we calculated before have the form $i\mathcal{M}(2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2)$ [1, p. 112]. Then

$$\mathcal{M} = -6\lambda \quad \text{for} \quad \Phi^1\Phi^1 \rightarrow \Phi^1\Phi^1, \quad \mathcal{M} = -2\lambda \quad \text{for} \quad \Phi^1\Phi^2 \rightarrow \Phi^1\Phi^2 \text{ and } \Phi^1\Phi^1 \rightarrow \Phi^2\Phi^2.$$

So the differential cross sections are, to leading order in λ ,

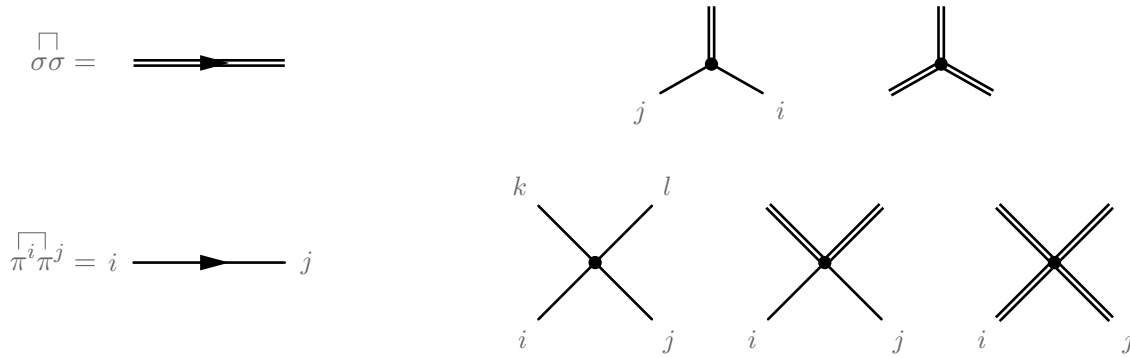
$$\begin{aligned} (\Phi^1\Phi^2 \rightarrow \Phi^1\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{|-2\lambda|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{\lambda^2}{16\pi^2 E_{\text{cm}}^2}, \\ (\Phi^1\Phi^2 \rightarrow \Phi^1\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{|-6\lambda|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{9\lambda^2}{16\pi^2 E_{\text{cm}}^2}, \\ (\Phi^1\Phi^1 \rightarrow \Phi^2\Phi^2) \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{|-6\lambda|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{9\lambda^2}{16\pi^2 E_{\text{cm}}^2}. \end{aligned}$$

1(b) Now consider the case $m^2 < 0$: $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V . By rotational invariance, we can consider this shift to be in the N th direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \quad i = 1, \dots, N-1, \quad \Phi^N(x) = v + \sigma(x)$$

where v is a constant chosen to minimize V . (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ massless pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \rightarrow 0$. Construct the Feynman rules by

assigning values to the propagators and vertices:



Solution. With the negative mass, Eq. (1) becomes

$$V(\Phi^2) = -\frac{1}{2}\mu^2(\Phi^i)^2 + \frac{\lambda}{4}((\Phi^i)^2)^2.$$

To find the minimum of V , we differentiate with respect to Φ^N . We stipulate

$$0 = \frac{\partial V}{\partial \Phi^N} = -\mu^2 \Phi^N + \lambda(\Phi \cdot \Phi) \Phi^N = -\mu^2 \Phi^N + \lambda(\Phi^N)^3,$$

where we have used the chain rule to evaluate the second term, and the fact that V is minimal for all $\Phi^i = 0$ with $i \neq N$. This implies $\Phi^N = 0$ or

$$(\Phi^N)^2 = \frac{\mu^2}{\lambda}$$

when V is minimal. Thus

$$v = \frac{\mu}{\sqrt{\lambda}}.$$

In order to determine the form of the theory, we need to rewrite $V(\Phi^2)$ in the new coordinates. Note that $\Phi = (\pi, v + \sigma)$. Then

$$\begin{aligned} V(\Phi^2) &= -\frac{1}{2}\mu^2 [\pi^2 + (v + \sigma)^2] + \frac{\lambda}{4} [\pi^2 + (v + \sigma)^2]^2 \\ &= -\frac{1}{2}\mu^2 \left(\pi^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2 \right) + \frac{\lambda}{4} \left(\pi^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2 \right)^2 \\ &= -\frac{1}{2}\mu^2 \left(\pi^2 + \frac{\mu^2}{\lambda} + 2\frac{\mu\sigma}{\sqrt{\lambda}} + \sigma^2 \right) \\ &\quad + \frac{\lambda}{4} \left((\pi^2)^2 + 2\frac{\pi^2\mu^2}{\lambda} + \frac{\mu^4}{\lambda^2} + 4\frac{\pi^2\mu\sigma}{\sqrt{\lambda}} + 4\frac{\mu^3\sigma}{\lambda^{3/2}} + 2\pi^2\sigma^2 + 6\frac{\mu^2\sigma^2}{\lambda} + 4\frac{\mu\sigma^3}{\sqrt{\lambda}} + \sigma^4 \right) \\ &= -\frac{\pi^2\mu^2}{2} - \frac{\mu^4}{2\lambda} - \frac{\mu^3\sigma}{\sqrt{\lambda}} - \frac{\mu^2\sigma^2}{2} + \frac{(\pi^2)^2\lambda}{4} + \frac{\pi^2\mu^2}{2} + \frac{\mu^4}{4\lambda} \\ &\quad + \pi^2\mu\sigma\sqrt{\lambda} + \frac{\mu^3\sigma}{\sqrt{\lambda}} + \frac{\pi^2\sigma^2\lambda}{2} + \frac{3\mu^2\sigma^2}{2} + \mu\sigma^3\sqrt{\lambda} + \frac{\sigma^4\lambda}{4} \\ &= -\frac{\mu^4}{4\lambda} + \mu^2\sigma^2 + \frac{(\pi^2)^2\lambda}{4} + \pi^2\mu\sigma\sqrt{\lambda} + \frac{\pi^2\sigma^2\lambda}{2} + \mu\sigma^3\sqrt{\lambda} + \frac{\sigma^4\lambda}{4}. \end{aligned}$$

This expression includes a $\mu^2\sigma^2$ term, which indicates a massive sigma field. Comparing with Eq. (3), the pion mass is $\sqrt{2}\mu$. However, there is no $\mu^2\pi^2$ term, which indicates that the pion field is massless. The terms of

$\mathcal{O}(\sqrt{\lambda})$ and $\mathcal{O}(\lambda)$ have factors of π^4 , $\pi^2\sigma$, $\pi^2\sigma^2$, σ^3 , and σ^4 ; these are all cubic and quartic factors. Since they are of $\mathcal{O}(\sqrt{\lambda})$ and $\mathcal{O}(\lambda)$, they become small as $\lambda \rightarrow 0$. This is what we wanted to show. \square

For the propagators, we can use (4.46) of Peskin & Schroeder:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}.$$

Then we can write

$$\text{---}\text{---}\text{---} = \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip \cdot (x-y)}}{p^2 - 2\mu^2 + i\epsilon}, \quad i \text{---}\text{---} j = \delta^{ij} \int \frac{d^4p}{(2\pi)^3} \frac{ie^{-ip \cdot (x-y)}}{p^2 - 2\mu^2 + i\epsilon}. \quad (4)$$

We can associate each of the vertices with a term in $V(\Psi^2)$. The symmetry factors for each of the terms are

$$\pi^2\mu\sigma\sqrt{\lambda} : 2! = 2, \quad \mu\sigma^3\sqrt{\lambda} : 3! = 6, \quad \frac{(\pi^2)^2\lambda}{4} : 4! = 24, \quad \frac{\pi^2\sigma^2\lambda}{2} : 2!2! = 4, \quad \frac{\sigma^4\lambda}{4} : 4! = 24.$$

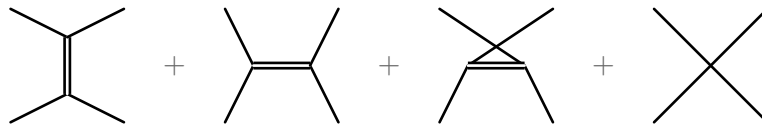
Then the vertices are

$$\begin{aligned} \text{---}\text{---}\text{---} &= -2i\mu\sqrt{\lambda}\delta^{ij}, & \text{---}\text{---}\text{---} &= -6i\mu\sqrt{\lambda}, \\ \text{---}\text{---}\text{---} &= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}), & \text{---}\text{---}\text{---} &= -2i\lambda\delta^{ij}, & \text{---}\text{---}\text{---} &= -6i\lambda. \end{aligned} \quad (5)$$

1(c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \rightarrow \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ . There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to *zero*. Show that, in the special case $N = 2$ (1 species of pion), the term $\mathcal{O}(p^2)$ also cancels.

Solution. Using the propagators and vertices of Eqs. (4) and (5), the contributions from each diagram are

$$\text{---}\text{---}\text{---} = (-2i\mu\sqrt{\lambda}\delta^{ij}) \frac{i}{(p_1 + p_2)^2 - 2\mu^2} (-2i\mu\sqrt{\lambda}\delta^{kl}) = -\frac{4i\mu^2\lambda}{(p_1 + p_2)^2 - 2\mu^2} \delta^{ij}\delta^{kl},$$

$$\begin{aligned}
& \begin{array}{c} k \quad l \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ i \quad j \end{array} = (-2i\mu\sqrt{\lambda}\delta^{ik}) \frac{i}{(p_1 - p_3)^2 - 2\mu^2} (-2i\mu\sqrt{\lambda}\delta^{jl}) = -\frac{4i\mu^2\lambda}{(p_1 - p_3)^2 - 2\mu^2} \delta^{ik} \delta^{jl}, \\
& \begin{array}{c} k \quad l \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ i \quad j \end{array} = (-2i\mu\sqrt{\lambda}\delta^{il}) \frac{i}{(p_1 - p_4)^2 - 2\mu^2} (-2i\mu\sqrt{\lambda}\delta^{jk}) = -\frac{4i\mu^2\lambda}{(p_1 - p_4)^2 - 2\mu^2} \delta^{il} \delta^{jk}, \\
& \begin{array}{c} k \quad l \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ i \quad j \end{array} = -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}),
\end{aligned}$$

so the total amplitude is

$$\begin{aligned}
i\mathcal{M} &= -\frac{4i\mu^2\lambda}{(p_1 + p_2)^2 - 2\mu^2} \delta^{ij}\delta^{kl} - \frac{4i\mu^2\lambda}{(p_1 - p_3)^2 - 2\mu^2} \delta^{ik}\delta^{jl} - \frac{4i\mu^2\lambda}{(p_1 - p_4)^2 - 2\mu^2} \delta^{il}\delta^{jk} - 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) \\
&= -4i\mu^2\lambda \left(\frac{\delta^{ij}\delta^{kl}}{(p_1 + p_2)^2 - 2\mu^2} + \frac{\delta^{ik}\delta^{jl}}{(p_1 - p_3)^2 - 2\mu^2} + \frac{\delta^{il}\delta^{jk}}{(p_1 - p_4)^2 - 2\mu^2} \right) - 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}),
\end{aligned}$$

where we have referred to (4.119) of Peskin & Schroeder.

When $\mathbf{p}_i = 0$, $p_i = 0$. This is because $p_i = (E_i, \mathbf{p}_i)$ and $m_\pi = 0$, so $E_i = 0$ at zero momentum. Then the amplitude is

$$\begin{aligned}
i\mathcal{M} &= 4i\mu^2\lambda \left(\frac{\delta^{ij}\delta^{kl}}{2\mu^2} + \frac{\delta^{ik}\delta^{jl}}{2\mu^2} + \frac{\delta^{il}\delta^{jk}}{2\mu^2} \right) - 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) \\
&= 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) - 2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) \\
&= 0
\end{aligned}$$

as we wanted to show. □

When there is only one species of pion, $i = j = k = l$. So the amplitude is

$$i\mathcal{M} = -4i\mu^2\lambda \left(\frac{1}{(p_1 + p_2)^2 - 2\mu^2} + \frac{1}{(p_1 - p_3)^2 - 2\mu^2} + \frac{1}{(p_1 - p_4)^2 - 2\mu^2} \right) - 6i\lambda.$$

Assuming that the pions have similar momenta, $(p_i \pm p - j)^2 \ll \mu^2$. Let $x = (p_i \pm p - j)^2/\mu^2$. Then the first three terms of $i\mathcal{M}$ each look like

$$\frac{1}{2\mu^2} \frac{1}{x - 1} = -\frac{1 + x}{2\mu^2} + \mathcal{O}(x^2),$$

where we have performed a Taylor series expansion about 0 to first order in x [2]. Then the amplitude is

$$\begin{aligned}
i\mathcal{M} &= -2i\lambda \left(-1 - \frac{(p_1 + p_2)^2}{2\mu^2} - 1 - \frac{(p_1 - p_3)^2}{2\mu^2} - 1 - \frac{(p_1 - p_4)^2}{2\mu^2} \right) - 6i\lambda + \mathcal{O}(p^4) \\
&= \frac{i\lambda}{\mu^2} [(p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2] + \mathcal{O}(p^4).
\end{aligned} \tag{6}$$

From Peskin & Schroeder (5.69), we can write the Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad t = (p_3 - p_1)^2 = (p_4 - p_2)^2, \quad u = (p_4 - p_1)^2 = (p_3 - p_2)^2.$$

For four particles of the same mass m [1, p. 159],

$$s + t + u = 4m^2.$$

The pion has mass $m_\pi = 0$, so $s + t + u = 0$ for the pions. Substituting the Mandelstam variables into Eq. (6), we have

$$i\mathcal{M} = \frac{i\lambda}{\mu^2} [s + t + u] + \mathcal{O}(p^4) = \mathcal{O}(p^4).$$

So we have shown that the $\mathcal{O}(p^2)$ terms cancel, as desired. □

References

- [1] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”. Perseus Books Publishing, 1995.
- [2] E. W. Weisstein, “Maclaurin Series.” From MathWorld—A Wolfram Web Resource.
<https://mathworld.wolfram.com/MaclaurinSeries.html>.