Problem 1. Verify that the functional

$$J[u] = \int_R \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy \tag{1}$$

is invariant under a rotation of the coordinate axis:

$$\tilde{x} = x \cos \epsilon + y \sin \epsilon, \qquad\qquad\qquad \tilde{y} = -x \sin \epsilon + y \cos \epsilon. \tag{2}$$

Solution. The functional is invariant if $J[u(x,y)] = J[u(\tilde{x}, \tilde{y})]$. By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial x} = \cos \epsilon \frac{\partial}{\partial \tilde{x}} - \sin \epsilon \frac{\partial}{\partial \tilde{y}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial y} = \sin \epsilon \frac{\partial}{\partial \tilde{x}} + \cos \epsilon \frac{\partial}{\partial \tilde{y}}.$$

The Jacobian for (2) is

$$A = \begin{bmatrix} \partial \tilde{x}/\partial x & \partial \tilde{x}/\partial y \\ \partial \tilde{y}/\partial x & \partial \tilde{y}/\partial y \end{bmatrix} = \begin{bmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{bmatrix} \implies \det(A) = \cos^2 \epsilon + \sin^2 \epsilon = 1,$$

and therefore

$$\int_R dx\, dy \mapsto \int_{\tilde{R}} d\tilde{x}\, d\tilde{y}\,.$$

Making these substitutions into (1), we have

$$J[u(x,y)] = \int_R \left[\left(\cos \epsilon \frac{\partial u}{\partial \tilde{x}} - \sin \epsilon \frac{\partial u}{\partial \tilde{y}} \right)^2 + \left(\sin \epsilon \frac{\partial u}{\partial \tilde{x}} + \cos \epsilon \frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy$$

$$= \int_R \left(\cos^2 \epsilon \frac{\partial^2 u}{\partial \tilde{x}^2} - 2 \cos \epsilon \sin \epsilon \frac{\partial^2 u}{\partial \tilde{x} \partial \tilde{y}} + \sin^2 \epsilon \frac{\partial^2 u}{\partial \tilde{y}^2} + \sin^2 \epsilon \frac{\partial^2 u}{\partial \tilde{x}^2} + 2 \cos \epsilon \sin \epsilon \frac{\partial^2 u}{\partial \tilde{x} \partial \tilde{y}} + \cos^2 \epsilon \frac{\partial^2 u}{\partial \tilde{y}^2} \right) dx\, dy$$

$$= \int_R \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy = \int_{\tilde{R}} \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] d\tilde{x}\, d\tilde{y}$$

$$= J[u(\tilde{x}, \tilde{y})]$$

as desired. $\qquad$ $\square$

Problem 2. Consider the real-valued Lagrangian density $\mathcal{L}$ depending on a complex-valued function $\phi(t, x, y)$:

$$\mathcal{L} = \frac{i}{2} \left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt} \phi \right) - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi, \tag{3}$$

where * is complex conjugation, and $\nabla \phi = (\partial \phi / \partial x\, , \partial \phi / \partial y)$. Treating ϕ and ϕ^* as independent objects, derive the Euler-Lagrange equations.

Solution. We will have two Euler-Lagrange equations; one for ϕ and one for ϕ^*. In general, they are given by

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \phi_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi_x} - \frac{d}{dy} \frac{\partial \mathcal{L}}{\partial \phi_y}, \qquad\qquad 0 = \frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \phi_t^*} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi_x^*} - \frac{d}{dy} \frac{\partial \mathcal{L}}{\partial \phi_y^*}.$$

Expanding out $\nabla \phi^* \cdot \nabla \phi$, (3) becomes

$$\mathcal{L} = \frac{i}{2} \left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt} \phi \right) - \frac{\partial \phi^*}{\partial x} \frac{\partial \phi}{\partial x} - \frac{\partial \phi^*}{\partial y} \frac{\partial \phi}{\partial y} - m^2 \phi^* \phi.$$

Then

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{i}{2}\phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x} = -\frac{\partial \phi^*}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \phi_y} = -\frac{\partial \phi^*}{\partial y},$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t^*} = -\frac{i}{2}\phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x^*} = -\frac{\partial \phi}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \phi_y^*} = -\frac{\partial \phi}{\partial y},$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{i}{2}\frac{d\phi^*}{dt}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x} = -\frac{\partial^2 \phi^*}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y} = -\frac{\partial^2 \phi^*}{\partial y^2},$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t^*} = -\frac{i}{2}\frac{d\phi}{dt}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x^*} = -\frac{\partial^2 \phi}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y^*} = -\frac{\partial^2 \phi}{\partial y^2}.$$

Then the Euler-Lagrange equations are

$$0 = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^* - \frac{i}{2}\frac{d\phi^*}{dt} + \frac{\partial^2 \phi^*}{\partial x^2} + \frac{\partial^2 \phi^*}{\partial y^2}, \qquad 0 = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi + \frac{i}{2}\frac{d\phi}{dt} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2},$$

which simplify to

$$0 = i\frac{d\phi^*}{dt} - \nabla^2\phi^* + m^2\phi^*, \qquad 0 = i\frac{d\phi}{dt} + \nabla^2\phi^* - m^2\phi^*.$$

Problem 3. The nondimensionalized, multidimensional Sine-Gordon equation,

$$\theta_{xx} + \theta_{yy} - \theta_{tt} = \sin\theta$$

for $\theta(x, y, t)$, is the Euler-Lagrange equation for the action integral

$$S[\theta] = \int_R \left\{ \frac{1}{2}\left[\theta_t^2 - (\nabla\theta)^2\right] - \sin\theta \right\} dx\, dy\, dt$$

with $\nabla\theta = (\partial\theta/\partial x\,,\, \partial\theta/\partial y)$. The functional $S[\theta]$ is invariant under translation of x, y, and t. Find the associated energy-momentum tensor and energy-momentum vector.

Solution. Expanding out $(\nabla\theta)^2$, the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) - \sin\theta. \tag{4}$$

The energy-momentum tensor is defined by

$$T_{ij} = \frac{\partial \mathcal{L}}{\partial \theta_{x_i}}\frac{\partial \theta}{\partial x_j} - \mathcal{L}\,\delta_{ij},$$

where $x_i \in \{x_0, x_1, x_2\} = \{t, x, y\}$. The diagonal elements of T are then

$$T_{00} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial t} - \mathcal{L} = \theta_t^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = \frac{1}{2}(\theta_t^2 + \theta_x^2 + \theta_y^2) + \sin\theta,$$

$$T_{11} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial x} - \mathcal{L} = -\theta_x^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 + \theta_x^2 - \theta_y^2) + \sin\theta,$$

$$T_{22} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial y} - \mathcal{L} = -\theta_y^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 - \theta_x^2 + \theta_y^2) + \sin\theta,$$

and the nondiagonal elements are

$$T_{01} = \frac{\partial \mathcal{L}}{\partial \theta_t} \frac{\partial \theta}{\partial x} = \theta_t \theta_x, \qquad T_{02} = \frac{\partial \mathcal{L}}{\partial \theta_t} \frac{\partial \theta}{\partial y} = \theta_t \theta_y, \qquad T_{12} = \frac{\partial \mathcal{L}}{\partial \theta_x} \frac{\partial \theta}{\partial y} = -\theta_x \theta_y,$$

$$T_{10} = \frac{\partial \mathcal{L}}{\partial \theta_x} \frac{\partial \theta}{\partial t} = -\theta_t \theta_x, \qquad T_{20} = \frac{\partial \mathcal{L}}{\partial \theta_y} \frac{\partial \theta}{\partial t} = -\theta_t \theta_y, \qquad T_{21} = \frac{\partial \mathcal{L}}{\partial \theta_y} \frac{\partial \theta}{\partial x} = -\theta_x \theta_y.$$

In matrix form, we have

$$T = \begin{bmatrix} (\theta_t^2 + \theta_x^2 + \theta_y^2)/2 + \sin\theta & \theta_t \theta_x & \theta_t \theta_y \\ -\theta_t \theta_x & -(\theta_t^2 + \theta_x^2 - \theta_y^2)/2 + \sin\theta & \theta_t \theta_y \\ -\theta_t \theta_y & -\theta_x \theta_y & -(\theta_t^2 - \theta_x^2 + \theta_y^2)/2 + \sin\theta \end{bmatrix}.$$

The energy-momentum vector is defined by

$$P_j = \int T_{0j} \, dx_1 \, dx_2 \,.$$

Its components are then

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2 + \theta_x^2 + \theta_y^2) + \sin\theta \right] dx \, dy \,, \qquad P_1 = \int \theta_t \theta_x \, dx \, dy \,, \qquad P_2 = \int \theta_t \theta_y \, dx \, dy \,.$$

Problem 4.   Extra credit

4.a Verify that the nondimensionalized, one-dimensional Sine-Gordon equation,

$$\theta_{xx} - \theta_{tt} = \sin\theta, \tag{5}$$

is also invariant under a Lorentz transformation on $(x_0 = t, x_1 = x)$. The transformation is given by

$$\begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix},$$

where $\gamma = 1/\sqrt{1 - \nu^2}$.

Solution.   Define $(\tilde{t}, \tilde{x})$ as the transformed coordinates. (5) is invariant if it has the same form under the substitution $\theta(t, x) \mapsto \theta(\tilde{t}, \tilde{x})$. The new coordinates are given by

$$\begin{bmatrix} \tilde{t} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma(t - \nu x) \\ \gamma(x - \nu t) \end{bmatrix},$$

or

$$\tilde{t} = \gamma(t - \nu x), \qquad\qquad \tilde{x} = \gamma(x - \nu t).$$

Proceeding similarly to problem 1, the chain rule gives us

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} + \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial t} = \gamma \left(\frac{\partial}{\partial \tilde{t}} - \nu \frac{\partial}{\partial \tilde{x}} \right), \qquad\qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial x} + \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} = \gamma \left(\frac{\partial}{\partial \tilde{x}} - \nu \frac{\partial}{\partial \tilde{t}} \right).$$

For the second derivatives,

$$\frac{\partial^2}{\partial t^2} = \gamma^2 \left(\frac{\partial}{\partial \tilde{t}} - \nu \frac{\partial}{\partial \tilde{x}}\right)^2 = \gamma^2 \left(\frac{\partial^2}{\partial \tilde{t}^2} - 2\nu \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2}{\partial \tilde{x}^2}\right),$$

$$\frac{\partial^2}{\partial x^2} = \gamma^2 \left(\frac{\partial}{\partial \tilde{x}} - \nu \frac{\partial}{\partial \tilde{t}}\right)^2 = \gamma^2 \left(\frac{\partial^2}{\partial \tilde{x}^2} - 2\nu \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2}{\partial \tilde{t}^2}\right).$$

Making these substitutions, (5) becomes

$$\sin\theta = \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial t^2}$$

$$= \gamma^2 \left(\frac{\partial^2 \theta}{\partial \tilde{x}^2} - 2\nu \frac{\partial^2 \theta}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2 \theta}{\partial \tilde{t}^2}\right) - \gamma^2 \left(\frac{\partial^2 \theta}{\partial \tilde{t}^2} - 2\nu \frac{\partial^2 \theta}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2 \theta}{\partial \tilde{x}^2}\right)$$

$$= \gamma^2 \left[(1 - \nu^2)\frac{\partial^2 \theta}{\partial \tilde{x}^2} - (1 - \nu^2)\frac{\partial^2 \theta}{\partial \tilde{t}^2}\right]$$

$$= \frac{\partial^2 \theta}{\partial \tilde{x}^2} - \frac{\partial^2 \theta}{\partial \tilde{t}^2},$$

because $\gamma^2 = 1/(1 - \nu^2)$. Thus, we have demonstrated the invariance of (5). $\qquad \square$

4.b Find the associated conserved quantity. Is it analogous to a common conserved quantity in classical mechanics?

Solution. By analogy to problem 3, the Lagrangian for this system is given by

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2) - \sin\theta$$

which is like (4), but with only one spatial dimension. Continuing the analogy, the components of the energy-momentum vector are

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2 + \theta_x^2) + \sin\theta\right] dx, \qquad\qquad P_1 = \int \theta_t \theta_x \, dx.$$

These are the conserved quantitites, or "currents." The component P_0 is analagous to the calssical Hamiltonian, or the total energy of the system. This corresponds to $\mathcal{L}$'s having no explicit t dependence. The component P_1 is like the momentum conjugate to x, since it corresponds to $\mathcal{L}$'s having no explicit x dependence. Since we are concerned with only one spatial dimension, P_1 is analogous to the classical total (linear) momentum of the system.

Problem 5. Interacting line vortices

A system of n vortices moving on a two-dimensional plane has the Hamiltonian

$$H = \sum_{j=1}^{n} \sum_{i=1}^{j-1} -\gamma^{(i)} \gamma^{(j)} \ln |\mathbf{r}_i - \mathbf{r}_j|,$$

where $\gamma^{(i)}$ is the strength of the ith line vortex, and $\mathbf{r}_i = (x_i, y_i)$ its position in the plane. Using the Poisson bracket structure

$$[f, g] = \sum_{i=1}^{n} \frac{1}{\gamma^{(i)}} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right),$$

Hamilton's equations for the vortex system simplify to

$$\dot{x}_i = [x_i, H], \qquad\qquad\qquad \dot{y}_i = [y_i, H].$$

Consider two vortices. Show that the equations of motion can be solved explicitly. Most importantly, show that the solution tells us the two vortices orbit each other with a frequency that is inversely proportional to the square of their separation.

Solution. For two vortices, the Hamiltonian reduces to

$$H = -\gamma^{(1)}\gamma^{(2)} \ln |\mathbf{r}_1 - \mathbf{r}_2|.$$

For $i, j \in \{1, 2\}$, note that

$$\frac{\partial x_i}{\partial x_j} = \frac{\partial y_i}{\partial y_j} = \delta_{ij}, \qquad\qquad \frac{\partial x_i}{\partial y_j} = \frac{\partial y_i}{\partial x_j} = 0.$$

Note also that

$$|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{x_1^2 - 2x_1 x_2 + x_2^2 + y_1^2 - 2y_1 y_2 + y_2^2},$$

and define $R \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ as the separation of the vortices. Define also

$$u \equiv \ln R, \qquad\qquad v \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 = R^2.$$

Then

$$\frac{\partial H}{\partial x_i} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial R}\frac{\partial R}{\partial v}\frac{\partial v}{\partial x_i} = -\gamma^{(i)}\gamma^{(j)}\frac{1}{R}\frac{1}{2R}(2x_i - 2x_j) = -\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{(x_1 - x_2)^2 + (y_1 - y_2)^2} = -\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{R^2},$$

$$\frac{\partial H}{\partial y_i} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial R}\frac{\partial R}{\partial v}\frac{\partial v}{\partial y_i} = -\gamma^{(i)}\gamma^{(j)}\frac{y_i - y_j}{R^2},$$

where $i \neq j$.

Combining the above, and again fixing $i \neq j$,

$$\dot{x}_i = \frac{1}{\gamma^{(i)}}\left(\frac{\partial x_i}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial x_i}{\partial y_i}\frac{\partial H}{\partial x_i}\right) + \frac{1}{\gamma^{(j)}}\left(\frac{\partial x_i}{\partial x_j}\frac{\partial H}{\partial y_j} - \frac{\partial x_i}{\partial y_j}\frac{\partial H}{\partial x_j}\right) = \frac{1}{\gamma^{(i)}}\frac{\partial H}{\partial y_i} = -\gamma^{(j)}\frac{y_i - y_j}{R^2},$$

$$\dot{y}_i = \frac{1}{\gamma^{(i)}}\left(\frac{\partial y_i}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial y_i}{\partial y_i}\frac{\partial H}{\partial x_i}\right) + \frac{1}{\gamma^{(j)}}\left(\frac{\partial y_i}{\partial x_j}\frac{\partial H}{\partial y_j} - \frac{\partial y_i}{\partial y_j}\frac{\partial H}{\partial x_j}\right) = -\frac{1}{\gamma^{(i)}}\frac{\partial H}{\partial x_i} = \gamma^{(j)}\frac{x_i - x_j}{R^2}.$$

Explicitly, the four equations of motion are

$$\dot{x}_1 = -\gamma^{(2)}\frac{y_1 - y_2}{R^2}, \qquad \dot{x}_2 = \gamma^{(1)}\frac{y_1 - y_2}{R^2}, \qquad \dot{y}_1 = \gamma^{(2)}\frac{x_1 - x_2}{R^2}, \qquad \dot{y}_2 = -\gamma^{(1)}\frac{x_1 - x_2}{R^2}. \qquad (6)$$

Note that

$$\frac{\partial R}{\partial x_i} = \frac{\partial R}{\partial v}\frac{\partial v}{\partial x_i} = \frac{x_i - x_j}{R^2}, \qquad\qquad \frac{\partial R}{\partial y_i} = \frac{y_i - y_j}{R^2}$$

so

$$[H, R] = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(\frac{\partial H}{\partial x_i}\frac{\partial v}{\partial y_i} - \frac{\partial H}{\partial y_i}\frac{\partial v}{\partial x_i}\right) = \sum_{i=1}^{n}\frac{1}{\gamma^{(i)}}\left(-\gamma^{(i)}\gamma^{(j)}\frac{x_i - x_j}{R^2}\frac{y_i - y_j}{R^2} + \gamma^{(i)}\gamma^{(j)}\frac{y_i - y_j}{R^2}\frac{x_i - x_j}{R^2}\right) = 0.$$

This means R is a constant.

Define $\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2 = (X, Y)$, where $|\mathbf{R}| = R$. Now we have two generalized coordinates X and Y, where $X = x_1 - x_2$ and $Y = y_1 - y_2$. This gives us the two equations of motion

$$\dot{X} = \dot{x}_1 - \dot{x}_2 = -(\gamma^{(1)} + \gamma^{(2)})\frac{y_1 - y_2}{R^2} = -\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}Y, \tag{7}$$

$$\dot{Y} = \dot{y}_1 - \dot{y}_2 = (\gamma^{(1)} + \gamma^{(2)})\frac{x_1 - x_2}{R^2} = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}X. \tag{8}$$

We can differentiate these to obtain two uncoupled second-order equations:

$$\ddot{X} = -\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}\dot{Y} = -\frac{(\gamma^{(1)} + \gamma^{(2)})^2}{R^4}X, \qquad \ddot{Y} = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}\dot{X} = -\frac{(\gamma^{(1)} + \gamma^{(2)})^2}{R^4}Y.$$

These equations have the solutions

$$X(t) = C_1\cos\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right) + C_2\sin\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right), \quad Y(t) = D_1\cos\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right) + D_2\sin\left(\frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}t\right),$$

where C_1, C_2, D_1, D_2 are constants. Define

$$\omega = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}$$

as the angular frequency. Then

$$\dot{X}(t) = -C_1\omega\sin\omega t + C_2\omega\cos\omega t = -\omega Y, \qquad \dot{Y}(t) = -D_1\omega\sin\omega t + D_2\omega\cos\omega t = \omega X,$$

where we have used (7) and (8). This fixes

$$C_1 = D_2, \qquad\qquad C_2 = -D_1, \qquad\qquad -D_1 = C_2, \qquad\qquad D_2 = C_1$$

which implies $C_1 = D_2 = 0$ and $C_2 = D_1 \equiv K$, where we have defined some new constant K. Finally, we have

$$X(t) = K\sin\omega t = x_1(t) - x_2(t), \qquad\qquad Y(t) = K\cos\omega t = y_1(t) - y_2(t).$$

Thus, we have shown that the equations of motion (6) can be solved explicitly. Since $\mathbf{R} = (X, Y)$ is the vector separating the vortices, we have also shown that they orbit each other. The orbital frequency

$$\omega = \frac{\gamma^{(1)} + \gamma^{(2)}}{R^2}$$

is clearly inversely proportional to R^2, where $R = |\mathbf{R}|$ is the magnitude of the vortices' separation.

Problem 6.   Conserved quantities for a system of line vortices

Now consider the general case of n vortices.

6.a   Verify that the "total linear momentum along x" and the "total linear momentum along y,"

$$P_x = \sum_{i=1}^{n}\gamma^{(i)}y_i, \qquad\qquad\qquad\qquad P_y = \sum_{i=1}^{n}-\gamma^{(i)}x_i,$$

are conserved.

6.b Verify that $[P_x, P_y]$ gives a conserved quantity.

Problem 7. Charged particle in a magnetic field

Suppose a charged particle moves in a two-dimensional plane while experiencing a magnetic field $\mathbf{B} = (0, 0, B)$. Use the vector potential $\mathbf{A} = (-By, 0, 0)$. The Hamiltonian for the particle is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{p_y^2}{2m}.$$

7.a Write down Hamilton's equations. Verify that by appropriate manipulation we have

$$p_y + \frac{eB}{c}x = a, \qquad\qquad p_x = m\dot{x} - \frac{eB}{c}y = b,$$

where a and b are constants.

Solution. Note that

$$H = \frac{1}{2m}\left(p_x^2 + 2\frac{eB}{c}p_x y + \frac{e^2 B^2}{c^2}y^2\right) + \frac{p_y^2}{2m} = \frac{p_x^2}{2m} + \frac{eB}{c}\frac{p_x y}{m} + \frac{e^2 B^2}{c^2}\frac{y^2}{2m} + \frac{p_y^2}{2m}$$

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{eB}{c}y\right), \tag{9}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \tag{10}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \tag{11}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{eB}{c}\frac{1}{m}\left(p_x + \frac{eB}{c}y\right). \tag{12}$$

Substituting (9) into (12),

$$\dot{p}_y = -\frac{eB}{c}\dot{x}.$$

By integrating with respect to t, we obtain

$$p_y = -\frac{eB}{c}\int \dot{x}\,dt = -\frac{eB}{c}x + a,$$

where a is some constant. Therefore, we have

$$p_y + \frac{eB}{c}x = a, \tag{13}$$

as desired.

From (9),

$$m\dot{x} = p_x + \frac{eB}{c}y \iff p_x = m\dot{x} - \frac{eB}{c}y,$$

and from (10),

$$p_x = \int 0\,dt = b,$$

where b is some constant. Combining these, we have

$$p_x = m\dot{x} - \frac{eB}{c}y = b \tag{14}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7.b Using the relations above and the equations of motion, verify that the charged particle moves in a circle in the (x, y) plane and that the circling frequency ω is given by

$$\omega = \frac{eB}{mc}.$$

This is called the *Larmor frequency.*

Solution. Substituting (14) into (9) yields

$$\dot{x} = \frac{1}{m}\left(b + \frac{eB}{c}y\right) = \frac{eB}{mc}\left(\frac{c}{eB}b + y\right). \tag{15}$$

Similarly, solving (13) for p_y and substituting into (11) gives us

$$\dot{y} = \frac{1}{m}\left(a - \frac{eB}{c}x\right) = \frac{eB}{mc}\left(\frac{c}{eB}a - x\right). \tag{16}$$

(15) and (16) are a system of two coupled first-order equations. Differentiating each by t, we obtain two uncoupled second-order equations:

$$\ddot{x} = \frac{eB}{mc}\dot{y} = -\frac{e^2B^2}{m^2c^2}\left(x - \frac{c}{eB}a\right), \qquad\qquad \ddot{y} = -\frac{eB}{mc}\dot{x} = -\frac{e^2B^2}{m^2c^2}\left(y + \frac{c}{eB}b\right). \tag{17}$$

Define

$$\tilde{x} \equiv x - \frac{c}{eB}a, \qquad\qquad\qquad\qquad \tilde{y} \equiv y + \frac{c}{eB}b.$$

Then

$$\frac{d\tilde{x}}{dt} = \dot{x}, \qquad\qquad \frac{d^2\tilde{x}}{dt^2} = \ddot{x}, \qquad\qquad \frac{d\tilde{y}}{dt} = \dot{y}, \qquad\qquad \frac{d^2\tilde{y}}{dt^2} = \ddot{y},$$

and the equations (17) can be rewritten in terms of $\tilde{x}$ and $\tilde{y}$:

$$\frac{d^2\tilde{x}}{dt^2} = -\frac{e^2B^2}{m^2c^2}\tilde{x}, \qquad\qquad\qquad\qquad \frac{d^2\tilde{y}}{dt^2} = -\frac{e^2B^2}{m^2c^2}\tilde{y}.$$

These equations have the solutions

$$\tilde{x}(t) = C_1\cos\left(\frac{eB}{mc}t\right) + C_2\sin\left(\frac{eB}{mc}t\right) \equiv C_1\cos(\omega t) + C_2\sin(\omega t),$$

$$\tilde{y}(t) = D_1\cos\left(\frac{eB}{mc}t\right) + D_2\sin\left(\frac{eB}{mc}t\right) \equiv D_1\cos(\omega t) + D_2\sin(\omega t),$$

where C_1, C_2, D_1, D_2 are constants, and we have defined

$$\omega \equiv \frac{eB}{mc}.$$

Note that

$$\frac{d\tilde{x}}{dt} = -C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t), \qquad\qquad \frac{d\tilde{y}}{dt} = -D_1\omega\sin(\omega t) + D_2\omega\cos(\omega t).$$

Applying (15) and (16),

$$-C_1\omega\sin(\omega t) + C_2\omega\cos(\omega t) = \omega\left[D_1\cos(\omega t) + D_2\sin(\omega t)\right],$$
$$-D_1\omega\sin(\omega t) + D_2\omega\cos(\omega t) = \omega\left[C_1\cos(\omega t) + C_2\sin(\omega t)\right]$$

which imply

$$-C_1 = D_2, \qquad\qquad C_2 = D_1, \qquad\qquad -D_1 = C_2, \qquad\qquad D_2 = C_1$$

7.c Now consider the limit where the B field can be made arbitrarily strong. Compare the Poisson bracket $[x, p_x]$ for the charged particle with the Poisson bracket relation

$$[x_i, y_i] = \frac{\delta_{ij}}{\gamma^{(i)}}$$

for the system of line vortices described in problems 5 and 6.

Solution. The Poisson bracket for the charged particle is given by

$$[x, p_x] = \frac{\partial x}{\partial x}\frac{\partial p_x}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial p_x}{\partial x} + \frac{\partial x}{\partial y}\frac{\partial p_x}{\partial p_y} - \frac{\partial x}{\partial p_y}\frac{\partial p_x}{\partial y}.$$

Note that

$$\frac{\partial x}{\partial x} = \frac{\partial p_x}{\partial p_x} = 1, \qquad \frac{\partial x}{\partial p_x} = \frac{\partial p_x}{\partial x} = \frac{\partial x}{\partial y} = \frac{\partial p_x}{\partial p_y} = 0, \qquad \frac{\partial x}{\partial p_y} = -\frac{c}{eB}, \qquad \frac{\partial p_x}{\partial y} = -\frac{eB}{c}.$$

Then

$$[x, p_x] = 1 - 1 = 0.$$

It's different?

While writing up these solutions, I consulted Gelfand and Fomin's *Calculus of Variations*, Goldstein's *Classical Mechanics*, Tong's *Classical Dynamics*, and Riley, Hobson, and Bence's *Mathematical Methods for Physics and Engineering*.