Problem 1. Consider a dielectric ball of radius R with dielectric constant ϵ. Obtain a multipole expansion for the field, $\phi(\mathbf{x})$, of a point charge q placed at a point $\mathbf{x}'$ with $|\mathbf{x}'| = d > R$ (so the charge is outside of the dielectric ball).

Hint: Follow the procedure we used in class to find the multipole expansion of a point charge without the dielectric, but now consider the three regions $r \leq R$, $R \leq r \leq d$, and $r \geq d$. Obtain the form of the solution in these regions and match suitably.

Solution. In class, we derived the multipole expansion for $|\mathbf{x}| \geq R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only within $|\mathbf{x}'| \leq R$. We can find an equivalent expression for the reverse situation (within $|\mathbf{x}| \leq R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only for $|\mathbf{x}'| \geq R$) using the spherical harmonic expansion of the Green's function $G(\mathbf{x}, \mathbf{x}')$ in Eq. (2.78):

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \begin{cases} \sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y^*_{lm}(\theta', \varphi')\, Y_{lm}(\theta, \varphi) & \text{if } r < r', \\ \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\theta', \varphi')\, Y_{lm}(\theta, \varphi) & \text{if } r > r'. \end{cases}$$

As in Eq. (2.79) in the course notes, we integrate and obtain

$$\phi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}')\, \rho(\mathbf{x}')\, d^3x' = \sum_{l,m} \frac{4\pi}{2l+1} r^l Y_{lm}(\theta, \varphi) \int \frac{\rho(\mathbf{x}')}{r'^{l+1}} Y^*_{lm}(\theta', \varphi')\, d^3x' \,.$$

Combining this with the result of Eq. (2.79), we have

$$\phi(\mathbf{x}) = \begin{cases} \sum_{l,m} \frac{4\pi}{2l+1} r^l\, q'_{lm}\, Y_{lm}(\theta, \varphi) & \text{if } r < r' \text{ and } \rho(\mathbf{x}')(r) = 0, \\ \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi) & \text{if } r > r' \text{ and } \rho(\mathbf{x}')(r) = 0, \end{cases} \tag{1}$$

where

$$q_{lm} \equiv \int \rho(\mathbf{x}')\, r'^l\, Y^*_{lm}(\theta', \varphi')\, d^3x' \,, \qquad q'_{lm} \equiv \int \frac{\rho(\mathbf{x}')}{r'^{l+1}} Y^*_{lm}(\theta', \varphi')\, d^3x' \,,$$

from Eq. (2.80) and our derivation. Additionally, the spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi},$$

and the associated Legendre polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1 - x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l.$$

Poisson's equation inside a dielectric medium is given by Eq. (3.22),

$$\nabla^2 \langle \phi \rangle = -\frac{4\pi}{\epsilon} \langle \rho_f \rangle \,,$$

where ρ_f is the free charge density. For this problem, $\rho_f = 0$ since the point charge is outside the dielectric.

Without loss of generality, we may choose the location of the point charge to be on the z axis at $z = d$, so $\mathbf{x}' = (r', 0, 0)$. We will begin inside the dielectric, where $r \leq R$. We need a solution to Laplace's equation, which is the first case of (1), with a factor of $1/\epsilon$ inserted to account for the dielectric constant:

$$\langle \phi \rangle(\mathbf{x}) = \frac{1}{\epsilon} \sum_{l,m} A_{lm} \frac{4\pi}{2l+1} r^l\, q'_{lm}\, Y_{lm}(\theta, \varphi) = \frac{1}{\epsilon} \sum_{l} A_l \frac{4\pi}{2l+1} r^l\, q'_{l0}\, Y_{l0}(\theta, \varphi) \quad \text{if } r \leq R, \tag{2}$$

where A_l are constants, and $m = 0$ because the system is azimuthally symmetric. Then

$$\begin{aligned}
q'_{l0} &= \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{l!}{l!}}\int \frac{\rho(\mathbf{x}')}{r'^{l+1}} P_l^0(\cos\theta' = 1)\, d^3x' \\
&= \sqrt{\frac{2l+1}{4\pi}}\frac{1}{2^l l!}\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty} \frac{\delta(d-r')}{r'^{l+1}}\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 r'^2\, dr'\, d(\cos\theta')\, d\varphi' \\
&= \sqrt{\frac{2l+1}{4\pi}}\frac{1}{2^l l!}\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 \int_0^{2\pi} d\varphi' \int_{-1}^{1} d(\cos\theta') \int_0^{\infty}\frac{\delta(d-r')}{r'^{l-1}}\, dr' \\
&= \sqrt{\frac{2l+1}{4\pi}}\frac{1}{2^l l!}\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 \left[\varphi\right]_0^{2\pi}\left[\cos\theta'\right]_{-1}^{1}\frac{1}{d^{l-1}} \\
&= \frac{\sqrt{4\pi(2l+1)}}{2^l l!}\frac{1}{d^{l-1}}\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 .
\end{aligned}$$

In the region $R \le d \le r$, we are in the same regime as the former situation with respect to the position of the charge. However, we now in free space, where $\langle\phi\rangle = \phi$ and we no longer need the factor of $1/\epsilon$. Then

$$\phi(\mathbf{x}) = \sum_l B_l \frac{4\pi}{2l+1} r^l\, q'_{l0}\, Y_{l0}(\theta,\varphi) \quad \text{if } R \le r \le d, \tag{3}$$

where B_l are constants.

In the region $r > d$, we need to use the second case of (1). Once again taking advantage of the the azimuthal symmetry, his gives us

$$\phi(\mathbf{x}) = \sum_l C_l \frac{4\pi}{2l+1}\frac{q_{l0}}{r^{l+1}} Y_{l0}(\theta,\varphi) \quad \text{if } r \ge d, \tag{4}$$

where C_l are constants, and

$$\begin{aligned}
q_{l0} &= \sqrt{\frac{2l+1}{4\pi}}\int \rho(\mathbf{x}')\, r'^l\, P_l^0(\cos\theta' = 1)\, d^3x' \\
&= \sqrt{\frac{2l+1}{4\pi}}\frac{1}{2^l l!}\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 \int_0^{2\pi} d\varphi' \int_{-1}^{1} d(\cos\theta') \int_0^{\infty}\delta(d-r')r'^{l+2}\, dr' \\
&= \frac{\sqrt{4\pi(2l+1)}}{2^l l!} d^{l+2}\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 .
\end{aligned}$$

Now we must match (2) and (3) at $r = R$. Evaluating (2), we have

$$\langle\phi\rangle(r=R) = \frac{4\pi}{\epsilon}\sum_l A_l \sqrt{\frac{4\pi}{2l+1}}\frac{1}{2^l l!}\frac{R^l}{d^{l-1}} Y_{l0}(\theta,\varphi)\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 ,$$

and for (3), we have

$$\phi(r=R) = 4\pi\sum_l B_l \sqrt{\frac{4\pi}{2l+1}}\frac{1}{2^l l!}\frac{R^l}{d^{l-1}} Y_{l0}(\theta,\varphi)\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 .$$

Equating these gives us $A_l = \epsilon B_l$.

We must also match (3) and (**??**) at $r = d$. Evaluating (3), we have

$$\phi(r=d) = 4\pi d\sum_l B_l \sqrt{\frac{4\pi}{2l+1}}\frac{1}{2^l l!} Y_{l0}(\theta,\varphi)\left[\frac{d^l}{d(\cos\theta')^l}(\cos^2\theta'-1)^l\right]_1 ,$$

and for (**??**), we have

$$\phi(r=d) = 4\pi d \sum_l C_l \sqrt{\frac{4\pi}{2l+1}} \frac{1}{2^l l!} Y_{l0}(\theta,\varphi) \left[\frac{d^l}{d(\cos\theta')^l} (\cos^2\theta' - 1)^l \right]_1 ,$$

which just seems plan wrong :(

Problem 2. A dielectric ball of radius R and dielectric constant ϵ is placed in the external electrostatic potential $\phi_0 = \alpha(2z^2 - x^2 - y^2)$ where α is a constant, with the center of the ball at $\mathbf{x} = 0$.

2.a Find the total electrostatic potential ϕ everywhere.

Hint: It is useful to note that the external potential is proportional to $r^2 Y_{20}(\theta, \varphi)$. This should allow you to determine/guess the form of the total potential inside and outside the dielectric up to unknown constants, which can then be determined by matching.

2.b Calculate the interaction energy between the field produced by the dielectric and the external field. Assume that the potential arises from "distant charges" so that the formula for $\mathcal{E}_{\text{int}}$ given in class and the notes can be used.

2.c Calculate the total force needed to hold the dielectric ball in place.

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics* while writing up these solutions.