1 Problem 1

Let's consider coherent states of a one-dimensional quantum particle with mass m confined in a one-dimensional harmonic potential $V(x) = m\omega^2 x^2/2$:

$$a\left|\lambda\right\rangle = \lambda\left|\lambda\right\rangle, \qquad \left|\lambda\right\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\exp\left(\lambda a^\dagger\right)\left|0\right\rangle.$$

Here, λ is a complex paramter.

1.1 Compute $\langle x|\lambda\rangle$.

Solution. In terms of the position and momentum operators X and P,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right),$$

so

$$\langle x|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\left\langle x\right|\exp\left(\lambda a^\dagger\right)\left|0\right\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\left\langle x\right|\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\}\left|0\right\rangle. \tag{1}$$

Note that for two operators A and B, $e^{A+B} = e^{-[A,B]/2}e^A e^B$ if $[A,B]$ commutes with each A and B. Note also that

$$\left[X, -\frac{iP}{m\omega}\right] = -\frac{i}{m\omega}[X,P] = \frac{\hbar}{m\omega}.$$

Thus,

$$\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\} = \exp\left(-\frac{\hbar\lambda}{2m\omega}\sqrt{\frac{m\omega}{2\hbar}}\right)\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)\exp\left(-\frac{i\lambda}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right)$$

so (1) becomes

$$\begin{aligned}\langle x|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(-\frac{\hbar\lambda}{2m\omega}\right)\left\langle x\right|\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)\exp\left(-\frac{i\lambda}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right)\left|0\right\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(-\frac{\hbar\lambda}{2m\omega}\right)\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}x\right)\exp\left(-\frac{\hbar\lambda}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}\frac{\partial}{\partial x}\right)\langle x|0\rangle\end{aligned}$$

From (2.3.30) in Sakurai,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

1.2 Compute $\langle\lambda|x|\lambda\rangle$, $\langle\lambda|p|\lambda\rangle$, $\langle\lambda|x^2|\lambda\rangle$, and $\langle\lambda|p^2|\lambda\rangle$. Also, compute $\left\langle(\Delta x)^2\right\rangle_\lambda\left\langle(\Delta p)^2\right\rangle_\lambda$ where $\Delta A = A\left\langle A\right\rangle$.