

1 Problem 1

Let's consider coherent states of a one-dimensional quantum particle with mass m confined in a one-dimensional harmonic potential $V(x) = m\omega^2 x^2/2$:

$$a|\lambda\rangle = \lambda|\lambda\rangle, \quad |\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right) \exp(\lambda a^\dagger)|0\rangle.$$

Here, λ is a complex parameter.

1.1 Compute $\langle x|\lambda\rangle$.

Solution. In terms of the position and momentum operators X and P ,

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(X + \frac{iP}{m\omega} \right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(X - \frac{iP}{m\omega} \right),$$

so

$$\begin{aligned} \langle x|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right) \langle x|\exp(\lambda a^\dagger)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \langle x|\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\}|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \langle x|\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)\exp\left(-\lambda\sqrt{\frac{m\omega}{2\hbar}}\frac{iP}{m\omega}\right)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}x\right) \langle x|\exp\left(-\frac{i\lambda}{\sqrt{2\hbar m\omega}}P\right)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right) \exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}x\right) \exp\left(-\frac{\hbar\lambda}{\sqrt{2\hbar m\omega}}\frac{\partial}{\partial x}\right) \langle x|0\rangle. \end{aligned}$$

From (2.3.30) in Sakurai,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right),$$

so

$$\langle x|\lambda\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{|\lambda|^2}{2}\right) \exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}x\right) \exp\left(-\frac{\hbar\lambda}{\sqrt{2\hbar m\omega}}\frac{\partial}{\partial x}\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

???

1.2 Compute $\langle \lambda|x|\lambda\rangle$, $\langle \lambda|p|\lambda\rangle$, $\langle \lambda|x^2|\lambda\rangle$, and $\langle \lambda|p^2|\lambda\rangle$. Also, compute $\langle (\Delta x)^2 \rangle_\lambda \langle (\Delta p)^2 \rangle_\lambda$ where $\Delta A = A \langle A \rangle$.