1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1}$$

1.1 Verify the following:

a. $i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle$,

b. $i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle$,

c. $i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle = -\dfrac{\hbar^2}{2m}\big(\langle\Phi(t)|x\rangle\, \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\big)$,

d. $\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle +\ \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle = \dfrac{\hbar}{i}\left(\langle\Phi(t)|x\rangle\, \partial_x \langle x|\Psi(t)\rangle - (\partial_x \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right)$

e. $\dfrac{\hbar}{i}\partial_x \left[\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle +\ \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - mel\Phi(t)p^2x \langle x|\Psi(t)\rangle$

Solution.

a. Beginning with Schrödinger's equation, note that

$$\begin{aligned} i\hbar\partial_t |\Psi(t)\rangle &= H |\Psi(t)\rangle & (2)\\ i\hbar\partial_t \langle x|\Psi(t)\rangle &= \langle x|H|\Psi(t)\rangle & (3)\\ (i\hbar\partial_t \langle x|\Psi(t)\rangle)^* &= (\langle x|H|\Psi(t)\rangle)^* & (4)\\ -i\hbar\partial_t \langle\Psi(t)|x\rangle &= \langle\Psi(t)|H|x\rangle & (5)\\ i\hbar\partial_t \langle\Psi(t)|x\rangle &= -\langle\Psi(t)|H|x\rangle\,, & (6) \end{aligned}$$

where in going to (5) we have used the fact that H is Hermitian. □

b. Beginning with what was proven in (a),

$$\begin{aligned} i\hbar\partial_t \langle\Phi(t)|x\rangle &= -\langle\Phi(t)|H|x\rangle & (7)\\ i\hbar(\partial_t \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle &= -\langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle\,. & (8) \end{aligned}$$

From (3), we can write

$$\langle\Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle\,. \tag{9}$$

Adding (15) and (17) yields

$$\begin{aligned} \langle\Phi(t)|x\rangle\, i\hbar\partial_t \langle x|\Psi(t)\rangle + i\hbar(\partial_t \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle & (10)\\ i\hbar\partial_t \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle -\ \langle\Phi(t)|H|x\rangle \langle x|\Psi(t)\rangle\,, & (11) \end{aligned}$$

where in going to (11) we have used the product rule of differentiation. □

c. Using (1), note that:

$$\begin{aligned}
\langle x|H|\Psi(t)\rangle &= \langle x|\left(\frac{p^2}{2m}+V(x)\right)|\Psi(t)\rangle && (12)\\
&= \frac{1}{2m}\langle x|p^2|\Psi(t)\rangle + \langle x|V(x)|\Psi(t)\rangle && (13)\\
&= \frac{(-i\hbar\partial_x)^2}{2m}\langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle && (14)\\
&= -\frac{\hbar^2}{2m}\partial_x^2\langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle\,, && (15)
\end{aligned}$$

where in going to (14) we have used the fact that

$$\langle x|p|\Psi(x)\rangle = -i\hbar\partial_x\langle x|\Psi(t)\rangle\,. \tag{16}$$

Similarly, note that

$$\langle\Phi(t)|H|x\rangle = -\frac{\hbar^2}{2m}\partial_x^2\langle\Phi(t)|x\rangle + V(x)\langle\Phi(t)|x\rangle \tag{17}$$

where we have used

$$\langle\Phi(t)|p|x\rangle = i\hbar\partial_x\langle\Phi(t)|x\rangle\,. \tag{18}$$

Making the substitutions (15) and (17) into what was proven in (b),

$$\begin{aligned}
i\hbar\partial_t\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle\left(-\frac{\hbar^2}{2m}\partial_x^2\langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle\right)\\
&\quad-\left(-\frac{\hbar^2}{2m}\partial_x^2\langle\Phi(t)|x\rangle + V(x)\langle\Phi(t)|x\rangle\right)\langle x|\Psi(t)\rangle && (19)\\
&= -\frac{\hbar^2}{2m}\big(\langle\Phi(t)|x\rangle\,\partial_x^2\langle\Phi(t)|x\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\big)\\
&\quad+V(x)\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle - V(x)\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle && (20)\\
&= -\frac{\hbar^2}{2m}\big(\langle\Phi(t)|x\rangle\,\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\big)\,, && (21)
\end{aligned}$$

as we sought to prove. □

d. Applying (18) and (16),

$$\begin{aligned}
\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle(-i\hbar\partial_x\langle x|\Psi(t)\rangle) + (i\hbar\partial_x\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle && (22)\\
&= \frac{\hbar}{i}\left[\langle\Phi(t)|x\rangle\,\partial_x\langle x|\Psi(t)\rangle - (\partial_x\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right] && (23)
\end{aligned}$$

as we sought to prove. □

e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle) = (\partial_x\langle\Phi(t)|x\rangle)\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\,\partial_x\langle x|p|\Psi(t)\rangle \tag{24}$$

Multiplying through by $\hbar/i$,

$$\begin{aligned}
\frac{\hbar}{i}\partial_x(\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle) &= (-i\hbar\partial_x\langle\Phi(t)|x\rangle)\langle x|p|\Psi(t)\rangle - \langle\Phi(t)|x\rangle\, i\hbar\partial_x\langle x|p|\Psi(t)\rangle && (25)\\
&= -\langle\Phi(t)|p|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle\,, && (26)
\end{aligned}$$

where in going to (26) we have used (18) and (16). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i}\partial_x(\,\langle\Phi(t)|p|x\rangle\,\langle x|\Psi(t)\rangle) = (-i\hbar\partial_x\,\langle\Phi(t)|p|x\rangle)\,\langle x|\Psi(t)\rangle - \,\langle\Phi(t)|p|x\rangle\,i\hbar\partial_x\,\langle x|\Psi(t)\rangle \tag{27}$$

$$= -\,\langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle + \,\langle\Phi(t)|p|x\rangle\,\langle x|p|\Psi(t)\rangle\,. \tag{28}$$

Adding the results of (**??**) and (**??**),

$$\frac{\hbar}{i}\partial_x\left[\langle\Phi(t)|x\rangle\,\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\,\langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle\,\langle x|p^2|\Psi(t)\rangle - \,\langle\Phi(t)|p|x\rangle\,\langle x|p|\Psi(t)\rangle$$

$$+\,\langle\Phi(t)|p|x\rangle\,\langle x|p|\Psi(t)\rangle - \,\langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle \tag{29}$$

$$= \langle\Phi(t)|x\rangle\,\langle x|p^2|\Psi(t)\rangle - \,\langle\Phi(t)|p^2|x\rangle\,\langle x|\Psi(t)\rangle \tag{30}$$

as we sought to prove. $\square$