Problem 1. Consider a dielectric ball of radius R with dielectric constant ϵ. Obtain a multipole expansion for the field, $\phi(\mathbf{x})$, of a point charge q placed at a point $\mathbf{x}'$ with $|\mathbf{x}'| = d > R$ (so the charge is outside of the dielectric ball).

Hint: Follow the procedure we used in class to find the multipole expansion of a point charge without the dielectric, but now consider the three regions $r \leq R$, $R \leq r \leq d$, and $r \geq d$. Obtain the form of the solution in these regions and match suitably.

Solution. The spherical harmonic expansion of the Green's function $G(\mathbf{x}, \mathbf{x}')$ is given by Eq. (2.78):

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \begin{cases} \displaystyle\sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y_{lm}^*(\theta', \varphi')\, Y_{lm}(\theta, \varphi) & \text{if } r < r', \\ \displaystyle\sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \varphi')\, Y_{lm}(\theta, \varphi) & \text{if } r > r'. \end{cases} \tag{1}$$

The spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi},$$

and the associated Legendre polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1 - x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l.$$

We will assume the dielectric is linear, homogeneous, and isotropic. Poisson's equation inside such a dielectric is given by Eq. (3.22) in the course notes,

$$\nabla^2 \langle \phi \rangle = -\frac{4\pi}{\epsilon} \langle \rho_f \rangle,$$

where ρ_f is the free charge density. Here, $\langle \rho_f \rangle = 0$ since there are no free charges within the dielectric, so this reduces to Laplace's equation. The general solution to Laplace's equation is given by Eq. (3.61) in Jackson,

$$\langle \phi \rangle(r, \theta, \varphi) = \sum_{l,m} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \varphi), \tag{2}$$

where A_{lm} and B_{lm} are constant coefficients.

We will begin inside the dielectric, where $r \leq R$. Here we must have $B_{lm} = 0$ because $1/r^{l+1}$ is undefined at the origin. Without loss of generality, we may choose the location of the point charge to be on the z axis at $z = d$, so $\mathbf{x}' = (d, 0, 0)$. Clearly, the system is azimuthally symmetric, so $m = 0$. This gives us the macroscopically averaged potential

$$\langle \phi \rangle(r, \theta, \varphi) = \sum_l A_l r^l\, Y_{l0}(\theta, \varphi) = \sum_l \sqrt{\frac{2l+1}{4\pi}} A_l r^l P_l(\cos\theta) \quad \text{if } r \leq R. \tag{3}$$

In the region $R \leq d \leq r$, we are in free space so $\langle \phi \rangle = \phi$. The point charge is at greater r, so we account for its contribution using the first case of (1). Additionally, there are multipole contributions from the dielectric at lesser r, which we must account for as well. This is similar to the problem of a point charge outside a conducting sphere, so we can use the method of images to find the dielectric contribution in this regime. However, since we are working with a dielectric and not a conductor, in this case the image charge will not have a charge of exactly q. Therefore, we need to assign coefficients to this contribution and match at the boundary to obtain the correct expression of the potential due to the dielectric.

The Dirichlet Green's function for a spherical cavity is Eq. (2.91) in the lecture notes:

$$G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{\alpha}{|\mathbf{x}-\mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}'\frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \alpha = -\frac{R}{|\mathbf{x}'|}.$$

Here, the second term is the same as the Green's function for an image charge inside the conducting sphere. We will use this term to find the dielectric ball's contribution to the potential. Adapting the second case of (1) to this case, we obtain

$$G'(\mathbf{x},\mathbf{x}') = \frac{\alpha}{|\mathbf{x}-\mathbf{x}''|} = -\sum_{l,m}\frac{4\pi}{2l+1}\frac{R^{2l+1}}{r'^{l+1}r^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi) \quad \text{if } r \le R.$$

Putting these together, and again taking advantage of the azimuthal symmetry, we have

$$\begin{aligned}\phi(r,\theta,\phi) &= q\sum_{l,m}\frac{4\pi}{2l+1}\frac{r^l}{d^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi) - \sum_{l,m}B_{lm}\frac{4\pi}{2l+1}\frac{R^{2l+1}}{d^{l+1}r^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\\ &= \sum_{l,m}\frac{4\pi}{2l+1}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\frac{1}{d^{l+1}}\left(qr^l - B_{lm}\frac{R^{2l+1}}{r^{l+1}}\right)\\ &= \sum_l \frac{P_l(\cos\theta)}{d^{l+1}}\left(qr^l - B_l\frac{R^{2l+1}}{r^{l+1}}\right) \qquad \text{if } R \le r \le d, \quad (4)\end{aligned}$$

where $\cos\theta' = 1$ due to our choice of coordinates, and $P_l(1) = 1$ because the Legendre polynomials are normalized.

In the region $r \ge d$, we account for the point charge using the second case of (1), and for the dielectric using the same methods as above. With the azimuthal symmetry, this gives us

$$\begin{aligned}\phi(r,\theta,\phi) &= q\sum_{l,m}\frac{4\pi}{2l+1}\frac{d^l}{r^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi) - \sum_{l,m}C_{lm}\frac{4\pi}{2l+1}\frac{R^{2l+1}}{d^{l+1}r^{l+1}}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\\ &= \sum_{l,m}\frac{4\pi}{2l+1}Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\frac{1}{r^{l+1}}\left(qd^l - C_l\frac{R^{2l+1}}{d^{l+1}}\right)\\ &= \sum_l \frac{P_l(\cos\theta)}{r^{l+1}}\left(qd^l - C_l\frac{R^{2l+1}}{d^{l+1}}\right) \qquad \text{if } r \ge d. \quad (5)\end{aligned}$$

Now we must match $\langle\phi\rangle$ at the boundaries of each region. Inspecting (4) and (5), it is obvious that $B_l = C_l$. For (3) and (4), we must match at $r = R$. Evaluating at this boundary,

$$\langle\phi\rangle(R,\theta,\phi) = \sum_l P_l(\cos\theta)R^l\begin{cases}\sqrt{\dfrac{2l+1}{4\pi}}A_l & \text{if } r \le R,\\[2ex] \dfrac{q-B_l}{d^{l+1}} & \text{if } R \le r \le d.\end{cases}$$

Equating these gives us

$$A_l = \sqrt{\frac{4\pi}{2l+1}}\frac{q-B_l}{d^{l+1}}. \qquad (6)$$

Here we must also match $\hat{\mathbf{n}} \cdot \langle \mathbf{D} \rangle$ at the boundary, where

$$\langle \mathbf{D} \rangle = \epsilon \langle \mathbf{E} \rangle \tag{7}$$

inside the dielectric, from Eq. (3.20) in the course notes. (In vacuum, $\mathbf{D} = \mathbf{E}$.) Here $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so we are only concerned with the r component of $\langle \mathbf{E} \rangle$. Applying $\langle \mathbf{E} \rangle = -\boldsymbol{\nabla} \langle \phi \rangle$ to (3) and (4) gives us

$$\langle E_r \rangle (R, \theta, \phi) = -\sum_l P_l(\cos\theta)\, R^{l-1} \begin{cases} \sqrt{\dfrac{2l+1}{4\pi}}\, l A_l & \text{if } r \leq R, \\ \dfrac{lq + (l+1)B_l}{d^{l+1}} & \text{if } R \leq r \leq d. \end{cases}$$

Then we stipulate that

$$\hat{\mathbf{r}} \cdot \langle \mathbf{D} \rangle (R, \theta, \phi) = -\epsilon \sum_l l \sqrt{\frac{2l+1}{4\pi}} A_l P_l(\cos\theta)\, R^{l-1} = -\sum_l l \frac{lq + (l+1)B_l}{d^{l+1}} P_l(\cos\theta)\, R^{l-1},$$

which implies

$$A_l = \frac{1}{\epsilon} \sqrt{\frac{4\pi}{2l+1}} \frac{lq + (l+1)B_l}{l d^{l+1}}. \tag{8}$$

By equating (6) and (8), we can solve for B_l:

$$q - B_l = \frac{lq + (l+1)B_l}{\epsilon l} \implies l(\epsilon - 1)q = (\epsilon l + l + 1)B_l \implies B_l = \frac{l(\epsilon - 1)}{\epsilon l + l + 1} q.$$

Feeding this back into (6),

$$A_l = \sqrt{\frac{4\pi}{2l+1}} \frac{1}{d^{l+1}} \left(1 - \frac{l(\epsilon - 1)}{\epsilon l + l + 1}\right) q = \sqrt{\frac{4\pi}{2l+1}} \frac{2l+1}{\epsilon l + l + 1} \frac{q}{d^{l+1}} = \frac{\sqrt{4\pi(2l+1)}}{\epsilon l + l + 1} \frac{q}{d^{l+1}}.$$

Substituting in all of the coefficients, (3), (4), and (5) can be written as

$$\langle \phi \rangle (r, \theta, \phi) = q \sum_l P_l(\cos\theta) \begin{cases} \dfrac{2l+1}{\epsilon l + l + 1} \dfrac{r^l}{d^{l+1}} & \text{if } r \leq R, \\ \dfrac{l(1-\epsilon)}{\epsilon l + l + 1} \dfrac{R^{2l+1}}{d^{l+1}} \dfrac{1}{r^{l+1}} + \dfrac{r^l}{d^{l+1}} & \text{if } R \leq r \leq d, \\ \dfrac{l(1-\epsilon)}{\epsilon l + l + 1} \dfrac{R^{2l+1}}{d^{l+1}} \dfrac{1}{r^{l+1}} + \dfrac{d^l}{r^{l+1}} & \text{if } r \geq d. \end{cases}$$

Problem 2. A dielectric ball of radius R and dielectric constant ϵ is placed in the external electrostatic potential $\phi_0 = \alpha(2z^2 - x^2 - y^2)$ where α is a constant, with the center of the ball at $\mathbf{x} = 0$.

2.a Find the total electrostatic potential ϕ everywhere.

Hint: It is useful to note that the external potential is proportional to $r^2\,Y_{20}(\theta, \varphi)$. This should allow you to determine/guess the form of the total potential inside and outside the dielectric up to unknown constants, which can then be determined by matching.

Solution. Firstly, note that

$$Y_{20}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1),$$

and so

$$\phi_0 = \alpha r^2(3\cos^2\theta - 1) = 4\alpha r^2\sqrt{\frac{\pi}{5}}Y_{20}(\theta,\varphi) \equiv \beta r^2\,Y_{20}(\theta,\varphi),$$

where we have defined $\beta \equiv 4\alpha\sqrt{\pi/5}$.

As in problem 1, $\langle\rho_f\rangle = 0$ so we need to solve Lapace's equation, which has general solutions given by (2). In the region $r < R$, we must have $B_{lm} = 0$ because $1/r^{l+1}$ is undefined at the origin. For the region $r > R$, we expect the external potential to dominate as $r \to \infty$, so may invoke the boundary condition at infinity:

$$\lim_{r\to\infty}\phi(r,\theta,\varphi) = \phi_0 = \beta r^2\,Y_{20}(\theta,\varphi).$$

This implies that the only nonzero A_{lm} here is $A_{20} = \beta$. Thus we have

$$\langle\phi\rangle(r,\theta,\varphi) = \begin{cases} \displaystyle\sum_{l,m} A_{lm}r^l\,Y_{lm}(\theta,\varphi) & \text{if } r \le R, \\ \displaystyle\beta r^2\,Y_{20}(\theta,\varphi) + \sum_{l,m}\frac{B_{lm}}{r^{l+1}}\,Y_{lm}(\theta,\varphi) & \text{if } r \ge R. \end{cases}$$

To solve for the remaining coefficients, we invoke the boundary conditions at $r = R$. Firstly, $\langle\phi\rangle$ must be continuous at the boundary. This gives us

$$\langle\phi\rangle(R,\theta,\varphi) = \sum_{l,m} A_{lm}R^l\,Y_{lm}(\theta,\varphi) = \beta R^2\,Y_{20}(\theta,\varphi) + \sum_{l,m}\frac{B_{lm}}{R^{l+1}}\,Y_{lm}(\theta,\varphi),$$

so

$$A_{20} = \beta + \frac{B_{20}}{R^5}, \qquad\qquad A_{lm} = \frac{B_{lm}}{R^{l+3}} \quad \text{for } (l,m) \neq (2,0). \tag{9}$$

Secondly, we require that $\hat{\mathbf{n}}\cdot\langle\mathbf{D}\rangle$ is also continuous at the boundary, where $\langle\mathbf{D}\rangle$ is defined in (7). Here $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so we are only concerned with the r component of $\langle\mathbf{E}\rangle$. Applying $\langle\mathbf{E}\rangle = -\boldsymbol{\nabla}\langle\phi\rangle$, we have

$$\langle E_r\rangle(r,\theta,\phi) = \begin{cases} \displaystyle\sum_{l,m} A_{lm}lr^{l-1}\,Y_{lm}(\theta,\varphi) & \text{if } r \le R, \\ \displaystyle 2\beta r\,Y_{20}(\theta,\varphi) - \sum_{l,m}(l+1)\frac{B_{lm}}{r^{l+2}}\,Y_{lm}(\theta,\varphi) & \text{if } r \ge R. \end{cases}$$

Then we need to satisfy

$$\hat{\mathbf{r}} \cdot \langle \mathbf{D} \rangle (R, \theta, \varphi) = \epsilon \sum_{l,m} A_{lm} l R^{l-1} \, Y_{lm}(\theta, \varphi) = 2\beta R \, Y_{20}(\theta, \varphi) - \sum_{l,m} (l+1) \frac{B_{lm}}{R^{l+2}} \, Y_{lm}(\theta, \varphi),$$

which stipulates

$$A_{20} = \frac{1}{\epsilon} \left(\beta - \frac{3}{2} \frac{B_{20}}{R^5} \right), \qquad A_{lm} = -\frac{1}{\epsilon} \frac{(l+1)}{l} \frac{B_{lm}}{R^{2l+1}} \quad \text{for } (l, m) \neq (2, 0). \tag{10}$$

Eliminating B_{lm} from (9) and (10), we obtain

$$A_{20} = \frac{5\beta}{2\epsilon + 3}, \qquad A_{lm} = 0 \quad \text{for } (l, m) \neq (2, 0),$$

and substituting back into (9) yields

$$B_{20} = 2\beta R^5 \frac{1 - \epsilon}{2\epsilon + 3}, \qquad B_{lm} = 0 \quad \text{for } (l, m) \neq (2, 0).$$

Finally, the total electrostatic potential everywhere is

$$\langle \phi \rangle (r, \theta, \varphi) = \alpha r^2 (3 \cos^2 \theta - 1) \times \begin{cases} \dfrac{5}{2\epsilon + 3} & \text{if } r \leq R, \\ 1 + 2 \dfrac{1 - \epsilon}{2\epsilon + 3} \dfrac{R^5}{r^5} & \text{if } r \geq R, \end{cases} \tag{11}$$

or, in Cartesian coordinates,

$$\langle \phi \rangle (x, y, z) = \alpha (2z^2 - x^2 - y^2) \times \begin{cases} \dfrac{5}{2\epsilon + 3} & \text{if } r \leq R, \\ 1 + 2 \dfrac{1 - \epsilon}{2\epsilon + 3} \dfrac{R^5}{\sqrt{x^2 + y^2 + z^2}} & \text{if } r \geq R. \end{cases}$$

2.b Calculate the interaction energy between the field produced by the dielectric and the external field. Assume that the potential arises from "distant charges" so that the formula for $\mathcal{E}_{\text{int}}$ given in class and the notes can be used.

Solution. Equation (3.34) in the lectures notes gives the interaction energy:

$$\mathcal{E}_{\text{int}} = \int (\langle \rho_f \rangle \phi_0 - \langle \mathbf{P} \rangle \cdot \mathbf{E}_0) \, d^3x \,,$$

where $\mathbf{E}_0$ is the electric field due to the external potential ϕ_0. Again, $\langle \rho_f \rangle = 0$. For our assumption of a linear, homogeneous, and isotropic dielectric,

$$\langle \mathbf{P} \rangle = \chi \langle \mathbf{E} \rangle \tag{12}$$

by Eq. (3.19), where

$$\epsilon = 1 + 4\pi\chi$$

from Eq. (3.21).

The gradient in spherical coordinates is

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\,\hat{\boldsymbol{\varphi}}. \tag{13}$$

Differentiating (11) for $r \leq R$,

$$\langle E_r \rangle = 2\alpha\frac{5}{2\epsilon+3}r(3\cos^2\theta - 1), \qquad \langle E_\theta \rangle = -6\alpha\frac{5}{2\epsilon+3}r\cos\theta\sin\theta, \qquad \langle E_\varphi \rangle = 0. \tag{14}$$

For the external field,

$$E_{0r} = 2\alpha r(3\cos^2\theta - 1), \qquad E_{0\theta} = -6\alpha r\cos\theta\sin\theta, \qquad E_{0\varphi} = 0. \tag{15}$$

Note that $\langle \mathbf{P} \rangle = (\epsilon - 1)\langle \mathbf{E} \rangle / 4\pi$, so

$$\langle \mathbf{P} \rangle \cdot \mathbf{E}_0 = 4\alpha^2\frac{\epsilon-1}{4\pi}\frac{5}{2\epsilon+3}r^2\left[(3\cos^2\theta-1)^2 + 9\cos^2\theta\sin^2\theta\right].$$

Then

$$\begin{aligned}
\mathscr{E}_{\text{int}} &= -\int \langle \mathbf{P} \rangle \cdot \mathbf{E}_0\, d^3x = 4\alpha^2\frac{1-\epsilon}{4\pi}\frac{5}{2\epsilon+3}\int_0^{2\pi} d\varphi \int_{-1}^{1}(3\cos^2\theta+1)\,d(\cos\theta)\int_0^R r^4\,dr \\
&= 4\alpha^2\frac{1-\epsilon}{4\pi}\frac{5}{2\epsilon+3}\Big[\varphi\Big]_0^{2\pi}\Big[\cos^3\theta+\cos\theta\Big]_{-1}^{1}\left[\frac{r^5}{5}\right]_0^R = 4\alpha^2\frac{1-\epsilon}{4\pi}\frac{5}{2\epsilon+3}(2\pi)(4)\frac{R^5}{5} \\
&= 8\alpha^2\frac{1-\epsilon}{2\epsilon+3}R^5.
\end{aligned}$$

2.c Calculate the total force needed to hold the dielectric ball in place.

Solution. Equation (3.26) in the lecture notes gives the total force on a dielectric:

$$\mathbf{F} = \int \left[\langle \rho_f \rangle \mathbf{E}_0 + (\langle \mathbf{P} \rangle \cdot \boldsymbol{\nabla})\mathbf{E}_0\right] d^3x\,,$$

where we note that there is no contribution from the dielectric's self field in electrostatics. Recall that $\langle \rho_f \rangle = 0$. Substituting in (12) and (7),

$$\mathbf{F} = \int (\chi\langle \mathbf{E} \rangle \cdot \boldsymbol{\nabla})\mathbf{E}_0\, d^3x = \int \frac{\chi}{\epsilon}(\boldsymbol{\nabla} \cdot \langle \mathbf{D} \rangle)\mathbf{E}_0\, d^3x = 0,$$

because

$$\boldsymbol{\nabla} \cdot \langle \mathbf{D} \rangle = -4\pi\langle \rho_f \rangle$$

according to Eq. (3.17).

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics* and Wolfram MathWorld while writing up these solutions.