Problem 1. **(Jackson 14.1)** Verify by explicit calculation that the Liénard-Wiechert expressions for *all* components of $\mathbf{E}$ and $\mathbf{B}$ for a particle moving with constant velocity agree with the ones obtained in the text by means of a Lorentz transformation. Follow the general method at the end of Section 14.1.

Solution. The Liénard-Wiechert expressions for the fields are given by Jackson (14.13–14):

$$\mathbf{B} = [\hat{\mathbf{n}} \times \mathbf{E}]_{\text{ret}}, \qquad \mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\text{ret}} + \frac{e}{c}\left[\frac{\hat{\mathbf{n}} \times \{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3 R}\right]_{\text{ret}}, \tag{1}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ with $\mathbf{v}$ being the particle's velocity, R is the distance from the observation point to the particle's position, and $\hat{\mathbf{n}}$ is a unit vector defined by $\mathbf{x} - \mathbf{r}(\tau) = R\,\hat{\mathbf{n}}$. Here, $\mathbf{r}(\tau)$ is the particle's present position and τ the proper time.

The expressions for the components of $\mathbf{E}$ and $\mathbf{B}$ obtained by a Lorentz transformation are given by Jackson (11.152):

$$E_1 = -\frac{e\gamma vt}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_2 = \frac{e\gamma b}{(b^2+\gamma^2v^2t^2)^{3/2}}, \qquad E_3 = B_1 = B_2 = 0, \qquad B_3 = \beta E_2, \tag{2}$$

where the particle is moving in the x_1 direction at impact parameter b on the x_2 axis, as shown in Fig. (1).

For a particle moving with constant velocity in the x_1 direction with velocity v as shown in Fig. (1), $\boldsymbol{\beta} = \beta\,\hat{\mathbf{x}}_1$ and $\dot{\boldsymbol{\beta}} = 0$. From Jackson (14.16), note that

$$(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2R^2 = b^2 + v^2t^2 - \beta^2b^2 = \frac{b^2+\gamma^2v^2t^2}{\gamma^2} \qquad \Longrightarrow \qquad (1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2 = \frac{(b^2+\gamma^2v^2t^2)^{3/2}}{R\gamma^3}.$$

This calculation comes from Fig. (2), where O is the observation point, P is the present position of the particle, and P' its retarded position. Also from Fig. 2,

$$\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}}_1 + \sin\theta\,\hat{\mathbf{x}}_2 = \frac{\beta R - vt}{R}\,\hat{\mathbf{x}}_1 + \frac{b}{R}\,\hat{\mathbf{x}}_2.$$

Making these substitutions in the expression for $\mathbf{E}(\mathbf{x}, t)$ in Eq. (1),

$$\mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\text{ret}} = e\left[\frac{(\beta - vt/R - \beta)\,\hat{\mathbf{x}}_1 + (b/R)\,\hat{\mathbf{x}}_2}{\gamma^2(b^2+\gamma^2v^2t^2)^{3/2}}R\gamma^3\right]_{\text{ret}} = e\gamma\frac{-vt\,\hat{\mathbf{x}}_1 + b\,\hat{\mathbf{x}}_2}{(b^2+\gamma^2v^2t^2)^{3/2}}. \tag{3}$$

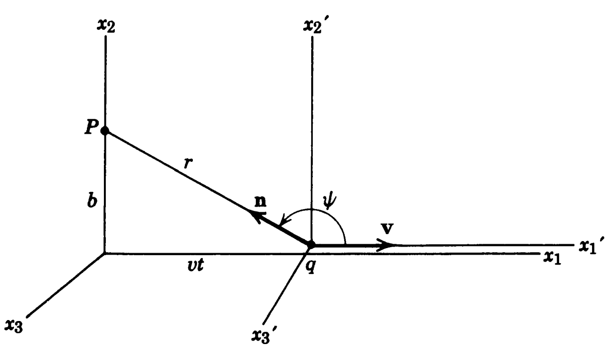

Figure 1: (Jackson Fig. 11.8) Particle of charge q moving at constant velocity $\mathbf{v}$ passes an observation point P at impact parameter b.

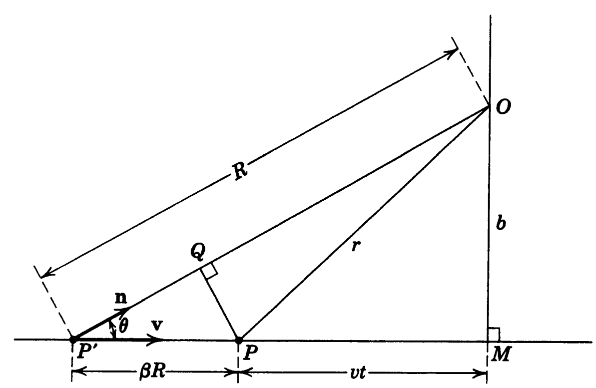

Figure 2: (Jackson Fig. 14.2) Present and retarded positions of a charge in uniform motion.

For $\mathbf{B}(\mathbf{x}, t)$, note that

$$\hat{\mathbf{n}} \times \mathbf{E} \propto \left(\frac{\beta R - vt}{R}\,\hat{\mathbf{x}}_1 + \frac{b}{R}\,\hat{\mathbf{x}}_2\right) \times (-vt\,\hat{\mathbf{x}}_1 + b\,\hat{\mathbf{x}}_2) = \left(b\frac{\beta R - vt}{R} + \frac{bvt}{R}\right)\hat{\mathbf{x}}_3 = \beta b,$$

so

$$\mathbf{B}(\mathbf{x}, t) = e\gamma \frac{\beta b\,\hat{\mathbf{x}}_3}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \tag{4}$$

Writing Eqs. (3) and (4) in component notation, we find

$$E_1 = -\frac{e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_2 = \frac{e\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad E_3 = 0,$$

$$B_1 = 0, \qquad B_2 = 0, \qquad B_3 = \frac{e\gamma \beta b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \beta E_2,$$

which are identical to Eq. (2) as was to be shown. □

Problem 2. (Jackson 14.3) The Heaviside-Feynman expression for the electric field of a particle of charge e in arbitrary motion, an alternative to the Liénard-Wiechert expression in Eq. (1), is

$$\mathbf{E} = e\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + e\left[\frac{R}{c}\right]_{\text{ret}} \frac{d}{dt}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + \frac{e}{c^2}\frac{d^2}{dt^2}[\hat{\mathbf{n}}]_{\text{ret}}, \tag{5}$$

where the time derivatives are with respect to the time at the observation point. Using the fact that the retarded time is $t' = t - R(t')/c$ and that, as a result,

$$\frac{dt}{dt'} = 1 - \boldsymbol{\beta}(t') \cdot \hat{\mathbf{n}}(t'),$$

show that the form above yields the expression for $\mathbf{E}$ in Eq. (1) when the time differentiations are performed.

Solution. One of the vector identities on the inside cover of Jackson is

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

From this,

$$\begin{aligned}
\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] &= (\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})(\hat{\mathbf{n}} - \boldsymbol{\beta}) - [\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} - \boldsymbol{\beta})]\dot{\boldsymbol{\beta}} = (\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \dot{\boldsymbol{\beta}})\boldsymbol{\beta} - \dot{\boldsymbol{\beta}} + (\hat{\mathbf{n}} \cdot \boldsymbol{\beta})\dot{\boldsymbol{\beta}} \\
&= \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{c} - \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v}}{c^2} - \frac{\dot{\mathbf{v}}}{c} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}}{c^2}.
\end{aligned}$$

Then Eq. (1) can be written

$$\begin{aligned}
\mathbf{E} &= e\left[\frac{1}{\gamma^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2}\left(\hat{\mathbf{n}} - \frac{\mathbf{v}}{c}\right)\right]_{\text{ret}} + \frac{e}{c}\left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R}\left(\frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{c} - \frac{(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v}}{c^2} - \frac{\dot{\mathbf{v}}}{c} + \frac{(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}}{c^2}\right)\right]_{\text{ret}} \\
&= e\left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2}\left(\frac{\hat{\mathbf{n}}}{\gamma^2} - \frac{\mathbf{v}}{c\gamma^2} + \frac{R(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}}}{c^2} - \frac{R(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v}}{c^3} - \frac{R\dot{\mathbf{v}}}{c^2} + \frac{R(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}}{c^3}\right)\right]_{\text{ret}} \\
&= \frac{e}{c^3}\left[\frac{1}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3 R^2}\left(\frac{c^3\hat{\mathbf{n}}}{\gamma^2} - \frac{c^2\mathbf{v}}{\gamma^2} + cR(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\hat{\mathbf{n}} - R(\hat{\mathbf{n}} \cdot \dot{\mathbf{v}})\mathbf{v} - cR\dot{\mathbf{v}} + R(\hat{\mathbf{n}} \cdot \mathbf{v})\dot{\mathbf{v}}\right)\right]_{\text{ret}}.
\end{aligned}$$

Since $R(t') = c(t-t')$, note that

$$\left[\frac{dR}{dt}\right]_{\text{ret}} = \frac{dR(t')}{dt'} = c\left(\frac{dt}{dt'} - 1\right) = c(1 - \boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t') - 1) = -[\hat{\mathbf{n}}\cdot\mathbf{v}]_{\text{ret}}.$$

Note that $[\hat{\mathbf{n}}]_{\text{ret}} = [(\mathbf{x}-\mathbf{r})/R]_{\text{ret}}$. Then

$$\begin{aligned}
\frac{d}{dt'}[\hat{\mathbf{n}}]_{\text{ret}} &= \left[\frac{d\hat{\mathbf{n}}}{dt}\right]_{\text{ret}} = \left[\frac{1}{R^2}\left(R\frac{d}{dt}(\mathbf{x}-\mathbf{r}) - (\mathbf{x}-\mathbf{r})\frac{dR}{dt}\right)\right]_{\text{ret}} = \left[\frac{1}{R^2}\left(-R\mathbf{v} + R(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}\right)\right]_{\text{ret}} \\
&= \left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R}\right]_{\text{ret}},
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt'}\left[\frac{\hat{\mathbf{n}}}{R}\right]_{\text{ret}} &= \left[\frac{d}{dt}\left(\frac{\hat{\mathbf{n}}}{R}\right)\right]_{\text{ret}} = \left[\frac{1}{R^2}\left(R\frac{d\hat{\mathbf{n}}}{dt} - \hat{\mathbf{n}}\frac{dR}{dt}\right)\right]_{\text{ret}} = \left[\frac{1}{R^2}\left\{R\left(\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R}\right) + (\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}\right\}\right]_{\text{ret}} \\
&= \left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v} + (\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} = \left[\frac{2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R^2}\right]_{\text{ret}},
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt'}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} &= \left[\frac{d}{dt}\left(\frac{\hat{\mathbf{n}}}{R^2}\right)\right]_{\text{ret}} = \left[\frac{1}{R^2}\left\{R\frac{d}{dt}\left(\frac{\hat{\mathbf{n}}}{R}\right) - \frac{\hat{\mathbf{n}}}{R}\frac{dR}{dt}\right\}\right]_{\text{ret}} = \left[\frac{1}{R^2}\left(R\frac{2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R^2} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R}\right)\right]_{\text{ret}} \\
&= \left[\frac{2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v} + (\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R^3}\right]_{\text{ret}} = \left[\frac{3(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R^3}\right]_{\text{ret}}.
\end{aligned}$$

For the second derivative of $\hat{\mathbf{n}}$, note that

$$\frac{d}{dt'}[\hat{\mathbf{n}}\cdot\mathbf{v}]_{\text{ret}} = \left[\frac{d}{dt}(\hat{\mathbf{n}}\cdot\mathbf{v})\right]_{\text{ret}} = \left[\hat{\mathbf{n}}\cdot\frac{d\mathbf{v}}{dt} + \frac{d\hat{\mathbf{n}}}{dt}\cdot\mathbf{v}\right]_{\text{ret}} = \left[\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R}\cdot\mathbf{v}\right]_{\text{ret}} = \left[\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - \mathbf{v}^2}{R}\right]_{\text{ret}},$$

so then

$$\begin{aligned}
\frac{d^2}{dt'^2}[\hat{\mathbf{n}}]_{\text{ret}} &= \left[\frac{d^2\hat{\mathbf{n}}}{dt^2}\right]_{\text{ret}} = \left[\frac{d}{dt}\left(\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R}\right)\right]_{\text{ret}} = \left[\frac{d}{dt}\left(\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R}\right) - \frac{d}{dt}\left(\frac{\mathbf{v}}{R}\right)\right]_{\text{ret}} \\
&= \left[\frac{\hat{\mathbf{n}}}{R}\frac{d}{dt}(\hat{\mathbf{n}}\cdot\mathbf{v}) + (\hat{\mathbf{n}}\cdot\mathbf{v})\frac{d}{dt}\left(\frac{\hat{\mathbf{n}}}{R}\right) - \frac{1}{R^2}\left(R\frac{d\mathbf{v}}{dt} - \mathbf{v}\frac{dR}{dt}\right)\right]_{\text{ret}} \\
&= \left[\frac{\hat{\mathbf{n}}}{R}\left(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - \mathbf{v}^2}{R}\right) + (\hat{\mathbf{n}}\cdot\mathbf{v})\left(\frac{2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v}}{R^2}\right) - \frac{1}{R^2}\left(R\dot{\mathbf{v}} + (\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}\right)\right]_{\text{ret}} \\
&= \left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}^2\hat{\mathbf{n}}}{R^2} + 2\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}}{R^2} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}}{R^2} - \frac{\dot{\mathbf{v}}}{R} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}}{R^2}\right]_{\text{ret}} \\
&= \left[\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R} + 3\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}^2\hat{\mathbf{n}}}{R^2} - 2\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}}{R^2} - \frac{\dot{\mathbf{v}}}{R}\right]_{\text{ret}}
\end{aligned}$$

By the chain rule,

$$\frac{d}{dt} = \frac{dt'}{dt}\frac{d}{dt'} = \frac{1}{1 - \boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\frac{d}{dt'} = \left[\frac{1}{1 - \boldsymbol{\beta}\cdot\hat{\mathbf{n}}}\right]_{\text{ret}}\frac{d}{dt'}.$$

For the second derivative, note that

$$\begin{aligned}
\frac{d}{dt'}\left[\frac{1}{1 - \boldsymbol{\beta}\cdot\hat{\mathbf{n}}}\right]_{\text{ret}} &= \left[\frac{d}{dt}\left(\frac{1}{\mathbf{v}\cdot\hat{\mathbf{n}}/c} - 1\right)\right]_{\text{ret}} = \left[\frac{1}{(1 - \mathbf{v}\cdot\hat{\mathbf{n}}/c)^2}\frac{d}{dt}\left(1 - \frac{\hat{\mathbf{n}}\cdot\mathbf{v}}{c}\right)\right]_{\text{ret}} \\
&= \frac{1}{c}\left[\frac{1}{(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2}\left(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - \mathbf{v}^2}{R}\right)\right]_{\text{ret}},
\end{aligned}$$

so

$$\begin{aligned}
\frac{d^2}{dt^2} &= \frac{d}{dt}\left(\frac{1}{1-\boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\frac{d}{dt'}\right) = \frac{1}{1-\boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\frac{d}{dt'}\left(\frac{1}{1-\boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\frac{d}{dt'}\right)\\
&= \frac{1}{1-\boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\left\{\frac{d}{dt'}\left(\frac{1}{1-\boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\right)\frac{d}{dt'} + \frac{1}{1-\boldsymbol{\beta}(t')\cdot\hat{\mathbf{n}}(t')}\frac{d^2}{dt'^2}\right\}\\
&= \frac{1}{c}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}\left(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2-\mathbf{v}^2}{R}\right)\right]_{\rm ret}\frac{d}{dt'} + \left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2}\right]_{\rm ret}\frac{d^2}{dt'^2}.
\end{aligned}$$

The first term of Eq. (5) can be written

$$\begin{aligned}
e\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\rm ret} &= e\left[\frac{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3\hat{\mathbf{n}}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\rm ret} = e\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\left(\hat{\mathbf{n}} - 3\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{c} + 3\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}}{c^2} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}}}{c^3}\right)\right]_{\rm ret}\\
&= \frac{e}{c^3}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{c^3\hat{\mathbf{n}} - 3c^2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}}\}\right]_{\rm ret}. \qquad (6)
\end{aligned}$$

The second term of Eq. (5) becomes

$$\begin{aligned}
e\left[\frac{R}{c}\right]_{\rm ret}\frac{d}{dt}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\rm ret} &= \frac{e}{c}\left[\frac{R}{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}\right]_{\rm ret}\frac{d}{dt'}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\rm ret} = \frac{e}{c}\left[\frac{3(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}-\mathbf{v}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})R^2}\right]_{\rm ret} = \frac{e}{c}\left[\frac{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2\{3(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}-\mathbf{v}\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\rm ret}\\
&= \frac{e}{c}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\left(1 - 2\frac{\hat{\mathbf{n}}\cdot\mathbf{v}}{c} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2}{c^2}\right)\{3(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}-\mathbf{v}\}\right]_{\rm ret}\\
&= \frac{e}{c}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\left(3(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - \mathbf{v} - 6\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}}{c} + 2\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}}{c} + 3\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}}}{c^2} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v}}{c^2}\right)\right]_{\rm ret}\\
&= \frac{e}{c^3}\left[\frac{3c^2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - c^2\mathbf{v} - 6c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} + 3(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\rm ret}. \qquad (7)
\end{aligned}$$

The third term of Eq. (5) becomes

$$\frac{e}{c^2}\frac{d^2}{dt^2}[\hat{\mathbf{n}}]_{\rm ret} = \frac{e}{c^3}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}\left(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2-\mathbf{v}^2}{R}\right)\right]_{\rm ret}\frac{d}{dt'}[\hat{\mathbf{n}}]_{\rm ret} + \frac{e}{c^2}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2}\right]_{\rm ret}\frac{d^2}{dt'^2}[\hat{\mathbf{n}}]_{\rm ret}. \qquad (8)$$

The first term of Eq. (8) is

$$\begin{aligned}
&\frac{e}{c^3}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}\left(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2-\mathbf{v}^2}{R}\right)\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}-\mathbf{v}}{R}\right]_{\rm ret}\\
&\quad= \frac{e}{c^3}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3}\left(\frac{(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R} + \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}}}{R^2} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}^2\hat{\mathbf{n}}}{R^2} - \frac{(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})\mathbf{v}}{R} - \frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v}}{R^2} + \frac{\mathbf{v}^2\mathbf{v}}{R^2}\right)\right]_{\rm ret}\\
&\quad= \frac{e}{c^3}\left[\frac{R(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}^2\hat{\mathbf{n}} - R(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})\mathbf{v} - (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v} + \mathbf{v}^2\mathbf{v}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\rm ret},
\end{aligned}$$

and the second term of Eq. (8) is

$$\begin{aligned}
&\frac{e}{c^2}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^2}\left(\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}}{R} + 3\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}}{R^2} - \frac{\mathbf{v}^2\hat{\mathbf{n}}}{R^2} - 2\frac{(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}}{R^2} - \frac{\dot{\mathbf{v}}}{R}\right)\right]_{\rm ret}\\
&\quad= \frac{e}{c^2}\left[\frac{1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{R(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + 3(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} - \mathbf{v}^2\hat{\mathbf{n}} - 2(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} - R\dot{\mathbf{v}}\}\right]_{\rm ret}\\
&\quad= \frac{e}{c^3}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{Rc(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} - c\mathbf{v}^2\hat{\mathbf{n}} - 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} - Rc\dot{\mathbf{v}}\right.\\
&\qquad\left. - R(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} - 3(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v}^2\hat{\mathbf{n}} + 2(\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v} + R(\hat{\mathbf{n}}\cdot\mathbf{v})\dot{\mathbf{v}}\}\right]_{\rm ret}.
\end{aligned}$$

Summing the two terms, we find

$$\frac{e}{c^2}\frac{d^2}{dt^2}[\hat{\mathbf{n}}]_{\text{ret}} = \frac{e}{c^3}\Bigg[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{R(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - 2(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - R(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})\mathbf{v} + (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v} + \mathbf{v}^2\mathbf{v} + Rc(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}$$
$$+ 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} - c\mathbf{v}^2\hat{\mathbf{n}} - 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} - Rc\dot{\mathbf{v}} - R(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + R(\hat{\mathbf{n}}\cdot\mathbf{v})\dot{\mathbf{v}}\}\Bigg]_{\text{ret}}. \quad (9)$$

Summing Eqs. (6) and (7),

$$e\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} + e\left[\frac{R}{c}\right]_{\text{ret}}\frac{d}{dt}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\text{ret}} = \frac{e}{c^3}\Bigg[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{c^3\hat{\mathbf{n}} - 3c^2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}}$$
$$+ 3c^2(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - c^2\mathbf{v} - 6c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} + 3(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v}\}\Bigg]_{\text{ret}}$$
$$= \frac{e}{c^3}\left[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{c^3\hat{\mathbf{n}} - 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + 2(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - c^2\mathbf{v} + 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} - (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v}\}\right]_{\text{ret}}. \quad (10)$$

Summing Eqs. (10) and (9),

$$\mathbf{E} = \frac{e}{c^3}\Bigg[\frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\{c^3\hat{\mathbf{n}} - 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + 2(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - c^2\mathbf{v} + 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} - (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v} + R(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}}$$
$$- 2(\hat{\mathbf{n}}\cdot\mathbf{v})^3\hat{\mathbf{n}} - R(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})\mathbf{v} + (\hat{\mathbf{n}}\cdot\mathbf{v})^2\mathbf{v} + \mathbf{v}^2\mathbf{v} + Rc(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} + 3c(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}}$$
$$- c\mathbf{v}^2\hat{\mathbf{n}} - 2c(\hat{\mathbf{n}}\cdot\mathbf{v})\mathbf{v} - Rc\dot{\mathbf{v}} - R(\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + R(\hat{\mathbf{n}}\cdot\mathbf{v})\dot{\mathbf{v}}\}\Bigg]_{\text{ret}}$$
$$= \frac{e}{c^3}\left[\frac{(c^3 - c\mathbf{v}^2)\hat{\mathbf{n}} + (\mathbf{v}^2 - c^2)\mathbf{v} + R\{(\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - (\hat{\mathbf{n}}\cdot\dot{\mathbf{v}})\mathbf{v} + c(\hat{\mathbf{n}}\cdot\mathbf{v})\hat{\mathbf{n}} - c\dot{\mathbf{v}} - (\hat{\mathbf{n}}\cdot\mathbf{v})^2\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{v})\dot{\mathbf{v}}\}}{(1-\boldsymbol{\beta}\cdot\hat{\mathbf{n}})^3R^2}\right]_{\text{ret}}$$

Problem 3. **(Jackson 14.4)** Using the Liénard-Wiechart fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a particle with charge e, moving as described below. Sketch the angular distribution of the radiation and determine the total power radiated in each case.

3(a) The particle is moving along the z axis with instantaneous position $z(t) = \alpha \cos \omega_0 t$.

Solution. For a nonrelativistic particle, the power radiated per unit solid angle is given by Jackson (14.21),

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2} |\dot{\mathbf{v}}|^2 \sin^2 \Theta, \tag{11}$$

where Θ is the angle between $\dot{\mathbf{v}}$ and $\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector pointing toward the observer. The total instantaneous power radiated is given by Jackson (14.22):

$$P = \frac{2}{3} \frac{e^2}{c^3} |\dot{\mathbf{v}}|^2. \tag{12}$$

In this case, we have

$$\mathbf{x}(t) = \alpha \cos \omega_0 t \, \hat{\mathbf{x}}_3, \qquad \mathbf{v}(t) = -\alpha \omega_0 \sin \omega_0 t \, \hat{\mathbf{x}}_3, \qquad \dot{\mathbf{v}}(t) = -\alpha \omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_3.$$

The system is azimuthally symmetric since $\dot{\mathbf{v}}$ always points along the z axis. Thus, $\Theta = \theta$ where θ is the polar angle in spherical coordinates. Equation (11) becomes

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^2} \left| -\alpha \omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_3 \right|^2 \sin^2 \theta = \frac{e^2 \alpha^2 \omega_0^4}{4\pi c^2} \cos^2 \omega_0 t \sin^2 \theta,$$

so the time-averaged power radiated per unit solid angle is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^2} \alpha^2 \omega_0^4 \left\langle \cos^2 \omega_0 t \right\rangle \sin^2 \theta = \frac{e^2 \alpha^2 \omega_0^4}{8\pi c^2} \sin^2 \theta. \tag{13}$$

A plot of the angular distribution of the radiation is shown in Fig. 3 in the xz plane, and in three dimensions in Fig. 5.

Equation (12) becomes

$$P = \frac{2}{3} \frac{e^2}{c^3} \left| -\alpha \omega_0^2 \cos \omega_0 t \, \hat{\mathbf{x}}_3 \right|^2 = \frac{2}{3} \frac{e^2 \alpha^2 \omega_0^4}{c^3} \cos^2 \omega_0 t,$$

so the time-averaged total power radiated is

$$\langle P \rangle = \frac{2}{3} \frac{e^2 \alpha^2 \omega_0^4}{c^3} \left\langle \cos^2 \omega_0 t \right\rangle = \frac{e^2 \alpha^2 \omega_0^4}{3c^3}.$$

3(b) The particle is moving in a circle of radius R in the xy plane with constant angular frequency ω_0.

Solution. For a charge moving counter-clockwise,

$$\begin{aligned}\mathbf{x}(t) &= R\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} - R\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2},\\ \mathbf{v}(t) &= -R\omega_0\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} - R\omega_0\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2},\\ \dot{\mathbf{v}}(t) &= -R\omega_0^2\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} + R\omega_0^2\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2}.\end{aligned}$$

This system is also azimuthally symmetric, so it is sufficient to restrict the position of the observer to the yz plane. In polar coordinates, $\hat{\mathbf{n}} = \sin\theta\,\hat{\mathbf{x}}_\mathbf{2} + \cos\theta\,\hat{\mathbf{x}}_\mathbf{3}$. Then $\sin^2\Theta$ can be found by

$$\sin^2\Theta = 1 - \cos^2\Theta = 1 - \frac{(\dot{\mathbf{v}}\cdot\hat{\mathbf{n}})^2}{\dot{v}^2} = 1 - \sin^2\theta\sin^2\omega_0 t.$$

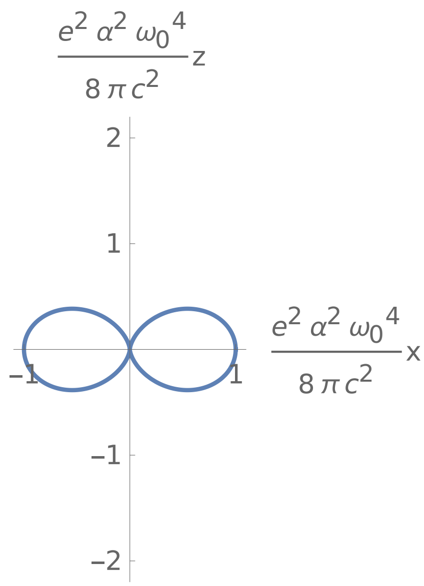

Figure 3: Plot of Eq. (13) in the xz plane.

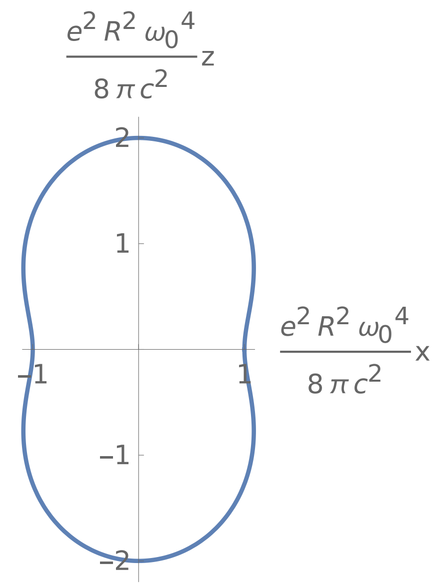

Figure 4: Plot of Eq. (14) in the xz plane.

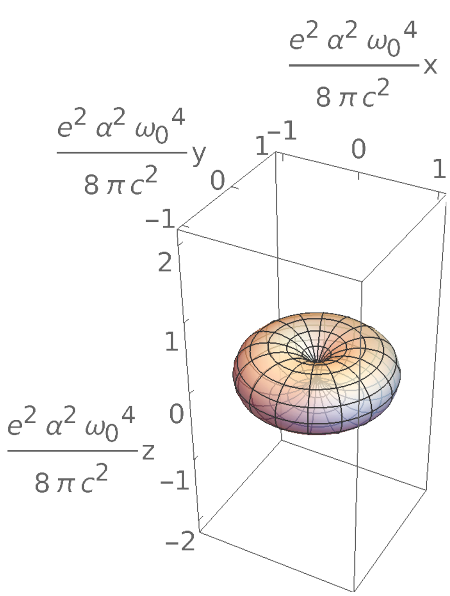

Figure 5: Three-dimensional plot of Eq. (13).

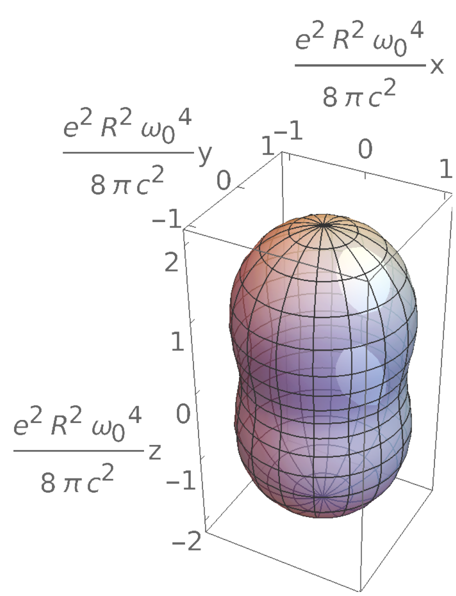

Figure 6: Three-dimensional plot of Eq. (14).

With these substitutions, Eq. (11) becomes

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{e^2}{4\pi c^2}\left|-R\omega_0^2\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} + R\omega_0^2\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2}\right|^2(1-\sin^2\theta\sin^2\omega_0 t)\\ &= \frac{e^2R^2\omega_0^4}{4\pi c^2}(\cos^2\omega_0 t+\sin^2\omega_0 t)(1-\sin^2\theta\sin^2\omega_0 t) = \frac{e^2R^2\omega_0^4}{4\pi c^2}(1-\sin^2\theta\sin^2\omega_0 t),\end{aligned}$$

giving us the time-averaged power radiated per unit solid angle:

$$\begin{aligned}\left\langle\frac{dP}{d\Omega}\right\rangle &= \frac{e^2R^2\omega_0^4}{4\pi c^2}(1-\sin^2\theta\left\langle\sin^2\omega_0 t\right\rangle) = \frac{e^2R^2\omega_0^4}{4\pi c^2}\left(1-\frac{\sin^2\theta}{2}\right) = \frac{e^2R^2\omega_0^4}{4\pi c^2}\left(1-\frac{1-\cos^2\theta}{2}\right)\\ &= \frac{e^2R^2\omega_0^4}{8\pi c^2}(1+\cos^2\theta).\end{aligned} \tag{14}$$

A plot of the angular distribution of the radiation is shown in Fig. 4, in the xz plane, and in three dimensions in Fig. 6.

From Eq. (12), we have

$$P = \frac{2}{3}\frac{e^2}{c^3}\left|-R\omega_0^2\cos\omega_0 t\,\hat{\mathbf{x}}_\mathbf{1} + R\omega_0^2\sin\omega_0 t\,\hat{\mathbf{x}}_\mathbf{2}\right|^2 = \frac{2}{3}\frac{e^2R^2\omega_0^4}{c^3}(\cos^2\omega_0 t+\sin^2\omega_0 t) = \frac{2}{3}\frac{e^2R^2\omega_0^4}{c^3} = \langle P\rangle\,.$$