1 Problem 1

Let's consider coherent states of a one-dimensional quantum particle with mass m confined in a one-dimensional harmonic potential $V(X) = m\omega^2 X^2/2$:

$$a|\lambda\rangle = \lambda|\lambda\rangle, \qquad |\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\exp\left(\lambda a^\dagger\right)|0\rangle.$$

Here, λ is a complex parameter.

1.1 Compute $\langle x|\lambda\rangle$.

Solution. In terms of the position and momentum operators X and P,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right),$$

so

$$\langle x|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\langle x|\exp\left(\lambda a^\dagger\right)|0\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\langle x|\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\}|0\rangle. \tag{1}$$

Note that for two operators A and B, $e^{A+B} = e^{-[A,B]/2}e^A e^B$ if $[A,B]$ commutes with each A and B. Note also that

$$\left[X, -\frac{iP}{m\omega}\right] = -\frac{i}{m\omega}[X,P] = \frac{\hbar}{m\omega}.$$

Thus,

$$\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\} = \exp\left(-\lambda\frac{\hbar}{2m\omega}\sqrt{\frac{m\omega}{2\hbar}}\right)\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)\exp\left(-\lambda\frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right).$$

Now, note that

$$\exp\left(-\lambda\frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right) = \exp\left(-\frac{i}{\hbar}\lambda\frac{\hbar}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right) = U\left(\lambda\sqrt{\frac{\hbar}{2m\omega}}\right) \equiv U(b), \tag{2}$$

where $U(b)$ is the translation operator, and we have defined b. So (1) becomes

$$\begin{aligned}\langle x|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(-\lambda\frac{\hbar}{2m\omega}\sqrt{\frac{m\omega}{2\hbar}}\right)\langle x|\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)U(b)|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(-\frac{b}{2}\right)\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}x\right)\langle x-b|0\rangle. \end{aligned}\tag{3}$$

From (2.3.30) in Sakurai,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \implies \langle x-b|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}(x-b)^2\right).$$

so (3) becomes

$$\begin{aligned}\langle x|\lambda\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{b}{2} + \lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega}{2\hbar}(x^2 - 2bx + b^2)\right) \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{b}{2} + \lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega}{2\hbar}x^2 + \lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega}{2\hbar}\lambda^2\frac{\hbar}{2m\omega}\right) \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{b}{2} - \frac{m\omega}{2\hbar}x^2 + 2\lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{\lambda^2}{4}\right) \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{m\omega}{2\hbar}x^2 + \lambda\sqrt{\frac{2m\omega}{\hbar}}x\right), \end{aligned}\tag{4}$$

where we have dropped a constant phase.

1.2 Compute $\langle\lambda|X|\lambda\rangle$, $\langle\lambda|P|\lambda\rangle$, $\langle\lambda|X^2|\lambda\rangle$, and $\langle\lambda|P^2|\lambda\rangle$. Also, compute $\langle\lambda|(\Delta X)^2|\lambda\rangle\ \langle\lambda|(\Delta P)^2|\lambda\rangle$ where $\Delta A = A - \langle A\rangle$.

Solution. Note that

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

Then for $\langle\lambda|X|\lambda\rangle$,

$$\begin{aligned}\langle\lambda|X|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger X a|\lambda\rangle \\ &= \frac{1}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|a^\dagger(a+a^\dagger)a|\lambda\rangle = \frac{1}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|(a^\dagger a^2 + a^{\dagger 2}a|\lambda\rangle = \frac{|\lambda|^2(\lambda^* + \lambda)}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}} \\ &= 2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar}{2m\omega}}, \end{aligned} \tag{5}$$

and for $\langle\lambda|P|\lambda\rangle$,

$$\begin{aligned}\langle\lambda|P|\lambda\rangle &= \frac{1}{\lambda^2}\langle\lambda|a^\dagger P a|\lambda\rangle \\ &= \frac{i}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}}\langle\lambda|a^\dagger(a^\dagger - a)a|\lambda\rangle = \frac{i}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}}\langle\lambda|a^{\dagger 2}a - a^\dagger a^2|\lambda\rangle = \frac{i|\lambda|^2(\lambda^* - \lambda)}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}} \\ &= 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar m\omega}{2}}. \end{aligned} \tag{6}$$

Note also that

$$X^2 = \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}), \qquad P^2 = -\frac{\hbar m\omega}{2}(a^{\dagger 2} - a^\dagger a - aa^\dagger + a^2).$$

Then for $\langle\lambda|X^2|\lambda\rangle$,

$$\begin{aligned}\langle\lambda|X^2|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger X^2 a|\lambda\rangle = \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|a^\dagger(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2})a|\lambda\rangle \\ &= \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger a^3 + a^\dagger a a^\dagger a + a^{\dagger 2}a^2 + a^{\dagger 3}a)|\lambda\rangle = \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger a^3 + a^\dagger a + 2a^{\dagger 2}a^2 + a^{\dagger 3}a)|\lambda\rangle \\ &= (\lambda^2 + 1 + 2|\lambda|^2 + \lambda^{*2})\frac{\hbar}{2m\omega} = \left(1 + 2\left[\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2\right] + 2\left[\mathrm{Re}(\lambda)^2 - \mathrm{Im}(\lambda)^2\right]\right)\frac{\hbar}{2m\omega} \\ &= [1 + 4\,\mathrm{Re}(\lambda)^2]\frac{\hbar}{2m\omega}, \end{aligned} \tag{7}$$

where we have used $[a, a^\dagger] = 1$. For $\langle\lambda|P^2|\lambda\rangle$,

$$\begin{aligned}\langle\lambda|P^2|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger P^2 a|\lambda\rangle = -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|a^\dagger(a^{\dagger 2} - a^\dagger a - aa^\dagger + a^2)a|\lambda\rangle \\ &= -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|(a^{\dagger 2}a - a^{\dagger 2}a^2 - a^\dagger a a^\dagger a + a^\dagger a^3|\lambda\rangle = -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|(a^{\dagger 3}a - a^\dagger a - 2a^{\dagger 2}a^2 + a^\dagger a^3|\lambda\rangle \\ &= -(\lambda^{*2} - 1 - 2|\lambda|^2 + \lambda^2)\frac{\hbar m\omega}{2} = \left(1 + 2\left[\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2\right] - 2[\mathrm{Re}(\lambda)^2 - \mathrm{Im}(\lambda)^2]\right)\frac{\hbar m\omega}{2} \\ &= [1 + 4\,\mathrm{Im}(\lambda)^2]\frac{\hbar m\omega}{2}. \end{aligned} \tag{8}$$

Note that $\left\langle(\Delta A)^2\right\rangle = \left\langle A^2\right\rangle - \langle A\rangle^2$. Then

$$\langle\lambda|(\Delta X)^2|\lambda\rangle = \langle\lambda|X^2|\lambda\rangle - \langle\lambda|X|\lambda\rangle^2 = [1+4\,\mathrm{Re}(\lambda)^2]\frac{\hbar}{2m\omega} - 4\,\mathrm{Re}(\lambda)^2\frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega},$$

where we have used (5) and (7), and

$$\langle\lambda|(\Delta P)^2|\lambda\rangle = \langle\lambda|P^2|\lambda\rangle - \langle\lambda|P|\lambda\rangle^2 = [1+4\,\mathrm{Im}(\lambda)^2]\frac{\hbar m\omega}{2} - 4\,\mathrm{Im}(\lambda)^2\frac{\hbar m\omega}{2} = \frac{\hbar m\omega}{2},$$

where we have used (6) and (8). Finally,

$$\langle\lambda|(\Delta X)^2|\lambda\rangle\ \langle\lambda|(\Delta P)^2|\lambda\rangle = \frac{\hbar^2}{4},$$

which shows that the coherent state $|\lambda\rangle$ satisfies the minimum uncertainty relation.

1.3 Starting from $|\psi(0)\rangle = |\lambda\rangle$ at $t=0$, we let $|\psi(t)\rangle$ evolve in time. What is the state $|\psi(t)\rangle$ for $t>0$?

Solution. From (2.3.43) in Sakurai,

$$a(t) = ae^{-i\omega t}, \qquad\qquad a^\dagger(t) = a^\dagger e^{i\omega t},$$

where $a = a(0)$ and $a^\dagger = a^\dagger(0)$. Equating the Schrödinger and Heisenberg pictures,

$$|\psi(0)\rangle = |\lambda\rangle = \frac{1}{\lambda}a\,|\lambda\rangle \implies |\psi(t)\rangle = \frac{1}{\lambda}a(t)\,|\lambda\rangle\,,$$

and so

$$|\psi(t)\rangle = \frac{1}{\lambda}ae^{-i\omega t}\,|\lambda\rangle = e^{-i\omega t}\,|\lambda\rangle\,.$$

1.4 Compute $\langle\psi(t)|X|\psi(t)\rangle$ and $\langle\psi(t)|P|\psi(t)\rangle$, and their time derivatives $d\,\langle X\rangle/dt$ and $d\,\langle P\rangle/dt$.

Solution. Firstly, we have

$$\begin{aligned}
\langle\psi(t)|X|\psi(t)\rangle &= \langle\lambda|\,e^{i\omega t}Xe^{-i\omega t}\,|\lambda\rangle = \langle\lambda|X|\lambda\rangle = 2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar}{2m\omega}},\\
\langle\psi(t)|P|\psi(t)\rangle &= \langle\lambda|\,e^{i\omega t}Pe^{-i\omega t}\,|\lambda\rangle = \langle\lambda|P|\lambda\rangle = 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar m\omega}{2}},
\end{aligned}$$

where we have used (5) and (6).

For the time derivatives, note that the harmonic oscillator Hamiltonian is given by

$$H = \frac{P^2}{2m} + \frac{m\omega^2X^2}{2}.$$

Then, using the Ehrenfest theorem and the other results of problem 4.1 of the previous homework,

$$\begin{aligned}
\frac{d\,\langle X\rangle}{dt} &= -\frac{i}{\hbar}\,\langle\psi(t)|[X,H]|\psi(t)\rangle = \frac{1}{m}\,\langle\psi(t)|P|\psi(t)\rangle = 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar\omega}{2m}},\\
\frac{d\,\langle P\rangle}{dt} &= -\frac{i}{\hbar}\,\langle\psi(t)|[P,H]|\psi(t)\rangle = -m\omega^2\,\langle\psi(t)|X|\psi(t)\rangle = -2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar m\omega^3}{2}},
\end{aligned}$$

which again are similar to the classical equations of motion.

1.5 Compute $\langle\lambda''|\exp(-iHt/\hbar)|\lambda'\rangle$.

Solution. Note that $U(t) = \exp(-iHt/\hbar)$ where $U(t)$ is the time evolution operator. From problem 1.3,

$$|\psi(t)\rangle = U(t)\,|\lambda\rangle \implies \exp\left(-\frac{iHt}{\hbar}\right)\left|\lambda'\right\rangle = e^{-i\omega t}\left|\lambda'\right\rangle,$$

so

$$\langle\lambda''|\exp\left(-\frac{iHt}{\hbar}\right)|\lambda'\rangle = e^{-i\omega t}\,\langle\lambda''|\lambda'\rangle\,.$$

Using the power series representation,

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n a^{\dagger n}}{n!}\,|0\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\,|n\rangle\,,$$

so

$$\langle\lambda''|\lambda'\rangle = \exp\left(-\frac{|\lambda''|^2}{2}\right)\exp\left(-\frac{|\lambda'|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\lambda''^{*}\lambda')^n}{n!}\,\langle n|n\rangle = \exp\left(-\frac{|\lambda''|^2}{2} + \lambda''^{*}\lambda' - \frac{|\lambda'|^2}{2}\right).$$

Finally,

$$\langle\lambda''|\exp\left(-\frac{iHt}{\hbar}\right)|\lambda'\rangle = \exp\left(-i\omega t - \frac{|\lambda''|^2}{2} + \lambda''^{*}\lambda' - \frac{|\lambda'|^2}{2}\right).$$