Problem 1.   **LCR circuit**   An electrical circuit consists of an inductance L, resistance R and capacitance C in series, driven by a voltage source $V(t) = V_0 \cos(\omega t)$.

1(a) Show that the charge $q(t)$ on the capacitor satisfies the equation

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V(t), \tag{1}$$

and use it to define the complex susceptibility from

$$q(\omega) = \chi(\omega)V(\omega). \tag{2}$$

Solution.   An example LCR circuit is shown in Fig. 1. We can use Kirchoff's loop rule to obtain the differential equation for this circuit. Beginning from the bottom left corner of the circuit and moving clockwise, we have [1, pp. 849, 1007]

$$0 = V(t) - IR - L\frac{dI}{dt} - \frac{q}{C},$$

where we have applied Ohm's law $V_{ab} = IR$, the potential difference across an inductor $V_{ab} = L\, dI/dt$, and the definition of capacitance $C = q/V_{ab}$ [1, pp. 782, 999]. The current $I(t) = dq(t)/dt$ and charge $q(t)$ are identical at all points in a series circuit. Feeding in $I = dq(t)/dt$, this relation becomes

$$V(t) = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

as we wanted to show.   □

For the complex susceptibility, we recall that differentiating in the time domain is equivalent to multiplying by $i\omega$ in the frequency domain [3]:

$$\mathcal{F}_x[f^{(n)}(x)](\omega) = (i\omega)^n \mathcal{F}[f(x)](\omega).$$

We Fourier transform both sides of Eq. (1):

$$V(\omega) = L(i\omega)^2 q(\omega) + R(i\omega)q(\omega) + \frac{q(\omega)}{C} = \left(i\omega R - \omega^2 L + \frac{1}{C}\right)q(\omega) \quad \Longrightarrow \quad q(\omega) = \frac{V(\omega)}{i\omega R - \omega^2 L + 1/C}.$$

Applying Eq. (2), we find

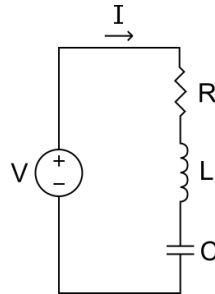$$\chi(\omega) = \frac{1}{i\omega R - \omega^2 L + 1/C}. \tag{3}$$

Figure 1: An LCR series circuit [2].

1(b) Show that the forced solution of this equation is

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}},$$

where

$$\tan(\phi) = \frac{\omega R}{\omega^2 L - 1/C}.$$

Solution. Equation (1) is an ODE representing forced damped motion of a mass-spring system. Its solution can be written as the sum of the homogeneous solution, which dies out with time, and a particular solution [4, pp. 38, 40, 50–51]. Rewriting Eq. (1) as

$$\frac{V_0}{L} \cos(\omega t) = \ddot{q} + 2p\dot{q} + \omega_0^2 q$$

where $p = R/2L$ and $\omega_0^2 = 1/LC$, the ansatz for the particular solution is

$$q(t) = A_c \cos(\omega t) + A_s \sin(\omega t).$$

Feeding this into the ODE and collecting terms yields

$$-A_c\omega^2 + 2pA_s\omega + \omega_0^2 A_c = \frac{V_0}{L}, \qquad\qquad -A_s\omega^2 - 2pA_c\omega + \omega_0^2 A_s = 0.$$

This system has the solutions [4, p. 51]

$$A_s = \frac{2p\omega V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}, \qquad\qquad A_c = \frac{(\omega^2 - \omega_0^2)V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}. \qquad (4)$$

If we define

$$A = \sqrt{A_c^2 + A_s^2} \qquad\qquad (5)$$

and write

$$q(t) = A\left(\frac{A_c}{A}\cos(\omega t) + \frac{A_s}{A}\sin(\omega t),\right)$$

there exists an angle ϕ such that $\cos(\phi) = A_c/A$, $\sin(\phi) = A_s/A$, and $\tan(\phi) = A_s/A_c$. Thus

$$q(t) = A[\cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega).$$

Using the identity

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta),$$

it follows that [4, pp. 39, 51]

$$q(t) = A\cos(\omega t - \phi).$$

The amplitude A is given by [4, p. 51]

$$A = \frac{CV_0}{\sqrt{(\nu^2 - 1)^2 + 4c^2\nu^2}}, \qquad\qquad \text{where} \quad c = \frac{R}{2\sqrt{L/C}}, \quad \nu = \frac{\omega}{\omega_0}.$$

Substituting back to the original quantities, this is

$$A = \frac{CV_0}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + (R^2C/L)(\omega^2/\omega_0^2)}} = \frac{CV_0}{\sqrt{(LC\omega^2 - 1)^2 + C^2 sR^2\omega^2}} = \frac{V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Additionally, from $\tan(\phi) = A_s/A_c$,

$$\tan(\phi) = \frac{2p\omega}{\omega^2 - \omega_0^2} = \frac{R\omega/L}{\omega^2 - 1/LC} = \frac{\omega R}{\omega^2 L - 1/C}.$$

Hence we have shown

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

1(c) Show that the mean rate of power dissipation is

$$W = \frac{1}{2} \frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Solution. The average power into a general AC circuit is [1, p. 1032]

$$P_{\text{av}} = \frac{1}{2} V I \sin(\phi),$$

where I is the current amplitude, V is the voltage amplitude, and ϕ is the phase angle determined in 1 [1, pp. 1028, 1032]. Assuming the circuit is perfectly efficient, the average power into the circuit is equal to the average power it dissipates, so $W = P_{\text{av}}$. Clearly $V = V_0$. For I,

$$I(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\left(\frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}\right) = -\frac{\omega V_0 \sin(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}, \qquad (6)$$

so

$$I = \frac{\omega V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}. \qquad (7)$$

Thus

$$W = \frac{1}{2} V I \sin(\phi) = \frac{1}{2} \frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}} \qquad (8)$$

as we wanted to show. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

1(d) Sketch the real and imaginary parts of χ as a function of frequency, for the cases $Q \ll 1$, $Q \approx 1$, and $Q \gg 1$, where $Q = \sqrt{L/C}/R$ is the "quality factor."

Where are the poles of χ in the complex ω plane?

Solution. In the case $Q \ll 1$, $\sqrt{L/C} \gg R$. For small R, Eq. (3) becomes

$$\chi(\omega) \approx \frac{1}{-\omega^2 L + 1/C}, \qquad (9)$$

which is purely real and diverges at $\pm 1/\sqrt{LC}$.

In the case $Q \approx 1$, $\sqrt{L/C} \approx R$. Then

$$\chi(\omega) \approx \frac{1}{i\omega - \omega^2 + 1}. \qquad (10)$$
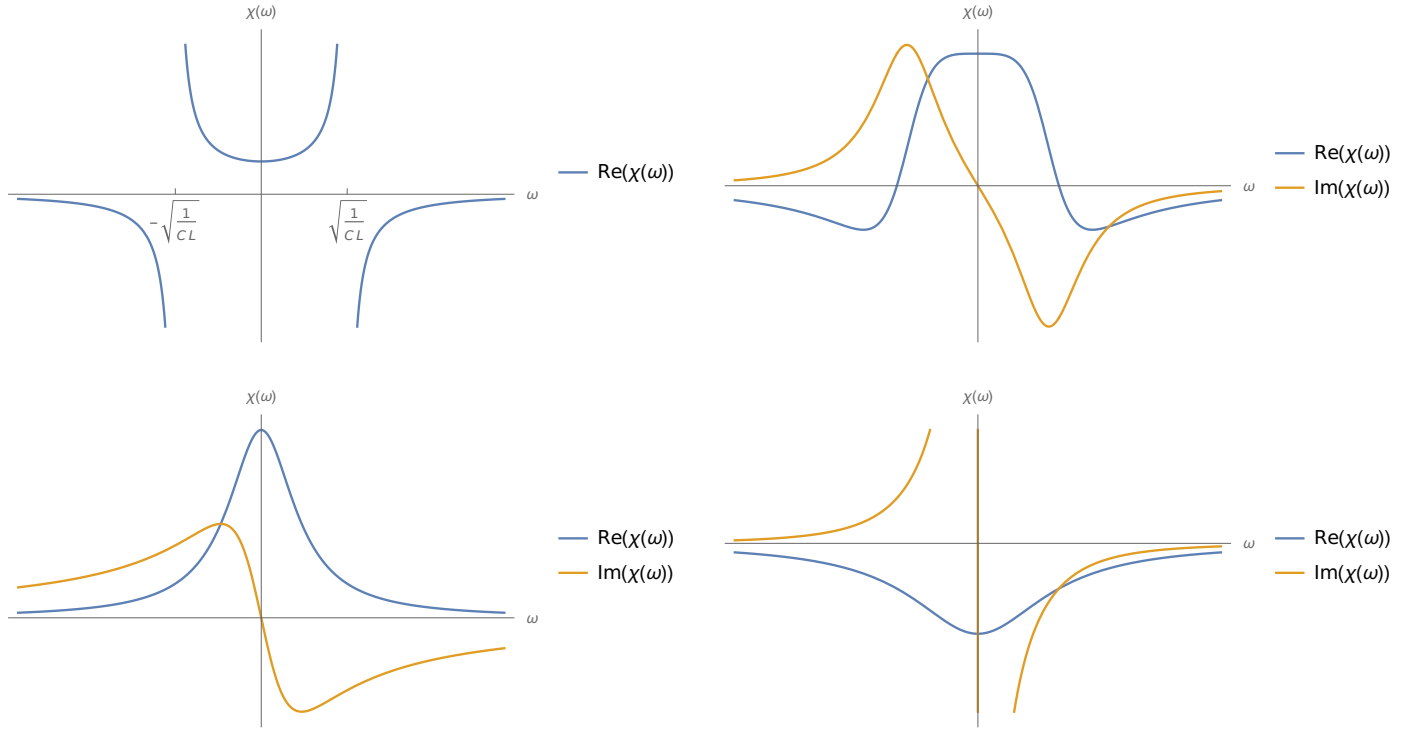
Figure 2: Real (blue line) and imaginary (gold line) parts of $\chi(\omega)$ in the cases $Q \ll 1$ (top left), $Q \approx 1$ (top right), $Q \gg 1$ with L small (bottom left), and $Q \gg 1$ with $1/C$ small (bottom right). These cases are given by Eqs. (9), (10), (11), and (12), respectively.

In the case $Q \gg 1$, $\sqrt{L} \ll R\sqrt{C}$ or $1/\sqrt{C} \ll R/\sqrt{L}$. For small L,

$$\chi(\omega) = \frac{1}{i\omega R + 1/C}. \tag{11}$$

For small $1/C$,

$$\chi(\omega) = \frac{1}{i\omega R - \omega^2 L}, \tag{12}$$

whose imaginary part diverges at 0.

Figure 2 shows plots of the real (blue line) and imaginary (gold line) parts of $\chi(\omega)$ in the cases $Q \ll 1$ (top left), $Q \approx 1$ (top right), $Q \gg 1$ with L small (bottom left), and $Q \gg 1$ with $1/C$ small (bottom right).

The function has poles at $\bar{\omega}$ such that $1/\chi(\bar{\omega}) = 0$ [5]:

$$0 = i\bar{\omega}R - \bar{\omega}^2 L + \frac{1}{C} \quad \implies \quad \bar{\omega} = \frac{iR \pm \sqrt{4L/C - R^2}}{2L}.$$

These poles are shown in Fig. 2 for the three Q regimes.

Problem 2. Landau theory of phase transitions   A ferroelectric crystal is one that supports a macroscopic polarization P, which usually arises because the underlying crystal structure does not have inversion symmetry. However, as temperature or pressure is changed, the crystal may recover the inversion symmetry. This can be modeled by Landau's theory of second order phase transitions, where we postulate a form for the free energy density (per unit volume)

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4 + \frac{c}{6}P^6 + \cdots, \tag{13}$$

where the coefficient $a = a_0(T - T_c)$ is temperature dependent and all the other coefficients are constant. Although the polarization P is of course a vector, we assume that it can point only in a symmetry direction of the crystal, and so it is replaced by a scalar.

2(a) Assume that $b > 0$ and $c = 0$. Use Eq. (13) to determine the form for the equilibrium $P(T)$.

Solution. When $b > 0$ and $c = 0$, Eq. (13) becomes

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4.$$

The equilibrium $P(T)$ occurs at the minima of $\mathcal{F}$, where $d\mathcal{F}/dP = 0$ [6]:

$$\frac{d\mathcal{F}}{dP} = aP + bP^3 = 0.$$

This implies

$$P = 0, \qquad\qquad\qquad P = \pm\sqrt{-\frac{a}{b}}.$$

Note, however, that $P = 0$ is a local maximum of $\mathcal{F}$ if $T < T_c$:

$$\left.\frac{d^2\mathcal{F}}{dP^2}\right|_{P=0} = \left[a + 2bP^2\right]_{P=0} = a_0(T - T_c) < 0 \quad \text{when } T < T_c.$$

However, the nonzero solution is imaginary for $T > T_c$. Thus the equilibrium $P(T)$ is given by

$$P(T) = \begin{cases} \pm\sqrt{\dfrac{a_0}{b}(T_c - T)} & T < T_c, \\[3mm] 0, \pm\sqrt{\dfrac{a_0}{b}(T_c - T)} & T > T_c. \end{cases} \tag{14}$$

2(b) Including in $\mathcal{F}$ the energy of the polarization coupled to an external electric field E, determine the dielectric susceptibility $\chi = dP/dE$ both above and below the critical temperature.

Solution. With the addition of the coupling term [6]:

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4 - EP.$$

Then

$$\frac{d\mathcal{F}}{dP} = aP + bP^3 - E = 0. \tag{15}$$

Differentiating both sides by E, we find

$$0 = a\frac{dP}{dE} + b\frac{dP^3}{dE} - 1 = a\frac{dP}{dE} + b\frac{dP^3}{dP}\frac{dP}{dE} - 1 = a\chi + 3bP^2\chi - 1,$$

which implies

$$\chi(T) = \begin{cases} \dfrac{1}{a + 3bP^2} = \dfrac{1}{a_0(T - T_c) + 3a_0(T_c - T)} = \dfrac{1}{2a_0(T_c - T)}, & T < T_c, \\[4mm] \dfrac{1}{a} = \dfrac{1}{a_0(T - T_c)}, & T > T_c, \end{cases} \tag{16}$$

where we have used $P(T)$ as defined in Eq. (14) and ignored the imaginary $P(T)$. Although this $P(T)$ is evaluated at $E = 0$, we assume the difference is negligible from $P(T)$ evaluated at small E, as mentioned on p. 79 of the lecture notes.
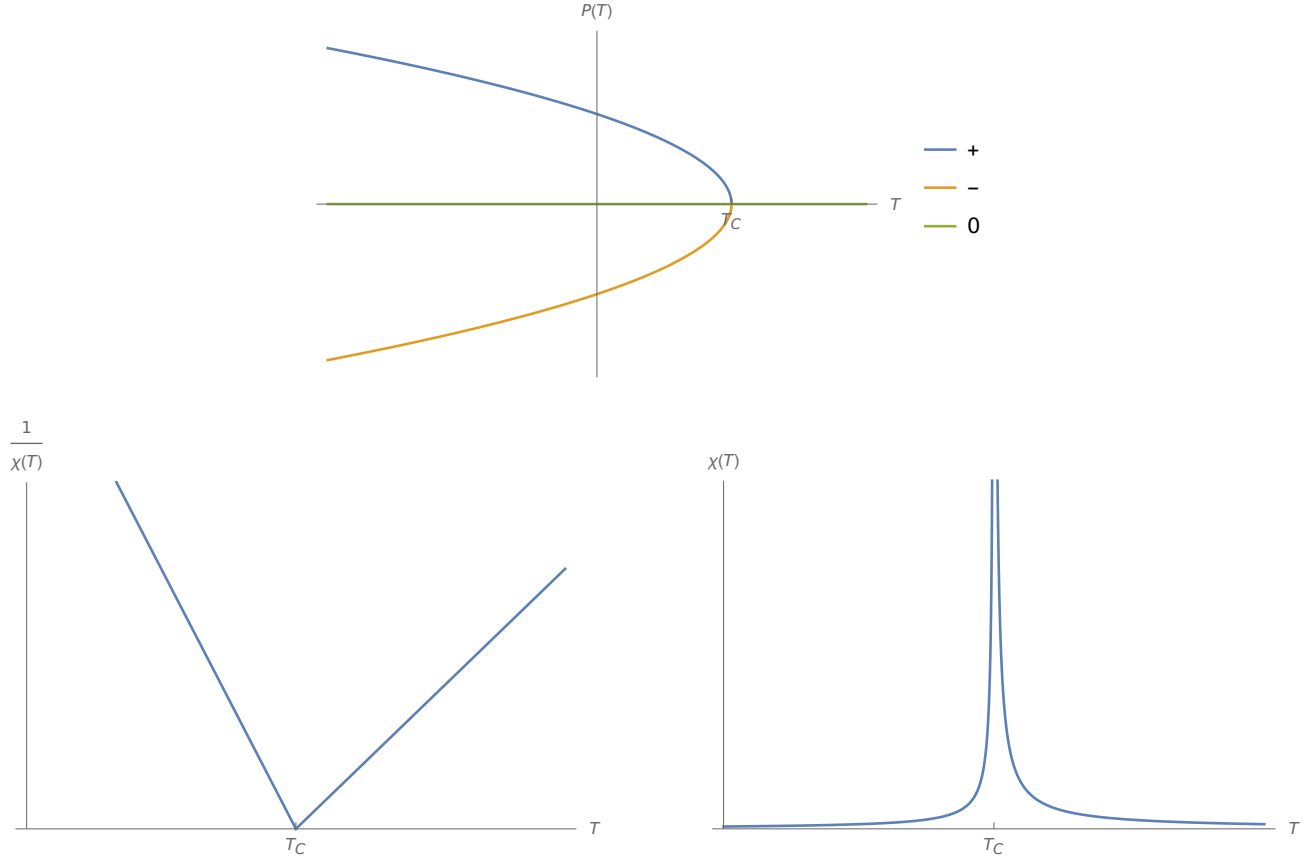
Figure 3: Plots of $P(T)$ (top), $\chi^{-1}(T)$ (bottom left), and $\chi(T)$ (bottom right), where $P(T)$ is given by Eq. (14) and $\chi(T)$ is given by Eq. (16). In the top figure, the blue (gold) line corresponds to the upper (lower) sign, and the green line to 0.

2(c) Sketch curves for $P(T)$, $\chi^{-1}(T)$, and $\chi(T)$.

Solution. Figure 3 shows the curves of $P(T)$ (top), $\chi^{-1}(T)$ (bottom left), and $\chi(T)$ (bottom right).

2(d) In a different material, the free energy is described by a similar form to Eq. (13), but with $b < 0$ and $c > 0$. By sketching $\mathcal{F}$ at different temperatures, discuss the behavior of the equilibrium polarization and the linear susceptibility, contrasting the results with those found in 2(c).

Solution. $\mathcal{F}$ is shown for at temperatures above, equal to, and below the critical temperature for the current case in Fig. 4 (left) and for the case of 2(c) in Fig. 4 (right).

Equilibrium polarizations occur at the local minima of $\mathcal{F}(P)$. In the $b < 0$, $c > 0$ case, there are three equilibrium polarizations when $T > T_c$ and two when $T \leq T_c$. In the $b > 0$, $c = 0$ case, there is one equilibrium polarization at $P = 0$ when $T \geq T_c$ and two when $T < T_c$. The behavior in the former case is signifies a first-order phase transition, since P acquires a nonzero value immediately below the critical temperature [7, p. 556].

In the $b > 0$, $c = 0$ case, the susceptibility diverges at the critical temperature as seen in Fig. 3 (bottom right). This is characteristic of a second-order phase transition as stated on p. 79 of the lecture notes. In the $b < 0$, $c > 0$ case, the susceptibility is discontinuous but does not diverge.
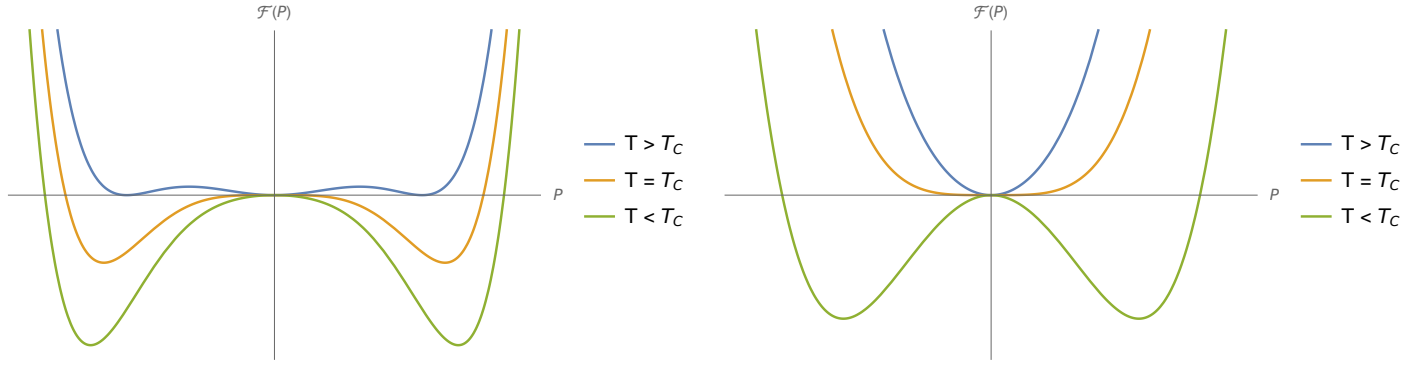
Figure 4: Plots of $\mathcal{F}(P)$ for the material of 2(d) with $b < 0$ and $c > 0$ (left) and for the material of 2(c) $b > 0$ and $c = 0$ (right). Curves are shown for $T > T_c$ (blue), $T = T_c$ (gold), and $T < T_c$ (green).

Problem 3.   **Reflectivity of metals**   The phase velocity of light in a conducting medium is the speed of light divided by the complex dielectric constant $N(\omega) = \sqrt{\epsilon(\omega)}$ where we may use for ϵ the Drude result

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega/\tau}. \tag{17}$$

In a good Drude metal, we have $1/\tau \ll \omega_p$.

3(a) Sketch curves of

(i) $\mathrm{Re}[\sigma(\omega)]$,

(ii) $\mathrm{Re}[\epsilon(\omega)]$,

(iii) $\mathrm{Im}[1/\epsilon(\omega)]$.

Solution.   The conductivity is defined in (5.25) of the lecture notes:

$$\sigma(\omega) = \frac{\omega_p^2}{4\pi(1/\tau - i\omega)}.$$

Thus

$$\mathrm{Re}[\sigma(\omega)] = \frac{\omega_p^2}{4\pi\tau} \frac{1}{1/\tau^2 + \omega^2}. \tag{18}$$

Note also that

$$\mathrm{Re}[\epsilon(\omega)] = 1 - \frac{\omega_p^2}{1/\tau^2 + \omega^2}. \tag{19}$$

and that

$$\mathrm{Im}[\epsilon(\omega)] = \frac{\omega_p^2}{\tau\omega} \frac{1}{1/\tau^2 + \omega^2}. \tag{20}$$

Also,

$$1/\epsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2 + i\omega/\tau}\right)^{-1} = \left(\frac{\omega^2 + i\omega/\tau - \omega_p^2}{\omega^2 + i\omega/\tau}\right)^{-1} = \frac{\omega^2 + i\omega/\tau}{\omega^2 + i\omega/\tau - \omega_p^2}$$

so

$$\mathrm{Im}[1/\epsilon(\omega)] = \frac{(\omega/\tau)(\omega^2 - \omega_p^2) - \omega^3/\tau}{(\omega^2 - \omega_p^2)^2 + \omega^2/\tau^2} = -\frac{\omega}{\tau} \frac{\omega_p^2}{(\omega^2 - \omega_p^2)^2 + \omega^2/\tau^2}. \tag{21}$$

Figure 5 shows plots of $\mathrm{Re}[\sigma(\omega)]$ (blue), $\mathrm{Re}[\epsilon(\omega)]$ (gold), and $\mathrm{Im}[1/\epsilon(\omega)]$ (green).
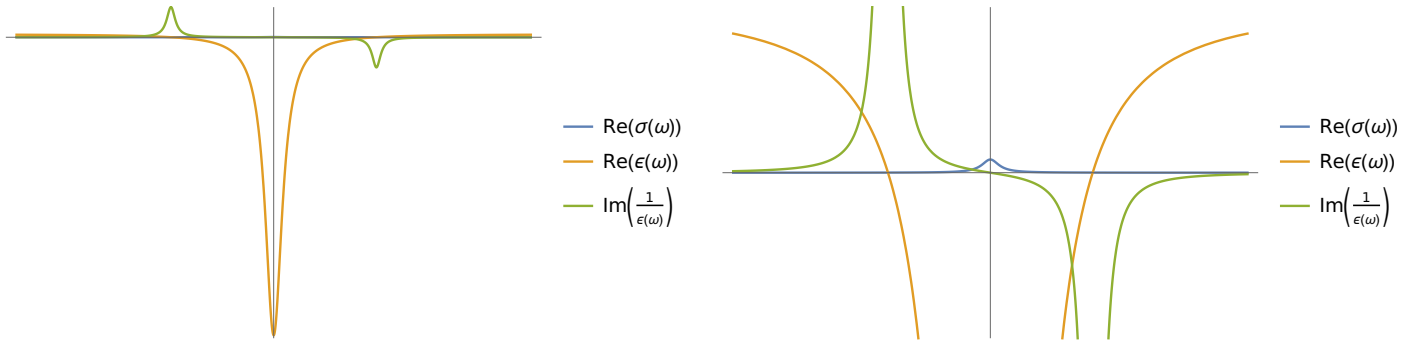
Figure 5: Plots of $\mathrm{Re}[\sigma(\omega)]$ (blue), $\mathrm{Re}[\epsilon(\omega)]$ (gold), and $\mathrm{Im}[1/\epsilon(\omega)]$ (green). These expressions are given by Eqs. (18), (19), and (21), respectively. The right figure is zoomed in to show the peak in $\mathrm{Re}[\sigma(\omega)]$.

3(b) Consider a light wave with the electric field polarized in the x direction at normal incidence from the vacuum on a good Drude metal occupying the region $z > 0$. In the vacuum ($z < 0$), the incident E_1 and reflected E_2 waves give rise to a field

$$E_x = E_1 e^{i\omega(z/c-t)} + E_2 e^{-i\omega(z/c+t)},$$

whereas in the medium, the electric field is

$$E_x = E_0 e^{i\omega[N(\omega)z/c-t]}.$$

Matching the electric and magnetic fields on the boundary, show that

$$E_0 = E_1 + E_2, \qquad\qquad\qquad NE_0 = E_1 - E_2,$$

and hence show that the reflection coefficient satisfies

$$R = \left|\frac{E_2}{E_1}\right|^2 = \left|\frac{1-N}{1+N}\right|^2.$$

Solution. The magnetic fields inside and outside the medium are, respectively,

$$B_y = \frac{N}{c}\left(E_1 e^{i\omega(z/c-t)} - E_2 e^{-i\omega(z/c+t)}\right), \qquad\qquad B_y = \frac{1}{c}E_0 e^{i\omega[N(\omega)z/c-t]},$$

since $v = c/N$ as stated in 3 [8, p. 403]. The electrodynamic boundary conditions are [8, p. 402]

$$\epsilon_{\mathrm{out}}E^{\perp}_{\mathrm{out}} = \epsilon_{\mathrm{in}}E^{\perp}_{\mathrm{in}}, \qquad B^{\perp}_{\mathrm{out}} = B^{\perp}_{\mathrm{in}}, \qquad E^{\|}_{\mathrm{out}} = E^{\|}_{\mathrm{in}}, \qquad \frac{B^{\|}_{\mathrm{out}}}{\mu_{\mathrm{out}}} = \frac{B^{\|}_{\mathrm{in}}}{\mu_{\mathrm{in}}},$$

where "in" represents the field, permittivity, or susceptibility inside the medium and "out" that outside the medium. Here, $\epsilon_{\mathrm{in}} = \epsilon(\omega)$, $\epsilon_{\mathrm{out}} = 1$, and $\mu_{\mathrm{in}} = \mu_{\mathrm{out}} = 1$. Since there are no components perpendicular to the surface, we only need the third and fourth equation. By the third,

$$E_1 + E_2 = E_0 \qquad\qquad\qquad\qquad (22)$$

as desired. By the fourth,

$$\frac{1}{c}(E_1 - E_2) = \frac{N}{c}E_0 \quad\implies\quad NE_0 = E_1 - E_2 \qquad\qquad\qquad (23)$$

as we wanted to show [8, p. 404].

Since the E_1 and E_0 terms of the field are propagating in the same direction, E_1 must be the amplitude of the incident wave and E_0 the amplitude of the transmitted wave. The E_2 component is propagating in the opposite direction, and so E_2 is the amplitude of the reflected wave. The reflection coefficient is defined by

$$R = \frac{I_R}{I_I},$$

where I_R and I_I are the intensities of the reflected and incident waves, respectively. The intensity is defined by [8, p. 402]

$$I = \frac{1}{2}\epsilon v E^2.$$

Thus

$$R = \frac{\epsilon v E_2^2/2}{\epsilon v E_1^2/2} = \left|\frac{E_2}{E_1}\right|^2.$$

Adding and subtracting Eq. (22) and (23), we find

$$(1+N)E_0 = 2E_1, \qquad\qquad (1-N)E_0 = 2E_2.$$

Then

$$R = \left|\frac{2E_2}{2E_1}\right|^2 = \left|\frac{(1-N)E_0}{(1+N)E_0}\right|^2 = \left|\frac{1-N}{1+N}\right|^2 \tag{24}$$

as we wanted to show. $\qquad\qquad\square$

3(c) Using the Drude formula above, show that

$$R \approx \begin{cases} 1 - 2\sqrt{\dfrac{\omega}{2\pi\sigma(0)}} & \omega \ll 1/\tau, \\[3mm] 1 - \dfrac{2}{\omega_p\tau} & 1/\tau \ll \omega \ll \omega_p, \\[3mm] 0 & \omega_p \ll \omega, \end{cases} \tag{25}$$

and sketch the reflectivity $R(\omega)$.

Solution. From Eq. (26), we can write

$$R = \left|\frac{1-N}{1+N}\right|^2 = \frac{(1-\mathrm{Re}[N])^2 + \mathrm{Im}[N]^2}{(1+\mathrm{Re}[N])^2 + \mathrm{Im}[N]^2}, \tag{26}$$

where we have used Ashcroft & Mermin (K.6).

When $\omega \ll 1/\tau$, from Eqs. (18) and (20),

$$\mathrm{Re}[\epsilon(\omega)] \approx 1 - \frac{\omega_p^2}{1/\tau^2} \approx 0, \qquad\qquad \mathrm{Im}[\epsilon(\omega)] = \frac{\omega_p^2}{\tau\omega}\frac{\tau^2}{1+\omega^2\tau^2} \approx \frac{\omega_p^2\tau}{\omega}.$$

Then Eq. (26) becomes (using Mathematica)

$$\mathrm{Im}[N] = \sqrt{\frac{\mathrm{Im}[\epsilon]}{2}} \approx \sqrt{\frac{\omega_p^2\tau}{2\omega}} = \mathrm{Re}[N],$$

so

$$R \approx \frac{(1 - \mathrm{Re}[N])^2 + \mathrm{Re}[N]^2}{(1 + \mathrm{Re}[N])^2 + \mathrm{Re}[N]^2}$$

$$= \frac{1 - 2\,\mathrm{Re}[N] + 2\,\mathrm{Re}[N]^2}{1 + 2\,\mathrm{Re}[N] + 2\,\mathrm{Re}[N]^2}$$

$$= \frac{1 + 2\,\mathrm{Re}[N] + 2\,\mathrm{Re}[N]^2 - 4\,\mathrm{Re}[N]}{1 + 2\,\mathrm{Re}[N] + 2\,\mathrm{Re}[N]^2}$$

$$= 1 - \frac{4\,\mathrm{Re}[N]}{1 + 2\,\mathrm{Re}[N] + 2\,\mathrm{Re}[N]^2}$$

$$= 1 - \frac{2}{\mathrm{Re}[N]}$$

$$= 1 - 2\sqrt{\frac{2\omega}{\omega_p^2 \tau}}$$

$$= 1 - 2\sqrt{\frac{\omega}{2\pi\sigma(0)}},$$

where we have used (5.27) in the lecture notes,

$$\sigma(0) = \frac{\omega_p^2 \tau}{4\pi}.$$

and the fact that

$$\mathrm{Re}[N]^2 = \frac{\omega_p^2 \tau^2}{2\omega\tau}$$

is very large. $\qquad \square$

When $1/\tau \ll \omega \ll \omega_p$,

$$\mathrm{Re}[\epsilon(\omega)] = 1 - \frac{\omega_p^2 \tau^2}{1 + \omega^2 \tau^2} \approx 1 - \frac{\omega_p^2}{\omega^2} \approx -\frac{\omega_p^2}{\omega^2}, \qquad\qquad \mathrm{Im}[\epsilon(\omega)] = \frac{\omega_p^2}{\tau\omega}\frac{\tau^2}{1 + \omega^2 \tau^2} \approx \frac{\omega_p^2}{\tau\omega^3}.$$

Then, since $\mathrm{Im}[\epsilon]$ is small,

$$N = \sqrt{-\frac{\omega_p^2}{\omega^2} + i\frac{\omega_p^2}{\tau\omega^3}} = i\frac{\omega_p}{\omega}\sqrt{1 - \frac{i}{\tau\omega}} \approx i\frac{\omega_p}{\omega}\left(1 - \frac{i}{2\tau\omega}\right) = \frac{\omega_p}{2\tau\omega^2} + i\frac{\omega_p}{\omega}.$$

Clearly $|N| \gg 1$. Then we can Taylor expand Eq. (26) for large $|N|$:

$$R = \frac{N-1}{1+N}\frac{N^*-1}{1+N^*} \approx \left(1 - \frac{2}{N}\right)\left(1 - \frac{2}{N^*}\right) \approx 1 - 2\left(\frac{1}{N} + \frac{1}{N^*}\right) = 1 - 2\frac{N+N^*}{|N|^2} = 1 - 2\frac{2\,\mathrm{Re}[N]}{|N|^2}.$$

Then

$$R = 1 - 2\frac{\omega_p/\tau\omega^2}{\omega_p^2/4\tau^2\omega^4 + \omega_p^2/\omega^2} \approx 1 - 2\frac{\omega_p/\tau\omega^2}{\omega_p^2/\omega^2} = 1 - \frac{2}{\omega_p\tau}$$

as desired. $\qquad \square$

When $\omega_p \ll \omega$, Eq. (17) becomes $\epsilon(\omega) \approx 1$, which implies $N = 1$, so $\mathrm{Re}[N] = 1$ and $\mathrm{Im}[N] = 0$. Then by Eq. (26),

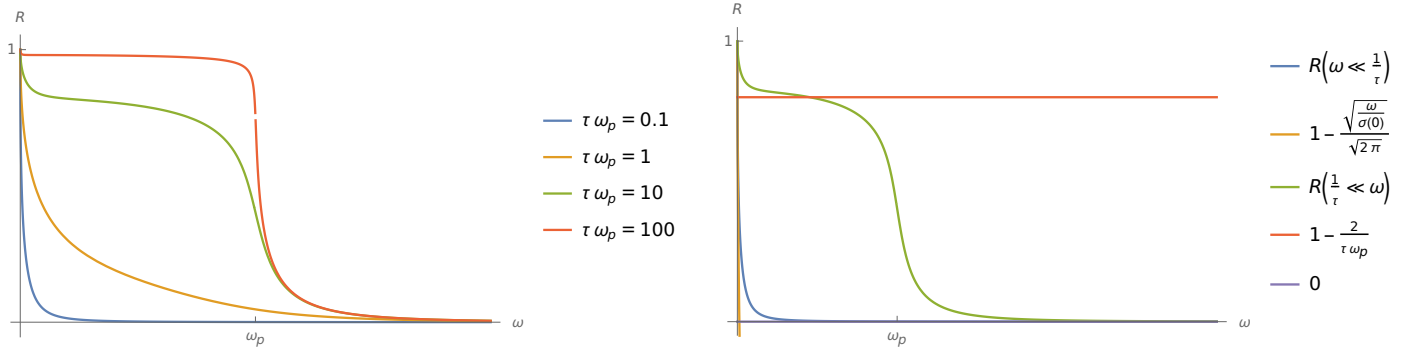$$R \approx \frac{(1-1)^2 + 0}{(1+1)^2 + 0} = \frac{0}{4} = 0$$

Figure 6: Left: exact expression for R in Eq. (26) for $\tau\omega_p = 0.1$ (blue), $\tau\omega_p = 1$ (gold), $\tau\omega_p = 10$ (green), and $\tau\omega_p = 100$ (red). Right: exact R for $\omega \ll 1/\tau$ (blue) and for $\omega \gg 1/\tau$ (gold) and Eq. (25) approximation for $\omega \ll 1/\tau$ (green), $1/\tau \ll \omega \ll \omega_p$ (red), and $\omega_p \ll \omega$ (violet).

as desired. $\qquad\square$

Figure 6 (left) shows the exact expression for R for four different values of $\tau\omega_p$: 0.1 (blue), 1 (gold), 10 (green), and 100 (red). Figure 6 (right) compares the exact expression for $\omega \ll 1/\tau$ (blue) and for $\omega \gg 1/\tau$ (gold) and to the approximations for $\omega \ll 1/\tau$ (green), $1/\tau \ll \omega \ll \omega_p$ (red), and $\omega_p \ll \omega$ (violet).

Problem 4.   Phonons   From Eq. (5.8) construct $\text{Im}[\chi]$ in the limit that $\gamma \to 0$. Use the Kramers–Krönig relation to then reconstruct $\text{Re}[\chi]$ from $\text{Im}[\chi]$ in the same limit.

Solution.   Equation (5.8) in the lecture notes is

$$\chi(\mathbf{q}, \omega) = \frac{1}{-\rho\omega^2 + i\gamma\omega + Kq^2}.$$

Define $\chi' = -\rho\chi$. Then

$$\chi' = \frac{1}{\omega^2 - i\gamma\omega/\rho - Kq^2/\rho} \equiv \frac{1}{\omega^2 - \omega_0^2 - ib\omega}$$

where we have defined $\omega_0^2 = Kq^2/\rho$ and $b = \gamma/\rho$. Then χ' has poles at

$$\omega = \frac{ib \pm \sqrt{4\omega_0^2 - b^2}}{2} \equiv i\alpha \pm \beta,$$

where we have defined $\alpha = b/2$ and $\beta = \sqrt{4\omega_0^2 - b^2}/2$. We note that β is real since $b \propto \gamma \to 0$. Then

$$\chi' = \frac{1}{(\omega - \beta - i\alpha)(\omega + \beta - i\alpha)}.$$

Note that as $\gamma \to 0$, $\alpha \to 0$ and $\beta \to 0$. Equation (5.85) in the lecture notes is

$$\lim_{\eta \to 0^+} \frac{1}{x + i\eta} = \text{PV}\frac{1}{x} - i\pi\delta(x).$$

Applying this,

$$\chi' = \left(\text{PV}\frac{1}{\omega - \beta} - i\pi\delta(\omega - \beta)\right)\left(\text{PV}\frac{1}{\omega + \beta} - i\pi\delta(\omega + \beta)\right)$$

$$= \text{PV}\frac{1}{\omega - \beta}\text{PV}\frac{1}{\omega + \beta} - i\pi\delta(\omega + \beta)\text{PV}\frac{1}{\omega - \beta} - i\pi\delta(\omega - \beta)\text{PV}\frac{1}{\omega + \beta} - \pi^2\delta(\omega + \beta)\delta(\omega - \beta)$$

$$= \text{PV}\frac{1}{\omega - \beta}\text{PV}\frac{1}{\omega + \beta} - i\pi\left(\delta(\omega + \beta)\text{PV}\frac{1}{\omega - \beta} + \delta(\omega - \beta)\text{PV}\frac{1}{\omega + \beta}\right).$$

Thus

$$\mathrm{Re}[\chi] = -\frac{1}{\rho(\omega - \beta)(\omega + \beta)}, \qquad \mathrm{Im}[\chi] = \frac{\pi}{\rho}\left(\frac{\delta(\omega + \beta)}{\omega - \beta} + \frac{\delta(\omega - \beta)}{\omega + \beta}\right) \qquad (27)$$

where

$$\beta = \frac{1}{2}\sqrt{\frac{4Kq^2}{\rho} - \frac{\gamma^2}{\rho^2}}. \qquad (28)$$

The relevant Kramers–Krönig relation is given by (5.39),

$$\mathrm{Re}[\kappa(\omega)] = \mathrm{PV}\int \frac{d\omega'}{\pi}\frac{\mathrm{Im}[\kappa(\omega')]}{\omega' - \omega}.$$

Then

$$\begin{aligned}
\mathrm{Re}[\chi] &= \frac{\pi}{\rho}\mathrm{PV}\int \frac{d\omega'}{\pi}\frac{1}{\omega' - \omega}\left(\frac{\delta(\omega' + \beta)}{\omega' - \beta} + \frac{\delta(\omega' - \beta)}{\omega' + \beta}\right) \\
&= \frac{\pi}{\rho}\mathrm{PV}\int \frac{d\omega'}{\pi}\left(\frac{\delta(\omega' + \beta)}{(\omega' - \omega)(\omega' - \beta)} + \frac{\delta(\omega' - \beta)}{(\omega' - \omega)(\omega' + \beta)}\right) \\
&= \frac{1}{\rho}\left(\frac{1}{(-\beta - \omega)(-\beta - \beta)} + \frac{1}{(\beta - \omega)(\beta + \beta)}\right) \\
&= \frac{1}{2\beta\rho}\left(\frac{\beta - \omega}{(\beta - \omega)(\beta + \omega)} + \frac{\beta + \omega}{(\beta - \omega)(\beta + \omega)}\right) \\
&= \frac{1}{2\beta\rho}\left(\frac{\beta - \omega + \beta + \omega}{(\beta - \omega)(\beta + \omega)}\right) \\
&= \frac{1}{2\beta\rho}\left(\frac{2\beta}{(\beta - \omega)(\beta + \omega)}\right) \\
&= -\frac{1}{\rho(\omega - \beta)(\omega + \beta)},
\end{aligned}$$

with β given by Eq. (28). This is the same as Eq. (27). □

Problem 5. Screened Coulomb interaction Consider a nucleus of charge Z producing a potential

$$V_{\mathrm{ext}}(\mathbf{q}) = -\frac{4\pi Ze^2}{q^2}.$$

Using the long-wavelength limit of the dielectric function, show that the screened potential satisfies

$$V_{\mathrm{scr}}(\mathbf{q} = 0) = -\frac{2}{3}\Omega E_F,$$

where Ω is the volume of the unit cell and E_F is the Fermi energy for Z free electrons per unit cell.

Solution. From (5.90) in the lecture notes,

$$\frac{V_{\mathrm{scr}}}{V_{\mathrm{ext}}} = \frac{1}{\epsilon(\mathbf{q}, \omega)}.$$

In the long-wavelength limit, the dielectric constant is given by (5.98),

$$\epsilon(\mathbf{q} \to 0, \omega) = 1 + \frac{q_{\mathrm{TF}}^2}{q^2},$$

where q_{TF} is the Thomas-Fermi wavevector, given by (4.134):

$$q_{\text{TF}}^2 = \frac{4}{\pi}\frac{me^2}{\hbar^2}k_F.$$

Thus

$$V_{\text{scr}} = \frac{V_{\text{ext}}}{\epsilon(\mathbf{q},\omega)} = -\frac{4\pi Ze^2}{q^2}\frac{1}{1 + q_{\text{TF}}^2/q^2} = -\frac{4\pi Ze^2}{q^2 + q_{\text{TF}}^2}.$$

When $\mathbf{q} = 0$,

$$V_{\text{scr}}(\mathbf{q} = 0) = -\frac{4\pi Ze^2}{q_{\text{TF}}^2} = -4\pi Ze^2\frac{\pi\hbar^2}{4me^2 k_F} = -\frac{\pi^2\hbar^2 Z}{mk_F}.$$

We note that Z represents the number of electrons per nucleus, so $Z = n\omega$ with n given by (2.9),

$$n = \frac{k_F^3}{3\pi^2}.$$

We note also that the Fermi energy is given by (2.3),

$$E_F = \frac{\hbar^2 k_F^2}{2m}.$$

With these substitutions, we obtain

$$V_{\text{scr}}(\mathbf{q} = 0) = -\frac{\pi^2\hbar^2 k_F^3\Omega}{3\pi^2 mk_F} = -\frac{\hbar^2 k_F^2\Omega}{3m} = -\frac{2}{3}\Omega E_F,$$

as we sought to prove. $\qquad\square$

Problem 6. Peierls transition By rewriting the term containing $n_{\mathbf{k}+\mathbf{q}}$ (replace $\mathbf{k} + \mathbf{q} \to -\mathbf{k}'$ and then relabel $\mathbf{k}'$ as $\mathbf{k}$), show that the static density response function can be written

$$\chi_0(\mathbf{q} = 0) = 2\sum_{k<k_F}\frac{1}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}.$$

In one dimension, make a linear approximation to the electronic dispersion near k_F, i.e. $\epsilon_{\mathbf{k}} = v_F|k|$, and consider the response for $q = 2k_F + p$, where $p \ll 2k_F$. By considering terms in the sum over k near $k \approx -k_F$, show that

$$\chi_0(2k_F + p) \approx \frac{1}{2\pi v_F}\ln\left|\frac{2k_F}{p}\right|.$$

Explain why this result suggests that a one-dimensional metal will be unstable to a lattice distortion with wavevector $2k_F$.

Solution. From (5.95) in the lecture notes,

$$\chi_0(\mathbf{q}) = 2\sum_{\mathbf{k}}\frac{n_{\mathbf{k}} - n_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}.$$

Following steps to get to (5.102),

$$\chi_0(\mathbf{q}) = 2\sum_{\mathbf{k}}\left(\frac{n_{\mathbf{k}}}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}} - \frac{n_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}\right) = 2\sum_{\mathbf{k}}\left(\frac{n_{\mathbf{k}}}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}} - \frac{n_{-\mathbf{k}}}{\epsilon_{-\mathbf{k}} - \epsilon_{-\mathbf{k}-\mathbf{q}}}\right),$$

where we have made the suggested substitution. We know from p. 91 of the lecture notes that

$$n(k) = \begin{cases} 1 & |k| < k_F, \\ 0 & \text{otherwise.} \end{cases}$$

We note also that $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$. Then

$$\chi_0(\mathbf{q}) = 2 \sum_{k < k_F} \left(\frac{1}{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}}} - \frac{1}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k+q}}} \right) = 2 \sum_{k < k_F} \left(\frac{1}{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}}} + \frac{1}{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}}} \right) = 4 \sum_{k < k_F} \frac{1}{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}}}$$

as desired (up to a factor of 2, which I think is incorrect in the problem statement). □

We approximate the sum as an integral and substitute $\epsilon_{\mathbf{k}} = v_F |k|$. When $k \approx -k_F$, we need only consider $k < 0$:

$$\begin{aligned}
\chi_0(\mathbf{q}) &= \frac{4}{2\pi} \int_0^{k_F} \frac{dk}{\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}}} \\
&= \frac{2}{\pi v_F} \int_0^{k_F} \frac{dk}{|k + 2k_F + p| - |k|} \\
&= \frac{2}{\pi v_F} \int_{-k_F}^{0} \frac{dk}{|k + 2k_F + p| - |k|} \\
&= -\frac{2}{\pi v_F} \int_{k_F}^{0} \frac{dk}{-k + 2k_F + p - k} \\
&= \frac{2}{\pi v_F} \int_0^{k_F} \frac{dk}{2k_F + p - 2k} \\
&= \frac{2}{\pi v_F} \frac{1}{2} \ln\left(\frac{2k_F + p}{2} \right) \\
&\approx \frac{1}{\pi v_F} \ln\left| \frac{k_F}{p} \right|,
\end{aligned}$$

since $p \ll k_F$. This is the desired result up to the same factor of 2 as before. □

This result suggests that the response function of a one-dimensional metal has a singularity at $q = 2k_F$. The charge density response function is given by (5.53) in the lecture notes,

$$\delta\rho(\mathbf{q}, \omega) = \chi(\mathbf{q}, \omega) V(\mathbf{q}, \omega).$$

This shows that a singularity in χ creates an infinite derivative of the charge density ρ at that value of $\mathbf{q}$. Therefore, a one-dimensional metal will have a divergent change in charge density when subject to a lattice distortion with wavevector $2k_F$, indicating an instability.

Problem 7. Optical properties Discuss why, at optical frequencies, glass is transparent and silver is shiny, while graphite appears black and powdered sugar is white.

Solution. Glass is an amorphous solid; that is, it does not have a crystal structure [9, pp. 573–674]. Only solids with crystal structure have an energy band structure [9, p. 161]. In order to absorb visible light, glass would need a band structure with the spacing between bands corresponding to optical frequencies. Glass does not have this (or any) band structure, and a large gap between energy levels, so it cannot absorb visible light. Moreover, glass is a nonmetal and so does not have free electrons, meaning it cannot appreciably reflect light.
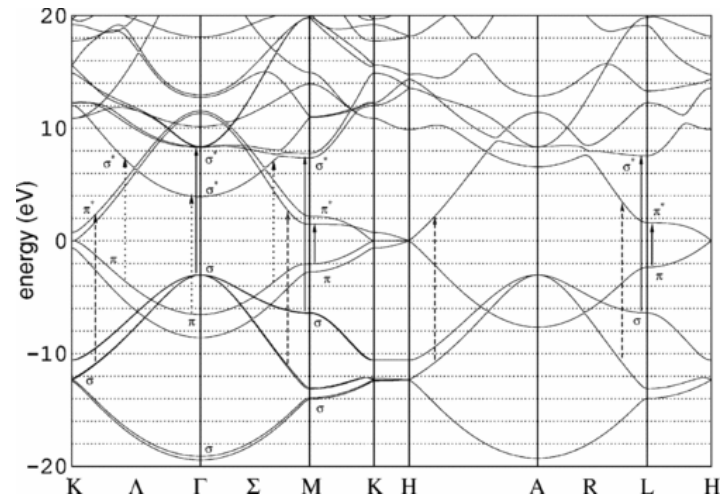
Figure 7: Band structure of graphite, from Ref. [10].

Finally, since glass is created by cooling a liquid without crystallization [9, pp. 573–574], it does not have structural impurities and thus does not scatter light. Since glass does not appreciably absorb, reflect, or scatter visible light, it must only transmit it; this makes it transparent.

Silver, in contrast, is a metal and therefore characterized by its abundance of free electrons in a partially filled energy band [7, p. 562]. This means it has a high plasma frequency $\omega_p^2 \propto n$ from (5.27), where n is the free electron concentration. We know from Fig. **??** (left) that a Drude metal reflects light of all frequencies up to the plasma frequency; since ω_p is large, it must be above the optical range. This enables silver to reflect much of the optical light that is incident upon it. In addition, the reflected waves interfere with light that would be transmitted. So silver, like other metals, appears shiny because it almost exclusively reflects light.

The band structure of graphite is shown in Fig. 7, which is from Ref. [10]. The de Broglie energy of visible light ranges from approximately 1.6 to 3.3 eV [1, p. 1051]. As the figure illustrates, graphite has many energy bands, and the spacing between them allows for band-to-band transmissions at nearly all optical frequencies. This means that nearly all visible light that is incident upon graphite is absorbed, creating its black appearance.

Table sugar is very pure, crystalline sucrose. Granulated sugar is made up of small, nearly-transparent sucrose crystals with smooth edges [11]. Powdered sugar is the result of grinding these crystals into a fine powder. Since the grinding process results in many structural imperfections and harsh edges on the surface of the powdered sugar particles, they scatter visible light. This gives powdered sugar its white appearance.

Problem 8. Metals and insulators Explain the differences between a metal and an insulator. Your discussion should include reference to single particle properties, screening of the Coulomb interaction, optical properties, and electrical and thermal properties.

Solution. An insulator has a completely filled set of bands and empty higher-energy bands. Thus, an insulator has a band gap. A metal, however, has a set of bands that is partially filled. The occupied states with highest energy sit at the Fermi surface in momentum space. The number of electrons per unit cell in a metal may be even or odd, although it is typically odd. In contrast, an insulator must have an even number of electrons per unit cell (as mentioned on p. 39 of the lecture notes).

In metals, the Coulomb interaction is screened; the external potential $V_{\text{ext}}(\mathbf{r}) = Q/r$ is "seen" by the electrons in the metal as $V(\mathbf{r}) = Qe^{-q_{\text{TF}}r}/r$ [7, p. 342]. This is because the charge distribution of the free electrons is

affected by the potential. Since insulators, not being conductors, do not have freely-flowing electrons within them, the effect of screening are small.

Plasma oscillations or "plasmons" occur within an electron gas (p. 75 of the lecture notes), meaning the free electrons in a metal. Metals are very reflective because they have a high plasma frequency due to their large number of free electrons, as explained in 7. Insulators, on the other hand, do not have such a freely-flowing electron gas; instead, they have band gaps that allow them to absorb light. If the band gaps are of the proper width to absorb optical frequencies, they will not be visibly reflective.

Metals readily conduct electricity thanks to their abundance of free electrons. Insulators, by definition, do not conduct electricity; their electrons are too limited in their range of motion. Insulators, however, can become electrically polarized. The specific heat of the electron gas is given by (2.16), where we see $c_v \propto T$. This means that, as the temperature of a metal increases, it costs more energy to increase its temperature.

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