Problem 1. LCR circuit An electrical circuit consists of an inductance L, resistance R and capacitance C in series, driven by a voltage source $V(t) = V_0 \cos(\omega t)$.

1(a) Show that the charge $q(t)$ on the capacitor satisfies the equation

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V(t), \tag{1}$$

and use it to define the complex susceptibility from

$$q(\omega) = \chi(\omega)V(\omega). \tag{2}$$

Solution. An example LCR circuit is shown in Fig. 1. We can use Kirchoff's loop rule to obtain the differential equation for this circuit. Beginning from the bottom left corner of the circuit and moving clockwise, we have [1, pp. 849, 1007]

$$0 = V(t) - IR - L\frac{dI}{dt} - \frac{q}{C},$$

where we have applied Ohm's law $V_{ab} = IR$, the potential difference across an inductor $V_{ab} = L\, dI/dt$, and the definition of capacitance $C = q/V_{ab}$ [1, pp. 782, 999]. The current $I(t) = dq(t)/dt$ and charge $q(t)$ are identical at all points in a series circuit. Feeding in $I = dq(t)/dt$, this relation becomes

$$V(t) = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

as we wanted to show. □

For the complex susceptibility, we recall that differentiating in the time domain is equivalent to multiplying by $i\omega$ in the frequency domain [3]:

$$\mathcal{F}_x[f^{(n)}(x)](\omega) = (i\omega)^n \mathcal{F}[f(x)](\omega).$$

We Fourier transform both sides of Eq. (1):

$$V(\omega) = L(i\omega)^2 q(\omega) + R(i\omega)q(\omega) + \frac{q(\omega)}{C} = \left(i\omega R - \omega^2 L^2 + \frac{1}{C}\right) q(\omega) \quad \Longrightarrow \quad q(\omega) = \frac{V(\omega)}{i\omega R - \omega^2 L^2 + 1/C}.$$

Applying Eq. (2), we find

$$\chi(\omega) = \frac{1}{i\omega R - \omega^2 L^2 + 1/C}. \tag{3}$$

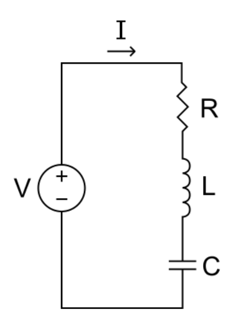

Figure 1: An LCR series circuit [2].

1(b) Show that the forced solution of this equation is

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}},$$

where

$$\tan(\phi) = \frac{\omega R}{\omega^2 L - 1/C}.$$

Solution. Equation (1) is an ODE representing forced damped motion of a mass-spring system. Its solution can be written as the sum of the homogeneous solution, which dies out with time, and a particular solution [4, pp. 38, 40, 50–51]. Rewriting Eq. (1) as

$$\frac{V_0}{L}\cos(\omega t) = \ddot{q} + 2p\dot{q} + \omega_0^2 q$$

where $p = R/2L$ and $\omega_0^2 = 1/LC$, the ansatz for the particular solution is

$$q(t) = A_c \cos(\omega t) + A_s \sin(\omega t).$$

Feeding this into the ODE and collecting terms yields

$$-A_c\omega^2 + 2pA_s\omega + \omega_0^2 A_c = \frac{V_0}{L}, \qquad -A_s\omega^2 - 2pA_c\omega + \omega_0^2 A_s = 0.$$

This system has the solutions [4, p. 51]

$$A_s = \frac{2p\omega V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}, \qquad A_c = \frac{(\omega^2 - \omega_0^2)V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}. \tag{4}$$

If we define

$$A = \sqrt{A_c^2 + A_s^2} \tag{5}$$

and write

$$q(t) = A\left(\frac{A_c}{A}\cos(\omega t) + \frac{A_s}{A}\sin(\omega t),\right)$$

there exists an angle ϕ such that $\cos(\phi) = A_c/A$, $\sin(\phi) = A_s/A$, and $\tan(\phi) = A_s/A_c$. Thus

$$q(t) = A[\cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega).$$

Using the identity

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta),$$

it follows that [4, pp. 39, 51]

$$q(t) = A\cos(\omega t - \phi).$$

The amplitude A is given by [4, p. 51]

$$A = \frac{CV_0}{\sqrt{(\nu^2 - 1)^2 + 4c^2\nu^2}}, \qquad \text{where} \quad c = \frac{R}{2\sqrt{L/C}}, \quad \nu = \frac{\omega}{\omega_0}.$$

Substituting back to the original quantities, this is

$$A = \frac{CV_0}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + (R^2C/L)(\omega^2/\omega_0^2)}} = \frac{CV_0}{\sqrt{(LC\omega^2 - 1)^2 + C^2 sR^2\omega^2}} = \frac{V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Additionally, from $\tan(\phi) = A_s/A_c$,

$$\tan(\phi) = \frac{2p\omega}{\omega^2 - \omega 0^2} = \frac{R\omega/L}{\omega^2 - 1/LC} = \frac{\omega R}{\omega^2 L - 1/C}.$$

Hence we have shown

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}$$

as desired. □

1(c) Show that the mean rate of power dissipation is

$$W = \frac{1}{2} \frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Solution. The average power into a general AC circuit is [1, p. 1032]

$$P_{\text{av}} = \frac{1}{2} V I \sin(\phi),$$

where I is the current amplitude, V is the voltage amplitude, and ϕ is the phase angle determined in 1 [1, pp. 1028, 1032]. Assuming the circuit is perfectly efficient, the average power into the circuit is equal to the average power it dissipates, so $W = P_{\text{av}}$. Clearly $V = V_0$. For I,

$$I(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\left(\frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}\right) = -\frac{\omega V_0 \sin(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}, \tag{6}$$

so

$$I = \frac{\omega V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}. \tag{7}$$

Thus

$$W = \frac{1}{2} V I \sin(\phi) = \frac{1}{2} \frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}} \tag{8}$$

as we wanted to show. □

1(d) Sketch the real and imaginary parts of χ as a function of frequency, for the cases $Q \ll 1$, $Q \approx 1$, and $Q \gg 1$, where $Q = \sqrt{L/C}/R$ is the "quality factor."

Where are the poles of χ in the complex ω plane?

Solution. From Eq. (3), we can write $\chi(\omega)$ in terms of Q as

$$\chi(\omega) = \frac{1}{i\omega Q\sqrt{C/L} + Q^2/R^2 L - \omega^2 Q^4 C^2/R^4}.$$

In the case $Q \ll 1$, the Taylor series expansion to $\mathcal{O}(Q)$ is

$$\chi(\omega) \approx \frac{L^2}{CR^2\omega^2} + i\left(\frac{QL^3}{\omega^3 R^4 C\sqrt{C/L}} - \frac{1}{\omega Q\sqrt{C/L}}\right) = \frac{L^2}{\omega^2 CR^2} + i\left(\frac{L^4}{\omega^3 R^5 C^2} - \frac{LR}{\omega C}\right),$$

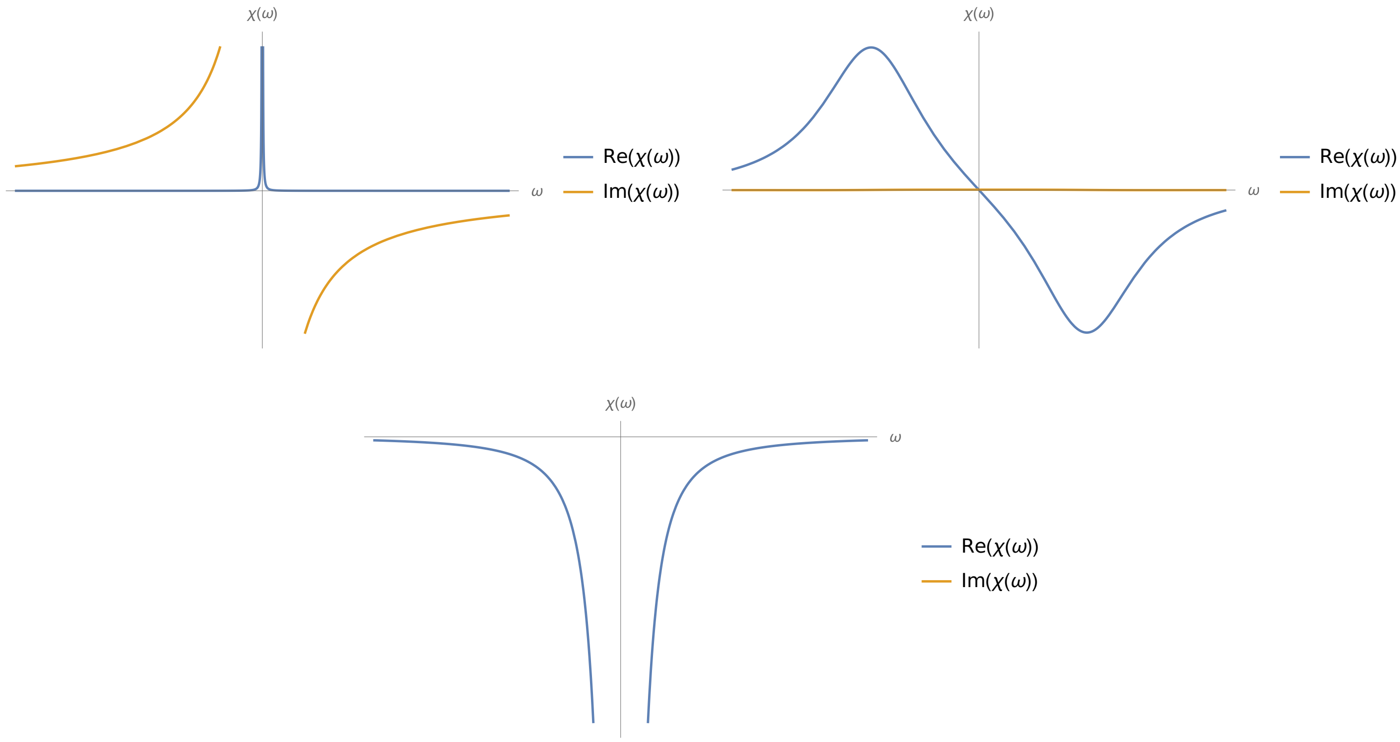

Figure 2: Real (blue line) and imaginary (gold line) parts of $\chi(\omega)$ in the cases $Q \ll 1$ (top left), $Q \approx 1$ (top right), and $Q \gg 1$ (bottom). These cases are given by Eqs. (6), (7), and (8), respectively.

which we have evaluated using Mathematica. In the case $Q \approx 1$,

$$\chi(\omega) \sim \frac{1}{i\omega R - \omega^2 L^2 + 1/C}$$

so

$$\mathrm{Re}[\chi(\omega)] = \frac{1/C - \omega^2 L^2}{(1/C - \omega^2 L^2)^2 + \omega^2 R^2}, \qquad \mathrm{Im}[\chi(\omega)] = -\frac{\omega R}{(1/C - \omega^2 L^2)^2 + \omega^2 R^2}.$$

In the case $Q \gg 1$, the Taylor series expansion to $\mathcal{O}(Q^4)$ is

$$\chi(\omega) \approx -\frac{R^4}{\omega^2 Q^2 C^2} = -\frac{R^2}{\omega^2 LC},$$

which is real.

Figure 2 shows plots of the real (blue line) and imaginary (gold line) parts of $\chi(\omega)$ in the cases $Q \ll 1$ (top left), $Q \approx 1$ (top right), and $Q \gg 1$ (bottom).

The function has poles at $\bar{\omega}$ such that $1/\chi(\bar{\omega}) = 0$ [5]:

$$0 = i\bar{\omega}R - \bar{\omega}^2 L^2 + \frac{1}{C} \quad \Longrightarrow \quad \bar{\omega} = \frac{iR \pm \sqrt{4L^2/C - R^2}}{2L^2}.$$

Problem 2. Landau theory of phase transitions A ferroelectric crystal is one that supports a macroscopic polarization P, which usually arises because the underlying crystal structure does not have inversion symmetry. However, as temperature or pressure is changed, the crystal may recover the inversion symmetry. This can be modeled by Landau's theory of second order phase transitions, where we postulate a form for the free energy density (per unit volume)

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4 + \frac{c}{6}P^6 + \cdots , \tag{9}$$

where the coefficient $a = a_0(T - T_c)$ is temperature dependent and all the other coefficients are constant. Although the polarization P is of course a vector, we assume that it can point only in a symmetry direction of the crystal, and so it is replaced by a scalar.

2(a) Assume that $b > 0$ and $c = 0$. Use Eq. (9) to determine the form for the equilibrium $P(T)$.

Solution. When $b > 0$ and $c = 0$, Eq. (9) becomes

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4.$$

The equilibrium $P(T)$ occurs at the minima of $\mathcal{F}$, where $d\mathcal{F}/dP = 0$ [6]:

$$\frac{d\mathcal{F}}{dP} = aP + bP^3 = 0.$$

This implies

$$P = 0, \qquad\qquad P = \pm\sqrt{-\frac{a}{b}}.$$

Note, however, that $P = 0$ is a local maximum of $\mathcal{F}$:

$$\left.\frac{d^2\mathcal{F}}{dP^2}\right|_{P=0} = \left[a + 2bP^2\right]_{P=0} = a_0(T - T_c) < 0 \quad \text{when } T < T_c,$$

which is the regime we are interested in for a ferroelectric [7, p. 556][6]. Thus the equilibrium $P(T)$ is given by

$$P(T) = \pm\sqrt{\frac{a_0}{b}(T_c - T)}. \tag{10}$$

2(b) Including in $\mathcal{F}$ the energy of the polarization coupled to an external electric field E, determine the dielectric susceptibility $\chi = dP/dE$ both above and below the critical temperature.

Solution. With the addition of the coupling term [6]:

$$\mathcal{F} = \frac{a}{2}P^2 + \frac{b}{4}P^4 - EP.$$

Then

$$\frac{d\mathcal{F}}{dP} = aP + bP^3 - E = 0. \tag{11}$$

Differentiating both sides by E, we find

$$0 = a\frac{dP}{dE} + b\frac{dP^3}{dE} - 1 = a\frac{dP}{dE} + b\frac{dP^3}{dP}\frac{dP}{dE} - 1 = a\chi + 3bP^2\chi - 1,$$

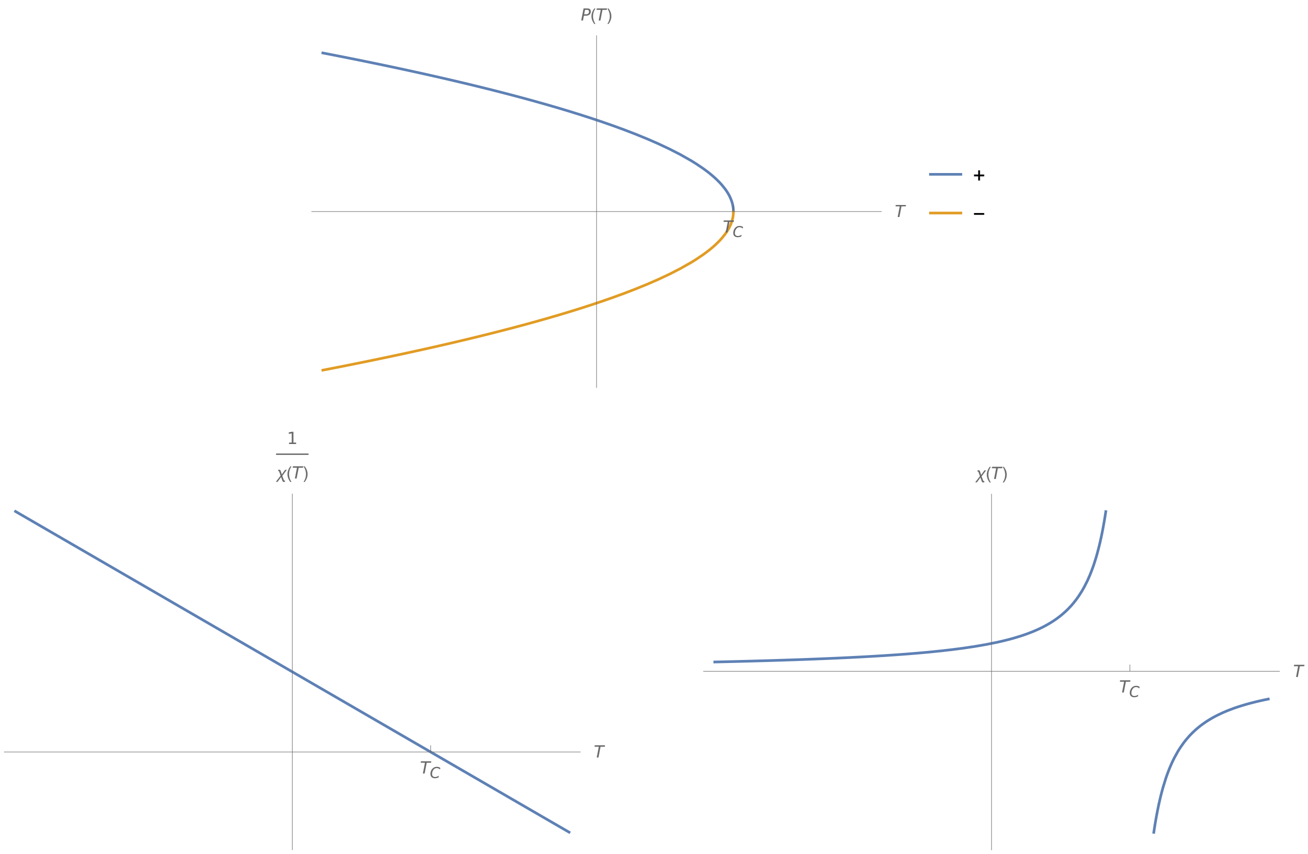

Figure 3: Plots of $P(T)$ (top), $\chi^{-1}(T)$ (bottom left), and $\chi(T)$ (bottom right), where $P(T)$ is given by Eq. (10) and $\chi(T)$ is given by Eq. (12). In the top figure, the blue (gold) line corresponds to the upper (lower) sign.

which implies

$$\chi(T) = \frac{1}{a + 3bP^2} = \frac{1}{a_0(T - T_c) + 3a_0(T_c - T)} = \frac{1}{2a_0(T_c - T)}, \tag{12}$$

where we have used $P(T)$ as defined in Eq. (10). Although this $P(T)$ is evaluated at $E = 0$, we assume the difference is negligible from $P(T)$ evaluated at small E, as mentioned on p. 79 of the lecture notes. This expression is valid above and below the critical temperature.

2(c) Sketch curves for $P(T)$, $\chi^{-1}(T)$, and $\chi(T)$.

Solution. Figure 3 shows the curves of $P(T)$ (top), $\chi^{-1}(T)$ (bottom left), and $\chi(T)$ (bottom right).

2(d) In a different material, the free energy is described by a similar form to Eq. (9), but with $b < 0$ and $c > 0$. By sketching $\mathcal{F}$ at different temperatures, discuss the behavior of the equilibrium polarization and the linear susceptibility, contrasting the results with those found in 2(c).

Solution. $\mathcal{F}$ is shown for at temperatures above, equal to, and below the critical temperature for the current case in Fig. 4 (left) and for the case of 2(c) in Fig. 4 (right).

Equilibrium polarizations occur at the local minima of $\mathcal{F}(P)$. In the $b < 0$, $c > 0$ case, there are three equilibrium polarizations when $T > T_c$ and two when $T \leq T_c$. In the $b > 0$, $c = 0$ case, there is one equilibrium

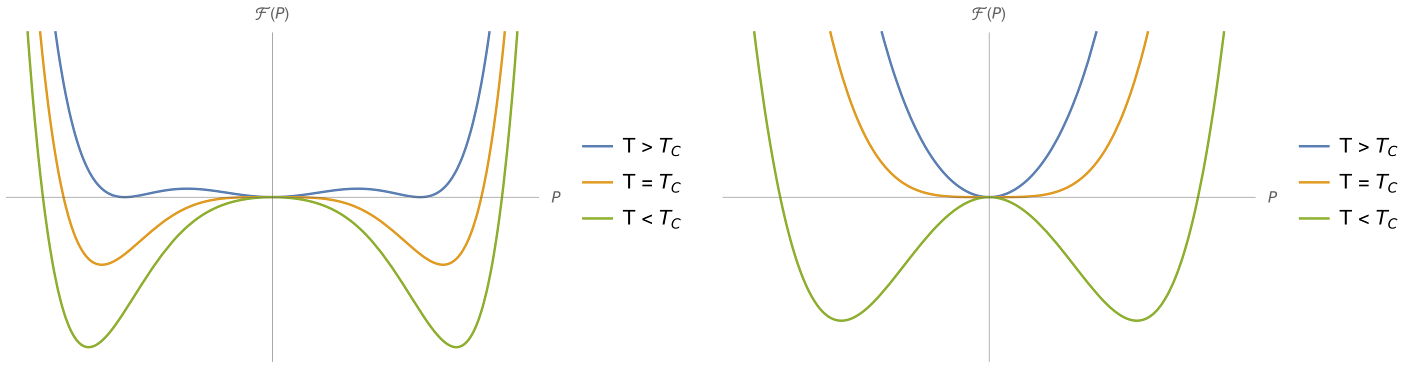

Figure 4: Plots of $\mathcal{F}(P)$ for the material of 2(d) with $b < 0$ and $c > 0$ (left) and for the material of 2(c) $b > 0$ and $c = 0$ (right). Curves are shown for $T > T_c$ (blue), $T = T_c$ (gold), and $T < T_c$ (green).

polarization at $P = 0$ when $T \geq T_c$ and two when $T < T_c$. The behavior in the former case is signifies a first-order phase transition, since P acquires a nonzero value immediately below the critical temperature [7, p. 556].

In the $b > 0$, $c = 0$ case, the susceptibility diverges at the critical temperature as seen in Fig. 3 (bottom right). This is characteristic of a second-order phase transition as stated on p. 79 of the lecture notes. In the $b < 0$, $c > 0$ case, the susceptibility is discontinuous but does not diverge.

Problem 3. Reflectivity of metals The phase velocity of light in a conducting medium is the speed of light divided by the complex dielectric constant $N(\omega) = \sqrt{\epsilon(\omega)}$ where we may use for ϵ the Drude result

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\omega/\tau}. \tag{13}$$

In a good Drude metal, we have $1/\tau \ll \omega_p$.

3(a) Sketch curves of

(i) $\mathrm{Re}[\sigma(\omega)]$,

(ii) $\mathrm{Re}[\epsilon(\omega)]$,

(iii) $\mathrm{Im}[1/\epsilon(\omega)]$.

Solution. The conductivity is defined in (5.25) of the lecture notes:

$$\sigma(\omega) = \frac{\omega_p^2}{4\pi(1/\tau - i\omega)}.$$

Thus

$$\mathrm{Re}[\sigma(\omega)] = \frac{\omega_p^2}{4\pi\tau}\frac{1}{1/\tau^2 + \omega^2}. \tag{14}$$

Note also that

$$\mathrm{Re}[\epsilon(\omega)] = 1 - \frac{\omega_p^2}{1/\tau^2 + \omega^2}. \tag{15}$$

and that

$$\mathrm{Im}[\epsilon(\omega)] = \frac{\omega_p^2}{\tau\omega}\frac{1}{1/\tau^2 + \omega^2}. \tag{16}$$

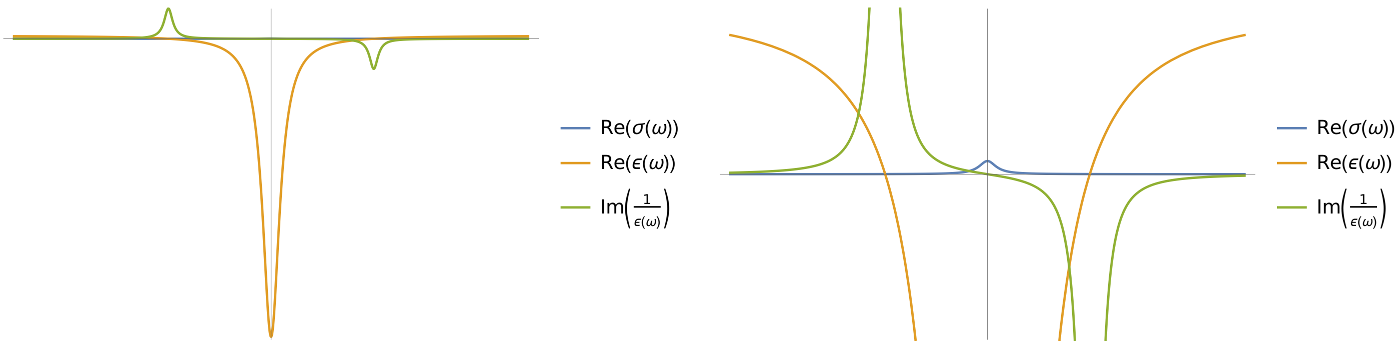

Figure 5: Plots of Re$[\sigma(\omega)]$ (blue), Re$[\epsilon(\omega)]$ (gold), and Im$[1/\epsilon(\omega)]$ (green). These expressions are given by Eqs. (14), (15), and (17), respectively. The right figure is zoomed in to show the peak in Re$[\sigma(\omega)]$.

Also,

$$1/\epsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2 + i\omega/\tau}\right)^{-1} = \left(\frac{\omega^2 + i\omega/\tau - \omega_p^2}{\omega^2 + i\omega/\tau}\right)^{-1} = \frac{\omega^2 + i\omega/\tau}{\omega^2 + i\omega/\tau - \omega_p^2}$$

so

$$\text{Im}[1/\epsilon(\omega)] = \frac{(\omega/\tau)(\omega^2 - \omega_p^2) - \omega^3/\tau}{(\omega^2 - \omega_p^2)^2 + \omega^2/\tau^2} = -\frac{\omega}{\tau}\frac{\omega_p^2}{(\omega^2 - \omega_p^2)^2 + \omega^2/\tau^2}. \tag{17}$$

Figure 5 shows plots of Re$[\sigma(\omega)]$ (blue), Re$[\epsilon(\omega)]$ (gold), and Im$[1/\epsilon(\omega)]$ (green).

3(b) Consider a light wave with the electric field polarized in the x direction at normal incidence from the vacuum on a good Drude metal occupying the region $z > 0$. In the vacuum ($z < 0$), the incident E_1 and reflected E_2 waves give rise to a field

$$E_x = E_1 e^{i\omega(z/c - t)} + E_2 e^{-i\omega(z/c + t)},$$

whereas in the medium, the electric field is

$$E_x = E_0 e^{i\omega[N(\omega)z/c - t]}.$$

Matching the electric and magnetic fields on the boundary, show that

$$E_0 = E_1 + E_2, \qquad NE_0 = E_1 - E_2,$$

and hence show that the reflection coefficient satisfies

$$R = \left|\frac{E_2}{E_1}\right|^2 = \left|\frac{1-N}{1+N}\right|^2.$$

Solution. The magnetic fields inside and outside the medium are, respectively,

$$B_y = \frac{N}{c}\left(E_1 e^{i\omega(z/c - t)} - E_2 e^{-i\omega(z/c + t)}\right), \qquad B_y = \frac{1}{c}E_0 e^{i\omega[N(\omega)z/c - t]},$$

since $v = c/N$ as stated in 3 [8, p. 403]. The electrodynamic boundary conditions are [8, p. 402]

$$\epsilon_{\text{out}} E_{\text{out}}^{\perp} = \epsilon_{\text{in}} E_{\text{in}}^{\perp}, \qquad B_{\text{out}}^{\perp} = B_{\text{in}}^{\perp}, \qquad E_{\text{out}}^{\parallel} = E_{\text{in}}^{\parallel}, \qquad \frac{B_{\text{out}}^{\parallel}}{\mu_{\text{out}}} = \frac{B_{\text{in}}^{\parallel}}{\mu_{\text{in}}},$$

where "in" represents the field, permittivity, or susceptibility inside the medium and "out" that outside the medium. Here, $\epsilon_{\text{in}} = \epsilon(\omega)$, $\epsilon_{\text{out}} = 1$, and $\mu_{\text{in}} = \mu_{\text{out}} = 1$. Since there are no components perpendicular to the surface, we only need the third and fourth equation. By the third,

$$E_1 + E_2 = E_0 \tag{18}$$

as desired. By the fourth,

$$\frac{1}{c}(E_1 - E_2) = \frac{N}{c}E_0 \quad \Longrightarrow \quad NE_0 = E_1 - E_2 \tag{19}$$

as we wanted to show [8, p. 404].

Since the E_1 and E_0 terms of the field are propagating in the same direction, E_1 must be the amplitude of the incident wave and E_0 the amplitude of the transmitted wave. The E_2 component is propagating in the opposite direction, and so E_2 is the amplitude of the reflected wave. The reflection coefficient is defined by

$$R = \frac{I_R}{I_I},$$

where I_R and I_I are the intensities of the reflected and incident waves, respectively. The intensity is defined by [8, p. 402]

$$I = \frac{1}{2}\epsilon v E^2.$$

Thus

$$R = \frac{\epsilon v E_2^2/2}{\epsilon v E_1^2/2} = \left|\frac{E_2}{E_1}\right|^2.$$

Adding and subtracting Eq. (18) and (19), we find

$$(1+N)E_0 = 2E_1, \qquad (1-N)E_0 = 2E_2.$$

Then

$$R = \left|\frac{2E_2}{2E_1}\right|^2 = \left|\frac{(1-N)E_0}{(1+N)E_0}\right|^2 = \left|\frac{1-N}{1+N}\right|^2 \tag{20}$$

as we wanted to show. □

3(c) Using the Drude formula above, show that

$$R \approx \begin{cases} 1 - 2\sqrt{\dfrac{\omega}{2\pi\sigma(0)}} & \omega \ll 1/\tau, \\ 1 - \dfrac{2}{\omega_p \tau} & 1/\tau \ll \omega \ll \omega_p, \\ 0 & \omega_p \ll \omega, \end{cases}$$

and sketch the reflectivity $R(\omega)$.

Solution. From Eq. (21), we can write

$$R = \left|\frac{1-N}{1+N}\right|^2 = \frac{(1-\text{Re}[N])^2 + \text{Im}[N]^2}{(1+\text{Re}[N])^2 + \text{Im}[N]^2}, \tag{21}$$

where we have used Ashcroft & Mermin (K.6).

When $\omega \ll 1/\tau$, from Eqs. (14) and (16),

$$\mathrm{Re}[\epsilon(\omega)] = 1 - \frac{\omega_p^2\tau^2}{1+\omega^2\tau^2} \approx 0, \qquad \mathrm{Im}[\epsilon(\omega)] = \frac{\omega_p^2}{\tau\omega}\frac{\tau^2}{1+\omega^2\tau^2} \approx \frac{\omega_p^2\tau}{\omega}.$$

Then (using Mathematica)

$$\mathrm{Im}[N] = \sqrt{\frac{\mathrm{Im}[\epsilon]}{2}} \approx \sqrt{\frac{\omega_p^2\tau}{2\omega}} = \mathrm{Re}[N],$$

so

$$\begin{aligned}
R &\approx \frac{(1-\mathrm{Re}[N])^2+\mathrm{Re}[N]^2}{(1+\mathrm{Re}[N])^2+\mathrm{Re}[N]^2} \\
&= \frac{1-2\,\mathrm{Re}[N]+2\,\mathrm{Re}[N]^2}{1+2\,\mathrm{Re}[N]+2\,\mathrm{Re}[N]^2} \\
&= \frac{1+2\,\mathrm{Re}[N]+2\,\mathrm{Re}[N]^2-4\,\mathrm{Re}[N]}{1+2\,\mathrm{Re}[N]+2\,\mathrm{Re}[N]^2} \\
&= 1 - \frac{4\,\mathrm{Re}[N]}{1+2\,\mathrm{Re}[N]+2\,\mathrm{Re}[N]^2} \\
&= 1 - \frac{2}{\mathrm{Re}[N]} \\
&= 1 - 2\sqrt{\frac{2\omega}{\omega_p^2\tau}} \\
&= 1 - 2\sqrt{\frac{\omega}{2\pi\sigma(0)}},
\end{aligned}$$

where we have used (5.27) in the lecture notes,

$$\sigma(0) = \frac{\omega_p^2\tau}{4\pi}.$$

and the fact that

$$\mathrm{Re}[N]^2 = \frac{\omega_p^2\tau^2}{2\omega\tau}$$

is very large. □

When $1/\tau \ll \omega \ll \omega_p$,

$$\mathrm{Re}[\epsilon(\omega)] = 1 - \frac{\omega_p^2\tau^2}{1+\omega^2\tau^2} \approx 1 - \frac{\omega_p^2}{\omega^2} \approx -\frac{\omega_p^2}{\omega^2}, \qquad \mathrm{Im}[\epsilon(\omega)] = \frac{\omega_p^2}{\tau\omega}\frac{\tau^2}{1+\omega^2\tau^2} \approx \frac{\omega_p^2}{\tau\omega^3}.$$

We can write $\epsilon = re^{i\theta}$, where

$$r = \sqrt{\frac{\omega_p^4}{\omega^4} + \frac{\omega_p^4}{\tau^2\omega^6}}, \qquad \theta = \tan^{-1}\left(-\frac{\omega_p^2}{\tau\omega^3}\frac{\omega^2}{\omega_p^2}\right) = \tan^{-1}\left(-\frac{1}{\tau\omega}\right) \approx 0.$$

Then

$$\mathrm{Re}[N] = \sqrt{r}\cos\left(\frac{\theta}{2}\right) \approx \sqrt{r}, \qquad \mathrm{Im}[N] = \sqrt{r}\sin\left(\frac{\theta}{2}\right) \approx 0,$$

so

$$R = \frac{(1-\sqrt{r})^2}{(1+\sqrt{r})^2} = \frac{1-2\sqrt{r}+r}{1+2\sqrt{r}+r} = \frac{1+2\sqrt{r}+r-4\sqrt{r}}{1+2\sqrt{r}+r} = 1 - \frac{4\sqrt{r}}{1+2\sqrt{r}+r} \approx 1 - \frac{4}{\sqrt{r}}.$$

Note that

$$\sqrt{r} = \left(\frac{\omega_p^4}{\omega^4} + \frac{\omega_p^4}{\tau^2\omega^6}\right)^{1/4} = \frac{\omega_p}{\omega}\sqrt{1 + \frac{1}{\tau^2\omega^2}} \approx \frac{\omega_p}{\omega}\left(1 + \frac{1}{4\tau^2\omega^2}\right) \approx 2\omega_p\tau,$$

so

$$R \approx 1 - \frac{2}{\omega_p\tau}$$

as we wanted to show. $\square$

When $\omega_p \ll \omega$, Eq. (13) becomes $\epsilon(\omega) \approx 1$, which implies $N = 1$, so $\text{Re}[N] = 1$ and $\text{Im}[N] = 0$. Then by Eq. (21),

$$R \approx \frac{(1-1)^2 + 0}{(1+1)^2 + 0} = \frac{0}{4} = 0,$$

as desired. $\square$

Problem 4. Phonons From Eq. (5.8) construct Im[χ] in the limit that $\gamma \to 0$. Use the Kramers–Krönig relation to then reconstruct Re[χ] from Im[χ] in the same limit.

Solution. Equation (5.8) in the lecture notes is

$$\chi(\mathbf{q}, \omega) = \frac{1}{-\rho\omega^2 + i\gamma\omega + Kq^2}.$$

Then

$$\mathrm{Re}[\chi] = \frac{Kq^2 - \rho\omega^2}{(Kq^2 - \rho\omega)^2 + \gamma^2\omega^2}, \qquad \mathrm{Im}[\chi] = -\frac{\gamma\omega}{(Kq^2 - \rho\omega)^2 + \gamma^2\omega^2}.$$

In the limit $\gamma \to 0$,

$$\mathrm{Re}[\chi] = \frac{Kq^2 - \rho\omega^2}{(Kq^2 - \rho\omega)^2}, \qquad \mathrm{Im}[\chi] = -\frac{\gamma\omega}{(Kq^2 - \rho\omega^2)^2}.$$

The relevant Kramers–Krönig relation is given by (5.39),

$$\mathrm{Re}[\kappa(\omega)] = \mathrm{PV}\int \frac{d\omega'}{\pi}\frac{\mathrm{Im}[\kappa(\omega')]}{\omega' - \omega}.$$

Then

$$\mathrm{Re}[\kappa(\omega)] = -\mathrm{PV}\int \frac{d\omega'}{\pi}\frac{1}{\omega' - \omega}\frac{\gamma\omega'}{(Kq^2 - \rho\omega'^2)^2} =$$

partial fraction decomposition???

Problem 5. Screened Coulomb interaction Consider a nucleus of charge Z producing a potential

$$V_{\text{ext}}(\mathbf{q}) = -\frac{4\pi Z e^2}{q^2}.$$

Using the long-wavelength limit of the dielectric function, show that the screened potential satisfies

$$V_{\text{scr}}(\mathbf{q}=0) = -\frac{2}{3}\Omega E_F,$$

where Ω is the volume of the unit cell and E_F is the Fermi energy for Z free electrons per unit cell.

Solution. From (5.90) in the lecture notes,

$$\frac{V_{\text{scr}}}{V_{\text{ext}}} = \frac{1}{\epsilon(\mathbf{q},\omega)}.$$

In the long-wavelength limit, the dielectric constant is given by (5.94),

$$\epsilon(0,\omega) = 1 - \frac{\omega_p^2}{\omega^2}.$$

Then

$$V_{\text{scr}} = -\frac{4\pi Z e^2}{q^2}\frac{1}{1-\omega_p^2/\omega^2}$$

Problem 6. Peierls transition By rewriting the term containing $n_{\mathbf{k}+\mathbf{q}}$ (replace $\mathbf{k}+\mathbf{q} \to -\mathbf{k}'$ and then relabel $\mathbf{k}'$ as $\mathbf{k}$), show that the static density response function can be written

$$\chi_0(\mathbf{q}=0) = 2\sum_{k<k_F} \frac{1}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}.$$

In one dimension, make a linear approximation to the electronic dispersion near k_F, i.e. $\epsilon_{\mathbf{k}} = v_F|k|$, and consider the response for $q = 2k_F + p$, where $p \ll 2k_F$. By considering terms in the sum over k near $k \approx -k_F$, show that

$$\chi_0(2k_F + p) \approx \frac{1}{2\pi v_F} \ln\left|\frac{k_F}{p}\right|.$$

Explain why this result suggests that a one-dimensional metal will be unstable to a lattice distortion with wavevector $2k_F$.

Solution. From (5.95) in the lecture notes,

$$\chi_0(\mathbf{q}) = 2\sum_{\mathbf{k}} \frac{n_{\mathbf{k}} - n_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}$$

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