Problem 1. Consider a dielectric ball of radius R with dielectric constant ϵ. Obtain a multipole expansion for the field, $\phi(\mathbf{x})$, of a point charge q placed at a point $\mathbf{x}'$ with $|\mathbf{x}'| = d > R$ (so the charge is outside of the dielectric ball).

Hint: Follow the procedure we used in class to find the multipole expansion of a point charge without the dielectric, but now consider the three regions $r \leq R$, $R \leq r \leq d$, and $r \geq d$. Obtain the form of the solution in these regions and match suitably.

Solution. The multipole expansion in spherical harmonics is given by Eq. (2.79) in the course notes,

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi), \tag{1}$$

where the spherical multipole moments q_{lm} are defined in Eq. (2.80),

$$q_{lm} \equiv \int \rho(\mathbf{x}')\, r'^l\, Y^*_{lm}(\theta', \phi')\, d^3x' \,.$$

Note that (1) is valid only for $|\mathbf{x}| \geq R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only within $|\mathbf{x}'| \leq R$, which is outside the dielectric.

The spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi},$$

and the associated Legendre polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l.$$

Problem 2. A dielectric ball of radius R and dielectric constant ϵ is placed in the external electrostatic potential $\phi_0 = \alpha(2z^2 - x^2 - y^2)$ where α is a constant, with the center of the ball at $\mathbf{x} = 0$.

2.a Find the total electrostatic potential ϕ everywhere.

Hint: It is useful to note that the external potential is proportional to $r^2\, Y_{20}(\theta, \phi)$. This should allow you to determine/guess the form of the total potential inside and outside the dielectric up to unknown constants, which can then be determined by matching.

Solution. Firstly, note that

$$Y_{20}(\theta, \phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1),$$

and

$$\phi_0 = \alpha r^2 (3\cos^2\theta - 1) = 4\alpha r^2 \sqrt{\frac{\pi}{5}} Y_{20}(\theta, \phi) \equiv \beta r^2\, Y_{20}(\theta, \phi),$$

where we have defined $\beta \equiv 4\alpha\sqrt{\pi/5}$.

We assume the dielectric is linear, homogeneous, and isotropic. Poisson's equation inside such a dielectric is given by Eq. (3.22) in the course notes,

$$\nabla^2 \langle \phi \rangle = -\frac{4\pi}{\epsilon} \langle \rho_f \rangle .$$

Here, $\langle \rho_f \rangle = 0$ since there are no free charges within the dielectric, so this reduces to Laplace's equation. The general solution to Laplace's equation is given by Eq. (3.61) in Jackson,

$$\langle \phi \rangle(r,\theta,\varphi) = \sum_{l,m} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta,\phi), \tag{2}$$

where A_{lm} and B_{lm} are constant coefficients.

In the region $r < R$, we must have $B_{lm} = 0$ because $1/r^{l+1}$ is undefined at the origin. In the region $r > R$, we may invoke the boundary condition at infinity:

$$\phi(r > R, \theta, \varphi) \to \phi_0 = \beta r^2 \, Y_{20}(\theta,\phi),$$

where we note that $\langle \phi \rangle = \phi$ for $r > R$. This implies that the only nonzero A_{lm} here is $A_{20} = \beta$. Thus we have

$$\langle \phi \rangle(r,\theta,\varphi) = \begin{cases} \sum_{l,m} A_{lm} r^l \, Y_{lm}(\theta,\phi) & \text{if } r \le R, \\ \beta r^2 \, Y_{20}(\theta,\phi) + \sum_{l,m} \frac{B_{lm}}{r^{l+1}} \, Y_{lm}(\theta,\phi) & \text{if } r \ge R. \end{cases}$$

To solve for the remaining coefficients, we invoke the boundary conditions at $r = R$. Firstly, $\langle \phi \rangle$ must be continuous at the boundary. This gives us

$$\langle \phi \rangle(R,\theta,\varphi) = \sum_{l,m} A_{lm} R^l \, Y_{lm}(\theta,\phi) = \beta R^2 \, Y_{20}(\theta,\phi) + \sum_{l,m} \frac{B_{lm}}{R^{l+1}} \, Y_{lm}(\theta,\phi),$$

so

$$A_{20} = \beta + \frac{B_{20}}{R^5}, \qquad A_{lm} = \frac{B_{lm}}{R^{l+3}} \quad \text{for } (l,m) \ne (2,0). \tag{3}$$

Secondly, we require that $\hat{\mathbf{n}} \cdot \langle \mathbf{D} \rangle$ is also continuous at the boundary, where

$$\langle \mathbf{D} \rangle = \epsilon \, \langle \mathbf{E} \rangle$$

inside the dielectric, from Eq. (3.20) in the course notes. (In vacuum, $\mathbf{D} = \mathbf{E}$.) Here we are only concerned with the r component of $\langle \mathbf{E} \rangle$. Applying $\langle \mathbf{E} \rangle = -\boldsymbol{\nabla} \langle \phi \rangle$, we have

$$\langle E_r \rangle(r,\theta,\phi) = \begin{cases} \sum_{l,m} A_{lm} l r^{l-1} \, Y_{lm}(\theta,\phi) & \text{if } r \le R, \\ 2\beta r \, Y_{20}(\theta,\phi) - \sum_{l,m} (l+1) \frac{B_{lm}}{r^{l+2}} \, Y_{lm}(\theta,\phi) & \text{if } r \ge R. \end{cases}$$

Then we need to satisfy

$$\langle \mathbf{D} \rangle(R,\theta,\varphi) = \epsilon \sum_{l,m} A_{lm} l R^{l-1} \, Y_{lm}(\theta,\phi) = 2\beta R \, Y_{20}(\theta,\phi) - \sum_{l,m} (l+1) \frac{B_{lm}}{R^{l+2}} \, Y_{lm}(\theta,\phi),$$

which stipulates

$$A_{20} = \frac{1}{\epsilon} \left(\beta - \frac{3}{2} \frac{B_{20}}{R^5} \right), \qquad A_{lm} = -\frac{1}{\epsilon} \frac{(l+1)}{l} \frac{B_{lm}}{R^{2l+1}} \quad \text{for } (l,m) \ne (2,0). \tag{4}$$

Eliminating B_{lm} from (3) and (4), we obtain

$$A_{20} = \frac{5\beta}{2\epsilon + 3}, \qquad A_{lm} = 0 \quad \text{for } (l, m) \neq (2, 0),$$

and substituting back into (3) yields

$$B_{20} = 2\beta R^5 \frac{1-\epsilon}{2\epsilon + 3}, \qquad B_{lm} = 0 \quad \text{for } (l, m) \neq (2, 0).$$

Finally, the total electrostatic potential everywhere is

$$\langle\phi\rangle(r, \theta, \varphi) = \alpha(3\cos^2\theta - 1)r^2 \times \begin{cases} \dfrac{5}{2\epsilon + 3} & \text{if } r \leq R, \\ 1 + 2\dfrac{1-\epsilon}{2\epsilon+3}\dfrac{R^5}{r^5} & \text{if } r \geq R, \end{cases} \tag{5}$$

or, in Cartesian coordinates,

$$\langle\phi\rangle(x, y, z) = \alpha(2z^2 - x^2 - y^2) \times \begin{cases} \dfrac{5}{2\epsilon + 3} & \text{if } r \leq R, \\ 1 + 2\dfrac{1-\epsilon}{2\epsilon+3}\dfrac{R^5}{\sqrt{x^2+y^2+z^2}} & \text{if } r \geq R. \end{cases}$$

2.b Calculate the interaction energy between the field produced by the dielectric and the external field. Assume that the potential arises from "distant charges" so that the formula for $\mathcal{E}_{\text{int}}$ given in class and the notes can be used.

Solution. Equation (3.34) in the lectures notes gives the interaction energy:

$$\mathscr{E}_{\text{int}} = \int (\langle\rho_f\rangle\phi_0 - \langle\mathbf{P}\rangle \cdot \mathbf{E}_0)\, d^3x\,,$$

where $\mathbf{E}_0$ is the electric field due to the external potential ϕ_0. Again, $\rho_f = 0$. For our assumption of a linear, homogeneous, and isotropic dielectric,

$$\langle\mathbf{P}\rangle = \chi\langle\mathbf{E}\rangle$$

by Eq. (3.19), where

$$\epsilon = 1 + 4\pi\chi$$

from Eq. (3.21).

The gradient in spherical coordinates is

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\,\hat{\boldsymbol{\varphi}}. \tag{6}$$

Differentiating (5) for $r \leq R$,

$$\langle E_r\rangle = 2\alpha\frac{5}{2\epsilon+3}r(3\cos^2\theta - 1), \qquad \langle E_\theta\rangle = -6\alpha\frac{5}{2\epsilon+3}r\cos\theta\sin\theta, \qquad \langle E_\varphi\rangle = 0. \tag{7}$$

For the external field,

$$E_{0r} = 2\alpha r(3\cos^2\theta - 1), \qquad E_{0\theta} = -6\alpha r\cos\theta\sin\theta, \qquad E_{0\varphi} = 0. \tag{8}$$

Note that $\langle \mathbf{P} \rangle = (\epsilon - 1)\langle \mathbf{E} \rangle / 4\pi$, so

$$\langle \mathbf{P} \rangle \cdot \mathbf{E}_0 = 4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} r^2 \left[(3\cos^2\theta - 1)^2 + 9\cos^2\theta \sin^2\theta \right]$$

Then

$$\begin{aligned}
\mathscr{E}_{\text{int}} &= -\int \langle \mathbf{P} \rangle \cdot \mathbf{E}_0 \, d^3x = 4\alpha^2 \frac{1-\epsilon}{4\pi} \frac{5}{2\epsilon + 3} \int_0^{2\pi} d\varphi \int_{-1}^{1} (3\cos^2\theta + 1)\, d(\cos\theta) \int_0^R r^4 \, dr \\
&= 4\alpha^2 \frac{1-\epsilon}{4\pi} \frac{5}{2\epsilon + 3} \Big[\varphi \Big]_0^{2\pi} \Big[\cos^3\theta + \cos\theta \Big]_{-1}^{1} \left[\frac{r^5}{5} \right]_0^R = 4\alpha^2 \frac{1-\epsilon}{4\pi} \frac{5}{2\epsilon + 3} (2\pi)(4) \frac{R^5}{5} = 8\alpha^2 \frac{1-\epsilon}{2\epsilon + 3} R^5.
\end{aligned}$$

2.c Calculate the total force needed to hold the dielectric ball in place.

Solution. Equation (3.26) in the lecture notes gives the total force on a dielectric:

$$\mathbf{F} = \int [\langle \rho_f \rangle \mathbf{E}_0 + (\langle \mathbf{P} \rangle \cdot \boldsymbol{\nabla}) \mathbf{E}_0] \, d^3x \, .$$

In electrostatics, there is no contribution from the dielectric's self field, and here $\rho_f = 0$. To find the force needed to hold the ball in place, we will need to insert a minus sign.

From (7) and (6), we have

$$\langle \mathbf{P} \rangle \cdot \boldsymbol{\nabla} = 2\alpha \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} \left[r(3\cos^2\theta - 1) \frac{\partial}{\partial r} - 3\cos\theta \sin\theta \frac{\partial}{\partial \theta} \right] .$$

From (8), note that

$$\begin{aligned}
\frac{\partial \mathbf{E}_0}{\partial r} &= \frac{\partial}{\partial r} \left[2\alpha r (3\cos^2\theta - 1)\, \hat{\mathbf{r}} - 6\alpha r \cos\theta \sin\theta \, \hat{\boldsymbol{\theta}} \right] = 2\alpha \left[(3\cos^2\theta - 1)\, \hat{\mathbf{r}} - 3\cos\theta \sin\theta \, \hat{\boldsymbol{\theta}} \right] , \\
\frac{\partial \mathbf{E}_0}{\partial \theta} &= \frac{\partial}{\partial \theta} \left[\alpha r (3\cos 2\theta + 1)\, \hat{\mathbf{r}} - 3\alpha r \sin 2\theta \, \hat{\boldsymbol{\theta}} \right] = -6\alpha r (\sin 2\theta \, \hat{\mathbf{r}} + \cos 2\theta \, \hat{\boldsymbol{\theta}}) \\
&= -6\alpha r \left[2\cos\theta \sin\theta \, \hat{\mathbf{r}} + (\cos^2\theta - \sin^2\theta)\, \hat{\boldsymbol{\theta}} \right] .
\end{aligned}$$

Then

$$\begin{aligned}
(\langle \mathbf{P} \rangle \cdot \boldsymbol{\nabla}) \mathbf{E}_0 &= 2\alpha \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} \left[r(3\cos^2\theta - 1) \frac{\partial \mathbf{E}_0}{\partial r} - 3\cos\theta \sin\theta \frac{\partial \mathbf{E}_0}{\partial \theta} \right] \\
&= 4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} r \Big[(3\cos^2\theta - 1) \Big((3\cos^2\theta - 1)\, \hat{\mathbf{r}} - 3\cos\theta \sin\theta \, \hat{\boldsymbol{\theta}} \Big) \\
&\qquad + 9\cos\theta \sin\theta \Big(2\cos\theta \sin\theta \, \hat{\mathbf{r}} + (\cos^2\theta - \sin^2\theta)\, \hat{\boldsymbol{\theta}} \Big) \Big] \\
&= 4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} r \Big[\big((3\cos^2\theta - 1)^2 - 18\cos^2\theta \sin^2\theta \big)\, \hat{\mathbf{r}} \\
&\qquad + 3\cos\theta \sin\theta \left(1 - 3\cos^2\theta + 3(\cos^2\theta - \sin^2\theta) \right) \hat{\boldsymbol{\theta}} \Big] \\
&= 4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} r \left[(-9\cos^4\theta + 12\cos^2\theta + 1)\, \hat{\mathbf{r}} + 3\cos\theta (-3\sin^2\theta + \sin\theta)\, \hat{\boldsymbol{\theta}} \right] ,
\end{aligned}$$

and the integral becomes

$$\begin{aligned}
\mathbf{F} &= -\int (\langle \mathbf{P} \rangle \cdot \boldsymbol{\nabla})\mathbf{E}_0 \, d^3x \\
&= -4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} \int_0^{2\pi} \int_{-1}^{1} \int_0^R r^3 \left[(-9\cos^4\theta + 12\cos^2\theta + 1)\, \hat{\mathbf{r}} + 3\cos\theta(-3\sin^2\theta + \sin\theta)\, \hat{\boldsymbol{\theta}} \right] dr \, d(\cos\theta) \, d\varphi \\
&= -4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} \int_0^{2\pi} d\varphi \int_{-1}^{1} \left[(-9\cos^4\theta + 12\cos^2\theta + 1)\, \hat{\mathbf{r}} + 3\cos\theta(-3\sin^2\theta + \sin\theta)\, \hat{\boldsymbol{\theta}} \right] d(\cos\theta) \int_0^R r^3 \, dr \, .
\end{aligned}$$

For the second integral, note that

$$\int_{-1}^{1} \cos\theta(-3\sin^2\theta + \sin\theta)\, d(\cos\theta) = \int_0^{\pi} \cos\theta \sin\theta(-3\sin^2\theta + \sin\theta)\, d\theta = \int_0^0 (-3\sin^3\theta + \sin^3\theta)\, d(\sin\theta) = 0.$$

Then we have

$$\begin{aligned}
\mathbf{F} &= -4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} \Big[\varphi \Big]_0^{2\pi} \left[-\frac{9}{5}\cos^5\theta + 4\cos^3\theta + \cos\theta \right]_{-1}^{1} \left[\frac{r^4}{4} \right]_0^R \hat{\mathbf{r}} = -4\alpha^2 \frac{\epsilon - 1}{4\pi} \frac{5}{2\epsilon + 3} (2\pi) \left(\frac{32}{5} \right) \frac{R^4}{4} \, \hat{\mathbf{r}} \\
&= -16\alpha^2 \frac{\epsilon - 1}{2\epsilon + 3} R^4 \, \hat{\mathbf{r}}.
\end{aligned}$$

In addition to the course lecture notes, I consulted Jackson's *Classical Electrodynamics* while writing up these solutions.