Problem 1. Supersymmetry (Peskin & Schroeder 3.5) It is possible to write field theories with continuous symmetries linking fermions and bosons; such transformations are called *supersymmetries*.

1(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, written in the form

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^* F.$$

Here F is an auxiliary complex scalar field whose field equation is $F = 0$. Show that this Lagrangian is invariant (up to a total divergence) under the infinitesimal transformation

$$\delta\phi = -i\epsilon^T\sigma^2\chi, \qquad \delta\chi = \epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^*, \qquad \delta F = -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi,$$

where the parameter ϵ_a is a 2-component spinor of Grassmann numbers.

Solution. Using the supplied transformations and dropping terms of $\mathcal{O}(\delta^2)$, we have

$$\begin{aligned}
\mathcal{L} &\to \partial_\mu(\phi^* + \delta\phi^*)\partial^\mu(\phi+\delta\phi) + (\chi^\dagger + \delta\chi^\dagger)i\bar{\sigma}\cdot\partial(\chi+\delta\chi) + (F^*\delta F^*)(F+\delta F)\\
&\approx \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \chi^\dagger\bar{\sigma}\cdot\partial\delta\chi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F + F^*\delta F + \delta F^* F\\
&= \mathcal{L} + \partial_\mu\phi^*\partial^\mu\delta\phi + \partial_\mu\delta\phi^*\partial^\mu\phi + \chi^\dagger\bar{\sigma}\cdot\partial\delta\chi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F + \delta F^* F. \qquad (1)
\end{aligned}$$

Note that Grassmann numbers satisfy $\alpha\beta = -\beta\alpha$ and $(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*$ for any α,β [1, p. 73]. Then

$$\begin{aligned}
\delta\phi^* &= i(\epsilon^T\sigma^2\chi)^* = i\epsilon^\dagger\sigma^{2*}\chi^* = -i\epsilon^\dagger\sigma^2\chi^* = i\chi^\dagger\sigma^2\epsilon^*,\\
\delta\chi^\dagger &= (\epsilon F)^\dagger + (\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*)^\dagger = F^*\epsilon^\dagger + \epsilon^T\sigma^{2\dagger}\partial_\mu\phi^*\sigma^{\mu\dagger} = F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu,\\
\delta F^* &= -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi = i(\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi)^* = -i\epsilon^T\bar{\sigma}^{\mu*}\partial_\mu\chi^* = i\partial_\mu\chi^\dagger\bar{\sigma}^{\mu\dagger}\epsilon,
\end{aligned}$$

where we have transposed as needed to obtain $\chi^\dagger$ or χ^*. So the $\mathcal{O}(\delta)$ terms in Eq. (1) are

$$\begin{aligned}
\partial_\mu\phi^*\partial^\mu\delta\phi &= -i\partial_\mu\phi^*\partial^\mu(\epsilon^T\sigma^2\chi), & \partial_\mu\delta\phi^*\partial^\mu\phi &= i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi,\\
\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\epsilon F + \sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*), & \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi &= i(F^*\epsilon^\dagger + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)\bar{\sigma}^\nu\partial_\nu\chi, \qquad (2)\\
F^*\delta F &= -iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi, & \delta F^* F &= i\partial_\mu\chi^\dagger\bar{\sigma}^{\mu\dagger}\epsilon F.
\end{aligned}$$

Adding the fourth and fifth terms above,

$$\delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F = iF^*\epsilon^\dagger\bar{\sigma}^\nu\partial_\nu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi - iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi = i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi.$$

Adding this to the first term of Eq. (2),

$$\partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F = -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\bar{\sigma}^\nu\partial_\nu\chi.$$

Note that

$$\sigma^\mu\bar{\sigma}^\nu = \frac{\sigma^\mu\bar{\sigma}^\nu + \bar{\sigma}^\nu\sigma^\mu + \sigma^\mu\bar{\sigma}^\nu - \bar{\sigma}^\nu\sigma^\mu}{2} = \frac{\{\sigma^\mu,\bar{\sigma}^\nu\}}{2} + \frac{[\sigma^\mu,\bar{\sigma}^\nu]}{2} = g^{\mu\nu} + \frac{[\sigma^\mu,\bar{\sigma}^\nu]}{2}$$

where we have used $\{\sigma^\mu,\bar{\sigma}^\nu\} = 2g^{\mu\nu}$ since $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ [2, p. 165]. Then

$$\begin{aligned}
\partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*\delta F &= -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^* g^{\mu\nu}\partial_\nu\chi + \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\\
&= -i\partial_\mu\phi^*\epsilon^T\sigma^2\partial^\mu\chi + i\epsilon^T\sigma^2\partial_\mu\phi^*\partial^\mu\chi + \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\\
&= \frac{i}{2}\epsilon^T\sigma^2\partial_\mu\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\\
&= \partial_\mu\left(\frac{i}{2}\epsilon^T\sigma^2\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu]\right). \qquad (3)
\end{aligned}$$

Adding the third and sixth terms of Eq. (2),

$$\begin{aligned}\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi + \delta F^* F &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\epsilon F) + i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) + i\partial_\mu\chi^\dagger\bar{\sigma}^{\mu\dagger}\epsilon F\\ &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) + i\bar{\sigma}^\mu\partial_\mu(\chi^\dagger\epsilon F)\\ &= i\chi^\dagger\bar{\sigma}^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) + \partial_\mu(i\bar{\sigma}^\mu\chi^\dagger\epsilon F)\end{aligned}$$

Adding this to the second term of Eq. (2),

$$\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi + \delta F^* F + \partial_\mu\delta\phi^*\partial^\mu\phi = i\chi^\dagger\bar{\sigma}^\mu\sigma^\nu\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) + i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu(i\bar{\sigma}^\mu\chi^\dagger\epsilon F).$$

Similar to before,

$$\bar{\sigma}^\mu\sigma^\nu = \frac{\bar{\sigma}^\mu\sigma^\nu + \sigma^\nu\bar{\sigma}^\mu + \bar{\sigma}^\mu\sigma^\nu - \sigma^\nu\bar{\sigma}^\mu}{2} = \frac{\{\bar{\sigma}^\mu,\sigma^\nu\}}{2} + \frac{[\bar{\sigma}^\mu,\sigma^\nu]}{2} = g^{\mu\nu} + \frac{[\bar{\sigma}^\mu,\sigma^\nu]}{2},$$

so

$$\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi + \delta F^* F + \partial_\mu\delta\phi^*\partial^\mu\phi = i\chi^\dagger g^{\mu\nu}\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) + \frac{i}{2}\chi^\dagger[\bar{\sigma}^\mu,\sigma^\nu]\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) + i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu(i\bar{\sigma}^\mu\chi^\dagger\epsilon F).$$

Note that

$$\chi^\dagger[\bar{\sigma}^\mu,\sigma^\nu]\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) = \chi^\dagger[\bar{\sigma}^\nu,\sigma^\mu]\partial_\nu(\partial_\mu\phi\sigma^2\epsilon^*) = -\chi^\dagger[\bar{\sigma}^\mu,\sigma^\nu]\partial_\mu(\partial_\nu\phi\sigma^2\epsilon^*) = 0,$$

where we have used $[\bar{\sigma}^\mu,\sigma^\nu] = -[\bar{\sigma}^\nu,\sigma^\mu]$, since $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ [2, p. 165]. Then

$$\begin{aligned}\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\delta\chi + \delta F^* F + \partial_\mu\delta\phi^*\partial^\mu\phi &= i\chi^\dagger\partial_\mu(\partial^\mu\phi\sigma^2\epsilon^*) + i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu(i\bar{\sigma}^\mu\chi^\dagger\epsilon F)\\ &= \partial_\mu(i\partial^\mu\chi^\dagger\sigma^2\epsilon^*\phi + i\bar{\sigma}^\mu\chi^\dagger\epsilon F).\end{aligned}\tag{4}$$

Finally, substituting Eqs. (3) and (4) into Eq. (1),

$$\mathcal{L} \to \mathcal{L} + \partial_\mu\left(\frac{i}{2}\epsilon^T\sigma^2\phi^*\partial_\nu\chi[\sigma^\mu,\bar{\sigma}^\nu] + i\partial^\mu\chi^\dagger\sigma^2\epsilon^*\phi + i\bar{\sigma}^\mu\chi^\dagger\epsilon F\right),$$

which is the same up to a total divergence. □

1(b) Show that the term

$$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2}im\chi^T\sigma^2\chi\right) + (\text{complex conjugate})$$

is also left invariant by the transformation given in 1(a). Eliminate F from the complete Lagrangian $\mathcal{L} + \delta\mathcal{L}$ by solving its field equation, and show that the fermion and boson fields ϕ and χ are given the same mass.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] J. J. Sakurai, "Modern Quantum Mechanics". Addison-Wesley Publishing Company, revised edition, 1994.