Problem 1. Alternative regulators in QED (Peskin & Schroeder 7.2) In Section. 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Paul-Villars regularization.

1(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

Solution. From (7.47) in Peskin & Schroeder,

$$\Gamma^\mu(q=0) = \frac{1}{Z_1}\gamma^\mu,$$

we can find an expression for δZ_1, similar to how (7.31) is obtained from (7.26). Roughly,

$$\frac{1}{Z_1+\delta Z_1}\gamma^\mu \approx Z_1(1-\delta Z_1)\gamma^\mu = \Gamma^\mu(q=0) + \delta\Gamma^\mu(q=0) \qquad \Longrightarrow \qquad \delta\Gamma^\mu(q=0) = -\delta Z_1\gamma^\mu. \tag{1}$$

According to (6.33),

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

We note that $\Gamma^\mu = \gamma^\mu$, $F_1 = 1$, and $F_2 = 0$ to lowest order [1, pp. 185–186]. Then we can write

$$\delta\Gamma^\mu(q=0) = \gamma^\mu\delta F_1(0) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\delta F_2(0). \tag{2}$$

Using this equation and the identity $\gamma^\mu\gamma_\mu = 4$ [2], Eq. (1) can be written

$$\delta Z_1 = -\frac{1}{4}\gamma_\mu\delta\Gamma^\mu(q=0) = -\delta F_1(0) - \gamma_\mu\frac{i\sigma^{\mu\nu}q_\nu}{8m}\delta F_2(0). \tag{3}$$

In order to find $\delta\Gamma^\mu$ we use (6.47):

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3}\\ \times\,\bar{u}(p')\left\{\gamma^\mu\left[-\frac{\ell^2}{2}+(1-x)(1-y)q^2+(1-4z+z^2)m^2\right]\right.\\ \left.+\frac{i\sigma^{\mu\nu}q_\nu}{2m}[2m^2z(1-z)]\right\}u(p),\end{aligned} \tag{4}$$

where $\Delta \equiv -xyq^2 + (1-z)^2m^2$ by (6.44), $\ell \equiv k + yq - zp$, and $D = \ell^2 - \Delta + i\epsilon$ [1, p. 191]. The momenta k and p are assigned in the Feynman diagram on p. 189 of Peskin & Schroeder, and x, y are Feynman parameters [1, p. 190].

The forms of the two integrals we need to compute are given by Peskin & Schroeder (6.49) and (6.50). Equation (6.49) is

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}. \tag{5}$$

Here $m=3$ because we have D^{-3} in Eq. (4). We can evaluate the left-hand side using the Euclidian 4-momentum defined in (6.48),

$$\ell^0 \equiv i\ell^0_E, \qquad\qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E. \tag{6}$$

Following the steps on p. 193, we have

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} = \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int \frac{d^4\ell_E}{(\ell_E^2+\Delta)^3} = \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^3},$$

where we have replaced the upper (infinite) bound of integration by a finite number Λ. Evaluating this integral using Mathematica and using $\int d\Omega_4 = 2\pi^2$ [1, p. 193], we find

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{\Lambda^4}{4\Delta(\Delta+\Lambda^2)^2}\\ &= -\frac{i}{32\pi^2}\frac{\Lambda^4}{\Delta(\Delta+\Lambda^2)^2}\\ &\approx -\frac{i}{32\pi^2}\frac{1}{\Delta} \equiv \alpha,\end{aligned} \tag{7}$$

where we have taken the limit $\Lambda \gg \Delta$ [1, p. 218] and defined α. Equation (6.50) in Peskin & Schroeder is

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}.$$

Following similar steps as for Eq. (4), the left-hand side is

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^3} &= \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3}\\ &= \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4\int_0^\Lambda d\ell_E\,\frac{\ell_E^5}{(\ell_E^2+\Delta)^3}\\ &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{1}{4}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right]\\ &= -\frac{i}{32\pi^2}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right]\\ &\approx -\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right) \equiv \beta,\end{aligned} \tag{8}$$

where we have defined β and ignored terms of $\mathcal{O}(\Lambda^{-2})$ [1, p. 218]. We also ignore constant terms since they do not diverge [1, p. 196].

We now set $q^2 = 0$, and define $\Delta_0 = (1-z)^2m^2$. Then $\Delta \to \Delta_0$ in our expression and $\alpha \to \alpha_0, \beta \to \beta_0$ (which are functions of Δ_0). Feeding in Eqs. (7) and (8), Eq. (4) can be written

$$\bar{u}(p')\delta\Gamma^\mu(q=0)u(p) = 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p')\int\left\{\gamma^\mu\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right]\right\}u(p).$$

Then

$$\begin{aligned}\delta F_1(0) &= 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right]\\ &= 2ie^2\int_0^1 dz\,(1-z)\left[-\beta_0 + 2m^2(1-4z+z^2)\alpha_0\right],\end{aligned}$$

$$\begin{aligned}\delta F_2(0) &= 8ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)m^2z(1-z)\alpha_0\\ &= 8ie^2\int_0^1 dz\,m^2z(1-z)^2\alpha_0.\end{aligned}$$

We ignore $\delta F_2(0)$ since it is not affected by the divergence [1, p. 196]. In order to avoid issues coming from the divergence in $\delta F_1(0)$, we add a $z\mu^2$ term to Δ_0 [1, p. 195]. So, feeding these results into Eq. (3), we obtain

$$\delta Z_1 = -2ie^2 \int_0^1 dz\,(1-z)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right], \tag{9}$$

where

$$\alpha_0 = -\frac{i}{32\pi^2}\frac{1}{\Delta_0}, \qquad \beta_0 = -\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta_0}\right), \qquad \Delta_0 = (1-z)^2m^2 + z\mu^2. \tag{10}$$

For δZ_2, we can use the first part of Peskin & Schroeder (7.31),

$$\delta Z_2 = \left.\frac{d\Sigma_2}{d\not{p}}\right|_{\not{p}=m}, \tag{11}$$

where Σ_2 is given by (7.17),

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\not{p}+4m_0}{(\ell^2-\Delta+i\epsilon)^2}, \tag{12}$$

where $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. We may once again follow the steps on p. 193 to evaluate the integral, now with $m = 2$. Changing the upper bound of integration to Λ once more, we have

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} &= \frac{i}{(-1)^2}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{1}{(\ell_E^2+\Delta)^2} \\
&= \frac{i(-1)^2}{(2\pi)^4}\int d\Omega_4\int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^2} \\
&= \frac{i}{(2\pi)^4}(2\pi^2)\frac{1}{2}\left[\frac{\Delta}{\Delta+\Lambda^2} + \ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] \\
&= \frac{i}{16\pi^2}\left[\frac{\Delta}{\Delta+\Lambda^2} + \ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] \\
&\approx \frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right),
\end{aligned}$$

where we have evaluated the integral using Mathematica, taken the large Λ limit, and dropped the irrelevant constant. Substituting back into Eq. (12), we find

$$\Sigma_2(p) = -ie^2\int_0^1 dx\,(-2x\not{p}+4m_0)\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right).$$

Note that

$$\begin{aligned}
\frac{d\Sigma_2}{d\not{p}} &= \frac{e^2}{16\pi^2}\frac{d}{d\not{p}}\left[\int_0^1 dx\,(-2x\not{p}+4m_0)\ln\left(\frac{\Lambda^2}{\Delta}\right)\right] \\
&= \frac{e^2}{16\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{\Delta}\right)\frac{d}{d\not{p}}(-2x\not{p}+4m_0) + (-2x\not{p}+4m_0)\frac{d}{d\not{p}}\ln\left(\frac{\Lambda^2}{\Delta}\right)\right] \\
&= \frac{e^2}{16\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{\Delta}\right)\frac{d}{d\not{p}}(-2x\not{p}+4m_0) + (-2x\not{p}+4m_0)\frac{d}{d\Delta}\ln\left(\frac{\Lambda^2}{\Delta}\right)\frac{d\Delta}{d\not{p}}\right].
\end{aligned} \tag{13}$$

Using $p^2 = \not{p}^2$ [1, p. 220], note that

$$\frac{d\Delta}{d\not{p}} = \frac{d}{d\not{p}}[-x(1-x)\not{p}^2 + x\mu^2 + (1-x)m_0^2] = -2x(1-x)\not{p}. \tag{14}$$

Also,

$$\frac{d}{d\not{p}}(-2x\not{p} + 4m_0) = -2x, \qquad \frac{d}{d\Delta}\left[\ln\left(\frac{\Lambda^2}{\Delta}\right)\right] = \frac{d}{d\Delta}\left[\ln(\Lambda^2) - \ln(\Delta)\right] = -\frac{1}{\Delta}. \tag{15}$$

Making these substitutions in Eq. (13),

$$\frac{d\Sigma_2}{d\not{p}} = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2x\ln\left(\frac{\Lambda^2}{\Delta}\right) - \frac{(2x\not{p} - 4m_0)[2x(1-x)\not{p}]}{\Delta}\right].$$

We now define

$$\Delta_m \equiv -x(1-x)m^2 + x\mu^2 + (1-x)m_0^2 \approx (1-x)^2m^2 + x\mu^2, \tag{16}$$

since $m \approx m_0$. Then Eq. (11) becomes

$$\delta Z_2 = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2x\ln\left(\frac{\Lambda^2}{\Delta}\right) - \frac{(2xm + 4m_0)[2x(1-x)m]}{\Delta_m}\right]. \tag{17}$$

Now we write out δZ_1 and δZ_2 fully, feeding Eqs. (10) and (16) into Eqs. (9) and (17), respectively. We also rename $x \to z$ in δZ_2:

$$\begin{aligned}\delta Z_1 &= -2ie^2\int_0^1 dz\,(1-z)\left[-\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right) + 2(1-4z+z^2)m^2\left(-\frac{i}{32\pi^2}\frac{1}{\Delta_0}\right)\right] \\ &= \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\ln\left(\frac{\Lambda^2}{(1-z)^2m^2 + z\mu^2}\right) - \frac{m^2(1-4z+z^2)}{(1-z)^2m^2 + z\mu^2}\right],\end{aligned}$$

$$\delta Z_2 = -\frac{e^2}{8\pi^2}\int_0^1 dz\left[z\ln\left(\frac{\Lambda^2}{(1-z)^2m^2 + z\mu^2}\right) + \frac{2zm^2(1-z)(2+z)}{(1-z)^2m^2 + z\mu^2}\right].$$

Clearly $\delta Z_1 \neq \delta Z_2$, as we wanted to show. □

1(b) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be 4×4 as usual, but note that, in d dimensions,

$$g_{\mu\nu}\gamma^\mu\gamma^\nu = d. \tag{18}$$

Show that, with this definition, $\delta Z_1 = \delta Z_2$.

Solution. To find δZ_1, we need to start at Peskin & Schroeder (6.38) for arbitrary dimension:

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int\frac{d^dk}{(2\pi)^d}\frac{-ig_{\nu\rho}}{(k-p)^2 + i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}' + m)}{k'^2 - m^2 + i\epsilon}\gamma^\mu\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}(-ie\gamma^\rho)u(p) \\ &= -ie^2\int\frac{d^dk}{(2\pi)^d}\bar{u}(p')\frac{\gamma^\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma_\nu}{[(k-p)^2 + i\epsilon](k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}u(p).\end{aligned}$$

Let N be the numerator of the integrand. Then

$$N = \bar{u}(p')\gamma^\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma_\nu u(p) = \bar{u}(p')\gamma^\nu(\not{k}'\gamma^\mu\not{k} + \not{k}'\gamma^\mu m + m\gamma^\mu\not{k} + m^2\gamma^\mu)\gamma_\nu u(p).$$

using the p identity thing, Note that

$$\gamma^\nu\not{k}'\gamma^\mu\not{k}\gamma_\nu = \gamma^\nu(2k'^\mu - \gamma^\mu\not{k}')(2k_\nu - \gamma_\nu\not{k}) = 4\gamma^\nu k'^\mu k_\nu - 2\gamma^\nu k'^\mu\gamma_\nu\not{k} - 2\gamma^\nu\gamma^\mu\not{k}'k_\nu + \gamma^\nu\gamma^\mu\gamma_\nu\not{k}'\not{k}$$

We need to fix Peskin & Schroeder (7.17) so it has arbitrary d instead of $d = 4$. We begin from (7.16), changing $4 \to d$:

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 + m_0^2 + i\epsilon} \gamma_\mu \frac{i}{(p-k)^2 - \mu^2 + i\epsilon}.$$

Following the procedure on pp. 217–218, we introduce the Feynman parameter $x=$ to combine the denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon} \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx \, \frac{1}{[k^2 - 2xk \cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}.$$

Let $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. Then

$$\begin{aligned}
-i\Sigma_2(p) &= (-ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \gamma^\mu \frac{i^2(\not{k} + m_0)}{[\ell^2 - \Delta + i\epsilon]^2} \gamma_\mu \\
&= -(-ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \gamma^\mu \frac{\not{\ell} + x\not{p} + m_0}{[\ell^2 - \Delta + i\epsilon]^2} \gamma_\mu \\
&= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \gamma^\mu \frac{2\ell^\mu - \gamma_\mu \not{\ell} + x(2p_\mu - \gamma_\mu \not{p}) + m_0\gamma_\mu}{[\ell^2 - \Delta + i\epsilon]^2} \\
&= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)x\not{p} + dm_0}{[\ell^2 - \Delta + i\epsilon]^2},
\end{aligned} \tag{19}$$

where we have applied Eq. (18) and $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}$ [1, p. 191], and we have dropped terms linear in ℓ [cite].

To evaluate the integral, we can write it in terms of the Euclidean 4-momentum defined in Eq. (6), as on p. 193 in Peskin & Schroeder:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(-1)^2} \frac{1}{(2\pi)^d} \int d^d\ell_E \, \frac{1}{(\ell_E^2 + \Delta)^2} = i \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2}.$$

Then we can apply (7.84), which takes the limit as $d \to 4$:

$$\int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} \to \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right),$$

where $\epsilon = 4 - d$ [1, p. 250]. Making these substitutions into Eq. (19), we find

$$\Sigma_2(p) = \frac{e^2}{16\pi^2} \int_0^1 dx \, [(2-d)x\not{p} + dm_0] \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right).$$

Then

$$\begin{aligned}
\frac{d\Sigma_2}{d\not{p}} &= \frac{e^2}{16\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) \frac{d}{d\not{p}} [(2-d)x\not{p} + dm_0] \\
&\qquad + [(2-d)x\not{p} + dm_0] \frac{d}{d\not{p}} \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) \\
&= \frac{e^2}{16\pi^2} \int_0^1 dx \left[\left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) (2-d)x + [(2-d)x\not{p} + dm_0] \frac{2x(1-x)\not{p}}{\Delta} \right]
\end{aligned}$$

where

$$\frac{d}{d\not{p}} (\ln \Delta) = \frac{d\Delta}{d\not{p}} \frac{d}{d\Delta} (\ln \Delta) = -\frac{2x(1-x)\not{p}}{\Delta}$$

from Eqs. (14) and (15). Then applying $m \approx m_0$,

$$\delta Z_2 = \frac{e^2}{16\pi^2} \int_0^1 dx \left[(2-d)x \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) + \frac{2x(1-x)[(2-d)x + d]m^2}{\Delta_m} \right].$$

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] Wikipedia contributors, "Gamma matrices." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Gamma_matrices`.