# 1 Problem 3

Consider a particle moving in one dimension with the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1}$$

### 1.1 Verify the following:

a. $i\hbar\partial_t \langle\Psi(t)|x\rangle = -\langle\Psi(t)|H|x\rangle$,

b. $i\hbar\partial_t \langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle\langle x|\Psi(t)\rangle$,

c. $i\hbar\partial_t \langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle = -\dfrac{\hbar^2}{2m}\left[\langle\Phi(t)|x\rangle\,\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right]$,

d. $\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle = \dfrac{\hbar}{i}\left[\langle\Phi(t)|x\rangle\,\partial_x\langle x|\Psi(t)\rangle - (\partial_x\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right]$

e. $\dfrac{\hbar}{i}\partial_x\left[\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle - mel\Phi(t)p^2x\langle x|\Psi(t)\rangle$

**Solution.**

a. Beginning with Schrödinger's equation, note that

$$\begin{aligned} i\hbar\partial_t|\Psi(t)\rangle &= H|\Psi(t)\rangle && (2)\\ i\hbar\langle x|\partial_t|\Psi(t)\rangle &= \langle x|H|\Psi(t)\rangle && (3)\\ (i\hbar\langle x|\partial_t|\Psi(t)\rangle)^\dagger &= (\langle x|H|\Psi(t)\rangle)^\dagger && (4)\\ -i\hbar\langle\Psi(t)|\partial_t|x\rangle &= \langle\Psi(t)|H|x\rangle && (5)\\ i\hbar\partial_t\langle\Psi(t)|x\rangle &= -\langle\Psi(t)|H|x\rangle\,, && (6)\end{aligned}$$

where in going to (5) we are assuming that $H$ is Hermitian. Note also that $\partial_t$ is Hermitian because $t$ is merely a parameter of the system. (6) is what we sought to prove. □

b. Rewriting what was proven in (a) with $\Psi \mapsto \Phi$ and then multiplying by $\Psi(x,t)$ on the right,

$$\begin{aligned} i\hbar\partial_t\langle\Phi(t)|x\rangle &= -\langle\Phi(t)|H|x\rangle && (7)\\ i\hbar(\partial_t\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle &= -\langle\Phi(t)|H|x\rangle\langle x|\Psi(t)\rangle\,. && (8)\end{aligned}$$

Multiplying (3) by $\Phi^*(x,t)$ on the left,

$$\langle\Phi(t)|x\rangle\, i\hbar\partial_t\langle x|\Psi(t)\rangle = \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle\,. \tag{9}$$

Adding (9) and (8) yields

$$\begin{aligned} \langle\Phi(t)|x\rangle\, i\hbar\partial_t\langle x|\Psi(t)\rangle + i\hbar(\partial_t\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle\langle x|\Psi(t)\rangle && (10)\\ i\hbar\partial_t\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle\ \langle x|H|\Psi(t)\rangle - \langle\Phi(t)|H|x\rangle\langle x|\Psi(t)\rangle\,, && (11)\end{aligned}$$

where in going to (11) we have used the product rule of differentiation on the left-hand side. (11) is what we sought to prove. □

c. Using (1), note that:

$$\begin{aligned}
\langle x|H|\Psi(t)\rangle &= \langle x|\left[\frac{p^2}{2m}+V(x)\right]|\Psi(t)\rangle && (12)\\
&= \frac{1}{2m}\langle x|p^2|\Psi(t)\rangle + \langle x|V(x)|\Psi(t)\rangle && (13)\\
&= \frac{(-i\hbar\partial_x)^2}{2m}\langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle && (14)\\
&= -\frac{\hbar^2}{2m}\partial_x^2\langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle\,, && (15)
\end{aligned}$$

where in going to (14) we have (twice) used the fact that

$$\langle x|p|\Psi(x)\rangle = -i\hbar\partial_x\langle x|\Psi(t)\rangle\,. \tag{16}$$

Similarly, note that

$$\langle\Phi(t)|H|x\rangle = -\frac{\hbar^2}{2m}\partial_x^2\langle\Phi(t)|x\rangle + V(x)\langle\Phi(t)|x\rangle \tag{17}$$

where we have (twice) used the adjoint of (16) with $\Psi \mapsto \Phi$,

$$\langle\Phi(t)|p|x\rangle = i\hbar\partial_x\langle\Phi(t)|x\rangle\,. \tag{18}$$

This follows because $p$ is Hermitian. Making the substitutions (15) and (17) into what was proven in (b),

$$\begin{aligned}
i\hbar\partial_t\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle\left[-\frac{\hbar^2}{2m}\partial_x^2\langle x|\Psi(t)\rangle + V(x)\langle x|\Psi(t)\rangle\right]\\
&\quad - \left[-\frac{\hbar^2}{2m}\partial_x^2\langle\Phi(t)|x\rangle + V(x)\langle\Phi(t)|x\rangle\right]\langle x|\Psi(t)\rangle && (19)\\
&= -\frac{\hbar^2}{2m}\left[\langle\Phi(t)|x\rangle\partial_x^2\langle\Phi(t)|x\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right]\\
&\quad + V(x)\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle - V(x)\langle\Phi(t)|x\rangle\langle x|\Psi(t)\rangle && (20)\\
&= -\frac{\hbar^2}{2m}\left[\langle\Phi(t)|x\rangle\partial_x^2\langle x|\Psi(t)\rangle - (\partial_x^2\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right]\,, && (21)
\end{aligned}$$

as we sought to prove. ☐

d. Applying (16) and (18) to the left-hand side of (d),

$$\begin{aligned}
\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle\langle x|\Psi(t)\rangle &= \langle\Phi(t)|x\rangle(-i\hbar\partial_x\langle x|\Psi(t)\rangle) + (i\hbar\partial_x\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle && (22)\\
&= \frac{\hbar}{i}\left[\langle\Phi(t)|x\rangle\partial_x\langle x|\Psi(t)\rangle - (\partial_x\langle\Phi(t)|x\rangle)\langle x|\Psi(t)\rangle\right] && (23)
\end{aligned}$$

as we sought to prove. ☐

e. Beginning with the first term of the left-hand side of the expression in (e), applying the product rule of differentiation yields

$$\partial_x(\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle) = (\partial_x\langle\Phi(t)|x\rangle)\langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\partial_x\langle x|p|\Psi(t)\rangle \tag{24}$$

Multiplying through by $\hbar/i$,

$$\begin{aligned}
\frac{\hbar}{i}\partial_x(\langle\Phi(t)|x\rangle\ \langle x|p|\Psi(t)\rangle) &= (-i\hbar\partial_x\langle\Phi(t)|x\rangle)\langle x|p|\Psi(t)\rangle - \langle\Phi(t)|x\rangle i\hbar\partial_x\langle x|p|\Psi(t)\rangle && (25)\\
&= -\langle\Phi(t)|p|x\rangle\ \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|x\rangle\ \langle x|p^2|\Psi(t)\rangle\,, && (26)
\end{aligned}$$

where in going to (26) we have used (16) and (18). Using a similar procedure for the second term of the left-hand side of (e),

$$\frac{\hbar}{i}\partial_x( \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle) = (-i\hbar\partial_x \langle\Phi(t)|p|x\rangle) \langle x|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle\, i\hbar\partial_x \langle x|\Psi(t)\rangle \tag{27}$$

$$= -\langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle. \tag{28}$$

Adding the results of (26) and (28),

$$\frac{\hbar}{i}\partial_x \left[\langle\Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle\right] = \langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle$$

$$+ \langle\Phi(t)|p|x\rangle \langle x|p|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle \tag{29}$$

$$= \langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle \tag{30}$$

as we sought to prove. □

## 1.2 Define

$$\rho(x,t) = \langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle, \tag{31}$$

$$J_x(x,t) = \frac{1}{2m}\left[\langle\Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle\right]. \tag{32}$$

Show that $\rho(x,t) + \partial_x J_x(x,t) = 0$.

**Solution.** From (31),

$$\partial_t\rho(x,t) = \partial_t(\langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle), \tag{33}$$

and from what was proven in 1(c),

$$\partial_t(\langle\Phi(t)|x\rangle \langle x|\Psi(t)\rangle) = -\frac{1}{i\hbar}\left[\langle\Phi(t)|x\rangle\, \partial_x^2 \langle x|\Psi(t)\rangle - (\partial_x^2 \langle\Phi(t)|x\rangle) \langle x|\Psi(t)\rangle\right] \tag{34}$$

$$= -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle\right], \tag{35}$$

where we have applied (16) and (18) in going to (35). Equating (33) and (35),

$$\partial_t\rho(x,t) = -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle\right]. \tag{36}$$

Beginning from (32),

$$\partial_x J_x(x,t) = \frac{1}{2m}\partial_x\left[\langle\Phi(t)|x\rangle \langle x|p|\Psi(t)\rangle + \langle\Phi(t)|p|x\rangle \langle x|\Psi(t)\rangle\right] \tag{37}$$

$$= \frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle\right], \tag{38}$$

where in going to (38) we have used what was proven in 1(e). Summing (36) and (38), we have

$$\partial_t\rho(x,t) + \partial_x J_x(x,t) = -\frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle\right]$$

$$+ \frac{1}{2m}\frac{i}{\hbar}\left[\langle\Phi(t)|x\rangle \langle x|p^2|\Psi(t)\rangle - \langle\Phi(t)|p^2|x\rangle \langle x|\Psi(t)\rangle\right] \tag{39}$$

$$= 0 \tag{40}$$

as we sought to prove. This is is the continuity equation for probability. □

# 2 Problem 4

Consider a particle moving in three dimensions. Consider an operator

$$U(\phi) = \exp\left(-\frac{i}{\hbar} L_3 \phi\right), \qquad L_3 = L_z = XP_y - YP_x, \tag{41}$$

where $X, Y$ and $P_x, P_y$ are position and momentum operators, respectively. Define new operators

$$X(\phi) = U^\dagger(\phi) X U(\phi), \qquad Y(\phi) = U^\dagger(\phi) Y U(\phi). \tag{42}$$

Note that $X(0) = Y(0) = 0$.

### 2.1 Derive the equation

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar} U^\dagger(\phi)[L_3, X]U(\phi) = -Y(\phi), \tag{43}$$

and a similar equation for $\mathrm{d}Y(\phi)/\mathrm{d}\phi$.

**Solution.** Using the definition of $X(\phi)$ in (42) and applying the product rule of differentiation,

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(U^\dagger X U\right) = \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi} XU + U^\dagger \frac{\mathrm{d}}{\mathrm{d}\phi}(XU) \tag{44}$$

$$= \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi} XU + U^\dagger \frac{\mathrm{d}X}{\mathrm{d}\phi} U + U^\dagger X \frac{\mathrm{d}U}{\mathrm{d}\phi}. \tag{45}$$

We know immediately that $\mathrm{d}X/\mathrm{d}\phi = 0$ because $\phi$ is not a parameter of the position operator $X$. Let $|l_{3,i}\rangle$ denote the eigenbasis of $L_3$ and $l_{3,i}$ its eigenvalues. $L_3$ is Hermitian so an orthonormal basis is guaranteed to exist. USE POWER SERIES INSTEAD. In this basis, $U(\phi)$ is diagonal and its nonzero matrix elements are given by

$$U_{ii} = \exp\left(-\frac{i}{\hbar} l_{3,i} \phi\right) \tag{46}$$

which implies

$$\frac{\mathrm{d}U_{ii}}{\mathrm{d}\phi} = -\frac{i}{\hbar} l_{3,i} \exp\left(-\frac{i}{\hbar} l_{3,i}\phi\right) = -\exp\left(-\frac{i}{\hbar} l_{3,i}\phi\right)\frac{i}{\hbar} l_{3,i} \tag{47}$$

$$= -\frac{i}{\hbar} l_{3,i} U_{ii} = -\frac{i}{\hbar} U_{ii} l_{3,i}. \tag{48}$$

The power-series representation of $e^x$ allows us to retrieve from (48) the operator relationships

$$\frac{\mathrm{d}U}{\mathrm{d}\phi} = -\frac{i}{\hbar} L_3 U = -\frac{i}{\hbar} U L_3, \tag{49}$$

which informs us that $[L_3, U] = 0$. In a similar fashion, note that

$$U^\dagger = \exp\left(\frac{i}{\hbar} L_3 \phi\right) \implies \frac{\mathrm{d}U^\dagger}{\mathrm{d}\phi} = \frac{i}{\hbar} L_3 \exp\left(\frac{i}{\hbar} L_3\phi\right) = \frac{i}{\hbar} L_3 U^\dagger = \frac{i}{\hbar} U^\dagger L_3 \tag{50}$$

and $[L_3, U^\dagger] = 0$ as well. Then (45) becomes

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar} U^\dagger L_3 X U - \frac{i}{\hbar} U^\dagger X L_3 U = \frac{i}{\hbar} U^\dagger (L_3 X - X L_3) U = \frac{i}{\hbar} U^\dagger(\phi)[L_3, X]U(\phi), \tag{51}$$

which is the first equality of what we wanted to show in (43).

From the definition of $L_3$ in (41),

$$[L_3, X] = L_3X - XL_3 = (XP_y - YP_x)X - X(XP_y - YP_x) \tag{52}$$
$$= XP_yX - YP_xX - XXP_y + XYP_x = YXP_x - YP_xX \tag{53}$$
$$= Y[X, P_x] = i\hbar Y \tag{54}$$

where in (53) we have used $[X, P_y] = [X, Y] = 0$, and in (54) we have used $[X, P_x] = i\hbar$. Making the substitution (54) into (51), we have

$$\frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)(i\hbar Y)U(\phi) = -U^\dagger(\phi)YU(\phi) = -Y(\phi), \tag{55}$$

where the last equality is from the definition of $Y(\phi)$ in (42). This is the second equality of what we wanted to show in (43), which completes the proof.

For $\mathrm{d}Y(\phi)/\mathrm{d}\phi$, we can make the substitutions $X(\phi) \mapsto Y(\phi), X \mapsto Y$ in (45) and (51) to obtain

$$\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)[L_3, Y]U(\phi). \tag{56}$$

Then making similar use of commutators $[Y, P_x] = [X, Y] = 0$ and $[Y, P_y] = i\hbar$ as for (53) and (54),

$$[L_3, Y] = L_3Y - YL_3 = (XP_y - YP_x)Y - Y(XP_y - YP_x) \tag{57}$$
$$= XP_yY - YP_xY - YXP_y + YYP_x = XP_yY - XYP_y \tag{58}$$
$$= X[P_y, Y] = -X[Y, P_y] = -i\hbar X. \tag{59}$$

Substituting (59) into (56),

$$\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)(-i\hbar X)U(\phi) = X(\phi), \tag{60}$$

and so we have derived

$$\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = \frac{i}{\hbar}U^\dagger(\phi)[L_3, Y]U(\phi) = X(\phi). \tag{61}$$

and (43) as desired. □

**2.2** Define $X_\pm(\phi) = X(\phi) \pm iY(\phi)$. From the results of previous parts, show $X_+(\phi) = e^{i\phi}X_+$ where $X_+ = X_+(0)$. Derive the similar expression for $X_-(\phi)$.

**Solution.** Differentiating $X_\pm(\phi)$ and making use of (43) and (61),

$$\frac{\mathrm{d}X_\pm(\phi)}{\mathrm{d}\phi} = \frac{\mathrm{d}X(\phi)}{\mathrm{d}\phi} \pm i\frac{\mathrm{d}Y(\phi)}{\mathrm{d}\phi} = -Y(\phi) \pm iX(\phi) = \pm i\left[X(\phi) \pm iY(\phi)\right] \tag{62}$$
$$= \pm iX_\pm(\phi). \tag{63}$$

The differential equation (63) has solutions given by exponential functions of $\pm i\phi$. We will make the ansatz

$$X_\pm(\phi) = e^{\pm i\phi}C_\pm, \tag{64}$$

where $C_\pm$ is an operator "constant" in $\phi$ (that is, independent of it) and is fixed by an initial condition. Inspecting (64), clearly $X_\pm(0) = C_\pm$ where it is defined $X_\pm(0) \equiv X_\pm$. All that remains is to show that (64) obeys the relation (63), as follows:

$$\frac{\mathrm{d}X_\pm(\phi)}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(e^{\pm i\phi}\right)C_\pm = \pm i e^{\pm i\phi} C_\pm = \pm i X_\pm(\phi). \tag{65}$$

Thus, we have derived

$$X_+(\phi) = e^{i\phi} X_+, \qquad X_-(\phi) = e^{-i\phi} X_- \tag{66}$$

as desired. □

**2.3** Show that $[L_3, X_+] = \hbar X_+$. Derive the similar expression for $[L_3, X_-]$.

**Solution.** Firstly, note that

$$X_\pm = X_\pm(0) = X(0) \pm iY(0) = U^\dagger(0) X U(0) \pm i U^\dagger(0) Y U(0) = X \pm iY \tag{67}$$

because $U(0) = U^\dagger(0) = I$. Also applying the definition of $L_3$ in (41), we have

$$\begin{aligned}
[L_3, X_\pm] &= [XP_y - YP_x, X \pm iY] = (XP_y - YP_x)(X \pm iY) - (X \pm iY)(XP_y - YP_x) && (68)\\
&= XP_yX \pm iXP_yY - YP_xX \mp iYP_xY - XXP_y + XYP_x \mp iYXP_y \pm iYYP_x && (69)\\
&= \pm iXP_yY - YP_xX + XYP_x \mp iYXP_y = \pm iX[P_y, Y] + Y[X, P_x] && (70)\\
&= \pm \hbar X + i\hbar Y = \pm \hbar[X \pm iY] = \pm \hbar X_\pm. && (71)
\end{aligned}$$

Thus, we have shown

$$[L_3, X_+] = \hbar X_+, \qquad [L_3, X_-] = -\hbar X_- \tag{72}$$

as desired. □