Problem 1. Renormalization of Yukawa theory (P&S 10.2) Consider the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi, \tag{1}$$

where ϕ is a real scalar field and ψ is a Dirac fermion. Notice that this Lagrangian is invariant under the parity transformation $\psi(t, \mathbf{x}) \to \gamma^0\psi(zt, -\mathbf{x})$, $\phi(t, \mathbf{x}) \to -\phi(t, -\mathbf{x})$, in which the field ϕ carries odd parity.

1(a) Determine the superficially divergent amplitudes and work out the Feynman rules for renormalized perturbation theory for this Lagrangian. Include all necessary counterterm vertices. Show that the theory contains a superficially divergent 4ϕ amplitude. This means that the theory cannot be renormalized unless one includes a scalar self-interaction,

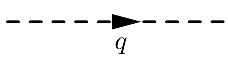
$$\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4, \tag{2}$$

and a counterterm of the same form. It is of course possible to set the renormalized value of this coupling to zero, but that is not a natural choice, since the counterterm will still be nonzero. Are any further interactions required?

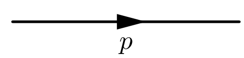
Solution. We write Eq. (1) explicitly in terms of the bare masses m_0, M_0 and the bare coupling constant g_0:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M_0)\psi - ig_0\bar{\psi}\gamma^5\psi\phi, \tag{3}$$

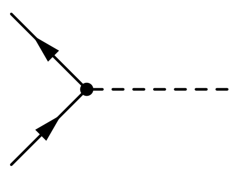
The Feynman rules for a pseudoscalar Yukawa theory are [1, pp. 24–25]



$= \dfrac{i}{q^2 - m_0^2 + i\epsilon}$



$= \dfrac{i(p\!\!\!/ + M_0)}{p^2 - M_0^2 + i\epsilon}$

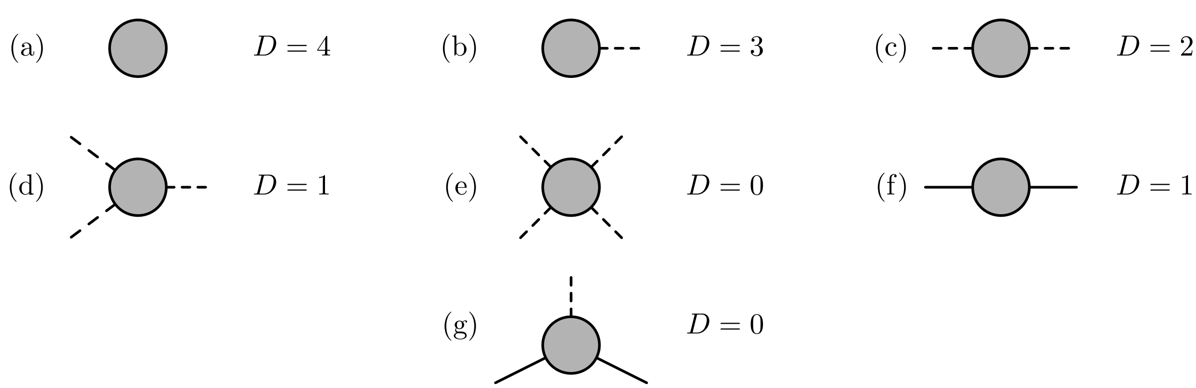

$= g_0\gamma^5$

These Feynman rules are similar enough to those for QED; that is, the powers of k are the same, each propagator has a momentum integral, each vertex has a delta function, and each vertex involves one ϕ line and two fermion lines [2, p. 316]. So we can adapt P&S (10.4) for the superficial degree of divergence:

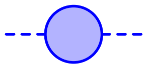
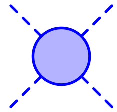
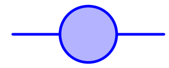
$$D = 4 - N_\phi - \frac{3}{2}N_f,$$

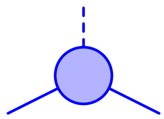
where N_ϕ is the number of external ϕ lines and N_f is the number of external fermion lines.

This means the superficially divergent amplitudes are a subset of those appearing in Fig. 10.2 of P&S, with the photon lines replaced by pseudoscalar lines:

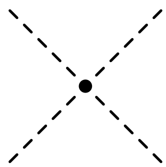


We ignore (a) since it is irrelevant to scattering processes [2, pp. 317–318]. Amplitudes (b) and (d) vanish because the theory is invariant under the parity transformation, which means all amplitudes with zero external fermion legs and an odd number of external ϕ legs vanish [2, pp. 318, 323–324]. So the superficially divergent amplitudes are

(c)  $D = 2$ (e)  $D = 0$ (f)  $D = 1$

(g)  $D = 0$

Note that amplitude (e) is a 4ϕ amplitude. Since it is superficially divergent, according to the problem statement we must introduce the scalar self-interaction given by Eq. (2). We subtract this term as in the ϕ^4 theory [2, p. 324]. The Feynman rule for this vertex is [2, p. 325]

 $= -i\lambda_0.$

With the addition of this new term, our Lagrangian in Eq. (3) becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M_0)\psi - ig_0\bar{\psi}\gamma^5\psi\phi - \frac{\lambda_0}{4!}\phi^4, \tag{4}$$

where λ_0 is the bare coupling constant for the scalar self-interaction. To work out the renormalized theory, we rescale the field as in P&S (10.15):

$$\phi = Z_1^{1/2}\phi_r.$$

The rescaling for the fermion is [2, p. 330]

$$\psi = Z_2^{1/2}\psi_r.$$

Feeding these into Eq. (4), we obtain the renormalized Lagrangian [2, p. 324]

$$\mathcal{L} = \frac{1}{2}Z_1(\partial_\mu \phi)^2 - \frac{1}{2}Z_1 m_0^2\phi^2 + Z_2\bar{\psi}(i\partial\!\!\!/ - M_0)\psi - iZ_1^{1/2}Z_2 g_0\bar{\psi}\gamma^5\psi\phi - \frac{\lambda_0}{4!}Z_1^2\phi^4. \tag{5}$$

Define [2, pp. 324, 331]

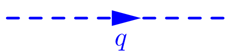
$$\begin{aligned} \delta_{Z_1} &= Z_1 - 1, & \delta_{Z_2} &= Z_2 - 1, & \delta_m &= m_0^2 Z_1 - m^2, \\ \delta_M &= M_0 Z_2 - M, & \delta_g &= (g_0/g)Z_1^{1/2}Z_2 - 1, & \delta_\lambda &= \lambda_0 Z_1^2 - \lambda \end{aligned}$$

Then Eq. (5) becomes

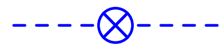
$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(1+\delta_{Z_1})(\partial_\mu \phi)^2 - \frac{1}{2}(m^2+\delta_m)\phi^2 + \bar{\psi}[i(\delta_{Z_2}+1)\partial\!\!\!/ - (M+\delta_M)]\psi - ig(1+\delta_g)\bar{\psi}\gamma^5\psi\phi + \frac{\lambda+\delta_\lambda}{4!}\phi^4 \\ &= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \\ &\qquad + \frac{1}{2}\delta_{Z_1}(\partial_\mu \phi)^2 - \frac{1}{2}\delta_m\phi^2 + \bar{\psi}(i\delta_{Z_2}\partial\!\!\!/ - \delta_M)\psi - ig\delta_g\bar{\psi}\gamma^5\psi\phi - \frac{\delta_\lambda}{4!}\phi^4. \end{aligned}$$

Here the first first five terms look like Eq. (4), but written in terms of the physical masses and couplings. The last five terms are the counterterms [2, p. 325].

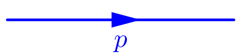
The Feynman rules for the renormalized theory are



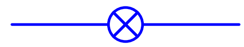
$$= \frac{i}{q^2 - m^2 + i\epsilon}$$



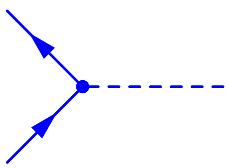
$$= i(p^2 \delta_{Z_1} - \delta_m)$$



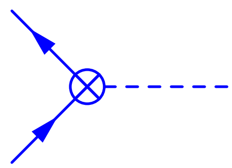
$$= \frac{i(\not{p} + M)}{p^2 - M^2 + i\epsilon}$$



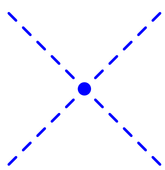
$$= i(\not{p}\delta_{Z_2} - \delta_M)$$



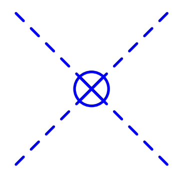
$$= g\gamma^5$$



$$= g\delta_g \gamma^5$$



$$= -i\lambda$$



$$= -i\delta_\lambda$$

why we don't need any more interactions

1(b) Compute the divergent part (the pole as $d \to 4$) of each counterterm, to the one-loop order of perturbation theory, implementing a sufficient set of renormalization conditions. You need not worry about finite parts of the counterterms. Since the divergent parts must have a fixed dependence on the external momenta, you can simplify this calculation by choosing the momenta in the simplest possible form.

Solution. To compute the divergent part of the fermion propagator counterterm to one-loop order, we include the fermion self-energy: (draw the diagram)

The fermion-self energy here looks similar to that in QED, so we may adapt P&S (7.16) for that term. Using our Feynman rules from 1(a), we have

$$\begin{aligned} -i\Sigma_2(p) &= i(\not{p}\delta_{Z_2} - \delta_M) + g^2 \int \frac{d^d k}{(2\pi)^2} \gamma^5 \frac{i(\not{k} + M)}{p^2 - M^2 + i\epsilon} \gamma^5 \frac{i}{(p-k)^2 - m^2 + i\epsilon} \\ &= i(\not{p}\delta_{Z_2} - \delta_M) + g^2 \int \frac{d^d k}{(2\pi)^2} \frac{\not{k} - M}{(p^2 - M^2 + i\epsilon)[(p-k)^2 - m^2 + i\epsilon]}, \end{aligned} \tag{6}$$

where we have used P&S (3.70), $(\gamma^5)^2 = 1$, and (3.71), $\{\gamma^5, \gamma^\mu\} = 0$, which implies $\gamma^5\gamma^\mu\gamma^5 = -\gamma^\mu$. Following the procedure on pp. 217–218, we introduce the Feynman parameter x to combine the denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon} \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx \, \frac{1}{[k^2 - 2xk \cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}. \tag{7}$$

Let $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + xm^2 + (1-x)M^2$. Then Eq. (6) can be written

$$-i\Sigma_2(p) = i(\not{p}\delta_{Z_2} - \delta_M) + g^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{(x\not{p} - M)}{[\ell^2 - \Delta + i\epsilon]^2}. \tag{8}$$

To evaluate the integral, we can write it in terms of the Euclidean 4-momentum defined by [2, p. 193]

$$\ell^0 \equiv i\ell_E^0, \qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E. \tag{9}$$

Then we can write

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2} = \frac{i}{(-1)^2}\frac{1}{(2\pi)^d}\int d^d\ell_E\,\frac{1}{(\ell_E^2+\Delta)^2} = i\int\frac{d^d\ell_E}{(2\pi)^2}\frac{1}{(\ell_E^2+\Delta)^2}.$$

Then we can apply (7.84), which takes the limit as $d \to 4$:

$$\int\frac{d^d\ell_E}{(2\pi)^2}\frac{1}{(\ell_E^2+\Delta)^2} \to \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma+\ln\left(\frac{4\pi}{\Delta}\right)\right) \approx \frac{1}{8\pi^2\epsilon},$$

where $\epsilon = 4-d$ [2, p. 250], and we have omitted the finite parts. Making these substitutions into Eq. (**??**), we find

$$\begin{aligned}
-i\Sigma_2(p) &= i(\not{p}\delta_{Z_2}-\delta_M) + \frac{ig^2}{8\pi^2\epsilon}\int_0^1 dx\,(x\not{p}-M) \\
&= i(\not{p}\delta_{Z_2}-\delta_M) + \frac{ig^2}{8\pi^2\epsilon}\left[\frac{x^2}{2}\not{p}-Mx\right]_0^1 \\
&= i(\not{p}\delta_{Z_2}-\delta_M) + \frac{ig^2}{8\pi^2\epsilon}\left(\frac{\not{p}}{2}-M\right) \\
&= i\not{p}\left(\delta_{Z_2}+\frac{g^2}{16\pi^2\epsilon}\right) - i\left(\delta_M+\frac{g^2}{8\pi^2\epsilon}M\right).
\end{aligned}$$

This implies

$$\delta_{Z_1} = -\frac{g^2}{16\pi^2\epsilon}, \qquad \delta_M = -\frac{g^2}{8\pi^2\epsilon}M$$

are the conditions to eliminate the divergence.

For the scalar-fermion vertex, we can adapt some of our work from problem 2 of Homework 1. We adapt Peskin & Schroeder (6.38) using the pseudoscalar field Feynman rules to write [2, p. 123]

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p,p')u(p) &= \bar{u}(p')g\delta_g\gamma^5u(p) + g^3\int\frac{d^dk}{(2\pi)^2}\bar{u}(p')\frac{\gamma^5(\not{k}'+M)\gamma^5(\not{k}+M)\gamma^5}{[(k-p)^2-m^2+i\epsilon](k'^2-M^2+i\epsilon)(k^2-M^2+i\epsilon)}u(p) \\
&= \bar{u}(p')g\delta_g\gamma^5u(p) + g^3\gamma^5\int\frac{d^dk}{(2\pi)^2}\bar{u}(p')\frac{(\not{k}'+M)(\not{k}-M)}{[(k-p)^2-m^2+i\epsilon](k'^2-M^2+i\epsilon)(k^2-M^2+i\epsilon)}u(p),
\end{aligned} \tag{10}$$

where we have once more used $(\gamma^5)^2 = 1$. We use Peskin & Schroeder (6.41) to write

$$\frac{1}{[(k-p)^2-m^2+i\epsilon](k'^2-M^2+i\epsilon)(k^2-M^2+i\epsilon)} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3}, \tag{11}$$

where [2, pp. 190–191]

$$D = k^2+2k(qy-pz)+z(p^2-m^2)-(1-z)M^2+i\epsilon = k^2-2kpz+z(p^2-m^2)-(1-z)M^2+i\epsilon \equiv \ell^2-\Delta+i\epsilon.$$

Here we have used $x+y+z=1$ and set $q=0$ (so $k'=k$) as in problem 2 of Homework 1. We have defined $\ell \equiv k - zp$ [2, p. 191], and

$$\Delta \equiv (1-z)^2M^2 + zm^2.$$

For the numerator of Eq. (10), we use $\ell \equiv k - zp$ [2, p. 191], and define

$$N \equiv \bar{u}(p')\gamma^5(\ell+zp+M)(\ell+zp-M)u(p) \tag{12}$$

References

[1] C. Blair, "Quantum Field Theory—Useful Formulae and Feynman Rules", May, 2010. `https://www.maths.tcd.ie/~cblair/notes/list.pdf`.

[2] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.