Problem 1. Alternative regulators in QED (Peskin & Schroeder 7.2) In Section. 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Paul-Villars regularization.

1(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

Solution. From (7.47) in Peskin & Schroeder,

$$\Gamma^\mu(q=0) = \frac{1}{Z_1}\gamma^\mu,$$

we can find an expression for δZ_1, similar to how (7.31) is obtained from (7.26). Roughly,

$$\frac{1}{Z_1+\delta Z_1}\gamma^\mu \approx Z_1(1-\delta Z_1)\gamma^\mu = \Gamma^\mu(q=0) + \delta\Gamma^\mu(q=0) \qquad \Longrightarrow \qquad \delta\Gamma^\mu(q=0) = -\delta Z_1\gamma^\mu. \tag{1}$$

According to (6.33),

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

We note that $\Gamma^\mu = \gamma^\mu$, $F_1 = 1$, and $F_2 = 0$ to lowest order [1, pp. 185–186]. Then we can write

$$\delta\Gamma^\mu(q=0) = \gamma^\mu\delta F_1(0) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\delta F_2(0). \tag{2}$$

Using this equation and the identity $\gamma^\mu\gamma_\mu = 4$ [2], Eq. (1) can be written

$$\delta Z_1 = -\frac{1}{4}\gamma_\mu\delta\Gamma^\mu(q=0) = -\delta F_1(0) - \gamma_\mu\frac{i\sigma^{\mu\nu}q_\nu}{8m}\delta F_2(0). \tag{3}$$

In order to find $\delta\Gamma^\mu$ we use (6.47):

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3} \\ \times\,\bar{u}(p')\left\{\gamma^\mu\left[-\frac{\ell^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right]\right. \\ \left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m}[2m^2z(1-z)]\right\}u(p),\end{aligned} \tag{4}$$

where $\Delta \equiv -xyq^2 + (1-z)^2m^2$ by (6.44), $\ell \equiv k + yq - zp$, and $D = \ell^2 - \Delta + i\epsilon$ [1, p. 191]. The momenta k and p are assigned in the Feynman diagram on p. 189 of Peskin & Schroeder, and x, y are Feynman parameters [1, p. 190].

The forms of the two integrals we need to compute are given by Peskin & Schroeder (6.49) and (6.50). Equation (6.49) is

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}. \tag{5}$$

Here $m = 3$ because we have D^{-3} in Eq. (4). We can evaluate the left-hand side using the Euclidian 4-momentum defined in (6.48),

$$\ell^0 \equiv i\ell_E^0, \qquad\qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E. \tag{6}$$

Following the steps on p. 193, we have

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} = \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int \frac{d^4\ell_E}{(\ell_E^2+\Delta)^3} = \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^3},$$

where we have replaced the upper (infinite) bound of integration by a finite number Λ. Evaluating this integral using Mathematica and using $\int d\Omega_4 = 2\pi^2$ [1, p. 193], we find

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{\Lambda^4}{4\Delta(\Delta+\Lambda^2)^2}\\ &= -\frac{i}{32\pi^2}\frac{\Lambda^4}{\Delta(\Delta+\Lambda^2)^2}\\ &\approx -\frac{i}{32\pi^2}\frac{1}{\Delta} \equiv \alpha,\end{aligned} \tag{7}$$

where we have taken the limit $\Lambda \gg \Delta$ [1, p. 218] and defined α. Equation (6.50) in Peskin & Schroeder is

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}.$$

Following similar steps as for Eq. (4), the left-hand side is

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^3} &= \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3}\\ &= \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4\int_0^\Lambda d\ell_E\,\frac{\ell_E^5}{(\ell_E^2+\Delta)^3}\\ &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{1}{4}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right]\\ &= -\frac{i}{32\pi^2}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right]\\ &\approx -\frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - \frac{3}{2}\right] \equiv \beta,\end{aligned} \tag{8}$$

where we have defined β and ignored terms of $\mathcal{O}(\Lambda^{-2})$ [1, p. 218].

We now set $q^2 = 0$, and define $\Delta_0 = (1-z)^2m^2$. Then $\Delta \to \Delta_0$ in our expression and $\alpha \to \alpha_0, \beta \to \beta_0$ (which are functions of Δ_0). Feeding in Eqs. (7) and (8), Eq. (4) can be written

$$\bar{u}(p')\delta\Gamma^\mu(q=0)u(p) = 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p')\int \left\{\gamma^\mu\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right]\right\}u(p).$$

Then

$$\begin{aligned}\delta F_1(0) &= 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right]\\ &= 2ie^2\int_0^1 dz\,(1-z)\left[-\beta_0 + 2m^2(1-4z+z^2)\alpha_0\right],\end{aligned}$$

$$\begin{aligned}\delta F_2(0) &= 8ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)m^2z(1-z)\alpha_0\\ &= 8ie^2\int_0^1 dz\,m^2z(1-z)^2\alpha_0.\end{aligned}$$

We ignore $\delta F_2(0)$ since it is not affected by the divergence [1, p. 196]. In order to avoid issues coming from the divergence in $\delta F_1(0)$, we add a $z\mu^2$ term to Δ_0 [1, p. 195]. So, feeding these results into Eq. (3), we obtain

$$\begin{aligned}\delta Z_1 &= -2ie^2\int_0^1 dz\,(1-z)\left[-\beta_0+2(1-4z+z^2)m^2\alpha_0\right]\\ &= -2ie^2\int_0^1 dz\,(1-z)\left\{\frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Delta_0+\Lambda^2}{\Delta_0}\right)-\frac{3}{2}\right]-2(1-4z+z^2)m^2\frac{i}{32\pi^2}\frac{1}{\Delta_0}\right\}\\ &= \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\ln\!\left(\frac{\Delta_0+\Lambda^2}{\Delta_0}\right)-\frac{3}{2}-\frac{m^2(1-4z+z^2)}{\Delta_0}\right],\end{aligned}\tag{9}$$

where

$$\Delta_0=(1-z)^2m^2+z\mu^2.\tag{10}$$

For δZ_2, we can use the first part of Peskin & Schroeder (7.31),

$$\delta Z_2=\left.\frac{d\Sigma_2}{d\not p}\right|_{\not p=m},\tag{11}$$

where Σ_2 is given by (7.17),

$$-i\Sigma_2(p)=-e^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{-2x\not p+4m_0}{(\ell^2-\Delta+i\epsilon)^2},\tag{12}$$

where $\Delta=-x(1-x)p^2+x\mu^2+(1-x)m_0^2$. We may once again follow the steps on p. 193 to evaluate the integral, now with $m=2$. Changing the upper bound of integration to Λ once more, we have

$$\begin{aligned}\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} &= \frac{i}{(-1)^2}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{1}{(\ell_E^2+\Delta)^2}\\ &= \frac{i(-1)^2}{(2\pi)^4}\int d\Omega_4\int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^2}\\ &= \frac{i}{(2\pi)^4}(2\pi^2)\frac{1}{2}\left[\frac{\Delta}{\Delta+\Lambda^2}+\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\\ &= \frac{i}{16\pi^2}\left[\frac{\Delta}{\Delta+\Lambda^2}+\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\\ &\approx \frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right],\end{aligned}$$

where we have evaluated the integral using Mathematica and dropped terms of $\mathcal{O}(\Lambda^{-2})$. Substituting back into Eq. (12), we find

$$\Sigma_2(p)=-ie^2\int_0^1 dx\,(-2x\not p+4m_0)\frac{i}{16\pi^2}\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right].$$

Note that

$$\begin{aligned}\frac{d\Sigma_2}{d\not p} &= \frac{e^2}{16\pi^2}\frac{d}{d\not p}\left\{\int_0^1 dx\,(-2x\not p+4m_0)\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\right\}\\ &= \frac{e^2}{16\pi^2}\int_0^1 dx\left\{\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\frac{d}{d\not p}(-2x\not p+4m_0)+(-2x\not p+4m_0)\frac{d}{d\not p}\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\right\}\\ &= \frac{e^2}{16\pi^2}\int_0^1 dx\left\{\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\frac{d}{d\not p}(-2x\not p+4m_0)+(-2x\not p+4m_0)\frac{d}{d\Delta}\left[\ln\!\left(\frac{\Delta+\Lambda^2}{\Delta}\right)-1\right]\frac{d\Delta}{d\not p}\right\}.\end{aligned}\tag{13}$$

Using $p^2 = \not{p}^2$ [1, p. 220], note that

$$\frac{d\Delta}{d\not{p}} = \frac{d}{d\not{p}}[-x(1-x)\not{p}^2 + x\mu^2 + (1-x)m_0^2] = -2x(1-x)\not{p}. \tag{14}$$

Also, ignoring terms of $\mathcal{O}(\Lambda^{-2})$,

$$\frac{d}{d\not{p}}(-2x\not{p} + 4m_0) = -2x, \tag{15}$$

$$\frac{d}{d\Delta}\left[\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] = \frac{d}{d\Delta}\left[\ln(\Delta+\Lambda^2) - \ln(\Delta) - 1\right] = \frac{1}{\Delta+\Lambda^2} - \frac{1}{\Delta} \approx -\frac{1}{\Delta}. \tag{16}$$

Making these substitutions in Eq. (13),

$$\frac{d\Sigma_2}{d\not{p}} = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2x\left[\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] - \frac{(2x\not{p} - 4m_0)[2x(1-x)\not{p}]}{\Delta}\right].$$

We now define

$$\Delta_m \equiv -x(1-x)m^2 + x\mu^2 + (1-x)m_0^2 \approx (1-x)^2m^2 + x\mu^2, \tag{17}$$

since $m \approx m_0$. Then Eq. (11) becomes

$$\begin{aligned}\delta Z_2 &= \frac{e^2}{16\pi^2}\int_0^1 dx\left[-x\left[\ln\left(\frac{\Delta_m+\Lambda^2}{\Delta_m}\right) - 1\right] + \frac{(2xm - 4m)[2x(1-x)m]}{\Delta_m}\right] \\ &= -\frac{e^2}{8\pi^2}\int_0^1 dx\left[x\ln\left(\frac{\Delta_m+\Lambda^2}{\Delta_m}\right) - x - \frac{2xm^2(2-x)(1-x)}{\Delta_m}\right]\end{aligned} \tag{18}$$

Now we rename $x \to z$ in δZ_2. This means $\Delta_0 = \Delta_m$ from Eqs. (10) and (17). Naming $\Delta \equiv \Delta_0 = \Delta_m$ in Eqs. (9) and (18), we have

$$\delta Z_1 = \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - \frac{3}{2} - \frac{m^2(1-4z+z^2)}{\Delta}\right],$$
$$\delta Z_2 = -\frac{e^2}{8\pi^2}\int_0^1 dx\left[x\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - x - \frac{2xm^2(2-x)(1-x)}{\Delta}\right],$$

where $\Delta = (1-z)^2m^2 + z\mu^2$. We see that $\delta Z_1 \neq \delta Z_2$, as we wanted to show. □

1(b) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be 4×4 as usual, but note that, in d dimensions,

$$g_{\mu\nu}\gamma^\mu\gamma^\nu = d. \tag{19}$$

Show that, with this definition, $\delta Z_1 = \delta Z_2$.

Solution. To find δZ_1, we need to start at Peskin & Schroeder (6.38) for arbitrary dimension:

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int\frac{d^dk}{(2\pi)^d}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p) \\ &= -ie^2\int\frac{d^dk}{(2\pi)^d}\bar{u}(p')\frac{\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{[(k-p)^2+i\epsilon](k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}u(p).\end{aligned}$$

Let N be the numerator of the integrand. Then

$$N = \bar{u}(p')\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu u(p) = \bar{u}(p')\gamma^\nu(\not{k}'\gamma^\mu\not{k} + \not{k}'\gamma^\mu m + m\gamma^\mu\not{k} + m^2\gamma^\mu)\gamma_\nu u(p).$$

using the p identity thing, Note that

$$\gamma^\nu\not{k}'\gamma^\mu\not{k}\gamma_\nu = \gamma^\nu(2k'^\mu - \gamma^\mu\not{k}')(2k_\nu - \gamma_\nu\not{k}) = 4\gamma^\nu k'^\mu k_\nu - 2\gamma^\nu k'^\mu\gamma_\nu\not{k} - 2\gamma^\nu\gamma^\mu\not{k}'k_\nu + \gamma^\nu\gamma^\mu\gamma_\nu\not{k}'\not{k}$$

We need to fix Peskin & Schroeder (7.17) so it has arbitrary d instead of $d = 4$. We begin from (7.16), changing $4 \to d$:

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 + m_0^2 + i\epsilon} \gamma_\mu \frac{i}{(p-k)^2 - \mu^2 + i\epsilon}.$$

Following the procedure on pp. 217–218, we introduce the Feynman parameter $x=$ to combine the denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon} \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx \, \frac{1}{[k^2 - 2xk \cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon]^2}.$$

Let $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. Then

$$\begin{aligned}
-i\Sigma_2(p) &= (-ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \gamma^\mu \frac{i^2(\not{k} + m_0)}{[\ell^2 - \Delta + i\epsilon]^2} \gamma_\mu \\
&= -(-ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \gamma^\mu \frac{\not{\ell} + x\not{p} + m_0}{[\ell^2 - \Delta + i\epsilon]^2} \gamma_\mu \\
&= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \gamma^\mu \frac{2\ell^\mu - \gamma_\mu \not{\ell} + x(2p_\mu - \gamma_\mu \not{p}) + m_0 \gamma_\mu}{[\ell^2 - \Delta + i\epsilon]^2} \\
&= -e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)x\not{p} + dm_0}{[\ell^2 - \Delta + i\epsilon]^2},
\end{aligned} \tag{20}$$

where we have applied Eq. (19) and $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$ [1, p. 191], and we have dropped terms linear in ℓ [cite].

To evaluate the integral, we can write it in terms of the Euclidean 4-momentum defined in Eq. (6), as on p. 193 in Peskin & Schroeder:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(-1)^2} \frac{1}{(2\pi)^d} \int d^d\ell_E \, \frac{1}{(\ell_E^2 + \Delta)^2} = i \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2}.$$

Then we can apply (7.84), which takes the limit as $d \to 4$:

$$\int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2 + \Delta)^2} \to \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right),$$

where $\epsilon = 4 - d$ [1, p. 250]. Making these substitutions into Eq. (20), we find

$$\Sigma_2(p) = \frac{e^2}{16\pi^2} \int_0^1 dx \, [(2-d)x\not{p} + dm_0] \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right).$$

Then

$$\begin{aligned}
\frac{d\Sigma_2}{d\not{p}} &= \frac{e^2}{16\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) \frac{d}{d\not{p}} [(2-d)x\not{p} + dm_0] \\
&\qquad + [(2-d)x\not{p} + dm_0] \frac{d}{d\not{p}} \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) \\
&= \frac{e^2}{16\pi^2} \int_0^1 dx \left[\left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) (2-d)x + [(2-d)x\not{p} + dm_0] \frac{2x(1-x)\not{p}}{\Delta} \right]
\end{aligned}$$

where

$$\frac{d}{d\not{p}} (\ln \Delta) = \frac{d\Delta}{d\not{p}} \frac{d}{d\Delta} (\ln \Delta) = -\frac{2x(1-x)\not{p}}{\Delta}$$

from Eqs. (14) and (15). Then applying $m \approx m_0$,

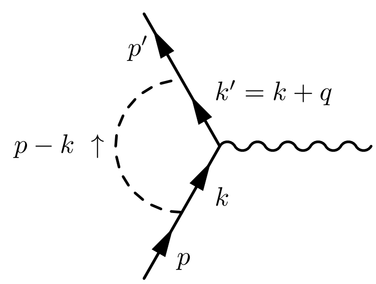
$$\delta Z_2 = \frac{e^2}{16\pi^2} \int_0^1 dx \left[(2-d)x \left(\frac{2}{\epsilon} - \ln(\Delta) - \gamma + \ln(4\pi) \right) + \frac{2x(1-x)[(2-d)x + d]m^2}{\Delta_m} \right].$$

Problem 2. (Peskin & Schroeder 7.3) Consider a theory of elementary fermions that couple both to QED and to a Yukawa field ϕ:

$$H_{\text{int}} = \int d^3x \, \frac{\lambda}{\sqrt{2}} \phi \bar{\psi}\psi + \int d^3x \, eA_\mu \bar{\psi}\gamma^\mu \psi.$$

2(a) Verify that the contribution to Z_1 from the vertex diagram with a virtual ϕ equals the contribution to Z_2 from the diagram with a virtual ϕ. Use dimensional regularization. Is the Ward identity generally true in this theory?

Solution. We are interested in the diagram



We also considered this diagram in Homework 6, with the Yukawa field identified as a Higgs field. There, we adapted Peskin & Schroeder (6.38) to write

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p,p')u(p) &= \int \frac{d^dk}{(2\pi)^d} \frac{i}{(k-p)^2 - m_\phi^2 + i\epsilon} \bar{u}(p') \left(-i\frac{\lambda}{\sqrt{2}}\right) \frac{i(\not{k}' + m_e)}{k'^2 - m_e^2 + i\epsilon} \gamma^\mu \frac{i(\not{k} + m_e)}{k^2 - m_e^2 + i\epsilon} \left(-i\frac{\lambda}{\sqrt{2}}\right) u(p) \\
&= i\frac{\lambda^2}{2} \int \frac{d^dk}{(2\pi)^d} \bar{u}(p') \frac{(\not{k}' + m_e)\gamma^\mu(\not{k} + m_e)}{[(k-p)^2 - m_\phi^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} u(p). \qquad (21)
\end{aligned}$$

We used Peskin & Schroeder (6.41) to write

$$\frac{1}{[(k-p)^2 - m_\phi^2 + i\epsilon](k'^2 - m_e^2 + i\epsilon)(k^2 - m_e^2 + i\epsilon)} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{D^3}, \qquad (22)$$

where [1, pp. 190–191]

$$D = k^2 + 2k(qy - pz) + z(p^2 - m_\phi^2) - (1-z)m_e^2 + i\epsilon \equiv \ell^2 - \Delta + i\epsilon,$$

where we used $x + y + z = 1$ and $k' = k + q$, and defined $\ell \equiv k + yq - zp$ [1, p. 191], and

$$\Delta \equiv -xyq^2 + (1-z)^2 m_e^2 + z m_\phi^2.$$

$$D = \ell^2 + xyq^2 - (1-z)^2 m_e^2 - m_\phi^2 z + i\epsilon. \qquad (23)$$

For the numerator of Eq. (21), we used $k' = k + q$ and $\ell \equiv k + yq - zp$ [1, p. 191], and defined

$$N \equiv \bar{u}(p')[\not{\ell} + (1-y)\not{q} + z\not{p} + m_e]\gamma^\mu(\not{\ell} - y\not{q} + z\not{p} + m_e)u(p). \qquad (24)$$

We know from Eqs. (2) and (3) that we only care about terms in Eq. (21) that are linear in γ^μ. So Eq. (24)

2(b) Now consider the renormalization of the $\phi\bar{\psi}\psi$ vertex. Show that the rescaling of this vertex at $q^2 = 0$ is *not* canceled by the correction to Z_2. (It suffices to compute the ultraviolet-divergent parts of the diagrams.) In this theory, the vertex and field-strength rescaling give additional shifts of the observable coupling constant relative to its bare value.

References

[1] M. E. Peskin and D. V. Schroeder, "An Introduction to Quantum Field Theory". Perseus Books Publishing, 1995.

[2] Wikipedia contributors, "Gamma matrices." From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Gamma_matrices`.