Problem 1. LCR circuit An electrical circuit consists of an inductance L, resistance R and capacitance C in series, driven by a voltage source $V(t) = V_0 \cos(\omega t)$.

1(a) Show that the charge $q(t)$ on the capacitor satisfies the equation

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V(t), \tag{1}$$

and use it to define the complex susceptibility from

$$q(\omega) = \chi(\omega)V(\omega). \tag{2}$$

Show that the forced solution of this equation is

$$q(t) = \frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}},$$

where

$$\tan(\phi) = \frac{\omega R}{\omega^2 L - 1/C}.$$

Solution. An example LCR circuit is shown in Fig. 1. We can use Kirchoff's loop rule to obtain the differential equation for this circuit. Beginning from the bottom left corner of the circuit and moving clockwise, we have [1, pp. 849, 1007]

$$0 = V(t) - IR - L\frac{dI}{dt} - \frac{q}{C},$$

where we have applied Ohm's law $V_{ab} = IR$, the potential difference across an inductor $V_{ab} = L\, dI/dt$, and the definition of capacitance $C = q/V_{ab}$ [1, pp. 782, 999]. The current $I(t) = dq(t)/dt$ and charge $q(t)$ are identical at all points in a series circuit. Feeding in $I = dq(t)/dt$, this relation becomes

$$V(t) = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

as we wanted to show. □

Equation (1) is an ODE representing forced damped motion of a mass-spring system. Its solution can be written as the sum of the homogeneous solution, which dies out with time, and a particular solution [3, pp. 38, 40, 50–51]. Rewriting Eq. (1) as

$$\frac{V_0}{L}\cos(\omega t) = \ddot{q} + 2p\dot{q} + \omega_0^2 q$$

where $p = R/2L$ and $\omega_0^2 = 1/LC$, the ansatz for the particular solution is

$$q(t) = A_c \cos(\omega t) + A_s \sin(\omega t).$$

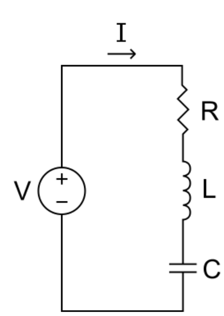

Figure 1: An LCR series circuit [2].

Feeding this into the ODE and collecting terms yields

$$-A_c\omega^2 + 2pA_s\omega + \omega_0^2 A_c = \frac{V_0}{L}, \qquad -A_s\omega^2 - 2pA_c\omega + \omega_0^2 A_s = 0.$$

This system has the solutions [3, p. 51]

$$A_s = \frac{2p\omega V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}, \qquad A_c = \frac{(\omega^2 - \omega_0^2)V_0/L}{4p^2\omega^2 + (\omega_0^2 - \omega^2)^2}. \tag{3}$$

If we define

$$A = \sqrt{A_c^2 + A_s^2} \tag{4}$$

and write

$$q(t) = A\left(\frac{A_c}{A}\cos(\omega t) + \frac{A_s}{A}\sin(\omega t),\right)$$

there exists an angle ϕ such that $\cos(\phi) = A_c/A$, $\sin(\phi) = A_s/A$, and $\tan(\phi) = A_s/A_c$. Thus

$$q(t) = A[\cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega).$$

Using the identity

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta),$$

it follows that [3, pp. 39, 51]

$$q(t) = A\cos(\omega t - \phi).$$

The amplitude A is given by [3, p. 51]

$$A = \frac{CV_0}{\sqrt{(\nu^2 - 1)^2 + 4c^2\nu^2}}, \qquad \text{where} \quad c = \frac{R}{2\sqrt{L/C}}, \quad \nu = \frac{\omega}{\omega_0}.$$

Substituting back to the original quantities, this is

$$A = \frac{CV_0}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + (R^2C/L)(\omega^2/\omega_0^2)}} = \frac{CV_0}{\sqrt{(LC\omega^2 - 1)^2 + C^2sR^2\omega^2}} = \frac{V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Additionally, from $\tan(\phi) = A_s/A_c$,

$$\tan(\phi) = \frac{2p\omega}{\omega^2 - \omega 0^2} = \frac{R\omega/L}{\omega^2 - 1/LC} = \frac{\omega R}{\omega^2 L - 1/C}.$$

Hence we have shown

$$q(t) = \frac{V_0\cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}$$

as desired. □

Finally, the complex susceptibility is defined by

$$\frac{V_0\cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}} = \chi(\omega)V_0\cos(\omega t),$$

where we have substituted into Eq. (2). This implies

$$\chi(\omega) = \frac{\cos(\omega t - \phi)}{\cos(\omega t)}\frac{1}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

1(b) Show that the mean rate of power dissipation is

$$W = \frac{1}{2} \frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Solution. The average power into a general AC circuit is [1, p. 1032]

$$P_{\text{av}} = \frac{1}{2} V I \sin(\phi),$$

where I is the current amplitude, V is the voltage amplitude, and ϕ is the phase angle determined in 1 [1, pp. 1028, 1032]. Assuming the circuit is perfectly efficient, the average power into the circuit is equal to the average power it dissipates, so $W = P_{\text{av}}$. Clearly $V = V_0$. For I,

$$I(t) = \frac{dq(t)}{dt} = \frac{d}{dt}\left(\frac{V_0 \cos(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}\right) = -\frac{\omega V_0 \sin(\omega t - \phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}},$$

so

$$I = \frac{\omega V_0}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}.$$

Thus

$$W = \frac{1}{2} V I \sin(\phi) = \frac{1}{2} \frac{\omega V_0^2 \sin(\phi)}{\sqrt{(-\omega^2 L + 1/C)^2 + (\omega R)^2}}$$

as we wanted to show. □

1(c) Sketch the real and imaginary parts of χ as a function of frequency, for the cases $Q \ll 1$, $Q \approx 1$, and $Q \gg 1$, where $Q = \sqrt{L/C}/R$ is the "quality factor."

Where are the poles of χ in the complex ω plane?

Solution. How is it even complex?

References

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