Problem 1. Consider the path integral for a single point particle, with the action

$$S = \int_0^1 dt \left[p_\mu(t)\dot{x}^\mu(t) + \frac{N(t)}{2}[p^2(t) - m^2 - i\epsilon] \right].$$

This represents the quantization of the coordinates and momenta of the particle, subject to the mass shell constraint $p^2 = m^2$ (together with the $i\epsilon$ prescription) imposed by the Lagrange multiplier N. This action admits the reparametrization symmetry $\delta x = \alpha p$, $\delta p = 0$, $\delta N = -\partial_t \alpha$ where $\alpha(t)$ is any function. This symmetry allows us to fix the gauge condition $N(t) = T$; the constant T must still be integrated over, however.

1(a) Path integrate over $x(t)$, subject to the boundary conditions $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$, yielding a delta function $\delta(\dot{p})$ along the path. Solve this constraint (find the set of functions that solve it) and path integrate over those $p(t)$ to find the quantum mechanical propagation amplitude

$$\langle y|x\rangle = D_F(x-y) = \int_0^\infty dT\,(2\pi i T)^{d/2} \exp\left[-\frac{i}{2}\left((m^2 - i\epsilon)T + \frac{(x-y)^2}{T} \right) \right],$$

where d is the number of spacetime dimensions.

1(b) Use this integral representation to show that D_F satisfies

$$(\delta^2 + m^2)D_F = i\delta^{(d)}(x-y).$$

1(c) Evaluate the T integral in terms of Bessel functions.

Problem 2. Quantum statistical mechanics (Peskin & Schroeder 9.2)

2(a) Evaluate the quantum statistical partition function

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

(where $\beta = 1/kT$) using the strategy of Section 9.1 for evaluating the matrix elements of e^{-iHt} in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

2(b) Evaluate this integral for a simple harmonic oscillator,

$$L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2,$$

by introducing a Fourier decomposition of $x(t)$:

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that, up to a (possibly divergent and β-dependent) constant, the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You may find the identity

$$\sinh z = z \prod_{n=1}^\infty \left(1 + \frac{z^2}{(n\pi)^2}\right)$$

useful.]

2(c) Generalize this construction to field theory. Show that the quantum statistical partition function for a free scalar field can be written in terms of a functional integral. The value of this integral is given formally by

$$[\det(-\partial^2 + m^2)]^{-1/2},$$

where the operator acts on functions on Euclidean space that are periodic in the time direction with periodicity β. As before, the β dependence of this expression is difficult to compute directly. However, the dependence on m^2 is unambiguous. Show that the determinant indeed reproduces the partition function for relativistic scalar particles.