Problem 1. (Jackson 9.8)

1(a) Show that a classical oscillating electric dipole $\mathbf{p}$ with fields given by

$$\mathbf{H} = \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right), \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{k^2(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}\frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}, \tag{1}$$

radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\mathbf{L}}{dt} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^* \times \mathbf{p}].$$

Solution. According to Jackson (9.20), the time-averaged angular momentum density is

$$\mathbf{l} = \frac{\mathrm{Re}[\mathbf{x} \times (\mathbf{E} \times \mathbf{H}^*)]}{2c^2}.$$

One of the vector identities on the inside cover of Jackson is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, so

$$\mathbf{l} = \frac{(\mathbf{x} \cdot \mathbf{H}^*)\mathbf{E} - (\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{2c^2}. \tag{2}$$

From Eq. (1), note that

$$\mathbf{x} \cdot \mathbf{H}^* \propto \mathbf{x} \cdot (\hat{\mathbf{n}} \times \mathbf{p}^*) = \mathbf{p}^* \cdot (\mathbf{x} \times \hat{\mathbf{n}}) = \mathbf{0},$$

where we have used the identity $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ and the fact that $\hat{\mathbf{n}}$ points in the $\mathbf{x}$ direction. For $\mathbf{x} \cdot \mathbf{E}$, note that

$$\begin{aligned}\mathbf{x} \cdot [(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}] &= -\mathbf{x} \cdot [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{p})] = -\mathbf{x} \cdot [(\hat{\mathbf{n}} \cdot \mathbf{p})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{p}] = -(\hat{\mathbf{n}} \cdot \mathbf{p})(\mathbf{x} \cdot \hat{\mathbf{n}}) + \mathbf{x} \cdot \mathbf{p} \\ &= -r(\hat{\mathbf{n}} \cdot \mathbf{p}) + \mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{p} = 0,\end{aligned}$$

$$\mathbf{x} \cdot [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] = 3(\mathbf{x} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3r(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3(\mathbf{x} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 2(\mathbf{x} \cdot \mathbf{p}),$$

since $|\mathbf{x}| = r$ and $\mathbf{x} = r\,\hat{\mathbf{n}}$. Then

$$\mathbf{x} \cdot \mathbf{E} = \frac{1}{2\pi\epsilon_0}(\mathbf{x} \cdot \mathbf{p})\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} = \frac{1}{2\pi\epsilon_0}(\hat{\mathbf{n}} \cdot \mathbf{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}.$$

With these substitutions, Eq. (2) becomes

$$\begin{aligned}\mathbf{l} &= -\frac{(\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{c^2} = -\frac{1}{4\pi\epsilon_0 c^2}(\hat{\mathbf{n}} \cdot \mathbf{p})\left(\frac{1}{r^2} - \frac{ik}{r}\right)e^{ikr}\frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p}^*)\frac{e^{-ikr}}{r}\left(1 + \frac{1}{ikr}\right) \\ &= -\frac{k^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{r^2} - \frac{ik}{r}\right)\left(1 - \frac{i}{kr}\right) = -\frac{k^2}{16\pi^2\epsilon_0 c}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{r^2} - \frac{i}{kr^3} - \frac{ik}{r} - \frac{1}{r^2}\right) \\ &= -\frac{ik^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{kr^3} + \frac{k}{r^2}\right) = \frac{ik^3}{16\pi^2\epsilon_0 cr^2}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)\left(\frac{1}{k^2r^2} + 1\right).\end{aligned}$$

Let $\mathbf{L}$ be the angular momentum radiated to a distance R. Then

$$\mathbf{L} = \int_R \mathbf{l}\,d^3x = \int_0^{\pi}\int_0^{2\pi}\int_0^R \mathbf{l}\,r^2\sin\theta\,dr\,d\phi\,d\theta,$$

and the time derivative is

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}\, r^2 \sin\theta\, dr\, d\phi\, d\theta\right) = \frac{dr}{dt}\frac{d}{dr}\left(\int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}\, r^2 \sin\theta\, dr\, d\phi\, d\theta\right) \\ &= c\int_0^\pi \int_0^{2\pi} \mathbf{l}\, r^2 \sin\theta\, d\phi\, d\theta = \frac{ik^3}{16\pi^2\epsilon_0}\left(\frac{1}{k^2r^2}+1\right)\int_0^\pi \int_0^{2\pi} (\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\times\mathbf{p}^*)\sin\theta\, d\phi\, d\theta. \qquad (3)\end{aligned}$$

Note that

$$[(\hat{\mathbf{n}}\cdot\mathbf{p})(\hat{\mathbf{n}}\times\mathbf{p}^*)]_i = \sum_{j=1}^3 n_j p_j (\hat{\mathbf{n}}\times\mathbf{p}^*)_i = \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \epsilon_{ikl} n_j p_j n_k p_l^*,$$

so

$$\frac{dL_i}{dt} \propto \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \int n_j p_k\, d\Omega = \sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \frac{4\pi}{3}\delta_{jk} = \frac{4\pi}{3}\epsilon_{ikl} p_k p_l^* = \frac{4\pi}{3}(\mathbf{p}\times\mathbf{p}^*)_i,$$

where we have used Jackson (9.47), $\int n_\beta n_\gamma\, d\Omega = 4\pi\delta_{\beta\gamma}/3$. Making this substitution into Eq. (3),

$$\frac{d\mathbf{L}}{dt} = \frac{ik^3}{6\pi\epsilon_0}\left(\frac{1}{k^2r^2}+1\right)(\mathbf{p}\times\mathbf{p}^*).$$

Taking the limit as $r\to\infty$, we find

$$\frac{d\mathbf{L}}{dt} = \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0}(\mathbf{p}\times\mathbf{p}^*)\right] = \mathrm{Re}\left[-\frac{ik^3}{12\pi\epsilon_0}(\mathbf{p}^*\times\mathbf{p})\right] = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}], \qquad (4)$$

as desired. □

1(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

Solution. According to Jackson (9.24), the total power radiated by an oscillating electric dipole $\mathbf{p}$ is

$$P = \frac{dE}{dt} = \frac{c^2 Z_0 k^4}{12\pi}|\mathbf{p}|^2.$$

Then the ratio of angular momentum radiated to energy radiated is

$$\frac{d\mathbf{L}/dt}{dE/dt} = \frac{k^3}{12\pi\epsilon_0}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}]\frac{12\pi}{c^2 Z_0 k^4|\mathbf{p}|^2} = \frac{1}{\epsilon_0}\,\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}]\frac{1}{c^2 Z_0 k|\mathbf{p}|^2} = \frac{\mathrm{Im}[\mathbf{p}^*\times\mathbf{p}]}{\omega|\mathbf{p}|^2},$$

where we have used $Z_0 = \sqrt{\mu_0/\epsilon_0} = 1/\sqrt{\epsilon_0^2 c^2} = 1/\epsilon_0 c$, $c^2 = 1/(\epsilon_0\mu_0)$, and $\omega = kc$.

In the limit of high frequency, $(d\mathbf{L}/dt)/(dE/dt)\to 0$. In this scenario, the energy radiated dominates over the angular momentum radiated. Likewise, in the limit of low frequency, $(d\mathbf{L}/dt)/(dE/dt)\to\infty$, meaning that angular momentum radiation dominates. This is sensible because rotational kinetic energy $E\propto\omega^2$, while angular momentum $L\propto\omega$.

1(c) For a charge e rotating in the xy plane at radius a and angular speed ω, show that there is only a z component of radiated angular momentum with magnitude $dL_z/dt = e^2k^3a^2/6\pi\epsilon_0$. What about a charge oscillating along the z axis?

Solution. We know from Homework 5 that the position of a point charge rotating counterclockwise in the xy plane is

$$\mathbf{x}(t) = a\cos(\omega t)\,\mathbf{x} + a\sin(\omega t)\,\hat{\mathbf{y}}.$$

Then the charge distribution is

$$\rho(\mathbf{x}, t) = e\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z).$$

According to Jackson (4.8), the dipole moment is defined

$$\mathbf{p} = \int \mathbf{x}'\,\rho(\mathbf{x}')\,d^3x'\,.$$

The components of $\mathbf{p}$ for the point charge are then

$$\begin{aligned}
p_x &= e\iiint x\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z)\,dx\,dy\,dz = ea\cos(\omega t),\\
p_y &= e\iiint y\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z)\,dx\,dy\,dz = ea\sin(\omega t),\\
p_z &= e\iiint z\,\delta[x - a\cos(\omega t)]\,\delta[y - a\sin(\omega t)]\,\delta(z)\,dx\,dy\,dz = 0,
\end{aligned}$$

so we can write $\mathbf{p} = ea\,e^{-i\omega t}(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}})$. Substituting into Eq. (4),

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0}e^2a^2e^{-i\omega t}e^{i\omega t}[(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}}) \times (\hat{\mathbf{x}} - i\,\hat{\mathbf{y}})]\right] = \mathrm{Re}\left[\frac{ie^2k^3a^2}{12\pi\epsilon_0}(-2i\,\hat{\mathbf{x}} \times \hat{\mathbf{y}})\right] = \mathrm{Re}\left[\frac{e^2k^3a^2}{6\pi\epsilon_0}\,\hat{\mathbf{z}}\right]\\
&= \frac{e^2k^3a^2}{6\pi\epsilon_0}\cos(\omega t)\,\hat{\mathbf{z}},
\end{aligned}$$

as desired. □

A charge oscillating along the z axis with amplitude a has the charge density

$$\rho(\mathbf{x}, t) = ea\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)],$$

which gives the dipole moment

$$\begin{aligned}
p_x &= ea\iiint x\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)]\,dx\,dy\,dz = 0,\\
p_y &= ea\iiint y\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)]\,dx\,dy\,dz = 0,\\
p_z &= ea\iiint z\,\delta(x)\,\delta(y)\,\delta[z - \cos(\omega t)]\,dx\,dy\,dz = ea\cos(\omega t).
\end{aligned}$$

In complex notation, $\mathbf{p} = ea\,e^{-i\omega t}\,\hat{\mathbf{z}}$. Substituting into Eq. (4), we find

$$\frac{d\mathbf{L}}{dt} = \mathrm{Re}\left[\frac{ik^3}{12\pi\epsilon_0}e^2a^2e^{-i\omega t}e^{i\omega t}(\hat{\mathbf{z}} \times \hat{\mathbf{z}})\right] = \mathbf{0}.$$

So we see that a charge undergoing linear motion does not lead to a radiated angular momentum, which is sensible.

1(d) What are the results corresponding to Probs. 1(a) and 1(b) for magnetic dipole radiation?

Solution. The radiation fields for a magnetic dipole are given by Jackson (19.35–36),

$$\mathbf{H} = \frac{1}{4\pi}\left\{k^2(\hat{\mathbf{n}}\times\mathbf{m})\times\hat{\mathbf{n}}\frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{m})-\mathbf{m}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}, \quad \mathbf{E} = -\frac{Z_0}{4\pi}k^2(\hat{\mathbf{n}}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right).$$

Comparing with Eq. (1), we see that $\mathbf{H}\to-\mathbf{E}/Z_0$, $\mathbf{E}\to Z_0\mathbf{H}$, and $\mathbf{p}\to\mathbf{m}/c$ as stated in the book [1, p. 413]. Making these substitutions, the results of Probs. 1.1(a) and (b) become

$$\frac{d\mathbf{L}}{dt} = \frac{\mu_0 k^3}{12\pi}\,\mathrm{Im}[\mathbf{m}^*\times\mathbf{m}], \qquad \frac{d\mathbf{L}/dt}{dE/dt} = \frac{\mathrm{Im}[\mathbf{m}^*\times\mathbf{m}]}{\omega|\mathbf{m}|^2}$$

where we have used $\mu = 1/\epsilon_0 c^2$.

Problem 2. (Jackson 10.1)

2(a) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{4} - |\boldsymbol{\epsilon}_0\cdot\hat{\mathbf{n}}|^2 - \frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}}_0\cdot\hat{\mathbf{n}}\right],$$

where $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{n}}$ are the directions of the incident and scattered radiations, respectively, while $\boldsymbol{\epsilon}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\boldsymbol{\epsilon}_0^*\cdot\boldsymbol{\epsilon}_0 = 1$; $\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0 = 0$).

Solution. Jackson (10.14) gives the differential cross section for scattering off a small, perfectly conducting sphere with initial polarization $\boldsymbol{\epsilon}_0$ and outgoing polarization $\boldsymbol{\epsilon}$:

$$\frac{d\sigma}{d\Omega}\hat{\mathbf{n}},\boldsymbol{\epsilon};\hat{\mathbf{n}}_0,\boldsymbol{\epsilon}_0) = k^4a^6\left|(\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0 - \frac{1}{2}(\hat{\mathbf{n}}\times\boldsymbol{\epsilon}^*)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2. \tag{5}$$

We will use the polarization vectors $\boldsymbol{\epsilon}^{(1)}$ and $\boldsymbol{\epsilon}^{(2)}$, which are defined in Fig. 1 [1, p. 458]. According to the figure,

$$\boldsymbol{\epsilon}^{(2)} = \frac{\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0}{|\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0|} = \frac{\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}},$$

$$\boldsymbol{\epsilon}^{(1)} = \boldsymbol{\epsilon}^{(2)}\times\hat{\mathbf{n}} = \frac{-\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0)}{\sin\theta} = \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sin\theta} = \frac{\hat{\mathbf{n}}_0 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}},$$

which are both real. In the denominator, we have used $\sin^2\theta = 1+\cos^2\theta = 1+(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2$. We also note that $\hat{\mathbf{n}}_0$, $\hat{\mathbf{n}}$, and $\boldsymbol{\epsilon}^{(1)}$ are in the same plane, and that $\hat{\mathbf{n}}\perp\boldsymbol{\epsilon}^{(1)}$.

The cross section summed over outgoing polarizations is then found by plugging $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(1)}$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(2)}$ into Eq. (5), and taking the sum. For the first term,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}},\boldsymbol{\epsilon}^{(1)};\hat{\mathbf{n}}_0,\boldsymbol{\epsilon}_0) &= k^4a^6\left|\boldsymbol{\epsilon}^{(1)^*}\cdot\boldsymbol{\epsilon}_0 - \frac{1}{2}(\hat{\mathbf{n}}\times\boldsymbol{\epsilon}^{(1)^*})\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2 \\
&= k^4a^6\left|\frac{\hat{\mathbf{n}}_0-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}}\cdot\boldsymbol{\epsilon}_0 - \frac{1}{2}\left(\hat{\mathbf{n}}\times\frac{\hat{\mathbf{n}}_0-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}}\right)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2 \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left|-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)(\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}}\times\hat{\mathbf{n}}_0)\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)\right|^2.
\end{aligned}$$

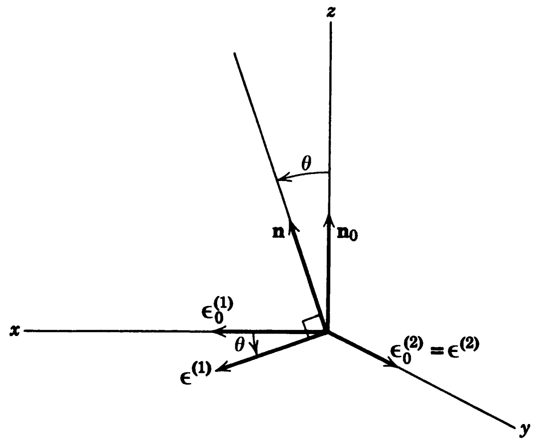

Figure 1: (Jackson 10.1) Polarization and propagation vectors for the incident and scattered radiation.

One of the vector identities on the inside cover of Jackson is $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$. Applying this, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(1)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) &= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) + \frac{1}{2}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}}_0 \cdot \boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0)(\hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}}_0) \right|^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0) \right|^2 = \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \left[(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) - \frac{1}{2} \right]^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} |\hat{\mathbf{n}} \cdot \boldsymbol{\epsilon}_0|^2 \left[(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{1}{4} \right].
\end{aligned}$$

For the second term,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(2)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) &= k^4 a^6 \left| \boldsymbol{\epsilon}^{(2)*} \cdot \boldsymbol{\epsilon}_0 - \frac{1}{2}(\hat{\mathbf{n}} \times \boldsymbol{\epsilon}^{(2)*}) \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\
&= k^4 a^6 \left| \frac{\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}} \cdot \boldsymbol{\epsilon}_0 - \frac{1}{2} \left(\hat{\mathbf{n}} \times \frac{\hat{\mathbf{n}} \times \hat{\mathbf{n}}_0}{\sqrt{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2}} \right) \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) - \frac{1}{2}[(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}_0] \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) - \frac{1}{2}[(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)\hat{\mathbf{n}} - \hat{\mathbf{n}}_0] \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) - \frac{1}{2}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) + \frac{1}{2}\epsilon_0 \cdot (\hat{\mathbf{n}}_0 \times \hat{\mathbf{n}}_0) \right|^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left| \left(1 - \frac{1}{2}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) \right) \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0) \right|^2 \\
&= \frac{k^4 a^6}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2} \left[1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0) + \frac{1}{4}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0)^2 \right] |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2.
\end{aligned}$$

Summing the two terms, we find

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(1)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) + \frac{d\sigma}{d\Omega}(\hat{\mathbf{n}}, \boldsymbol{\epsilon}^{(2)}; \hat{\mathbf{n}}_0, \boldsymbol{\epsilon}_0) \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left\{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\left[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0) + \frac{1}{4}\right] + \left[1 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0) + \frac{1}{4}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right]|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2\right\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left\{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2 - |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0) + \frac{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2}{4} + |\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2\right. \\
&\qquad\qquad \left. - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4}\right\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left\{\frac{5|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + \frac{5|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2}{4} - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right. \\
&\qquad\qquad \left. - \frac{|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} - \frac{|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2}{4} + \frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2|\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left\{\left[\frac{5}{4} - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right]\left[|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right] - \left[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right]\left[\frac{|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right]\right\} \\
&= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left[\frac{5}{4} - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right]\left[|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right] - k^4a^6\left[\frac{|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2}{4} + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right]. \qquad (6)
\end{aligned}$$

Since $\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0 = 0$, we note that

$$\hat{\mathbf{n}} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)\,\hat{\mathbf{n}}_0 + (\hat{\mathbf{n}}\cdot\epsilon_0)\,\boldsymbol{\epsilon}_0 + [\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)]\,(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0) \quad\Longrightarrow\quad 1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2 + |\hat{\mathbf{n}}\cdot\epsilon_0|^2 + |\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\cdot\boldsymbol{\epsilon}_0)|^2. \qquad (7)$$

Substituting into Eq. (6),

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{k^4a^6}{1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2}\left[\frac{5}{4} - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right]\left[1-(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0)^2\right] - k^4a^6\left[\frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 + |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2\right] \\
&= k^4a^6\left[\frac{5}{4} - |\hat{\mathbf{n}}\cdot\boldsymbol{\epsilon}_0|^2 - \frac{1}{4}|\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0\right], \qquad (8)
\end{aligned}$$

as we sought to prove. □

2(b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi)\right],$$

where $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of linear polarization.

Solution. We choose coordinates as in Fig. 1, such that the direction of linear polarization $\boldsymbol{\epsilon}_0$ points along the x axis and $\hat{\mathbf{n}}_0$ points along the z axis. Then $\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0$ points along the y axis. In spherical coordinates, Eq. (7) becomes

$$\hat{\mathbf{n}} = \cos\phi\sin\theta\,\boldsymbol{\epsilon}_0 + \sin\phi\sin\theta\,(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0) + \cos\theta\,\hat{\mathbf{n}}_0, \qquad (9)$$

which implies

$$\cos\phi\sin\theta = \hat{\mathbf{n}}\cdot\epsilon_0, \qquad \sin\phi\sin\theta = \hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0), \qquad \cos\theta = \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}_0.$$

Making these substitutions in Eq. (8), we obtain

$$\begin{aligned}
\frac{d\sigma}{d\Omega}(\theta,\phi) &= k^4a^6\left[\frac{5}{4} - \cos^2\phi\sin^2\theta - \frac{1}{4}\sin^2\phi\sin^2\theta - \cos\theta\right] \\
&= k^4a^6\left[\frac{5}{4} - \frac{1}{2}[1+\cos(2\phi)]\sin^2\theta - \frac{1}{8}[1-\cos(2\phi)]\sin^2\theta - \cos\theta\right] \\
&= k^4a^6\left[\frac{5}{4} - \frac{1}{2}(1-\cos^2\theta) - \frac{1}{2}\cos(2\phi)\sin^2\theta - \frac{1}{8}(1-\cos^2\theta) + \frac{1}{8}\cos(2\phi)\sin^2\theta - \cos\theta\right] \\
&= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos(2\phi)\right],
\end{aligned}$$

where we have used the identities $2\sin^2\phi = 1-\cos(2\phi)$, $2\cos^2\phi = 1+\cos(2\phi)$, and $\cos^2\theta+\sin^2\theta = 1$. $\square$

2(c) What is the ratio of scattered intensities at $\theta=\pi/2$, $\phi=0$ and $\theta=\pi/2$, $\phi=\pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

Solution. Firstly, we note that

$$\frac{d\sigma}{d\Omega}(\pi/2,0) = k^4a^6\left[\frac{5}{8}-\frac{3}{8}\right] = \frac{k^4a^6}{4}, \qquad \frac{d\sigma}{d\Omega}(\pi/2,\pi/2) = k^4a^6\left[\frac{5}{8}+\frac{3}{8}\right] = k^4a^6,$$

so the ratio is

$$\frac{d\sigma/d\Omega\ (\pi/2,0)}{d\sigma/d\Omega\ (\pi/2,\pi/2)} = \frac{1}{4}.$$

According to Jackson (10.12–13), the electric and magnetic dipole moments of a perfectly conducting sphere are, respectively,

$$\mathbf{p} = 4\pi\epsilon_0 a^3\mathbf{E}_{\rm inc}, \qquad \mathbf{m} = 2\pi a^3\mathbf{H}_{\rm inc},$$

where $\mathbf{E}_{\rm inc}$ and $\mathbf{H}_{\rm inc}$ are the incident fields. They are given by Jackson (10.1), wherein

$$\mathbf{E}_{\rm inc} = \boldsymbol{\epsilon}_0 E_o e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}}, \qquad \mathbf{H}_{\rm inc} = \hat{\mathbf{n}}_0\times\mathbf{E}_{\rm inc}/Z_0.$$

The scattered fields are given by Jackson (10.2),

$$\mathbf{E}_{\rm sc} = \frac{1}{4\pi\epsilon_0}k^2\frac{e^{ikr}}{r}\left[(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}} - \hat{\mathbf{n}}\times\frac{\mathbf{m}}{c}\right], \qquad \mathbf{H}_{\rm sc} = \hat{\mathbf{n}}\times\frac{\mathbf{E}_{\rm sc}}{Z_0}.$$

When $\phi=0$, Eq. (9) indicates that $\hat{\mathbf{n}} = \boldsymbol{\epsilon}_0$. Applying the relations above, $\hat{\mathbf{n}}$ and $\mathbf{p}$ therefore point in the same direction. This means $\hat{\mathbf{n}}\times\mathbf{p} = \mathbf{0}$, so $\mathbf{E}_{\rm sc}$ only has a contribution from the magnetic dipole. However, when $\phi=\pi/2$, $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0\times\boldsymbol{\epsilon}_0$ and therefore $\hat{\mathbf{n}}$ points in the same direction as $\mathbf{m}$. This means $\hat{\mathbf{n}}\times\mathbf{m} = \mathbf{0}$, so $\mathbf{E}_{\rm sc}$ only has a contribution from the electric dipole. The ratio $1/4$ indicates that the strength of radiation from a purely electric dipole is four times that from a purely magnetic dipole.

Problem 3. (Jackson 12.15) Consider the Proca equation for a localized steady-state distribution of current that has only a static magnetic moment. This model can be used to study the observable effects of a finite photon mass on the earth's magnetic field. Note that if the magnetization is $\boldsymbol{\mathcal{M}}(\mathbf{x})$ the current density can be written as $\mathbf{J} = c\,(\boldsymbol{\nabla} \times \boldsymbol{\mathcal{M}})$.

3(a) Show that if $\boldsymbol{\mathcal{M}} = \mathbf{m}\, f(\mathbf{x})$, where $\mathbf{m}$ is a fixed vector and $f(\mathbf{x})$ is a localized scalar function, the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m} \times \boldsymbol{\nabla} \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \,.$$

Solution. The Proca equations of motion in the static limit are given by the equation immediately following Jackson (12.93),

$$\nabla^2 A_\alpha - \mu^2 A_\alpha = -\frac{4\pi}{c} J_\alpha,$$

which implies

$$\nabla^2 \mathbf{A} - \mu^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J}. \tag{10}$$

We will proceed by finding the Green's function for this equation, which satisfies

$$(\nabla^2 - \mu^2)\, G(\mathbf{x}) = \delta^3(\mathbf{x}). \tag{11}$$

We can Fourier transform $G(\mathbf{x})$ so long as $\mathbf{A}$ and its derivatives vanish at infinity [2]. The Fourier transform expression and its inverse in one dimension are, according to Jackson (2.45–46),

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \,, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk \,.$$

Generalizing to three dimensions, the Green's function transforms as [2]

$$G(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int G(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x \,.$$

Transforming both sides of Eq. (11) in the same way, we have [2]

$$\begin{aligned}
\frac{1}{(2\pi)^{3/2}} \int (\nabla^2 - \mu^2)\, G(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x &= \frac{1}{(2\pi)^{3/2}} \int \nabla^2\, G(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x - \frac{\mu^2}{(2\pi)^{3/2}} \int G(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x \\
&= \frac{k^2 - \mu^2}{(2\pi)^{3/2}} \int G(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x \,,
\end{aligned}$$

$$\frac{1}{(2\pi)^{3/2}} \int \delta^3(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3x = \frac{1}{(2\pi)^{3/2}},$$

so

$$(k^2 - \mu^2)\, G(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \qquad \Longrightarrow \qquad G(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}(k^2 - \mu^2)}.$$

Then we can find $G(\mathbf{x})$ using the inverse Fourier transform [2]:

$$\begin{aligned}
G(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int G(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3k = \frac{1}{(2\pi)^3} \int \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2 - \mu^2}\, d^3k = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{e^{ik|\mathbf{x}|}}{k^2 - \mu^2} k^2 \sin\theta\, dk\, d\theta\, d\phi \\
&= \frac{4\pi}{(2\pi)^3|\mathbf{x}|} \int_0^{\infty} \frac{k \sin(k|\mathbf{x}|)}{k^2 - \mu^2} k^2\, dk = \frac{e^{-\mu|\mathbf{x}|}}{4\pi|\mathbf{x}|}.
\end{aligned}$$

Then the solution to Eq. (10) is

$$\mathbf{A}(\mathbf{x}) = \frac{4\pi}{c}\int \mathbf{J}(\mathbf{x}')\,G(\mathbf{x}-\mathbf{x}')\,d^3x' = \frac{1}{c}\int \mathbf{J}(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'\,.$$

Note that

$$\frac{\mathbf{J}}{c} = \boldsymbol{\nabla}\times[\mathbf{m}\,f(\mathbf{x})] = \boldsymbol{\nabla} f(\mathbf{x})\times\mathbf{m} + f(\mathbf{x})\,\boldsymbol{\nabla}\times\mathbf{m} = \boldsymbol{\nabla} f(\mathbf{x})\times\mathbf{m} = -\mathbf{m}\times\boldsymbol{\nabla} f(\mathbf{x}),$$

where we have used the identity $\boldsymbol{\nabla}\times(\psi\mathbf{a}) = \boldsymbol{\nabla}\psi\times\mathbf{a} + \psi\boldsymbol{\nabla}\times\mathbf{a}$. Making this substitution,

$$\mathbf{A}(\mathbf{x}) = -\int \mathbf{m}\times\boldsymbol{\nabla}' f(\mathbf{x}')\,\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' = -\mathbf{m}\times\int \boldsymbol{\nabla}' f(\mathbf{x}')\,\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'\,. \tag{12}$$

Integrating by parts,

$$\int \boldsymbol{\nabla}' f(\mathbf{x}')\,\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' = \left[f(\mathbf{x}')\boldsymbol{\nabla}'\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\right]_{-\infty}^{\infty} - \int f(\mathbf{x}')\,\boldsymbol{\nabla}'\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' = -\int f(\mathbf{x}')\,\boldsymbol{\nabla}'\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'\,,$$

since $f(\mathbf{x})$ is a localized scalar function and therefore vanishes at infinity. Since the Green's function depends only on $|\mathbf{x}-\mathbf{x}'|$, we can replace $\boldsymbol{\nabla}'$ by $-\boldsymbol{\nabla}$:

$$-\int f(\mathbf{x}')\,\boldsymbol{\nabla}'\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' = \int f(\mathbf{x}')\,\boldsymbol{\nabla}\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' = \boldsymbol{\nabla}\int f(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'\,.$$

Making this substitution in Eq. (12),

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m}\times\boldsymbol{\nabla}\int f(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x'\,, \tag{13}$$

as desired. $\square$

3(b) If the magnetic dipole is a point dipole at the origin [$f(\mathbf{x}) = \delta(\mathbf{x})$], show that the magnetic field away from the origin is

$$\mathbf{B}(\mathbf{x}) = [3\,\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}]\left(1 + \mu r + \frac{\mu^2 r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2\mathbf{m}\frac{e^{-\mu r}}{r}.$$

Solution. Setting $f(\mathbf{x}) = \delta(\mathbf{x})$ in Eq. (13),

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m}\times\boldsymbol{\nabla}\int \delta(\mathbf{x}')\frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' = -\mathbf{m}\times\boldsymbol{\nabla}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}.$$

Note that

$$\boldsymbol{\nabla}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} = \left(-\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2} - \frac{\mu e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}\right)\mathbf{x} = -(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2}\hat{\mathbf{x}} = -(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}\mathbf{x},$$

so

$$\mathbf{A}(\mathbf{x}) = (\mathbf{m}\times\mathbf{x})(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2}.$$

The magnetic field is given by $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$:

$$\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times\left[(\mathbf{m}\times\mathbf{x})(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}\right] = \boldsymbol{\nabla}\left[(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}\right]\times(\mathbf{m}\times\mathbf{x}) + (1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}\boldsymbol{\nabla}\times(\mathbf{m}\times\mathbf{x}).$$

For the first term,

$$\boldsymbol{\nabla}\left[(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}\right] = \left[-\mu\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} + (1+\mu|\mathbf{x}|)\left(-\frac{\mu e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} - \frac{3e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^4}\right)\right]\hat{\mathbf{x}} = -(3+3\mu|\mathbf{x}|+\mu^2|\mathbf{x}|^2)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^4}\,\hat{\mathbf{x}}.$$

Then, using the identity $\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b}$,

$$\boldsymbol{\nabla}\times(\mathbf{m}\times\mathbf{x}) = \mathbf{m}(\boldsymbol{\nabla}\cdot\mathbf{x}) - \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{m}) + (\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{m} - (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{x} = \mathbf{m}(\boldsymbol{\nabla}\cdot\mathbf{x}) - (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{x}.$$

Making these substitutions,

$$\begin{aligned}
\mathbf{B}(\mathbf{x}) &= -(3+3\mu|\mathbf{x}|+\mu^2|\mathbf{x}|^2)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}\,\hat{\mathbf{x}}\times(\mathbf{m}\times\hat{\mathbf{x}}) + (1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}[\mathbf{m}(\boldsymbol{\nabla}\cdot\hat{\mathbf{x}}) - (\mathbf{m}\cdot\boldsymbol{\nabla})\hat{\mathbf{x}}] \\
&= -(3+3\mu|\mathbf{x}|+\mu^2|\mathbf{x}|^2)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}[(\hat{\mathbf{x}}\cdot\hat{\mathbf{x}})\mathbf{m} - (\hat{\mathbf{x}}\cdot\mathbf{m})\hat{\mathbf{x}}] + 2(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2}\mathbf{m} \\
&= -3\left(1+\mu|\mathbf{x}|+\frac{\mu^2|\mathbf{x}|^2}{3}\right)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3}[\mathbf{m} - (\hat{\mathbf{x}}\cdot\mathbf{m})\hat{\mathbf{x}}] + 2(1+\mu|\mathbf{x}|)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^2}\mathbf{m} \\
&= [3\,\hat{\mathbf{x}}(\hat{\mathbf{x}}\cdot\mathbf{m}) - \mathbf{m}]\left(1+\mu|\mathbf{x}|+\frac{\mu^2|\mathbf{x}|^2}{3}\right)\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|^3} - \frac{2}{3}\mu^2\mathbf{m}\frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}.
\end{aligned}$$

Letting $\hat{\mathbf{x}}\to\hat{\mathbf{r}}$ and $|\mathbf{x}|\to r$, we have

$$\mathbf{B}(\mathbf{x}) = [3\,\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}]\left(1+\mu r+\frac{\mu^2 r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2\mathbf{m}\frac{e^{-\mu r}}{r} \tag{14}$$

as we sought to show. □

3(c) The result of Prob. 3(b) shows that at fixed $r = R$ (on the surface of the earth), the earth's magnetic field will appear as a dipole angular distribution, plus an added constant magnetic field (an apparently external field) antiparallel to $\mathbf{m}$. Satellite and surface observations lead to the conclusion that the "external" field is less than 4×10^{-3} times the dipole field at the magnetic equator. Estimate a lower limit on μ^{-1} in earth radii and an upper limit on the photon mass in grams from this datum.

Solution. The earth's magnetic field is parallel to its surface at the magnetic equator [3]. Figure 2 shows the field of a magnetic dipole $\mathbf{m}$ that points in the z direction [4, pp. 245–246]. Thus, at the magnetic equator, $\mathbf{m}$ is also parallel to the field, and so the spherical unit vector $\hat{\mathbf{r}}$ is perpendicular to $\mathbf{m}$. So Eq. (14) becomes

$$\mathbf{B}(\mathbf{x}) = -\mathbf{m}\left(1+\mu R+\frac{\mu^2R^2}{3}\right)\frac{e^{-\mu R}}{R^3} - \frac{2}{3}\mu^2\mathbf{m}\frac{e^{-\mu R}}{R} \equiv \mathbf{B}_{\text{dip}} + \mathbf{B}_{\text{ext}},$$

where we have defined $\mathbf{B}_{\text{dip}}$ as the dipole field and $\mathbf{B}_{\text{ext}}$ as the "external' field. The identification is made by comparing with Jackson (5.56), which gives the magnetic field due to a dipole (in SI units):

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{m})}{|\mathbf{x}|^3}.$$

As stated in the problem, $|\mathbf{B}_{\text{ext}}| < (4\times 10^{-3})|\mathbf{B}_{\text{dip}}|$. This gives us

$$\frac{2}{3}\mu^2\frac{e^{-\mu R}}{R} < (4\times 10^{-3})\left(1+\mu R+\frac{\mu^2R^2}{3}\right)\frac{e^{-\mu R}}{R^3} \quad\Longrightarrow\quad \frac{2}{3}\mu^2R^2 < (4\times 10^{-3})\left(1+\mu R+\frac{\mu^2R^2}{3}\right).$$

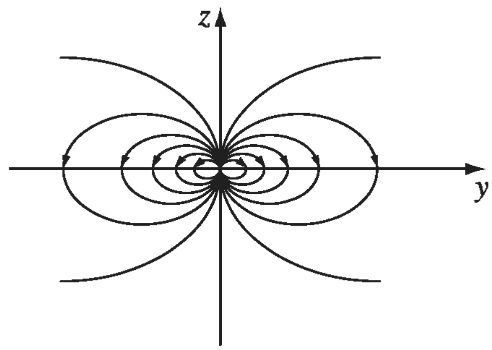

Figure 2: [Griffiths 5.55(a)] The field of a "pure" dipole.

Solving the second expression with Mathematica gives the positive solution $\mu = 0.0806/R$. Thus, the lower limit on μ^{-1} is $\mu^{-1} > 12R$.

The parameter μ is defined by $\mu = m_\gamma c/\hbar$, where m_γ is the photon mass [1, p. 600]. The earth's radius is $R = 6.37 \times 10^6\,\mathrm{m}$ [5], so $\mu < 1.267 \times 10^{-8}\,\mathrm{m}^{-1}$. Since $c = 2.998 \times 10^8\,\mathrm{m\,s^{-1}}$ and $\hbar = h/(2\pi) = (6.63 \times 10^{-34}\,\mathrm{J\,s})/(2\pi) = 1.055 \times 10^{-34}\,\mathrm{J\,s}$ [5], we have

$$m_\gamma = \frac{\hbar\mu}{c} < \frac{(1.055 \times 10^{-34}\,\mathrm{J\,s})(1.267 \times 10^{-8}\,\mathrm{m^{-1}})}{2.998 \times 10^8\,\mathrm{m\,s^{-1}}} \approx 4 \times 10^{-51}\,\mathrm{kg} = 4 \times 10^{-48}\,\mathrm{g}.$$

So the upper bound on the photon mass is $m_\gamma < 4 \times 10^{-48}\,\mathrm{g}$.

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