Problem 1. A spherical shell of radius R has a total charge Q uniformly spread over the shell. The shell is now put into uniform rotation about the z axis with angular velocity ω. Find the vector potential $\mathbf{A}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ everywhere, i.e., both inside and outside of the shell.

Solution. Let $\rho(\mathbf{x})$ be the charge density everywhere in space, so

$$\rho(\mathbf{x}) = \frac{1}{4\pi}\frac{Q}{R^2}\delta(r-R).$$

The linear velocity of the moving charge everywhere is

$$\mathbf{v}(\mathbf{x}) = \omega r\,\delta(r-R)\,\hat{\boldsymbol{\varphi}}.$$

Then the current density $\mathbf{J}$ is simply the product of charge density and the linear velocity of the charge:

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\,\mathbf{v}(\mathbf{x}) = \frac{Q\omega}{4\pi}\frac{r}{R^2}\delta(r-R)\,\hat{\boldsymbol{\varphi}}.$$

From Eq. (4.21) in the lecture notes, $\mathbf{A}(\mathbf{x})$ everywhere is given by

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x})'}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,.$$

The integral we need to evaluate is then

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi c}\frac{Q\omega}{R^2}\int \frac{r'\,\delta(r'-R)}{|\mathbf{x}-\mathbf{x}'|}\,d^3\mathbf{x}'\,.$$

The problem is azimuthally symmetric, so we will rotate our coordinate system such that $\mathbf{x}$ points along the z axis. In the new coordinate system,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{\mathbf{x}^2 - 2\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'^2}} = \frac{1}{\sqrt{r^2 - 2rr'\cos\theta' + r'^2}}.$$

Let $\boldsymbol{\omega}$ be the angular velocity vector (that lay along the z axis of the original coordinate system), which we choose to lie in the xz plane. Let α be the angle between $\boldsymbol{\omega}$ and the z axis. Then the linear velocity of the moving charge is

$$\begin{aligned}\mathbf{v}(\mathbf{x}') = \boldsymbol{\omega}\times\mathbf{x}' &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\alpha & 0 & \omega\cos\alpha \\ r'\sin\theta'\cos\varphi' & r'\sin\theta'\sin\varphi' & r'\cos\theta' \end{vmatrix} \\ &= -\omega r'(\cos\alpha\sin\theta'\sin\varphi')\,\hat{\mathbf{x}} + \omega r'(\cos\alpha\sin\theta'\cos\varphi' - \sin\alpha\cos\theta')\,\hat{\mathbf{y}} + \omega r'(\sin\alpha\sin\theta'\sin\varphi')\,\hat{\mathbf{z}},\end{aligned}$$

so in the new coordinate system,

$$\mathbf{J}(\mathbf{x}') = \frac{Q}{4\pi}\frac{\boldsymbol{\omega}\times\mathbf{x}'}{R^2}\,\delta(r'-R) = \frac{Q\omega}{4\pi}\frac{r'}{R^2}(\hat{\boldsymbol{\omega}}\times\hat{\mathbf{x}}')\,\delta(r'-R),$$

where

$$\hat{\boldsymbol{\omega}}\times\hat{\mathbf{x}}' = -(\cos\alpha\sin\theta'\sin\varphi')\,\hat{\mathbf{x}} + (\cos\alpha\sin\theta'\cos\varphi' - \sin\alpha\cos\theta')\,\hat{\mathbf{y}} + (\sin\alpha\sin\theta'\sin\varphi')\,\hat{\mathbf{z}}.$$

The integral we need to evaluate becomes

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi c}\frac{Q\omega}{R^2}\int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty}\frac{r'^3(\hat{\boldsymbol{\omega}}\times\hat{\mathbf{x}}')\,\delta(r'-R)}{\sqrt{r^2-2rr'\cos\theta'+r'^2}}\,dr'\,d(\cos\theta')\,d\varphi\,.$$

Evaluating the radial integral, we have

$$\mathbf{A}(\mathbf{x}) = \frac{Q\omega R}{4\pi c}\int_0^{2\pi}\int_{-1}^{1}\frac{\hat{\boldsymbol{\omega}}\times\hat{\mathbf{x}}'}{\sqrt{r^2-2Rr\cos\theta'+R^2}}\,d(\cos\theta')\,d\varphi\,.$$

For the angular integrals, the $\hat{\mathbf{x}}$ term is

$$-\cos\alpha\,\hat{\mathbf{x}}\int_{-1}^{1}\frac{\sin\theta'}{\sqrt{r^2-2Rr\cos\theta'+R^2}}\,d(\cos\theta')\int_0^{2\pi}\sin\varphi'\,d\varphi \propto \Big[-\cos\varphi'\Big]_0^{2\pi} = 0.$$

Similarly, the $\hat{\mathbf{z}}$ term is

$$\sin\alpha\,\hat{\mathbf{z}}\int_{-1}^{1}\frac{\sin\theta'}{\sqrt{r^2-2Rr\cos\theta'+R^2}}\,d(\cos\theta')\int_0^{2\pi}\sin\varphi'\,d\varphi \propto \Big[-\cos\varphi'\Big]_0^{2\pi} = 0.$$

There are two $\hat{\mathbf{y}}$ terms. For the first,

$$\cos\alpha\,\hat{\mathbf{y}}\int_{-1}^{1}\frac{\sin\theta'}{\sqrt{r^2-2Rr\cos\theta'+R^2}}\,d(\cos\theta')\int_0^{2\pi}\cos\varphi'\,d\varphi \propto \Big[\sin\varphi'\Big]_0^{2\pi} = 0.$$

For the second,

$$\begin{aligned}
-\sin\alpha\,\hat{\mathbf{y}}\int_{-1}^{1}\frac{\cos\theta'}{\sqrt{r^2-2Rr\cos\theta'+R^2}}\,d(\cos\theta')\int_0^{2\pi}d\varphi &= -2\pi\sin\alpha\,\hat{\mathbf{y}}\int_{-1}^{1}\frac{\cos\theta'}{\sqrt{r^2-2Rr\cos\theta'+R^2}}\,d(\cos\theta') \\
&= -2\pi\sin\alpha\,\hat{\mathbf{y}}\left(\left[-\frac{\cos\theta'\sqrt{r^2-2Rr\cos\theta'+R^2}}{Rr}\right]_{-1}^{1} + \frac{1}{Rr}\int_{-1}^{1}\sqrt{r^2-2Rr\cos\theta'+R^2}\,d(\cos\theta')\right) \\
&= -2\pi\sin\alpha\,\hat{\mathbf{y}}\left(\left[-\frac{\cos\theta'\sqrt{r^2-2Rr\cos\theta'+R^2}}{Rr}\right]_{-1}^{1} + \frac{1}{Rr}\left[-\frac{(r^2-2Rr\cos\theta'+R^2)^{3/2}}{3Rr}\right]_{-1}^{1}\right) \\
&= -2\pi\sin\alpha\,\hat{\mathbf{y}}\left(-\frac{\sqrt{r^2+2Rr+R^2}}{Rr}+\frac{\sqrt{r^2-2Rr+R^2}}{Rr}-\frac{(r^2-2Rr+R^2)^{3/2}}{3R^2r^2}+\frac{(r^2+2Rr+R^2)^{3/2}}{3R^2r^2}\right) \\
&= 2\pi\sin\alpha\frac{3Rr\sqrt{(r+R)^2}-3Rr\sqrt{(r-R)^2}+[(r-R)^2]^{3/2}-[(r+R)^2]^{3/2}}{3R^2r^2}\hat{\mathbf{y}} \\
&= 2\pi\sin\alpha\frac{3Rr|r+R|-3Rr|r-R|+(r-R)^2|r-R|-(r+R)^2|r+R|}{3R^2r^2}\hat{\mathbf{y}} \\
&= 2\pi\sin\alpha\frac{(r^2+Rr+R^2)|r-R|-(r^2-Rr+R^2)(r+R)}{3R^2r^2}\hat{\mathbf{y}} \\
&= \frac{2\pi\sin\alpha\,\hat{\mathbf{y}}}{3R^2r^2}\begin{cases}(r^2+Rr+R^2)(R-r)-(r^2-Rr+R^2)(r+R) & r<R,\\ (r^2+Rr+R^2)(r-R)-(r^2-Rr+R^2)(r+R) & r>R\end{cases} \\
&= -\frac{4}{3}\pi\sin\alpha\,\hat{\mathbf{y}}\begin{cases}\dfrac{r}{R^2} & r<R,\\[2mm] \dfrac{R}{r^2} & r>R.\end{cases}
\end{aligned}$$

Finally, in the new coordinate system we have

$$\mathbf{A}(\mathbf{x}) = -\frac{Q\omega}{3c}\sin\alpha\,\hat{\mathbf{y}}\begin{cases}\dfrac{r}{R} & r<R,\\[2mm] \dfrac{R^2}{r^2} & r>R.\end{cases}$$

Transforming back to the old coordinate system, $\sin\alpha \to -\sin\theta$. Since the original system is azimuthally symmetric, $\varphi = 0$ so $\hat{\mathbf{y}} = \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$. Thus we have

$$\mathbf{A}(\mathbf{x}) = \frac{Q\omega}{3c}\sin\theta\,\hat{\boldsymbol{\varphi}}\begin{cases}\dfrac{r}{R} & r < R,\\[2ex] \dfrac{R^2}{r^2} & r > R.\end{cases}$$

The magnetic field is given by Eq. (1.7),

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{1}$$

In spherical coordinates,

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}},$$

so

$$\mathbf{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\hat{\boldsymbol{\theta}}.$$

For $r < R$,

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \frac{Q\omega}{3c}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{r}{R}\sin^2\theta\right)\hat{\mathbf{r}} - \frac{\partial}{\partial r}\left(\frac{r^2}{R}\sin\theta\right)\hat{\boldsymbol{\theta}}\right] = \frac{Q\omega}{3c}\frac{1}{r}\left(\frac{r}{R}\frac{2\cos\theta\sin\theta}{\sin\theta}\,\hat{\mathbf{r}} - \frac{2r}{R}\sin\theta\,\hat{\boldsymbol{\theta}}\right)\\ &= \frac{2}{3}\frac{Q\omega}{cR}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\frac{Q\omega}{cR}\,\hat{\mathbf{z}}.\end{aligned}$$

For $r > R$,

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= \frac{Q\omega}{3c}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{R^2}{r^2}\sin^2\theta\right)\hat{\mathbf{r}} - \frac{\partial}{\partial r}\left(\frac{R^2}{r}\sin\theta\right)\hat{\boldsymbol{\theta}}\right] = \frac{Q\omega}{3c}\frac{1}{r}\left(\frac{R^2}{r^2}\frac{2\cos\theta\sin\theta}{\sin\theta}\,\hat{\mathbf{r}} + 2\frac{R^2}{r^2}\sin\theta\,\hat{\boldsymbol{\theta}}\right)\\ &= \frac{2}{3}\frac{Q\omega}{c}\frac{R^2}{r^3}(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).\end{aligned}$$

In summary,

$$\mathbf{B}(\mathbf{x}) = \frac{2}{3}\frac{Q\omega}{c}\begin{cases}\dfrac{\hat{\mathbf{z}}}{R} & r < R,\\[2ex] \dfrac{R^2}{r^3}(\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) & r > R.\end{cases}$$

Problem 2. If an electric and magnetic field are both present, the momentum density carried by the electromagnetic field is given by Poynting's formula

$$\mathcal{P} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B}).$$

Consider a bounded distribution of time-independent charges and currents, i.e., $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are time independent and vanish when $|\mathbf{x}| > R$ for some R.

2.a Show that the total momentum can be written as

$$\mathbf{P} \equiv \int \mathcal{P}(\mathbf{x})\, d^3x = \int \phi(\mathbf{x})\, \mathbf{J}(\mathbf{x})\, d^3x\,.$$

Solution. Applying (1),

$$\mathbf{E} \times \mathbf{B} = \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{A}).$$

Vector identity (4) in Griffiths is

$$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{b}) + \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{a}) + (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a},$$

which allows us to write

$$\mathbf{E} \times \mathbf{B} = \boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{E}) - \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{E}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{E} - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{A} = \boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{E}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{E} - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{A},$$

since $\boldsymbol{\nabla} \times E = 0$ in electrostatics by Eq. (1.4) in the lecture notes. Now using component notation with implied sums,

$$(\mathbf{A} \cdot \boldsymbol{\nabla})E_i = A_j \frac{\partial E_i}{\partial x_j} = \frac{\partial}{\partial x_j}(A_j E_i) - E_i \frac{\partial A_j}{\partial x_j} = \frac{\partial}{\partial x_j}(A_j E_i).$$

Here we have used the product rule in addition to Eq. (4.20), which states that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in the Coulomb gauge, which we may choose without loss of generality. Similarly,

$$(\mathbf{E} \cdot \boldsymbol{\nabla})A_i = E_j \frac{\partial A_i}{\partial x_j} = \frac{\partial}{\partial x_j}(E_j A_i) - A_i \frac{\partial E_j}{\partial x_j} = \frac{\partial}{\partial x_j}(E_j A_i) + A_i \nabla^2 \phi,$$

where we have used Eq. (2.2), $\mathbf{E} = -\boldsymbol{\nabla}\phi$, which holds in the electrostatic case. Putting this all together, we have

$$(\mathbf{E} \times \mathbf{B})_i = \frac{\partial}{\partial x_i}(A_j E_j) - \frac{\partial}{\partial x_j}(A_j E_i) - \frac{\partial}{\partial x_j}(E_j A_i) - A_i \nabla^2 \phi,$$

and so

$$\int (\mathbf{E} \times \mathbf{B})_i\, d^3x = \int \left(\frac{\partial}{\partial x_i}(A_j E_j) - \frac{\partial}{\partial x_j}(A_j E_i) - \frac{\partial}{\partial x_j}(E_j A_i) - A_i \nabla^2 \phi \right) d^3x\,.$$

Let $L \geq R$. Note that

$$\int f(\mathbf{x})\, d^3x = \lim_{L \to \infty} \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} f(\mathbf{x})\, dx\, dy\, dz\,.$$

Then for the first term, integrating with respect to x_i by parts gives us

$$\lim_{L \to \infty} \int_{-L}^{L} \frac{\partial}{\partial x_i}(A_j E_j)\, dx_i = \Big[A_j E_j \Big]_{-L}^{L} = 0,$$

since both $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ vanish for $|\mathbf{x}| > R$. This means $\mathbf{E} \to 0$ and $\mathbf{A} \to 0$ as $|\mathbf{x}| \to \infty$. Applying similar logic to the second and third terms,

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{\partial}{\partial x_j}(A_j E_i)\, dx_j = \Big[A_j E_i\Big]_{-L}^{L} = 0, \qquad \lim_{L\to\infty} \int_{-L}^{L} \frac{\partial}{\partial x_j}(E_j A_i)\, dx_j = \Big[E_j A_i\Big]_{-L}^{L} = 0,$$

where there are no implied sums over the derivatives. Now we have

$$\int (\mathbf{E} \times \mathbf{B})_i \, d^3x = -\int A_i \nabla^2 \phi \, d^3x \,.$$

Green's theorem is given by Eq. (2.96),

$$\int_S \hat{\mathbf{n}} \cdot (\phi_1 \boldsymbol{\nabla} \phi_2 - \phi_2 \boldsymbol{\nabla} \phi_1)\, dS = -4\pi \int_{\mathcal{V}} (\phi_1 \rho_2 - \phi_2 \rho_1)\, d^3x = \int_{\mathcal{V}} (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1)\, d^3x \,,$$

where the final equality comes from the proof in Eq. (2.97). Let $\mathcal{V}$ be a cube of side length $2L$ centered at the origin. Applying Green's theorem gives us

$$\int_{\mathcal{V}} (\mathbf{E} \times \mathbf{B})_i \, d^3x = \int_S \hat{\mathbf{n}} \cdot (\phi \boldsymbol{\nabla} A_i - A_i \boldsymbol{\nabla} \phi)\, dS - \int_{\mathcal{V}} \phi \nabla^2 A_i \, d^3x \,.$$

Note that

$$\lim_{L\to\infty} \int_S \hat{\mathbf{n}} \cdot (\phi \boldsymbol{\nabla} A_i - A_i \boldsymbol{\nabla} \phi)\, dS \propto \lim_{L\to\infty} \int_S \frac{1}{|\mathbf{x}|^3}\, dS = 0$$

since $\phi, A_i \propto 1/|\mathbf{x}|$ and $\boldsymbol{\nabla}\phi, \boldsymbol{\nabla} A_i \propto 1/|\mathbf{x}|^2$. Now we have

$$\int (\mathbf{E} \times \mathbf{B})\, d^3x = -\int \phi \nabla^2 \mathbf{A}\, d^3x \,.$$

Vector identity (11) in Griffiths states that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2 \mathbf{a},$$

which gives us

$$\int (\mathbf{E} \times \mathbf{B})\, d^3x = \int \phi [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})]\, d^3x = \frac{4\pi}{c} \int \phi \mathbf{J}\, d^3x \,,$$

where we have once again used the Coulomb gauge condition, and that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = 4\pi \mathbf{J}/c$ from Eq. (4.4). Thus, we have proven

$$\mathbf{P} = \int \mathcal{P}(\mathbf{x})\, d^3x = \frac{c}{4\pi} \int (\mathbf{E} \times \mathbf{B})\, d^3x = \int \phi(\mathbf{x})\, \mathbf{J}(\mathbf{x})\, d^3x \,,$$

as desired. $\square$

2.b Give an example of a stationary, bounded charge and current distribution for which $\mathbf{P} \neq 0$.

Solution. Consider a toroid in the xy plane centered on the origin, with N total turns and current I. Consider also a point charge of charge Q at the origin. In cylindrical coordinates (s, φ, z), the magnetic field due to the solenoid is given by Eq. (5.60) in Griffiths:

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \dfrac{2NI}{s}\hat{\boldsymbol{\varphi}} & \text{inside,} \\ 0 & \text{outside.} \end{cases}$$

The electric field due to the point charge is simply

$$\mathbf{E} = \frac{Q}{s^2 + z^2}\hat{\mathbf{r}}.$$

Let T denote the volume of the toroid, and note that $\mathbf{E} \times \mathbf{B} \neq 0$ only within this finite volume. Then

$$\mathbf{P} = \frac{c}{4\pi} \int (\mathbf{E} \times \mathbf{B})\, d^3x = \frac{c}{4\pi} 2NIQ \int_T \frac{\hat{\mathbf{s}} \times \hat{\boldsymbol{\varphi}}}{s(s^2 + z^2)}\, d^3x = \frac{c}{2\pi} NIQ\, \hat{\mathbf{z}} \int_T \frac{d^3x}{s(s^2 + z^2)},$$

which is nonzero.

Problem 3. The angular momentum density of the electromagnetic field is given by

$$\mathbf{l} = \mathbf{x} \times \mathcal{P} = \frac{1}{4\pi c}\mathbf{x} \times (\mathbf{E} \times \mathbf{B}).$$

Consider a source free ($\rho = 0$, $\mathbf{J} = 0$) solution to Maxwell's equations in electrodynamics with $\mathbf{E}$ and $\mathbf{B}$ vanishing rapidly as $|\mathbf{x}| \to \infty$, so the total angular momentum

$$\mathbf{L} = \int \mathbf{l}\, d^3x$$

is well defined. Show that $\mathbf{L}$ is conserved, i.e., independent of time.

In addition to the course lecture notes, I consulted Griffiths's *Introduction to Electrodynamics*, Jackson's *Classical Electrodynamics*, and Kirk McDonald's electromagnetism notes while writing up these solutions.