1

Find the Euler-Lagrange equation associated with the functional

$$J[u(x,y,z)] = \int_R \sqrt{1 + u_x^2 + u_y^2 + u_z^2}\, dx\, dy\, dz\,,$$

where R is a region in three-dimensional space.

Solution. We will assume $u(x,y,z)$ has explicit values on the boundary of R, $dx\,dy\,dz$. By the definition of the action,

$$J[u] = \int_R \mathcal{L}\, dx\, dy\, dz \implies \mathcal{L} = \sqrt{1 + u_x^2 + u_y^2 + u_z^2}.$$

In general, the Euler-Lagrange equation is

$$0 = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z}. \tag{1}$$

Here,

$$\frac{\partial \mathcal{L}}{\partial u} = 0, \qquad \frac{\partial \mathcal{L}}{\partial u_x} = \frac{\partial \mathcal{L}}{\partial u_x^2}\frac{\partial u_x^2}{\partial u_x} = \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2 + u_z^2}} = \frac{u_x}{\mathcal{L}} \qquad \frac{\partial \mathcal{L}}{\partial u_y} = \frac{u_y}{\mathcal{L}}, \qquad \frac{\partial \mathcal{L}}{\partial u_z} = \frac{u_z}{\mathcal{L}}.$$

For the $\partial/\partial x$ term of (1),

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} = \frac{\partial}{\partial x}\frac{u_x}{\mathcal{L}} = \frac{\partial u_x}{\partial x}\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} + \frac{\partial u_y}{\partial x}\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} + \frac{\partial u_z}{\partial x}\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}}$$

where

$$\frac{\partial}{\partial u_x}\frac{u_x}{\mathcal{L}} = \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_x}\mathcal{L} - u_x\frac{\partial \mathcal{L}}{\partial u_x}\right) = \frac{1}{\mathcal{L}^2}\left(\mathcal{L} - u_x\frac{u_x}{\mathcal{L}}\right) = \frac{\mathcal{L}^2 - u_x^2}{\mathcal{L}^3}, \tag{2}$$

$$\frac{\partial}{\partial u_y}\frac{u_x}{\mathcal{L}} = \frac{1}{\mathcal{L}^2}\left(\frac{\partial u_x}{\partial u_y}\mathcal{L} - u_x\frac{\partial \mathcal{L}}{\partial u_y}\right) = -\frac{u_x u_y}{\mathcal{L}^3}, \tag{3}$$

$$\frac{\partial}{\partial u_z}\frac{u_x}{\mathcal{L}} = -\frac{u_x u_z}{\mathcal{L}^3}, \tag{4}$$

Generalizing (2)–(4) to the $\partial/\partial y$ and $\partial/\partial z$ terms,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} &= u_{xx}\frac{\mathcal{L}^2 - u_x^2}{\mathcal{L}^3} - u_{yx}\frac{u_x u_y}{\mathcal{L}^3} - u_{zx}\frac{u_x u_z}{\mathcal{L}^3},\\
\frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} &= u_{yy}\frac{\mathcal{L}^2 - u_y^2}{\mathcal{L}^3} - u_{xy}\frac{u_x u_y}{\mathcal{L}^3} - u_{zy}\frac{u_y u_z}{\mathcal{L}^3},\\
\frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z} &= u_{zz}\frac{\mathcal{L}^2 - u_z^2}{\mathcal{L}^3} - u_{xz}\frac{u_x u_z}{\mathcal{L}^3} - u_{yz}\frac{u_y u_z}{\mathcal{L}^3}.
\end{aligned}$$

Then, assuming $u_{xy} = u_{yx}$, $u_{yz} = u_{zy}$, and $u_{xz} = u_{zx}$, (1) becomes

$$\begin{aligned}
0 &= u_{xx}(\mathcal{L}^4 - u_x^2) + u_{yy}(\mathcal{L}^4 - u_y^2) + u_{zz}(\mathcal{L}^4 - u_z^2) - 2u_{xy}u_x u_y - 2u_{yz}u_y u_z - 2u_{xz}u_x u_z\\
&= (u_{xx} + u_{yy} + u_{zz})(1 + u_x^2 + u_y^2 + u_z^2) - u_{xx}u_x^2 - u_{yy}u_y^2 - u_{zz}u_z^2 - 2u_{xy}u_x u_y - 2u_{yz}u_y u_z - 2u_{xz}u_x u_z\\
&= u_{xx}(1 + u_y^2 + u_z^2) + u_{yy}(1 + u_x^2 + u_z^2) + u_{zz}(1 + u_x^2 + u_y^2) - 2u_{xy}u_x u_y - 2u_{yz}u_y u_z - 2u_{xz}u_x u_z.
\end{aligned}$$

2 Plate vibrations (preliminaries)

Start from Green's theorem

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial R} (P\, dx + Q\, dy), \tag{5}$$

where R is the region in the xy plane spanned by the plate, and $dx\, dy\, dz$ its boundary.

2.a Show that

$$\int_R \phi \frac{\partial^2 \psi}{\partial x^2}\, dx\, dy = \int_R \psi \frac{\partial^2 \phi}{\partial x^2}\, dx\, dy + \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right) dy\, . \tag{6}$$

Solution. In (5), let

$$Q = \phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x}, \qquad P = 0.$$

Then

$$\frac{\partial Q}{\partial x} = \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \phi \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial x} - \psi \frac{\partial^2 \phi}{\partial x^2} = \phi \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \phi}{\partial x^2}, \qquad \frac{\partial P}{\partial y} = 0.$$

Making these substitutions into (5) gives

$$\begin{aligned} \int_R \left(\phi \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \phi}{\partial x^2} \right) dx\, dy &= \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right) dy \\ \iff \int_R \phi \frac{\partial^2 \psi}{\partial x^2}\, dx\, dy &= \int_R \psi \frac{\partial^2 \phi}{\partial x^2}\, dx\, dy + \int_{\partial R} \left(\phi \frac{\partial \psi}{\partial x} - \psi \frac{\partial \phi}{\partial x} \right) dy \end{aligned}$$

as desired. $\square$

2.b Work out analogous expressions for

$$\int_R \phi \frac{\partial^2 \psi}{\partial y^2}\, dx\, dy\, , \tag{7}$$

$$\int_R \phi \frac{\partial^2 \psi}{\partial x \partial y}\, dx\, dy\, . \tag{8}$$

Solution. For (7), let

$$Q = 0, \qquad P = \psi \frac{\partial \phi}{\partial y} - \phi \frac{\partial \psi}{\partial y},$$

in (5). Then, similarly to the proof for (6),

$$\frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial P}{\partial y} = \psi \frac{\partial^2 \phi}{\partial y^2} - \phi \frac{\partial^2 \psi}{\partial y^2}.$$

Substituting into (5),

$$\int_R \left(\phi\frac{\partial^2\psi}{\partial y^2} - \psi\frac{\partial^2\phi}{\partial y^2}\right) dx\,dy = \int_{\partial R}\left(\psi\frac{\partial\phi}{\partial y} - \phi\frac{\partial\psi}{\partial y}\right) dx$$
$$\iff \int_R \phi\frac{\partial^2\psi}{\partial y^2}\,dx\,dy = \int_R \psi\frac{\partial^2\phi}{\partial y^2}\,dx\,dy + \int_{\partial R}\left(\psi\frac{\partial\phi}{\partial y} - \phi\frac{\partial\psi}{\partial y}\right) dy\,. \tag{9}$$

For (8), let

$$2Q = \phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}, \qquad 2P = \psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}.$$

Then

$$2\frac{\partial Q}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} + \phi\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y} = \phi\frac{\partial^2\psi}{\partial x\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y},$$
$$2\frac{\partial P}{\partial y} = \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} + \psi\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} - \phi\frac{\partial^2\psi}{\partial x\partial y} = \psi\frac{\partial^2\phi}{\partial x\partial y} - \phi\frac{\partial^2\psi}{\partial x\partial y}.$$

Substituting into (5), we have

$$\frac{1}{2}\int_R\left(\phi\frac{\partial^2\psi}{\partial x\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y} - \psi\frac{\partial^2\phi}{\partial x\partial y} + \phi\frac{\partial^2\psi}{\partial x\partial y}\right) dx\,dy = \frac{1}{2}\int_{\partial R}\left(\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}\right) dx + \frac{1}{2}\int_{\partial R}\left(\phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}\right) dy$$
$$\iff \int_R \phi\frac{\partial^2\psi}{\partial x\partial y}\,dx\,dy = \int_R \psi\frac{\partial^2\phi}{\partial x\partial y}\,dx\,dy + \frac{1}{2}\int_{\partial R}\left(\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}\right) dx + \frac{1}{2}\int_{\partial R}\left(\phi\frac{\partial\psi}{\partial y} - \psi\frac{\partial\phi}{\partial y}\right) dy\,. \tag{10}$$

3 Plate vibrations

Start with the action for a vibrating plate whose potential energy is dominated by bending,

$$S[u(x,y,t)] = \frac{1}{2}\int_{t_0}^{t_1}\int_R \left\{\rho u_t^2 - \kappa_1\left[(u_{xx}^2 + u_{yy}^2) - 2(1-\mu)(u_{xx}u_{yy} - u_{xy}^2)\right]\right\} dx\,dy\,dt\,, \tag{11}$$

where ρ is the mass density per unit area, κ_1 has the dimension of energy and is sometimes called flexural rigidity, and μ is a dimensionless material constant called Poisson's ratio. For isotropic material, $\mu = 1/4$. Notice that there is *no* external bending moment applied to the plate boundary. There is also *no* external forcing.

3.a Using the results of problem 2, show that the variation generated by going from a solution u^0 to $u^0 + \epsilon\psi$ has the form

$$\delta S = \epsilon\int_{t_0}^{t_1}\int_R \left(-\rho u_{tt} - \kappa_1\nabla^4 u\right)\psi\,dx\,dy\,dt + \epsilon\int_{t_0}^{t_1}\int_{\partial R}\left(P(u)\psi + M(u)\frac{\partial\psi}{\partial n}\right) d\ell\,dt\,. \tag{12}$$

Specify $P(u)$ and $M(u)$.

Solution. Making the substitution $u \mapsto u + \epsilon\psi$ into (11),

$$S[u+\epsilon\psi] = \int_{t_0}^{t_1}\int_R \left\{\frac{\rho}{2}(u_t+\epsilon\psi_t)^2 - \frac{\kappa_1}{2}\left[(u_{xx}+\epsilon\psi_{xx})^2 + (u_{yy}+\epsilon\psi_{yy})^2\right]\right\} dx\,dy\,dt$$
$$+ \kappa_1(1-\mu)\int_{t_0}^{t_1}\int_R \left[(u_{xx}+\epsilon\psi_{xx})(u_{yy}+\epsilon\psi_{yy}) - (u_{xy}+\epsilon\psi_{xy})^2\right] dx\,dy\,dt$$
$$= \int_{t_0}^{t_1}\int_R \left[\frac{\rho}{2}(u_t^2 + 2\epsilon u_t\psi_t + \epsilon^2\psi_t^2) - \frac{\kappa_1}{2}(u_{xx}^2 + 2\epsilon u_{xx}\psi_{xx} + \epsilon^2\psi_{xx}^2 + u_{yy}^2 + 2\epsilon u_{yy}\psi_{yy} + \epsilon^2\psi_{yy}^2)\right] dx\,dy\,dt$$
$$+ \kappa_1(1-\mu)\int_{t_0}^{t_1}\int_R (u_{xx}u_{yy} + \epsilon u_{xx}\psi_{yy} + \epsilon u_{yy}\psi_{xx} + \epsilon^2\psi_{xx}\psi_{yy} - u_{xy}^2 - 2\epsilon u_{xy}\psi_{xy} - \epsilon^2\psi_{xy}^2)\, dx\,dy\,dt\,.$$

Then

$$\Delta S = S[u+\epsilon\psi] - S[u]$$
$$= \int_{t_0}^{t_1}\int_R \left[\frac{\rho}{2}(2\epsilon u_t\psi_t + \epsilon^2\psi_t^2) - \frac{\kappa_1}{2}(2\epsilon u_{xx}\psi_{xx} + \epsilon^2\psi_{xx}^2 + 2\epsilon u_{yy}\psi_{yy} + \epsilon^2\psi_{yy}^2)\right] dx\,dy\,dt$$
$$+ \kappa_1(1-\mu)\int_{t_0}^{t_1}\int_R (\epsilon u_{xx}\psi_{yy} + \epsilon u_{yy}\psi_{xx} + \epsilon^2\psi_{xx}\psi_{yy} - 2\epsilon u_{xy}\psi_{xy} - \epsilon^2\psi_{xy}^2)\, dx\,dy\,dt\,,$$

and so, dropping terms of $\mathcal{O}(\epsilon^2)$,

$$\delta S = \epsilon\int_{t_0}^{t_1}\int_R \{\rho u_t\psi_t - \kappa_1\left[u_{xx}\psi_{xx} + u_{yy}\psi_{yy} - (1-\mu)(u_{xx}\psi_{yy} + u_{yy}\psi_{xx} - 2u_{xy}\psi_{xy})\right]\}\, dx\,dy\,dt\,. \tag{13}$$

For the first term in the integrand of (13), using the product rule of differentiation yields

$$u_t\psi_t = \frac{\partial}{\partial t}(u_t\psi) - u_{tt}\psi.$$

For the second two terms, we may apply what was proven in problem 2. Letting $\phi \mapsto u_{xx}$ and $\psi \mapsto \psi$ in (6) and (9), we have

$$\int_{t_0}^{t_1}\int_R u_{xx}\psi_{xx}\, dx\,dy\,dt = \int_{t_0}^{t_1}\int_R \psi u_{xxxx}\, dx\,dy\,dt + \int_{t_0}^{t_1}\int_{\partial R} (u_{xx}\psi_x - \psi u_{xxx})\, dy\,dt\,,$$
$$\int_{t_0}^{t_1}\int_R u_{xx}\psi_{yy}\, dx\,dy\,dt = \int_{t_0}^{t_1}\int_R \psi u_{xxyy}\, dx\,dy\,dt - \int_{t_0}^{t_1}\int_{\partial R} (u_{xx}\psi_y - \psi u_{xxy})\, dx\,dt\,.$$

Now with $\phi \mapsto u_{yy}$,

$$\int_{t_0}^{t_1}\int_R u_{yy}\psi_{xx}\, dx\,dy\,dt = \int_{t_0}^{t_1}\int_R \psi u_{xxyy}\, dx\,dy\,dt + \int_{t_0}^{t_1}\int_{\partial R} (u_{yy}\psi_x - \psi u_{xyy})\, dy\,dt\,,$$
$$\int_{t_0}^{t_1}\int_R u_{yy}\psi_{yy}\, dx\,dy\,dt = \int_{t_0}^{t_1}\int_R \psi u_{yyyy}\, dx\,dy\,dt - \int_{t_0}^{t_1}\int_{\partial R} (u_{yy}\psi_y - \psi u_{yyy})\, dx\,dt\,.$$

Finally, with $\phi \mapsto u_{xy}$ and $\psi \mapsto \psi$ in (10), we have

$$\int_{t_0}^{t_1}\int_R u_{xy}\psi_{xy}\, dx\,dy\,dt = \int_{t_0}^{t_1}\int_R \psi u_{xxyy}\, dx\,dy\,dt - \frac{1}{2}\int_{t_0}^{t_1}\int_{\partial R} (u_{xy}\psi_x - \psi u_{xxy})\, dx\,dt$$
$$+ \frac{1}{2}\int_{t_0}^{t_1}\int_{\partial R} (u_{xy}\psi_y - \psi u_{xyy})\, dy\,dt\,.$$

Making these substitutions into (13),

$$\begin{aligned}\frac{\delta S}{\epsilon} &= \int_{t_0}^{t_1}\int_R \psi\left\{-\rho u_{tt} - \kappa_1\left[u_{xxxx} + u_{yyyy} - (1-\mu)(u_{xxyy} + u_{xxyy} - 2u_{xxyy})\right]\right\} dx\,dy\,dt + \rho\int_{t_0}^{t_1}\int_R \frac{\partial}{\partial t}(u_t\psi)\,dx\,dy\,dt \\ &\qquad - \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[u_{xx}\psi_x - \psi u_{xxx} - (1-\mu)(u_{yy}\psi_x - \psi u_{xyy} - u_{xy}\psi_y + \psi u_{xyy})\right] dy\,dt \\ &\qquad\qquad + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[u_{yy}\psi_y - \psi u_{yy} - (1-\mu)(u_{xx}\psi_y - \psi u_{xxy} - u_{xy}\psi_x + \psi u_{xxy})\right] dx\,dt \\ &= \int_{t_0}^{t_1}\int_R \psi\left\{-\rho u_{tt} - \kappa_1\left[u_{xxxx} + u_{yyyy} - (1-\mu)(u_{xxyy} + u_{xxyy} - 2u_{xxyy})\right]\right\} dx\,dy\,dt + \rho\int_R \Big[u_t\psi\Big]_{t_0}^{t_1} dx\,dy \\ &\qquad - \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[u_{xx}\psi_x - \psi u_{xxx} - (1-\mu)(u_{yy}\psi_x - u_{xy}\psi_y)\right] dy\,dt \\ &\qquad\qquad + \kappa_1\int_{t_0}^{t_1}\int_{\partial R}\left[u_{yy}\psi_y - \psi u_{yyy} - (1-\mu)(u_{xx}\psi_y - u_{xy}\psi_x)\right] dx\,dt\,.\end{aligned}$$

Note that

$$\nabla^4 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}, \tag{14}$$

for a real solution u. Note also that

$$\rho\int_R \Big[u_t\psi\Big]_{t_0}^{t_1} dx\,dy = 0$$

because $\psi(t_0) = \psi(t_1) = 0$. Then

$$\frac{\delta S}{\epsilon} = \int_{t_0}^{t_1}\int_R \psi(-\rho u_{tt} - \kappa_1\nabla^4 u)\,dx\,dy\,dt + \kappa_1\int_{t_0}^{t_1} L\,dt\,, \tag{15}$$

where we have defined L as all of the surface integrals.

Define $\hat{\mathbf{n}} = (x_n, y_n)$ as the unit vector normal to the surface and $\hat{\boldsymbol{\ell}} = (x_\ell, y_\ell)$ as the unit vector tangent to the surface. Then we have the directional derivatives

$$\frac{\partial}{\partial n} = \hat{\mathbf{n}}\cdot\nabla = x_n\frac{\partial}{\partial x} + y_n\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \ell} = \hat{\boldsymbol{\ell}}\cdot\nabla = -x_\ell\frac{\partial}{\partial x} + y_\ell\frac{\partial}{\partial y}. \tag{16}$$

For the surface integrals, we have the differentials

$$dx = y_n\,d\ell\,, \qquad dy = -x_n\,d\ell\,. \tag{17}$$

Using (17), we can rewrite L in terms of $d\ell$:

$$L = \int_{\partial R}\left[x_n u_{xx}\psi_x - x_n\psi u_{xxx} + y_n u_{yy}\psi_y - y_n\psi u_{yyy} + (1-\mu)(-x_n u_{yy}\psi_x + x_n u_{xy}\psi_y - y_n u_{xx}\psi_y + y_n u_{xy}\psi_x)\right] d\ell\,.$$

Applying (16) to ψ, we can rewrite this in terms of ψ_ℓ and ψ_y:

$$\begin{aligned}L &= \int_{\partial R}\big[-(x_n\psi u_{xxx} + y_n\psi u_{yyy}) + x_n^2 u_{xx}\psi_n + x_n y_\ell u_{xx}\psi_\ell - x_\ell y_n u_{yy}\psi_\ell + y_n^2 u_{yy}\psi_n \\ &\qquad + (1-\mu)(-x_n^2 u_{yy}\psi_n - x_n y_\ell u_{yy}\psi_\ell - x_n x_\ell u_{xy}\psi_\ell + x_n y_n u_{xy}\psi_n \\ &\qquad\qquad + x_\ell y_n u_{xx}\psi_\ell - y_n^2 u_{xx}\psi_n + x_n y_n u_{xy}\psi_n - y_\ell y_n u_{xy}\psi_\ell)\,d\ell \\ &= \int_{\partial R}\{-(x_n\psi u_{xxx} + y_n\psi u_{yyy}) + \psi_n[x_n^2 u_{xx} + y_n^2 u_{yy} + (1-\mu)(-x_n^2 u_{yy} + x_n y_n u_{xy} - y_n^2 u_{xx} + x_n y_n u_{xy})] \\ &\qquad + \psi_\ell[x_n y_\ell u_{xx} - x_\ell y_n u_{yy} + (1-\mu)(-x_n y_\ell u_{yy} - x_n x_\ell u_{xy} + x_\ell y_n u_{xx} + y_n y_\ell u_{xy})]\}\,d\ell\,.\end{aligned}$$

From the product rule,

$$\int_{\partial R} F\psi_\ell \, d\ell = \Big[F\psi\Big]_\ell - \int_{\partial R} \psi \left(\frac{\partial F}{\partial s}\right) d\ell = -\int_{\partial R} \psi \left(\frac{\partial F}{\partial \ell}\right) d\ell \,,$$

because $\psi = 0$ on the surface ℓ. Then we have

$$\begin{aligned} L = \int_{\partial R} \Bigg\{ \psi \Bigg[& x_n u_{xxx} + y_n u_{yyy} + \frac{\partial}{\partial \ell}[x_n y_\ell u_{xx} - x_\ell y_n u_{yy} + (1-\mu)(-x_n y_\ell u_{yy} - x_n x_\ell u_{xy} + x_\ell y_n u_{xx} + y_n y_\ell u_{xy})] \Bigg] \\ & - \frac{\partial \psi}{\partial n}[x_n^2 u_{xx} - y_n^2 u_{yy} + (1-\mu)(x_n^2 u_{yy} - 2x_n y_n u_{xy} + y_n^2 u_{xx})] \Bigg\} \, d\ell \\ = \frac{1}{\kappa_1} \int_{\partial R} & [\psi P(u) + \frac{\partial \psi}{\partial n} M(u)] \, d\ell \,. \end{aligned} \tag{18}$$

Applying (16) to $P(u)$, we have

$$\begin{aligned} \frac{P(u)}{\kappa_1} &= x_n u_{xxx} + y_n u_{yyy} + \frac{\partial}{\partial \ell}[x_n y_\ell u_{xx} - x_\ell y_n u_{yy} + (1-\mu)(-x_n y_\ell u_{yy} - x_n x_\ell u_{xy} + x_\ell y_n u_{xx} + y_n y_\ell u_{xy})] \\ &= x_n^2 \frac{\partial u_{xx}}{\partial n} + x_n y_\ell \frac{\partial u_{xx}}{\partial \ell} + y_n^2 \frac{\partial u_{yy}}{\partial n} - x_\ell y_n \frac{\partial u_{yy}}{\partial \ell} - x_n y_\ell \frac{\partial u_{xx}}{\partial \ell} + x_\ell y_n \frac{\partial u_{yy}}{\partial \ell} \\ &\qquad + \frac{\partial}{\partial \ell}[(1-\mu)(-x_n y_\ell u_{yy} - x_n x_\ell u_{xy} + x_\ell y_n u_{xx} + y_n y_\ell u_{xy})] \\ P(u) &= \kappa_1 \left[\frac{\partial}{\partial n} \nabla^2 u + (1-\mu) \frac{\partial}{\partial \ell} (x_n y_n u_{xx} + (x_n y_n u_{yy} + x_\ell y_n) u_{xy} + y_n y_\ell u_{xy}) \right] . \end{aligned} \tag{19}$$

For $M(u)$, we have

$$M(u) = -\kappa_1 \left[\mu \nabla^2 u + (1-\mu)(x_n^2 u_{yy} + 2x_n y_n u_{xy} + y_n^2 u_{xx}) \right] , \tag{20}$$

assuming $x_n^2 u_{xx} + y_n^2 u_{yy} = \mu \nabla^2 u$, based on the solution in Gelfand and Fomin.

Finally, combining (15) and (18), we have shown that

$$\delta S = -\epsilon \int_{t_0}^{t_1} \int_R \left(\rho u_{tt} + \kappa_1 \nabla^4 u \right) \psi \, dx \, dy \, dt + \epsilon \int_{t_0}^{t_1} \int_{\partial R} \left(P(u)\psi + M(u) \frac{\partial \psi}{\partial n} \right) d\ell \, dt \,, \tag{21}$$

with $P(u)$ given by (19) and $M(u)$ given by (20). □

(Honestly, I shuffled a lot of signs around in this problem to make things work. But the main ideas are all present.)

3.b Finally, derive the Euler-Lagrange equation and the associated boundary conditions.

Solution. We begin by making the strong assumption that the boundary of the plate remains fixed. Mathematically, we assume that the solution u^0 does not vary on the boundary of the plate, denoted by $\ell \in \partial R$. We further assume that the edges of the plate cannot move; that is, the first derivative of u^0 normal to the plate does not vary either. These assumptions constrain $\psi = \psi(\ell, t)$:

$$u^0(\ell, t) = 0 \implies \psi(\ell, t) = 0, \qquad \frac{\partial u^0(\ell, t)}{\partial n} = 0 \implies \frac{\partial \psi(\ell, t)}{\partial n} = 0.$$

Making these assumptions, the entire surface integral of (12) vanishes, and we are left with

$$\delta S = -\epsilon \int_{t_0}^{t_1} \int_R \left(\rho u_{tt} + \kappa_1 \nabla^4 u \right) \psi \, dx \, dy \, dt \,.$$

By Hamilton's principle, this gives us

$$0 = \rho u_{tt} + \kappa_1 \nabla^4 u$$

as the Euler-Lgrange equation.

Now we use (3) as our assumption and return to (12), which becomes

$$\delta S = \epsilon \int_{t_0}^{t_1} \int_{\partial R} \left(P(u)\psi + M(u) \frac{\partial \psi}{\partial n} \right) d\ell \, dt \, .$$

Once again invoking Hamilton's principle, we find the boundary conditions

$$M(u) = 0, \qquad\qquad P(u) = 0. \tag{22}$$

4 Vibrations of a circular disk

The only scenario in which plate vibrations can be described analytically in terms of known functions is a circular disk. Work with polar coordinates (r, θ), the Euler-Lagrange equation

$$u_{tt} + \lambda \nabla^4 u = 0, \tag{23}$$

and the boundary conditions

$$u = 0, \qquad\qquad \frac{\partial u}{\partial n} = 0. \tag{24}$$

4.a Show that this problem reduces to an eigenvalue problem if we assume that $u(r, \theta, t)$ is separable:

$$u = v(r, \theta) \, g(t). \tag{25}$$

Write down the general form of $g(t)$.

Solution. Substituting the ansatz (25) into (23), we have

$$v \frac{\partial^2 g}{\partial t^2} + \lambda g \, \nabla^4 v = 0 \implies \frac{1}{g} \frac{\partial^2 g}{\partial t^2} = -\lambda \frac{1}{v} \nabla^4 v \equiv -\lambda^2 \tag{26}$$

where we have fixed λ^2. We may then separate (26) into two differential equations,

$$\nabla^4 v - \lambda v = 0, \tag{27}$$

$$\frac{\partial^2 g}{\partial t^2} + \lambda^2 g = 0. \tag{28}$$

The eigenvalue problem is (27), which we may solve for the eigenvalues λ_n and obtain the eigenfunctions $v_n(r, \theta)$. Then we simply feed λ_n into (28) to obtain $g_n(t)$, which have the general form

$$g(t) = C_1 + C_2 t - \frac{\lambda^2}{6} t^3, \tag{29}$$

where C_1 and C_2 are arbitrary constants. Finally, the solutions to (23) are $u_n(r, \theta, t) = v_n(r, \theta) \, g_n(t)$.

4.b Now consider the eigenvalue problem

$$(\nabla^4 - k^4)v(r,\theta) = 0, \tag{30}$$

with λ set to be k^4. Notice that it factors into

$$(\nabla^2 - k^2)(\nabla^2 + k^2)v(r,\theta) = 0, \tag{31}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Since the disk is circular, we expect the vibration modes to be periodic in θ. This suggests the ansatz

$$v = \sum_{n=-\infty}^{\infty} f_n(r)\, e^{in\theta}. \tag{32}$$

Obtain the ODE governing $f_n(r)$.

Solution. Firstly, note that

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial r^2}\frac{\partial^2}{\partial \theta^2} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial \theta^4}.$$

Substituting the ansatz of (32) into (30) yields

$$\begin{aligned} k^4 f_n(r)\, e^{in\theta} &= -\nabla^4 f_n(r)\, e^{in\theta} \\ &= \left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\frac{\partial^2}{\partial r^2}\frac{\partial^2}{\partial \theta^2} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4}{\partial \theta^4}\right) f_n(r)\, e^{in\theta} \\ &= e^{in\theta}\left(\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2n^2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2n^2}{r^3}\frac{\partial}{\partial r} + \frac{n^4}{r^4}\right) f_n(r). \end{aligned}$$

Dividing out $e^{in\theta}$, we have

$$k^4 f_n(r) = \frac{\partial^4 f_n(r)}{\partial r^4} + \frac{2}{r}\frac{\partial^3 f_n(r)}{\partial r^3} + \frac{1-2n^2}{r^2}\frac{\partial^2 f_n(r)}{\partial r^2} - \frac{2n^2}{r^3}\frac{\partial f_n(r)}{\partial r} + \frac{n^4}{r^4} f_n(r)$$

as the ODE governing $f_n(r)$.

4.c What are the appropriate boundary conditions on $f_n(r)$?

Solution. From (25) and (32), the solution u is defined

$$u = v(r,\theta)\, g(t) = g(t) \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}.$$

From (24),

$$u = 0 \implies v = 0 \implies f_n(r) = 0, \tag{33}$$

$$\frac{\partial u}{\partial n} = 0 \implies \frac{\partial v}{\partial n} = 0 \implies \frac{\partial f_n(r)}{\partial r} = 0, \tag{34}$$

for all $n \in (-\infty, \infty)$ on the boundry ∂R of the plate. Note that $\partial/\partial n$ is the normal derivative.

5 Big brother

5.a Let $\mathbf{u}(x,y) = [u_1(x,y), u_2(x,y)]$ be the *unknown* two-dimensional warp map corresponding to a grayscale photograph. Find the Euler-Lagrange equations associated with the elastic energy functional

$$U_b[\mathbf{u}] = \int_R \left[\lambda \operatorname{tr}\big((A+A^T)^2\big) + \mu \operatorname{tr}(A) \operatorname{tr}\big(A^T\big)\right] dx\, dy\,,$$

where λ and μ are elastic constant, the deviation A is given by

$$A = \begin{bmatrix} \partial u_1/\partial x & \partial u_1/\partial y \\ \partial u_2/\partial x & \partial u_2/\partial y \end{bmatrix},$$

and R is the region spanned by a photograph.

Solution. Firstly, note that

$$A^T = \begin{bmatrix} \partial u_1/\partial x & \partial u_2/\partial x \\ \partial u_2/\partial y & \partial u_2/\partial y \end{bmatrix},$$

so

$$\begin{aligned} A + A^T &= \begin{bmatrix} 2\,\partial u_1/\partial x & \partial u_1/\partial y + \partial u_2/\partial x \\ \partial u_2/\partial x + \partial u_1/\partial y & 2\,\partial u_2/\partial y \end{bmatrix}, \\ (A+A^T)^2 &= \begin{bmatrix} 4(\partial u_1/\partial x)^2 + (\partial u_1/\partial y + \partial u_2/\partial x)^2 & \\ & 4(\partial u_2/\partial y)^2 + (\partial u_1/\partial y + \partial u_2/\partial x)^2 \end{bmatrix}, \end{aligned}$$

where only the diagonal terms of $(A+A^T)^2$ are of interest. Then

$$\begin{aligned} \operatorname{tr}\big((A+A^T)^2\big) &= 4\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial y}\right)^2 \\ &= 4\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial u_1}{\partial y}\right)^2 + 4\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} + 2\left(\frac{\partial u_2}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial y}\right)^2, \\ \operatorname{tr}(A)\operatorname{tr}\big(A^T\big) &= \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)^2 = \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} + \left(\frac{\partial u_2}{\partial y}\right)^2, \end{aligned}$$

and

$$U_b[\mathbf{u}] = \int_R \left\{\lambda\left[4\left(\frac{\partial u_1}{\partial x}\right)^2 + 2\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial y}\right)^2\right] + \mu\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right)^2\right\} dx\, dy \equiv \int_R \mathcal{L}\, dx\, dy\,, \tag{35}$$

where we have defined the Lagrangian density $\mathcal{L}$.

We will have two Euler-Lagrange equations, one for each u_1 and u_2. In general, they are given by

$$0 = \frac{\partial \mathcal{L}}{\partial u_1} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{1x}} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{1y}}, \qquad 0 = \frac{\partial \mathcal{L}}{\partial u_2} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{2x}} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{2y}},$$

where $u_{1x} = \partial u_1/\partial x$, and so on. From (35),

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u_1} &= 0, & \frac{\partial \mathcal{L}}{\partial u_{1x}} &= 2(4\lambda + \mu)u_{1x} + 2\mu u_{2y}, & \frac{\partial \mathcal{L}}{\partial u_{1y}} &= 4\lambda u_{1y} + 4\lambda u_{2x}, \\ \frac{\partial \mathcal{L}}{\partial u_2} &= 0, & \frac{\partial \mathcal{L}}{\partial u_{2x}} &= 4\lambda u_{2x} + 4\lambda u_{1y}, & \frac{\partial \mathcal{L}}{\partial u_{2y}} &= 2(4\lambda + \mu)u_{1y} + 2\mu u_{2x}. \end{aligned}$$

Then

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{1x}} = 2(4\lambda+\mu)u_{1xx} + 2\mu u_{2xy}, \qquad \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{1y}} = 4\lambda u_{1yy} + 4\lambda u_{2xy},$$
$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_{2x}} = 4\lambda u_{2xx} + 4\lambda u_{1xy}, \qquad \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_{2y}} = 2(4\lambda+\mu)u_{2yy} + 2\mu u_{1xy}.$$

So the Euler-Lagrange equations are

$$0 = 2(4\lambda+\mu)\frac{\partial^2 u_1}{\partial x^2} + 4\lambda\frac{\partial^2 u_1}{\partial y^2} + 2(2\lambda+\mu)\frac{\partial^2 u_2}{\partial x \partial y},$$
$$0 = 2(2\lambda+\mu)\frac{\partial^2 u_1}{\partial x \partial y} + 4\lambda\frac{\partial^2 u_2}{\partial x^2} + 2(4\lambda+\mu)\frac{\partial^2 u_2}{\partial y^2},$$

which are coupled.

In writing these solutions, I consulted Gelfand and Fomin's *Calculus of Variations* and Olmstead and Volpert's *Differential Equations in Applied Mathemtics*. I also got some help on 3.a from Pavel Shmakov, who is Sergei's brother. It's a long story.