Problem 1. Show that for an arbitrary spatially bound charge-current source, the electric dipole moment $\mathbf{p}$ satisfies

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{J}\, d^3x\,.$$

Solution. The electric dipole moment $\mathbf{p}$ is defined by Eq. (2.36),

$$\mathbf{p} = \int \mathbf{x}\,\rho(x)\, d^3x\,.$$

Differentiating both sides with respect to t, we find

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int \mathbf{x}\rho\, d^3x = \int \frac{d}{dt}(\mathbf{x}\rho)\, d^3x = \int \mathbf{x}\frac{\partial\rho}{\partial t}\, d^3x\,, \tag{1}$$

because $\mathbf{x}$ is simply the point at which we are evaluating the potential, and is therefore independent of time.

The charge-current conservation law is given by Eq. (5.8),

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0.$$

Multiplying by $\mathbf{x}$ on both sides and integrating over all space, we obtain

$$\int \mathbf{x}\frac{\partial\rho}{\partial t}\, d^3x + \int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = 0.$$

Applying (1), we have

$$\frac{d\mathbf{p}}{dt} = -\int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x\,. \tag{2}$$

It remains to be shown that the right side is equal to the integral of $\mathbf{J}$ over all space.

Vector identity (5) in Griffiths is

$$\boldsymbol{\nabla}\cdot(f\mathbf{a}) = f(\boldsymbol{\nabla}\cdot\mathbf{a}) + \mathbf{a}\cdot(\boldsymbol{\nabla}f).$$

Writing the right side of (2) in component notation and applying the identity gives us

$$-\int x_i(\boldsymbol{\nabla}\cdot\mathbf{J})\, d^3x = \int \mathbf{J}\cdot(\boldsymbol{\nabla}x_i)\, d^3x - \int \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x\,. \tag{3}$$

Gauss's theorem is given by Eq. (2.6),

$$\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\mathbf{v}\, d^3x = \int_S \mathbf{v}\cdot\hat{\mathbf{n}}\, dS\,,$$

Here, let $\mathcal{V}$ be a ball of radius R, with R large enough that the entire charge-current source is enclosed. Then S is the surface of $\mathcal{V}$, and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Applying Gauss's theorem to the second integral on the right side of (3), we have

$$\int \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x = \lim_{R\to\infty}\int_{\mathcal{V}} \boldsymbol{\nabla}\cdot(x_i\mathbf{J})\, d^3x = \lim_{R\to\infty}\int_S x_i\mathbf{J}\cdot\hat{\mathbf{r}}\, dS = 0,$$

since $\mathbf{J}$ is bounded, and therefore $\mathbf{J}$ evaluated on S reaches zero well before x_i becomes very large.

Returning to (3), we now have

$$-\int x_i(\boldsymbol{\nabla}\cdot\mathbf{J})\,d^3x = \int \mathbf{J}\cdot(\boldsymbol{\nabla}x_i)\,d^3x = \sum_j \int J_j\partial_j x_i\,d^3x = \sum_j \int J_j\delta_{ij}\,d^3x = \int J_i\,d^3x\,,$$

where we have followed the proof in Eq. (4.24) of the course notes. Finally, (2) becomes

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{J}\,d^3x$$

as desired. □

Problem 2. A particle of charge q_1 moves with velocity v in a circular orbit of radius R about the origin in the xy plane, such that its φ coordinate varies as $\varphi = \omega t$, with $\omega = v/R$. Assume that $v \ll c$. Another particle of charge q_2 is at rest at point $\mathbf{x}$, where $|\mathbf{x}| \gg R$. To order $1/|\mathbf{x}|$, find the force $\mathbf{F}$ on the particle of charge q_2 at time t.

Problem 3. An "antenna" is a segment of conducting wire in which a current flows (driven by an external power supply). Suppose an antenna of length L is placed on the z axis between $z = 0$ and $z = L$, and suppose that the current in the antenna is

$$\mathbf{J}(t,z) = I_0 \sin\left(\frac{\pi z}{L}\right)\cos(\omega t)\,\delta(x)\,\delta(y)\,\hat{\mathbf{z}}.$$

3.a Find the charge density $\rho(t, z)$ in the antenna.

3.b Assume that $\omega L \ll c$. Find the electric and magnetic fields, $\mathbf{E}(t, z)$ and $\mathbf{B}(t, z)$, at large distances from the antenna (valid to order $1/|\mathbf{x}|$).

In addition to the course lecture notes, I consulted Griffiths's *Introduction to Electrodynamics*, Jackson's *Classical Electrodynamics*, and K. T. McDonald's and D. K. Ghosh's notes on electromagnetism while writing up these solutions.