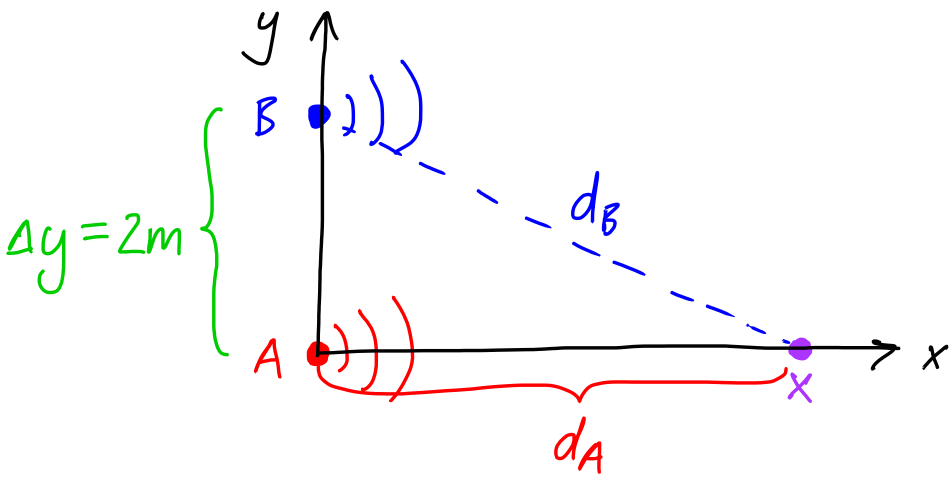
Problem 1. Consider two speakers emitting sound at the same volume with frequency $f = 800\,\mathrm{Hz}$. One speaker is located at the origin, and the other on the y axis at $y = 2\,\mathrm{m}$. At what locations on the positive x axis is the interference completely constructive? At what points is it completely destructive?

Now we decrease f until there are no longer any points of completely destructive interference on the positive x axis. How low must f be for this to occur?

Solution. Consider the setup shown below:



The path difference d (which is called Δx in the lecture slides) is given by

$$d = d_B - d_A = d_A - x,$$

and from trigonometry,

$$d_B^2 = x^2 + (\Delta y)^2 \qquad \Longrightarrow \qquad d_B = \sqrt{x^2 + (\Delta y)^2},$$

where Δy is the distance between the two speakers. Putting these together, we can write

$$d = \sqrt{x^2 + (\Delta y)^2} - x.$$

Completely constructive interference occurs where the interference pattern of the speakers has a maximum, which is when

$$d = n\lambda, \qquad\qquad n = 0, \pm 1, \pm 2, \ldots.$$

Completely destructive interference occurs where it has a minimum, and

$$d = \left(n + \frac{1}{2}\right)\lambda, \qquad\qquad n = 0, \pm 1, \pm 2, \ldots.$$

Recall that the wavelength $\lambda = v/f$, where $v = 344\,\mathrm{m\,s^{-1}}$ is the speed of sound in air. For this problem,

$$\lambda = \frac{344\,\mathrm{m\,s^{-1}}}{800\,\mathrm{Hz}} = 0.43\,\mathrm{m}.$$

Constructive interference will occur at x when

$$n\lambda = \sqrt{x^2 + (\Delta y)^2} - x.$$

Solving for x,

$$(x+n\lambda)^2 = x^2 + (\Delta y)^2 \implies x^2 + n\lambda x + n^2\lambda^2 = x^2 + (\Delta y)^2 \implies n\lambda x = (\Delta y)^2 - n^2\lambda^2,$$

which implies

$$x = \frac{(\Delta y)^2}{n\lambda} - n\lambda. \tag{1}$$

Now we can plug in numerical quantities and $n = 0, \pm1, \pm2, \dots$ into Eq. (1) to find

$$\begin{aligned}
x(n=1) &= \frac{(2\,\mathrm{m})^2}{0.43\,\mathrm{m}} - (0.43\,\mathrm{m}) = 8.87\,\mathrm{m}, \\
x(n=2) &= \frac{(2\,\mathrm{m})^2}{2(0.43\,\mathrm{m})} - 2(0.43\,\mathrm{m}) = 3.79\,\mathrm{m}, \\
x(n=3) &= \frac{(2\,\mathrm{m})^2}{3(0.43\,\mathrm{m})} - 3(0.43\,\mathrm{m}) = 1.81\,\mathrm{m}, \\
x(n=4) &= \frac{(2\,\mathrm{m})^2}{4(0.43\,\mathrm{m})} - 4(0.43\,\mathrm{m}) = 0.61\,\mathrm{m}.
\end{aligned}$$

Note that x is undefined for $n = 0$ and is negative for $n > 4$. Plugging in $n = -1, -2, -3, \dots$ would also give us negative values. None of these makes sense since we are interested only in the positive x axis.

For destructive interference, we have to satisfy

$$\left(n + \frac{1}{2}\lambda\right) = \sqrt{x^2 + (\Delta y)^2} - x,$$

and solving for x in the same manner as before gives us

$$x = \frac{(\Delta y)^2}{(n+1/2)\lambda} - \left(n + \frac{1}{2}\right)\lambda. \tag{2}$$

Plugging in numerical quantities and $n = 0, 1, 2, \dots$ into Eq. (2),

$$\begin{aligned}
x(n=0) &= \frac{(2\,\mathrm{m})^2}{(1/2)(0.43\,\mathrm{m})} - \frac{1}{2}(0.43\,\mathrm{m}) = 18.4\,\mathrm{m}, \\
x(n=1) &= \frac{(2\,\mathrm{m})^2}{(3/2)(0.43\,\mathrm{m})} - \frac{3}{2}(0.43\,\mathrm{m}) = 5.56\,\mathrm{m}, \\
x(n=2) &= \frac{(2\,\mathrm{m})^2}{(5/2)(0.43\,\mathrm{m})} - \frac{5}{2}(0.43\,\mathrm{m}) = 2.65\,\mathrm{m}, \\
x(n=3) &= \frac{(2\,\mathrm{m})^2}{(7/2)(0.43\,\mathrm{m})} - \frac{7}{2}(0.43\,\mathrm{m}) = 1.15\,\mathrm{m}, \\
x(n=4) &= \frac{(2\,\mathrm{m})^2}{(9/2)(0.43\,\mathrm{m})} - \frac{9}{2}(0.43\,\mathrm{m}) = 0.13\,\mathrm{m}.
\end{aligned}$$

Again, $x < 0$ for $n < 0$ and $n > 4$, which are not sensible.

In order to find the frequency for which there is no destructive interference on the x axis, we should look at $n = 0$, since this gives us the point with the largest value of x. If we plug $n = 0$ into Eq. (2) and set $x = 0$, we are requiring that destructive interference can only occur at the origin. Solving for the wavelength λ tells us the smallest wavelength at which there is still destructive interference. We find

$$0 = \frac{(\Delta y)^2}{\lambda/2} - \frac{\lambda}{2} \implies \frac{\lambda}{2} = \frac{(\Delta y)^2}{\lambda/2} \implies \frac{\lambda^2}{4} = (\Delta y)^2 \implies \lambda = 2\Delta y.$$

But if $\lambda > 2\Delta y$, then

$$\frac{(\Delta y)^2}{\lambda/2} < \frac{1}{2}\lambda,$$

and Eq. (2) tells us

$$x = \frac{(\Delta y)^2}{\lambda/2} - \frac{1}{2}\lambda < 0.$$

This means there is no destructive interference on the x axis. Thus, we need to satisfy

$$\lambda = \frac{v}{f} > 2\Delta y \qquad \Longrightarrow \qquad f < \frac{v}{2\Delta y}.$$

Plugging in numbers, we find

$$f < \frac{344\,\mathrm{m\,s^{-1}}}{2(2\,\mathrm{m})} = 86\,\mathrm{Hz}.$$

Problem 2. A police car is waiting on the shoulder of Lake Shore Drive. In order to catch speeding commuters, the officer uses a radar gun that emits sound of frequency 10.5 GHz. There is a single speeding car on the road, which is directly in front of or behind her. When she fires the gun, the frequency that returns from the speeder's car is 12.5 GHz. What is the speed of the car? Is it moving toward the officer, or away from her?

A few moments later, the officers takes off at $35\,\mathrm{m\,s^{-1}}$ in pursuit of the speeder, who does not change his speed. If the officer fires the gun now, what frequency will she receive?

Solution. This is a problem about the Doppler effect. In general, we have

$$f_L = \frac{v \pm v_L}{v \pm v_S} f_S, \tag{3}$$

where f_L and f_S are the frequenceis in the frames of the listener and source, respectively, v_L and v_S are the velocities of the listener and source, and v is the speed of sound. (The speed of sound in air is $v = 344\,\mathrm{m\,s^{-1}}$.) Written in this form, we choose a $+$ sign in the numerator if the listener is moving toward the source, and a $-$ sign if it is moving away. The convention is flipped for the denominator: we choose a $-$ sign if the *source* is moving toward the *listener*, and a $+$ sign if it is moving away.

It is easiest to answer the question about the direction of the car first. Since the frequency that the officer hears is greater than the frequency she emitted, we know the speeding car is moving toward her.

In order to find the speed of the car, we need to apply (3) twice. Let f be the 10.5 GHz frequency emitted by the radar gun, and f' the frequency that is "heard" by the speeding car. Equation (3) becomes

$$f' = \frac{v + v_{\mathrm{car}}}{v} f, \tag{4}$$

where v_{car} is the velocity of the speeding car, which is the "listener" in this scenario, and is moving toward the officer. The officer, who is the "source," is stationary. Now we can use f' to find the 12.5 GHz frequency that returns to the officer, which we will call f''. This is found by writing Eq. (3) as

$$f'' = \frac{v}{v - v_{\mathrm{car}}} f', \tag{5}$$

where the stationary officer is now the "listener," and the speeding car is now the "source," which again is moving toward the officer.

Substituting Eq. (4) into Eq. (5), we find

$$f'' = \frac{v}{v - v_{\mathrm{car}}} \frac{v + v_{\mathrm{car}}}{v} f = \frac{v + v_{\mathrm{car}}}{v - v_{\mathrm{car}}} f.$$

Now we can solve for v_{car}:

$$f''(v - v_{\mathrm{car}}) = f(v + v_{\mathrm{car}}) \quad \Longrightarrow \quad (f'' - f)v = (f'' + f)v_{\mathrm{car}} \quad \Longrightarrow \quad v_{\mathrm{car}} = \frac{f'' - f}{f'' + f} v.$$

Finally, we can plug in known quantities to obtain

$$v_{\mathrm{car}} = \frac{(12.5\,\mathrm{GHz}) - (10.5\,\mathrm{GHz})}{(12.5\,\mathrm{GHz}) + (10.5\,\mathrm{GHz})}(344\,\mathrm{m\,s^{-1}}) = \frac{2}{22.5}(344\,\mathrm{m\,s^{-1}}) = 30.6\,\mathrm{m\,s^{-1}},$$

which is over 68 mph! (The speed limit on Lake Shore Drive is 40–45 mph.)

When the officer pursues the car, the equivalent of Eq. (4) is

$$f' = \frac{v - v_{\text{car}}}{v - v_{\text{off}}} f,$$

where v_{off} is the speed of the officer's police car. Here we use a $-$ sign in the numerator since the speeder is moving away from the officer, and a $-$ sign in the denominator because the officer is moving toward the speeder. Likewise, the equivalent of Eq. (5) is

$$f'' = \frac{v + v_{\text{off}}}{v + v_{\text{car}}} f'.$$

Once again, we may substitute in f' to find

$$f'' = \frac{v + v_{\text{off}}}{v + v_{\text{car}}} \frac{v - v_{\text{car}}}{v - v_{\text{off}}} f.$$

Finally, plugging in quantities,

$$\begin{aligned} f'' &= \frac{(344\,\mathrm{m\,s^{-1}}) + (35\,\mathrm{m\,s^{-1}})}{(344\,\mathrm{m\,s^{-1}}) + (30.6\,\mathrm{m\,s^{-1}})} \frac{(344\,\mathrm{m\,s^{-1}}) - (30.6\,\mathrm{m\,s^{-1}})}{(344\,\mathrm{m\,s^{-1}}) - (35\,\mathrm{m\,s^{-1}})} (10.5\,\mathrm{GHz}) = \frac{379}{374.6} \frac{313.4}{309} (10.5\,\mathrm{GHz}) \\ &= 1.03(10.5\,\mathrm{GHz}) = 10.8\,\mathrm{GHz}. \end{aligned}$$