Problem 1. Thermodynamics of a relativistic gas

1.1 Find the statistical distribution of a relativistic gas in momentum space, and in energies. Discuss the relativistic corrections compared to the Maxwell distribution.

1.2 Now take the ultra-relativistic limit. Find the mean energy $\langle E \rangle$ and the second moment of energy $\langle E^2 \rangle$. Find the free energy and the entropy in the limits of high and low temperature.

1.3 In the non-relativistic Maxwell distribution, the different translational degrees of freedom are independent as the kinetic energy is the sum of three independent terms $K = \sum_{i=1}^{3} p_i^2/2m$. This is not so in the relativistic case. For the ultra-relativistic gas compute the quantities

$$a_{ij} = \frac{\langle p_i^2 p_j^2 \rangle}{3\langle p_i^2 \rangle \langle p_j^2 \rangle}, \qquad r_{ij} = \frac{\langle p_i^2 p_j^2 \rangle}{\sqrt{\langle p_i^4 \rangle \langle p_j^4 \rangle}},$$

in spatial dimensions $d = 2, 3$ (here i, j enumerate spatial dimensions). Compare them to the non-relativistic case. Discuss their meaning and dependence on d (at least based on $d = 2, 3$).

Problem 2. Collision frequency and pressure Consider an ideal relativistic gas in a container. Given the rate of the collisions of molecules with the wall of the container per unit area per unit time, find the pressure of the gas in the relativistic, non-relativistic, and ultra-relativistic cases, and compare the results.

Problem 3. Boltzmann distribution Consider an ideal gas consisting of N identical one-dimensional quantum harmonic oscillators with Hamiltonian $H(p,q) = p^2/2m + m\omega q^2/2$. Determine the total number of oscillators in states with energies $\epsilon \geq \epsilon_1 = \hbar\omega(n_1 + 1/2)$.

Problem 4. Boltzmann H-function The equilibrium distribution function $f(p,q)$ of a non-interacting gas is a Maxwell-Boltzmann distribution. Show that the entropy of such a system satisfies $S = -k_B H + \text{const.}$, where $H = \int f \ln f \, d\Gamma$ is the Boltzmann H-function.

Problem 5. BBGKY Consider for simplicity a 1D system (a system on a circle) of N particles with an arbitrary two-body interaction:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_i U(x_i) + \sum_{i>j} V(x_i - x_j).$$

Give a derivation of the first equation of the BBGKY hierarchy at equilibrium for this system, which is a relation between the 1-point and 2-point distribution (correlation) functions.

Problem 6. Partition function as a generating functional Consider the Gibbs distribution of the system described in Problem 5. For simplicity neglect the kinetic energy. Let $n(x) = \sum_i \delta(x - x_i)$ be the density, and $\langle n(x) \rangle$ its expectation value. Let $C(x,y) = \langle \delta n(x) \, \delta n(y) \rangle$, where $\delta n(x) = n(x) - \langle n \rangle$, be the two-point correlation function.

6.1 Show that $\langle n(x) \rangle = -T \, \delta \ln Z / \delta U(x)$, where $Z[U(x)]$ is the partition function of the Gibbs distribution treated as a functional of the potential U.

6.2 Show that

$$C(x,y) = T^2 \frac{\delta \ln Z}{\delta U(x)\, \delta U(y)} = -T \frac{\delta \langle n(x) \rangle}{\delta U(y)} = -T \frac{\delta \langle n(y) \rangle}{\delta U(x)}.$$