Problem 1. Verify that the functional

$$J[u] = \int_R \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy \tag{1}$$

is invariant under a rotation of the coordinate axis:

$$\tilde{x} = x \cos \epsilon + y \sin \epsilon, \qquad \tilde{y} = -x \sin \epsilon + y \cos \epsilon. \tag{2}$$

Solution. The functional is invariant if $J[u(x,y)] = J[u(\tilde{x},\tilde{y})]$. By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial x} = \cos \epsilon \frac{\partial}{\partial \tilde{x}} - \sin \epsilon \frac{\partial}{\partial \tilde{y}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial y} = \sin \epsilon \frac{\partial}{\partial \tilde{x}} + \cos \epsilon \frac{\partial}{\partial \tilde{y}}.$$

The Jacobian for (2) is

$$A = \begin{bmatrix} \partial \tilde{x}/\partial x & \partial \tilde{x}/\partial y \\ \partial \tilde{y}/\partial x & \partial \tilde{y}/\partial y \end{bmatrix} = \begin{bmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{bmatrix} \implies \det(A) = \cos^2 \epsilon + \sin^2 \epsilon = 1,$$

and therefore

$$\int_R dx\, dy \mapsto \int_{\tilde{R}} d\tilde{x}\, d\tilde{y}\,.$$

Making these substitutions into (1), we have

$$\begin{aligned} J[u(x,y)] &= \int_R \left[\left(\cos \epsilon \frac{\partial u}{\partial \tilde{x}} - \sin \epsilon \frac{\partial u}{\partial \tilde{y}} \right)^2 + \left(\sin \epsilon \frac{\partial u}{\partial \tilde{x}} + \cos \epsilon \frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy \\ &= \int_R \left(\cos^2 \epsilon \frac{\partial^2 u}{\partial \tilde{x}^2} - 2 \cos \epsilon \sin \epsilon \frac{\partial^2 u}{\partial \tilde{x} \partial \tilde{y}} + \sin^2 \epsilon \frac{\partial^2 u}{\partial \tilde{y}^2} + \sin^2 \epsilon \frac{\partial^2 u}{\partial \tilde{x}^2} + 2 \cos \epsilon \sin \epsilon \frac{\partial^2 u}{\partial \tilde{x} \partial \tilde{y}} + \cos^2 \epsilon \frac{\partial^2 u}{\partial \tilde{y}^2} \right) dx\, dy \\ &= \int_R \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy = \int_{\tilde{R}} \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] d\tilde{x}\, d\tilde{y} \\ &= J[u(\tilde{x},\tilde{y})] \end{aligned}$$

as desired. □

Problem 2. Consider the real-valued Lagrangian density $\mathcal{L}$ depending on a complex-valued function $\phi(t,x,y)$:

$$\mathcal{L} = \frac{i}{2} \left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt} \phi \right) - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi, \tag{3}$$

where * is complex conjugation, and $\nabla \phi = (\partial \phi / \partial x\ ,\ \partial \phi / \partial y)$. Treating ϕ and ϕ^* as independent objects, derive the Euler-Lagrange equations.

Solution. We will have two Euler-Lagrange equations; one for ϕ and one for ϕ^*. In general, they are given by

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \phi_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi_x} - \frac{d}{dy} \frac{\partial \mathcal{L}}{\partial \phi_y}, \qquad 0 = \frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \phi_t^*} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \phi_x^*} - \frac{d}{dy} \frac{\partial \mathcal{L}}{\partial \phi_y^*}.$$

Expanding out $\nabla \phi^* \cdot \nabla \phi$, (3) becomes

$$\mathcal{L} = \frac{i}{2} \left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt} \phi \right) - \frac{\partial \phi^*}{\partial x} \frac{\partial \phi}{\partial x} - \frac{\partial \phi^*}{\partial y} \frac{\partial \phi}{\partial y} - m^2 \phi^* \phi.$$

Then

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{i}{2}\phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x} = -\frac{\partial \phi^*}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \phi_y} = -\frac{\partial \phi^*}{\partial y},$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t^*} = -\frac{i}{2}\phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x^*} = -\frac{\partial \phi}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \phi_y^*} = -\frac{\partial \phi}{\partial y},$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{i}{2}\frac{d\phi^*}{dt}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x} = -\frac{\partial^2 \phi^*}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y} = -\frac{\partial^2 \phi^*}{\partial y^2},$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t^*} = -\frac{i}{2}\frac{d\phi}{dt}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x^*} = -\frac{\partial^2 \phi}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y^*} = -\frac{\partial^2 \phi}{\partial y^2}.$$

Then the Euler-Lagrange equations are

$$0 = -\frac{i}{2}\frac{d\phi^*}{dt} - m^2\phi^* - \frac{i}{2}\frac{d\phi^*}{dt} + \frac{\partial^2 \phi^*}{\partial x^2} + \frac{\partial^2 \phi^*}{\partial y^2}, \qquad 0 = \frac{i}{2}\frac{d\phi}{dt} - m^2\phi + \frac{i}{2}\frac{d\phi}{dt} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2},$$

which simplify to

$$0 = i\frac{d\phi^*}{dt} - \nabla^2\phi^* + m^2\phi^*, \qquad 0 = i\frac{d\phi}{dt} + \nabla^2\phi^* - m^2\phi^*.$$

Problem 3. The nondimensionalized, multidimensional Sine-Gordon equation,

$$\theta_{xx} + \theta_{yy} - \theta_{tt} = \sin\theta$$

for $\theta(x, y, t)$, is the Euler-Lagrange equation for the action integral

$$S[\theta] = \int_R \left\{ \frac{1}{2}\left[\theta_t^2 - (\nabla\theta)^2\right] - \sin\theta \right\} dx\, dy\, dt$$

with $\nabla\theta = (\partial\theta/\partial x\,,\, \partial\theta/\partial y)$. The functional $S[\theta]$ is invariant under translation of x, y, and t. Find the associated energy-momentum tensor and energy-momentum vector.

Solution. Expanding out $(\nabla\theta)^2$, the Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) - \sin\theta. \tag{4}$$

The energy-momentum tensor is defined by

$$T_{ij} = \frac{\partial \mathcal{L}}{\partial \theta_{x_i}}\frac{\partial \theta}{\partial x_j} - \mathcal{L}\,\delta_{ij},$$

where $x_i \in \{x_0, x_1, x_2\} = \{t, x, y\}$. The diagonal elements of T are then

$$\begin{aligned}
T_{00} &= \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial t} - \mathcal{L} = \theta_t^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = \frac{1}{2}(\theta_t^2 + \theta_x^2 + \theta_y^2) + \sin\theta,\\
T_{11} &= \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial x} - \mathcal{L} = -\theta_x^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 + \theta_x^2 - \theta_y^2) + \sin\theta,\\
T_{22} &= \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial y} - \mathcal{L} = -\theta_y^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 - \theta_x^2 + \theta_y^2) + \sin\theta,
\end{aligned}$$

and the nondiagonal elements are

$$T_{01} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial x} = \theta_t\theta_x, \qquad T_{02} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial y} = \theta_t\theta_y, \qquad T_{12} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial y} = -\theta_x\theta_y,$$
$$T_{10} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial t} = -\theta_t\theta_x, \qquad T_{20} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial t} = -\theta_t\theta_y, \qquad T_{21} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial x} = -\theta_x\theta_y.$$

In matrix form, we have

$$T = \begin{bmatrix} (\theta_t^2+\theta_x^2+\theta_y^2)/2+\sin\theta & \theta_t\theta_x & \theta_t\theta_y \\ -\theta_t\theta_x & -(\theta_t^2+\theta_x^2-\theta_y^2)/2+\sin\theta & -\theta_x\theta_y \\ -\theta_t\theta_y & -\theta_x\theta_y & -(\theta_t^2-\theta_x^2+\theta_y^2)/2+\sin\theta \end{bmatrix}.$$

The energy-momentum vector is defined by

$$P_j = \int T_{0j}\, dx_1\, dx_2\,.$$

Its components are then

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2+\theta_x^2+\theta_y^2)+\sin\theta\right] dx\,dy\,, \qquad P_1 = \int \theta_t\theta_x\,dx\,dy\,, \qquad P_2 = \int \theta_t\theta_y\,dx\,dy\,.$$

Problem 4. Extra credit

4.a Verify that the nondimensionalized, one-dimensional Sine-Gordon equation,

$$\theta_{xx} - \theta_{tt} = \sin\theta, \tag{5}$$

is also invariant under a Lorentz transformation on $(x_0 = t, x_1 = x)$. The transformation is given by

$$\begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix},$$

where $\gamma = 1/\sqrt{1-\nu^2}$.

Solution. Define $(\tilde{t}, \tilde{x})$ as the transformed coordinates. (5) is invariant if it has the same form under the substitution $\theta(t,x) \mapsto \theta(\tilde{t},\tilde{x})$. The new coordinates are given by

$$\begin{bmatrix} \tilde{t} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\nu \\ -\gamma\nu & \gamma \end{bmatrix}\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma(t-\nu x) \\ \gamma(x-\nu t) \end{bmatrix},$$

or

$$\tilde{t} = \gamma(t-\nu x), \qquad \tilde{x} = \gamma(x-\nu t).$$

Proceeding similarly to problem 1, the chain rule gives us

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial t} + \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial t} = \gamma\left(\frac{\partial}{\partial \tilde{t}} - \nu\frac{\partial}{\partial \tilde{x}}\right), \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{t}}\frac{\partial \tilde{t}}{\partial x} + \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} = \gamma\left(\frac{\partial}{\partial \tilde{x}} - \nu\frac{\partial}{\partial \tilde{t}}\right).$$

For the second derivatives,

$$\frac{\partial^2}{\partial t^2} = \gamma^2 \left(\frac{\partial}{\partial \tilde{t}} - \nu \frac{\partial}{\partial \tilde{x}} \right)^2 = \gamma^2 \left(\frac{\partial^2}{\partial \tilde{t}^2} - 2\nu \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2}{\partial \tilde{x}^2} \right),$$
$$\frac{\partial^2}{\partial x^2} = \gamma^2 \left(\frac{\partial}{\partial \tilde{x}} - \nu \frac{\partial}{\partial \tilde{t}} \right)^2 = \gamma^2 \left(\frac{\partial^2}{\partial \tilde{x}^2} - 2\nu \frac{\partial^2}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2}{\partial \tilde{t}^2} \right).$$

Making these substitutions, (5) becomes

$$\begin{aligned}
\sin\theta &= \frac{\partial^2\theta}{\partial x^2} - \frac{\partial^2\theta}{\partial t^2} \\
&= \gamma^2 \left(\frac{\partial^2\theta}{\partial \tilde{x}^2} - 2\nu \frac{\partial^2\theta}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2\theta}{\partial \tilde{t}^2} \right) - \gamma^2 \left(\frac{\partial^2\theta}{\partial \tilde{t}^2} - 2\nu \frac{\partial^2\theta}{\partial \tilde{t} \partial \tilde{x}} + \nu^2 \frac{\partial^2\theta}{\partial \tilde{x}^2} \right) \\
&= \gamma^2 \left[(1-\nu^2) \frac{\partial^2\theta}{\partial \tilde{x}^2} - (1-\nu^2) \frac{\partial^2\theta}{\partial \tilde{t}^2} \right] \\
&= \frac{\partial^2\theta}{\partial \tilde{x}^2} - \frac{\partial^2\theta}{\partial^2},
\end{aligned}$$

because $\gamma^2 = 1/(1-\nu^2)$. Thus, we have demonstrated the invariance of (5). □

4.b Find the associated conserved quantity. Is it analogous to a common conserved quantity in classical mechanics?

Solution. By analogy to problem 3, the Lagrangian for this system is given by

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2) - \sin\theta$$

which is like (4), but with only one spatial dimension. Continuing the analogy, the components of the energy-momentum vector are

$$P_0 = \int \left[\frac{1}{2}(\theta_t^2 + \theta_x^2) + \sin\theta \right] dx\,, \qquad P_1 = \int \theta_t \theta_x \, dx\,.$$

These are the conserved quantitites, or "currents." The component P_0 is analagous to the calssical Hamiltonian, or the total energy of the system. This corresponds to $\mathcal{L}$'s having no explicit t dependence. The component P_1 is like the momentum conjugate to x, since it corresponds to $\mathcal{L}$'s having no explicit x dependence. Since we are concerned with only one spatial dimension, P_1 is analogous to the classical total (linear) momentum of the system.

While writing up these solutions, I consulted Gelfand and Fomin's *Calculus of Variations* and Goldstein's *Classical Mechanics*.