

Problem 1. Consider the charge density $\rho(\mathbf{x})$ given by

$$\rho(\mathbf{x}) = \begin{cases} (R - r)(1 - \cos \theta)^2 & \text{for } |\mathbf{x}| \leq R, \\ 0 & \text{for } |\mathbf{x}| \geq R. \end{cases} \quad (1)$$

Find the electrostatic potential, $\phi(\mathbf{x})$, of this charge distribution at all \mathbf{x} with $|\mathbf{x}| \geq R$.

Solution. The multipole expansion in spherical harmonics is given by Eq. (2.79) in the course notes,

$$\phi(\mathbf{x}) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \phi), \quad (2)$$

where the spherical multipole moments q_{lm} are defined in Eq. (2.80),

$$q_{lm} \equiv \int \rho(\mathbf{x}') r'^l Y_{lm}^*(\theta', \phi') d^3x'.$$

Note that (2) is valid only for $|\mathbf{x}| \geq R$ when the charge distribution $\rho(\mathbf{x}')$ is nonzero only within $|\mathbf{x}'| \leq R$, which is the regime we are interested in here.

The spherical harmonics Y_{lm} are given by Eq. (2.58),

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi},$$

and the Lagrange polynomials P_l^m are given by Eq. (2.59),

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l,$$

although in practice I am taking all spherical harmonics from the table in Jackson.

We can write the angular component of $\rho(\mathbf{x})$ as an expansion of spherical harmonics. Inspecting (1), we will only have terms of $l = 0, 1, 2$ and $m = 0$. The relevant spherical harmonics are

$$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{20}(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right).$$

Then we have

$$\begin{aligned} \rho(r, \theta, \phi) &= (R - r)(1 - 2 \cos \theta + \cos^2 \theta) \\ &= (R - r) \left(\frac{2}{3} \sqrt{\frac{4\pi}{5}} Y_{20}(\theta, \phi) - 2 \sqrt{\frac{4\pi}{3}} Y_{10}(\theta, \phi) + 4 \frac{\sqrt{4\pi}}{3} Y_{00}(\theta, \phi) \right). \end{aligned}$$

The only nonzero q_{lm} are q_{00} , q_{10} , and q_{20} :

$$\begin{aligned}
 q_{00} &= \int_0^{2\pi} \int_{-1}^1 \int_0^R \rho(\mathbf{x}') r'^0 Y_{00}^*(\theta', \phi') r' dr' d(\cos \theta') d\phi' \\
 &= 4 \frac{\sqrt{4\pi}}{3} \int_0^{2\pi} \int_{-1}^1 Y_{00}^*(\theta', \phi') Y_{00}(\theta', \phi') d(\cos \theta') d\phi' \int_0^R (R - r') r' dr' \\
 &= 4 \frac{\sqrt{4\pi}}{3} \left[\frac{Rr'^2}{2} - \frac{r'^3}{3} \right]_0^R = 4 \frac{\sqrt{4\pi}}{3} \frac{R^3}{6} = \frac{4\sqrt{\pi}}{9} R^3, \\
 q_{10} &= \int_0^{2\pi} \int_{-1}^1 \int_0^R \rho(\mathbf{x}') r'^1 Y_{10}^*(\theta', \phi') r' dr' d(\cos \theta') d\phi' \\
 &= -2 \sqrt{\frac{4\pi}{3}} \int_0^{2\pi} \int_{-1}^1 Y_{10}^*(\theta', \phi') Y_{10}(\theta', \phi') d(\cos \theta') d\phi' \int_0^R (R - r') r'^2 dr' \\
 &= -2 \sqrt{\frac{4\pi}{3}} \left[\frac{Rr'^3}{3} - \frac{r'^4}{4} \right]_0^R = -2 \sqrt{\frac{4\pi}{3}} \frac{R^4}{12} = -\frac{1}{3} \sqrt{\frac{\pi}{3}} R^4, \\
 q_{20} &= \int_0^{2\pi} \int_{-1}^1 \int_0^R \rho(\mathbf{x}') r'^2 Y_{20}^*(\theta', \phi') r' dr' d(\cos \theta') d\phi' \\
 &= \frac{2}{3} \sqrt{\frac{4\pi}{5}} \int_0^{2\pi} \int_{-1}^1 Y_{20}^*(\theta', \phi') Y_{20}(\theta', \phi') d(\cos \theta') d\phi' \int_0^R (R - r') r'^3 dr' \\
 &= \frac{2}{3} \sqrt{\frac{4\pi}{5}} \left[\frac{Rr'^4}{4} - \frac{r'^5}{5} \right]_0^R = \frac{2}{3} \sqrt{\frac{4\pi}{5}} \frac{R^5}{20} = \frac{1}{15} \sqrt{\frac{\pi}{5}} R^5.
 \end{aligned}$$

Then ϕ is given by

$$\begin{aligned}
 \phi(\mathbf{x}) &= \frac{4\pi}{1} \frac{q_{00}}{r^1} Y_{00}(\theta, \phi) + \frac{4\pi}{2+1} \frac{q_{10}}{r^2} Y_{10}(\theta, \phi) + \frac{4\pi}{5} \frac{q_{20}}{r^3} Y_{20}(\theta, \phi) \\
 &= (4\pi) \frac{4\sqrt{\pi}}{9} \frac{R^3}{r} \frac{1}{\sqrt{4\pi}} - \frac{4\pi}{3} \frac{1}{3} \sqrt{\frac{\pi}{3}} \frac{R^4}{r^2} \sqrt{\frac{3}{4\pi}} \cos \theta + \frac{4\pi}{5} \frac{1}{15} \sqrt{\frac{\pi}{5}} \frac{R^5}{r^3} \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \\
 &= \frac{8\pi}{9} \frac{R^3}{r} - \frac{2\pi}{9} \frac{R^4}{r^2} \cos \theta + \frac{\pi}{75} \frac{R^5}{r^3} (2 \cos^2 \theta - 1).
 \end{aligned}$$

Problem 2. Let \mathcal{V} be an arbitrary bounded region of space and suppose that a total charge Q is to be distributed in \mathcal{V} in an arbitrary way, with $\rho = 0$ outside of \mathcal{V} . Show that the total energy is minimized if the charge is distributed the way that it would be if \mathcal{V} were a conductor, so that $\phi = \text{const.}$ within \mathcal{V} (and thus, in particular, all of the charge lies on the boundary of \mathcal{V}).

Hint: Let $\phi_0(\mathbf{x})$ be the potential one would obtain if \mathcal{V} were filled by a conducting body. Consider the energy of $\phi_0 + \phi'$, where the source ρ' of ϕ' vanishes outside of \mathcal{V} and has no net charge within \mathcal{V} .

Solution. Let $S = \partial\mathcal{V}$ denote the boundary of \mathcal{V} . We separate space into three mutually exclusive regions: \mathcal{V} , S , and the region outside (in which we are not interested). By the superposition principle, we may write

$$\rho = \rho_0 + \rho', \quad \phi = \phi_0 + \phi',$$

where ρ_0 is the charge of a conducting body filling \mathcal{V} , ϕ_0 is the electrostatic potential due to ρ_0 , ρ' is the charge distribution within \mathcal{V} , and ϕ' is the electrostatic potential due to ρ' . In order to eliminate ambiguity on the boundary, we require

$$\rho_0|_{\mathcal{V}} = 0, \quad \rho'|_S = 0. \quad (3)$$

That is, $\rho_0 = 0$ inside the conductor by definition, and ρ' vanishes on the boundary where ρ_0 is nonzero. For the entire body to have charge Q , we need

$$\int \rho_0 d^3x = \int_{\mathcal{V}} \rho' d^3x + \int_S \rho_0 dS = Q.$$

From (3), it follows that

$$\phi_0|_{\mathcal{V}} = \phi_0|_S = \text{const.}$$

The total energy is given by Eq. (2.25) in the course notes,

$$\mathcal{E} = \frac{1}{2} \int \phi \rho d^3x.$$

So

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \int (\phi_0 + \phi')(\rho_0 + \rho') d^3x = \frac{1}{2} \left(\int \phi_0(\rho_0 + \rho') d^3x + \int \phi'(\rho_0 + \rho') d^3x \right) \\ &= \frac{1}{2} \left(\phi_0 Q + \int_{\mathcal{V}} \phi' \rho' d^3x + \int_S \phi' \rho_0 dS \right). \end{aligned}$$

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Problem 3. Charge is distributed on a (nonconducting) sphere of radius R , i.e., the charge density throughout space is of the form $\rho(\mathbf{x}) = \sigma(\theta, \phi) \delta(r - R)$. The surface charge distribution σ on the sphere is chosen in such a way that the electrostatic potential on the sphere is $\phi(r = R, \theta, \varphi) = \alpha \cos \theta$, where α is a constant.

3.a Find the electrostatic potential $\phi(\mathbf{x})$ at all $r \leq R$.

Solution. This is a Dirichlet boundary value problem. We are seeking the solution to Poisson's equation $\nabla^2 \phi = -4\pi\rho$ subject to $\phi|_S = \psi = \alpha \cos \theta$. Equation (2.100) in the lecture notes gives the general solution,

$$\phi(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x' - \frac{1}{4\pi} \int_S \psi(\mathbf{x}') \hat{\mathbf{n}} \cdot \nabla_{\mathbf{x}'} G_D(\mathbf{x}', \mathbf{x}) dS_{\mathbf{x}'} . \quad (4)$$

The Green's function for a sphere is given by Eq. (2.91),

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + \frac{\beta}{|\mathbf{x} - \mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}' \frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \beta = -\frac{R}{|\mathbf{x}'|}.$$

The Green's function for unbounded space, $G(\mathbf{x}, \mathbf{x}')$, can be expanded in spherical harmonics according to Eq. (2.78):

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \begin{cases} \sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) & \text{if } r < r', \\ \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) & \text{if } r > r'. \end{cases}$$

Expanding $G_D(\mathbf{x}, \mathbf{x}')$ in spherical harmonics in the region $r < r'$ gives us

$$\begin{aligned} G_D(\mathbf{x}, \mathbf{x}') &= \sum_{l,m} \left(\frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) + \beta \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} Y_{lm}^*(\theta'', \phi'') Y_{lm}(\theta, \phi) \right) \\ &= \sum_{l,m} \frac{4\pi}{2l+1} r^l Y_{lm}(\theta, \phi) \left(\frac{1}{r'^{l+1}} Y_{lm}^*(\theta', \phi') + \frac{\beta}{r'^{l+1}} Y_{lm}^*(\theta'', \phi'') \right) \\ &= \sum_{l,m} \frac{4\pi}{2l+1} \frac{r^l}{R} Y_{lm}(\theta, \phi) \left[\frac{R}{r'^{l+1}} Y_{lm}^*(\theta', \phi') - \frac{r'^l}{R^{2l}} Y_{lm}^* \left(\frac{R^2 \theta'}{r'^2}, \frac{R^2 \phi'}{r'^2} \right) \right]. \end{aligned}$$

We can also write ψ in terms spherical harmonics:

$$\psi(r = R, \theta, \varphi) = \alpha \sqrt{\frac{4\pi}{3}} Y_{10}(\theta, \phi) = \alpha \sqrt{\frac{4\pi}{3}} Y_{10}^*(\theta, \phi).$$

Since charge is distributed only on the surface of the sphere, the first integral in (4) is 0. For the second integral, note that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ for a sphere. Making the substitutions from the problem statement, we have

$$\phi(\mathbf{x}) = -\frac{\alpha}{4\pi} \int_S \cos \theta' \hat{\mathbf{r}}' \cdot \nabla_{\mathbf{x}'} G_D(\mathbf{x}', \mathbf{x}) dS_{x'} = -\frac{\alpha}{4\pi} \int_0^{2\pi} \int_{-1}^1 \int_0^R \cos \theta' \frac{\partial G_D(\mathbf{x}', \mathbf{x})}{\partial r'} \delta(r' - R) dr' d(\cos \theta') d\varphi'. \quad (5)$$

Note that

$$\frac{\partial G_D(\mathbf{x}', \mathbf{x})}{\partial r'} = \sum_{l,m} 4\pi \frac{l}{2l+1} \frac{r'^{l-1}}{R} Y_{lm}(\theta', \phi') \left[\frac{R}{r'^{l+1}} Y_{lm}^*(\theta, \phi) - \frac{r'^l}{R^{2l}} Y_{lm}^* \left(\frac{R^2 \theta}{r'^2}, \frac{R^2 \varphi}{r'^2} \right) \right],$$

and that (5) is only nonzero for the $l = 1, m = 0$ term of $\partial G_D(\mathbf{x}', \mathbf{x})/\partial r'$, which is

$$\frac{4\pi}{3} \frac{1}{R} Y_{10}(\theta', \phi') \left[\frac{R}{r'^2} Y_{10}^*(\theta, \phi) - \frac{r'}{R^2} Y_{10}^* \left(\frac{R^2 \theta}{r'^2}, \frac{R^2 \varphi}{r'^2} \right) \right] = \frac{4\pi}{3} Y_{10}(\theta', \phi') \left[\frac{1}{r'^2} Y_{10}^*(\theta, \phi) - \frac{r'}{R^3} Y_{10}^* \left(\frac{R^2 \theta}{r'^2}, \frac{R^2 \varphi}{r'^2} \right) \right].$$

So (5) becomes

$$\phi(\mathbf{x}) = -\frac{\alpha}{3} \sqrt{\frac{4\pi}{3}} \left[\frac{1}{r'^2} Y_{10}^*(\theta, \phi) - \frac{r'}{R^3} Y_{10}^* \left(\frac{R^2 \theta}{r'^2}, \frac{R^2 \varphi}{r'^2} \right) \right] = \frac{\alpha}{3} \left[\frac{r}{R^3} \cos \left(\frac{R^2 \theta}{r'^2} \right) - \frac{1}{r'^2} \cos \theta \right].$$

3.b Find the electrostatic potential $\phi(\mathbf{x})$ at all $r \geq R$.

Solution. Expanding $G_D(\mathbf{x}, \mathbf{x}')$ in spherical harmonics in the region $r > r'$ gives us

$$G_D(\mathbf{x}, \mathbf{x}') = \sum_{l,m} \frac{4\pi}{2l+1} \frac{1}{r'^{l+1} r'} Y_{lm}(\theta, \phi) \left[r'^{l+1} Y_{lm}^*(\theta', \phi') - \frac{R^{2l+1}}{r'^l} Y_{lm}^* \left(\frac{R^2 \theta'}{r'^2}, \frac{R^2 \phi'}{r'^2} \right) \right].$$

In this region,

$$\frac{\partial G_D(\mathbf{x}', \mathbf{x})}{\partial r'} = \sum_{l,m} -4\pi \frac{l+1}{2l+1} \frac{1}{r'^{l+2} r} Y_{lm}(\theta, \phi) \left[r'^{l+1} Y_{lm}^*(\theta, \phi) - \frac{R^{2l+1}}{r'^l} Y_{lm}^* \left(\frac{R^2 \theta}{r'^2}, \frac{R^2 \varphi}{r'^2} \right) \right].$$

We may once again use (5) with the $l = 1, m = 0$ term of $\partial G_D(\mathbf{x}', \mathbf{x})/\partial r'$, which is

$$-\frac{8\pi}{3} \frac{1}{r'^3 r} Y_{10}(\theta', \phi') \left[r^2 Y_{10}^*(\theta, \phi) - \frac{R^3}{r} Y_{10}^* \left(\frac{R^2 \theta}{r^2}, \frac{R^2 \varphi}{r^2} \right) \right] = -\frac{8\pi}{3} \frac{1}{r'^3} Y_{10}(\theta', \phi') \left[r Y_{10}^*(\theta, \phi) - \frac{R^3}{r^2} Y_{10}^* \left(\frac{R^2 \theta}{r^2}, \frac{R^2 \varphi}{r^2} \right) \right].$$

Now (5) becomes

$$\phi(\mathbf{x}) = -\frac{2\alpha}{3} \sqrt{\frac{4\pi}{3}} \frac{1}{r^3} \left[r Y_{10}^*(\theta, \phi) - \frac{R^3}{r^2} Y_{10}^* \left(\frac{R^2 \theta}{r^2}, \frac{R^2 \varphi}{r^2} \right) \right] = \frac{2\alpha}{3} \left[\frac{R^3}{r^5} \cos \left(\frac{R^2 \theta}{r^2} \right) - \frac{1}{r^2} \cos \theta \right].$$

3.c Find the surface charge density $\sigma(\theta, \varphi)$ that was required in order to produce this potential ϕ .

Solution. The surface charge density can be found from Eq. (2.86),

$$\mathbf{E} \cdot \hat{\mathbf{n}} = 4\pi\sigma, \quad (6)$$

where $\mathbf{E} = -\nabla\phi$ in electrostatics. The gradient in spherical coordinates is given by

$$\nabla = \frac{\partial}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \hat{\boldsymbol{\varphi}}.$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ for the inner surface of a sphere, we are interested in only the r component of the field. We will find this using the solution to (a):

$$E_r = \frac{\alpha}{3} \left[\frac{1}{R^3} \cos \left(\frac{R^2 \theta}{r^2} \right) - \frac{r}{R^3} \frac{2R^2 \theta}{r^3} \sin \left(\frac{R^2 \theta}{r^2} \right) + \frac{2}{r^3} \cos \theta \right] = \frac{\alpha}{3} \left[\frac{1}{R^3} \cos \left(\frac{R^2 \theta}{r^2} \right) - \frac{2\theta}{Rr^2} \sin \left(\frac{R^2 \theta}{r^2} \right) + \frac{2}{r^3} \cos \theta \right].$$

On the surface of the sphere, the field is $E_r(r = R) \hat{\mathbf{r}}$. So we have

3.d Find the total electrostatic energy.

Problem 4. A point charge of charge q is placed at point \mathbf{x}' inside a conducting spherical shell of radius R . There is no net charge on the conductor. The potential inside the sphere is thus given by $q G_D(\mathbf{x}, \mathbf{x}')$, where the explicit formula for $G_D(\mathbf{x}, \mathbf{x}')$ for a spherical cavity is given in the lecture notes.

4.a Find the surface charge density $\sigma(\theta, \varphi)$ on the conducting shell.

Solution. The Green's function for a spherical cavity is given by Eq. (2.91),

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + \frac{\alpha}{|\mathbf{x} - \mathbf{x}''|} \quad \text{where} \quad \mathbf{x}'' = \mathbf{x}' \frac{R^2}{|\mathbf{x}'|^2} \quad \text{and} \quad \alpha = -\frac{R}{|\mathbf{x}'|}.$$

The surface charge density can be found from Eq. (2.86),

$$\mathbf{E} \cdot \hat{\mathbf{n}} = 4\pi\sigma, \quad (7)$$

where $\mathbf{E} = -\nabla\phi$ in electrostatics.

We will begin by finding \mathbf{E} . We will orient our coordinate system such that \mathbf{x}' (and consequently \mathbf{x}'') points along the z axis. Note that

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{R}{|\mathbf{x}'| \left| \mathbf{x} - \frac{R^2}{|\mathbf{x}'|^2} \mathbf{x}' \right|} = \frac{1}{\sqrt{\mathbf{x}^2 - 2\mathbf{x} \cdot \mathbf{x}' + \mathbf{x}'^2}} - \frac{R}{|\mathbf{x}'| \sqrt{\mathbf{x}^2 - 2\frac{R^2}{\mathbf{x}'^2} \mathbf{x} \cdot \mathbf{x}' + \frac{R^4}{\mathbf{x}'^4} \mathbf{x}'^2}}.$$

In spherical coordinates, we have

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{r^2 - 2rr' \cos \theta + r'^2}} - \frac{R}{r'} \frac{1}{\sqrt{r^2 - 2R^2 r \cos \theta / r' + R^4 / r'^2}},$$

where we note that θ is the angle between \mathbf{x} and the z axis. The gradient in spherical coordinates is given by

$$\nabla = \frac{\partial}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \hat{\boldsymbol{\varphi}}.$$

The r component of the electric field inside the conductor is then

$$E_r(\mathbf{x}) = -q \frac{\partial G_D(\mathbf{x}, \mathbf{x}')}{\partial r} = q \left(\frac{r - r' \cos \theta}{(r^2 - 2rr' \cos \theta + r'^2)^{3/2}} - \frac{R}{r'} \frac{r - R^2 \cos \theta / r'}{(r^2 - 2R^2 r \cos \theta / r' + R^4 / r'^2)^{3/2}} \right).$$

Since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ for the inner surface of a sphere, we are interested in only the r component of the field. On the surface of the sphere, the field is $E_r(r = R) \hat{\mathbf{r}}$. So we have

$$\begin{aligned} E_r(r = R) &= q \left(\frac{R - r' \cos \theta}{(R^2 - 2Rr' \cos \theta + r'^2)^{3/2}} - \frac{R}{r'} \frac{R - R^2 \cos \theta / r'}{(R^2 - 2R^3 \cos \theta / r' + R^4 / r'^2)^{3/2}} \right) \\ &= q \left(\frac{R - r' \cos \theta}{r'^3 (R^2 / r'^2 - 2R \cos \theta / r' + 1)^{3/2}} - \frac{R}{r'} \frac{R - R^2 \cos \theta / r'}{R^3 (1 - 2R \cos \theta / r' + R^2 / r'^2)^{3/2}} \right) \\ &= \frac{q}{r'} \frac{R^3 - R^2 r' \cos \theta - R r'^2 + R^2 r' \cos \theta}{R^2 r'^2 (R^2 / r'^2 - 2R \cos \theta / r' + 1)^{3/2}} = \frac{q}{R r'^3} \frac{R^2 - r'^2}{(R^2 / r'^2 - 2R \cos \theta / r' + 1)^{3/2}}. \end{aligned}$$

Finally, feeding this into (7),

$$\sigma = -\frac{\mathbf{E} \cdot \hat{\mathbf{r}}}{4\pi} = \frac{q}{4\pi R r'^3} \frac{r'^2 - R^2}{(R^2 / r'^2 - 2R \cos \theta / r' + 1)^{3/2}} = \frac{q}{4\pi R |\mathbf{x}'|^3} \frac{|\mathbf{x}'|^2 - R^2}{(R^2 / |\mathbf{x}'|^2 - 2R \cos \theta / |\mathbf{x}'| + 1)^{3/2}}.$$

4.b Find the force \mathbf{F} that must be exerted on the point charge in order to hold it in place.

Solution. The total force on a charge distribution arises only from the external electric field \mathbf{E}_0 , and is given by Eq. (2.42) in the lecture notes:

$$\mathbf{F} = \int \rho(\mathbf{x}) \mathbf{E}_0(\mathbf{x}) d^3x.$$

We now need the θ component of the field inside the conductor, which is

$$E_\theta(\mathbf{x}) = -\frac{q}{r} \frac{\partial G_D(\mathbf{x}, \mathbf{x}')}{\partial \theta} = -q \left(\frac{r' \sin \theta}{(r^2 - 2rr' \cos \theta + r'^2)^{3/2}} - \frac{R^3 \sin \theta}{r'^2 (r^2 - 2R^2 r \cos \theta / r' + R^4 / r'^2)^{3/2}} \right).$$

The charge density for a point charge located at \mathbf{x}' is given by $\rho(\mathbf{x}) = q \delta(\mathbf{x} - \mathbf{x}')$. Evaluating the integral, we have

$$\mathbf{F} = \int q \delta(\mathbf{x} - \mathbf{x}') \mathbf{E}(\mathbf{x}) d^3x = q \mathbf{E}(\mathbf{x}').$$

Recall that we chose \mathbf{x}' to point along the z axis, so $\theta' = 0$. The θ component of \mathbf{F} is then 0, and the r component is

$$\begin{aligned} F_r &= q^2 \left(\frac{r' - r'}{(r'^2 - 2r'^2 + r'^2)^{3/2}} - \frac{R}{r'} \frac{r' - R^2/r'}{(r'^2 - 2R^2 + R^4/r'^2)^{3/2}} \right) = q^2 R r'^2 \frac{R^2/r' - r'}{(r'^4 - 2R^2 r'^2 + R^4)^{3/2}} \\ &= q^2 R r'^2 \frac{(R^2 - r'^2)/r'}{(R^2 - r'^2)^3} = q^2 \frac{R r'}{(R^2 - r'^2)^2}. \end{aligned}$$

Since only the r component of \mathbf{F} is nonzero, it points in the z direction, which we chose to be equivalent to the unit vector $\mathbf{x}'/|\mathbf{x}'|$. Therefore,

$$\mathbf{F} = q^2 \frac{R|\mathbf{x}'|}{(R^2 - |\mathbf{x}'|^2)^2} \frac{\mathbf{x}'}{|\mathbf{x}'|} = q^2 \frac{R}{(R^2 - |\mathbf{x}'|^2)^2} \mathbf{x}'.$$

Problem 5. The “mean value theorem” is stated as follows: For any solution ϕ to $\nabla^2 \phi = 0$, the value of ϕ at \mathbf{x} is equal to the average value of ϕ on a sphere of radius R (for any R) centered at \mathbf{x} .

5.a Prove the mean value theorem. Hint: Apply Green’s theorem to ϕ and $1/|\mathbf{x} - \mathbf{x}'|$ for a suitable choice of region and a suitable choice of \mathbf{x}' .

5.b Use this result to show that a point charge can never be in stable equilibrium if placed in an electric field \mathbf{E} that is source free in a neighborhood of the charge—and, indeed, it can be in neutral equilibrium only if $\mathbf{E} = 0$ in this neighborhood.