1 Problem 1

Let's consider coherent states of a one-dimensional quantum particle with mass m confined in a one-dimensional harmonic potential $V(X) = m\omega^2 X^2/2$:

$$a\,|\lambda\rangle = \lambda\,|\lambda\rangle\,, \qquad |\lambda\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\exp\left(\lambda a^\dagger\right)|0\rangle\,.$$

Here, λ is a complex parameter.

1.1 Compute $\langle x|\lambda\rangle$.

Solution. In terms of the position and momentum operators X and P,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + \frac{iP}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right),$$

so

$$\langle x|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\langle x|\exp\left(\lambda a^\dagger\right)|0\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\langle x|\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\}|0\rangle\,. \tag{1}$$

Note that for two operators A and B, $e^{A+B} = e^{-[A,B]/2}e^A e^B$ if $[A,B]$ commutes with each A and B. Note also that

$$\left[X, -\frac{iP}{m\omega}\right] = -\frac{i}{m\omega}[X,P] = \frac{\hbar}{m\omega}.$$

Thus,

$$\exp\left\{\lambda\sqrt{\frac{m\omega}{2\hbar}}\left(X - \frac{iP}{m\omega}\right)\right\} = \exp\left(-\lambda\frac{\hbar}{2m\omega}\sqrt{\frac{m\omega}{2\hbar}}\right)\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)\exp\left(-\lambda\frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right).$$

Now, note that

$$\exp\left(-\lambda\frac{i}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right) = \exp\left(-\frac{i}{\hbar}\lambda\frac{\hbar}{m\omega}\sqrt{\frac{m\omega}{2\hbar}}P\right) = U\left(\lambda\sqrt{\frac{\hbar}{2m\omega}}\right) \equiv U(b), \tag{2}$$

where $U(b)$ is the translation operator, and we have defined b. So (1) becomes

$$\begin{aligned}\langle x|\lambda\rangle &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(-\lambda\frac{\hbar}{2m\omega}\sqrt{\frac{m\omega}{2\hbar}}\right)\langle x|\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}X\right)U(b)\,|0\rangle \\ &= \exp\left(-\frac{|\lambda|^2}{2}\right)\exp\left(-\frac{b}{2}\right)\exp\left(\lambda\sqrt{\frac{m\omega}{2\hbar}}x\right)\langle x-b|0\rangle\,. \end{aligned}\tag{3}$$

From (2.3.30) in Sakurai,

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \implies \langle x-b|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}(x-b)^2\right).$$

so (3) becomes

$$\begin{aligned}\langle x|\lambda\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{b}{2} + \lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega}{2\hbar}(x^2 - 2bx + b^2)\right) \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{b}{2} + \lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega}{2\hbar}x^2 + \lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{m\omega}{2\hbar}\lambda^2\frac{\hbar}{2m\omega}\right) \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{b}{2} - \frac{m\omega}{2\hbar}x^2 + 2\lambda\sqrt{\frac{m\omega}{2\hbar}}x - \frac{\lambda^2}{4}\right) \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{|\lambda|^2}{2} - \frac{m\omega}{2\hbar}x^2 + \lambda\sqrt{\frac{2m\omega}{\hbar}}x\right), \end{aligned}\tag{4}$$

where we have dropped a constant phase.

1.2 Compute $\langle\lambda|X|\lambda\rangle$, $\langle\lambda|P|\lambda\rangle$, $\langle\lambda|X^2|\lambda\rangle$, and $\langle\lambda|P^2|\lambda\rangle$. Also, compute $\langle\lambda|(\Delta X)^2|\lambda\rangle\,\langle\lambda|(\Delta P)^2|\lambda\rangle$ where $\Delta A = A - \langle A\rangle$.

Solution. From (2.3.24) in Sakurai,

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a). \tag{5}$$

Then for $\langle\lambda|X|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|X|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger X a|\lambda\rangle \\
&= \frac{1}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|a^\dagger(a + a^\dagger)a|\lambda\rangle = \frac{1}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|(a^\dagger a^2 + a^{\dagger 2}a|\lambda\rangle = \frac{|\lambda|^2(\lambda^* + \lambda)}{|\lambda|^2}\sqrt{\frac{\hbar}{2m\omega}} \\
&= 2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar}{2m\omega}},
\end{aligned} \tag{6}$$

and for $\langle\lambda|P|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|P|\lambda\rangle &= \frac{1}{\lambda^2}\langle\lambda|a^\dagger P a|\lambda\rangle \\
&= \frac{i}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}}\langle\lambda|a^\dagger(a^\dagger - a)a|\lambda\rangle = \frac{i}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}}\langle\lambda|a^{\dagger 2}a - a^\dagger a^2|\lambda\rangle = \frac{i|\lambda|^2(\lambda^* - \lambda)}{|\lambda|^2}\sqrt{\frac{\hbar m\omega}{2}} \\
&= 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar m\omega}{2}}.
\end{aligned} \tag{7}$$

From (5), note that

$$X^2 = \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}), \qquad P^2 = -\frac{\hbar m\omega}{2}(a^{\dagger 2} - a^\dagger a - aa^\dagger + a^2).$$

Then for $\langle\lambda|X^2|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|X^2|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger X^2 a|\lambda\rangle = \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|a^\dagger(a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2})a|\lambda\rangle \\
&= \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger a^3 + a^\dagger aa^\dagger a + a^{\dagger 2}a^2 + a^{\dagger 3}a)|\lambda\rangle = \frac{1}{|\lambda|^2}\frac{\hbar}{2m\omega}\langle\lambda|(a^\dagger a^3 + a^\dagger a + 2a^{\dagger 2}a^2 + a^{\dagger 3}a)|\lambda\rangle \\
&= (\lambda^2 + 1 + 2|\lambda|^2 + \lambda^{*2})\frac{\hbar}{2m\omega} = \left(1 + 2\left[\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2\right] + 2\left[\mathrm{Re}(\lambda)^2 - \mathrm{Im}(\lambda)^2\right]\right)\frac{\hbar}{2m\omega} \\
&= [1 + 4\,\mathrm{Re}(\lambda)^2]\frac{\hbar}{2m\omega},
\end{aligned} \tag{8}$$

where we have used $[a, a^\dagger] = 1$. For $\langle\lambda|P^2|\lambda\rangle$,

$$\begin{aligned}
\langle\lambda|P^2|\lambda\rangle &= \frac{1}{|\lambda|^2}\langle\lambda|a^\dagger P^2 a|\lambda\rangle = -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|a^\dagger(a^{\dagger 2} - a^\dagger a - aa^\dagger + a^2)a|\lambda\rangle \\
&= -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|(a^{\dagger 2}a - a^{\dagger 2}a^2 - a^\dagger aa^\dagger a + a^\dagger a^3|\lambda\rangle = -\frac{1}{|\lambda|^2}\frac{\hbar m\omega}{2}\langle\lambda|(a^{\dagger 3}a - a^\dagger a - 2a^{\dagger 2}a^2 + a^\dagger a^3|\lambda\rangle \\
&= -(\lambda^{*2} - 1 - 2|\lambda|^2 + \lambda^2)\frac{\hbar m\omega}{2} = \left(1 + 2\left[\mathrm{Re}(\lambda)^2 + \mathrm{Im}(\lambda)^2\right] - 2[\mathrm{Re}(\lambda)^2 - \mathrm{Im}(\lambda)^2]\right)\frac{\hbar m\omega}{2} \\
&= [1 + 4\,\mathrm{Im}(\lambda)^2]\frac{\hbar m\omega}{2}.
\end{aligned} \tag{9}$$

From (1.4.51) in Sakurai, $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$. Then

$$\langle\lambda|(\Delta X)^2|\lambda\rangle = \langle\lambda|X^2|\lambda\rangle - \langle\lambda|X|\lambda\rangle^2 = [1+4\,\mathrm{Re}(\lambda)^2]\frac{\hbar}{2m\omega} - 4\,\mathrm{Re}(\lambda)^2\frac{\hbar}{2m\omega} = \frac{\hbar}{2m\omega},$$

where we have used (6) and (8), and

$$\langle\lambda|(\Delta P)^2|\lambda\rangle = \langle\lambda|P^2|\lambda\rangle - \langle\lambda|P|\lambda\rangle^2 = [1+4\,\mathrm{Im}(\lambda)^2]\frac{\hbar m\omega}{2} - 4\,\mathrm{Im}(\lambda)^2\frac{\hbar m\omega}{2} = \frac{\hbar m\omega}{2},$$

where we have used (7) and (9). Finally,

$$\langle\lambda|(\Delta X)^2|\lambda\rangle\,\langle\lambda|(\Delta P)^2|\lambda\rangle = \frac{\hbar^2}{4},$$

which shows that the coherent state $|\lambda\rangle$ satisfies the minimum uncertainty relation.

1.3 Starting from $|\psi(0)\rangle = |\lambda\rangle$ at $t=0$, we let $|\psi(t)\rangle$ evolve in time. What is the state $|\psi(t)\rangle$ for $t>0$?

Solution. From (2.3.43) in Sakurai,

$$a(t) = ae^{-i\omega t}, \qquad a^\dagger(t) = a^\dagger e^{i\omega t},$$

where $a = a(0)$ and $a^\dagger = a^\dagger(0)$. Equating the Schrödinger and Heisenberg pictures,

$$|\psi(0)\rangle = |\lambda\rangle = \frac{1}{\lambda}a\,|\lambda\rangle \implies |\psi(t)\rangle = \frac{1}{\lambda}a(t)\,|\lambda\rangle\,,$$

and so

$$|\psi(t)\rangle = \frac{1}{\lambda}ae^{-i\omega t}\,|\lambda\rangle = e^{-i\omega t}\,|\lambda\rangle\,.$$

1.4 Compute $\langle\psi(t)|X|\psi(t)\rangle$ and $\langle\psi(t)|P|\psi(t)\rangle$, and their time derivatives $d\langle X\rangle/dt$ and $d\langle P\rangle/dt$.

Solution. Firstly, we have

$$\langle\psi(t)|X|\psi(t)\rangle = \langle\lambda|\,e^{i\omega t}Xe^{-i\omega t}\,|\lambda\rangle = \langle\lambda|X|\lambda\rangle = 2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar}{2m\omega}},$$
$$\langle\psi(t)|P|\psi(t)\rangle = \langle\lambda|\,e^{i\omega t}Pe^{-i\omega t}\,|\lambda\rangle = \langle\lambda|P|\lambda\rangle = 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar m\omega}{2}},$$

where we have used (6) and (7).

For the time derivatives, note that the harmonic oscillator Hamiltonian is given by

$$H = \frac{P^2}{2m} + \frac{m\omega^2X^2}{2}.$$

Then, using the Ehrenfest theorem and the other results of problem 4.1 of the previous homework,

$$\frac{d\langle X\rangle}{dt} = -\frac{i}{\hbar}\langle\psi(t)|[X,H]|\psi(t)\rangle = \frac{1}{m}\langle\psi(t)|P|\psi(t)\rangle = 2\,\mathrm{Im}(\lambda)\sqrt{\frac{\hbar\omega}{2m}},$$
$$\frac{d\langle P\rangle}{dt} = -\frac{i}{\hbar}\langle\psi(t)|[P,H]|\psi(t)\rangle = -m\omega^2\langle\psi(t)|X|\psi(t)\rangle = -2\,\mathrm{Re}(\lambda)\sqrt{\frac{\hbar m\omega^3}{2}},$$

which again are similar to the classical equations of motion.

1.5 Compute $\langle \lambda'' | \exp(-iHt/\hbar) | \lambda' \rangle$.

Solution. Note that $U(t) = \exp(-iHt/\hbar)$ where $U(t)$ is the time evolution operator. From problem 1.3,

$$|\psi(t)\rangle = U(t)\,|\lambda\rangle \implies \exp\left(-\frac{iHt}{\hbar}\right)|\lambda'\rangle = e^{-i\omega t}\,|\lambda'\rangle\,,$$

so

$$\langle \lambda'' | \exp\left(-\frac{iHt}{\hbar}\right) | \lambda' \rangle = e^{-i\omega t}\,\langle \lambda'' | \lambda' \rangle\,.$$

Using the power series representation,

$$|\lambda\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n a^{\dagger n}}{n!}\,|0\rangle = \exp\left(-\frac{|\lambda|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\,|n\rangle\,,$$

so

$$\langle \lambda'' | \lambda' \rangle = \exp\left(-\frac{|\lambda''|^2}{2}\right)\exp\left(-\frac{|\lambda'|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\lambda''^*\lambda')^n}{n!}\,\langle n|n\rangle = \exp\left(-\frac{|\lambda''|^2}{2} + \lambda''^*\lambda' - \frac{|\lambda'|^2}{2}\right).$$

Finally,

$$\langle \lambda'' | \exp\left(-\frac{iHt}{\hbar}\right) | \lambda' \rangle = \exp\left(-i\omega t - \frac{|\lambda''|^2}{2} + \lambda''^*\lambda' - \frac{|\lambda'|^2}{2}\right).$$

2 Problem 2

Consider a quantum system which has coordinate X_1 and momentum P_1, and another system which has coordinate X_2 and momentum P_2. (An operator from the first system always commutes with an operator of the second system.) We think of the second system as a "probe" which we can use to detect the properties of the first system. For a short time T, the two systems are coupled by a coupling Hamiltonian H_c, given by

$$H_c = \frac{X_1 P_2}{T}.$$

The coupling between the two systems disturbs the momentum of the first system. The disturbance operator is defined to be

$$D \equiv P_1(T) - P_1(0). \tag{10}$$

The probe introduces measurement error or "noise" into the system. The noise operator is defined by

$$N \equiv X_2(T) - X_1(0).$$

The stste of the system at $t = 0$ is $|\Psi(0)\rangle = |\phi_1(0)\,\phi_2(0)\rangle$, and all expectation values are taken in this state.

2.1 With H_c as the Hamiltonian, find the Heisenberg operators $X_1(t)$, $P_1(t)$, $X_2(t)$, and $P_2(t)$ in terms of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$. Time is restricted to the range $t \in [0, T]$.

Solution. In general, a Heisenberg operator $O(t)$ is defined by

$$O(t) = U^\dagger(t)\, O(0)\, U(t),$$

where $U(t)$ is the time evolution operator. For H_c, it is given by

$$U(t) = \exp\left(-\frac{iH_c t}{\hbar}\right) = \exp\left(-\frac{it}{\hbar T}\, X_1(0)\, P_2(0)\right).$$

(2.2.23b) in Sakurai gives the commutation relations

$$[X_i, F(\mathbf{P})] = i\hbar \frac{\partial F}{\partial P_i} \qquad\qquad [P_i, G(\mathbf{X})] = -i\hbar \frac{\partial G}{\partial X_i}.$$

Using these, we have

$$\begin{aligned}
&[X_1(0), U(t)] = 0,\\
&[X_2(0), U(t)] = i\hbar\left(-\frac{it}{\hbar T}\, X_1(0)\right) U(t) = \frac{t}{T}\, X_1(0)\, U(t) = \frac{t}{T}\, U(t)\, X_1(0),\\
&[P_1(0), U(t)] = -i\hbar\left(-\frac{it}{\hbar T}\, P_2(0)\right) U(t) = -\frac{t}{T}\, P_2(0)\, U(t) = -\frac{t}{T}\, U(t)\, P_2(0),\\
&[P_2(0), U(t)] = 0.
\end{aligned}$$

Then

$$X_1(t) = U^\dagger(t)\, X_1(0)\, U(t) = X_1(0), \tag{11}$$

$$P_1(t) = U^\dagger(t)\, P_1(0)\, U(t) = U^\dagger(t)\left(U(t)\, P_1(0) - \frac{t}{T} U(t)\, P_2(0)\right) = P_1(0) - \frac{t}{T}\, P_2(0), \tag{12}$$

$$X_2(t) = U^\dagger(t)\, X_2(0)\, U(t) = U^\dagger(t)\left(U(t)\, X_2(0) + \frac{t}{T}\, U(t)\, X_1(0)\right) = X_2(0) + \frac{t}{T}\, X_1(0), \tag{13}$$

$$P_2(t) = U^\dagger(t)\, P_2(0)\, U(t) = P_2(0). \tag{14}$$

2.2 Derive an expression for $\sigma(D)$ which involves only the standard deviations of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$. Here, we denote the standard deviation of an operator O as $\sigma(O) = \sqrt{\langle (O - \langle O\rangle)^2\rangle}$.

Solution. Substituting (14) into (10),

$$D = P_1(0) - \frac{T}{T} P_2(0) - P_2(0) = -P_2(0).$$

Note that for an operator O,

$$\sigma(-O) = \sqrt{\langle (-O - \langle -O\rangle)^2\rangle} = \sqrt{\langle (\langle O\rangle - O)^2\rangle} = \sigma(O),$$

so

$$\sigma(D) = \sigma\big(P_2(0)\big). \tag{15}$$

2.3 Derive an expression for $\sigma(N)$ which involves only the standard deviations of $X_1(0)$, $P_1(0)$, $X_2(0)$, and $P_2(0)$.

Solution. Substituting (13) into (10),

$$N = X_2(0) + \frac{T}{T} X_1(0) - X_1(0) = X_2(0).$$

which implies

$$\sigma(N) = \sigma\big(X_2(0)\big). \tag{16}$$

2.4 Now consider the product $\sigma(N)\,\sigma(D)$. Assume

$$\sigma\big(X_1(0)\big)\,\sigma\big(P_1(0)\big) \geq \frac{\hbar}{2}, \qquad \sigma\big(X_2(0)\big)\,\sigma\big(P_2(0)\big) \geq \frac{\hbar}{2}$$

both hold. Is $\sigma(N)\,\sigma(D) \geq \hbar/2$ satisfied? What conditions are required for equality?

Solution. From (15) and (16),

$$\sigma(N)\,\sigma(D) = \sigma\big(P_2(0)\big)\,\sigma\big(X_2(0)\big) \geq \frac{\hbar}{2},$$

where the final inequality is satisfied by assumption. For equality, we would need

$$\sigma\big(X_2(0)\big)\,\sigma\big(P_2(0)\big) = \frac{\hbar}{2}.$$

3 Problem 3

Answer the following questions about the angular momentum operator L_i.

3.1 Calculate $[L_i, \mathbf{r}]$ where $i = x, y, z$.

Solution. Firstly, note that

$$L_x = YP_z - ZP_y, \qquad L_y = ZP_x - XP_z, \qquad L_z = XP_y - YP_x,$$

where the expression for L_z was given in problem 2 of Homework 1, and L_x and L_y are cyclic permutations. Then

$$\begin{aligned}
[L_x, X] &= (YP_z - ZP_y)X - X(YP_z - ZP_y) = 0, \\
[L_x, Y] &= (YP_z - ZP_y)Y - Y(YP_z - ZP_y) = YP_zY - ZP_yY - YYP_z + YZP_y = [Y, P_y]Z = i\hbar Z, \\
[L_x, Z] &= (YP_z - ZP_y)Z - Z(YP_z - ZP_y) = YP_zZ - ZP_yZ - ZYP_z + ZZP_y = -[Z, P_z]Y = -i\hbar Y.
\end{aligned}$$

Generalizing these results to L_y and L_z,

$$[L_x, \mathbf{r}] = i\hbar \begin{bmatrix} 0 \\ Z \\ -Y \end{bmatrix}, \qquad [L_y, \mathbf{r}] = i\hbar \begin{bmatrix} -Z \\ 0 \\ X \end{bmatrix}, \qquad [L_z, \mathbf{r}] = i\hbar \begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix},$$

where $\mathbf{r} = \begin{bmatrix} X & Y & Z \end{bmatrix}^T$.

3.2 Let us now compare the above results with classical mechanics. Rotations around the x, y, and z axes by an angle θ in three-dimensional Cartesian space are represented by the following matrices:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate $R_i(\theta)\,\mathbf{r}$. Then expand $R_i(\theta)\,\mathbf{r}$ for a small angle θ and consider $\mathbf{r} - R_i(\theta)\,\mathbf{r}$ to first order in θ,

$$\mathbf{r} - R_i(\theta)\,\mathbf{r} = \theta M_i \mathbf{r} + \mathcal{O}\big(\theta^2\big).$$

Calculate the matrices M_i.

Solution. For $R_i(\theta)\,\mathbf{r}$, we have

$$R_x(\theta)\,\mathbf{r} = \begin{bmatrix} X \\ \cos\theta Y - \sin\theta Z \\ \sin\theta Y + \cos\theta Z \end{bmatrix}, \qquad R_y(\theta)\,\mathbf{r} = \begin{bmatrix} \cos\theta X + \sin\theta Z \\ Y \\ \cos\theta Z - \sin\theta X \end{bmatrix}, \qquad R_z(\theta)\,\mathbf{r} = \begin{bmatrix} \cos\theta X - \sin\theta Y \\ \sin\theta X + \cos\theta Y \\ Z \end{bmatrix},$$

In the small angle approximation, to first order $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. In this approximation,

$$R_x(\theta)\,\mathbf{r} \approx \begin{bmatrix} X \\ Y - \theta Z \\ \theta Y + Z \end{bmatrix}, \qquad R_y(\theta)\,\mathbf{r} \approx \begin{bmatrix} X + \theta Z \\ Y \\ Z - \theta X \end{bmatrix}, \qquad R_z(\theta)\,\mathbf{r} \approx \begin{bmatrix} X - \theta Y \\ \theta X + Y \\ Z \end{bmatrix},$$

and so

$$\mathbf{r} - R_x(\theta)\,\mathbf{r} \approx \begin{bmatrix} 0 \\ \theta Z \\ -\theta Y \end{bmatrix}, \qquad \mathbf{r} - R_y(\theta)\,\mathbf{r} \approx \begin{bmatrix} -\theta Z \\ 0 \\ \theta X \end{bmatrix}, \qquad \mathbf{r} - R_z(\theta)\,\mathbf{r} \approx \begin{bmatrix} \theta Y \\ -\theta X \\ 0 \end{bmatrix}.$$

These results suggest the matrices

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad M_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad M_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.3 Calculate the matrix elements of the angular momentum operator L_i in the basis ket $|l, m\rangle$ when $l = 1$ and $l = 2$. Here, $|l, m\rangle$ is the simultaneous eigenket of L^2 and L_z with the eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$, respectively.

Solution. The ladder operators are defined by (3.5.5) in Sakurai:

$$J_\pm = L_x \pm i L_y.$$

Clearly,

$$L_x = \frac{J_+ + J_-}{2}, \qquad L_y = \frac{J_+ - J_-}{2i}$$

From (3.5.39) and (3.5.40),

$$J_+ \,|l, m\rangle = \sqrt{(l-m)(l+m+1)}\hbar\,|l, m+1\rangle\,, \qquad J_- \,|l, m\rangle = \sqrt{(l+m)(l-m+1)}\hbar\,|l, m-1\rangle\,.$$

Then the matrix elements of L_x are given by

$$\begin{aligned}
\langle 1, m'|L_x|1, m\rangle &= \langle 1, m'|\frac{J_+ + J_-}{2}|1, m\rangle = \frac{\hbar}{2}\left(\delta_{m+1,m'}\sqrt{2 - m - m^2} + \delta_{m-1,m'}\sqrt{2 + m - m^2}\right), \\
\langle 1, m'|L_x|2, m\rangle &= 0, \\
\langle 2, m'|L_x|1, m\rangle &= 0, \\
\langle 2, m'|L_x|2, m\rangle &= \langle 1, m'|\frac{J_+ + J_-}{2}|1, m\rangle = \frac{\hbar}{2}\left(\delta_{m+1,m'}\sqrt{6 - m - m^2} + \delta_{m-1,m'}\sqrt{6 + m - m^2}\right),
\end{aligned}$$

where the integers $m, m' \in [-l, l]$. For $l = l' = 1$, they are either 0 or $\hbar/\sqrt{2}$. For $l = l' = 2$, they are either 0, $\hbar$, or $\hbar\sqrt{3/2}$.

The matrix elements of L_y are given by

$$\begin{aligned}
\langle 1, m'|L_y|1, m\rangle &= \langle 1, m'|\frac{J_+ - J_-}{2i}|1, m\rangle = -\frac{i\hbar}{2}\left(\delta_{m+1,m'}\sqrt{2 - m - m^2} - \delta_{m-1,m'}\sqrt{2 + m - m^2}\right), \\
\langle 1, m'|L_y|2, m\rangle &= 0, \\
\langle 2, m'|L_y|1, m\rangle &= 0, \\
\langle 2, m'|L_y|2, m\rangle &= \langle 1, m'|\frac{J_+ - J_-}{2i}|1, m\rangle = -\frac{i\hbar}{2}\left(\delta_{m+1,m'}\sqrt{6 - m - m^2} - \delta_{m-1,m'}\sqrt{6 + m - m^2}\right),
\end{aligned}$$

where again $m, m' \in [-l, l]$. For $l = l' = 1$, they are either 0 or $-i\hbar/\sqrt{2}$. For $l = l' = 2$, they are either 0, $-i\hbar$, or $-i\hbar\sqrt{3/2}$.

Since $|l, m\rangle$ are eigenkets of L_z, its matrix elements are given by

$$\langle l', m'|L_y|l, m\rangle = \hbar m\, \delta_{m,m'}\, \delta_{l,l'},$$

where $l, l' \in \{1, 2\}$ and $m, m' \in [-l, l]$.