1 Problem 1

A particle of mass m is moving on a sphere of radius a. Its wave function is given by $\psi(\theta, \phi)$ where θ and ϕ parameterize the sphere $(x, y, z) = a(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta)$. The Hamiltonian of the system is $H = \mathbf{L}^2/2ma^2$, where $\mathbf{L}^2$ is the square of the angular momentum operator, and is given by

$$\mathbf{L}^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right).$$

The eigenfunctions of H are spherical harmonics Y^l_m with energies

$$E_l = \frac{\hbar^2 l(l+1)}{2ma^2}. \tag{1}$$

1.1 The wave function of the system at $t = 0$ is given by

$$\psi(\theta, \phi, 0) = A \sin^2\theta\,\cos^2\phi,$$

where A is a constant. This wave function can be expanded in spherical harmonics:

$$\psi(\theta, \phi, 0) = \sum_{l,m} a^l_m Y^l_m(\theta, \phi).$$

Find all nonzero a^l_m.

Solution. We will look for nonzero a^l_m by comparing the θ and ϕ dependence of Y^l_m and $\psi(\theta, \phi, 0)$. From (3.6.36) in Sakurai, the spherical harmonic functions are given by

$$Y^l_m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)}{4\pi} \frac{(l+m)!}{(l-m)!}} \frac{e^{im\phi}}{sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l}$$

for $m \geq 0$. From (3.6.37),

$$Y^l_{-m}(\theta, \phi) = (-1)^m\, {Y^l_m}^*(\theta, \phi)$$

for $m < 0$. Beginning with the ϕ dependence of $\psi(\theta, \phi, 0)$, note that

$$\psi(\theta, \phi, 0) \propto \cos^2\phi = \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right)^2 = \frac{1}{2} + \frac{e^{i2\phi}}{4} + \frac{e^{-i2\phi}}{4}, \tag{2}$$

which implies that the only nonzero a^l_m correspond to $m \in \{0, \pm 2\}$.

For the θ dependence, we have $\psi(\theta, \phi, 0) \propto \sin^2\theta$. Looking at Y^l_m, note that $(\sin\theta)^{2l} = (1 - \cos^2\theta)^l$, so

$$Y^l_m \propto \frac{1}{\sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (1 - \cos^2\theta)^l.$$

For $m = 0$,

$$Y^l_0 \propto \frac{d^l}{d(\cos\theta)^l} (1 - \cos^2\theta)^l.$$

Plugging in the first few values of l,

$$\begin{aligned}
Y_0^0 &\propto \frac{d^0}{d(\cos\theta)^0}(1-\cos^2\theta)^0 = 1,\\
Y_0^1 &\propto \frac{d}{d(\cos\theta)}(1-\cos^2\theta) = -2\cos\theta,\\
Y_0^2 &\propto \frac{d^2}{d(\cos\theta)^2}(1-2\cos^2\theta+\cos^4\theta) = \frac{d}{d(\cos\theta)}(-4\cos\theta+4\cos^3\theta) = -4+12\cos^2\theta = 8-12\sin^2\theta,
\end{aligned}$$

so we know $a_0^1 = 0$. Inspecting the above, we deduce that Y_0^l with $l > 2$ contain mixed terms of $\sin\theta$ and $\cos\theta$ and higher powers of $\sin\theta$, so $a_0^l = 0$ for $l > 2$.

For $m = \pm 2$,

$$Y_{\pm 2}^l \propto \frac{1}{\sin^2\theta}\sin^2\theta\frac{d^{l-2}}{d(\cos\theta)^{l-2}}(1-\cos^2\theta)^l.$$

Plugging in $l = 2$,

$$Y_{\pm 2}^2 \propto \frac{1}{\sin^2\theta}\frac{d^0}{d(\cos\theta)^0}(1-\cos^2\theta)^2 = \frac{\sin^4\theta}{\sin^2\theta} = \sin^2\theta.$$

Again, by inspection $Y_{\pm 2}^l$ with $l > 2$ contain terms that are not in $\psi(\theta,\phi,0)$, so $a_{\pm 2}^l = 0$ for $l > 2$ as well.

Thus, only a_0^0, a_0^2, and $a_{\pm 2}^2$ are nonzero; that is,

$$\psi(\theta,\phi,0) = a_0^0 Y_0^0 + a_0^2 Y_0^2 + a_2^2 Y_2^2 + a_{-2}^2 Y_{-2}^2.$$

The relevant spherical harmonics are

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \qquad Y_0^2 = \sqrt{\frac{5}{16\pi}}(2-3\sin^2\theta), \qquad Y_{\pm 2}^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi}. \tag{3}$$

Expanding out $\psi(\theta,\phi,0)$ as in (2),

$$\psi(\theta,\phi,0) = \frac{A}{2}\sin^2\theta + \frac{A}{4}\sin^2\theta e^{i2\phi} + \frac{A}{4}\sin^2\theta e^{-i2\phi}.$$

Then we can deduce the nonzero a_m^l:

$$\begin{aligned}
\frac{A}{4}\sin^2\theta e^{\pm i2\phi} = a_{\pm 2}^2\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\phi} &\implies a_{\pm 2}^2 = A\sqrt{\frac{2\pi}{15}},\\
\frac{A}{2}\sin^2\theta = a_0^0\sqrt{\frac{1}{4\pi}} + a_0^2\sqrt{\frac{5}{16\pi}}(2-3\sin^2\theta) &\implies a_0^2 = -\frac{2}{3}A\sqrt{\frac{\pi}{5}},\ a_0^0 = \frac{2}{3}A\sqrt{\pi}.
\end{aligned}$$

1.2 Now consider the wave function at nonzero time t. Use your results from 1.1 and the expressions for spherical harmonics to derive an explicit expression in terms of sines and cosines of θ and ϕ for $\psi(\theta,\phi,t)$.

Solution. From 1.1, we have

$$\psi(\theta,\phi,0) = \frac{2}{3}A\sqrt{\pi}\,Y_0^0 - \frac{2}{3}A\sqrt{\frac{\pi}{5}}\,Y_0^2 + A\sqrt{\frac{2\pi}{15}}\,Y_2^2 + A\sqrt{\frac{2\pi}{15}}\,Y_{-2}^2. \tag{4}$$

We can evaluate the time evolution for each spherical harmonic term in (4) individually, and sum them up to find $\psi(\theta,\phi,t)$:

$$\psi(\theta,\phi,t)=U(t)\psi(\theta,\phi,0)=\frac{2}{3}A\sqrt{\pi}\,U(t)Y_0^0-\frac{2}{3}A\sqrt{\frac{\pi}{5}}\,U(t)Y_0^2+A\sqrt{\frac{2\pi}{15}}\,U(t)Y_2^2+A\sqrt{\frac{2\pi}{15}}\,U(t)Y_{-2}^2$$

The time evolution operator is given by $U(t)=e^{-iHt/\hbar}$. From (1), the relevant eigenvalues are

$$E_0=0, \qquad E_2=3\frac{\hbar^2}{ma^2},$$

so

$$U(t)Y_0^0=\exp\left(-\frac{i}{\hbar}E_0t\right)Y_0^0=Y_0^0, \qquad U(t)Y_m^2=\exp\left(-\frac{i}{\hbar}E_2t\right)Y_m^2=\exp\left(-3i\frac{\hbar}{ma^2}t\right)Y_m^2.$$

Then, using the explicit Y_m^l from (3),

$$\begin{aligned}\psi(\theta,\phi,t)&=\frac{2}{3}A\sqrt{\pi}\sqrt{\frac{1}{4\pi}}-\frac{2}{3}A\sqrt{\frac{\pi}{5}}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sqrt{\frac{5}{16\pi}}(2-3\sin^2\theta)+A\sqrt{\frac{2\pi}{15}}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}\\&\qquad+A\sqrt{\frac{2\pi}{15}}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\phi}\\&=\frac{A}{3}-\frac{A}{6}\exp\left(-3i\frac{\hbar}{ma^2}t\right)(2-3\sin^2\theta)+\frac{A}{4}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta e^{2i\phi}+\frac{A}{4}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta e^{-2i\phi}\\&=\frac{A}{3}-\frac{A}{3}\exp\left(-3i\frac{\hbar}{ma^2}t\right)+\frac{A}{2}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta+\frac{A}{2}\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta\cos 2\phi\\&=\frac{A}{3}\left[1-\exp\left(-3i\frac{\hbar}{ma^2}t\right)\right]+A\exp\left(-3i\frac{\hbar}{ma^2}t\right)\sin^2\theta\cos^2\phi.\end{aligned}\tag{5}$$

1.3 Use your results from 1.2 to derive expressions for the expected values of L_x, L_y, and L_z as functions of time.

Solution. From (3.6.23) in Sakurai, $\langle\theta,\phi|l,m\rangle=Y_m^l(\theta,\phi)$ and therefore $\psi(\theta,\phi,t)=\langle\theta,\phi|\psi(t)\rangle$. Using the result of 1.2, this implies

$$|\psi(t)\rangle=a_0^0\,|0,0\rangle+a_0^2\exp\left(-3i\frac{\hbar}{ma^2}t\right)|2,0\rangle+a_2^2\exp\left(-3i\frac{\hbar}{ma^2}t\right)|2,2\rangle+a_{-2}^2\exp\left(-3i\frac{\hbar}{ma^2}t\right)|2,-2\rangle.$$

Then the time-dependent expectation value of an operator O is given by

$$\begin{aligned}\langle\psi(t)|O|\psi(t)\rangle&={a_0^0}^2\langle 0,0|O|0,0\rangle+a_0^0a_0^2U(t)\langle 0,0|O|2,0\rangle+a_0^0a_2^2U(t)\langle 0,0|O|2,2\rangle+a_0^0a_{-2}^2U(t)\langle 0,0|O|2,-2\rangle\\&\quad+a_0^0a_0^2U^\dagger(t)\langle 2,0|O|0,0\rangle+{a_0^2}^2\langle 2,0|O|2,0\rangle+a_0^2a_2^2\langle 2,0|O|2,2\rangle+a_0^2a_{-2}^2\langle 2,0|O|2,-2\rangle\\&\quad+a_0^0a_2^2U^\dagger(t)\langle 2,2|O|0,0\rangle+a_0^2a_2^2\langle 2,2|O|2,0\rangle+{a_2^2}^2\langle 2,2|O|2,2\rangle+a_2^2a_{-2}^2\langle 2,2|O|2,-2\rangle\\&\quad+a_0^0a_{-2}^2U^\dagger(t)\langle 2,-2|O|0,0\rangle+a_0^2a_{-2}^2\langle 2,-2|O|2,0\rangle+a_2^2a_{-2}^2\langle 2,-2|O|2,2\rangle+{a_{-2}^2}^2\langle 2,-2|O|2,-2\rangle,\end{aligned}$$

where $U(t)=e^{-3i\hbar t/ma^2}$ and $U^\dagger(t)=e^{3i\hbar t/ma^2}$.

From the results of 3.3 on the previous homework,

$$0=\langle 2,-2|L_i|2,-2\rangle=\langle 2,-2|L_i|2,0\rangle=\langle 2,-2|L_i|2,2\rangle$$

$$
\begin{aligned}
&= \langle 2,0|L_i|2,-2\rangle = \langle 2,0|L_i|2,0\rangle = \langle 2,0|L_i|2,2\rangle \\
&= \langle 2,2|L_i|2,-2\rangle = \langle 2,2|L_i|2,0\rangle = \langle 2,2|L_i|2,2\rangle
\end{aligned}
$$

for $i \in \{x, y, z\}$. For $(l, m) = (0, 0)$, a similar procedure to the one used for 3.3 yields

$$
\begin{aligned}
\big\langle l', m'\big|L_x\big|0,0\big\rangle &= \big\langle 0,0\big|L_x\big|l', m'\big\rangle = \frac{\hbar}{2}\delta_{0,l'}\,\delta_{1,m'}\sqrt{l^2+l} = 0, \\
\big\langle l', m'\big|L_y\big|0,0\big\rangle &= \big\langle 0,0\big|L_y\big|l', m'\big\rangle = -\frac{i\hbar}{2}\delta_{0,l'}\,\delta_{1,m'}\sqrt{l^2+l} = 0, \\
\big\langle l', m'\big|L_z\big|0,0\big\rangle &= \big\langle 0,0\big|L_z\big|l', m'\big\rangle = 0,
\end{aligned}
$$

where the last result comes from the eigenvalues of L_z being $\hbar m$. Thus, we find

$$
\langle \psi(t)|L_x|\psi(t)\rangle = \langle \psi(t)|L_y|\psi(t)\rangle = \langle \psi(t)|L_z|\psi(t)\rangle = 0.
$$

2 Problem 3

Consider a spin 1/2 state $|\psi\rangle = c_1\,|{\uparrow}\rangle + c_2\,|{\downarrow}\rangle$, where $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ are the S_z eigenstates with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

2.1 Consider the operator $\rho = |\psi\rangle\langle\psi|$. Write down the matrix elements of ρ is the basis $\{|{\uparrow}\rangle\,, |{\downarrow}\rangle\}$.

Solution. From the definition of $|\psi\rangle$,

$$
\langle{\uparrow}|\psi\rangle = c_1, \qquad \langle\psi|{\uparrow}\rangle = c_1^*, \qquad \langle{\downarrow}|\psi\rangle = c_2, \qquad \langle\psi|{\downarrow}\rangle = c_2^*.
$$

Using these,

$$
\begin{aligned}
&\langle{\uparrow}|\rho|{\uparrow}\rangle = \langle{\uparrow}|\psi\rangle\,\langle\psi|{\uparrow}\rangle = c_1 c_1^* = |c_1|^2, &&\langle{\uparrow}|\rho|{\downarrow}\rangle = \langle{\uparrow}|\psi\rangle\,\langle\psi|{\downarrow}\rangle = c_1 c_2^*, \\
&\langle{\downarrow}|\rho|{\uparrow}\rangle = \langle{\downarrow}|\psi\rangle\,\langle\psi|{\uparrow}\rangle = c_1^* c_2, &&\langle{\downarrow}|\rho|{\downarrow}\rangle = \langle{\downarrow}|\psi\rangle\,\langle\psi|{\downarrow}\rangle = c_2 c_2^* = |c_2|^2.
\end{aligned}
$$

2.2 In the S_z eigenbasis, express ρ by using the Pauli matrices. That is, write ρ as

$$
\rho = \frac{s_0}{2}I + \frac{1}{2}\mathbf{s}\cdot\boldsymbol{\sigma},
$$

and express s_0, s_1, s_2, s_3 in terms of c_1 and c_2.

In writing up these solutions, I consulted Sakurai's *Modern Quantum Mechanics* and Shankar's *Principles of Quantum Mechanics.*