1 Problem 1

The motion of a particle in a cubic potential is governed by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \frac{k^2}{2}q^2 - \frac{A}{3}q^3. \tag{1}$$

Here m is the particle mass, k is the spring constant, and A is a positive dimensional constant.

1.a Sketch the potential and the contours of H. Identify any fixed points (mechanical equilibrium states) that exist. Classify them as stable (elliptic) or unstable (hyperbolic).

Solution. Define the potential of (1) as

$$V(q) \equiv \frac{k^2}{2}q^2 - \frac{A}{3}q^3 \equiv g(q) + g(q), \tag{2}$$

where we have defined $f(q) = k^2q^2/2$ and $g(q) = -Aq^3/3$. Figures 1 and 2 and show sketches of $f(q)$ and $g(q)$, respectively. Their sum $V(q)$ may be obtained by summing them graphically, and is shown in figure 3.

Fixed points are located where $dV/dq\,|_{q^*} = 0$. They are stable where $V(q)$ has a local minimum ($d^2V\big/dq^2\,|_{q^*} > 0$) and unstable where $V(q)$ has a local maximum ($d^2V\big/dq^2\,|_{q^*} < 0$). There are two fixed points, indicated by circles in figure 3. The stable (unstable) fixed point is indicated by a closed (open) circle.

Hamilton's equations for (1) are given by

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \implies p = m\dot{q}, \\ \dot{p} &= -\frac{\partial H}{\partial q} = k^2q - Aq^2. \end{aligned} \tag{3}$$

Fixed points occur where $\dot{q} = \dot{p} = 0$; that is, the solutions of the equation

$$p^* = k^2q^* - Aq^{*2}.$$

From (3), $\dot{q} = 0 \implies \dot{p} = 0$. Thus, the stable fixed point is located at $(q^*, p^*) = 0$, and the unstable fixed point is located at $(q^*, p^*) = (k^2/A, 0)$.

Contours are curves in the phase plane for which H is constant. Several contours are shown in figure 4.

1.b Sketch qualitatively both representative and interesting trajectories in the phase space. If there is a separatrix, a trajectory separating qualitatively different types of motion, specify the equation governing its shape.

Solution. Trajectories lie along contours of H. The directions of the trajectories may be deduced by (3), which indicates that time evolution flows in the $+q$ ($-q$) direction when $p > 0$ (< 0). This corresponds to the top (bottom) half of the phase plane. Representative trajectories corresponding to some of the contours in figure 4 are shown in figure 5.

There is a separatrix in figure 5, shown in red. The separatrix passes through the unstable fixed point at $(q^*, p^*) = (k^2/A, 0)$. Feeding these values into (1), we obtain

$$E \equiv \frac{k^2}{2}\left(\frac{k^2}{A}\right)^2 - \frac{A}{3}\left(\frac{k^2}{A}\right)^3 = \frac{1}{6}\frac{k^6}{A^2}$$

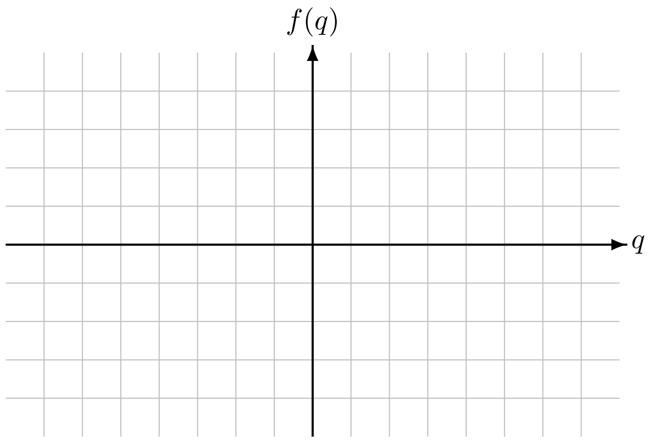

Figure 1: Sketch of $f(q)$ as defined in (2).

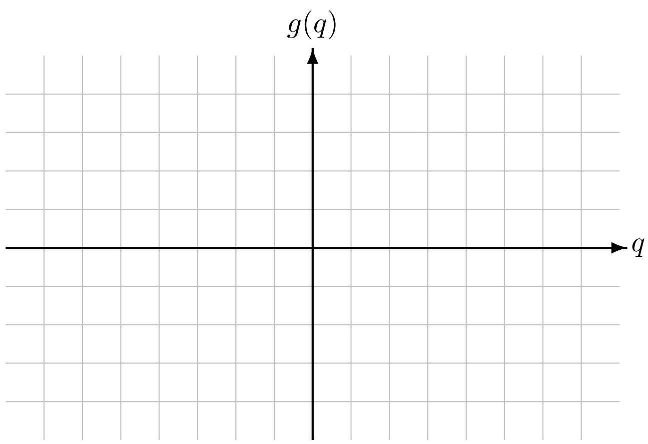

Figure 2: Sketch of $g(q)$ as defined in (2).

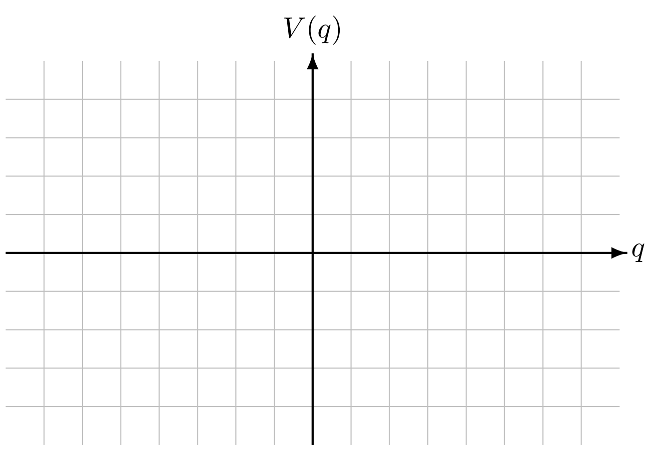

Figure 3: Sketch of $V(q)$ obtained by summing $f(q)$ and $g(q)$. The stable (unstable) fixed point is represented by a closed (open) circle.

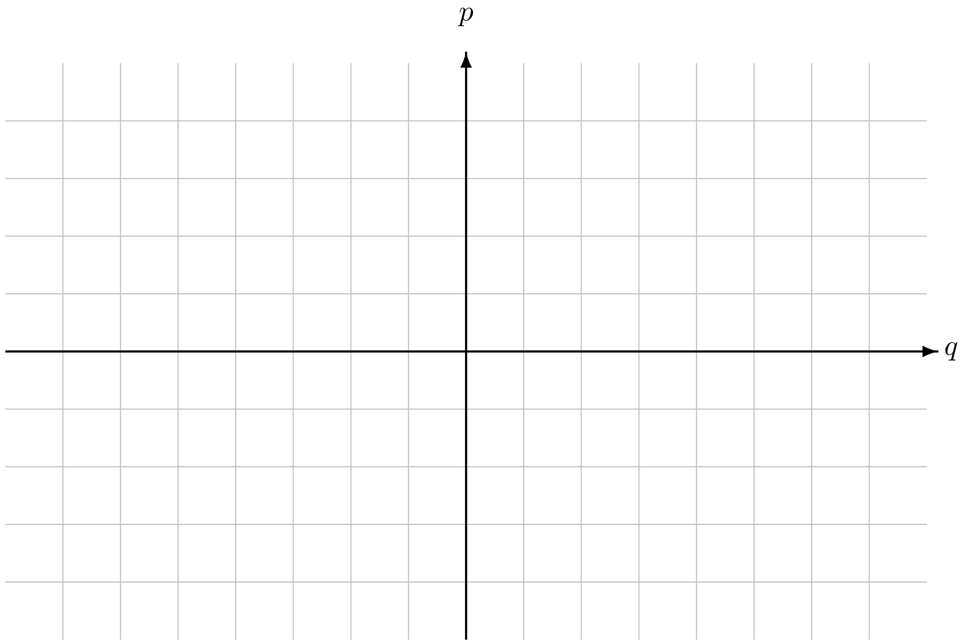

Figure 4: Contours of H. The stable (unstable) fixed point is represented by a closed (open) circle.

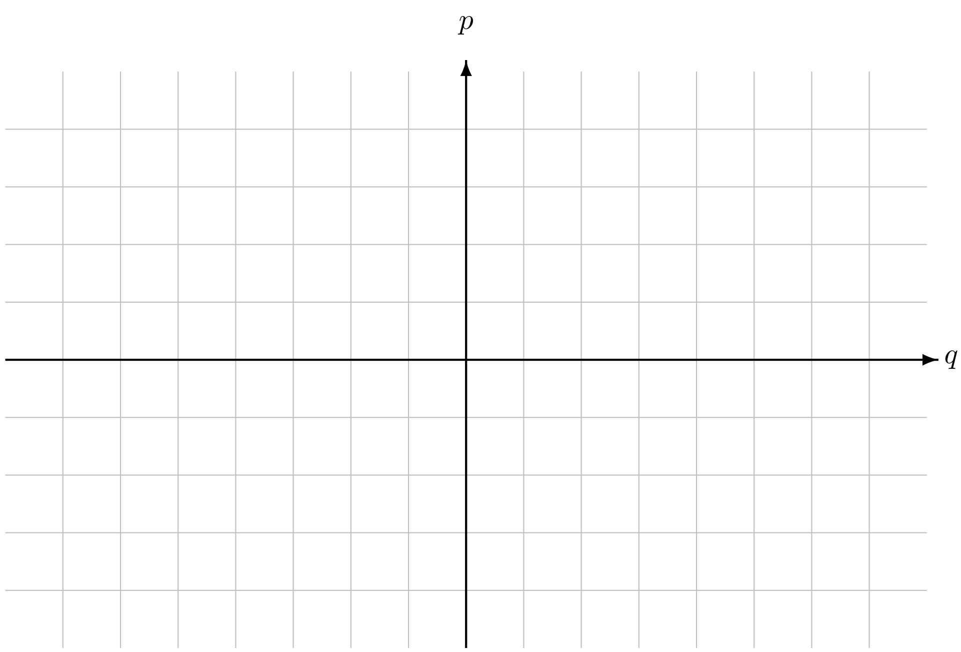

Figure 5: Trajectories of H, with the direction of time evolution indicated by arrows. The stable (unstable) fixed point is represented by a closed (open) circle. The separatrix is drawn in red.

as the constant energy of the separatrix. Substituting once more into (1) yields

$$\frac{1}{6}\frac{k^6}{A^2} = \frac{p^2}{2m} + \frac{k^2}{2}q^2 - \frac{A}{3}q^3 \iff p^3 = m\left(\frac{1}{3}\frac{k^6}{A^2} - k^2q^2 + \frac{2}{3}Aq^3\right)$$

as the equation governing the shape of the separatrix.

2 Problem 2

A particle in three spatial dimensions moves in a force field give by the Yukawa potential

$$U(r) = -\frac{k}{r}\exp\left(-\frac{r}{a}\right),$$

where k and a are positive, and r is the radial distance between the particle and the origin.

2.a Show that this central force problem can be reduced to an equivalent one-dimensional problem with an effective potential. Specify the effective potential.

Solution. We will show that the problem can be reduced to one dimension by showing that the system has two independent conserved quantities.

$U(r)$ is easily written in the spherical coordinates (r, θ, ϕ) where ϕ is the azimuthal. In these coordinates, the Lagrangian is given by

$$L = T - U = \frac{m}{2}(\dot{r}^2 + r^2\sin^2\theta\,\dot{\phi}^2 + r^2\dot{\theta}^2) + \frac{k}{r}\exp\left(-\frac{r}{a}\right). \tag{4}$$

Firstly, L has no explicit time dependence, so the total energy of the system $H = T + U$ is conserved.

Secondly, L has no explicit ϕ dependence. From Noether's theorem, this implies a second conserved quantity, given by

$$mr^2(\sin^2\theta\,\dot{\phi} + \dot{\theta}) \equiv J$$

L can be rewritten in terms of J as

$$L = \frac{mr^2}{2}\dot{\theta}^2 + \frac{1}{2}\frac{J^2}{mr^2\sin^2\theta} + \frac{k}{r}\exp\left(-\frac{r}{a}\right) \equiv \frac{mr^2}{2}\dot{\theta}^2 - U_{\text{eff}},$$

where we have defined the effective potential U_{eff} by

$$U_{\text{eff}}(r) = -\frac{1}{2}\frac{J^2}{mr^2\sin^2\theta} - \frac{k}{r}\exp\left(-\frac{r}{a}\right).$$

2.b Describe qualitatively the different types of motion possible as the system parameters are varied. If you think a sketch clarifies your answer, include it.