

**Problem 1. (Jackson 9.8)**

**1(a)** Show that a classical oscillating electric dipole  $\mathbf{p}$  with fields given by

$$\mathbf{H} = \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p}) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right), \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \left\{ k^2(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}} \frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\}, \quad (1)$$

radiates electromagnetic angular momentum to infinity at the rate

$$\frac{d\mathbf{L}}{dt} = \frac{k^3}{12\pi\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}].$$

**Solution.** From Prob. 3 of Homework 4 for Physics 322, the angular momentum density is  $\mathbf{l} = \mathbf{x} \times \mathbf{g}$ , where  $\mathbf{g}$  is the linear momentum density in SI units. According to Jackson 6.118,  $\mathbf{g} = (\mathbf{E} \times \mathbf{H})/c^2$ . However, according to Jackson (9.20), the time-averaged power radiated per unit solid angle by  $\mathbf{p}$  is given by

$$\frac{dP}{d\Omega} = \frac{1}{2} \text{Re}[r^2 \hat{\mathbf{n}} \cdot \mathbf{E} \times \mathbf{H}^*],$$

which indicates that we should send  $\mathbf{H} \rightarrow \mathbf{H}^*$  for a sensible result. Thus, the time-averaged angular momentum density is

$$\mathbf{l} = \frac{\mathbf{E} \times \mathbf{H}^*}{2c^2}.$$

One of the vector identities on the inside cover of Jackson is  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ , so

$$\mathbf{l} = \frac{(\mathbf{x} \cdot \mathbf{H}^*)\mathbf{E} - (\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{2c^2}. \quad (2)$$

From Eq. (1), note that

$$\mathbf{x} \cdot \mathbf{H}^* \propto \mathbf{x} \cdot (\hat{\mathbf{n}} \times \mathbf{p}^*) = \mathbf{p}^* \cdot (\mathbf{x} \times \hat{\mathbf{n}}) = 0,$$

where we have used the identity  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$  and the fact that  $\hat{\mathbf{n}}$  points in the  $\mathbf{x}$  direction. For  $\mathbf{x} \cdot \mathbf{E}$ , note that

$$\begin{aligned} \mathbf{x} \cdot [(\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}}] &= -\mathbf{x} \cdot [\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{p})] = -\mathbf{x} \cdot [(\hat{\mathbf{n}} \cdot \mathbf{p})\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\mathbf{p}] = -(\hat{\mathbf{n}} \cdot \mathbf{p})(\mathbf{x} \cdot \hat{\mathbf{n}}) + \mathbf{x} \cdot \mathbf{p} \\ &= -r(\hat{\mathbf{n}} \cdot \mathbf{p}) + \mathbf{x} \cdot \mathbf{p} = \mathbf{x} \cdot \mathbf{p} - \mathbf{x} \cdot \mathbf{p} = 0, \end{aligned}$$

$$\mathbf{x} \cdot [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{p}] = 3(\mathbf{x} \cdot \hat{\mathbf{n}})(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3r(\hat{\mathbf{n}} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 3(\mathbf{x} \cdot \mathbf{p}) - \mathbf{x} \cdot \mathbf{p} = 2(\mathbf{x} \cdot \mathbf{p}),$$

since  $|\mathbf{x}| = r$  and  $\mathbf{x} = r\hat{\mathbf{n}}$ . Then

$$\mathbf{x} \cdot \mathbf{E} = \frac{1}{2\pi\epsilon_0}(\mathbf{x} \cdot \mathbf{p}) \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} = \frac{1}{2\pi\epsilon_0}(\hat{\mathbf{n}} \cdot \mathbf{p}) \left( \frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr}.$$

With these substitutions, Eq. (2) becomes

$$\begin{aligned} \mathbf{l} &= -\frac{(\mathbf{x} \cdot \mathbf{E})\mathbf{H}^*}{c^2} = -\frac{1}{4\pi\epsilon_0 c^2}(\hat{\mathbf{n}} \cdot \mathbf{p}) \left( \frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr} \frac{ck^2}{4\pi}(\hat{\mathbf{n}} \times \mathbf{p}^*) \frac{e^{-ikr}}{r} \left( 1 + \frac{1}{ikr} \right) \\ &= -\frac{k^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{r^2} - \frac{ik}{r} \right) \left( 1 - \frac{i}{kr} \right) = -\frac{k^2}{16\pi^2\epsilon_0 c}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{r^2} - \frac{i}{kr^3} - \frac{ik}{r} - \frac{1}{r^2} \right) \\ &= -\frac{ik^2}{16\pi^2\epsilon_0 cr}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{kr^3} + \frac{k}{r^2} \right) = \frac{ik^3}{16\pi^2\epsilon_0 cr^2}(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \left( \frac{1}{k^2 r^2} + 1 \right). \end{aligned}$$

Let  $\mathbf{L}$  be the angular momentum radiated to a distance  $R$ . Then

$$\mathbf{L} = \int_R \mathbf{l}(r) d^3x = \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r) r^2 \sin \theta dr d\phi d\theta,$$

and the time derivative is

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt} \left( \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r) r^2 \sin \theta dr d\phi d\theta \right) = \frac{dr}{dt} \frac{d}{dr} \left( \int_0^\pi \int_0^{2\pi} \int_0^R \mathbf{l}(r) r^2 \sin \theta dr d\phi d\theta \right) \\ &= c \int_0^\pi \int_0^{2\pi} \mathbf{l}(r) r^2 \sin \theta d\phi d\theta = \frac{ik^3}{16\pi^2\epsilon_0} \left( \frac{1}{k^2 r^2} + 1 \right) \int_0^\pi \int_0^{2\pi} (\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*) \sin \theta d\phi d\theta. \end{aligned} \quad (3)$$

Note that

$$[(\hat{\mathbf{n}} \cdot \mathbf{p})(\hat{\mathbf{n}} \times \mathbf{p}^*)]_i = \sum_{j=1}^3 n_j p_j (\hat{\mathbf{n}} \times \mathbf{p}^*)_i = \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \epsilon_{ikl} n_j p_j n_k p_l^*,$$

so

$$\frac{dL_i}{dt} \propto \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \int n_j p_k d\Omega = \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \epsilon_{ikl} p_j p_l^* \frac{4\pi}{3} \delta_{jk} = \frac{4\pi}{3} \epsilon_{ikl} p_k p_l^* = \frac{4\pi}{3} (\mathbf{p} \times \mathbf{p}^*)_i,$$

where we have used Jackson (9.47),  $\int n_\beta n_\gamma d\Omega = 4\pi \delta_{\beta\gamma}/3$ . Making this substitution into Eq. (3),

$$\frac{d\mathbf{L}}{dt} = \frac{ik^3}{6\pi\epsilon_0} \left( \frac{1}{k^2 r^2} + 1 \right) (\mathbf{p} \times \mathbf{p}^*).$$

Taking the limit as  $r \rightarrow \infty$ , we find

$$\frac{d\mathbf{L}}{dt} = \text{Re} \left[ \frac{ik^3}{12\pi\epsilon_0} (\mathbf{p} \times \mathbf{p}^*) \right] = \text{Re} \left[ -\frac{ik^3}{12\pi\epsilon_0} (\mathbf{p}^* \times \mathbf{p}) \right] = \frac{k^3}{12\pi\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}], \quad (4)$$

as desired. □

**1(b)** What is the ratio of angular momentum radiated to energy radiated? Interpret.

**Solution.** According to Jackson (9.24), the total power radiated by an oscillating electric dipole  $\mathbf{p}$  is

$$P = \frac{dE}{dt} = \frac{c^2 Z_0 k^4}{12\pi} |\mathbf{p}|^2.$$

Then the ratio of angular momentum radiated to energy radiated is

$$\frac{d\mathbf{L}/dt}{dE/dt} = \frac{k^3}{12\pi\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}] \frac{12\pi}{c^2 Z_0 k^4 |\mathbf{p}|^2} = \frac{1}{\epsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}] \frac{1}{c^2 Z_0 k |\mathbf{p}|^2} = \frac{\text{Im}[\mathbf{p}^* \times \mathbf{p}]}{\omega |\mathbf{p}|^2},$$

where we have used  $Z_0 = \sqrt{\mu_0/\epsilon_0} = 1/\sqrt{\epsilon_0^2 c^2} = 1/\epsilon_0 c$ ,  $c^2 = 1/(\epsilon_0 \mu_0)$ , and  $\omega = kc$ .

In the limit of high frequency,  $(d\mathbf{L}/dt)/(dE/dt) \rightarrow 0$ . In this scenario, the energy radiated dominates over the angular momentum radiated. Likewise, in the limit of low frequency,  $(d\mathbf{L}/dt)/(dE/dt) \rightarrow \infty$ , meaning that angular momentum radiation dominates. This is sensible because rotational kinetic energy  $E \propto \omega^2$ , while angular momentum  $L \propto \omega$ .

**1(c)** For a charge  $e$  rotating in the  $xy$  plane at radius  $a$  and angular speed  $\omega$ , show that there is only a  $z$  component of radiated angular momentum with magnitude  $dL_z/dt = e^2 k^3 a^2 / 6\pi\epsilon_0$ . What about a charge oscillating along the  $z$  axis?

**Solution.** We know from Homework 5 that the position of a point charge rotating counterclockwise in the  $xy$  plane is

$$\mathbf{x}(t) = a \cos(\omega t) \hat{\mathbf{x}} + a \sin(\omega t) \hat{\mathbf{y}}.$$

Then the charge distribution is

$$\rho(\mathbf{x}, t) = e \delta[x - a \cos(\omega t)] \delta[y - a \sin(\omega t)] \delta(z).$$

According to Jackson (4.8), the dipole moment is defined

$$\mathbf{p} = \int \mathbf{x}' \rho(\mathbf{x}') d^3x'.$$

The components of  $\mathbf{p}$  for the point charge are then

$$p_x = e \iiint x \delta[x - a \cos(\omega t)] \delta[y - a \sin(\omega t)] \delta(z) dx dy dz = ea \cos(\omega t),$$

$$p_y = e \iiint y \delta[x - a \cos(\omega t)] \delta[y - a \sin(\omega t)] \delta(z) dx dy dz = ea \sin(\omega t),$$

$$p_z = e \iiint z \delta[x - a \cos(\omega t)] \delta[y - a \sin(\omega t)] \delta(z) dx dy dz = 0,$$

so we can write  $\mathbf{p} = ea e^{-i\omega t}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$ . Substituting into Eq. (4),

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \text{Re} \left[ \frac{ik^3}{12\pi\epsilon_0} e^2 a^2 e^{-i\omega t} e^{i\omega t} [(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \times (\hat{\mathbf{x}} - i\hat{\mathbf{y}})] \right] = \text{Re} \left[ \frac{ie^2 k^3 a^2}{12\pi\epsilon_0} (-2i \hat{\mathbf{x}} \times \hat{\mathbf{y}}) \right] = \text{Re} \left[ \frac{e^2 k^3 a^2}{6\pi\epsilon_0} \hat{\mathbf{z}} \right] \\ &= \frac{e^2 k^3 a^2}{6\pi\epsilon_0} \cos(\omega t) \hat{\mathbf{z}}, \end{aligned}$$

as desired. □

A charge oscillating along the  $z$  axis with amplitude  $a$  has the charge density

$$\rho(\mathbf{x}, t) = ea \delta(x) \delta(y) \delta[z - \cos(\omega t)],$$

which gives the dipole moment

$$p_x = ea \iiint x \delta(x) \delta(y) \delta[z - \cos(\omega t)] dx dy dz = 0,$$

$$p_y = ea \iiint y \delta(x) \delta(y) \delta[z - \cos(\omega t)] dx dy dz = 0,$$

$$p_z = ea \iiint z \delta(x) \delta(y) \delta[z - \cos(\omega t)] dx dy dz = ea \cos(\omega t).$$

In complex notation,  $\mathbf{p} = ea e^{-i\omega t} \hat{\mathbf{z}}$ . Substituting into Eq. (4), we find

$$\frac{d\mathbf{L}}{dt} = \text{Re} \left[ \frac{ik^3}{12\pi\epsilon_0} e^2 a^2 e^{-i\omega t} e^{i\omega t} (\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \right] = \mathbf{0}.$$

So we see that a charge undergoing linear motion does not lead to a radiated angular momentum, which is sensible.

**1(d)** What are the results corresponding to Probs. 1(a) and 1(b) for magnetic dipole radiation?

**Solution.** The radiation fields for a magnetic dipole are given by Jackson (19.35–36),

$$\mathbf{H} = \frac{1}{4\pi} \left\{ k^2 (\hat{\mathbf{n}} \times \mathbf{m}) \times \hat{\mathbf{n}} \frac{e^{ikr}}{r} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{m}) - \mathbf{m}] \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\}, \quad \mathbf{E} = -\frac{Z_0}{4\pi} k^2 (\hat{\mathbf{n}} \times \mathbf{m}) \frac{e^{ikr}}{r} \left( 1 - \frac{1}{ikr} \right).$$

Comparing with Eq. (1), we see that  $\mathbf{H} \rightarrow -\mathbf{E}/Z_0$ ,  $\mathbf{E} \rightarrow Z_0 \mathbf{H}$ , and  $\mathbf{p} \rightarrow \mathbf{m}/c$  as stated in the book [?, p. 413]. Making these substitutions, the results of Probs. 1.1(a) and (b) become

$$\frac{d\mathbf{L}}{dt} = \frac{\mu_0 k^3}{12\pi} \text{Im}[\mathbf{m}^* \times \mathbf{m}], \quad \frac{d\mathbf{L}/dt}{dE/dt} = \frac{\text{Im}[\mathbf{m}^* \times \mathbf{m}]}{\omega |\mathbf{m}|^2}$$

where we have used  $\mu = 1/\epsilon_0 c^2$ .

## Problem 2. (Jackson 10.1)

**2(a)** Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius  $a$ , summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[ \frac{5}{4} - |\boldsymbol{\epsilon}_0 \cdot \hat{\mathbf{n}}|^2 - \frac{1}{4} |\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}}_0 \times \boldsymbol{\epsilon}_0)|^2 - \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}} \right],$$

where  $\hat{\mathbf{n}}_0$  and  $\hat{\mathbf{n}}$  are the directions of the incident and scattered radiations, respectively, while  $\boldsymbol{\epsilon}_0$  is the (perhaps complex) unit polarization vector of the incident radiation ( $\boldsymbol{\epsilon}_0^* \cdot \boldsymbol{\epsilon}_0 = 1$ ;  $\hat{\mathbf{n}}_0 \cdot \boldsymbol{\epsilon}_0 = 0$ ).

**2(b)** If the incident radiation is linearly polarized, show that the cross section is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[ \frac{5}{8} (1 + \cos^2 \theta) - \cos \theta - \frac{3}{8} \sin^2 \theta \cos(2\phi) \right],$$

where  $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_0 = \cos \theta$  and the azimuthal angle  $\phi$  is measured from the direction of linear polarization.

**2(c)** What is the ratio of scattered intensities at  $\theta = \pi/2$ ,  $\phi = 0$  and  $\theta = \pi/2$ ,  $\phi = \pi/2$ ? Explain physically in terms of the induced multipoles and their radiation patterns.

**Problem 3. (Jackson 12.15)** Consider the Proca equation for a localized steady-state distribution of current that has only a static magnetic moment. This model can be used to study the observable effects of a finite photon mass on the earth's magnetic field. Note that if the magnetization is  $\mathcal{M}(\mathbf{x})$  the current density can be written as  $\mathbf{J} = c(\nabla \times \mathcal{M})$ .

**3(a)** Show that if  $\mathcal{M} = \mathbf{m}f(\mathbf{x})$ , where  $\mathbf{m}$  is a fixed vector and  $f(\mathbf{x})$  is a localized scalar function, the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\mathbf{m} \times \nabla \int f(\mathbf{x}') \frac{e^{-\mu|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} d^3x'.$$

**3(b)** If the magnetic dipole is a point dipole at the origin [ $f(\mathbf{x}) = \delta(\mathbf{x})$ ], show that the magnetic field away from the origin is

$$\mathbf{B}(\mathbf{x}) = [3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}] \left( 1 + \mu r + \frac{\mu^2 r^2}{3} \right) \frac{e^{-\mu r}}{r^3} - \frac{2}{3} \mu^2 \mathbf{m} \frac{e^{-\mu r}}{r}.$$

**3(c)** The result of Prob. 3(b) shows that at fixed  $r = R$  (on the surface of the earth), the earth's magnetic field will appear as a dipole angular distribution, plus an added constant magnetic field (an apparently external field) antiparallel to  $\mathbf{m}$ . Satellite and surface observations lead to the conclusion that the “external” field is less than  $4 \times 10^{-3}$  times the dipole field at the magnetic equator. Estimate a lower limit on  $\mu^{-1}$  in earth radii and an upper limit on the photon mass in grams from this datum.