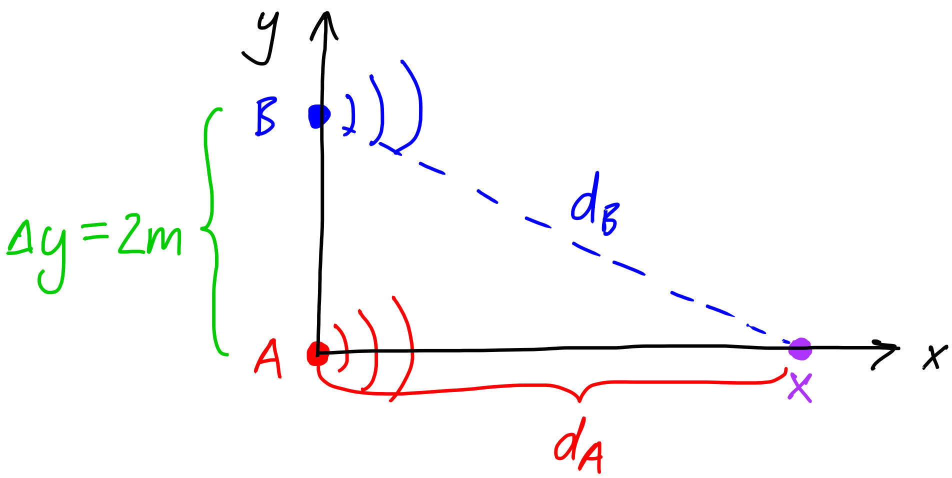
Problem 1. Consider two speakers emitting sound at the same volume with frequency $f = 800\,\mathrm{Hz}$. One speaker is located at the origin, and the other on the y axis at $y = 2\,\mathrm{m}$. At what locations on the positive x axis is the interference completely constructive? At what points is it completely destructive?

Now we decrease f until there are no longer any points of completely destructive interference on the positive x axis. How low must f be for this to occur?

Solution. Consider the setup shown below:



The path difference d (which is called Δx in the lecture slides) is given by

$$d = d_B - d_A = d_A - x,$$

and from trigonometry,

$$d_B^2 = x^2 + (\Delta y)^2 \qquad \Longrightarrow \qquad d_B = \sqrt{x^2 + (\Delta y)^2},$$

where Δy is the distance between the two speakers. Putting these together, we can write

$$d = \sqrt{x^2 + (\Delta y)^2} - x.$$

Completely constructive interference occurs where the interference pattern of the speakers has a maximum, which is when

$$d = n\lambda, \qquad\qquad n = 0, \pm 1, \pm 2, \ldots.$$

Completely destructive interference occurs where it has a minimum, and

$$d = \left(n + \frac{1}{2}\right)\lambda, \qquad\qquad n = 0, \pm 1, \pm 2, \ldots.$$

Recall that the wavelength $\lambda = v/f$, where $v = 344\,\mathrm{m\,s^{-1}}$ is the speed of sound in air. For this problem,

$$\lambda = \frac{344\,\mathrm{m\,s^{-1}}}{800\,\mathrm{Hz}} = 0.43\,\mathrm{m}.$$

Constructive interference will occur at x when

$$n\lambda = \sqrt{x^2 + (\Delta y)^2} - x.$$

Solving for x,

$$(x+n\lambda)^2 = x^2 + (\Delta y)^2 \quad \Longrightarrow \quad x^2 + 2n\lambda x + n^2\lambda^2 = x^2 + (\Delta y)^2 \quad \Longrightarrow \quad 2n\lambda x = (\Delta y)^2 - n^2\lambda^2,$$

which implies

$$x = \frac{(\Delta y)^2}{2n\lambda} - \frac{n\lambda}{2}. \tag{1}$$

Now we can plug in numerical quantities and $n = 0, \pm 1, \pm 2, \ldots$ into Eq. (1) to find

$$\begin{aligned}
x(n=1) &= \frac{(2\,\mathrm{m})^2}{2(0.43\,\mathrm{m})} - \frac{1}{2}(0.43\,\mathrm{m}) = 4.44\,\mathrm{m},\\
x(n=2) &= \frac{(2\,\mathrm{m})^2}{4(0.43\,\mathrm{m})} - (0.43\,\mathrm{m}) = 1.90\,\mathrm{m},\\
x(n=3) &= \frac{(2\,\mathrm{m})^2}{6(0.43\,\mathrm{m})} - \frac{3}{2}(0.43\,\mathrm{m}) = 0.91\,\mathrm{m},\\
x(n=4) &= \frac{(2\,\mathrm{m})^2}{8(0.43\,\mathrm{m})} - 2(0.43\,\mathrm{m}) = 0.31\,\mathrm{m}.
\end{aligned}$$

Note that x is undefined for $n = 0$ and is negative for $n > 4$. Plugging in $n = -1, -2, -3, \ldots$ would also give us negative values. None of these makes sense since we are interested only in the positive x axis.

For destructive interference, we have to satisfy

$$\left(n + \frac{1}{2}\lambda\right) = \sqrt{x^2 + (\Delta y)^2} - x,$$

and solving for x in the same manner as before gives us

$$x = \frac{(\Delta y)^2}{(2n+1)\lambda} - \frac{(n+1/2)\lambda}{2}. \tag{2}$$

Plugging in numerical quantities and $n = 0, 1, 2, \ldots$ into Eq. (2),

$$\begin{aligned}
x(n=0) &= \frac{(2\,\mathrm{m})^2}{0.43\,\mathrm{m}} - \frac{1}{4}(0.43\,\mathrm{m}) = 9.19\,\mathrm{m},\\
x(n=1) &= \frac{(2\,\mathrm{m})^2}{3(0.43\,\mathrm{m})} - \frac{3}{4}(0.43\,\mathrm{m}) = 2.78\,\mathrm{m},\\
x(n=2) &= \frac{(2\,\mathrm{m})^2}{5(0.43\,\mathrm{m})} - \frac{5}{4}(0.43\,\mathrm{m}) = 1.32\,\mathrm{m},\\
x(n=3) &= \frac{(2\,\mathrm{m})^2}{7(0.43\,\mathrm{m})} - \frac{7}{4}(0.43\,\mathrm{m}) = 0.58\,\mathrm{m},\\
x(n=4) &= \frac{(2\,\mathrm{m})^2}{9(0.43\,\mathrm{m})} - \frac{9}{4}(0.43\,\mathrm{m}) = 0.07\,\mathrm{m}.
\end{aligned}$$

Again, $x < 0$ for $n < 0$ and $n > 4$, which are not sensible.

In order to find the frequency for which there is no destructive interference on the x axis, we should look at $n = 0$, since this gives us the point with the largest value of x. If we plug $n = 0$ into Eq. (2) and set $x = 0$, we are requiring that destructive interference can only occur at the origin. Solving for the wavelength λ tells us the smallest wavelength at which there is still destructive interference. We find

$$0 = \frac{(\Delta y)^2}{\lambda} - \frac{\lambda}{4} \quad \Longrightarrow \quad \frac{\lambda^2}{4} = (\Delta y)^2 \quad \Longrightarrow \quad \lambda = 2\Delta y.$$

But if $\lambda > 2\Delta y$, then

$$\frac{(\Delta y)^2}{\lambda/2} < \frac{1}{2}\lambda,$$

and Eq. (2) tells us

$$x = \frac{(\Delta y)^2}{\lambda/2} - \frac{1}{2}\lambda < 0.$$

This means there is no destructive interference on the x axis. Thus, we need to satisfy

$$\lambda = \frac{v}{f} > 2\Delta y \qquad \Longrightarrow \qquad f < \frac{v}{2\Delta y}.$$

Plugging in numbers, we find

$$f < \frac{344\,\mathrm{m\,s^{-1}}}{2(2\,\mathrm{m})} = 86\,\mathrm{Hz}.$$