Problem 1. Verify that the functional

$$J[u] = \int_R \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy \tag{1}$$

is invariant under a rotation of the coordinate axis:

$$\tilde{x} = x\cos\epsilon + y\sin\epsilon, \qquad \tilde{y} = -x\sin\epsilon + y\cos\epsilon. \tag{2}$$

Solution. The functional is invariant if $J[u(x,y)] = J[u(\tilde{x},\tilde{y})]$. By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = \cos\epsilon\frac{\partial}{\partial \tilde{x}} - \sin\epsilon\frac{\partial}{\partial \tilde{y}}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = \sin\epsilon\frac{\partial}{\partial \tilde{x}} + \cos\epsilon\frac{\partial}{\partial \tilde{y}}.$$

The Jacobian for (2) is

$$A = \begin{bmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial y \\ \partial\tilde{y}/\partial x & \partial\tilde{y}/\partial y \end{bmatrix} = \begin{bmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{bmatrix} \implies \det(A) = \cos^2\epsilon + \sin^2\epsilon = 1,$$

and therefore

$$\int_R dx\, dy \mapsto \int_{\tilde{R}} d\tilde{x}\, d\tilde{y}\,.$$

Making these substitutions into (**??**), we have

$$\begin{aligned}
J[u(x,y)] &= \int_R \left[\left(\cos\epsilon\frac{\partial u}{\partial \tilde{x}} - \sin\epsilon\frac{\partial u}{\partial \tilde{y}} \right)^2 + \left(\sin\epsilon\frac{\partial u}{\partial \tilde{x}} + \cos\epsilon\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy \\
&= \int_R \left(\cos^2\epsilon\frac{\partial^2 u}{\partial \tilde{x}^2} - 2\cos\epsilon\sin\epsilon\frac{\partial^2 u}{\partial\tilde{x}\partial\tilde{y}} + \sin^2\epsilon\frac{\partial^2 u}{\partial \tilde{y}^2} + \sin^2\epsilon\frac{\partial^2 u}{\partial \tilde{x}^2} + 2\cos\epsilon\sin\epsilon\frac{\partial^2 u}{\partial\tilde{x}\partial\tilde{y}} + \cos^2\epsilon\frac{\partial^2 u}{\partial \tilde{y}^2} \right) dx\, dy \\
&= \int_R \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] dx\, dy = \int_{\tilde{R}} \left[\left(\frac{\partial u}{\partial \tilde{x}} \right)^2 + \left(\frac{\partial u}{\partial \tilde{y}} \right)^2 \right] d\tilde{x}\, d\tilde{y} \\
&= J[u(\tilde{x},\tilde{y})]
\end{aligned}$$

as desired.

Problem 2. Consider the real-valued Lagrangian density $\mathcal{L}$ depending on a complex-valued function $\phi(t,x,y)$:

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt}\phi \right) - \nabla\phi^* \cdot \nabla\phi - m\phi^*\phi, \tag{3}$$

where * is complex conjugation, and $\nabla\phi = (\partial\phi/\partial x\,,\, \partial\phi/\partial y)$. Treating ϕ and ϕ^* as independent objects, derive the Euler-Lagrange equations.

Solution. We will have two Euler-Lagrange equations; one for ϕ and one for ϕ^*. In general, they are given by

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi_t} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi_x} - \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi_y}, \qquad 0 = \frac{\partial\mathcal{L}}{\partial\phi^*} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\phi^*_t} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi^*_x} - \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\phi^*_y}.$$

Expanding out $\nabla\phi^* \cdot \nabla\phi$, (3) becomes

$$\mathcal{L} = \frac{i}{2}\left(\phi^* \frac{d\phi}{dt} - \frac{d\phi^*}{dt}\phi \right) - \frac{\partial\phi^*}{\partial x}\frac{\partial\phi}{\partial x} - \frac{\partial\phi^*}{\partial y}\frac{\partial\phi}{\partial y} - m\phi^*\phi.$$

Then

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{i}{2}\frac{d\phi^*}{dt} - m\phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{i}{2}\phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x} = -\frac{\partial \phi^*}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \phi_y} = -\frac{\partial \phi^*}{\partial y},$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = \frac{i}{2}\frac{d\phi}{dt} - m\phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi_t^*} = -\frac{i}{2}\phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi_x^*} = -\frac{\partial \phi}{\partial x}, \qquad \frac{\partial \mathcal{L}}{\partial \phi_y^*} = -\frac{\partial \phi}{\partial y},$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{i}{2}\frac{d\phi^*}{dt}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x} = -\frac{\partial^2 \phi^*}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y} = -\frac{\partial^2 \phi^*}{\partial y^2},$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \phi_t^*} = -\frac{i}{2}\frac{d\phi}{dt}, \qquad \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_x^*} = -\frac{\partial^2 \phi}{\partial x^2}, \qquad \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \phi_y^*} = -\frac{\partial^2 \phi}{\partial y^2}.$$

Then the Euler-Lagrange equations are

$$0 = -\frac{i}{2}\frac{d\phi^*}{dt} - m\phi^* - \frac{i}{2}\frac{d\phi^*}{dt} + \frac{\partial^2 \phi^*}{\partial x^2} + \frac{\partial^2 \phi^*}{\partial y^2}, \qquad 0 = \frac{i}{2}\frac{d\phi}{dt} - m\phi + \frac{i}{2}\frac{d\phi}{dt} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2},$$

which simplify to

$$0 = i\frac{d\phi^*}{dt} - \nabla^2\phi^* + m\phi^*, \qquad 0 = i\frac{d\phi}{dt} + \nabla^2\phi^* - m\phi^*.$$

Problem 3. The nondimensionalized, multidimensional Sine-Gordon equation,

$$\theta_{xx} + \theta_{yy} - \theta_{tt} = \sin\theta$$

for $\theta(x, y, t)$, is the Euler-Lagrange equation for the action integral

$$S[\theta] = \int_R \left\{ \frac{1}{2}\left[\theta_t^2 - (\nabla\theta)^2\right] - \sin\theta \right\} dx\, dt \tag{4}$$

with $\nabla\theta = (\partial\theta/\partial x\, ,\, \partial\theta/\partial y)$. The functional $S[\theta]$ is invariant under translation of x, y, and t. Find the associated energy-momentum tensor and energy-momentum vector.

Solution. The energy-momentum tensor is defined by

$$T_{ij} = \frac{\partial \mathcal{L}}{\partial \theta_{x_i}}\frac{\partial \theta}{\partial x_j} - \mathcal{L}\,\delta_{ij},$$

where $x_i \in \{x_1, x_2, x_3\} = \{x, y, t\}$, and

$$\mathcal{L} = \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) - \sin\theta.$$

The diagonal elements of T are then

$$T_{11} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial x} - \mathcal{L} = -\theta_x^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 + \theta_x^2 - \theta_y^2) + \sin\theta,$$

$$T_{22} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial y} - \mathcal{L} = -\theta_y^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = -\frac{1}{2}(\theta_t^2 - \theta_x^2 + \theta_y^2) + \sin\theta,$$

$$T_{33} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial t} - \mathcal{L} = \theta_t^2 - \frac{1}{2}(\theta_t^2 - \theta_x^2 - \theta_y^2) + \sin\theta = \frac{1}{2}(\theta_t^2 + \theta_x^2 + \theta_y^2) + \sin\theta,$$

and the nondiagonal elements are

$$T_{12} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial y} = -\theta_x\theta_y, \qquad T_{21} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial x} = -\theta_x\theta_y, \qquad T_{31} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial x} = \theta_x\theta_t,$$
$$T_{13} = \frac{\partial \mathcal{L}}{\partial \theta_x}\frac{\partial \theta}{\partial t} = -\theta_x\theta_t, \qquad T_{23} = \frac{\partial \mathcal{L}}{\partial \theta_y}\frac{\partial \theta}{\partial t} = -\theta_y\theta_t, \qquad T_{32} = \frac{\partial \mathcal{L}}{\partial \theta_t}\frac{\partial \theta}{\partial y} = \theta_y\theta_t.$$

While writing up these solutions, I consulted Gelfand and Fomin's *Calculus of Variations*.