Problem 1. Alternative regulators in QED (Peskin & Schroeder 7.2) In Section. 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Paul-Villars regularization.

1(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

Solution. From (7.47) in Peskin & Schroeder,

$$\Gamma^\mu(q=0) = \frac{1}{Z_1}\gamma^\mu,$$

we can find an expression for δZ_1, similar to how (7.31) is obtained from (7.26). Roughly,

$$\frac{1}{Z_1 + \delta Z_1}\gamma^\mu \approx Z_1(1-\delta Z_1)\gamma^\mu = \Gamma^\mu(q=0) + \delta\Gamma^\mu(q=0) \quad \Longrightarrow \quad \delta\Gamma^\mu(q=0) = -\delta Z_1 \gamma^\mu. \tag{1}$$

According to (6.33),

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2).$$

We note that $\Gamma^\mu = \gamma^\mu$, $F_1 = 1$, and $F_2 = 0$ to lowest order [1, pp. 185–186]. Then we can write

$$\delta\Gamma^\mu(q=0) = \gamma^\mu \delta F_1(0) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\delta F_2(0). \tag{2}$$

Using this equation and the identity $\gamma^\mu\gamma_\mu = 4$ [2], Eq. (1) can be written

$$\delta Z_1 = -\frac{1}{4}\gamma_\mu \delta\Gamma^\mu(q=0) = -\delta F_1(0) - \gamma_\mu\frac{i\sigma^{\mu\nu}q_\nu}{8m}\delta F_2(0). \tag{3}$$

In order to find $\delta\Gamma^\mu$ we use (6.47):

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = 2ie^2 \int \frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3} \\ \times \bar{u}(p')\left\{\gamma^\mu\left[-\frac{\ell^2}{2} + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right]\right. \\ \left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m}[2m^2 z(1-z)]\right\}u(p),\end{aligned} \tag{4}$$

where $\Delta \equiv -xyq^2 + (1-z)^2m^2$ by (6.44), $\ell \equiv k + yq - zp$, and $D = \ell^2 - \Delta + i\epsilon$ [1, p. 191]. The momenta k and p are assigned in the Feynman diagram on p. 189 of Peskin & Schroeder, and x, y are Feynman parameters [1, p. 190].

The forms of the two integrals we need to compute are given by Peskin & Schroeder (6.49) and (6.50). Equation (6.49) is

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}. \tag{5}$$

Here $m = 3$ because we have D^{-3} in Eq. (4). We can evaluate the left-hand side using the Euclidian 4-momentum defined in (6.48),

$$\ell^0 \equiv \ell_E^0, \qquad\qquad \boldsymbol{\ell} = \boldsymbol{\ell}_E.$$

Following the steps on p. 193, we have

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} = \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int \frac{d^4\ell_E}{(\ell_E^2+\Delta)^3} = \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^3},$$

where we have replaced the upper (infinite) bound of integration by a finite number Λ. Evaluating this integral using Mathematica and using $\int d\Omega_4 = 2\pi^2$ [1, p. 193], we find

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{\Lambda^4}{4\Delta(\Delta+\Lambda^2)^2} \\ &= -\frac{i}{32\pi^2}\frac{\Lambda^4}{\Delta(\Delta+\Lambda^2)^2} \\ &\approx -\frac{i}{32\pi^2}\frac{1}{\Delta} \equiv \alpha, \end{aligned} \tag{6}$$

where we have taken the limit $\Lambda \gg \Delta$ [1, p. 218] and defined α. Equation (6.50) in Peskin & Schroeder is

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}.$$

Following similar steps as for Eq. (4), the left-hand side is

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^3} &= \frac{i}{(-1)^3}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3} \\ &= \frac{i(-1)^3}{(2\pi)^4}\int d\Omega_4\int_0^\Lambda d\ell_E\,\frac{\ell_E^5}{(\ell_E^2+\Delta)^3} \\ &= \frac{i(-1)^3}{(2\pi)^4}(2\pi^2)\frac{1}{4}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right] \\ &= -\frac{i}{32\pi^2}\left[\frac{\Delta(3\Delta+4\Lambda^2)}{(\Delta+\Lambda^2)^2} + 2\ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 3\right] \\ &\approx -\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right) \equiv \beta, \end{aligned} \tag{7}$$

where we have defined β and ignored terms of $\mathcal{O}(\Lambda^{-2})$ [1, p. 218]. We also ignore constant terms since they do not diverge [1, p. 196].

We now set $q^2 = 0$, and define $\Delta_0 = (1-z)^2 m^2$. Then $\Delta \to \Delta_0$ in our expression and $\alpha \to \alpha_0, \beta \to \beta_0$ (which are functions of Δ_0). Feeding in Eqs. (6) and (7), Eq. (4) can be written

$$\bar{u}(p')\delta\Gamma^\mu(q=0)u(p) = 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\bar{u}(p')\int\left\{\gamma^\mu\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right]\right\}u(p).$$

Then

$$\begin{aligned}\delta F_1(0) &= 2ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right] \\ &= 2ie^2\int_0^1 dz\,(1-z)\left[-\beta_0 + 2m^2(1-4z+z^2)\alpha_0\right],\end{aligned}$$

$$\begin{aligned}\delta F_2(0) &= 8ie^2\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)m^2 z(1-z)\alpha_0 \\ &= 8ie^2\int_0^1 dz\,m^2 z(1-z)^2\alpha_0.\end{aligned}$$

We ignore $\delta F_2(0)$ since it is not affected by the divergence [1, p. 196]. In order to avoid issues coming from the divergence in $\delta F_1(0)$, we add a $z\mu^2$ term to Δ_0 [1, p. 195]. So, feeding these results into Eq. (3), we obtain

$$\delta Z_1 = -2ie^2 \int_0^1 dz\,(1-z)\left[-\beta_0 + 2(1-4z+z^2)m^2\alpha_0\right], \tag{8}$$

where

$$\alpha_0 = -\frac{i}{32\pi^2}\frac{1}{\Delta_0}, \qquad \beta_0 = -\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta_0}\right), \qquad \Delta_0 = (1-z)^2m^2 + z\mu^2. \tag{9}$$

For δZ_2, we can use the first part of Peskin & Schroeder (7.31),

$$\delta Z_2 = \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p = m}, \tag{10}$$

where Σ_2 is given by (7.17),

$$-i\Sigma_2(p) = -e^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\not p + 4m_0}{(\ell^2-\Delta+i\epsilon)^2}, \tag{11}$$

where $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. We may once again follow the steps on p. 193 to evaluate the integral, now with $m = 2$. Changing the upper bound of integration to Λ once more, we have

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^2} &= \frac{i}{(-1)^2}\frac{1}{(2\pi)^4}\int d^4\ell_E\,\frac{1}{(\ell_E^2+\Delta)^2} \\
&= \frac{i(-1)^2}{(2\pi)^4}\int d\Omega_4 \int_0^\Lambda d\ell_E\,\frac{\ell_E^3}{(\ell_E^2+\Delta)^2} \\
&= \frac{i}{(2\pi)^4}(2\pi^2)\frac{1}{2}\left[\frac{\Delta}{\Delta+\Lambda^2} + \ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] \\
&= \frac{i}{16\pi^2}\left[\frac{\Delta}{\Delta+\Lambda^2} + \ln\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - 1\right] \\
&\approx \frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right),
\end{aligned}$$

where we have evaluated the integral using Mathematica, taken the large Λ limit, and dropped the irrelevant constant. Substituting back into Eq. (11), we find

$$\Sigma_2(p) = -ie^2\int_0^1 dx\,(-2x\not p + 4m_0)\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right).$$

Note that

$$\begin{aligned}
\frac{d\Sigma_2}{d\not p} &= \frac{e^2}{16\pi^2}\frac{d}{d\not p}\left[\int_0^1 dx\,(-2x\not p + 4m_0)\ln\left(\frac{\Lambda^2}{\Delta}\right)\right] \\
&= \frac{e^2}{16\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{\Delta}\right)\frac{d}{d\not p}(-2x\not p + 4m_0) + (-2x\not p + 4m_0)\frac{d}{d\not p}\ln\left(\frac{\Lambda^2}{\Delta}\right)\right] \\
&= \frac{e^2}{16\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{\Delta}\right)\frac{d}{d\not p}(-2x\not p + 4m_0) + (-2x\not p + 4m_0)\frac{d}{d\Delta}\ln\left(\frac{\Lambda^2}{\Delta}\right)\frac{d\Delta}{d\not p}\right].
\end{aligned} \tag{12}$$

Using $p^2 = \not{p}^2$ [1, p. 220], note that

$$\frac{d\Delta}{d\not{p}} = \frac{d}{d\not{p}}[-x(1-x)\not{p}^2 + x\mu^2 + (1-x)m_0^2] = -2x(1-x)\not{p}.$$

Also,

$$\frac{d}{d\not{p}}(-2x\not{p} + 4m_0) = -2x, \qquad \frac{d}{d\Delta}\left[\ln\left(\frac{\Lambda^2}{\Delta}\right)\right] = \frac{d}{d\Delta}\left[\ln(\Lambda^2) - \ln(\Delta)\right] = -\frac{1}{\Delta}.$$

Making these substitutions in Eq. (12),

$$\frac{d\Sigma_2}{d\not{p}} = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2x\ln\left(\frac{\Lambda^2}{\Delta}\right) - \frac{(2x\not{p} - 4m_0)[2x(1-x)\not{p}]}{\Delta}\right].$$

We now define

$$\Delta_m \equiv -x(1-x)m^2 + x\mu^2 + (1-x)m_0^2 \approx (1-x)^2m^2 + x\mu^2, \tag{13}$$

since $m \approx m_0$. Then Eq. (10) becomes

$$\delta Z_2 = \frac{e^2}{16\pi^2}\int_0^1 dx\left[-2x\ln\left(\frac{\Lambda^2}{\Delta}\right) - \frac{(2xm + 4m_0)[2x(1-x)m]}{\Delta_m}\right]. \tag{14}$$

Now we write out δZ_1 and δZ_2 fully, feeding Eqs. (9) and (13) into Eqs. (8) and (14), respectively. We also rename $x \to z$ in δZ_2:

$$\begin{aligned}
\delta Z_1 &= -2ie^2\int_0^1 dz\,(1-z)\left[-\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{\Delta}\right) + 2(1-4z+z^2)m^2\left(-\frac{i}{32\pi^2}\frac{1}{\Delta_0}\right)\right] \\
&= \frac{e^2}{8\pi^2}\int_0^1 dz\,(1-z)\left[\ln\left(\frac{\Lambda^2}{(1-z)^2m^2 + z\mu^2}\right) - \frac{m^2(1-4z+z^2)}{(1-z)^2m^2 + z\mu^2}\right],
\end{aligned}$$

$$\delta Z_2 = -\frac{e^2}{8\pi^2}\int_0^1 dz\left[z\ln\left(\frac{\Lambda^2}{(1-z)^2m^2 + z\mu^2}\right) + \frac{2zm^2(1-z)(2+z)}{(1-z)^2m^2 + z\mu^2}\right].$$

Clearly $\delta Z_1 \neq \delta Z_2$, as we wanted to show. □

References

[1] M. E. Peskin and D. V. Schroeder, “An Introduction to Quantum Field Theory”. Perseus Books Publishing, 1995.

[2] Wikipedia contributors, “Gamma matrices.” From Wikipedia, the Free Encyclopedia. `https://en.wikipedia.org/wiki/Gamma_matrices`.