Problem 1. A spherical shell of radius R has a total charge Q uniformly spread over the shell. The shell is now put into uniform rotation about the z axis with angular velocity ω. Find the vector potential $\mathbf{A}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ everywhere, i.e., both inside and outside of the shell.

Solution. Let $\rho(\mathbf{x})$ be the charge density everywhere in space, so

$$\rho(\mathbf{x}) = \frac{1}{4\pi} \frac{Q}{R^2} \delta(r - R).$$

The linear velocity of the moving charge everywhere is

$$\mathbf{v}(\mathbf{x}) = \omega r \, \delta(r - R) \, \hat{\boldsymbol{\varphi}}.$$

Then the current density $\mathbf{J}$ is simply the product of charge density and the linear velocity of the charge:

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x}) \, \mathbf{v}(\mathbf{x}) = \frac{Q\omega}{4\pi} \frac{r}{R^2} \delta(r - R) \, \hat{\boldsymbol{\varphi}}.$$

From Eq. (4.21) in the lecture notes, $\mathbf{A}(\mathbf{x})$ everywhere is given by

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{x})'}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \,.$$

The integral we need to evaluate is then

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi c} \frac{Q\omega}{R^2} \int \frac{r' \, \delta(r' - R)}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \,.$$

The problem is azimuthally symmetric, so we will rotate our coordinate system such that $\mathbf{x}$ points along the z axis. In the new coordinate system,

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{\sqrt{\mathbf{x}^2 - 2\mathbf{x} \cdot \mathbf{x}' + \mathbf{x}'^2}} = \frac{1}{\sqrt{r^2 - 2rr' \cos\theta' + r'^2}}.$$

Let $\boldsymbol{\omega}$ be the angular velocity vector (that lay along the z axis of the original coordinate system), which we choose to lie in the xz plane. Let α be the angle between $\boldsymbol{\omega}$ and the z axis. Then the linear velocity of the moving charge is

$$\mathbf{v}(\mathbf{x}') = \boldsymbol{\omega} \times \mathbf{x}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\alpha & 0 & \omega \cos\alpha \\ r' \sin\theta' \cos\varphi' & r' \sin\theta' \sin\varphi' & r' \cos\theta' \end{vmatrix}$$
$$= -\omega r' (\cos\alpha \sin\theta' \sin\varphi') \, \hat{\mathbf{x}} + \omega r' (\cos\alpha \sin\theta' \cos\varphi' - \sin\alpha \cos\theta') \, \hat{\mathbf{y}} + \omega r' (\sin\alpha \sin\theta' \sin\varphi') \, \hat{\mathbf{z}},$$

so in the new coordinate system,

$$\mathbf{J}(\mathbf{x}') = \frac{Q}{4\pi} \frac{\boldsymbol{\omega} \times \mathbf{x}'}{R^2} \delta(r' - R) = \frac{Q\omega}{4\pi} \frac{r'}{R^2} (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}') \, \delta(r' - R),$$

where
$$\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}' = -(\cos\alpha \sin\theta' \sin\varphi') \, \hat{\mathbf{x}} + (\cos\alpha \sin\theta' \cos\varphi' - \sin\alpha \cos\theta') \, \hat{\mathbf{y}} + (\sin\alpha \sin\theta' \sin\varphi') \, \hat{\mathbf{z}}.$$

The integral we need to evaluate becomes

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi c} \frac{Q\omega}{R^2} \int_0^{2\pi} \int_{-1}^1 \int_0^\infty \frac{r'^3 (\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}') \, \delta(r' - R)}{\sqrt{r^2 - 2rr' \cos\theta' + r'^2}} \, dr' \, d(\cos\theta') \, d\varphi' \,.$$

Evaluating the radial integral, we have

$$\mathbf{A}(\mathbf{x}) = \frac{Q\omega R}{4\pi c} \int_0^{2\pi} \int_{-1}^1 \frac{\hat{\boldsymbol{\omega}} \times \hat{\mathbf{x}}'}{\sqrt{r^2 - 2Rr\cos\theta' + R^2}}\, d(\cos\theta')\, d\varphi\,.$$

For the angular integrals, the $\hat{\mathbf{x}}$ term is

$$-\cos\alpha\,\hat{\mathbf{x}} \int_{-1}^1 \frac{\sin\theta'}{\sqrt{r^2 - 2Rr\cos\theta' + R^2}}\, d(\cos\theta') \int_0^{2\pi} \sin\varphi'\, d\varphi \propto \left[-\cos\varphi'\right]_0^{2\pi} = 0.$$

Similarly, the $\hat{\mathbf{z}}$ term is

$$\sin\alpha\,\hat{\mathbf{z}} \int_{-1}^1 \frac{\sin\theta'}{\sqrt{r^2 - 2Rr\cos\theta' + R^2}}\, d(\cos\theta') \int_0^{2\pi} \sin\varphi'\, d\varphi \propto \left[-\cos\varphi'\right]_0^{2\pi} = 0.$$

There are two $\hat{\mathbf{y}}$ terms. For the first,

$$\cos\alpha\,\hat{\mathbf{y}} \int_{-1}^1 \frac{\sin\theta'}{\sqrt{r^2 - 2Rr\cos\theta' + R^2}}\, d(\cos\theta') \int_0^{2\pi} \cos\varphi'\, d\varphi \propto \left[\sin\varphi'\right]_0^{2\pi} = 0.$$

For the second,

$$-\sin\alpha\,\hat{\mathbf{y}} \int_{-1}^1 \frac{\cos\theta'}{\sqrt{r^2 - 2Rr\cos\theta' + R^2}}\, d(\cos\theta') \int_0^{2\pi} d\varphi = -2\pi\sin\alpha\,\hat{\mathbf{y}} \int_{-1}^1 \frac{\cos\theta'}{\sqrt{r^2 - 2Rr\cos\theta' + R^2}}\, d(\cos\theta')$$

$$= -2\pi\sin\alpha\,\hat{\mathbf{y}} \left(\left[-\frac{\cos\theta'\sqrt{r^2 - 2Rr\cos\theta' + R^2}}{Rr}\right]_{-1}^1 + \frac{1}{Rr} \int_{-1}^1 \sqrt{r^2 - 2Rr\cos\theta' + R^2}\, d(\cos\theta') \right)$$

$$= -2\pi\sin\alpha\,\hat{\mathbf{y}} \left(\left[-\frac{\cos\theta'\sqrt{r^2 - 2Rr\cos\theta' + R^2}}{Rr}\right]_{-1}^1 + \frac{1}{Rr} \left[-\frac{(r^2 - 2Rr\cos\theta' + R^2)^{3/2}}{3Rr}\right]_{-1}^1 \right)$$

$$= -2\pi\sin\alpha\,\hat{\mathbf{y}} \left(-\frac{\sqrt{r^2 + 2Rr + R^2}}{Rr} + \frac{\sqrt{r^2 - 2Rr + R^2}}{Rr} - \frac{(r^2 - 2Rr + R^2)^{3/2}}{3R^2r^2} + \frac{(r^2 + 2Rr + R^2)^{3/2}}{3R^2r^2} \right)$$

$$= 2\pi\sin\alpha \frac{3Rr\sqrt{(r+R)^2} - 3Rr\sqrt{(r-R)^2} + [(r-R)^2]^{3/2} - [(r+R)^2]^{3/2}}{3R^2r^2}\hat{\mathbf{y}}$$

$$= 2\pi\sin\alpha \frac{3Rr|r+R| - 3Rr|r-R| + (r-R)^2|r-R| - (r+R)^2|r+R|}{3R^2r^2}\hat{\mathbf{y}}$$

$$= 2\pi\sin\alpha \frac{(r^2 + Rr + R^2)|r-R| - (r^2 - Rr + R^2)(r+R)}{3R^2r^2}\hat{\mathbf{y}}$$

$$= \frac{2\pi\sin\alpha\,\hat{\mathbf{y}}}{3R^2r^2} \begin{cases} (r^2 + Rr + R^2)(R-r) - (r^2 - Rr + R^2)(r+R) & r < R, \\ (r^2 + Rr + R^2)(r-R) - (r^2 - Rr + R^2)(r+R) & r > R \end{cases}$$

$$= -\frac{4}{3}\pi\sin\alpha\,\hat{\mathbf{y}} \begin{cases} \dfrac{r}{R^2} & r < R, \\ \dfrac{R}{r^2} & r > R. \end{cases}$$

Finally, in the new coordinate system we have

$$\mathbf{A}(\mathbf{x}) = -\frac{Q\omega}{3c}\sin\alpha\,\hat{\mathbf{y}} \begin{cases} \dfrac{r}{R} & r < R, \\ \dfrac{R^2}{r^2} & r > R. \end{cases}$$

Transforming back to the old coordinate system, $\sin \alpha \to -\sin \theta$. Since the original system is azimuthally symmetric, $\varphi = 0$ so $\hat{\mathbf{y}} = \sin \theta \sin \varphi \, \hat{\mathbf{r}} + \cos \theta \sin \varphi \, \hat{\boldsymbol{\theta}} + \cos \varphi \, \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$. Thus we have

$$\mathbf{A}(\mathbf{x}) = \frac{Q\omega}{3c} \sin \theta \, \hat{\boldsymbol{\varphi}} \begin{cases} \dfrac{r}{R} & r < R, \\[2mm] \dfrac{R^2}{r^2} & r > R. \end{cases}$$

The magnetic field is given by Eq. (1.7),
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1}$$

In spherical coordinates,

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\sin \theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi) \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\varphi}},$$

so

$$\mathbf{B} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\varphi) \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\varphi) \hat{\boldsymbol{\theta}}.$$

For $r < R$,

$$\mathbf{B}(\mathbf{x}) = \frac{Q\omega}{3c} \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{r}{R} \sin^2 \theta \right) \hat{\mathbf{r}} - \frac{\partial}{\partial r} \left(\frac{r^2}{R} \sin \theta \right) \hat{\boldsymbol{\theta}} \right] = \frac{Q\omega}{3c} \frac{1}{r} \left(\frac{r}{R} \frac{2 \cos \theta \sin \theta}{\sin \theta} \hat{\mathbf{r}} - \frac{2r}{R} \sin \theta \, \hat{\boldsymbol{\theta}} \right)$$

$$= \frac{2}{3} \frac{Q\omega}{cR} (\cos \theta \, \hat{\mathbf{r}} - \sin \theta \, \hat{\boldsymbol{\theta}}) = \frac{2}{3} \frac{Q\omega}{cR} \hat{\mathbf{z}}.$$

For $r > R$,

$$\mathbf{B}(\mathbf{x}) = \frac{Q\omega}{3c} \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{R^2}{r^2} \sin^2 \theta \right) \hat{\mathbf{r}} - \frac{\partial}{\partial r} \left(\frac{R^2}{r} \sin \theta \right) \hat{\boldsymbol{\theta}} \right] = \frac{Q\omega}{3c} \frac{1}{r} \left(\frac{R^2}{r^2} \frac{2 \cos \theta \sin \theta}{\sin \theta} \hat{\mathbf{r}} + 2 \frac{R^2}{r^2} \sin \theta \, \hat{\boldsymbol{\theta}} \right)$$

$$= \frac{2}{3} \frac{Q\omega}{c} \frac{R^2}{r^3} (\cos \theta \, \hat{\mathbf{r}} + \sin \theta \, \hat{\boldsymbol{\theta}}).$$

In summary,

$$\mathbf{B}(\mathbf{x}) = \frac{2}{3} \frac{Q\omega}{c} \begin{cases} \dfrac{\hat{\mathbf{z}}}{R} & r < R, \\[3mm] \dfrac{R^2}{r^3} (\cos \theta \, \hat{\mathbf{r}} + \sin \theta \, \hat{\boldsymbol{\theta}}) & r > R. \end{cases}$$

Problem 2. If an electric and magnetic field are both present, the momentum density carried by the electromagnetic field is given by Poynting's formula

$$\mathcal{P} = \frac{1}{4\pi c}(\mathbf{E} \times \mathbf{B}).$$

Consider a bounded distribution of time-independent charges and currents, i.e., $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ are time independent and vanish when $|\mathbf{x}| > R$ for some R.

2.a Show that the total momentum can be written as

$$\mathbf{P} \equiv \int \mathcal{P}(\mathbf{x})\, d^3x = \int \phi(\mathbf{x})\, \mathbf{J}(\mathbf{x})\, d^3x\,.$$

Solution. Applying (1),

$$\mathbf{E} \times \mathbf{B} = \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{A}).$$

Vector identity (4) in Griffiths is

$$\boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{b}) + \mathbf{b} \times (\boldsymbol{\nabla} \times \mathbf{a}) + (\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b} + (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{a},$$

which allows us to write

$$\mathbf{E} \times \mathbf{B} = \boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{E}) - \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{E}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{E} - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{A} = \boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{E}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{E} - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{A},$$

since $\boldsymbol{\nabla} \times E = 0$ in electrostatics by Eq. (1.4) in the lecture notes. Now using component notation with implied sums,

$$(\mathbf{A} \cdot \boldsymbol{\nabla})E_i = A_j \frac{\partial E_i}{\partial x_j} = \frac{\partial}{\partial x_j}(A_j E_i) - E_i \frac{\partial A_j}{\partial x_j} = \frac{\partial}{\partial x_j}(A_j E_i).$$

Here we have used the product rule in addition to Eq. (4.20), which states that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in the Coulomb gauge, which we may choose without loss of generality. Similarly,

$$(\mathbf{E} \cdot \boldsymbol{\nabla})A_i = E_j \frac{\partial A_i}{\partial x_j} = \frac{\partial}{\partial x_j}(E_j A_i) - A_i \frac{\partial E_j}{\partial x_j} = \frac{\partial}{\partial x_j}(E_j A_i) + A_i \nabla^2 \phi,$$

where we have used Eq. (2.2), $\mathbf{E} = -\boldsymbol{\nabla}\phi$, which holds in the electrostatic case. Putting this all together, we have

$$(\mathbf{E} \times \mathbf{B})_i = \frac{\partial}{\partial x_i}(A_j E_j) - \frac{\partial}{\partial x_j}(A_j E_i) - \frac{\partial}{\partial x_j}(E_j A_i) - A_i \nabla^2 \phi,$$

and so

$$\int (\mathbf{E} \times \mathbf{B})_i\, d^3x = \int \left(\frac{\partial}{\partial x_i}(A_j E_j) - \frac{\partial}{\partial x_j}(A_j E_i) - \frac{\partial}{\partial x_j}(E_j A_i) - A_i \nabla^2 \phi \right) d^3x\,.$$

Let $L \geq R$. Note that

$$\int f(\mathbf{x})\, d^3x = \lim_{L\to\infty} \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} f(\mathbf{x})\, dx\, dy\, dz\,.$$

Then for the first term, integrating with respect to x_i by parts gives us

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{\partial}{\partial x_i}(A_j E_j)\, dx_i = \Big[A_j E_j \Big]_{-L}^{L} = 0,$$

since both $\rho(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ vanish for $|\mathbf{x}| > R$. This means $\mathbf{E} \to 0$ and $\mathbf{A} \to 0$ as $|\mathbf{x}| \to \infty$. Applying similar logic to the second and third terms,

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{\partial}{\partial x_j}(A_j E_i)\, dx_j = \left[A_j E_i\right]_{-L}^{L} = 0, \qquad \lim_{L\to\infty} \int_{-L}^{L} \frac{\partial}{\partial x_j}(E_j A_i)\, dx_j = \left[E_j A_i\right]_{-L}^{L} = 0,$$

where there are no implied sums over the derivatives. Now we have

$$\int (\mathbf{E} \times \mathbf{B})_i\, d^3x = -\int A_i \nabla^2 \phi\, d^3x\,.$$

Green's theorem is given by Eq. (2.96),

$$\int_S \hat{\mathbf{n}} \cdot (\phi_1 \boldsymbol{\nabla} \phi_2 - \phi_2 \boldsymbol{\nabla} \phi_1)\, dS = -4\pi \int_{\mathcal{V}} (\phi_1 \rho_2 - \phi_2 \rho_1)\, d^3x = \int_{\mathcal{V}} (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1)\, d^3x\,,$$

where the final equality comes from the proof in Eq. (2.97). Let $\mathcal{V}$ be a cube of side length $2L$ centered at the origin. Applying Green's theorem gives us

$$\int_{\mathcal{V}} (\mathbf{E} \times \mathbf{B})_i\, d^3x = \int_S \hat{\mathbf{n}} \cdot (\phi \boldsymbol{\nabla} A_i - A_i \boldsymbol{\nabla} \phi)\, dS - \int_{\mathcal{V}} \phi \nabla^2 A_i\, d^3x\,.$$

Note that

$$\lim_{L\to\infty} \int_S \hat{\mathbf{n}} \cdot (\phi \boldsymbol{\nabla} A_i - A_i \boldsymbol{\nabla} \phi)\, dS \propto \lim_{L\to\infty} \int_S \frac{1}{|\mathbf{x}|^3}\, dS = 0$$

since $\phi, A_i \propto 1/|\mathbf{x}|$ and $\boldsymbol{\nabla}\phi, \boldsymbol{\nabla} A_i \propto 1/|\mathbf{x}|^2$. Now we have

$$\int (\mathbf{E} \times \mathbf{B})\, d^3x = -\int \phi \nabla^2 \mathbf{A}\, d^3x\,.$$

Vector identity (11) in Griffiths states that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{a}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{a}) - \nabla^2 \mathbf{a},$$

which gives us

$$\int (\mathbf{E} \times \mathbf{B})\, d^3x = \int \phi[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})]\, d^3x = \frac{4\pi}{c} \int \phi \mathbf{J}\, d^3x\,,$$

where we have once again used the Coulomb gauge condition, and that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = 4\pi \mathbf{J}/c$ from Eq. (4.4). Thus, we have proven

$$\mathbf{P} = \int \mathcal{P}(\mathbf{x})\, d^3x = \frac{1}{4\pi c} \int (\mathbf{E} \times \mathbf{B})\, d^3x = \frac{1}{c^2} \int \phi(\mathbf{x})\, \mathbf{J}(\mathbf{x})\, d^3x\,,$$

as desired, except for the factor of $1/c^2$. □

2.b Give an example of a stationary, bounded charge and current distribution for which $\mathbf{P} \neq 0$.

Solution. Consider a toroid in the xy plane centered on the origin, with N total turns and current I. Consider also a point charge of charge Q at the origin. In cylindrical coordinates (s, φ, z), the magnetic field due to the solenoid is given by Eq. (5.60) in Griffiths:

$$\mathbf{B}(\mathbf{x}) = \begin{cases} \dfrac{2NI}{s}\hat{\boldsymbol{\varphi}} & \text{inside,} \\[2mm] 0 & \text{outside.} \end{cases}$$

The electric field due to the point charge is simply

$$\mathbf{E} = \frac{Q}{s^2 + z^2}\hat{\mathbf{r}}.$$

Let T denote the volume of the toroid, and note that $\mathbf{E} \times \mathbf{B} \neq 0$ only within this finite volume. Then

$$\mathbf{P} = \frac{1}{4\pi c}\int (\mathbf{E} \times \mathbf{B})\, d^3x = \frac{2NIQ}{4\pi c}\int_T \frac{\hat{\mathbf{s}} \times \hat{\boldsymbol{\varphi}}}{s(s^2+z^2)}\, d^3x = \frac{NIQ}{2\pi c}\hat{\mathbf{z}}\int_T \frac{d^3x}{s(s^2+z^2)},$$

which is nonzero.

Problem 3. The angular momentum density of the electromagnetic field is given by

$$\mathbf{l} = \mathbf{x} \times \mathcal{P} = \frac{c}{4\pi}\mathbf{x} \times (\mathbf{E} \times \mathbf{B}).$$

Consider a source free $(\rho = 0, \mathbf{J} = 0)$ solution to Maxwell's equations in electrodynamics with $\mathbf{E}$ and $\mathbf{B}$ vanishing rapidly as $|\mathbf{x}| \to \infty$, so the total angular momentum

$$\mathbf{L} = \int \mathbf{l}\, d^3x$$

is well defined. Show that $\mathbf{L}$ is conserved, i.e., independent of time.

Solution. We want to show that

$$\frac{d\mathbf{L}}{dt} = 0.$$

From Eq. (4.12) in the lecture notes,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{2}$$

Then

$$\mathbf{l} = (\mathbf{x} \cdot \mathbf{B})\mathbf{E} - (\mathbf{x} \cdot \mathbf{E})\mathbf{B}.$$

Differentiating with respect to time and applying the product rule, we find

$$\frac{\partial \mathbf{l}}{\partial t} = \mathbf{E}\frac{\partial}{\partial t}(\mathbf{x} \cdot \mathbf{B}) + (\mathbf{x} \cdot \mathbf{B})\frac{\partial \mathbf{E}}{\partial t} - \mathbf{B}\frac{\partial}{\partial t}(\mathbf{x} \cdot \mathbf{E}) + (\mathbf{x} \cdot \mathbf{E})\frac{\partial \mathbf{B}}{\partial t}$$

$$= \mathbf{E}\left(\frac{\partial \mathbf{x}}{\partial t} \cdot \mathbf{B} + \mathbf{x} \cdot \frac{\partial \mathbf{B}}{\partial t}\right) + (\mathbf{x} \cdot \mathbf{B})\frac{\partial \mathbf{E}}{\partial t} - \mathbf{B}\left(\frac{\partial \mathbf{x}}{\partial t} \cdot \mathbf{E} + \mathbf{x} \cdot \frac{\partial \mathbf{E}}{\partial t}\right) + (\mathbf{x} \cdot \mathbf{E})\frac{\partial \mathbf{B}}{\partial t}$$

For a source-free solution, Maxwell's equations (1.1)–(1.4) in the lecture notes become

$$\boldsymbol{\nabla}\cdot\mathbf{E}=0, \qquad \boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=0, \qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0, \qquad \boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0.$$

This allows us to write

$$\frac{\partial\mathbf{l}}{\partial t}=\mathbf{E}[\mathbf{v}\cdot\mathbf{B}-c\mathbf{x}\cdot(\boldsymbol{\nabla}\times\mathbf{E})]+c(\mathbf{x}\cdot\mathbf{B})(\boldsymbol{\nabla}\times\mathbf{B})-\mathbf{B}[\mathbf{v}\cdot\mathbf{E}+c\mathbf{x}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]-c(\mathbf{x}\cdot\mathbf{E})(\boldsymbol{\nabla}\times\mathbf{E})$$

$$=\mathbf{E}(\mathbf{v}\cdot\mathbf{B})-\mathbf{B}(\mathbf{v}\cdot\mathbf{E})-c\mathbf{E}[\mathbf{x}\cdot(\boldsymbol{\nabla}\times\mathbf{E})]-c\mathbf{B}[\mathbf{x}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]+c(\mathbf{x}\cdot\mathbf{B})(\boldsymbol{\nabla}\times\mathbf{B})-c(\mathbf{x}\cdot\mathbf{E})(\boldsymbol{\nabla}\times\mathbf{E})$$

where $\mathbf{v}=\partial\mathbf{x}/\partial t$. Vector calculus identity (6) in Griffiths is

$$\boldsymbol{\nabla}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\boldsymbol{\nabla}\times\mathbf{a})-\mathbf{a}\cdot(\boldsymbol{\nabla}\times\mathbf{b}).$$

Then

$$\mathbf{x}\cdot(\boldsymbol{\nabla}\times\mathbf{E})=\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{x})+\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{x})=\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{x}),$$

and similarly for $\mathbf{E}\to\mathbf{B}$. So

$$\frac{\partial\mathbf{l}}{\partial t}=\mathbf{E}(\mathbf{v}\cdot\mathbf{B})-\mathbf{B}(\mathbf{v}\cdot\mathbf{E})-c\mathbf{E}[\boldsymbol{\nabla}\cdot(\mathbf{x}\times\mathbf{E})]-c\mathbf{B}[\boldsymbol{\nabla}\cdot(\mathbf{x}\times\mathbf{B})]+c(\mathbf{x}\cdot\mathbf{B})(\boldsymbol{\nabla}\times\mathbf{B})-c(\mathbf{x}\cdot\mathbf{E})(\boldsymbol{\nabla}\times\mathbf{E})$$

In component notation,

$$\frac{\partial l_i}{\partial t}=E_i[v_jB_j-cx_j(\boldsymbol{\nabla}\times\mathbf{E})_j]+cx_jB_j(\boldsymbol{\nabla}\times\mathbf{B})_i-B_i[v_jE_j+cx_j(\boldsymbol{\nabla}\times\mathbf{B})_j]-cx_jE_j(\boldsymbol{\nabla}\times\mathbf{E})_i$$

$$=E_iv_jB_j-cE_ix_j(\boldsymbol{\nabla}\times\mathbf{E})_j+cx_jB_j(\boldsymbol{\nabla}\times\mathbf{B})_i-B_iv_jE_j+cB_ix_j(\boldsymbol{\nabla}\times\mathbf{B})_j-cx_jE_j(\boldsymbol{\nabla}\times\mathbf{E})_i$$

In addition to the course lecture notes, I consulted Griffiths's *Introduction to Electrodynamics*, Jackson's *Classical Electrodynamics*, and Kirk McDonald's electromagnetism notes while writing up these solutions.