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Homework 2

Computational Physics
October 11, 2019

Problem 1

10/10

a)

Proof. Starting point in overrelaxation, iteration uses the form

$$x' = (1+\omega)f(x) - \omega x$$

where $f(x)$ is the equation we are trying to solve for. If you call the next step x' as another function $g(x)$, then we have $g(x) = (1+\omega)f(x) - \omega x$. We can now expand $g(x)$ into a Taylor series of first order,

$$x' = g(x) = g(x^*) + (x - x^*)g'(x^*)$$

,

where $x*$ is the true solution. Because x^* is the solution, $g(x^*) = x^*$. Subtracting x^* from both sides gives

$$x' - x^* = (x - x^*)g'(x^*)$$

If we define our error in x as the difference between x and the true solution x^*, i.e. $\epsilon = x - x^*$, then our equation now reads as

$$\epsilon' = \epsilon g'(x^*)$$

✓

We now assume that we our value x relatively close to the true solution x^*, therefore, the functions $g(x)$ and $g(x^*)$ and their derivatives must be close to each other and we can replace one for the other in our equation. So we now have,

$$\epsilon' = \epsilon g'(x) = \epsilon[(1+\omega)f'(x) - \omega]$$

Thus we have now derived an equivalent of Eq. (6.81) for the overrelaxation method.

We will now show that the error on x' is given by

$$\epsilon' \simeq \frac{x - x'}{1 - 1/[(1+\omega)f'(x) - \omega]}$$

We start with the error on x and x' given by

$$x^* = x' + \epsilon' = x + \epsilon$$

Substituting our derived value for ϵ', we get

$$x' + \epsilon' = x + \frac{\epsilon'}{(1+\omega)f'(x) - \omega}$$

Some algebra will bring us to

$$\epsilon' - \frac{\epsilon'}{(1+\omega)f'(x)-\omega} = x - x'$$

$$\epsilon'(1 - \frac{1}{(1+\omega)f'(x)-\omega}) = x - x'$$

$$\epsilon' = \frac{x-x'}{1-1/[(1+\omega)f'(x)-\omega]}$$

$\square$

b) Number of iterations it takes for relaxation to converge to a solution of $x = 1 - e^{-cx}$ accurate to 10^{-6} is: 17

c) Results after trial and error and deciding on an overrelaxation parameter of $\omega = 0.7$

```
Overrelaxation with omega = 0.7
0.581330281775
0.761558295583
0.796255961834
0.79681701144
0.796812085284
0.79681213043
Overrelaxation number of iterations: 6
```

d)

Yes, there are circumstances in which an overrelaxation parameter of $\omega < 0$ could help find a solution faster. One potential circumstance is when our estimate keeps overshooting the true/correct value. With an $\omega > 0$ our estimate will continue to overestimate the answer as it oscillates between the two sides of the answer. However, with an $\omega < 0$, our answer will not overstep the true value as badly and may even underestimate it. Our estimate would then be able to converge faster toward the correct solution.

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Problem 2

Planck's radiation law tells us that the intensity of radiation per unit area and per unit wavelength λ from a black body of temperature T is

$$I(\lambda) = \frac{2\pi hc^2\lambda^{-5}}{\mathrm{e}^{hc/\lambda k_B T}-1}$$

a) Differentiating I w.r.t. λ gives via the chain rule

$$\frac{dI}{d\lambda} = \frac{-10\pi hc^2\lambda^{-6}}{\mathrm{e}^{hc/\lambda k_B T}-1} + \frac{hc\mathrm{e}^{hc/\lambda k_B T}}{\lambda^2 k_B T}\frac{2\pi hc^2\lambda^{-5}}{(\mathrm{e}^{hc/\lambda k_B T}-1)^2}$$

multiplying by $\frac{\lambda^6}{2\pi hc^2}$

$$\frac{dI}{d\lambda} = \frac{-5}{\mathrm{e}^{hc/\lambda k_B T}-1} + \frac{hc}{\lambda k_B T}\frac{\mathrm{e}^{hc/\lambda k_B T}}{(\mathrm{e}^{hc/\lambda k_B T}-1)^2}$$

Multiplying the numerator and denominator by $\mathrm{e}^{hc/\lambda k_B T} - 1$ gives

$$\frac{dI}{d\lambda} = \frac{-5(\mathrm{e}^{hc/\lambda k_B T}-1)+(hc/\lambda k_B T)\mathrm{e}^{hc/\lambda k_B T}}{(\mathrm{e}^{hc/\lambda k_B T}-1)^2}$$

Now, since we want to find the maximum wavelength λ for this function $I(\lambda)$, we can set $dI/d\lambda = 0$. Thus we can just set the numerator of fraction we derived to zero. So

$$-5(\mathrm{e}^{hc/\lambda k_B T} - 1) + \frac{hc}{\lambda k_B T}\mathrm{e}^{hc/\lambda k_B T} = 0$$

Or by multiplying by $\mathrm{e}^{-hc/\lambda k_B T}$, we get

$$5\mathrm{e}^{-hc/\lambda k_B T} + \frac{hc}{\lambda k_B T} - 5 = 0$$

.

Now we can make the substitution $x = hc/\lambda k_B T$ so that our equation now reads as

$$5\mathrm{e}^{-x} + x - 5 = 0$$

.

The solution to this equation optimizes $I(\lambda)$ and thus gives us a value for which the wavelength of maximum radiation obeys the *Wien displacement law*:

$$\lambda = \frac{b}{T}$$

where the so-called *Wien displacement constant* is $b = hc/k_B x$.

b) Solving this equation using binary search to an accuracy of $\epsilon = 10^{-6}$, the value for the displacement constant is

$$b = 0.00289777200602 \text{ [m} \cdot \text{K]}$$

c) From my equations and my value for b, my estimate of the surface temperature of the sun is $T = 5772.45419526$ [K].

are all these
digits significant?

Problem 3

Goal: Use gradient descent to fit the Schechter function:

$$n(M_{gal}) = \phi^* \big(\tfrac{M_{gal}}{M_*}\big)^{\alpha+1} \exp\big(-\tfrac{M_{gal}}{M_*}\big) \ln(10)$$

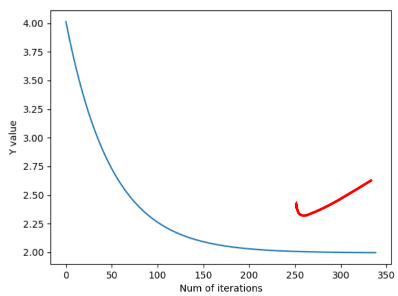
.

First, I tested an implementation of gradient descent to minimize the function

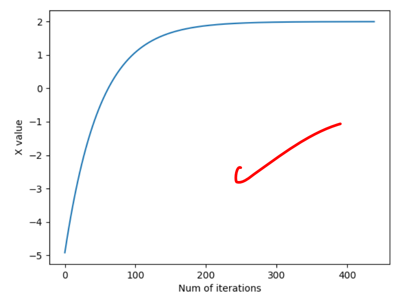
$$f(x, y) = (x-2)^2 + (y-2)^2$$

.

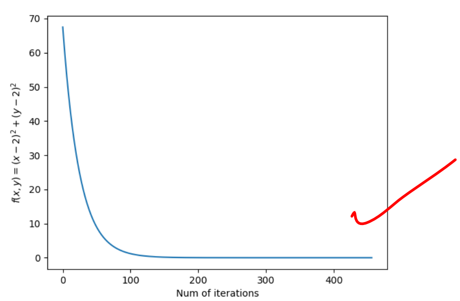
Using numerical derivatives, here are plots of my results:



(a) Y value convergence.



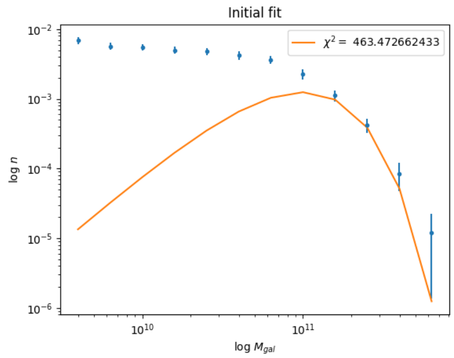
(b) X value convergence.



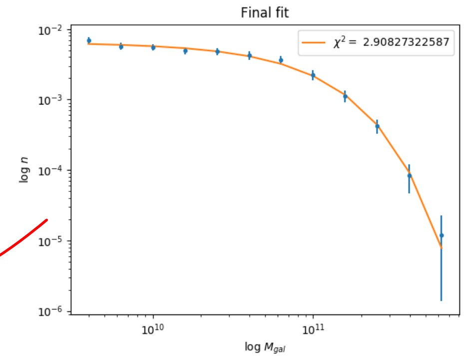
(c) $f(x, y)$ convergence.

Figure 1: As our gradient descent runs longer, our x and y values tend to 2 and our function $f(x, y)$ tends towards its minimum 0.

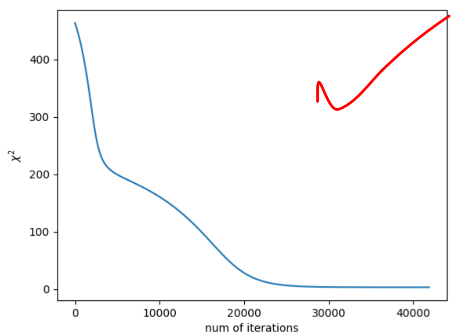
Now, applying gradient descent to the problem of fitting the Schechter function by minimizing the χ^2 of the model given some data, I am able to find the following results:



(a) Initial fit.



(b) Final fit.



(c) χ^2 as a function of step i.

Figure 2: Gradient descent minimizes the χ^2 model for the Schechter function to provide a fit with a chi-squared value of $\chi^2 = 2.9082$.