1. (+20) **Fourier transforms**. Evaluate the Fourier transform of the following functions by hand. Use the definitions I provided (includes $\frac{1}{\sqrt{2\pi}}$, this is common in physics but also now the default used in WolframAlpha - a powerful math AI tool) as well as the definition for Dirac delta I used if needed.

(a) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

(b) $f(t) = sin(\omega_0 t)$, ω_0 constant

(c) $f(x) = e^{-a|x|}$ and $a > 0$

(d) (distribution) $f(t) = \delta(t)$

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w')\, e^{iw't}\, dw'$$

a) $$\hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, e^{-iwx}\, dx$$

$$= \frac{1}{2\sigma\pi} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, e^{-iwx}\, dy \longrightarrow \text{Fourier Transform of zero-mean Gaussian}$$

$$= \frac{1}{2\sigma\pi} \cdot \left(\sqrt{2\pi}\,\sigma\, e^{-\frac{\sigma^2 w^2}{2}}\right) = \underline{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{\sigma^2 w^2}{2}} \cdot e^{-iw\mu}} \; /\!\!/$$

b) $f(t) = \sin(W_0 t)$, W_0 constant

$$\sin(W_0 t) = \frac{e^{iW_0 t} - e^{-iW_0 t}}{2i}$$

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(W_0 t)\, e^{-iwt}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{iW_0 t} - e^{-iW_0 t}}{2i}\, e^{-iwt}\, dt$$

$$\int e^{iW_0 t}\, e^{-iwt}\, dt = \int e^{-i(w-W_0)t}\, dt = 2\pi\,\delta(w-W_0)$$

$$\int e^{-iW_0 t}\, e^{-iwt}\, dt = \int e^{-i(w+W_0)t}\, dt = 2\pi\,\delta(w+W_0)$$

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{2i}\left[2\pi\,\delta(w-W_0) - 2\pi\,\delta(w+W_0)\right] = \underline{\frac{\sqrt{2\pi}}{2i}\left[\delta(w-W_0) - \delta(w+W_0)\right]} \; /\!\!/$$

c) $f(x) = e^{-a|x|}$, $a > 0$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{ax} e^{-iwx} \, dx + \int_{0}^{\infty} e^{-ax} e^{-iwx} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a-iw} + \frac{1}{a+iw} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{a+iw+a-iw}{a^2+w^2} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{a^2+w^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \, \frac{a}{a^2+w^2}$$ //

d) $f(t) = \delta(t)$

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-iwt} \, dt = \frac{1}{\sqrt{2\pi}}$$

2. (+10) **Correlation**. By definition, *correlation* is $p \odot q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p^*(\tau) q(t+\tau) d\tau$, and measures how similar one signal or data function is to another. Let $p(\tau) = \langle p \rangle + \delta_p(\tau)$ and $q(\tau) = \langle q \rangle + \delta_q(\tau)$, where $\langle\rangle$ and $\delta()$ denote the mean values and fluctuation functions (deviations about the mean). Two functions are defined to be *uncorrelated* when $p \odot q = \langle p \rangle \langle q \rangle$. Evaluate $p \odot q$ of the following functions:

$$p(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}, \quad q(t) = \begin{cases} 0 & t < 0 \\ 1 - t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

$p(\tau) q(t+\tau)$ is non-zero when $\begin{cases} 0 < \tau < 1 & (0,1) \\ 0 < t+\tau < 1 & (-t, 1-t) \end{cases}$

$\tau \in [\max(0,-t), \min(1, 1-t)]$

① If $t < -1$, $\max \geq 1$, $\min = 1$, no overlap, so $p \odot q = 0$

② If $-1 \leq t \leq 0$, then

$\max(0,-t) = -t$
$\min(1, 1-t) = 1$

$$p \cdot q = \frac{1}{\sqrt{2\pi}} \int_{\tau=-t}^{1} (1-t-\tau)\, d\tau = \frac{1-t^2}{2\sqrt{2\pi}}$$

③ If $0 \leq t \leq 1$, then
$\max(0,-t) = 0$
$\min(1, 1-t)$ $\quad 0 \leq t \leq 1$

$$p \cdot q = \frac{1}{\sqrt{2\pi}} \int_{\tau=0}^{1-t} (1-(t+\tau))\, d\tau = \frac{1}{\sqrt{2\pi}} \int_0^{1-t} 1-t-\tau\, d\tau$$

$$= \frac{(1-t)^2}{2\sqrt{2\pi}}$$

④ If $t > 1$

$\max(0,-t) = 0$
$\min(1, 1-t) = 1-t < 0$ $\Rightarrow$ No overlap, so $p \cdot q = 0$

$$p \cdot q = \begin{cases} \frac{1-t^2}{2\sqrt{2\pi}}, & -1 \leq t \leq 0 \\ \frac{(1-t)^2}{2\sqrt{2\pi}}, & 0 \leq t \leq 1 \\ 0, & |t| > 1 \end{cases}$$

t

3. (+10) **Autocorrelation.** Aside, periodic functions exhibit pronounced *autocorrelations* as shifting such functions by their period puts the function directly on itself. Alternatively, random functions or noise is characterized as being uncorrelated. Evaluate the autocorrelation $p \odot p$ of the following function:

$$p(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

$$p \cdot p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(\tau) p(t+\tau) \, d\tau$$

$\tau \in (0,1) \cap (-t, 1-t)$

Non-empty

① $-1 \le t \le 0$

$-t \ge 0$, $\max(0, -t) = -t$
$1-t \ge 1$, $\min(1, 1-t) = 1$

$\tau \in [-t, 1] \Rightarrow p(\tau) = 1$, $p(\tau + t) = 1$

$$p \cdot p = \frac{1}{\sqrt{2\pi}} \int_{-t}^{1} 1 \, d\tau = \frac{1+t}{\sqrt{2\pi}}, \quad -1 \le t \le 0$$

② $0 \le t \le 1$

$-t \le 0$, $\max(0, -t) = 0$
$1-t \le 1$, $\min(1, 1-t) = 1-t$

$\tau \in [0, 1-t] \Rightarrow p(\tau) = p(\tau + t) = 1$

$$p \cdot p = \frac{1}{\sqrt{2\pi}} \int_{0}^{1-t} 1 \, d\tau = \frac{1-t}{\sqrt{2\pi}}, \quad 0 \le t \le 1$$

③ $|t| > 1$

Doesn't overlap, so $p \cdot p = 0$

$$p \cdot p = \begin{cases} \frac{1+t}{\sqrt{2\pi}}, & -1 \le t \le 0 \\ \frac{1-t}{\sqrt{2\pi}}, & 0 \le t \le 1 \\ 0, & |t| > 1 \end{cases}$$

4. (+20) **Fourier transform diffusion equation solve.** Consider the diffusion equation $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$ where $T(x,t)$ describes the temperature profile of a long metal rod.

(a) Assume you know $T(x,0)$ and define the Fourier transform of $T(x,t)$ to be $\tau(k,t)$. Transform the original equation and initial conditions into k-space. Solve the resulting equation. Inverse transform the result to obtain the solution in terms of the original variables.

(b) Find the temperature in the rod given initial conditions $\kappa = 10^3 \frac{m^2}{s}$ and

$$T(x,0) = \begin{cases} 0 & |x| > 1m \\ 100^o \text{ C} & |x| \le 1m \end{cases} .$$

a) $\frac{\partial}{\partial t} T(x,t) \xrightarrow{F_x} \frac{\partial}{\partial t} \tau(k,t)$, $\quad \frac{\partial^2}{\partial x^2} T(x,t) \xrightarrow{F_x} -k^2 \tau(k,t)$

$$\frac{\partial \tau}{\partial t}(k,t) = -\kappa k^2 \tau(k,t) \qquad \tau(k,t) = \tau(k,0)\, e^{-\kappa k^2 t}$$

$$\tau(k,0) = \int_{-\infty}^{\infty} T(x,0)\, e^{-ikx}\, dx$$

$$T(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tau(k,0)\, e^{-\kappa k^2 t}\, e^{ikx}\, dk \qquad G(x,t) = \frac{1}{\sqrt{4\pi \kappa t}}\, e^{-\frac{x^2}{4\kappa t}}$$

$$T(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, T(y,0)\, dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-\frac{(x-y)^2}{4\kappa t}}\, T(y,0)\, dy$$

b) $G(x,t) = \frac{1}{\sqrt{4\pi \kappa t}}\, e^{-\frac{x^2}{4\kappa t}}$, $\quad t > 0$

$$T(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, T(y,0)\, dy$$

$$T(y,0) = 100 \Rightarrow T(x,y) = 100 \int_{-1}^{1} \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-\frac{(x-y)^2}{4\kappa t}}\, dy$$

Let $u = \frac{x-y}{2\sqrt{\kappa t}}$ $\quad dy = -2\sqrt{\kappa t}\, du$

$$\frac{1}{\sqrt{4\pi\kappa t}}\, dy = \frac{-2\sqrt{\kappa t}}{\sqrt{4\pi\kappa t}}\, du = -\frac{1}{\sqrt{\pi}}\, du$$

$$T(x,t) = 100\int_{\frac{x+1}{2\sqrt{kt}}}^{\frac{x-1}{2\sqrt{kt}}} e^{-u^2} \cdot -\frac{1}{\sqrt{\pi}}\, du = 100\,\frac{1}{\sqrt{\pi}}\int_{\frac{x-1}{2\sqrt{kt}}}^{\frac{x+1}{2\sqrt{kt}}} e^{-u^2}\, du$$

$$= 50\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{10^3 t}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{10^3 t}}\right)\right]$$