1. (+10) Given matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, evaluate the action of A on the unit balls of $\mathbb{R}^2$ defined by the 1-norm, 2-norm, and ∞-norm (induced matrix norms). Submit your work and drawings.

1-norm: $\max\{(1+0), (2+2)\} = \max\{1, 4\} = 4$

2-norm: Largest eval of $\sqrt{A^T A}$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

$$A^T A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{bmatrix}$$

$$\det(A^T A - \lambda I) = (1-\lambda)(8-\lambda) - 4$$
$$= \lambda^2 - 9\lambda + 4$$

$$\lambda = \sqrt{\frac{9 \pm \sqrt{81-16}}{2}} = \sqrt{\frac{9 \pm \sqrt{65}}{2}} \approx 2.920 \ \& \ 0.685$$

2-norm $= 2.920$

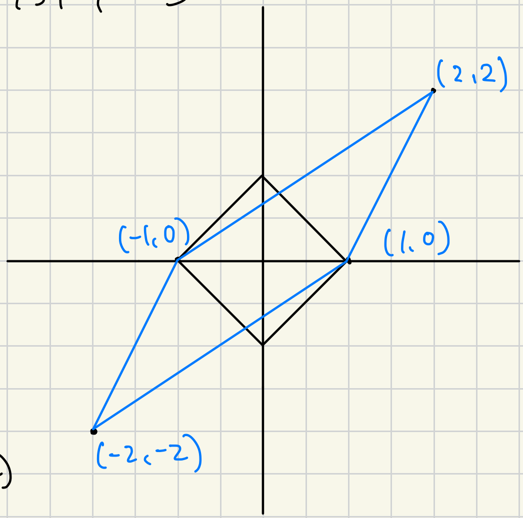
Infinity norm: $\max\{(1+2), (0+2)\} = \max\{3, 2\} = 3$

$$A(1,0) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,0)$$

$$A(0,1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = (2,2)$$

$$A(-1,0) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (-1,0)$$

$$A(0,-1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = (-2,-2)$$

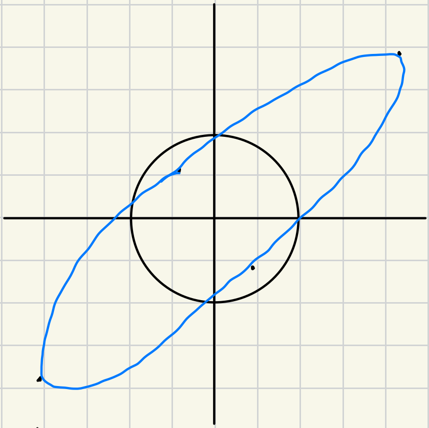


2-norm unit ball

endpoint = $A\hat{v}_i$

$$v_i = \begin{pmatrix} \frac{2}{\lambda_i - 1} \\ 1 \end{pmatrix}$$

$$\hat{v}_1 = \begin{pmatrix} \frac{2}{\lambda_1 - 1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2564 \\ 0.9656 \end{pmatrix}$$



$$A\hat{v}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0.2564 \\ 0.9656 \end{pmatrix} = \begin{pmatrix} 2.1876 \\ 1.9312 \end{pmatrix}$$

$$A\hat{v}_2 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -0.9662 \\ 0.2572 \end{pmatrix} = \begin{pmatrix} -0.4538 \\ 0.5144 \end{pmatrix}$$

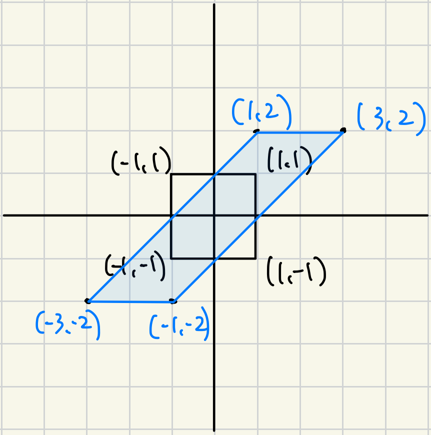
Infinity - norm unit-ball

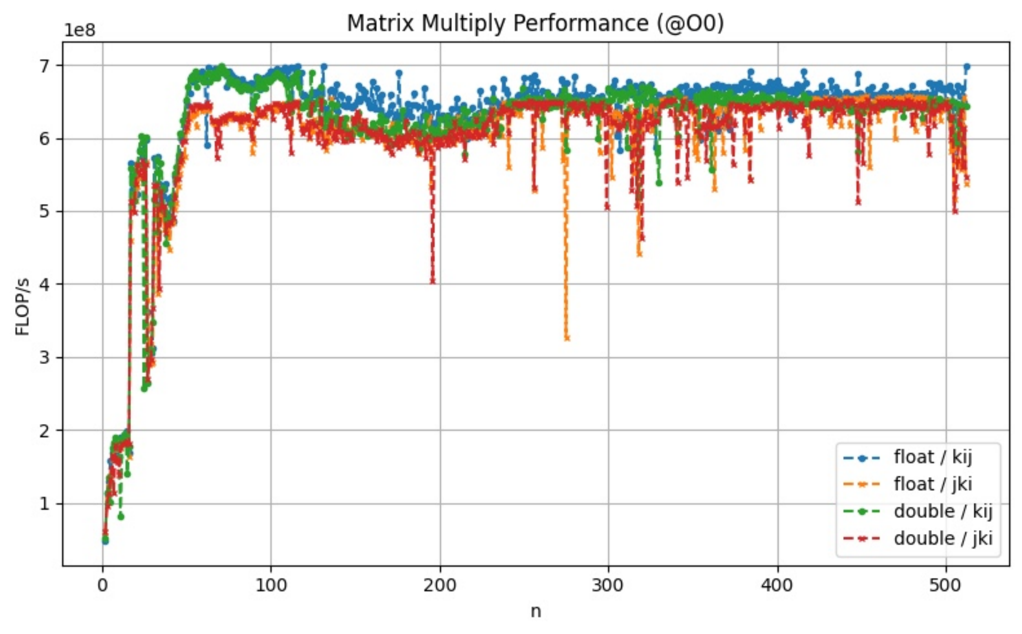
$$A(1,1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = (3,2)$$

$$A(-1,1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1,2)$$

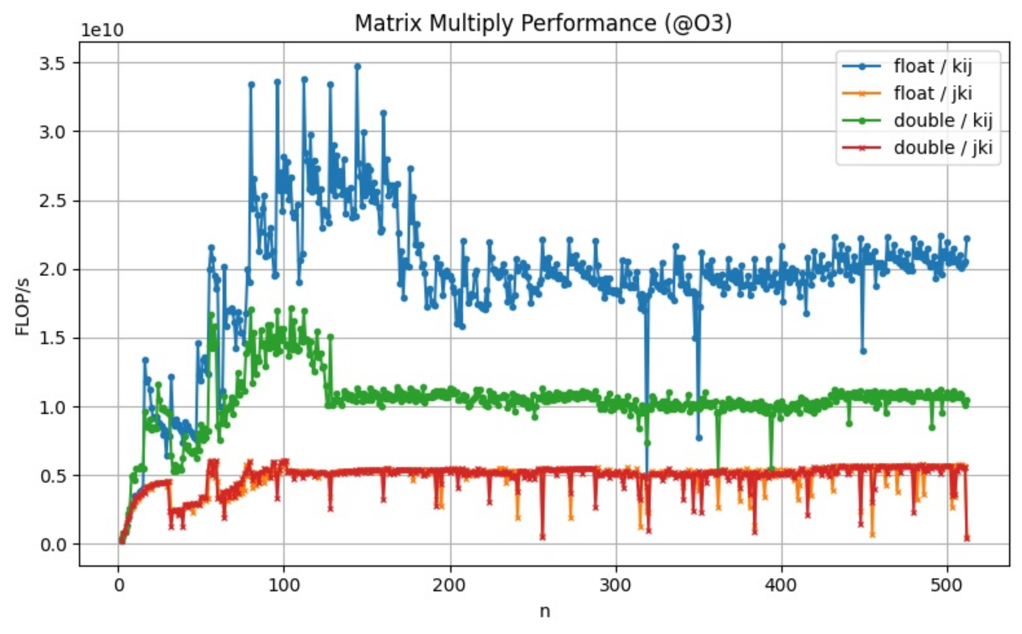
$$A(-1,-1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} = (-3,-2)$$

$$A(1,-1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1,-2)$$





Matrix Multiply Performance (@O0)
1e8
FLOP/s
n
float / kij
float / jki
double / kij
double / jki



Matrix Multiply Performance (@O3)
1e10
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
FLOP/s
0
100
200
300
400
500
n
float / kij
float / jki
double / kij
double / jki

4. (+10) **Extremum.** Consider the surface defined by $xy + 2xz = 5\sqrt{5}$ and $(x, y, z) \in \mathbb{R}^3$. Find (a) the coordinate instance(s) affiliated with the minimum distance from a point on the surface to the origin, and (b) the value of the minimum distance. You will find (may safely assume) the domain $[-3, 3] \times [-3, 3] \times [-3, 3] \in \mathbb{R}^3$ holds the correct coordinate instance(s).

Minimize squared distance $f(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = xy+2xz-5\sqrt{5} = 0$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2x = \lambda(y+2z) \\ 2y = \lambda x \\ 2z = 2\lambda x \end{cases}$$

$\lambda = \frac{2y}{x}$ $\quad \frac{2y}{x} = \frac{z}{x}$ $\quad z = 2y$

$\lambda = \frac{z}{x}$

$2x = \frac{2y}{x}(y+4y)$

$2x = \frac{10y^2}{x} \Rightarrow 2x^2 = 10y^2 \quad x^2 = 5y^2 \quad x = \pm\sqrt{5}\,y$

$xy + 2xz = xy + 4xy = 5xy = 5(\pm\sqrt{5}y)y = \pm 5\sqrt{5}\,y^2$

$$\pm 5\sqrt{5}\,y^2 = 5\sqrt{5} \begin{cases} 5\sqrt{5}\,y^2 = 5\sqrt{5} \Rightarrow y = \pm 1 \\ -5\sqrt{5}\,y^2 = 5\sqrt{5} \Rightarrow \text{no solution in real space} \end{cases}$$

Therefore $y = \pm 1$, Plug info equations, we get

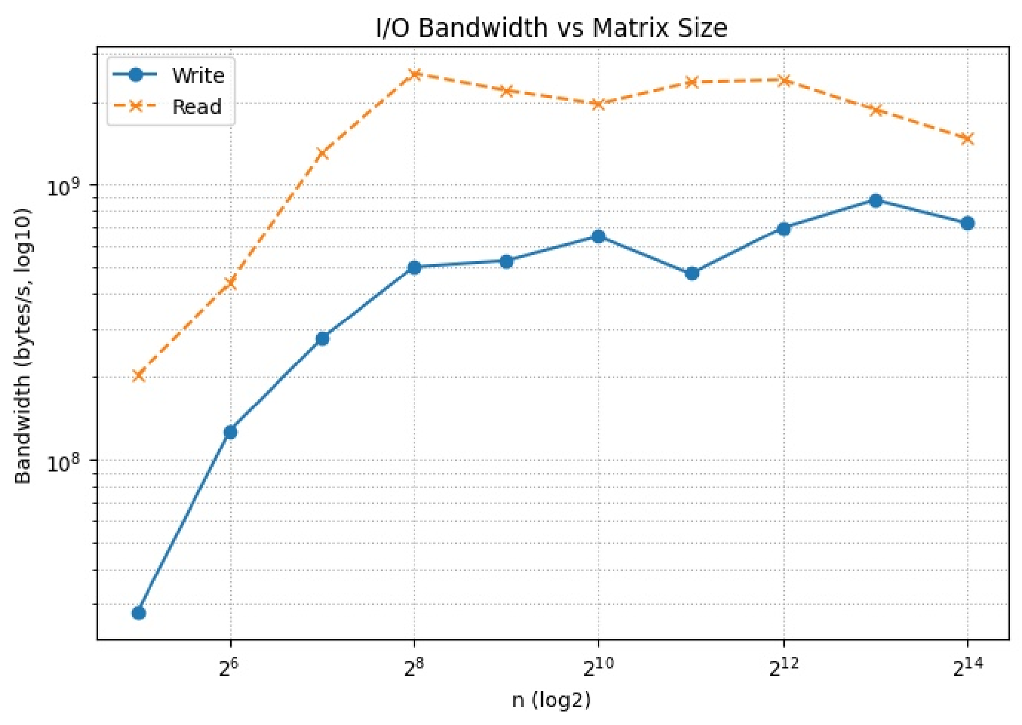
$x = \sqrt{5}\,y$

$z = 2y$

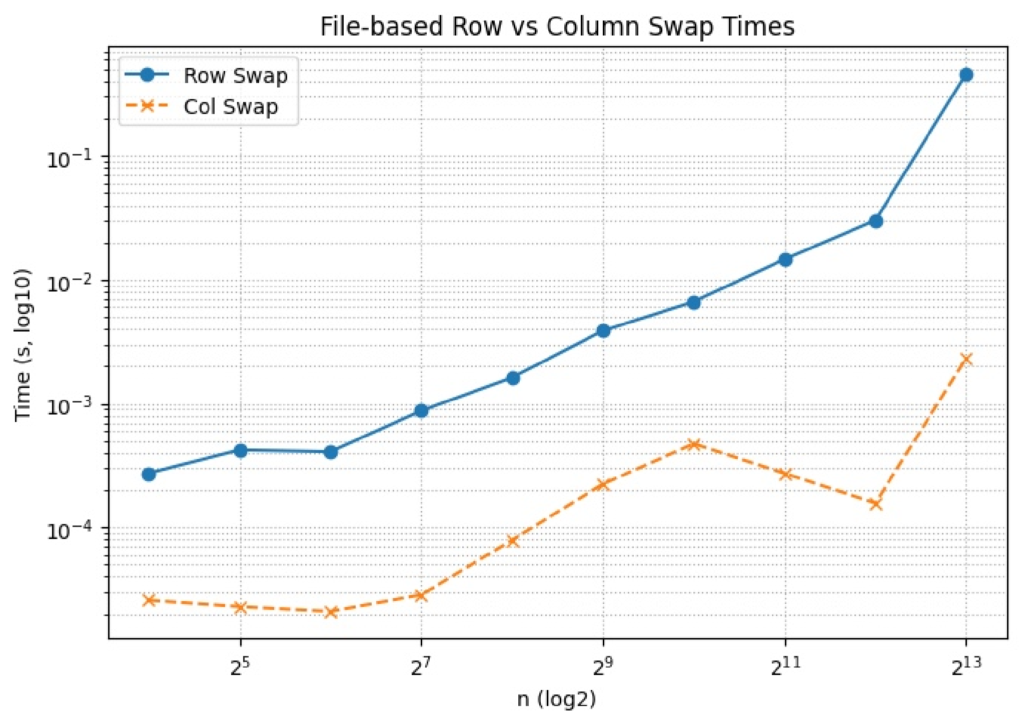
Possible points within $x, y, z \in [-3, 3]$ are

$(x,y,z) = (\sqrt{5}, 1, 2)$ and $(-\sqrt{5}, -1, -2)$ //

Distance to origin is $\sqrt{(\sqrt{5})^2 + (1)^2 + (2)^2} = \sqrt{10}$ //



I/O Bandwidth vs Matrix Size
Write
Read
Bandwidth (bytes/s, log10)
10^9
10^8
2^6
2^8
2^{10}
2^{12}
2^{14}
n (log2)



File-based Row vs Column Swap Times
Row Swap
Col Swap
Time (s, log10)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
2^5
2^7
2^9
2^{11}
2^{13}
n (log2)