1. (+15) Integrate the following integrals. Show work.

(a) $\int_0^{\frac{\pi}{2}} ln\, sin(x)\, dx$.

(b) $\int_0^{\pi} \frac{x\, sin(x)}{1+cos^2(x)}\, dx$. Use $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$.

(c) $\int_0^{\frac{\pi}{2}} \frac{1}{1+(tan(x))^{\sqrt{2}}}\, dx$.

a) $$\int_0^{\frac{\pi}{2}} \ln(\sin(x)) = \int_0^{\pi/2} \ln(\sin(\tfrac{\pi}{2}-x))\, dx = \int_0^{\pi/2} \ln(\cos(x))\, dx$$

Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\, dx$

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\, dx + \int_0^{\frac{\pi}{2}} \ln(\cos(x))\, dx = \int_0^{\frac{\pi}{2}} \ln(\sin(x) + \cos(x))\, dx$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\tfrac{1}{2}\sin(2x)\right) dx = \int_0^{\frac{\pi}{2}} \ln\left(\tfrac{1}{2}\right) + \ln(\sin(2x))\, dx$$

$$= \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) + \int_0^{\frac{\pi}{2}} \ln(\sin(2x))\, dx$$

$u = 2x$

$du = 2$

$$= \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) + \frac{1}{2}\int_0^{\pi} \ln(\sin(u))\, du$$

$$= \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) + \frac{1}{2}\left(\int_0^{\pi/2} \ln(\sin(u))\, du + \int_{\pi/2}^{\pi} \ln(\sin(u))\, du\right)$$

$\ln(\sin(u))$ is symmetric from $[0, \pi]$

$$= \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) + \int_0^{\pi/2} \ln(\sin(u))\, du$$

$$2\int_0^{\pi/2} \ln(\sin(x))\, dx = \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) + \int_0^{\pi/2} \ln(\sin(x))\, dx$$

$$\int_0^{\pi/2} \ln(\sin(u))\, dx = \frac{\pi}{2}\ln\left(\tfrac{1}{2}\right) = \underline{-\frac{\pi}{2}\ln(2)}$$

ii) $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} = \int_0^{\pi} \frac{(\pi-x)\sin(x)}{1+\cos^2(x)}\,dx$

$$f(x) = \frac{x \sin x}{1+\cos^2 x} \qquad f(\pi-x) = \frac{(\pi-x)\sin x}{1+\cos^2(x)}$$

$$f(x)+f(\pi-x) = \frac{\pi \sin(x)}{1+\cos^2 x}, \quad 2\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = 2I$$

$$2I = \int_0^{\pi} \frac{\pi \sin(x)}{1+\cos^2(x)}\,dx = \pi \int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}\,dx$$

$u = \cos x$

$du = -\sin(x)\,dx$

$$= \pi \int_1^{-1} \frac{-1}{1+u^2}\,du = \pi \int_{-1}^{1} \frac{1}{1+u^2}\,du = \pi\left[\tan^{-1}(u)\right]_{-1}^{1}$$

$$= \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi^2}{2}$$

$$I = \frac{1}{4}\pi^2, \quad \underline{\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{1}{4}\pi^2}$$

c) $\displaystyle\int_0^{\pi/2} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx$ $\qquad \tan\left(\frac{\pi}{2}-x\right) = \cot x = \frac{1}{\tan(x)}$

$$f(x) = \frac{1}{1+(\tan(x))^{\sqrt{2}}} \qquad f\left(\frac{\pi}{2}-x\right) = \frac{1}{1+\left(\frac{1}{\tan(x)}\right)^{\sqrt{2}}} = \frac{1}{1+(\tan(x))^{-\sqrt{2}}}$$

$$f(x)+f\left(\frac{\pi}{2}-x\right) = \frac{1}{1+(\tan(x))^{\sqrt{2}}} + \frac{1}{1+(\tan(x))^{-\sqrt{2}}} \qquad \text{Let } y = (\tan(x))^{\sqrt{2}}$$

$$= \frac{1}{1+y} + \frac{1}{1+\frac{1}{y}} = \frac{1}{1+y} + \frac{y}{1+y} = \frac{1+y}{1+y} = 1$$

$$2\int_0^{\pi/2} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$

$$\int_0^{\pi/2} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx = \frac{\pi}{4}$$

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2. (+15) Show work.

(a) Find $a, b \in \mathbb{R} \ni (1+i\sqrt{3})^{11} = a+ib$.

(b) Find values of $(1+i\sqrt{3})^{\frac{1}{5}}$.

(c) Solve for $w \in \mathbb{C}$ given $w^{\frac{4}{3}} + 2i = 0$.

a) $(1+i\sqrt{3})^{11} = a+ib$  Convert to polar form

$z = 1+i\sqrt{3}$  $|z| = \sqrt{1^2+(\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$  $z = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

De Moivre's theorem

$z^{11} = 2^{11}\left(\cos\left(\frac{11\pi}{3}\right) + i\sin\left(\frac{11\pi}{3}\right)\right)$  $\cos\left(\frac{11\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right)$

$= 2048\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

$= 2048\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 1024 - 1024i\sqrt{3}$ //

b) $(1+i\sqrt{3})^{\frac{1}{5}}$  $z = 1+i\sqrt{3}$

$z^{\frac{1}{5}} = 2^{\frac{1}{5}}\left(\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right)$

$= 2^{\frac{1}{5}}\left(\cos\left(\frac{\pi}{15} + \frac{2\pi k}{5}\right) + i\sin\left(\frac{\pi}{15} + \frac{2\pi k}{5}\right)\right)$

$= 2^{\frac{1}{5}}\left(\cos\left(\frac{\pi + 6\pi k}{15}\right) + i\sin\left(\frac{\pi + 6\pi k}{15}\right)\right)$  for $k = 0, 1, 2, 3, 4$ //

c) $w^{4/3} = -2i$  Convert $-2i$ to polar form

$-2i = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = w^{4/3}$

$w = 2^{\frac{3}{4}} \cdot \left(\cos\left(\frac{3}{4}\cdot\left(-\frac{\pi}{2} + 2\pi k\right)\right) + i\sin\left(\frac{3}{4}\cdot\left(-\frac{\pi}{2} + 2\pi k\right)\right)\right)$

$= 2^{\frac{3}{4}}\left(\cos\left(\frac{-3\pi + 12\pi k}{8}\right) + i\sin\left(\frac{-3\pi + 12\pi k}{8}\right)\right)$  for $k = 0, 1, 2,$ //

3. (+15) Write the following as the ratio of integers. Show work.

(a) $1 + 10^{-2} + 10^{-4} + 10^{-6} + \cdots$

(b) $376.376376\ldots$

(c) $.999\overline{9}$

a) $1 + \frac{1}{100} + \frac{1}{10000} + \cdots$ $\qquad \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

$r = \frac{1}{100} \qquad a = 1$

$$\sum_{k=0}^{\infty} ar^k = \frac{1}{1-\frac{1}{100}} = \frac{1}{\frac{99}{100}} = \frac{100}{99}$$ //

b) $x = 376.376376$ $\qquad 999x = 376000.000\ldots$

$1000x = 376376.376$ $\qquad x = \frac{376000}{999}$, $\quad 376.376376\ldots = \frac{376000}{999}$

c) $x = 0.9999\ldots$ $\qquad 9x = 9 \qquad x = 1$

$10x = 9.9999\ldots$

$0.9999\ldots = 1$ //

4. (+10) Estimate the following as the ratio of integers using the secant approximation. Show work.

(a) $(1.1)^{\frac{1}{3}}$

(b) $\sqrt{8.5}$

Let $x = (1.1)^{\frac{1}{3}}$ $\qquad x^3 = 1.1$, $\qquad$ Let $f(x) = x^3 - 1.1 = 0$

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \qquad x_0 = 1, \quad x_1 = 2$$

$$x_2 = 2 - f(2)\frac{2-1}{f(2) - f(1)} = 2 - 6.9 \cdot \frac{1}{6.9 - (-0.1)}$$

$$= 2 - 6.9 \cdot \frac{1}{7} = 2 - \frac{69}{70} = \frac{71}{70}$$

$(1.1)^{\frac{1}{3}} \approx \frac{71}{70}$ //

h) $x = \sqrt{8.5}$ $f(x) = x^2 - 8.5 = 0$

$x_0 = 2$ $f(x_0) = -4.5$

$x_1 = 3$ $f(x_1) = 0.5$

$$x_2 = 3 - 0.5 \cdot \frac{3-2}{0.5-(-4.5)} = 3 - \frac{1}{2} \cdot \frac{1}{5}$$

$$= 3 - \frac{1}{10} = \frac{29}{10}$$

$$\sqrt{8.5} \approx \frac{29}{10}$$

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5. (+10) Given $(x + y + z)^7$, find the expansion coefficients of the following terms. Show work.
   (a) $x^2y^2z^3$
   (b) $x^3z^4$

$(x+y+z)^n$ $x^a y^b z^c$ with $a+b+c=n$ $\frac{n!}{a!b!c!}$

$(x+y+z)^7$, where $a+b+c = 7 \Rightarrow \frac{7!}{a!b!c!}$

a) $x^2y^2z^3 \Rightarrow \frac{7!}{2!2!3!} = 210$ $210x^2y^2z^3$ //

b) $x^3z^4 \Rightarrow \frac{7!}{3!\,0!\,4!} = 35$ $35x^3z^4$ //

6. (+5) Given $(x + 2y - 3z + 2w + 5)^{16}$, find the expansion coefficient of the following term. Show work.
   (a) $x^2y^3z^2w^5$

$x^a\, 2y^b\, (-3z)^c\, (2w)^d\, (5)^e$ $a+b+c+d+e = 16$, $e = 4$

Multinomial coeff: $\frac{16!}{2!\,3!\,2!\,5!\,4!} = 302702400$

Numerical multiplier: $2^3 \cdot (-3)^2 \cdot (2)^5 \cdot 5^4 = 1440000$

$302702400 \cdot 1440000$ //

7. (+10) Two numbers $a, b \in \mathbb{Z}$ are relatively prime when $gcd(a,b) = 1$, or $\exists x, y \in \mathbb{Z}$ with $ax + by = 1$. Recall, $c = gcd(a,b)$ when $c|a$ and $c|b$, and for any other divisor $d$ of $a, b$ then $d|c$.

(a) For any $n \in \mathbb{Z}^+$, prove $8n+3$ and $5n+2$ are relatively prime. Show work. Hint: try Euclid's algorithm

(b) Find the $gcd(250, 111)$ and show result as linear combination of these integers. Show work.

Show $gcd(8n+3, 5n+2) = 1$

$(8n+3) - (5n+2) = 3n+1$

$gcd(8n+3, 5n+2) = gcd(5n+2, 3n+1)$

$gcd(5n+2, 3n+1) = gcd(3n+1, 2n+1)$

$gcd(3n+1, 2n+1) = gcd(2n+1, n)$

$gcd(2n+1, n) = gcd(n, 1) = 1$

Hence $gcd(8n+3, 5n+2) = 1$ //

b) $gcd(250, 111)$

$250 = (111) \cdot 2 + 28$

$111 = 28 \cdot 3 + 27$

$28 = 27 \cdot 1 + \underline{1}$ $\Rightarrow$ Reminder is 1, $gcd(250, 111) = 1$

$\Downarrow$

$1 = 28 - 27 \cdot 1 = 28 - (111 - 28 \cdot 3) \cdot 1 = 4 \cdot 28 - 111$

$27 = 111 - 28 \cdot 3$

$28 = 250 - 111 \cdot 2$

$= 4 \cdot (250 - 111 \cdot 2) - 111$

$= 4 \cdot 250 - 9 \cdot 111$

$1 = 4 \cdot 250 - 9 \cdot 111$ //

8. (+5) Write the prime factorization of 980220.

$2\,)\,\underline{980220}$

$2\,)\,\underline{490110}$

$5\,)\,\underline{245055}$

$3\,)\,\underline{49011}$

$17\,)\,\underline{16337}$

$31\,)\,\underline{961}$

$31$

$$980220 = 2^2 \cdot 3 \cdot 5 \cdot 17 \cdot 31^2$$

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