Modeling the Effect of Environmental Factors on the Dynamics of Rabbit-Sheep Competition in the Lotka-Volterra Competition Model

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Abstract

In this study, we used the Lotka-Volterra competition model to investigate the competitive interaction between rabbit and sheep populations under different environmental conditions. We introduced an environmental variable (denoted by c) to investigate its effect on the stability of these populations. Positive and negative c indicate the role of environmental conditions, such as climatic variability or resource availability, which can promote or inhibit population growth. Through systematic analysis, our findings suggest that the change in environmental conditions can greatly affect the dynamics of these populations. By modifying environmental parameters c, we explored a range of scenarios to analyze how environmental factors affect two species. The results show that when c is negative, it simulates unfavorable environmental conditions that lead to declines in both populations, which lead to extinction in the long term. Conversely, when c is positive, it represents favorable environmental conditions and the model predicts that the populations will move towards a steady state at the fixed point $(0, 2 + c)$. This scenario suggests that sheep populations will extinction due to competitive pressure, while rabbit populations will stabilize and potentially increase due to the easing of competition. These findings indicate the significant impact of environmental changes on population dynamics. This research contributes to a better understanding of the critical

role of environmental conditions in biodiversity conservation.

Introduction

The Lotka-Volterra model was originally developed to describe the dynamics of biological systems in which two species interact, particularly through competition. Formulated independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926, the model represents one of the earliest mathematical approaches to understanding ecological interactions. Its formulation was driven by the need to understand how species populations evolve over time in the presence of biological interactions such as competition, predation, and mutualism.

Building on this foundational model, we sought to innovate by introducing a new variable to represent an environmental factor (c) and to explore how changes in (c) affect the dynamics of two interacting populations, rabbits and sheep. Environmental factors play a crucial role in ecological dynamics, influencing resource availability, habitat conditions, and overall species fitness. By incorporating an environmental parameter, our modified Lotka-Volterra competition model aims to provide a more nuanced understanding of how these external conditions impact population stability and interaction.

The history of the Lotka-Volterra model traces back to the early 20th century when Lotka and Volterra independently developed the equations to describe competitive dynamics. Lotka, an American biophysicist, was interested in chemical reactions and their similarities to biological processes, while Volterra, an Italian mathematician, was motivated by ecological observations. The classic competition model consists of two differential equations:

$$\dot{x} = \alpha x - \beta xy$$

$$\dot{y} = \delta xy - \gamma y$$

where x and y represent the populations of two competing species, respectively. The parameters α, β, γ, and δ correspond to the intrinsic growth rates of the species and the competition coefficients.

In our study, we utilize the Lotka-Volterra competition model(from Strogatz [2]) to examine the effects of environmental change on these populations. Our modified equations are as follows:

$$\dot{x} = x(3 - x - 2y) + cx$$

$$\dot{y} = y(2 - x - y) + cy$$

Strogatz's underlying theory of nonlinear dynamics provides a solid theoretical foundation for our analysis. His work, particularly in the book "Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering" [2], emphasizes the importance of understanding how small changes in parameters can lead to significant shifts in system dynamics, particularly in ecological models. Strogatz's framework allows us to systematically assess how the environmental factors included in parameter (c) affect population interactions and stability.

Furthermore, the study by Tahara et al. [3] investigated the stability of the modified Lotka-Volterra model. Their findings serve as a benchmark for assessing population resilience under different environmental conditions. They explored how modifications to the traditional Lotka-Volterra equations could impact the stability and behavior of populations, providing valuable insights into the dynamics of competing species in varying environments.

Additionally, Murray's contributions to mathematical biology [1] contextualize these dynamics within broader ecological implications. Murray's work highlights how mathematical models can contribute to environmental policy-making and species conservation efforts. By applying mathematical biology principles, we can predict and understand the potential impacts of environmental changes on biodiversity and ecosystem health.

In this review, we aim to show that even small changes in habitat, represented by the parameter (c), can lead to significant changes in population dynamics. Our findings suggest that when c is negative, it simulates unfavorable environmental conditions that lead to declines in both populations, potentially resulting in extinction in the long term. Conversely, when c is positive, it represents favorable environmental conditions, and the model predicts that the populations will move towards a steady state at the fixed point $(0, 2+c)$. This scenario suggests that sheep populations will face extinction due to competitive pressure, while rabbit populations will stabilize and potentially increase due to the easing of competition.

Through a systematic analysis, our study provides a comprehensive understanding of the critical role of environmental factors in ecological studies. By modifying environmental parameters (c), we explored a range of scenarios to analyze how environmental factors affect two species. Our research contributes to a better understanding of the dynamics of these populations, emphasizing the significance of environmental changes in shaping ecological dynamics.

Model Explanation and Parameters

This project delves deeper into mathematical modeling by examining specific parameters and their biological interpretations, a method crucial for understanding how theoretical applications can be translated into practical conservation strategies.

In this model, equations are formed as follows:

$$\dot{x} = x(3 - x - 2y) + cx$$

$$\dot{y} = y(2 - x - y) + cy$$

Where:

- x and y: Represent the population sizes of sheep and rabbits, respectively.
- $3 - x - 2y$: This term shows that the growth of the sheep population is affected by its own population pressure (x) and competition with rabbits ($2y$). The coefficient 3 indicates the natural growth rate of sheep in the absence of competition and population pressure; the coefficient 2 indicates that rabbits exert significant competitive pressure on sheep.
- $2 - x - y$: Indicates that the growth of the rabbit population is also affected by its own population pressure (y) and competition with sheep (x). The coefficient 2 is the natural growth rate of rabbits in the absence of competition and population pressure.
- c: Represents external factors (such as environmental changes, and resource fluctuations) that affect both populations. It reflects how external conditions, whether beneficial or harmful,

impact the populations.

Model Dynamics:

- When $c = 0$, the model focuses solely on the impact of internal population competition.
- When c is positive, external conditions support population growth, potentially leading to rapid population increases.
- When c is negative, environmental pressures reduce, resulting in population decline and potentially leading to species extinction.

Thesis Statement

This paper uses the Lotka-Volterra competition model to explore the population change of rabbits and sheep due to competition under varying environmental conditions by adding environmental factor c into the model. Our research aims to demonstrate how changes in environmental conditions can significantly impact population dynamics and competition, potentially leading to scenarios of population stability or extinction.

First we will provide a detailed introduction to the model's background and theoretical basis, explaining the main parameters and their biological significance. Then, through phase plane analysis methods and bifurcation methods, we will investigate how variations in the environmental parameter c affect the outcomes of competition between rabbit and sheep populations. Finally, we will discuss the ecological implications and potential applications of our findings.

Data and Methods

Finding fixed points:

$$\frac{dx}{dt} = x(3 - x - 2y) + cx = 3x - x^2 - 2xy + cx$$
$$\frac{dy}{dt} = y(2 - x - y) + cy = 2y - xy - y^2 + cy$$

$$\frac{dx}{dt} = x(3 - x - 2y) + cx = 0 \tag{1}$$
$$\frac{dy}{dt} = y(2 - x - y) + cy = 0 \tag{2}$$

Let $x_1 = 0$, plug $x_1 = 0$ into equation (2)

$$y(2 - y + c) = 0$$
$$y_1 = 0$$
$$2 - y + c = 0$$
$$y_2 = -1 - c$$
$$\boxed{(0, 0), (0, 2 + c)}$$

Let $3 - x - 2y + c = 0 \Rightarrow x = 3 - 2y + c$

plug $x = 3 - 2y + c$ into equation (2)

$$y(2 - 3 + 2y - c - y + c) = 0$$
$$y(-1 + y) = 0$$
$$\begin{cases} x_1 = 3 + c & y_1 = 0 \\ x_2 = 1 + c & y_2 = 1 \end{cases}$$
$$\boxed{(3 + c, 0), (1 + c, 1)}$$

Hence, we obtain the following four fixed points:

$$\boxed{(0, 0), (0, 2 + c), (3 + c, 0), (1 + c, 1)}$$

Then, we formulate the Jacobian matrix.

$$J = \begin{bmatrix} f_1 x & f_1 y \\ f_2 x & f_2 y \end{bmatrix} = \begin{bmatrix} 3 - 2x - 2y + c & -2x \\ -y & 2 - x - 2y + c \end{bmatrix}$$

For $(x, y) = (0, 0)$

Jacobian Matrix and Eigenvalues:

$$J = \begin{bmatrix} 3+c & 0 \\ 0 & 2+c \end{bmatrix}$$

Determinant and Eigenvalues:

$$\det(J - \lambda I) = (3 + c - \lambda)(2 + c - \lambda) = 0$$

$$\lambda_1 = 3 + c, \quad \lambda_2 = 2 + c$$

$$\lambda_1 = 0 \Rightarrow c = -3, \quad \lambda_2 = 0 \Rightarrow c = -2$$

Stability Conditions:

- When $c < -3$: Both eigenvalues are negative, leading to a Stable Node.
- When $c = -3$: $\lambda_1 = 0$, $\lambda_2 < 0$ (Stable Node).
- When $-3 < c < -2$: $\lambda_1 > 0$, $\lambda_2 < 0$ (Saddle Node, unstable).
- When $c = -2$: $\lambda_1 > 0$, $\lambda_2 = 0$ (Unstable).
- When $c > -2$: $\lambda_1 > \lambda_2 > 0$ (Unstable Node).

Summary Table:

$c < -3$	$c = -3$	$-3 < c < -2$	$c = -2$	$c > -2$
Stable Node	Stable Node	Saddle Node	Unstable	Unstable

Table 1: Stability of Fixed Point $(0, 0)$ for Different Values of c

For $(x, y) = (0, 2 + c)$

Jacobian Matrix and Eigenvalues:

$$J = \begin{bmatrix} -1-c & 0 \\ 2+c & -2-c \end{bmatrix}$$

Determinant and Eigenvalues:

$$\det(J - \lambda I) = (-1 - c - \lambda)(-2 - c - \lambda) = 0$$

$$\lambda_1 = -c - 1, \quad \lambda_2 = -2 - c$$

$$\lambda_1 = 0 \Rightarrow c = -1, \quad \lambda_2 = 0 \Rightarrow c = -2$$

Stability Conditions:

- When $c < -2$: $\lambda_1 = -c - 1 > 0, \lambda_2 = -2 - c > 0$ (Unstable Node)
- When $c = -2$: $\lambda_1 = -3 < 0, \lambda_2 = 0$ (Stable)
- When $-2 < c < -1$: $\lambda_1 = -c - 1 > 0, \lambda_2 = -2 - c < 0$ (Saddle Node, unstable)
- When $c = -1$: $\lambda_1 = 0, \lambda_2 = -3 < 0$ (Stable)
- When $c > -1$: $\lambda_1 = -c - 1 < 0, \lambda_2 = -2 - c < 0$ (Stable Node)

Summary Table:

$c < -2$	$c = -2$	$-2 < c < -1$	$c = -1$	$c > -1$
Unstable Node	Stable	Saddle Node	Stable	Stable Node

Table 2: Stability of Fixed Point $(0, 2 + c)$ for Different Values of c

For $(x, y) = (3 + c, 0)$

Jacobian Matrix and Eigenvalues:

$$J = \begin{bmatrix} 3 - 2(3 + c) + c & -2(3 + c) \\ 0 & 2 - 3 - c + c \end{bmatrix} = \begin{bmatrix} -3 - c & -6 - 2c \\ 0 & -1 \end{bmatrix}$$

Determinant and Eigenvalues:

$$\det(J - \lambda I) = (-3 - c - \lambda)(-1 - \lambda) = 0$$

$$\lambda_1 = -3 - c, \quad \lambda_2 = -1$$

$$\lambda_1 = 0 \Rightarrow c = -3, \quad \lambda_1 = \lambda_2 \Rightarrow c = -2$$

Stability Conditions:

- When $c < -3$: $\lambda_1 = -3 - c > 0$, $\lambda_2 = -1$ (Unstable Node)
- When $c = -3$: $\lambda_1 = 0$, $\lambda_2 = -1 < 0$ (Stable)
- When $-3 < c < -2$: $\lambda_1 = -3 - c < 0$, $\lambda_2 = -1$ (Stable Node)
- When $c = -2$: $\lambda_1 = -1$, $\lambda_2 = -1 < 0$ (Stable)
- When $c > -2$: $\lambda_1 = -3 - c < 0$, $\lambda_2 = -1 < 0$ (Stable Node)

$c < -3$	$c = -3$	$-3 < c < -2$	$c = -2$	$c > -2$
Unstable Node	Stable	Stable Node	Stable	Stable Node

Table 3: Stability of Fixed Point $(3 + c, 0)$ for Different Values of c

For $(x, y) = (1 + c, 1)$

Jacobian Matrix and Eigenvalues:

$$J = \begin{bmatrix} 3 - 2(1 + c) - 2 + c & -2(1 + c) \\ -1 & 2 - 1 - c - 2 + c \end{bmatrix} = \begin{bmatrix} -1 - c & -2 - 2c \\ -1 & -1 \end{bmatrix}$$

Determinant and Eigenvalues:

$$\det(J - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} -1-c-\lambda & -2-2c \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-c-\lambda)(-1-\lambda) + 2 + 2c = 0$$

$$\lambda^2 + (2+c)\lambda + (-1-c) = 0$$

Analysis based on the discriminant Δ:

$$\Delta = (2+c)^2 - 4(-1-c) = c^2 + 8c + 8$$

$$\lambda = \frac{-2-c \pm \sqrt{c^2+8c+8}}{2} \Rightarrow \lambda_1 = \frac{-2-c+\sqrt{c^2+8c+8}}{2}, \quad \lambda_2 = \frac{-2-c-\sqrt{c^2+8c+8}}{2}$$

- **Case $\Delta > 0$ (Real and Distinct Roots):** This indicates that the eigenvalues are real and distinct.

$$\Delta > 0 \Rightarrow c^2 + 8c + 8 > 0 \Rightarrow c > -4 + 2\sqrt{2} \text{ or } c < -4 - 2\sqrt{2}$$

  - **Sub-Case 1:** $\lambda_1 > \lambda_2 > 0$

    To make sure $\lambda_2 > 0$

$$\lambda_2 = \frac{-2-c-\sqrt{c^2+8c+8}}{2} > 0$$

$$-2-c > \sqrt{c^2+8c+8}$$

    We can get the interval of c is in following two range:

$$c < -4 - 2\sqrt{2} \text{ or } -4 + 2\sqrt{2} < c < -1$$

    Then $\lambda_1 > \lambda_2 > 0$, fixed point is Unstable Node.

  - **Sub-Case 2:** $\lambda_1 > 0$**,** $\lambda_2 < 0$

    For $\lambda_1 > 0$ and $\lambda_2 < 0$, we need:

$$\lambda_1 = \frac{-2-c+\sqrt{c^2+8c+8}}{2} > 0 \text{ and } \lambda_2 = \frac{-2-c-\sqrt{c^2+8c+8}}{2} < 0$$

which get

$$\begin{cases} \lambda_1 > 0 \Rightarrow c < -1 \\ \lambda_2 < 0 \Rightarrow c > -1 \end{cases}$$

Such c vallue is not possible, hence there is no such cases where $\lambda_1 > 0$, $\lambda_2 < 0$.

- **Sub-Case 3:** $0 > \lambda_1 > \lambda_2$ For this case we need:

$$\lambda_1 = \frac{-2 - c + \sqrt{c^2 + 8c + 8}}{2} < 0$$

$$-2 - c + \sqrt{c^2 + 8c + 8} < 0$$

$$c > -1 > -4 + 2\sqrt{2}$$

Hence for $c > -1$, $0 > \lambda_1 > \lambda_2$, the fixed point is Stable Node.

- **Case $\Delta = 0$ (Repeated Roots):** This indicates that the eigenvalues are repeated.

$$\Delta = 0 \Rightarrow c^2 + 8c + 8 = 0 \Rightarrow c_1 = -4 + 2\sqrt{2} \text{ or } c_2 = -4 - 2\sqrt{2}$$

$$\lambda_1 = \lambda_2 = -1 - \frac{c}{2}$$

- **Sub-Case 1:** $c_1 = -4 + 2\sqrt{2}$

$$\lambda_1 = \lambda_2 = -1 - \frac{c_1}{2} = 1 - \sqrt{2} < 0$$

Hence for $c_1 = -4 + 2\sqrt{2}$, the fixed point is Stable.

- **Sub-Case 2:** $c_2 = -4 - 2\sqrt{2}$

$$\lambda_1 = \lambda_2 = -1 - \frac{c_2}{2} = 1 + \sqrt{2} > 0$$

Hence for $c_1 = -4 - 2\sqrt{2}$, the fixed point is Unstable.

- **Case $\Delta < 0$ (Complex Conjugate Roots):** This indicates that the eigenvalues are complexed.

$$\Delta < 0 \Rightarrow c^2 + 8c + 8 < 0 \Rightarrow -4 - 2\sqrt{2} < c < -4 + 2\sqrt{2}$$

$$\lambda_1 = a + bi, \quad \lambda_2 = a - bi \quad (a = \frac{-2-c}{2}, b \neq 0)$$

- **Sub-Case 1:** $a > 0$ For this case we need:

$$a = \frac{-2-c}{2} > 0 \Rightarrow c < -2$$

$$\because -4 - 2\sqrt{2} < c < -4 + 2\sqrt{2}$$

$$\therefore -4 - 2\sqrt{2} < c < -2$$

Hence for $-4 - 2\sqrt{2} < c < -2$, the fixed point is Unstable spiral.

- **Sub-Case 2:** $a < 0$ For this case we need:

$$a = \frac{-2-c}{2} < 0 \Rightarrow c > -2$$

$$\because -4 - 2\sqrt{2} < c < -4 + 2\sqrt{2}$$

$$\therefore -2 < c < -4 + 2\sqrt{2}$$

Hence for $-2 < c < -4 + 2\sqrt{2}$, the fixed point is Stable spiral.

- **Sub-Case 3:** $a = 0$ For this case we need:

$$a = \frac{-2-c}{2} = 0 \Rightarrow c = -2$$

Hence for $c = -2$, the fixed point is Center, the stability is semi-stable.

$c < -4 - 2\sqrt{2}$	$c = -4 - 2\sqrt{2}$	$-4 - 2\sqrt{2} < c < -2$	$c = -2$	$-2 < c < -4 + 2\sqrt{2}$
Unstable	Unstable	Unstable Spiral	Center	Stable Spiral
$c = -4 + 2\sqrt{2}$	$-4 + 2\sqrt{2} < c < -1$	$c = -1$	$c > -1$	
Stable	Stable	Stable	Saddle	

Table 4: Stability of Fixed Point $(1 + c, 1)$ for Different Values of c

Changes in the stability of fixed points for different values of c

This section examines how the stability of four specific fixed points in the Lotka-Volterra model changes as the environmental parameter c varies. We use specific values of c to illustrate different stability scenarios.

For $(x, y) = (0, 0)$

c	**Stability**
$c < -3$	Stable
$c = -3$	Stable
$-3 < c < -2$	Unstable
$c = -2$	Unstable
$c > -2$	Unstable

Consider $c = -3, -2$ here for further detailed analysis.

For $(x, y) = (0, 2 + c)$

c	**Stability**
$c < -2$	Unstable
$c = -2$	Stable
$-2 < c < -1$	Unstable
$c = -1$	Stable
$c > -1$	Stable

Consider $c = -2, -1$ here for detailed phase plane analysis.

For $(x, y) = (3 + c, 0)$

c	**Stability**
$c < -3$	Unstable
$c = -3$	Stable
$-3 < c < -2$	Stable
$c = -2$	Stable
$c > -2$	Stable

Consider $c = -3, -2$ here for simulations.

For $(x, y) = (1 + c, 1)$

c	**Stability**
$c < -4 - 2\sqrt{2}$	Unstable
$c = -4 - 2\sqrt{2}$	Unstable
$-4 - 2\sqrt{2} < c < -2$	Unstable spiral
$c = -2$	Center
$-2 < c < -4 + 2\sqrt{2}$	Stable spiral
$c = -4 + 2\sqrt{2}$	Stable
$-4 + 2\sqrt{2} < c < -1$	Unstable
$c > -1$	Stable

Consider $c = -4 + 2\sqrt{2}, -4 - 2\sqrt{2}, -2, -1$ here for further analysis.

Strategic selection of c values

To systematically explore the stability across the model, we select c values around critical points where stability transitions occur:

- Key transitions at $c = -3, -2, -1, -4 + 2\sqrt{2}, -4 - 2\sqrt{2}$.
- Additional points are selected to capture the behavior around these transitions: $-20, -5, -2.5, -1.5, -1.1, 20$.

This strategic selection enables a comprehensive analysis of stability under varying environmental conditions, providing insights into the model's dynamics across a range of scenarios.

c	**(0, 0)**	**(0, 2+c)**	**(3+c, 0)**	**(1+c, 1)**
-20.00	Stable	Unstable	Saddle	Unstable
$-4-2\sqrt{2}$	Stable	Unstable	Saddle	Unstable
-5.00	Stable	Unstable	Saddle	Unstable Spiral
-3.00	Stable	Unstable	Stable	Unstable Spiral
-2.50	Saddle	Unstable	Stable	Unstable Spiral
-2.00	Unstable	Stable	Stable	Center
-1.50	Unstable	Saddle	Stable	Stable Spiral
$-4+2\sqrt{2}$	Unstable	Saddle	Stable	Stable
-1.10	Unstable	Saddle	Stable	Stable
-1.00	Unstable	Stable	Stable	Saddle
20.00	Unstable	Stable	Stable	Unstable

Table 5: Stability of Fixed Points for Different Values of c

Results

Graphical Analysis

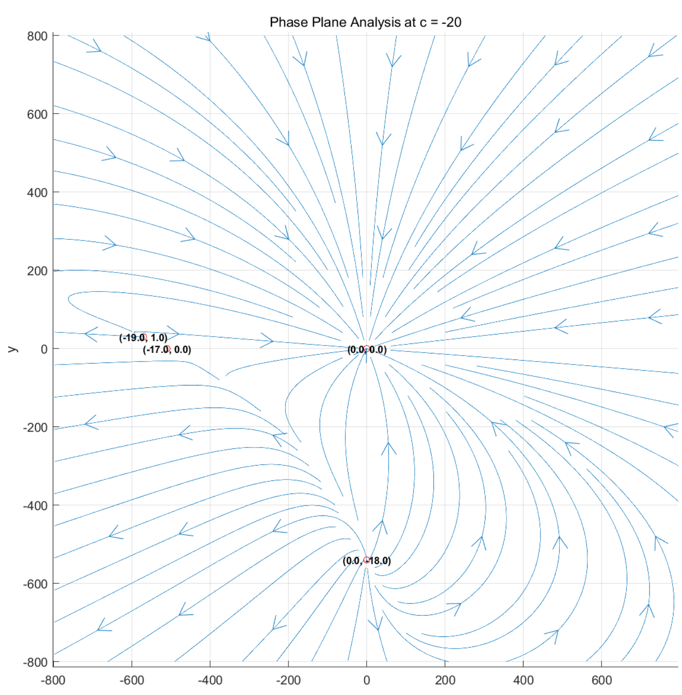

Figure 1: Phase Analysis at c = -20

Fixation points and stability.

- **(0, 0) - Stable**
  - As seen in the figure, this point is a stable node from which the streamlines converge.
- **(0, -18) - Unstable**
  - As seen in the figure, this point is an unstable node from which the streamlines diverge.
- **(-17, 0) - Unstable**
  - This point likewise shows an unstable nature, and the streamlines diverge from this point.
- **(-19, 1) - Unstable**
  - This point likewise shows an unstable nature, and the streamlines diverge from this point.

Long-term behavior.

- Since the population size is non-negative, we don't need to consider the 3 fixed points in this case, and secondly, the 3 fixed points are outwardly dispersed, and when the time tends to be infinite, they all converge to the fixed point of (0, 0), so no matter whether it is the rabbits or the goats in this intial condition their population will finally converge to 0, eventually leading to extinction.
- In real biological systems, this may mean that populations of both species may tend towards extinction, or that system dynamics are affected by other factors not considered in current models (e.g., environmental carrying capacity, additional biological or environmental interventions).

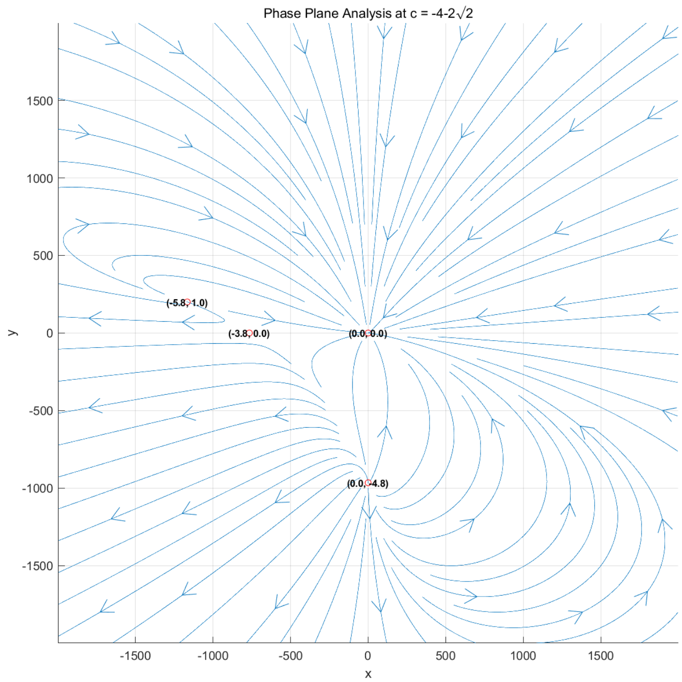

Figure 2: Phase Plane Analysis at $c = -4 - 2\sqrt{2}$

Fixed-point stabilization analysis

- **(-5.8, 1) and (-3.8, 0) - Unstable**
  - Both fixed points are located on the negative half-axis of the x-axis, indicating that the number of sheep is negative in these configurations, which is biologically meaningless. Therefore, these two points should not be considered in practical ecological modeling.
- **(0, -4.8) - Unstable**
  - This point indicates that there are no sheep and the number of rabbits is negative, which again is not biologically feasible. Therefore, this point should also not be considered in the actual analysis.
- **(0, 0) - Stable**
  - This point usually represents an stable node because the streamlines converge from this point. This means that regardless of the initial number of sheep and rabbits, the final number of sheep and rabbits will eventually lead to extinction.

Long-term behavioral analysis

- Since the three fixed points are biologically infeasible (since they indicate negative population sizes), and the only biologically feasible fixed point (0, 0) is stable. This suggests that under the environmental pressure $c = -4 - 2\sqrt{2}$, no matter what the intial condition of the sheep and rabbits is at the beginning, and as time tends to infinity, eventually the numbers of sheep and rabbits end up converging to 0.
- Such extreme external conditions could lead to rapid population declines or even extinction of both species.

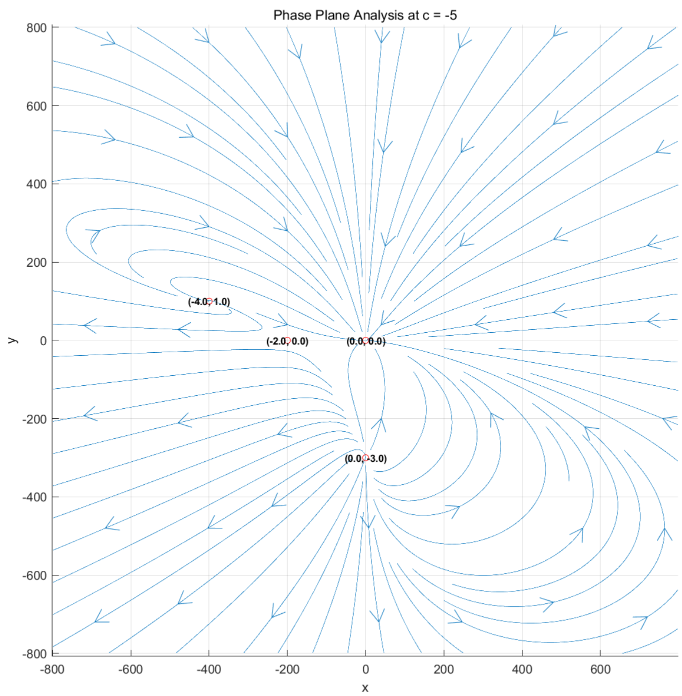

Figure 3: Phase Plane Analysis at c = -5

Fixed point analysis

- **(-4, 1) - Unstable spiral**
  - This point may indicate a negative number of sheep, which is biologically unacceptable. Therefore, this point should be ignored biologically, although it exists mathematically.
- **(-2, 0) - Unstable**

- Again, this point indicates that the number of sheep is negative and biologically unacceptable.

- **(0, -3) - Unstable**
  - This point indicates a negative number of rabbits, again not biologically feasible.

- **(0, 0) - Stable**
  - This point usually represents an stable node because the streamlines converge from this point. This means that regardless of the initial number of sheep and rabbits, the final number of sheep and rabbits will eventually lead to extinction.

Long-term behavior

- Since the three fixed points are biologically infeasible (since they indicate negative population sizes), and the only biologically feasible fixed point (0, 0) is stable. This suggests that under the environmental pressure c = -5, no matter what the intial condition of the sheep and rabbits is at the beginning, and as time tends to infinity, eventually the number of sheep and rabbits eventually tends to 0.

- Such extreme external conditions could lead to rapid population declines or even extinction of both species.

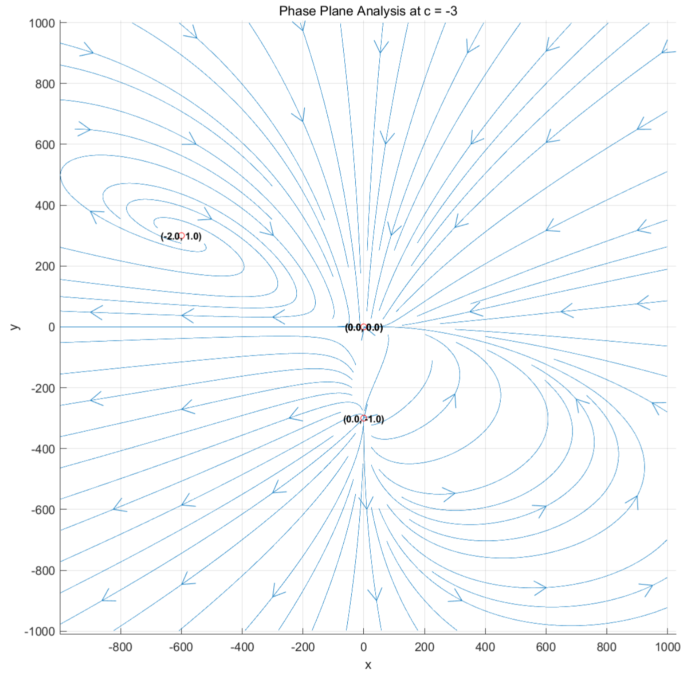

Figure 4: Phase Plane Analysis at c = -3

Fixed point analysis

- **(-2, 1) - Unstable spiral**
  - This point may indicate a negative number of sheep, which is biologically unacceptable. Therefore, this point should be ignored biologically, although it exists mathematically.
- **(0, -1) - Unstable**
  - This point indicates a negative number of rabbits, again not biologically feasible.
- **(0, 0) - Stable**
  - This point usually represents a stable node because the streamlines converge from this point. This means that regardless of the initial number of sheep and rabbits, the final number of sheep and rabbits will eventually lead to extinction.

Long-term behavior

- Since all three fixed points are biologically infeasible (since they indicate negative population sizes), the only biologically feasible fixed point (0, 0) is stable. This suggests that, under environmental pressure c = -3, the numbers of sheep and rabbits eventually converge to 0 over time, regardless of their initial conditions.

- Such extreme external conditions could lead to rapid population declines or even extinction of both species.

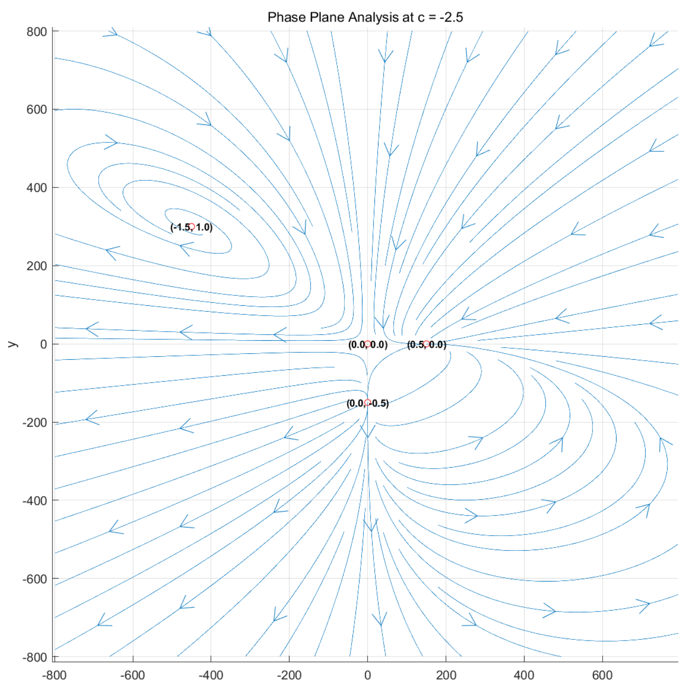

Figure 5: Phases Plane Analysis at c = -2.5

Fixed point analysis

- **(-1.5, 1) - Stable spiral**
  - This point is shown in the graph as stable spiral. but the number of sheep is negative, which is biologically impossible, so we do not need to consider this point.

- **(0, 0) - Unstable**

- The fact that the streamline diverges from this point indicates that it is an unstable node. This means that if the initial numbers of both sheep and rabbits are very close to zero, the system will quickly deviate from this state.

- **(0, -0.5) - Unstable**
  - This point is shown in the graph as unsatble node .Corresponding to the number of negative rabbits, this may be a biologically infeasible state and therefore should not be considered in practical applications.

- **(0.5, 0) - Stable**
  - The aggregation of streamlines around this point suggests that this may also be a stabilization point, and if the point is considered to be a point where the number of rabbits is zero and the number of sheep is 0.5, this suggests that the number of sheep may be at a lower level of stabilization in this case.

Long-term behavioral analysis

- **Stability point analysis.** Of all the fixed points, only (-1.5, 1) and (0.5, 0) are stable, so we only need to consider these two fixed points as time tends to infinity. However, since the first fixed point (-1.5, 1) is biologically impossible, we only need to consider this fixed point (0.5, 0). So as time tends to infinity, eventually both the sheep and the rabbit will converge to the point (0.5, 0).

- **System Dynamics.** Under such conditions, the sheep population may be at a low and stable level, while the rabbit population disappears. This could mean that under the environmental stress of $c = -2.5$, the survival of rabbits is threatened, while sheep show better adaptation.

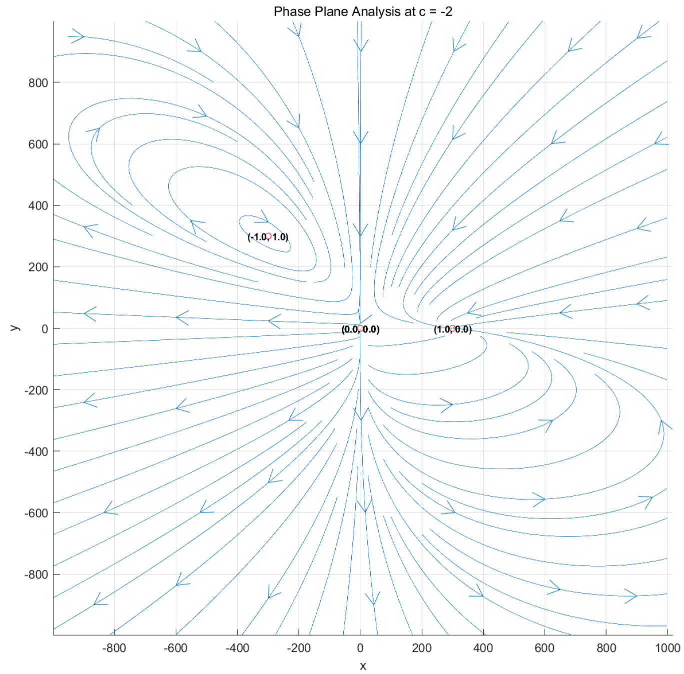

Figure 6: Phase Plane Analysis at c = -2

Analysis of the relevance of fixed points

a. **(-1, 1) - Center point, semi-stable.**

   - While this point shows semi-stability mathematically, it is biologically meaningless to have a negative number of sheep. Therefore, even though the mathematical model provides such a result, we should ignore the relevance of this point when interpreting and applying it.

b. **(0, 0) - Unstable**

   - This point represents a point where the populations are all at zero, effectively meaning that the population is extinct. The instability of this state suggests that any small perturbation will move the population rapidly away from this state, either towards recovery or further deterioration.

c. **(1, 0) - Stable**

   - This point indicates that the sheep population is stabilized at 1 and the rabbit population is zero. This is a biologically feasible steady state, indicating that under specific environmental conditions, sheep can independently maintain the stability of their populations.

Long-term ecological impacts

- **Stability point analysis**: Of all the fixed points, only (1,0) is stable. Secondly one has to consider the population under the factor that it cannot be negative. So only when time tends to infinity do both sheep and rabbits eventually converge to the point (1,0).
- **Extinction of a population**: In this modeling setup, we know from the stability point analysis that no matter where the intial condition of the sheep and the rabbit is, they will eventually converge to the point (1, 0), which means that the sheep will survive with c = -2 and the rabbit will perish.
- **Ecological management strategy**: Such analysis is essential for the development of targeted conservation measures and management strategies. If rabbits are unable to survive under certain conditions, conservation measures may need to focus on improving their survival conditions, or special attention may need to be paid to the ecological needs of rabbits when considering population recovery programs.

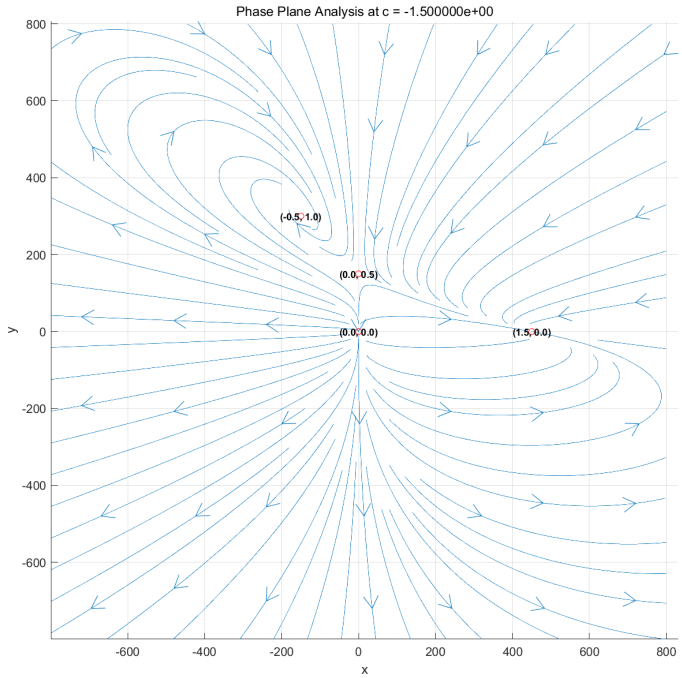

Figure 7: Phase Plane Analysis at c = -1.5

Fixed point analysis

a. **(-0.5, 1) - Stable spiral**

- This point indicates a stable spiral state in which the number of sheep is -0.5 and the number of rabbits is 1. Although this state is biologically unrealistic, the stability of the rabbits is indicative of the environmental or ecological factors that may be present to support rabbit survival under these conditions.

b. **(0, 0.5) - Unstable**

- Rabbit populations are at a point where growth is close to stabilization, but because of their instability, small perturbations may cause rapid changes in rabbit populations that deviate from this state.

c. **(0, 0) - Unstable**

- A state with a complete absence of sheep and rabbits is shown as an unstable node. This indicates that in the absence of sheep and rabbits, any small perturbation in the system can lead to rapid changes in the population, either in the direction of recovery or further deterioration.

d. **(1.5, 0) - Stable**

- This point indicates that the sheep population is stabilized at 1.5 with no rabbits. This is a biologically viable steady state, showing that sheep can maintain their populations independently under specific environmental conditions.

Long-term ecological impacts

- **Stability point analysis**: Of all the fixed points, only (-0.5, 1.0) and (1.5, 0) are stable, so we only need to consider these two fixed points when time tends to infinity. However, since the first fixed point (-0.5, 1.0) is biologically impossible, we only need to consider this fixed point (1.5, 0). Thus, as time tends to infinity, both the sheep and the rabbit eventually converge to the point (1.5, 0).

- **Extinction of a population**: In this modeling setup, we know from the stability point analysis that no matter where the initial condition of the sheep and the rabbit is, they will eventually converge to the point (1.5, 0), which means that the sheep will survive with c = -1.5. and the rabbit will perish.

- **Ecological management strategy**: Such analysis is essential for the development of targeted conservation measures and management strategies. If rabbits are unable to survive under certain conditions, conservation measures may need to focus on improving their survival conditions, or special attention may need to be paid to the ecological needs of rabbits when considering population recovery programs.

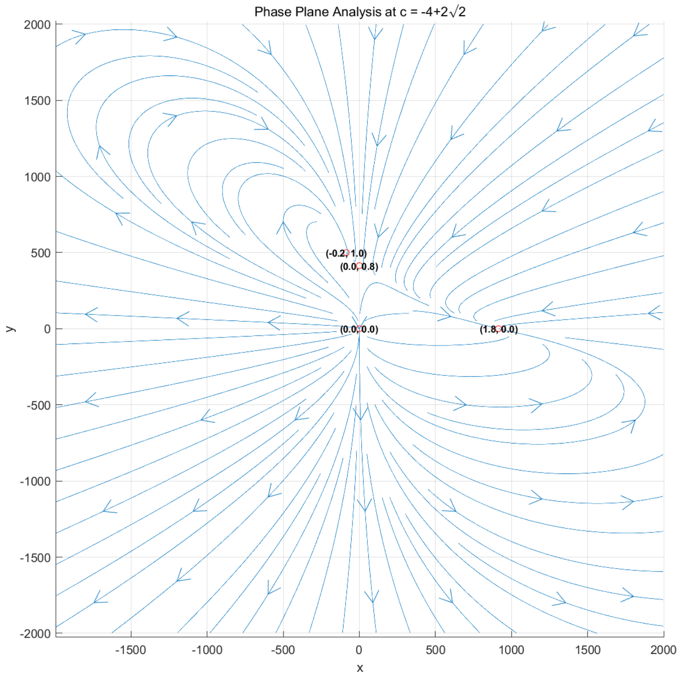

Figure 8: Phase Plane Analysis at $c = -4 + 2\sqrt{2}$

Fixed point analysis

a. **(-0.2, 1) - Stable:**

- This point indicates a stabilization point where the number of rabbits is less than half and the number of sheep is 1. Although the tiny number of rabbits cannot be biologically meaningless, it shows that rabbits can maintain a stable population size under certain environmental conditions.

b. **(0, 0.8) - Unstable:**

- This is indicated by the fact that the rabbit population is close to stabilizing, but due to its instability, small perturbations may cause the population to change rapidly and deviate from this state.

c. **(0, 0) - Unstable:**

- A state with a complete absence of sheep and rabbits is shown as an unstable node. This indicates that in the absence of sheep and rabbits, any small perturbation in the system can lead to rapid changes in the population, which may result in population recovery or further decline.

d. **(1.8, 0) - Stable:**

- This point indicates that the sheep population is stabilized at 1.8 with no rabbits. This is a biologically viable steady state, showing that sheep can maintain their populations independently under specific environmental conditions.

Long-term ecological impacts

- **Stability point analysis**: Of all the fixed points, only (-0.2, 1.0) and (1.8, 0) are stable, so we only need to consider these two fixed points when time tends to infinity. However, since the first fixed point (-0.2, 1.0) is biologically impossible, we only need to consider this fixed point (1.8, 0). Thus, as time tends to infinity, both the sheep and the rabbit eventually converge to the point (1.8, 0).

- **Population dynamics**: Under such conditions, rabbits may not be able to maintain their populations due to unsupportive environmental conditions, while sheep may survive and stabilize without the competitive pressures of rabbits.

- **Ecological management strategy**: Such analysis is essential for the development of targeted conservation measures and management strategies. In particular, given that rabbits may require specific environmental conditions to survive, conservation measures may need to focus on improving their survival conditions, or special attention may need to be paid to the ecological needs of rabbits when considering population recovery programs.

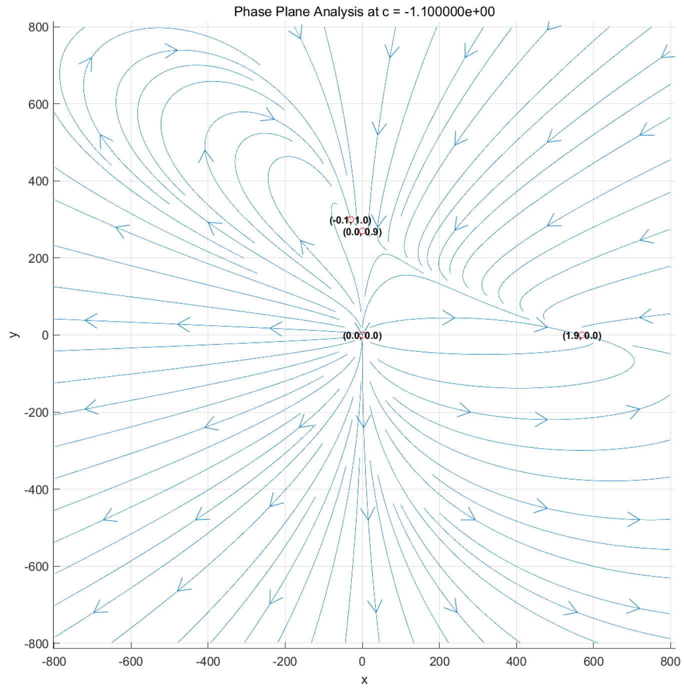

Figure 9: Phase Plane Analysis at c = -1.1

Fixed point analysis

a. **(-0.1, 1) - Stable:**

- This point indicates a stabilizing focus with fewer sheep (slightly less than one unit) and one rabbit. This indicates an increase in the stability of sheep under conditions of striving for fewer resources to the detriment of rabbits.

b. **(0, 0.9) - Unstable:**

- This point shows an unstable point where the number of rabbits is close to 1 but slightly less. This means that at any small perturbation could make the system unstable and unable to maintain stability at this point.

c. **(0, 0) - Unstable:**

- The state without any animals is shown as a point of instability. This indicates that in the absence of any animals, any small perturbation of the system could lead to an imbalance in this stability, possibly leading to the collapse or further evolution of the system.

d. **(1.9, 0) - Stable:**

- Sheep far outnumber rabbits, and there are no rabbits. This is a persistent upward stabilization point, showing that rabbits are less stable and that sheep can maintain their populations independently.

Long-term trends and possible impacts on ecosystems

- **Stability point analysis**: Of all the fixed points, only (-0.1, 1.0) and (1.9, 0) are stable, so we only need to consider these two fixed points when time tends to infinity. However, since the first fixed point (-0.1, 1.0) is biologically impossible, we only need to consider this fixed point (1.9, 0). Thus, as time tends to infinity, both the sheep and the rabbit eventually converge to the point (1.9, 0).

- **Population dynamics and environmental adaptation:**

  - This modeling analysis learns that the stability of rabbits interacts with the adaptability of environmental conditions. Higher stability implies environmental adaptation, while lower stability drives the need for more adaptive adjustments to the environment.

- **Ecological management strategies.**

  - For such ecosystems, conservation measures should consider promoting environmental adaptation to support more unstable ecological equilibria, while increasing the balance of environmental impacts due to excessive competition.

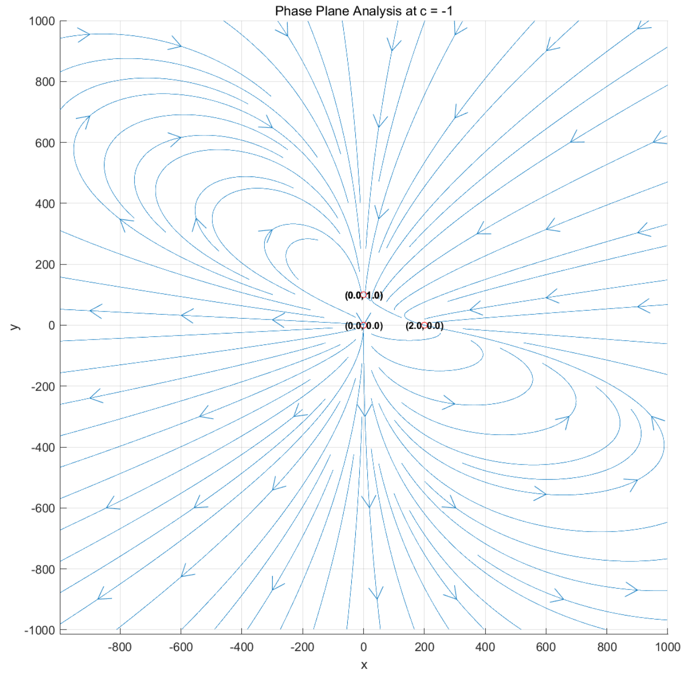

Figure 10: Phase Plane Analysis at c = -1

Fixed Point Analysis

- **(0,1) - Stable**
  - This fixed point exhibits a stable spiral behavior, suggesting a resilient ecosystem where sheep thrive due to favorable conditions, temporarily overshadowing the rabbit population.
- **(0,0) - Unstable**
  - Represents a point of instability, possibly a critical transition state in the ecosystem where neither sheep nor rabbits have a definitive advantage, leading to unpredictable dynamics.
- **(2,0) - Stable:**
  - Indicates a stable node where the rabbit population stabilizes, potentially due to a decrease in sheep population allowing for more resources for rabbits.

Long-Term Ecological Implications

- Ecological sustainability:
  - The stability at $(0, 1)$ and $(2, 0)$ suggests long-term sustainable states for sheep and rabbits respectively, with potential shifts depending on environmental changes or interventions.
- Risk of collapse:
  - The instability at $(0, 0)$ poses a risk of collapse, where slight changes in external conditions could lead to significant ecological shifts, underlining the delicate balance in predator-prey dynamics.
- Conservation strategies:
  - Suggests a focused conservation effort on maintaining the balance at these critical points to ensure the health and viability of both populations.

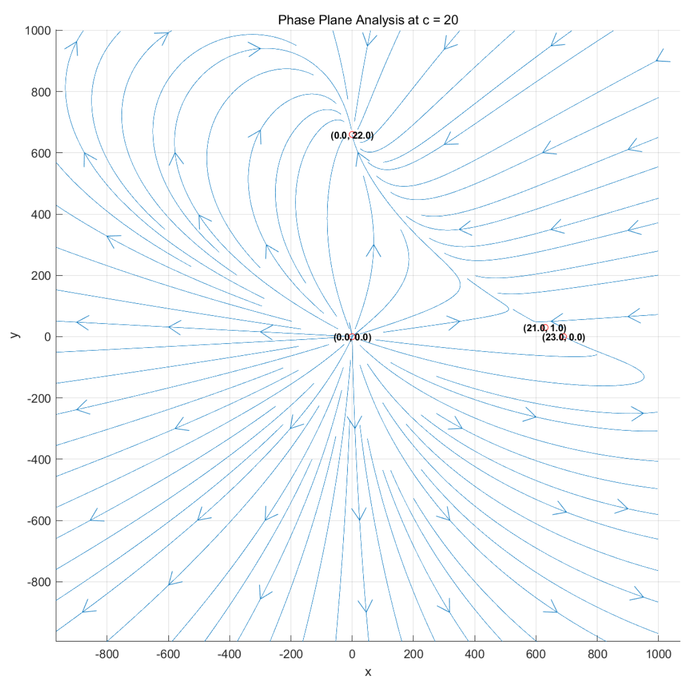

Figure 11: Phasa Plane Analysis at c = 20

Fixed Point Analysis

a. **(0, 22) - Stable**

- This fixed point suggests a stable state where rabbits are able to thrive significantly, reaching a high population due to possibly abundant resources and minimal competition from sheep. The high stability here implies a strong ecological niche for rabbits.

b. **(0,0) - Unstable**

- Represents an unstable equilibrium where both sheep and rabbit populations are zero. This point typically represents a barren or devastated ecosystem where neither species can sustain a population, potentially due to harsh environmental conditions or overexploitation.

c. **(21,1) and (23,0) - Stable**

- These points indicate stable conditions where sheep dominate the ecosystem. The slight presence of rabbits at $(21, 1)$ suggests minimal competition, but sheep have clearly adapted better or faced less environmental pressure. The point at $(23, 0)$ further solidifies the dominance of sheep when rabbits are completely absent, possibly due to sheep having a more robust nature against environmental or competitive pressures at this value of c.

Ecological Implications

- **Dominance and Adaptation**: The dominance of one species in multiple stable points suggests that certain adaptations or traits have allowed sheep to outcompete rabbits significantly at this high value of c. This might be indicative of competitive exclusion where one species (sheep) effectively monopolizes resources.

- **Resource Distribution**: The different stable states imply that resource distribution and environmental factors heavily favor sheep at this parameter setting, leading to their dominance in the ecosystem.

- **Conservation Concerns**: Management strategies may need to focus on rebalancing these populations to prevent the rabbit population from collapsing completely, ensuring biodiversity and ecosystem stability.

Discussion

In our research, we applied the Lotka-Volterra competition model, modified by adding the environmental parameter c, to examine the impact of environmental changes on the competition dynamics between rabbit and sheep populations. Our results, founded on rigorous phase plane and bifurcation analyses, demonstrate significant shifts in population stability and competition outcomes due to influenced by environmental conditions.

Interpretation of Results

The mathematical model's dynamics are governed by the parameter c, which encapsulates external environmental influences such as climate variability and resource availability on population sizes of sheep (x) and rabbits (y). Physically, our findings indicate that negative values of c simulate harsh environmental conditions leading to the decline and potential extinction of both populations, which is highlighted by the stability analysis showing the unstable nature of most fixed points for negative c values (Table 5). Conversely, positive c values foster conditions conducive to the stabilization and growth of the rabbit population while leading to the extinction of sheep. This response to environmental conditions is crucial, demonstrating how minor variations in c lead to vastly different ecological outcomes, which is critical for understanding species resilience and management.

Reliability and Significance

The reliability of our results stems from the application of validated mathematical methods and ecological theories, ensuring that our conclusions are robust within the assumptions of the model. The significance of this study lies in its contribution to ecological modeling, highlighting the profound impact environmental factors have on biodiversity and species survival, particularly under competitive pressures.

Comparison with Other Models

Our findings align with contemporary ecological models that emphasize the role of environmental factors in species competition and stability. However, unlike simpler deterministic models that may not account for the dynamic nature of environmental changes, our model integrates these variations directly into the competition dynamics, offering a more nuanced understanding of how species might react to real-world environmental fluctuations.

Methodological Advancements

Our approach utilizing phase plane analysis and bifurcation methods allowed us to explore the stability of the system under varying conditions systematically. These methods provide the flexibility to examine critical transitions between different population dynamics, a capability not always possible with more static modeling approaches.

Limitations and Future Directions

While our model provides significant insights, it assumes constant environmental conditions represented by a single parameter c. Future research could enhance model realism by incorporating time-varying environmental factors or stochastic elements to reflect more accurately the unpredictable nature of real-world ecosystems. Additionally, exploring interactions beyond two species or including factors such as predation, migration, and genetic variations could offer deeper insights into community dynamics and species adaptation strategies.

By extending our analysis to different ecological settings, such as aquatic systems, and comparing these results with those obtained from terrestrial models, future studies could further validate the applicability of the Lotka-Volterra model across diverse biological contexts. This would broaden our understanding of ecological dynamics and refine conservation strategies tailored to specific environmental and biological conditions.

Conclusions

In this research, we analyze the dynamics of sheep and rabbit populations under various environmental conditions and competitive pressure by using the Lotka-Volterra competition model with an additional environmental parameter c. Equations in our model explicitly take into account interspecific competition, and it reacts to environmental changes by considering external environmental elements that impact these dynamics. With this method, we can investigate methodically how environmental factors impact the survival and stability of these two rival species. This study's specific goal was to clarify how changes in environmental conditions might significantly impact population stability or cause population extinction.

We began with an introduction to the model, detailing the biological significance of the main parameters. Using phase plane analysis and bifurcation methods, we systematically explored how variations in the environmental parameter c influence the competition between rabbits and sheep and how their populations' numbers response. Our findings indicate that a negative c leads to the eventual extinction of both species, while a positive c results in the extinction of sheep and the stabilization of rabbit populations at an increased level. These outcomes demonstrate the critical impact of environmental conditions on species competition and survival, supporting our thesis that environmental changes play an important role in determining population dynamics.

Our study has important ecological implications, suggesting that the model can be applied equally well to the study of other species' interactions and competition. The sensitivity of population dynamics to environmental change emphasizes the need for careful management of ecosystems and protection environment.

In order to simulate more realistic environmental variations, environmental factors should be taken into account for their changes over time. In subsequent studies, a more detailed model can be constructed to reflect the environment's ability for self-repairing. Furthermore, adding genetic variables or adaptive features of species to the model might provide more understanding of how species adjust to environmental shifts and competitive pressures.

These extended models might be investigated in a later study in various ecological contexts, like aquatic ecosystems, where species interactions and environmental factors differ greatly from those in terrestrial systems. This will improve our knowledge of ecological dynamics across a larger range of habitats and validate the models' applicability in various biological contexts.

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