

CS 162 Notes (Winter 2017)

Justin Lubin

Contents

1	Asymptotic Analysis	5
1.1	Big-O Notation	5
1.1.1	Definition	5
1.1.2	Assumption: Constant Operations (Approximations)	5
1.1.3	Non-Constant Operations	5
1.2	Reducing Big-O Expressions	6
1.2.1	What to Eliminate	6
1.2.2	Examples	6
1.3	Linear Search	6
1.3.1	Code	6
1.3.2	Analysis	6
1.4	Binary Search	7
1.4.1	Code	7
1.4.2	Analysis	7
1.5	Methods of Asymptotic Analysis	7
1.5.1	Expansion Method	7
1.5.2	Substitution Method	8
1.6	The Towers of Hanoi	9
1.6.1	Gameplay	9
1.6.2	Algorithmic Solution	9
1.7	Mergesort	10
1.7.1	Description	10
1.7.2	Example	10
1.7.3	Analysis	10
2	Algorithm Correctness	13
2.1	Key Parts of an Algorithm	13
2.2	Proving Correctness	13
2.2.1	Basic Example	13
2.2.2	Loop Invariants	14
2.2.3	Strong Induction	14
2.2.4	Mergesort	14
3	Data Structures	17
3.1	Dictionaries	17
3.1.1	Description	17

3.1.2	Implementations	17
3.1.3	Better Implementations	18
3.2	Trees	18
3.2.1	Terminology	18
3.3	Binary Trees	18
3.3.1	Description	18
3.3.2	Binary Tree Traversal	19
3.4	AVL Trees	20
3.4.1	Motivation	20
3.4.2	Example	21
3.4.3	Maintaining the AVL Condition	21
3.5	Hashtables	23
3.5.1	Description	23
3.5.2	Example Hash Function	23
3.5.3	Collisions	23
3.5.4	Load Factor	24
3.5.5	Linear Probing	24
3.5.6	Quadratic Probing	24

Chapter 1

Asymptotic Analysis

1.1 Big-O Notation

1.1.1 Definition

We have that a function $f \in O(g(n))$ if and only if there exists constants c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.

We have that a function $f \in \Omega(g(n))$ if and only if there exists constants c and n_0 such that $f(n) \leq cg(n)$ for all $n \leq n_0$.

We have that a function $f \in \Theta(g(n))$ if and only if $f \in O(g(n))$ and $f \in \Omega(g(n))$.

This notation can be used to analyze the best-case, average-case, and worse-case efficiency of an algorithm, but this class typically concerns the worst-case efficiency of an algorithm.

Note that efficiency can be a measure of time, space, or even power complexity.

1.1.2 Assumption: Constant Operations (Approximations)

- Arithmetic (fixed width)
- Assignment
- Access any array element

1.1.3 Non-Constant Operations

Control Flow	Time
Consecutive statements	Sum of time of each statement
Conditional	Time of test + time of the slower branch
Loop	Number of iterations * time of body
Function call	Time of function body

Control Flow	Time
Recursion	Solve recurrence relation

1.2 Reducing Big-O Expressions

1.2.1 What to Eliminate

- Eliminate low-order terms
- Eliminate coefficients

1.2.2 Examples

- $4n + 5 \in O(n)$
- $\frac{1}{2}n \log n + 2n + 7 \in O(n \log n)$
- $n^3 + 2^n + 3n \in O(2^n)$
- $n \log(10n^2) + 2n \log n \in O(n \log n)$
 - Note that $n \log(10n^2) = 2n \log(10n)$

1.3 Linear Search

1.3.1 Code

```
int find(int[] arr, int arr_length, int k) {
    for (int i = 0; i < arr_length; ++i) {
        if (arr[i] == k) {
            return 1;
        }
    }
    return 0;
}
```

1.3.2 Analysis

- Best case: approximately six steps
 - $O(1)$
- Worst case: $6 * \text{arr_length}$ steps
 - $n = \text{arr_length}$, so $O(n)$

1.4 Binary Search

1.4.1 Code

```
int find(int[] arr, int k, int lo, int hi) {
    return help(arr, k, 0, arr_length);
}

int find(int[] arr, int arr_length, int k) {
    int mid = (hi + lo) / 2;
    if (lo == hi) {
        return 0;
    }
    if (arr[mid] == k) {
        return 1;
    }
    if (arr[mid] < k) {
        return help(arr, k, mid + 1, k);
    } else {
        return help(arr, k, lo, mid);
    }
}
```

1.4.2 Analysis

Let $T(n)$ be the efficiency of find. Then, because each split takes approximately ten operations, we have that:

$$\begin{aligned}
 T(n) &= 10 + T\left(\frac{n}{2}\right) \\
 &= 10 + \left(10 + T\left(\frac{n}{4}\right)\right) \\
 &= 10 + \left(10 + \left(10 + T\left(\frac{n}{8}\right)\right)\right) \\
 &= 10k + T\left(\frac{n}{2^k}\right).
 \end{aligned}$$

To solve this, there are a couple methods.

1.5 Methods of Asymptotic Analysis

1.5.1 Expansion Method

We know $T(1)$, so we should try to express this formula in terms of $T(1)$. To do so, let $\frac{n}{2^k} = 1$, so then $k = \log n$. Then

$$\begin{aligned}
T(n) &= 10 \log n + T\left(\frac{n}{2^{\log n}}\right) \\
&= 10 \log n + T(1) \\
&= 10 \log n + 10 \\
&\in O(\log n).
\end{aligned}$$

However, this method actually gives you a big-theta approximation for T ; in other words, not only is $T \in O(\log n)$, we also have that $T \in \Theta(\log n)$.

1.5.2 Substitution Method

IMPORTANT NOTE: This method is *not* recommended. Because it actually requires an inductive proof (not covered).

Guess $O(?)$, then check. For example (in this case), guess $\log n$ because we have something like $\frac{n}{2^n}$ in the formula. Then:

$$\begin{aligned}
T(n) &= 10 + T(n/2) \\
&= 10 + \log(n/2)
\end{aligned}$$

Because we have guessed that $T \in O(\log n)$, we have that $T(n) \leq c \log n$ for all $n \geq n_0$, for some constants c, n_0 .

$$\begin{aligned}
T(n) &\leq c \log n \\
10 + \log(n/2) &\leq c \log n \\
10 + \log(n) - \log(2) &\leq c \log n \\
\frac{10 + \log(n) - \log(2)}{\log n} &\leq c \\
\frac{10}{\log n} + 1 - \frac{\log(2)}{\log n} &\leq c \\
\frac{10}{\log n} + 1 - \frac{1}{\log n} &\leq c
\end{aligned}$$

Now take $n_0 = 2$, so $n \geq 2$ and thus $\log n \geq 1$ (\log is base two). We then have:

$$\begin{aligned}
c &\geq \frac{10}{1} + 1 - \frac{1}{1} \\
c &\geq 10
\end{aligned}$$

Therefore, $T \in O(\log n)$ because with $c = 10$ and $n_0 = 2$, we have that $T(n) \leq c \log n$ for all $n \geq n_0$.

Note that the substitution method is more general than the **Expansion Method**, but it does *not* give you a big-theta approximation (unlike the **Expansion Method**).

1.6 The Towers of Hanoi

1.6.1 Gameplay

The goal of the Towers of Hanoi is to move all disks to goal peg, with the following rules:

- You can only move one disk at a time
- You can only move the top-most disk in a pile
- You cannot put a larger disk on top of a smaller one

1.6.2 Algorithmic Solution

```

if n = 1:                                => T(1)
    move to goal (base case)              = 1
else:                                     => T(n)
    move top n-1 disks to temporary peg   = T(n - 1)
    move bottom disk to goal              + T(1)
    move the n-1 disks to goal            + T(n - 1)

```

Thus, $T(n) = T(n - 1) + T(1) + T(n - 1)$, with $T(1) = 1$. So:

$$\begin{aligned}
 T(n) &= 2T(n - 1) + 1 \\
 &= 2(2T(n - 2) + 1) + 1 \\
 &= 4T(n - 2) + 3 \\
 &= 4(2T(n - 3) + 1) + 3 \\
 &= 8T(n - 3) + 7.
 \end{aligned}$$

By inspection, we have:

$$T(n) = 2^{n-1}T(1) + (2^{n-1} - 1),$$

but $T(1) = 1$, so we have:

$$\begin{aligned}
 T(n) &= 2^{n-1} + 2^{n-1} - 1 \\
 &= 2^n - 1 \\
 &\in \Theta(2^n).
 \end{aligned}$$

1.7 Mergesort

1.7.1 Description

To mergesort a list, split the list into two and sort the sublists. To merge them back together, interleave the elements. Interleaving / merging is $O(n)$ and there are $O(\log n)$ splits, so mergesort is $O(n \log n)$.

Mergesort is based on the trick that it is really easy to interleave two sorted lists.

1.7.2 Example

```

8 2 9 4 5 3 1 6
=> 8 2 9 4
    => 8 2
        => 8
        => 2
        merge: 2 8
    => 9 4
        => 9
        => 4
        merge: 4 9
    merge: 2 4 8 9
=> 5 3 1 6
    => 5 3
        => 5
        => 3
        merge: 3 5
    => 1 6
        => 1
        => 6
        merge: 1 6
    merge: 1 3 5 6
merge: 1 2 3 4 5 6 8 9

```

1.7.3 Analysis

We have that $T(1) = 1$ and $T(n) = 2T(n/2) + n$ (split into two sublists, then mergesort them, then merge / interleave them). We then have that:

$$\begin{aligned}T(n) &= 2T(n/2) + n \\&= 2(T(n/4) + n/2) + n \\&= 2T(n/4) + 2n \\&= 4(2T(n/8) + n/8) + 2n \\&= 8T(n/8) + 3n \\&= 2^k T(n/2^k) + kn\end{aligned}$$

Set $n/2^k = 1$ (because we are using the **Expansion Method**), so $k = \log n$. Then:

$$\begin{aligned}T(n) &= 2^{\log n} T(n/2^{\log n}) + (\log n)n \\&= nT(1) + n \log n \\&= n + n \log n \\&\in \Theta(n \log n)\end{aligned}$$

Chapter 2

Algorithm Correctness

2.1 Key Parts of an Algorithm

There are a couple key things that every algorithm needs:

- Inputs
- Outputs
- Preconditions (restrictions on input)
- Postconditions (restrictions on output)
- Step-by-step process specification (either in natural language or pseudocode)

Therefore, we can define a “correct” algorithm to be one that, given any input data that satisfies the precondition, produces output data that satisfies the postcondition *and* terminates (stops) in finite time.

2.2 Proving Correctness

2.2.1 Basic Example

Consider the following pseudocode to swap two variables:

```
swap(x, y)
  aux := x
  x := y
  y := x
```

The following is a proof of correctness of `swap`:

1. Precondition: $x = a$ and $y = b$.
2. Postcondition: $x = b$ and $y = a$.
3. `aux := x` implies `aux := a`.
4. `x := y` implies `x := b`.
5. `y := aux` implies `y := a`.

6. Thus, $x := b$ and $y := a$, so the postcondition is satisfied.

2.2.2 Loop Invariants

A *loop invariant* is a logical predicate that, if true before any single iteration of a loop, then is also true after the iteration of the loop. It is called an “invariant” because it is always true.

When using induction to prove the correctness of an algorithm, the loop invariant is the inductive hypothesis.

Consider the following pseudocode to sum the first N elements of a list a :

```
sum_list(a, N)
  s := 0
  k := 0
  while (k < N):
    s := s + a[k]
    k := k + 1
```

To prove this algorithm is correct, we must show:

1. The loop invariant (that at step k , s = sum of first k numbers in a) holds true; and
2. The algorithm terminates.

We want to show that, at step k , s = sum of first k numbers in a . Hence, we will induct on k .

1. At $k = 0$, we have that $s = 0$, so the algorithm is correct for $k = 0$.
2. We will assume that the algorithm holds for some arbitrary k .
3. We will now prove that the algorithm holds for $k + 1$. In the $(k + 1)$ -th iteration of the loop, we set $s = s + a[k + 1]$, so s is the sum of the first k numbers (because, by the induction hypothesis, before the iteration of the loop we have that s is the sum of the first k numbers) plus $a[k + 1]$; i.e., it is the sum of the first $k + 1$ numbers.

Hence, the loop invariant holds for all k . However, we must also prove that the algorithm terminates. For each iteration, we have that $k := k + 1$, so after N iterations, $k = N$, and hence the loop will terminate.

2.2.3 Strong Induction

Strong induction is just like regular induction, but instead of assuming the inductive hypothesis for some k , we assume it for all values less than or equal to k . We will use strong induction in the proof of correctness of mergesort.

2.2.4 Mergesort

Consider the following pseudocode of mergesort:

```
mergesort(A, l, r):
  if l < r:
    m = floor((l + r) / 2)
```

```

mergesort(A, l, m)
mergesort(A, m + 1, r)
merge(A, l, m, r)

```

Note that the precondition of `merge` is that $A[l \dots m]$ and $A[m + 1 \dots r]$ are sorted, and the postcondition is that A is sorted.

The precondition for this algorithm is that A has at least 1 element between the indices l and r (otherwise, this code doesn't have anything to do). The postcondition is that the elements between l and r are sorted.

To prove the correctness of this algorithm, we must show that the postcondition holds. To do so, we will prove that `mergesort` can sort n elements (and so it will therefore be able to sort $n = r - l + 1$ elements).

The base case of $n = 1$ is true because A is simply a one-element list (which is always sorted).

We will let the inductive hypothesis be that `mergesort` correctly sorts $n = 1 \dots k$ elements (i.e., everything less than or equal to k). This is strong induction.

To show the inductive step ($n = k + 1$) true, consider the two recursive calls in the function.

1. For the first recursive call, we have that `mergesort` is sorting $m - l + 1 = (k + 1) / 2 \leq k$ elements, so by the inductive hypothesis, `mergesort` holds. Hence, $A[l \dots m]$ is sorted.
2. For the second recursive call, we have that `mergesort` is sorting $r - (m + 1) + 1 = r - m = (k + 1) / 2 \leq k$ elements, so by the inductive hypothesis, `mergesort` holds. Hence, $A[m + 1 \dots r]$ is sorted.

We therefore have that the precondition of `merge` is upheld, so the postcondition of `merge` must also be upheld. Therefore, we have that A is sorted, and so the postcondition of `mergesort` is upheld.

Lastly, we must show that `mergesort` terminates. Note that for each recursive call, the length of the subarray between p and q decreases, so eventually it must reach a point where there is only one element in the array, in which case we reach termination.

Chapter 3

Data Structures

3.1 Dictionaries

3.1.1 Description

A *dictionary* is a set of (key, value) pairs. The keys must be comparable. A dictionary has the following operations:

- insert(key, value)
- find(key)
- delete(key)

3.1.2 Implementations

Here are some possible underlying data structures for the dictionary:

	insert	find	delete
Unsorted linked list	$O(1)$	$O(n)$	$O(n)$
Unsorted array	$O(1)$	$O(n)$	$O(n)$
Sorted linked list	$O(n)$	$O(n)$	$O(n)$
Sorted array	$O(n)$	$O(\log n)$	$O(n)$
Balanced tree	$O(\log n)$	$O(\log n)$	$O(\log n)$
“Magic array”	$O(1)$	$O(1)$	$O(1)$

A “magic” array is a hashtable!

3.1.2.1 Side Note: Lazy Deletion

Rather than actually deleting and shifting the contents of an array, you can just “mark” it as deleted. This is simpler, faster, and you can do removal in batches, but takes up extra space and may complicate the implementation.

3.1.3 Better Implementations

The following are some better implementations for a dictionary:

1. Binary trees
2. AVL trees (guaranteed to be balanced)
3. B-trees (guaranteed to be balanced)
4. Hash tables

We will implement a dictionary as a binary tree where the data of each node is a (key, value), and the left child has key that is less than (or equal to) that of the parent node, and the right child has a key that is greater than that of the parent node. Note that such an ordering must be defined because we require that the keys are comparable.

3.2 Trees

3.2.1 Terminology

Here is some terminology for trees:

- Root: top of tree
- Leaves: at bottom of tree
- Children: the nodes directly below (and attached to) another node
- Parent: the node to which children are attached
- Siblings: two nodes are siblings if they are children to the same parent
- Descendants: subtree with a particular node at root; includes children, children of children, etc.
- Depth (of a node): distance to root
- Height (of a tree): distance from root to furthest leaf
- Branching factor: how many children each node can have (for a binary tree, this is 2)
- A *balanced* tree has height $\in O(\log n)$, where n is the number of nodes.

3.3 Binary Trees

3.3.1 Description

Here are some facts that are true of any binary tree with height h :

- The maximum number of leaves is 2^h

- The maximum number of nodes is $2^h + (2^h - 1) = 2^{h+1} - 1$
- The minimum number of leaves is 1
- The maximum number of leaves is $h + 1$

Here is a function to determine the height of a tree:

```
int tree_height(Node *root) {
    if (root == NULL) {
        return -1;
    } else {
        return 1 + max(tree_height(root->left, root->right));
    }
}
```

3.3.2 Binary Tree Traversal

Suppose that we have a binary tree defined as follows:

```
+
 *
  2
  4
 5
```

Then we can process t either in-order, pre-order, or post-order.

- In-order: 2 * 4 + 5
- Pre-order: + * 2 4 5
- Post-order: 2 4 * 5 +

Here is a slightly more abstract example of ordering. Consider the following tree:

```
A
 B
  D
  E
 C
  F
  G
```

Then:

- In-order: DBEAFCG
- Pre-order: ABDECFG
- Post-order: DEBFGCA

Note that, given an order trace, one cannot determine the structure of the original tree (but given a pre/post-order trace *and* an in-order trace, one can do so).

The following code performs an in-order traversal of a tree:

```
void in_order_traversal(Node *t) {
    if (t != NULL) {
```

```

        // for pre-order, process here instead
        in_order_traversal(t->left);
        process(t->data);
        in_order_traversal(t->right);
        // for post-order, process here instead
    }
}

```

3.4 AVL Trees

3.4.1 Motivation

For a binary tree, we have that:

- Each node has less than or equal to two children
- Operations are simple
- Order property
 - All keys in left subtree are smaller
 - All keys in right subtree are larger
 - This gives us $O(\log n)$ find — **but only if balanced**

This is what an unbalanced tree looks like:

```

A
  B
    C
      D
        E

```

It basically is a linked list, so `find`, `insert`, and `delete` are all $\Theta(n)$. If you wanted to build this tree (with n items), it would be $\Theta(n^2)$. This is bad, so we want to keep the tree *balanced*.

We must keep balance not just at build time, but also every time we insert or delete. To do this, we will need to define what constitutes a “balanced” tree. However, the conditions cannot be too weak (useless) or too strong (impossible). The sweet spot is the **AVL condition**.

Here are some possible conditions, and why they are not really that good.

1. Left + right subtrees of the root have the same number of nodes.
 - Too weak because the left and right subtrees can just themselves be really unbalanced.
2. Left + right subtrees of the root have the same height.
 - Too weak because the left and right subtrees might just both be linear (i.e. look like linked lists).
3. Left + right subtrees of **every node** have equal number of nodes.
 - This ensures that the tree will always be perfect, but is too strong because this is extremely difficult/expensive to maintain.
4. Left + right subtrees of **every node** have equal height.
 - Same thing as previous condition; too strong.

So, what is the solution? The AVL condition:

- Left + right subtrees of every node differ by at most 1.

3.4.2 Example

The following is a valid AVL tree:

```

6 (3)
 4 (1)
  1 (0)

 8 (2)
 7 (0)
11 (1)
 10 (0)
 12 (0)

```

The numbers in parentheses are the height of that particular node. The height is defined as the maximum distance to the bottom. It can recursively be defined as $\max(\text{height}(\text{child1}), \text{height}(\text{child2})) + 1$; i.e., one more than the maximum height of the children.

3.4.3 Maintaining the AVL Condition

There are four cases that we have to handle when updating the AVL tree to make sure that it is valid. Fixing invalid AVL trees is done via a technique known as *rotation*.

Note that, in the following examples, [x] denotes an arbitrary (sub)tree.

3.4.3.1 Left-Left

```

a (h+3)
 b (h+2)
  [x] (h+1; originally h)
   *added element*
  [y] (h)
 [z] (h)

```

becomes

```

b (h+2)
 [x] (h+1)
  *added element*
 a (h+1)
  [y] (h)
  [z] (h)

```

3.4.3.2 Right-Right

```

a (h+3)
  [x] (h)
  b (h+2)
    [y] (h)
    [z] (h+1; originally h)
      *added element*

```

becomes

```

b (h+2)
  a (h+1)
    [x] (h)
    [y] (h)
  [z] (h+1)
    *added element*

```

3.4.3.3 Right-Left

This utilizes a **double rotation**, but it is helpful to just think of moving the problematic node to its grandparent position, then just putting everything else back in how it fits (which will be unique).

```

a (h+3)
  [x] (h)
  b (h+2)
    c (h+1)
      [u] (h; originally h-1)
      *added element*
      [v] (h-1)
    [z] (h)

```

becomes

```

c (h+3)
  a (h+2)
    [x] (h+1)
    [u] (h)
    *added element*
  b (h+1)
    [v] (h-1)
    [z] (h)

```

3.5 Hashtables

3.5.1 Description

A hashtable maps keys to indices in an array via a hashing function. Keys can be anything, so long as the hash function maps it to an integer.

For insert, find, and delete, hashtables are really fast. However, the hashtable is just a random mapping between keys and indices, so there is no ordering. Hence, things like `findmin`, `findmax`, `predecessor`, and `successor` would all be $O(n)$. AVL trees would be a better choice if you need these functions.

Hashtables are useful if the key-space is very large, but the actual number of keys that you use is small. Note that the array backing a hashtable is almost always much smaller than the key-space.

Here are some examples of when it would be good to use a hashtable:

- Compiler: all possible identifiers for a variable vs. those used in a program. This is a large key-space with a small number of keys actually used, so a hashtable would be useful here.
- Database: all possible student names vs. the students in a class.
- AI: all possible chess configurations vs. ones seen in a single game

3.5.2 Example Hash Function

Here is a simple hash function:

$$h(k) = k \% n,$$

where k is the key and n is the table size.

If you have a bunch of numbers that are multiples of each other as your key, then you would want the table size to be prime (or just coprime to the multiple in question) in order to minimize collisions.

3.5.3 Collisions

A *collision* occurs when two keys map to the same index. One way to resolve this is to utilize *chaining*. Chaining is when you store a linked list at each index of the array (instead of just the key). Then, once you hash a key, you can look in / append to the linked list for it. If you have a really bad hashing function and you chain, then your hashtable just becomes a linked list.

3.5.4 Load Factor

When chaining, we can define λ to be the *load factor* of a hashtable, which corresponds to the average number of elements in each *bucket* (index in the array). We have that $\lambda = k/n$, where k is the number of elements and n is the table size.

One application of this is that we can be more precise in our analysis of `find`. We have that `find` is in $O(\lambda)$ in the unsuccessful case, and $O(\lambda/2)$ in the successful case (so it is all really $O(\lambda)$).

3.5.5 Linear Probing

Instead of having a linked list at each index (which takes more memory and is dynamically allocated), we can use a technique called *linear probing*. If there is a collision, then just store the key in the next available spot in the array. However, you have to be careful when finding and deleting. When you are looking for an element, you have to hash it, then do linear search until you either find the key or you find an empty spot. However, when deleting, you must use lazy deletion, otherwise you would mess up the linear probing search.

This is not usually used in practice. Quadratic probing is preferred.

3.5.6 Quadratic Probing

Quadratic probing is just like linear probing, but instead of storing the colliding key in the next index, you store it in the second, fourth, sixteenth (and so on) index away from the original index. This ensures that the indices are more spread out, so clusters don't arise (the goal of a hashtable is to have a good spread of indices).

Quadratic probing avoids the dynamic allocation of memory present in the chaining technique, while at the same time improving upon the naive linear probing method.