$$(S1)\quad x+0=x \qquad\qquad (P1)\quad x\cdot 0 = 0$$
$$(S2)\quad x+s(y)=s(x+y) \qquad\qquad (P2)\quad x\cdot s(y) = x\cdot y + x$$

$\forall x \forall y\, (x + s(y) = s(x) + y)$ $\boxed{\text{Lema}}$

Indução em y.

<u>Base $(y=0)$</u>: $\forall x\, (x+s(0) = s(x)+0)$ ✓

$$x+s(0) \overset{S2}{=} s(x+0) \overset{S1}{=} s(x) \overset{S1}{=} s(x)+0$$

<u>Passo $(\varphi(y)\to\varphi(s(y)))$</u>

<u>Hp.:</u> $\forall x\, (x+s(y) = s(x)+y)$ $[\varphi(y)]$

<u>Tese:</u> $\forall x\, (x+ss(y) = s(x)+s(y))$

$\varphi(y)\to\varphi(s(y))$

$$x+ss(y) \overset{S2}{=} s(x+s(y)) \overset{HP}{=} s(s(x)+y) \overset{S2}{=} s(x)+s(y)$$

Pelo princípio de indução, segue $\forall x \forall y\, (x+s(y) = s(x)+y)$ $(*)$

Vamos ver o que acontece fazendo indução em x.

<u>Base</u> $\forall y\, (0+s(y) = \underbrace{s(0)+y})$

$$0+s(y) \overset{S2}{=} s(0+y)$$

Propriedade comutativa da soma

$$\forall x \forall y \, (x + y = y + x)$$

Ind. em y.

Base $(y=0)$ $\forall x \, (x + 0 = 0 + x)$

Ind. em x

Base $(x=0)$ $0+0 = 0+0$ ✓

Passo

Hp.: $(x+0=0+x) \rightarrow$ Tese $(s(x)+0 = 0+s(x))$

$$s(x)+0 \overset{S1}{=} s(x) \overset{S1}{=} s(x+0) \overset{HP}{=} s(0+x) \overset{S2}{=} 0+s(x)$$

Passo:

Hp $(\forall x \, (x+y=y+x)) \rightarrow$ Tese $(\forall x \, (x+s(y) = s(y)+x))$

$$x+s(y) \overset{S2}{=} s(x+y) \overset{HP}{=} s(y+x) \overset{S2}{=} y+s(x) \overset{(*)}{=} s(y)+x \quad .$$

Propriedade cancelativa da soma

$$\forall x \forall y \forall z \left(x+y = x+z \rightarrow y=z \right)$$

Ind. em x

$(PA2)\ \forall x \forall y \left(s(x)=s(y) \rightarrow x=y \right)$

Base $(x=0)$ $\forall y \forall z \left(0+y = 0+z \rightarrow y=z \right)$

$$y \overset{S1}{=} y+0 \overset{C}{=} 0+y = 0+z \overset{C}{=} z+0 \overset{S1}{=} z$$

Passo

H_p $\left(\forall y \forall z \left(x+y = x+z \rightarrow y=z \right)\right) \rightarrow \left(\forall y \forall z \left(s(x)+y = s(x)+z \rightarrow y=z \right)\right)_{Tese}$

$$s(x)+y \overset{(*)}{=} x+s(y) \overset{S2}{=} s(x+y)$$

$$s(x)+z \overset{(*)}{=} x+s(z) \overset{S2}{=} s(x+z)$$

$$s(x+y) = s(x+z) \overset{PA2}{\longrightarrow} x+y = x+z \overset{HP}{\longrightarrow} y=z.$$

$$s(x+y) \overset{S2}{=} x+s(y) \overset{(*)}{=} s(x)+y = s(x)+z \overset{(*)}{=} x+s(z) \overset{S2}{=} s(x+z)$$

$$\downarrow PA2$$

$$x+y = x+z$$

$$\downarrow HP$$

$$y=z$$

Elemento neutro do produto

$$\forall x \left(x \cdot s(0) = x \right)$$

$$x \cdot s(0) \overset{P2}{=} x \cdot 0 + x \overset{P1}{=} 0 + x \overset{C}{=} x + 0 \overset{S1}{=} x$$

Prop. distributiva dir. do produto

$$\forall x \forall y \forall z \left((x+y) \cdot z = x \cdot z + y \cdot z \right)$$

Ind. em z

Base $(z=0)$ $\forall x \forall y \left((x+y) \cdot 0 = x \cdot 0 + y \cdot 0 \right)$

$$(x+y) \cdot 0 \overset{P1}{=} 0 \overset{S1}{=} 0+0 \overset{P1}{=} x \cdot 0 + y \cdot 0$$

Passo

Hp $\forall x \forall y \left((x+y) \cdot z = x \cdot z + y \cdot z \right) \rightarrow \forall x \forall y \left((x+y) \cdot s(z) = x \cdot s(z) + y \cdot s(z) \right)$ Tese

$$(x+y) \cdot s(z) \overset{P2}{=} (x+y) \cdot z + (x+y) \overset{HP}{=} (x \cdot z + y \cdot z) + (x+y) \overset{A}{=}$$

$$= x \cdot z + (y \cdot z + x) + y \overset{C}{=} x \cdot z + (x + y \cdot z) + y \overset{A}{=} (x \cdot z + x) + (y \cdot z + y) \overset{P2}{=}$$

$$= x \cdot s(z) + y \cdot s(z) . \quad \checkmark$$

Lema

$$(**)\quad \forall x\, \forall y\, (s(x)\cdot y = x\cdot y + y)$$

Ind. em y

Base $(y=0)$ $\forall x\, (s(x)\cdot 0 = x\cdot 0 + 0)$

$$s(x)\cdot 0 \overset{P1}{=} 0 \overset{P1}{=} x\cdot 0 \overset{S1}{=} x\cdot 0 + 0 \quad ✓$$

Passo

$$\text{Hp } \forall x (s(x)\cdot y = x\cdot y + y) \rightarrow \forall x (s(x)\cdot s(y) = x\cdot s(y) + s(y)) \text{ Tese}$$

$$s(x)\cdot s(y) \overset{P2}{=} s(x)\cdot y + s(x) \overset{HP}{=} (x\cdot y + y) + s(x) \overset{A}{=} x\cdot y + (y + s(x)) \overset{(*)}{=}$$

$$= x\cdot y + (s(y) + x) \overset{C}{=} x\cdot y + (x + s(y)) \overset{A}{=} (x\cdot y + x) + s(y) \overset{P2}{=}$$

$$= x\cdot s(y) + s(y) \quad ✓$$

Propr. comutativa do produto

$$\forall x \forall y \, (x \cdot y = y \cdot x)$$

Ind. em y

Base (y=0): $\forall x \, (x \cdot 0 = 0 \cdot x)$

Ind em x Base (x=0) $0 \cdot 0 = 0 \cdot 0$ ✓

Passo Hp: $(x \cdot 0 = 0 \cdot x) \rightarrow (s(x) \cdot 0 = 0 \cdot s(x))$ Tese

$$0 \cdot s(x) \overset{P2}{=} 0 \cdot x + 0 \overset{HP}{=} x \cdot 0 + 0 \overset{S1}{=} x \cdot 0 \overset{P1}{=} 0 \overset{P1}{=} s(x) \cdot 0$$

Passo

Hp $\forall x \, (x \cdot y = y \cdot x) \rightarrow \forall x \, (x \cdot s(y) = s(y) \cdot x)$ Tese.

$$x \cdot s(y) \overset{P2}{=} x \cdot y + x \overset{HP}{=} y \cdot x + x \overset{(**)}{=} s(y) \cdot x$$ ✓

Corolário $\forall x \forall y \forall z \, (x(y+z) = x \cdot y + x z)$ Distr. esq.

$$x(y+z) \overset{C}{=} (y+z)x \overset{Dd}{=} yx + zx \overset{C}{=} xy + xz$$

Propr. associativa do produto

$$\forall x \forall y \forall z \left((x \cdot y) \cdot z = x \cdot (y \cdot z) \right)$$

Ind. em z

Base $(z=0)$ $\forall x \forall y \left((x \cdot y) \cdot 0 = x \cdot (y \cdot 0) \right)$

$$(x \cdot y) \cdot 0 \overset{P1}{=} 0 \quad e \quad x \cdot (y \cdot 0) \overset{P1}{=} x \cdot 0 \overset{P1}{=} 0 \qquad \checkmark$$

Passo

Hp $\forall x \forall y \left((x \cdot y) \cdot z = x \cdot (y \cdot z) \right) \rightarrow \forall x \forall y \left((x \cdot y) \cdot s(z) = x \cdot (y \cdot s(z)) \right)$ Tese

$$(x \cdot y) \cdot s(z) = (x \cdot y) \cdot z + x \cdot y \overset{HP}{=} x \cdot (y \cdot z) + x \cdot y \overset{De}{=} x \left(y \cdot z + y \right) \overset{P2}{=}$$

$$= x \left(y \cdot s(z) \right). \qquad \checkmark$$

<u>Exercício</u> Prove a seguinte:

$$\forall x \forall y \forall z \left(\left(\neg (z=0) \wedge (xz=yz) \right) \rightarrow x=y \right)$$

$\neg$ = NOT

$\wedge$ = AND

$\vee$ = OR

$\rightarrow$ = IF ... THEN