**Summary of Computing Effective Annual Rates**[1]

1. Suppose we are told that our investment in a bank will grow at the rate of 8% per annum, compounded quarterly. Say we invest some amount $P$. What this means is that in 1 year, we will have

$$P\left(1+\frac{.08}{4}\right)^4 = F.$$

   $F$ is the future value of our investment. Here, 0.08 is called the **quoted rate** and $0.08/4 = 0.02$ is called the **period rate**. In this example, our investment earns 2% for 4 quarters.

   More generally, say $r$ is the stated annual interest rate (SAIR), and $r$ is compounded $n$ times a year. Then the following describes how our initial investment of $P$ grows up to be $F$:

$$P\left(1+\frac{r}{n}\right)^n = F.$$

   The periodic rate here is $r/n$. That's the rate we earn per period. In general, if we invest for $t$ years, we have

$$P\left(1+\frac{r}{n}\right)^{tn} = F.$$

   We could also use this formula for discounting:

$$P = F\left(1+\frac{r}{n}\right)^{-tn}.$$

   This last formula describes what it is worth for us to have $F$ dollars in $t$ years, compounded $n$ times a year, with an SAIR of $r$.

2. What could $n$ be? Besides semiannual compounding ($n = 2$) and quarterly compounding ($n = 4$), we could have monthly compounding ($n = 12$), daily compounding, hourly compounding, and on and on. As $n$ gets larger, we approach something

[1]Notes for Finance 604 & 612 prepared by Jessica A. Wachter.

known as continuous compounding. In fact

$$\left(1 + \frac{r}{n}\right)^n \to e^r$$

as $n$ gets large. Under continuous compounding,

$$Pe^{rt} = F$$

and,

$$P = Fe^{-rt}$$

Thus $e^{-rt}$ is a *discount factor*, just like $1/(1+r)$ in the annual compounding case.

3. To see how different rates of compounding affect the present discounted value, consider the following question. What is \$100 one year from now worth today?

$$\begin{aligned} \text{Annual Compounding:} &\quad \tfrac{\$100}{1.06} = \$94.3 \\ \text{Monthly Compounding:} &\quad \tfrac{\$100}{(1+.06/12)^{12}} = \$94.19 \\ \text{Continuous Compounding:} &\quad \$100e^{-0.06} = \$94.18 \end{aligned}$$

This exercise illustrates an interesting point. As the compounding frequency increases, the present value falls. Why is the present value of \$100 *less* when compounding is continuous than when compounding is annual or monthly?

To answer this question, think about the meaning of present value. The present value is the amount you must put away today to have \$100 in one year. Under continuous compounding, you can put away fewer dollars today because the money you invest will grow at a faster rate.

4. We would like to have some way of comparing these different rates of compounding. This is where the effective annual rate, the EAR, comes in. The EAR is defined to

be the rate that makes the following statement true:

$$1 + \text{EAR} = \left(1 + \frac{r}{n}\right)^n.$$

In words, this equation says that **the EAR is the interest rate that, when compounded annually, produces the same value as $r$, compounded $n$ times a year.**

Rearranging the equation

$$\text{EAR} = \left(1 + \frac{r}{n}\right)^n - 1$$

and for continuous compounding,

$$\text{EAR} = e^r - 1.$$

Using the EAR, we are able to translate everything into annual rates. We can translate monthly, quarterly, even continuous compounding into compounding at an annual rate.

Examples of calculating the EAR, assuming an SAIR rate of 6%:

$$\begin{array}{rc} \text{semiannual} & \left(1 + \frac{.06}{2}\right)^2 - 1 = 0.0609 \\ \text{quarterly} & \left(1 + \frac{.06}{4}\right)^4 - 1 = 0.0614 \\ \text{continuously} & e^{0.06} - 1 = 0.0618 \end{array}$$

In the first case, the EAR is 6.09%, in the second case its 6.13%, and in the last case its 6.18%. Notice that the EAR is larger, the greater the rate of compounding.

5. There is another name for the stated annual interest rate $r$. It is also called the Annual Periodic Rate, or APR. The APR represents simple interest and therefore is the *incorrect* way to measure annual returns. Nonetheless, credit cards and others

making loans to consumers are often required to report it. For example, your credit card company probably wrote something like this on your statement:

$$\begin{aligned}\text{periodic rate} &= 1.5\% \text{ per month}\\ \text{APR } &= 12 * 1.5\% = 18\% \text{ per year}\end{aligned}$$

Let's say we miss a payment on our credit card bill. For simplicity, let's say we missed a payment of \$1. In one month, the credit card company will charge us \$(1 + .015). In two months, we will be charged $\$(1+.015)^2$, because the credit card company will earn interest on our interest. If a year goes by, we will be charged $\$(1+.015)^{12}$. Thus,

$$\text{EAR} = (1+.015)^{12} - 1 = .1956$$

Notice the connection to the definition above:

$$\text{EAR} = \left(1+\frac{.18}{12}\right)^{12} - 1.$$

Thus the EAR is 19.56%. That's what the credit card company *should* be reporting.