CSE6643 Numerical Linear Algebra Midterm

December 7, 2016

1

For this problem, you do not need to consider the efficiency, cost, nor stability of your procedure.

(1)Let $\{v_1, v_2, \cdots, v_l\}$ be a given set of linearly independent vectors, and $\{q_1, q_2, \cdots, q_k\}$ be a given set of orthonormal vectors in R^n, with $n \geq l > k$. Derive a formula for a non-zero vector q_{k+1} in $span(v_1, v_2 \cdots, v_l)$ such that $\{q_1, q_2, \cdots, q_k\}$ is an orthonormal set.

Solution:

Assume X is a non-zero $l \times 1$ vector, V is the space spanned by $\{v_1, v_2, \cdots, v_l\}$, let $q_{k+1} = \frac{VX}{||VX||}$, then q_{k+1} is in $span\{v_1, v_2 \cdots, v_l\}$. To derive a q_{k+1} that is orthonormal to $Q = \{q_1, q_2, \cdots, q_k\}$, then we just need to solve the following equation:

$$Q^T V X = (Q^T V)X = 0$$

Since dimension of $Q^T V$ is $k \times l$ and dimension of X is $l \times 1$, besides $l > k$, thus there are non-zero solutions and $l - k$ linearly independent bases of X. The solution set of this linear system is a vector space. Assume $X_1, X_2, \cdots X_{l-k}$ are $l - k$ solution vectors of $(Q^T V)X = 0$ and $a_1, a_2, \cdots a_{l-k}$ are arbitrary constants, then $X = a_1X_1 + a_2X_2 + \cdots + a_{l-k}X_{l-k}$. Therefore,

$$q_{k+1} = \frac{VX}{||VX||} \quad \texttt{and} \quad X = a_1X_1 + a_2X_2 + \cdots + a_{l-k}X_{l-k}$$

(2)Given A an $n \times n$ real matrix and D a diagonal $n \times n$ matrix. Derive a procedure to construct an orthogonal $n \times n$ matrix Q(that is the columns of Q form an orthonormal basis) such that

$$AQ - QD = L$$

with L a lower-triangular matrix. Hint: identify the columns of the left and right of the above equation.

Solution: Solve by identifying the column of the left and right of the above equation.

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

$AQ - QD =$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix} - \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

Assume $L = AQ - QD$, then $l_{ij} = \sum_{k=1}^{n} a_{ik}q_{kj} - q_{ij}d_{jj}$. Since matrix L is a lower-triangular matrix, to construct matrix L, notice the zero entries in each column. For example, the entry $l_{12} = 0$.

$$l_{12} = \sum_{k=1}^{n} a_{1k}q_{k2} - q_{12}d_{22} = 0$$

$$\Rightarrow \begin{bmatrix} a_{11} - d_{22} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} q_{12} \\ q_{22} \\ \vdots \\ q_{n2} \end{bmatrix} = 0$$

Thus the second column of Q is the solution of this linear equation. And it has Similarly, we can obtain the rest of 0 entries by the same way, such that we can create $n-1$ linear equations. For the last column of L, we can obtain:

$$\Rightarrow \begin{bmatrix} a_{11} - d_{nn} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} - d_{nn} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} - d_{nn} & a_{1n} \end{bmatrix} \begin{bmatrix} q_{1n} \\ q_{2n} \\ \vdots \\ q_{nn} \end{bmatrix} = HX = 0$$

In the $(n-i)th$ equation set, we have $n-i$ equations and i unknown, then $r(H) = n - i < n (i \neq 0)$, it must have non-zero solution of homogeneous linear equation. Also, there are $n-i$ basis for the solution space.

Therefore, for the $(n-1)th$ homogeneous equation, since the system has infinitely many solutions. Select one solution as $q_n = \begin{bmatrix} q_{1n} \\ q_{2n} \\ \vdots \\ q_{nn} \end{bmatrix}$. Similarly for the $(n-2)th$ homogeneous linear equation set, we can construct two basic solution set $\{v_1, v_2\}$.(linearly independent) According to the first part of this problem, we can obtain a non-zero vector $q_{n-1} = \begin{bmatrix} q_{1,n-1} \\ q_{2,n-1} \\ \vdots \\ q_{n,n-1} \end{bmatrix}$ in $span(v_1, v_2)$ such that it is orthonormal to q_n. In the same analogy from this point, we can generate the q_{n-2} from the $(n-3)th$ homogeneous equation which is orthonormal to $\{q_{n-1}, q_n\}$ set. Following this logic of deduction, we can obtain a $\{q_1, q_2, \cdots, q_{n-1}, q_n\}$ orthonormal set, Q. Therefore, given A and D and Q is known, we can produce L.

2

Let $b = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $A = \begin{bmatrix}1 & 1\\ \epsilon & 0\\ 0 & \epsilon\end{bmatrix}$ with $\epsilon = 10^{-10}$. Solve the least square problem $Ax = b$ by hand. What happen if you solve the same problem by Matlab with double precision using the exact same method that you did by hand. What would be a good method to solve this problem if you have to use Matlab?

Solution: According to the normal equation, solve the least equation problem by $x = (A^TA)^{-1}A^Tb$

$$b = \begin{bmatrix}1\\1\\1\end{bmatrix}, A = \begin{bmatrix}1 & 1\\ \epsilon & 0\\ 0 & \epsilon\end{bmatrix} \Rightarrow A^T = \begin{bmatrix}1 & \epsilon & 0\\ 1 & 0 & \epsilon\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1 & 1\\ \epsilon & 0\\ 0 & \epsilon\end{bmatrix}\begin{bmatrix}1 & \epsilon & 0\\ 1 & 0 & \epsilon\end{bmatrix} = \begin{bmatrix}1+\epsilon^2 & 1\\ 1 & 1+\epsilon^2\end{bmatrix}$$

$$det(A^TA) = (1+\epsilon^2)^2 - 1 = \epsilon^4 + 2\epsilon^2 > 0 \quad A^TA \texttt{ is invertible.}$$

Thus,

$$\begin{aligned}
(A^TA)^{-1} &= \frac{1}{(1+\epsilon^2)(1+\epsilon^2)-1}\begin{bmatrix}1+\epsilon^2 & -1\\ -1 & 1+\epsilon^2\end{bmatrix}\\
(A^TA)^{-1}A^T &= \frac{1}{\epsilon^4+2\epsilon^2}\begin{bmatrix}1+\epsilon^2 & -1\\ -1 & 1+\epsilon^2\end{bmatrix}\begin{bmatrix}1 & \epsilon & 0\\ 1 & 0 & \epsilon\end{bmatrix}\\
&= \frac{1}{\epsilon^4+2\epsilon^2}\begin{bmatrix}\epsilon^2 & \epsilon^3+\epsilon & -\epsilon\\ \epsilon^2 & -\epsilon & \epsilon+\epsilon^3\end{bmatrix}\\
&= \frac{1}{\epsilon^3+2\epsilon}\begin{bmatrix}\epsilon & \epsilon^2+1 & -1\\ \epsilon & -1 & 1+\epsilon^2\end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
x = (A^TA)^{-1}A^Tb &= \frac{1}{\epsilon^3+2\epsilon}\begin{bmatrix}\epsilon & \epsilon^2+1 & -1\\ \epsilon & -1 & 1+\epsilon^2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\\
&= \frac{1}{\epsilon^3+2\epsilon}\begin{bmatrix}\epsilon^2+\epsilon\\ \epsilon^2+\epsilon\end{bmatrix}\\
&= \frac{1+\epsilon}{2+\epsilon^2}\begin{bmatrix}1\\1\end{bmatrix}\\
&= \frac{1+10^{-10}}{2+10^{-20}}\begin{bmatrix}1\\1\end{bmatrix}
\end{aligned}$$

When $\epsilon = 10^{-10}$, the result calculated by Matlab is NaN. And there is a warning said, Matrix is singular to working precision. In Matlab, the double precision can represent the minimum value is $2^{-53} > 10^{-20}$. Therefore, Matlab can't compute 10^{-20} in double precision. If I use Matlab to solve this problem, a good method is directly calculate the result by:$x = A\backslash b$, and the result is $\begin{bmatrix}0.5\\0.5\end{bmatrix}$, which is close to the result calculated by hand.

3

Suppose A is a 501×501 matrix with $||A||_2 = 100$ and $||A||_F = 100$. Find the sharpest possible lower bound of $\kappa_2(A)$. Show all the details.

Solution:

For a singular matrix A, then it is customary to write $\kappa_2(A) = \infty$.

For a non-singular matrix, note that if $||\cdot|| = ||\cdot||_2$, then $||A||_2 = \sigma_1 = 100$ and $||A^{-1}|| = 1/\sigma_m$. Since $\sigma_1 \geq \sigma_2 \cdots \sigma_{501}$. In order to find the sharpest lower bound of condition number, let:

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_{501}}$$

Since $||A||_F = \sqrt{tr(A^T A)} = \sqrt{\sum_{i=1}^{501} \sigma_i^2}$, besides since $\sigma_1 \geq \sigma_2 \cdots \sigma_{501}$, then we can obtain:

$$501\sigma_{501}^2 \leq ||A||_F^2 = \sigma_1^2 + \sigma_2^2 \cdots \sigma_{501}^2 = 110$$

$$500\sigma_{501}^2 \leq ||A||_F^2 - \sigma_1^2 = 110^2 - 100^2$$

$$\Rightarrow \sigma_{501} = \sqrt{\frac{110^2 - 100^2}{500}} = \frac{\sqrt{105}}{5}$$

Therefore the sharpest possible lower bound is:

$$\kappa_2(A) = \frac{100}{\sqrt{105}/5} = \frac{100\sqrt{105}}{21}$$

4

Let $A = (a_{i,j})$ be an $n \times n$ strictly column diagonally dominant matrix, that is

$$|a_{kk}| > \sum_{j \neq k} |a_{jk}|.$$

Show that A is non-singular.

Proof by contradiction:
Assume A is a singular matrix. That means exist non-zero vector $x \in R^n$ that satisfies $Ax = 0$. Select the max element of vector x, $x_k > 0$, since vector x is non-zero.

$$Ax = 0 \Rightarrow \sum_{j=1}^{n} a_{jk}x_j = a_{kk}x_k + \sum_{j=1, j\neq k}^{n} a_{jk}x_j = 0$$

$$\Rightarrow a_{kk}x_k = -\sum_{j=1, j\neq k}^{n} a_{jk}x_j$$

$$\Rightarrow |a_{kk}x_k| = |a_{kk}||x_k| = \sum_{j=1, j\neq k}^{n} |a_{jk}x_j| \leq |x_k| \sum_{j=1, j\neq k}^{n} |a_{jk}| \quad \texttt{Divide } x_k \texttt{ from both side.} \quad (x_k > 0)$$

$$\Rightarrow |a_{kk}| \leq \sum_{j=1, j\neq k}^{n} |a_{jk}|$$

Therefore it is contradict with to the definition of strictly column diagonally dominant matrix.

5 Problem 12.3

Answer:

(a)Figure 1 is the eigenvalues results of 100 random matrix with size $m = 8, 16, 32, 64, 128, 256$. From the figure we can notice the eigenvalues are gradually converging along with the increasing of size. The distributed pattern of the eigenvalues are asymptotic two a upper bound and lower bound. The maximum eigenvalues of each size are decreasing, the minimum eigenvalues of each size are increasing. But when it comes to bigger number of size, the velocity is decreasing.

The first 10 spectral radius $\rho(A)$ of the 100 random matrices with $m = 32$:

```
1.202055
1.015311
1.013898
0.9443638
1.235847
1.133478
0.9392387
1.078432
1.181174
1.011920
```

Figure 2 is the average spectral radius $\rho(A)$ of 100 random matrix with size $m = 8, 16, 32, 64$. From the figure we can notice, when $m \to \infty$, the spectral radius grows up very fast at first, and is oscillating around from 1.01 to 1.02.

(b)Figure 3 is the 2-norm average results of 100 random matrix with size $m = 8, 16, 32, 64, 128, 256$. From the figure we can notice, when $m \to \infty$, it also grows up very fast at first, then it gradually slows down and its value is close to a upperbound.

Figure 4 is result of comparison of $\rho(A)$, $||A||$ and their difference. From the figure we can notice, $\rho(A) \leq ||A||$, the inequality doesn't appear to approach an equality as $m \to \infty$ according to the red line. The difference between them is stable to specific value, when $m \to \infty$.

(c)Figure 5 is the result of average smallest singular value σ_{min} of 100 random matrices. From the figure, we can notice that σ_{min} is gradually decreasing when $m \to \infty$.
Figure 6 is the result of average proportion of σ_{min}. From the figure we can notice, all the lines reach 1 in the end. However, there are some differences. For small m, the proportion scale of σ_m grows slowly to 1, for bigger m, the proportion scale of σ_m grows very fast to 1. The tail of the probability distribution of smallest singular values looks like obeying the normal distribution.

The first 10 smallest singular value σ_{min} of the 100 random matrices with $m = 64$:

```
0.02362232
0.04597737
0.02201776
0.01826399
0.01569841
0.02447862
0.005611243
```

0.005084121
0.05986563
0.03627594

(d)When we change the matrix to upper-triangular matrices whose entries are samples from the same distribution as above. Figure 7 to 12 are the results of upper-triangular. According to the graphs, we can say that if we consider random triangular instead of full matrices, the results are similar.

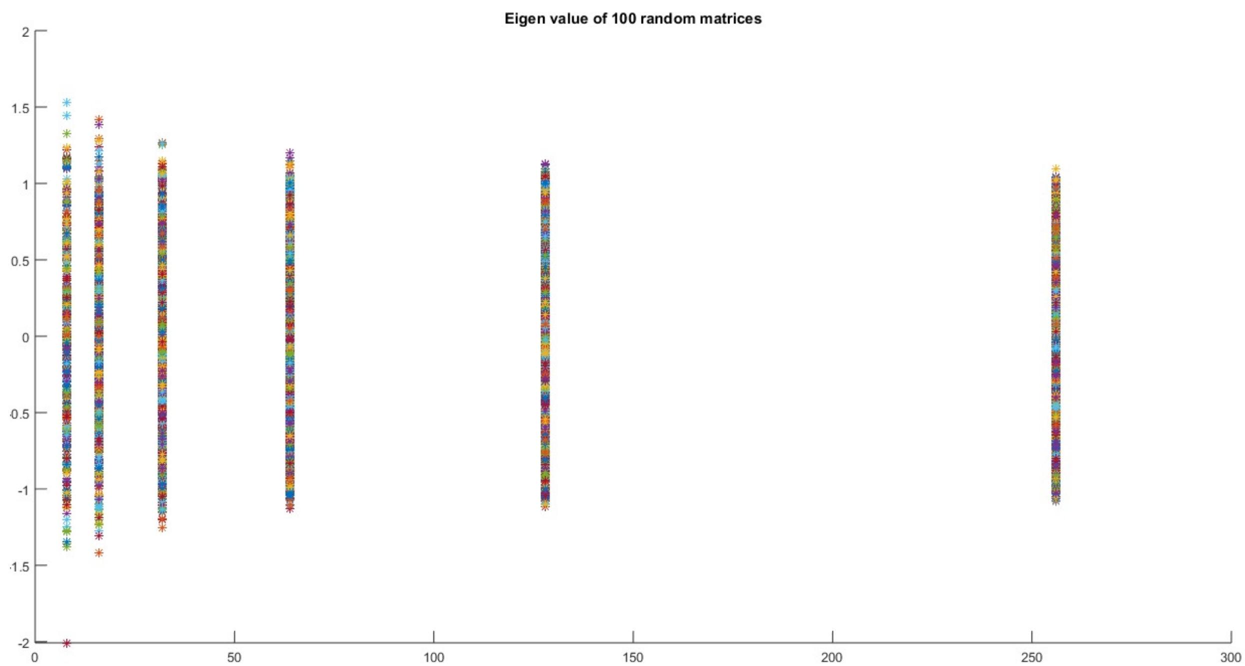

Figure 1: Result of eigenvalue of 100 random matrices

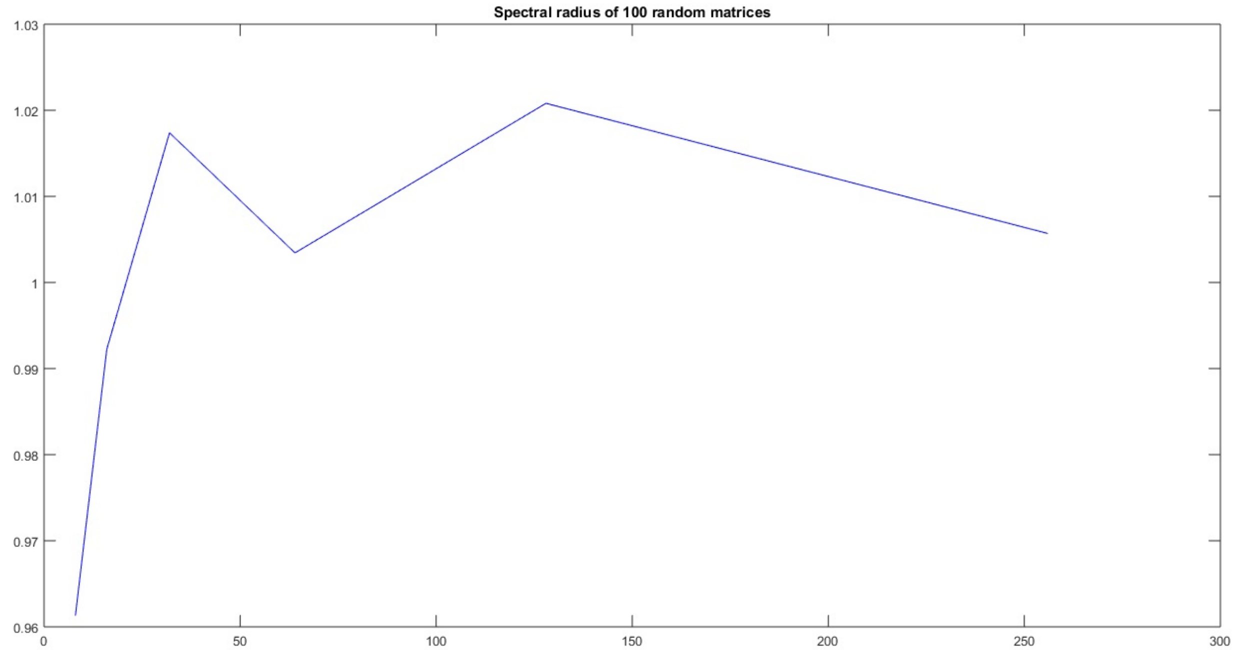

Figure 2: Result of average spectral radius of 100 random matrices

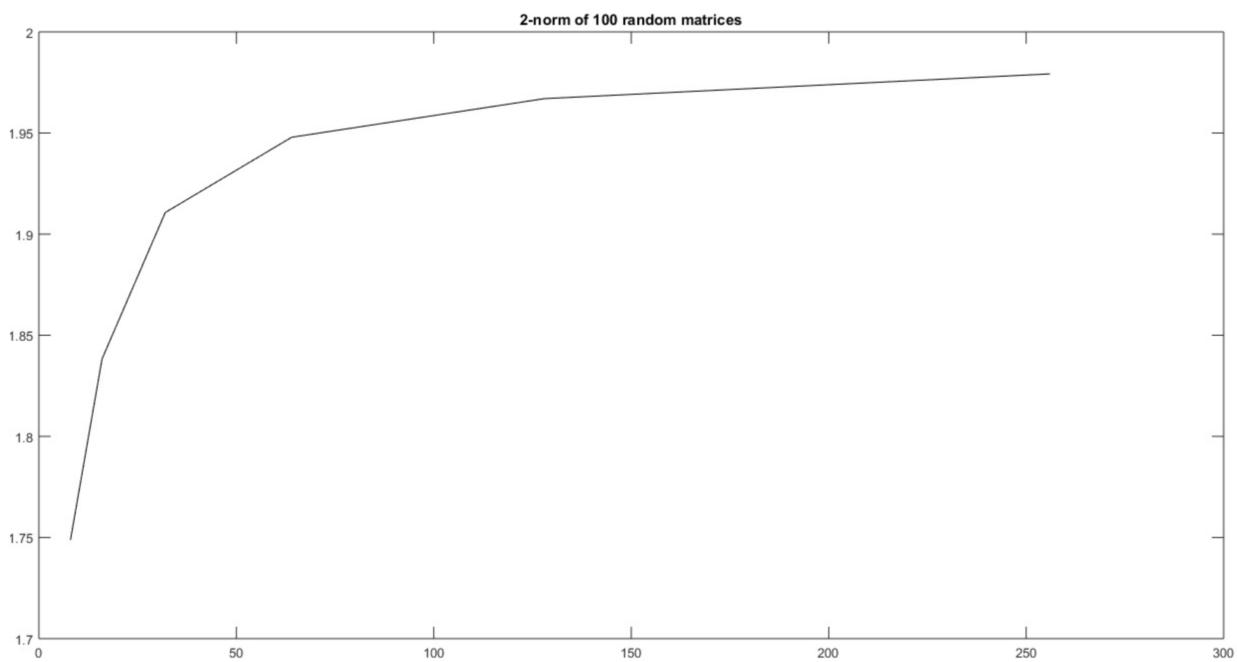

Figure 3: Result of average 2-norm of 100 random matrices

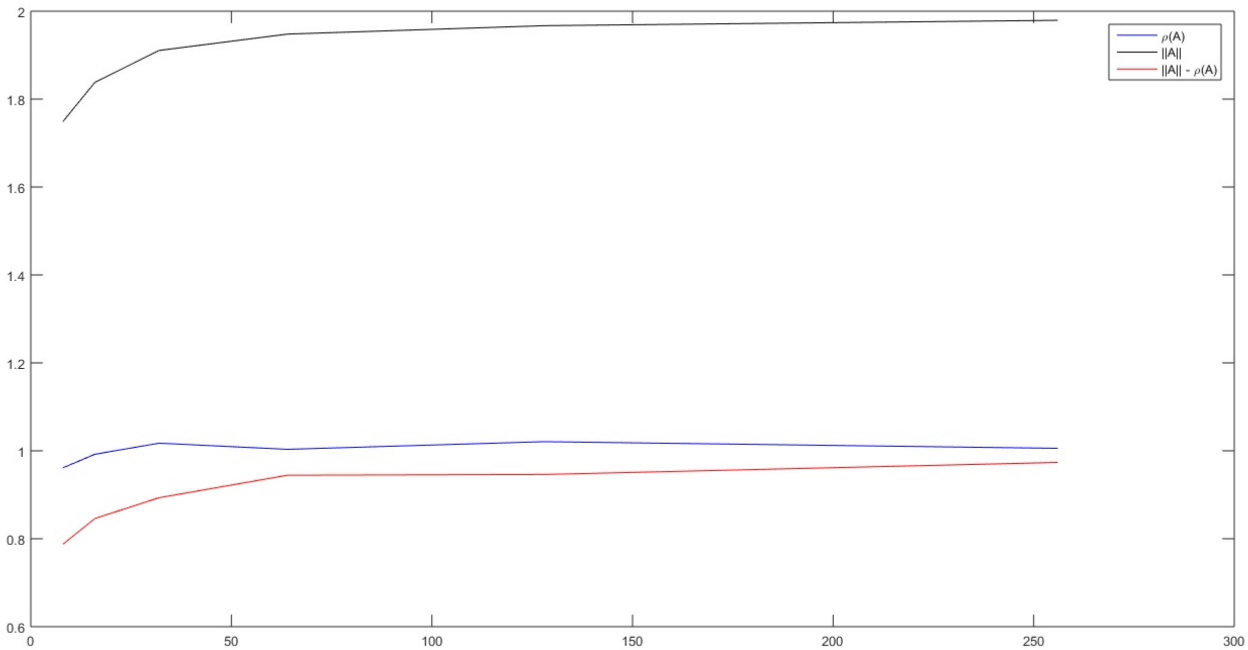

Figure 4: Result of comparison of $\rho(A), ||A||$ and their difference

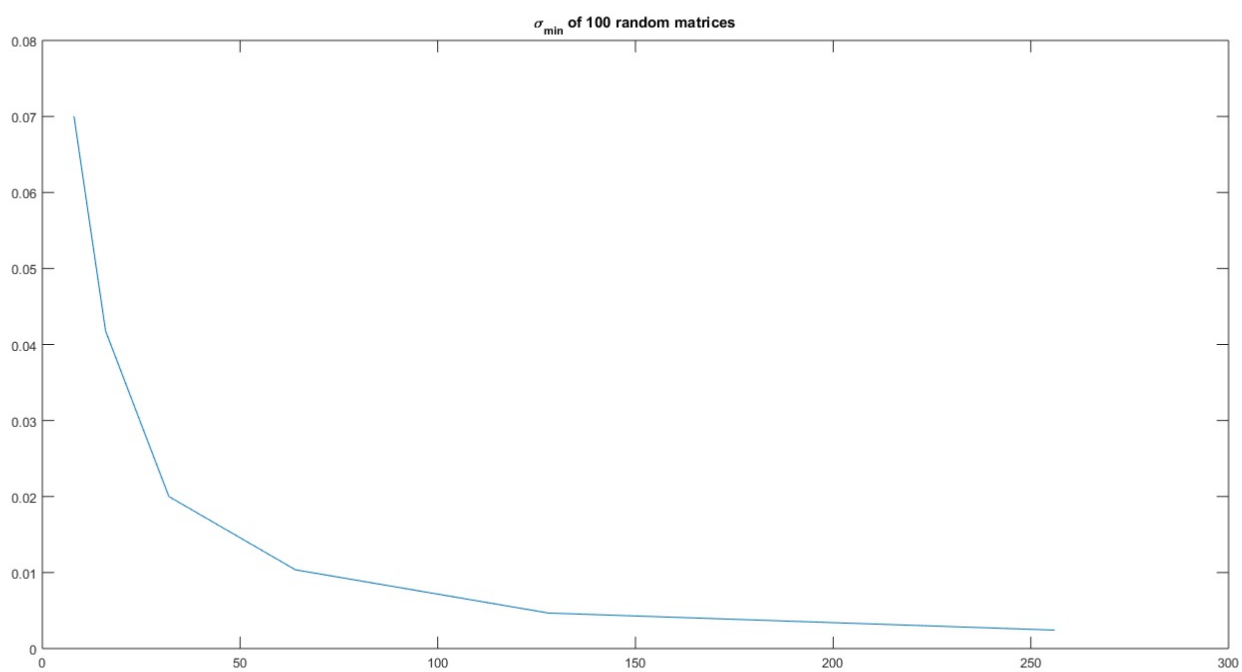

Figure 5: Result of average σ_{min} of 100 random matrices

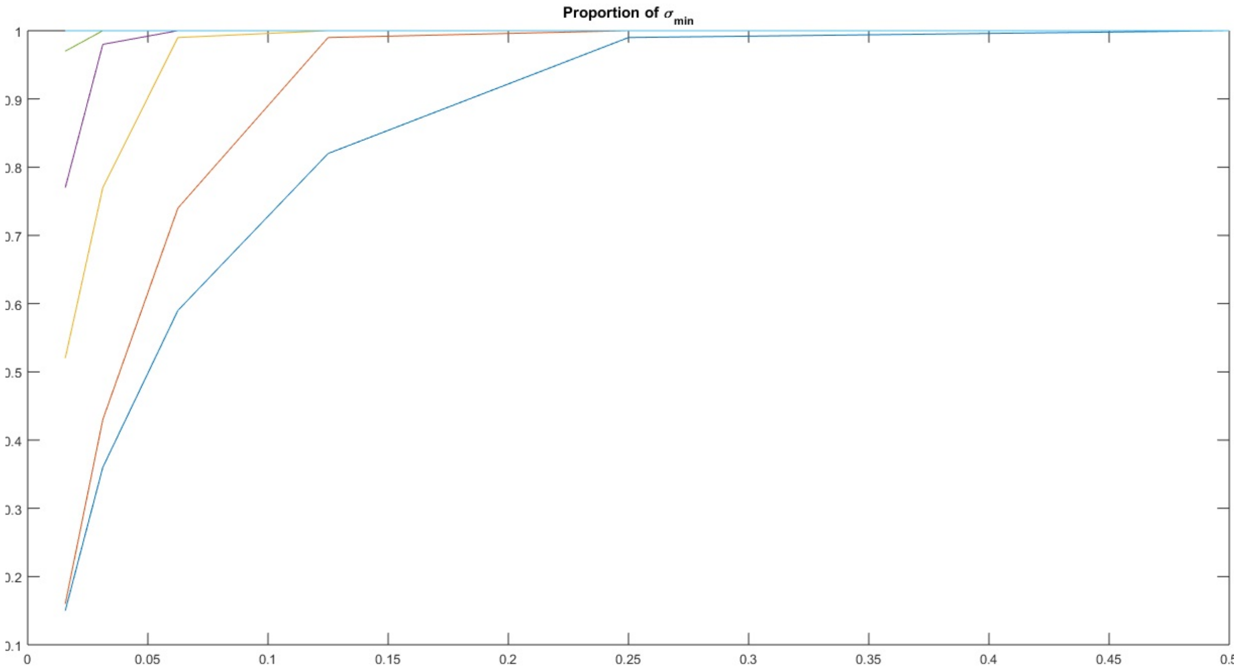

Figure 6: Result of average proportion of σ_{min}

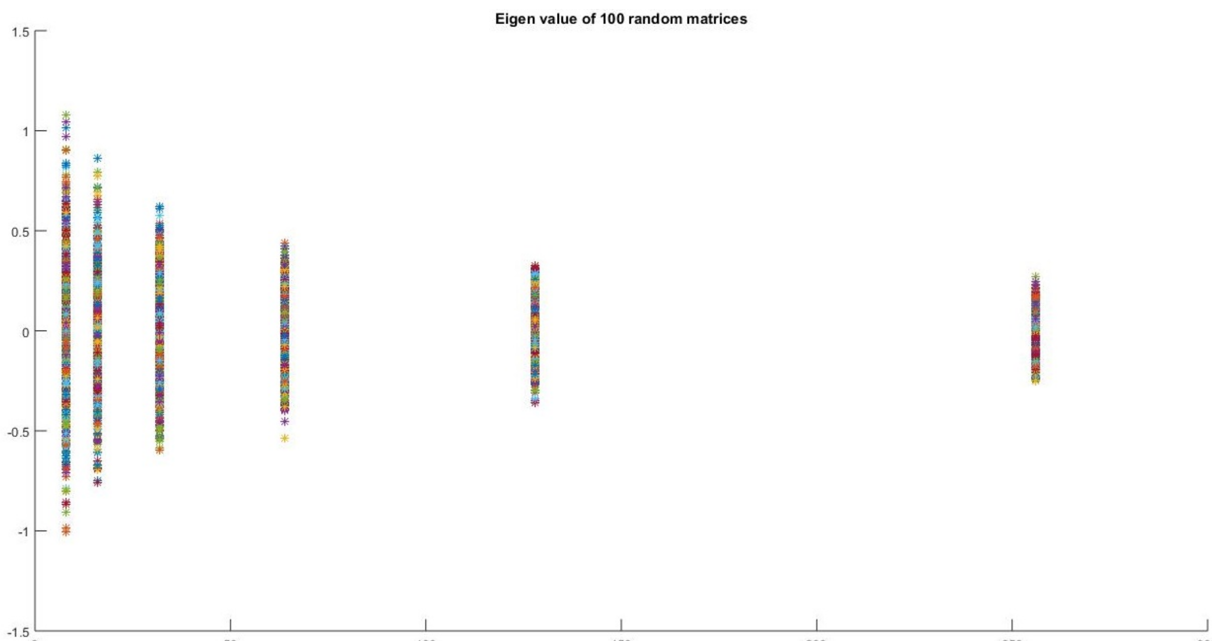

Figure 7: Result of eigenvalue of 100 random matrices

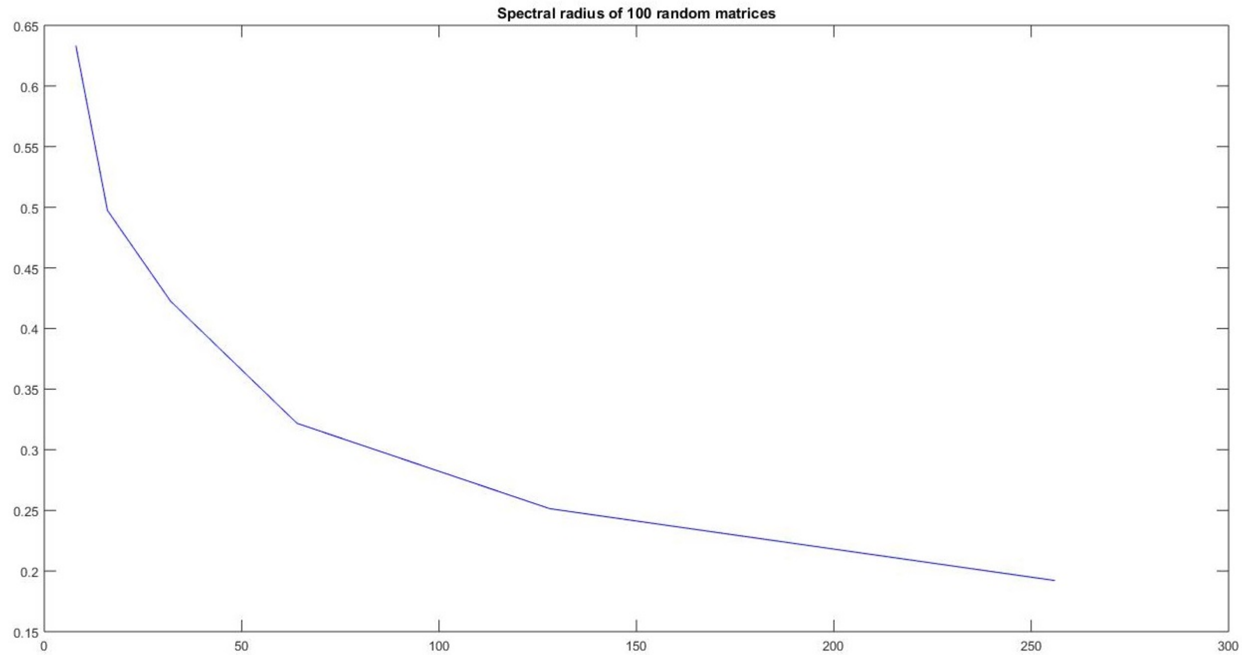

Figure 8: Result of average spectral radius of 100 random matrices

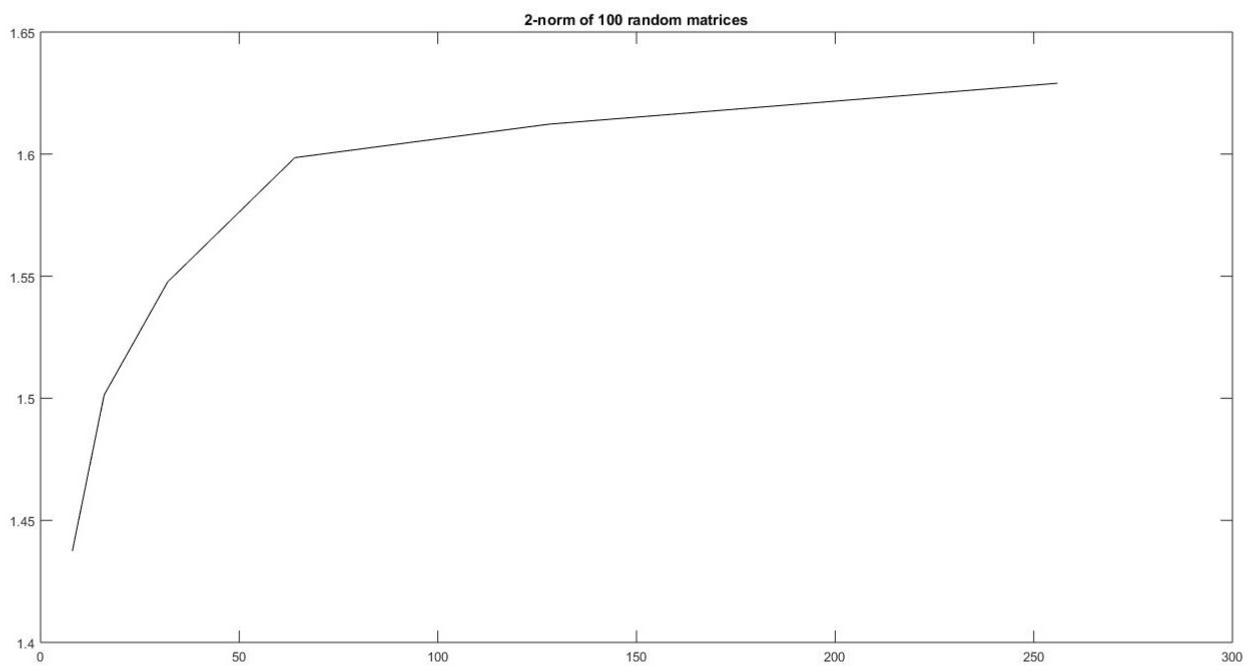

Figure 9: Result of average 2-norm of 100 random matrices

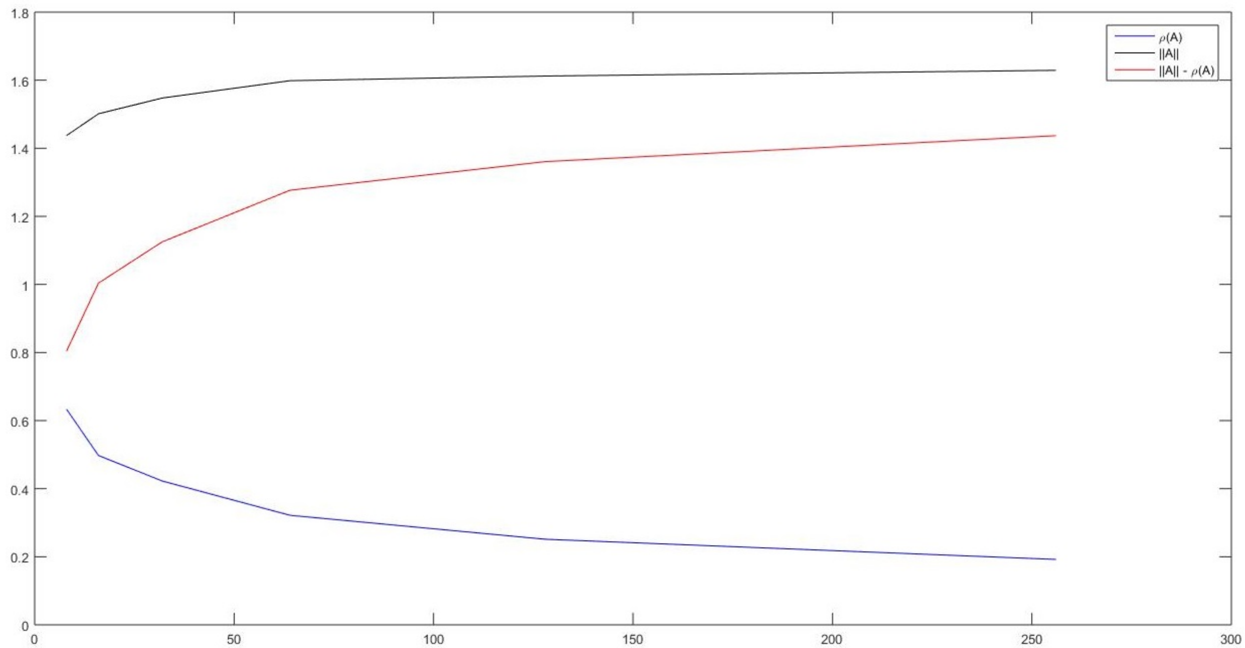

Figure 10: Result of comparison of $\rho(A), ||A||$ and their difference

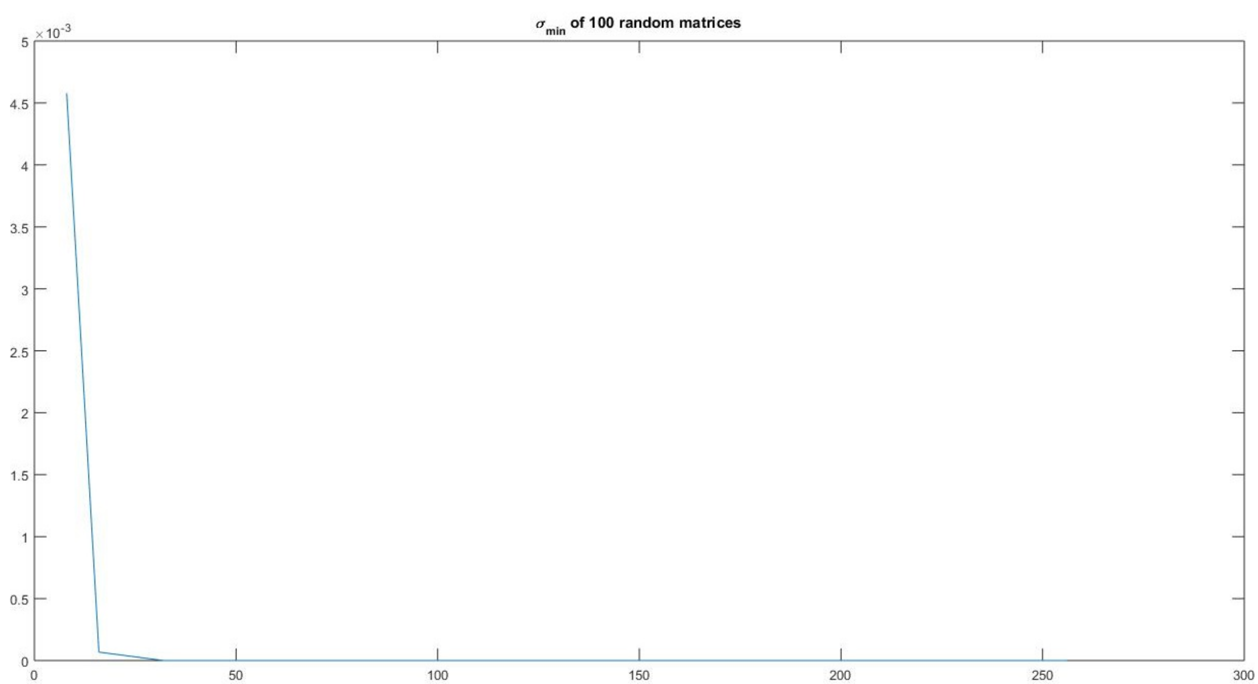

Figure 11: Result of average σ_{min} of 100 random matrices

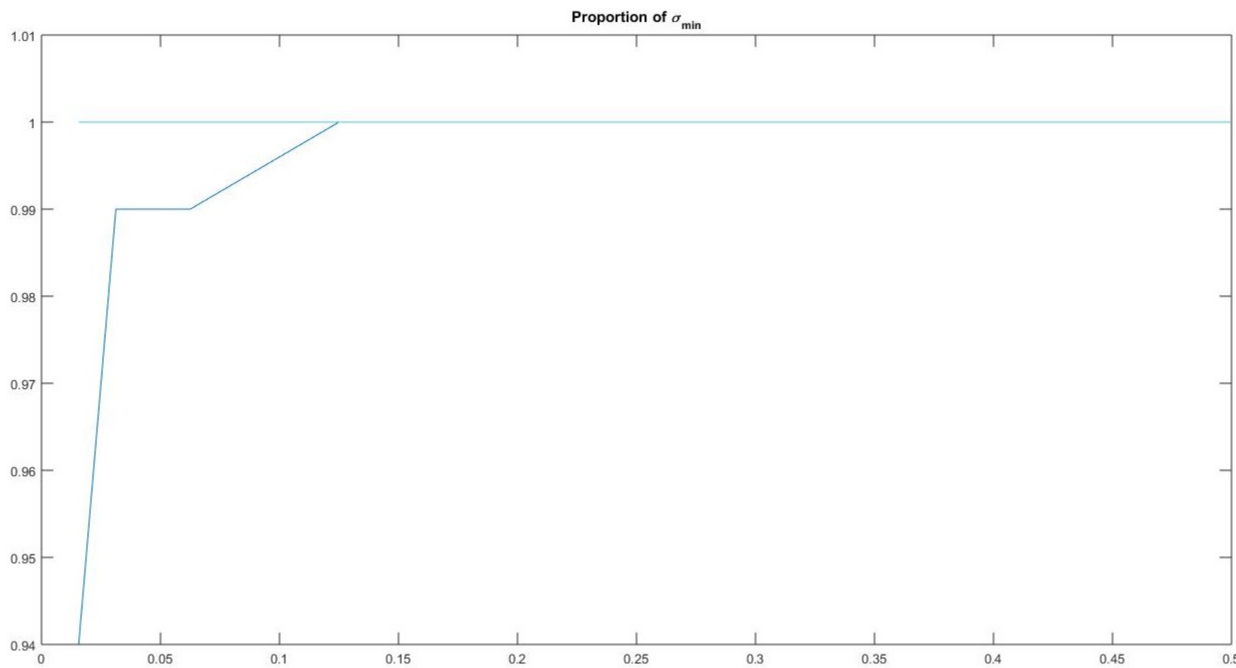

Figure 12: Result of average proportion of σ_{min}