# CSE6643 Numerical Linear Algebra HW3

November 21, 2016

## 1 Exercise 21.6

Suppose $A \in C^{m\times m}$ is strictly column diagonally dominant, which means that for each $k$,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|.$$

Show that if Gaussian elimination with partial pivoting is applied to $A$, no row interchanges take place.

**Proof:** Since in practice, it is common to pick as pivot the largest number among a set of entries being considered as candidates, which can improve stability. Therefore, the pivot at each step is chosen as the largest of the $m-k+1$ subdiagonal entries in column $k$.

In step one, since $|a_{11}| > \sum_{j\neq k} |a_{j1}|$, then no row interchanges take place. Assume after the firs step Gaussian elimination, the matrix is $A^{(1)}$. According to implementation of pure form Gaussian elimination, the rest of steps, we wish to subtract $l_{jk}$ times row $k$ from row $j$, where $l_{jk}$ is the multiplier:

$$l_{jk} = \frac{x_{jk}}{x_{kk}} \quad (k < j \leq m)$$

Let $A^{(1)}$ be:

$$A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22}^{(1)} & ... & a_{2n}^{(1)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & ... & a_{nn}^{(1)} \end{pmatrix} = T$$

To prove there is no row interchanges, we just need to prove $A^{(1)}$ is strictly column diagonally dominant, then in step two we just need to prove $|a_{kk}^{(1)}| > \sum_{j\geq 3, j\neq k}^{n} |a_{jk}^{(1)}|$

$$\begin{aligned}
\sum_{i\neq k}^{n} |a_{ik}^{(1)}| = \sum_{i\neq k}^{n} |a_{ik} - \frac{a_{i1}}{a_{11}} a_{1k}| &\leq \sum_{i\neq k}^{n} |a_{ik}| + \frac{|a_{1k}|}{|a_{11}|} \sum_{i\neq k}^{n} |a_{i1}| \\
&< |a_{kk}| - |a_{1k}| + \frac{|a_{1k}|}{|a_{11}|}(|a_{11}| - |a_{k1}|) \\
&< |a_{kk}| - \frac{|a_{1k}|}{|a_{11}|}|a_{k1}| < |a_{kk} - \frac{a_{1k}}{a_{11}} a_{k1}| = |a_{kk}^{(1)}|
\end{aligned}$$

## 2 Exercise 22.1

Show that for Gaussian elimination with partial pivoting applied to any matrix $A \in C^{m\times m}$, the growth factor(22.2) satisfies $\rho \le 2^{m-1}$.

**Proof:** For Gaussian elimination with partial pivoting, it will keep the maximum entry of each column as pivot, such that $max_{ij}|a_{ij}|$ will remain the same.

Assume $A^{(i)}(1 \le i \le m)$ denotes the matrix after $i$th step of Gaussian elimination. In each step, we do the partial pivoting, which requires to find the maximum entry in a column and then do the Gaussian elimination again. To do this we wish to subtract $l_{jk}$ times row $k$ from row $j$, where $l_{jk}$ is the multiplier:

$$l_{jk} = \frac{a_{jk}}{a_{kk}}(k < j \le m)$$

and $\frac{|a_{jk}|}{|a_{kk}|} \le 1$ since $a_{kk}$ is the maximum.

For the first step:

$$|a_{ij}^{(2)}| = |a_{i1}^{(1)} - \frac{a_{i1}^{(1)}}{a_{11}^{(1)}}a_{1j}^{(1)}| \le |a_{i1}^{(1)}| + |\frac{a_{i1}^{(1)}}{a_{11}^{(1)}}a_{1j}^{(1)}| < |a_{i1}^{(1)}| + |a_{1j}^{(1)}| < 2\max_{i,j}|a_{ij}^{(0)}| = 2\max_{i,j}|a_{ij}|$$

To form $U$ we need to do $m$ step of implementation, therefore:

$$\begin{aligned} |a_{ij}^{(2)}| &\le 2\max_{i,j}|a_{ij}^{(1)}| = 2\max_{i,j}|a_{ij}| \\ |a_{ij}^{(3)}| &\le 2\max_{i,j}|a_{ij}^{(2)}| = 2^2\max_{i,j}|a_{ij}| \\ \vdots \quad & \quad \vdots \quad \quad \vdots \quad \quad \vdots \\ |u_{i,j}| = |a_{ij}^{(m)}| &\le 2\max_{i,j}|a_{ij}^{(m-1)}| = 2^{m-1}\max_{i,j}|a_{ij}| \end{aligned}$$

Therefore

$$\rho = \frac{\max_{i,j}|u_{ij}|}{\max_{i,j}|a_{ij}|} \le 2^{m-1}$$

## 3

Write your own code for the LU decomposition with partial pivoting to implement the Algorithm 21.1([TB, p.160]). Generate random matrices $A$ of size $m = 32, 64, 128, \cdots$, either with uniform distribution over [-1,1] or with normal distribution with mean 0 and standard deviation $\sqrt{m}$.

Compare the growth factor of $A$, the size of entries of $L_{ij}^{-1}$, and computation time for carrying out the LU decomposition for different sizes of $A$. (You should run your program for at least 100 matrices of each size.) You are encouraged to consult Trefethen, pp. 167-170, and provide similar plots to support your claims.

**Answer:** In this question, my algorithm generated random matrices $A$ of size $m = 32, 64, 128, 256, 512$ with normal distribution with mean 0 and standard deviation $\sqrt{m}$. The following graphs are the results. Along with the increasing of size of matrix $A$, the growth factor grows gradually and the size of entries of $L_{ij}^{-1}$ and computation time grow faster than the the growth of growth factor. Since $\rho$ is large, then $L^{-1}$ is large too. The stability of Gaussian elimination with partial pivoting is highly unstable for certain matrices $A$.

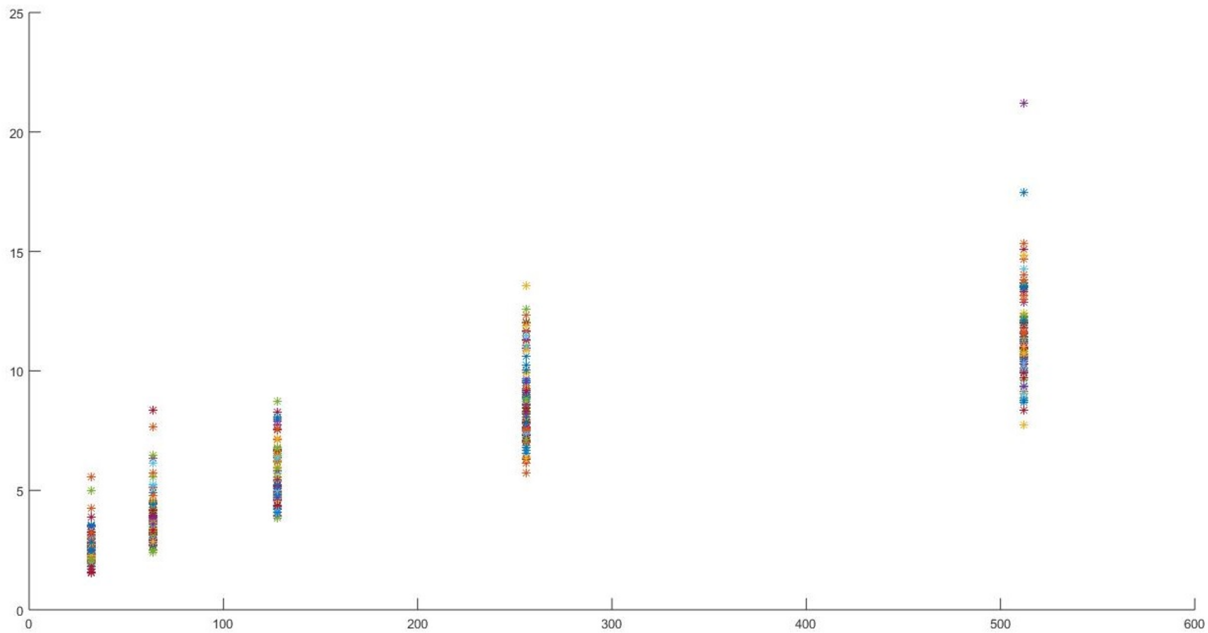


Figure 1: Result of growth of factor $A$

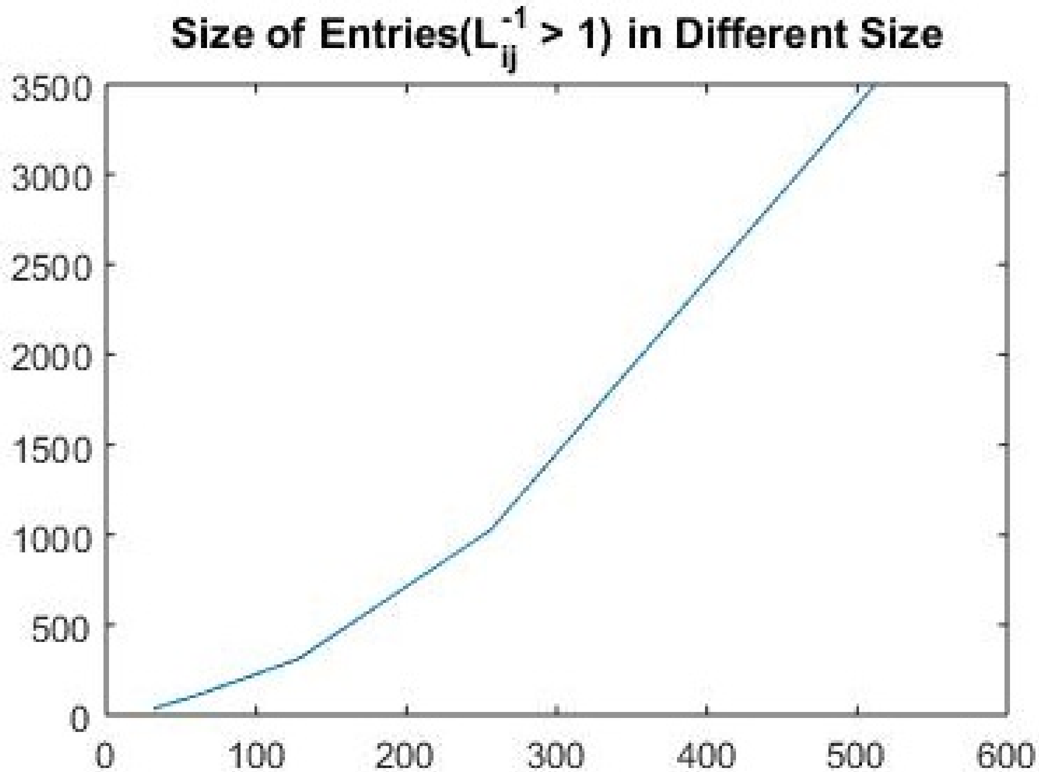


Figure 2: Result of size of entries of $L_{ij}^{-1}$

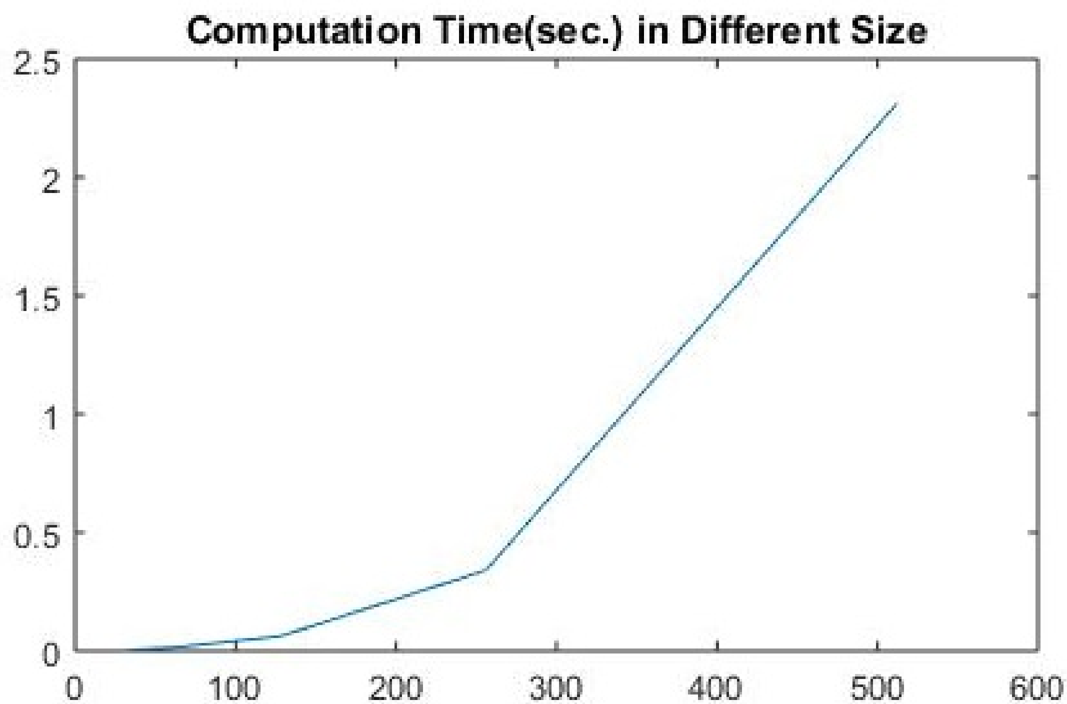


Figure 3: Result of computation time(*sec.*)