

CSE6643 Numerical Linear Algebra HW3

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October 13, 2016

1 Exercise 21.6

Suppose $A \in \mathbb{C}^{m \times m}$ is strictly column diagonally dominant, which means that for each k ,

$$|a_{kk}| > \sum_{j \neq k} |a_{jk}|.$$

Show that if Gaussian elimination with partial pivoting is applied to A , no row interchanges take place.

Proof: Since in practice, it is common to pick as pivot the largest number among a set of entries being considered as candidates, which can improve stability. Therefore, the pivot at each step is chosen as the largest of the $m - k + 1$ subdiagonal entries in column k .

In step one, since $|a_{11}| > \sum_{j \neq 1} |a_{j1}|$, then no row interchanges take place. Assume after the first step Gaussian elimination, the matrix is $A^{(1)}$. According to implementation of pure form Gaussian elimination, the rest of steps, we wish to subtract l_{jk} times row k from row j , where l_{jk} is the multiplier:

$$l_{jk} = \frac{x_{jk}}{x_{kk}} \quad (k < j \leq m)$$

Let $A^{(1)}$ be:

$$A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{pmatrix} = T$$

To prove there is no row interchanges, we just need to prove $A^{(1)}$ is strictly column diagonally dominant, then in step two we just need to prove $|a_{kk}^{(1)}| > \sum_{j \geq 3, j \neq k} |a_{jk}^{(1)}|$

$$\begin{aligned} \sum_{i \neq k} |a_{ik}^{(1)}| &= \sum_{i \neq k} |a_{ik} - \frac{a_{i1}}{a_{11}} a_{1k}| \leq \sum_{i \neq k} |a_{ik}| + \frac{|a_{1k}|}{|a_{11}|} \sum_{i \neq k} |a_{i1}| \\ &< |a_{kk}| - |a_{1k}| + \frac{|a_{1k}|}{|a_{11}|} (|a_{11}| - |a_{k1}|) \\ &< |a_{kk}| - \frac{|a_{1k}|}{|a_{11}|} |a_{k1}| < |a_{kk} - \frac{a_{1k}}{a_{11}} a_{k1}| = |a_{kk}^{(1)}| \end{aligned}$$

2 Exercise 22.1

Show that for Gaussian elimination with partial pivoting applied to any matrix $A \in C^{m \times m}$, the growth factor(22.2) satisfies $\rho \leq 2^{m-1}$.

Proof: For Gaussian elimination with partial pivoting, it will keep the maximum entry of each column as pivot, such that $\max_{ij} |a_{ij}|$ will remain the same.

Assume $A^{(i)} (1 \leq i \leq m)$ denotes the matrix after i th step of Gaussian elimination. In each step, we do the partial pivoting, which requires to find the maximum entry in a column and then do the Gaussian elimination again. To do this we wish to subtract l_{jk} times row k from row j , where l_{jk} is the multiplier:

$$l_{jk} = \frac{a_{jk}}{a_{kk}} (k < j \leq m)$$

and $\frac{|a_{jk}|}{|a_{kk}|} \leq 1$ since a_{kk} is the maximum.

For the first step:

$$|a_{ij}^{(2)}| = |a_{i1}^{(1)} - \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} a_{1j}^{(1)}| \leq |a_{i1}^{(1)}| + \frac{|a_{i1}^{(1)}|}{|a_{11}^{(1)}|} |a_{1j}^{(1)}| < |a_{i1}^{(1)}| + |a_{1j}^{(1)}| < 2 \max_{i,j} |a_{ij}^{(0)}| = 2 \max_{i,j} |a_{ij}|$$

To form U we need to do m step of implementation, therefore:

$$\begin{aligned} |a_{ij}^{(2)}| &\leq 2 \max_{i,j} |a_{ij}^{(1)}| = 2 \max_{i,j} |a_{ij}| \\ |a_{ij}^{(3)}| &\leq 2 \max_{i,j} |a_{ij}^{(2)}| = 2^2 \max_{i,j} |a_{ij}| \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ |u_{i,j}| = |a_{ij}^{(m)}| &\leq 2 \max_{i,j} |a_{ij}^{(m-1)}| = 2^{m-1} \max_{i,j} |a_{ij}| \end{aligned}$$

Therefore

$$\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|} \leq 2^{m-1}$$

3

Write your own code for the LU decomposition with partial pivoting to implement the Algorithm 21.1([TB, p.160]). Generate random matrices A of size $m = 32, 64, 128, \dots$, either with uniform distribution over $[-1,1]$ or with normal distribution with mean 0 and standard deviation \sqrt{m} .

Compare the growth factor of A , the size of entries of L_{ij}^{-1} , and computation time for carrying out the LU decomposition for different sizes of A . (You should run your program for at least 100 matrices of each size.) You are encouraged to consult Trefethen, pp. 167-170, and provide similar plots to support your claims.

Answer: In this question, my algorithm generated random matrices A of size $m = 32, 64, 128, 256, 512$ with normal distribution with mean 0 and standard deviation \sqrt{m} . The following graphs are the results. Along with the increasing of size of matrix A , the growth factor grows gradually and the size of entries of L_{ij}^{-1} and computation time grow faster than the the growth of growth factor. Since ρ is large, then L^{-1} is large too. The stability of Gaussian elimination with partial pivoting is highly unstable for certain matrices A .

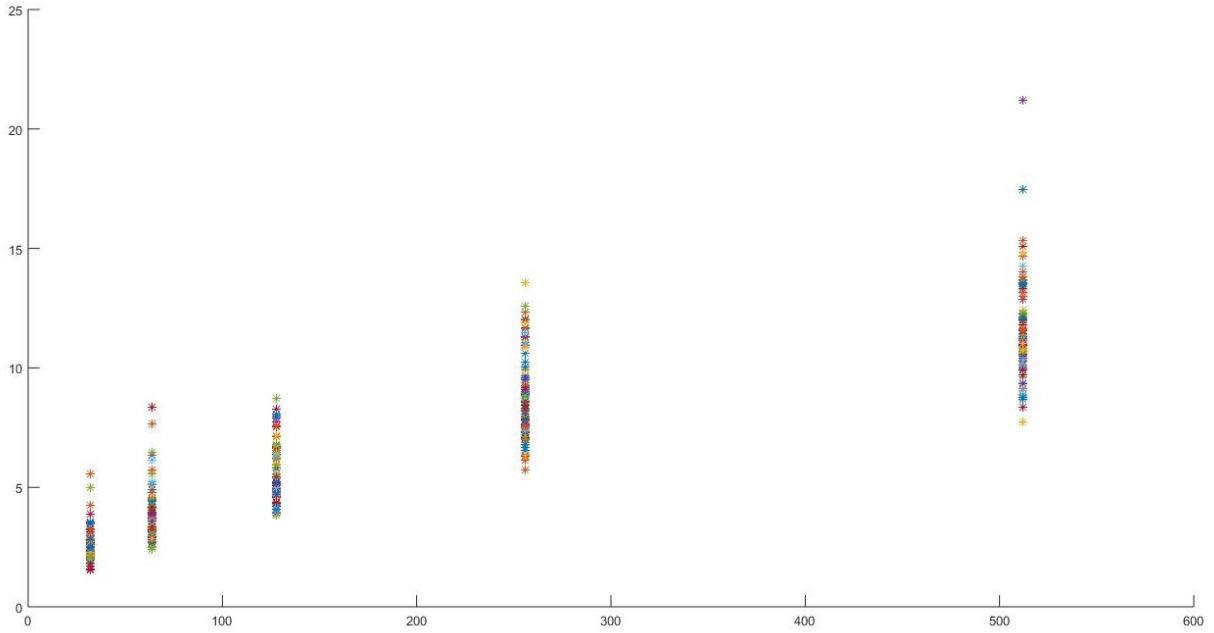


Figure 1: Result of growth of factor A

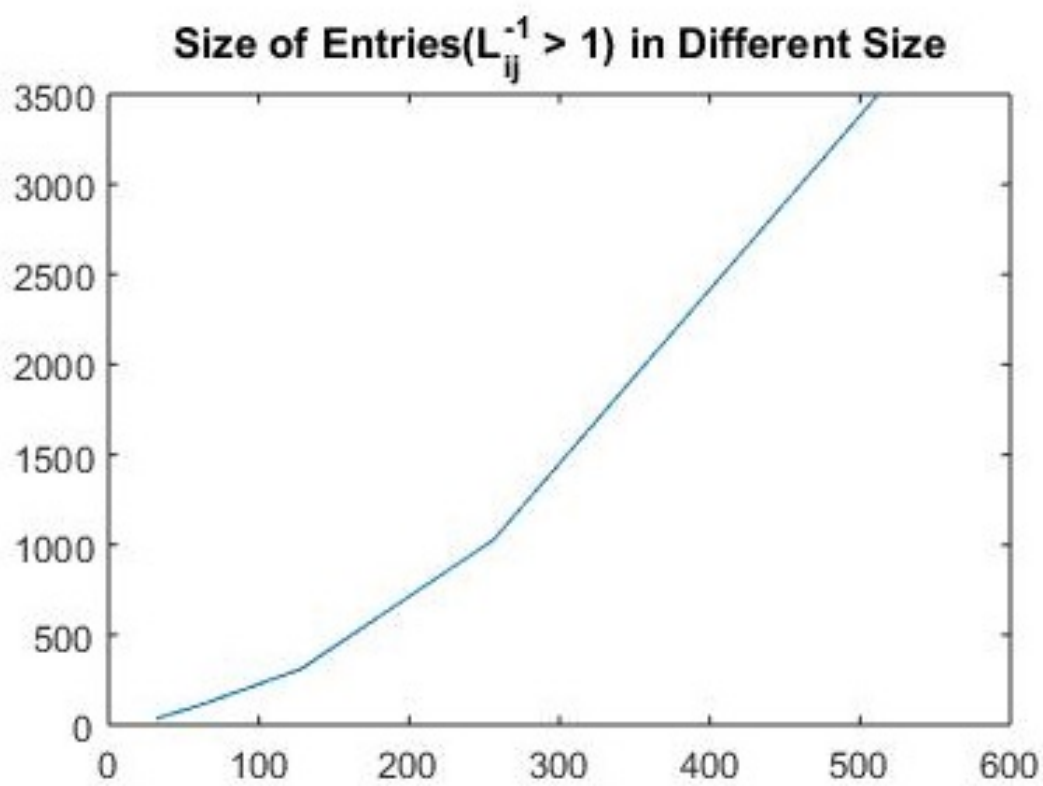


Figure 2: Result of size of entries of L_{ij}^{-1}

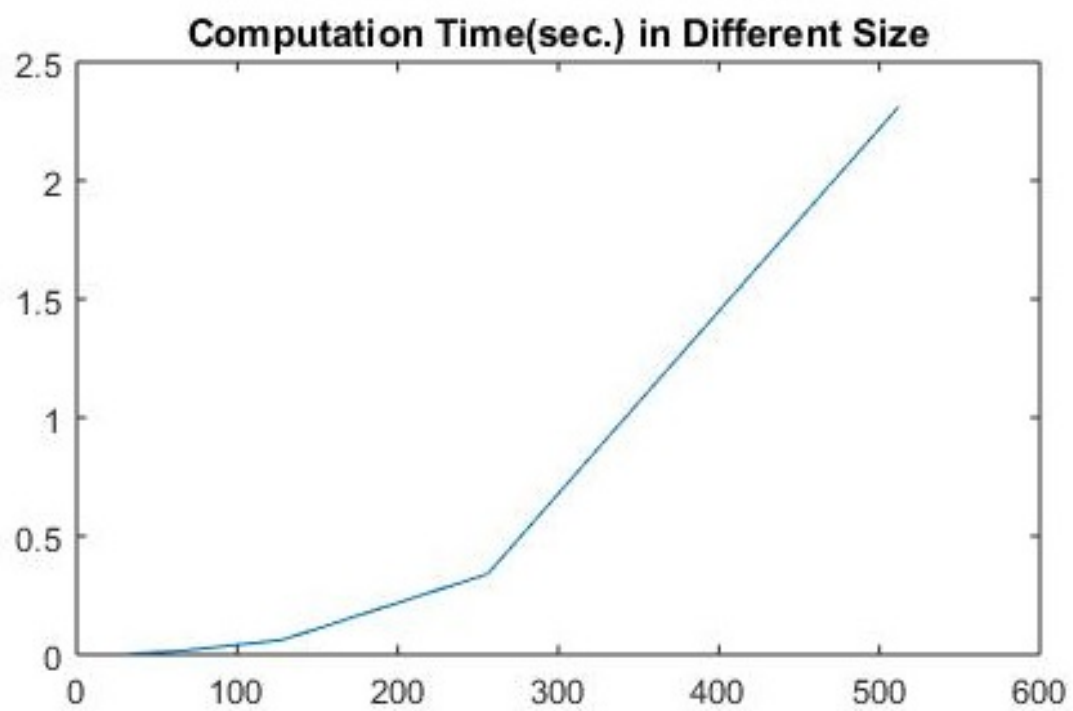


Figure 3: Result of computation time(*sec.*)