CSE6643 Numerical Linear Algebra HW5

November 28, 2016

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Let A be a real $m \times n$ matrix with $m \geq n$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let B

$$B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

- (1) Show that A, A^T, and $W_1 A W_2$ have the same set singular values, where W_1 and W_2 are any $m \times m$ and $n \times n$ orthogonal matrices respectively.
- (2) Show that eigenvalues of B are exactly $\sigma_1, \sigma_2, \cdots, \sigma_n, -\sigma_n, \cdots, -\sigma_2, -\sigma_1$ and $m-n$ of additional 0's.

(1)Solution:

Since A can be decomposed as $A = U\Sigma V^T$, U is $m \times m$ unitary matrix, V is $n \times n$ unitary matrix and Σ is $m \times n$ diagonal matrix. Then for matrix A^T, we can obtain:

$$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$$

Thus the singular values of A^T are the diagonal entries in Σ^T. The singular values are the same set of A.

For matrix $W_1 A W_2$, since W_1 is orthogonal matrix, then we have $W_1^T W_1 = 1$, $W_2^T W_2 = I$, similarly we have $U^T U = I$, $V^T V = I$. Then

$$(W_1 U)^T (W_1 U) = U^T (W_1^T W_1) U = U^T U = I$$

$$(V^T W_2)^T (V^T W_2) = W_2^T (V V^T) W_2 = W_2^T W_2 = I$$

Thus, matrix $W_1 U$ and $V^T W_2$ are orthogonal matrix, Σ is still diagonal matrix. Thus:

$$W_1 A W_2 = W_1 (U\Sigma V^T) W_2 = (W_1 U)\Sigma(V^T W_2)$$

Therefore, the singular values of matrix $W_1 A W_2$ are the diagonal entries in Σ which are the same set in matrix A.

(2)Solution:

For a $m \times n (m \geq n)$ matrix A, according to the full SVD, we can have $AV = U_0 \Sigma_0$. Also, we can rewrite the equations like:

$$\begin{aligned} AV &= U_0\Sigma_0 \\ \Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} V &= \begin{bmatrix} U & 0 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \\ \Rightarrow A_1V = U\Sigma, &\quad A_2V = 0 \end{aligned}$$

Also, we have $A^T(U_0^T)^{-1} = V\Sigma_0^T \rightarrow A^TU_0^T = V\Sigma_0$, which can be represented by:

$$\begin{aligned} \begin{bmatrix} A_1^T & A_2^T \end{bmatrix} \begin{bmatrix} U \\ 0 \end{bmatrix} &= V \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \\ \Rightarrow A_1^TU &= V\Sigma \end{aligned}$$

Consider the $(m+n)\times(m+n)$ matrix B, we have the block $(m+n)\times(m+n)$ equations which amounts to an eigenvalue decomposition of B.

$$\begin{bmatrix} 0 & 0 & A_1 \\ 0 & 0 & A_2 \\ A_1^T & A_2^T & 0 \end{bmatrix} \begin{bmatrix} U & 0 & -U \\ 0 & 0 & 0 \\ V & 0 & V \end{bmatrix} = \begin{bmatrix} U & 0 & -U \\ 0 & 0 & 0 \\ V & 0 & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Sigma \end{bmatrix}$$

In here, A_1 is a $n\times n$ matrix, and A_2 is a $(m-n)\times n$ matrix, they are submatrix of matrix A.

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Similarly, A_1^T is a $n\times n$ matrix, and A_2^T is a $n\times(m-n)$ matrix, they are submatrix of matrix A^T.

$$A^T = \begin{bmatrix} A_1^T & A_2^T \end{bmatrix}$$

Also, matrix U,V and Σ are $n\times n$ matrices.
To verify the equality of the equation,

$$P = \begin{bmatrix} 0 & 0 & A_1 \\ 0 & 0 & A_2 \\ A_1^T & A_2^T & 0 \end{bmatrix} \begin{bmatrix} U & 0 & -U \\ 0 & 0 & 0 \\ V & 0 & V \end{bmatrix} = \begin{bmatrix} A_1V & 0 & A_1V \\ A_2V & 0 & A_2V \\ A_1^TU & 0 & -A_1^TU \end{bmatrix}$$

$$Q = \begin{bmatrix} U & 0 & -U \\ 0 & 0 & 0 \\ V & 0 & V \end{bmatrix} \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Sigma \end{bmatrix} = \begin{bmatrix} U\Sigma & 0 & U\Sigma \\ 0 & 0 & 0 \\ V\Sigma & 0 & -V\Sigma \end{bmatrix}$$

$$P - Q = \begin{bmatrix} A_1V - U\Sigma & 0 & A_1V - U\Sigma \\ A_2V - 0 & 0 & A_2V - 0 \\ A_1^TU - V\Sigma & 0 & V\Sigma - A_1^TU \end{bmatrix}$$

Since

$$A_1V = U\Sigma, \quad A_2V = 0, \quad A_1^TU = V\Sigma$$

Then

$$P - Q = 0$$

Thus the matrix block equation is valid. Then we can tell the eigenvalues of B are in the matrix below:

$$\begin{bmatrix} \Sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Sigma \end{bmatrix}$$

which are exactly $\sigma_1, \sigma_2, \cdots, \sigma_n, -\sigma_n, \cdots, -\sigma_2, -\sigma_1$ and $m-n$ of additional 0's.

2

Let A be a $m \times n$ matrix with rank(A) $= k$, $m \geq n$. Let $A = U\Sigma V^T$ be the singular value decomposition of A, with U and V are $m \times m$ and $n \times n$ orthogonal matrices respectively.

- (1) Show that the last $n - k$ columns of V form an orthonormal basis for the null space of A.
- (2) Let B be a $p \times n$ matrix. Derive a simple procedure (by using MATLAB command) to find an orthonormal basis for the intersection of null space of A and B. (Hint: use part(1).)

(1)Solution:

Singular value decomposition of matrix A is:

$$A = U\Sigma V^T \Rightarrow U^T A = \Sigma V^T \Rightarrow U^T A V = \Sigma$$

Since rank(A)=k, then the number of nonzero singular values of matrix A is k according to matrix property via the SVD (Theorem 5.1 Page 33). And the singular values are numbered in descending order, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$ Then we can obtain the following:

$$\begin{bmatrix} & & \\ & U^T A & \\ & & \end{bmatrix} V = \Sigma$$

$$\Rightarrow \begin{bmatrix} & & \\ & U^T A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & & | & & | \\ v_1 & v_2 & \cdots & v_k & \cdots & v_n \\ | & | & & | & & | \end{bmatrix} = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_k & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

According to the above deduction, we can obtain:

$$\sigma_i = U^T A v_i \quad (i = 1, 2, \cdots n)$$

Thus for the last $n - k$ columns of V, we have:

$$\begin{aligned} & \sigma_i = U^T A v_i = 0 \\ \Rightarrow \quad & U\sigma_i = A v_i = 0 \quad (i = k+1, k+2, \cdots n) \end{aligned}$$

Therefore, the last $n - k$ columns of V form an orthonormal basis for the null space of A, null(A)=$< v_{k+1}, v_{k+2}, \cdots, v_n >$.

(2)Solution:

Suppose V is the null space of A and W is the null space of B. then $V \cap W$ is the smaller null space of the larger matrix C:

$$C = \begin{bmatrix} A \\ B \end{bmatrix}$$

$Cx = 0$ requires both $Ax = 0$ and $Bx = 0$, so x has to be in both null spaces.

Then, we can find an orthonormal basis for the intersection of null space of A and B by forming an orthonormal basis for the null space of C. Besides, use the result from part 1, the last $n-k$ columns of V form the orthonormal basis. So here is the simple procedure:

- Combine matrix A and B to form $C = [A; B]$;
- Calculate the rank of matrix C: $k = rank(C)$;
- Implement singular value decomposition on matrix C: $[U, S, V] = svd(C)$;
- Choose the last $n-k$ columns of V to form an orthonormal basis for the null space of C.

MATLAB Code:

```
function [X]=p_2(A,B)
 % For example:
 % A=[1 2 3 4; 5 6 7 8; 2 4 6 8];
 % B=[7 8 5 2; 3 6 9 12];
 C=[A;B];
 [m, n]=size(C);
 k=rank(C);
 [U,S,V]=svd(C);
 X=V(:,k+1:n);
end
```

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Discretize

$$\begin{cases} -u'' + \alpha u = \sin x, & x \in (0,1) \\ u(0) = 0, u(1) = 1, \end{cases} \tag{1}$$

using centered difference scheme. Write your own code to solve this equation by using Jacobi, Gauss-Seidel, SOR(with $\omega = 0.5, 1.1, 1.7$). Vary $\alpha = 1, 10, 100$. Using $n = 100$ and $n = 1000$ (mesh points). Compare the convergence of the methods by plotting the number of iterations required. Discuss what you observe. (Hint: be careful of what you choose to stop the iteration.)

Solution:

According to the central difference approximation of the second derivative, we can obtain:

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Assume we divided space $(0,1)$ into n parts by n evenly finite difference, where $h = \frac{1}{n}$ is the fixed evenly finite difference between 0 and 1.

Substitute u'' into the formula, we can obtain:

$$-\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + \alpha u(x) = \sin x$$
$$\Rightarrow -u(x-h) + (\alpha h^2 + 2)u(x) - u(x+h) = h^2 sin(x)$$

Thus we can represent this formula by matrix notation.

$$A = \begin{bmatrix} \alpha h^2+2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & \alpha h^2+2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & \alpha h^2+2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & -1 & \alpha h^2+2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & \alpha h^2+2 \end{bmatrix}$$
$$u = \begin{bmatrix} u(h) \\ u(2h) \\ u(3h) \\ \vdots \\ u((n-2)h) \\ u((n-1)h) \end{bmatrix} \quad b = \begin{bmatrix} h^2 sin(h) \\ h^2 sin(2h) \\ h^2 sin(3h) \\ \vdots \\ h^2 sin((n-2)h) \\ h^2 sin((n-1)h) + 1 \end{bmatrix}$$

Then we can obtain:

$$Au = b$$

A is a $(n-1) \times (n-1)$ matrix, u and b is an $n \times 1$ vector. Then we can solve this equation use Jacobi, Gauss-Seidel and SOR.

Discussion:

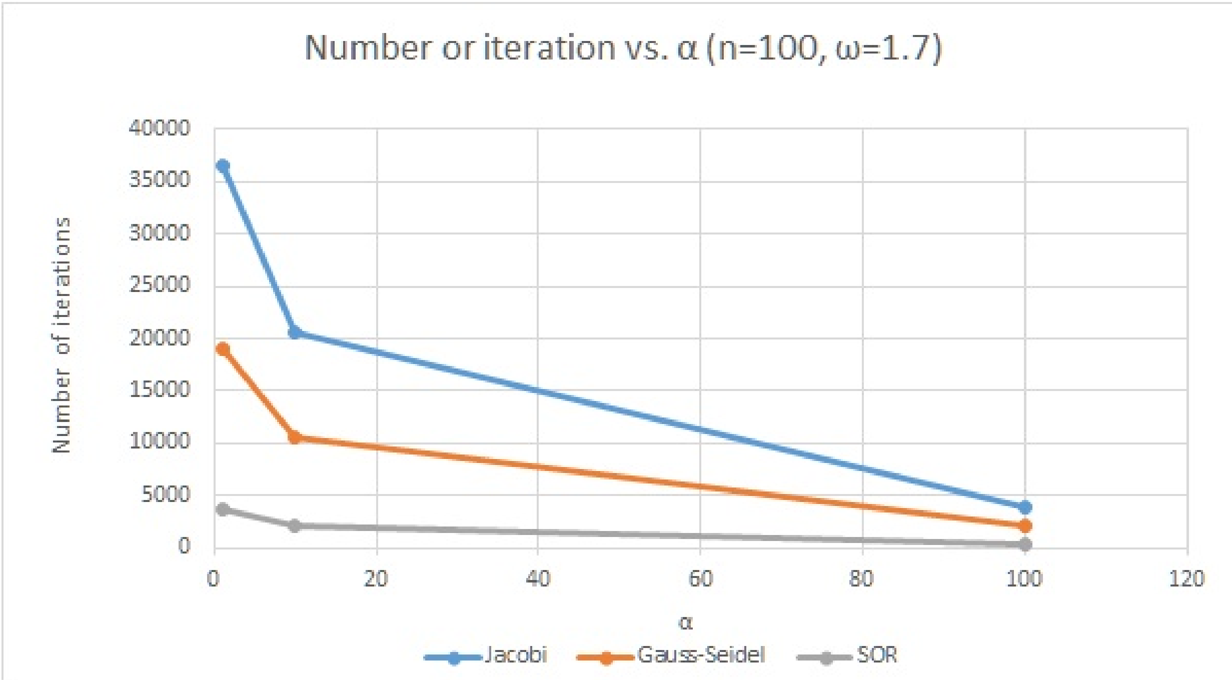

Figure 1: Number of iteration vs. α $(n = 100, \omega = 1.7)$

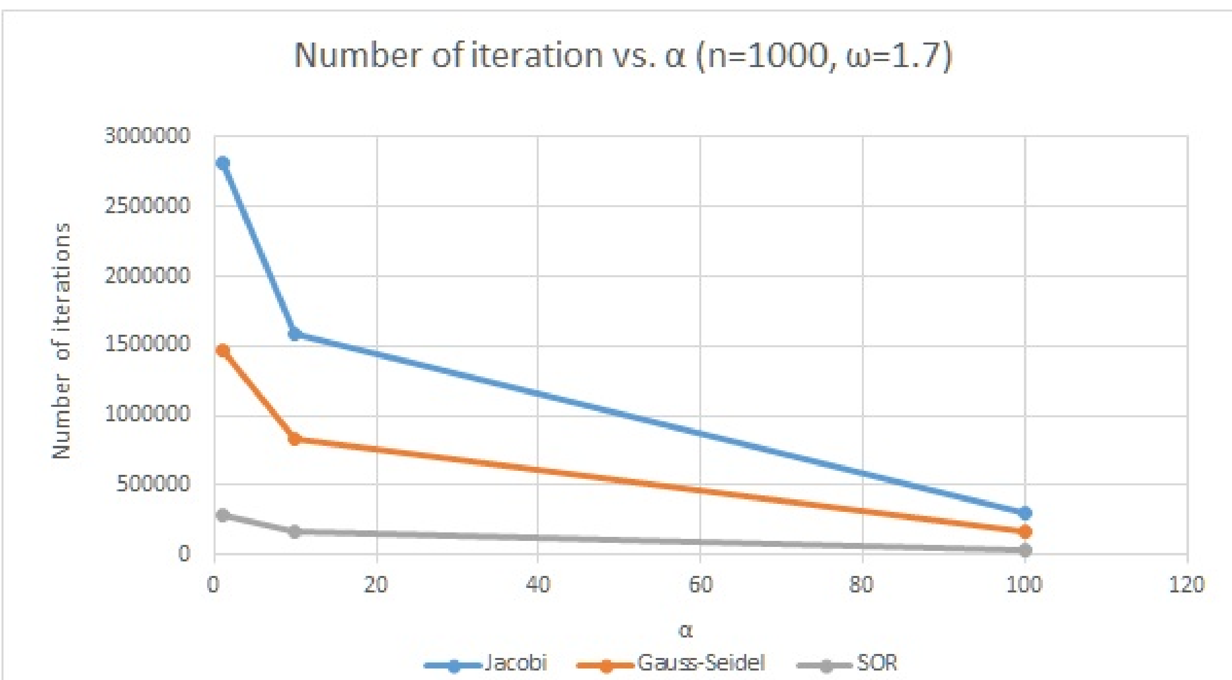

Figure 2: Number of iteration vs. α $(n = 1000, \omega = 1.7)$

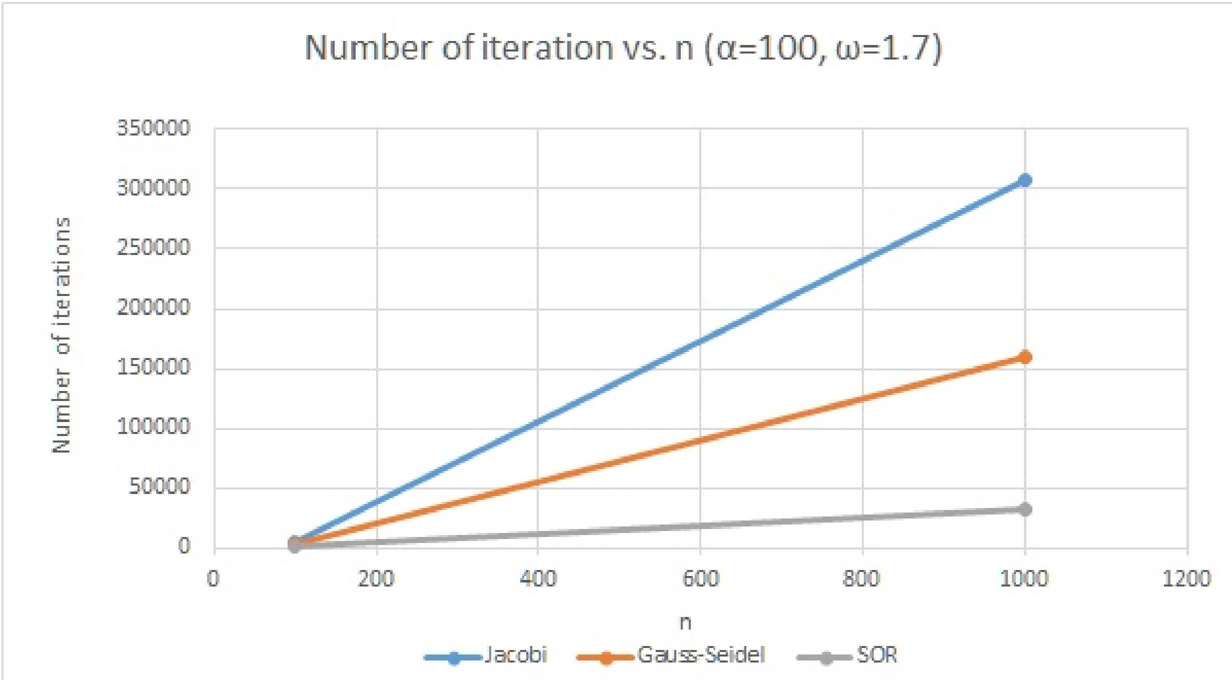

Figure 3: Number of iteration vs. n ($\alpha = 100, \omega = 1.7$)

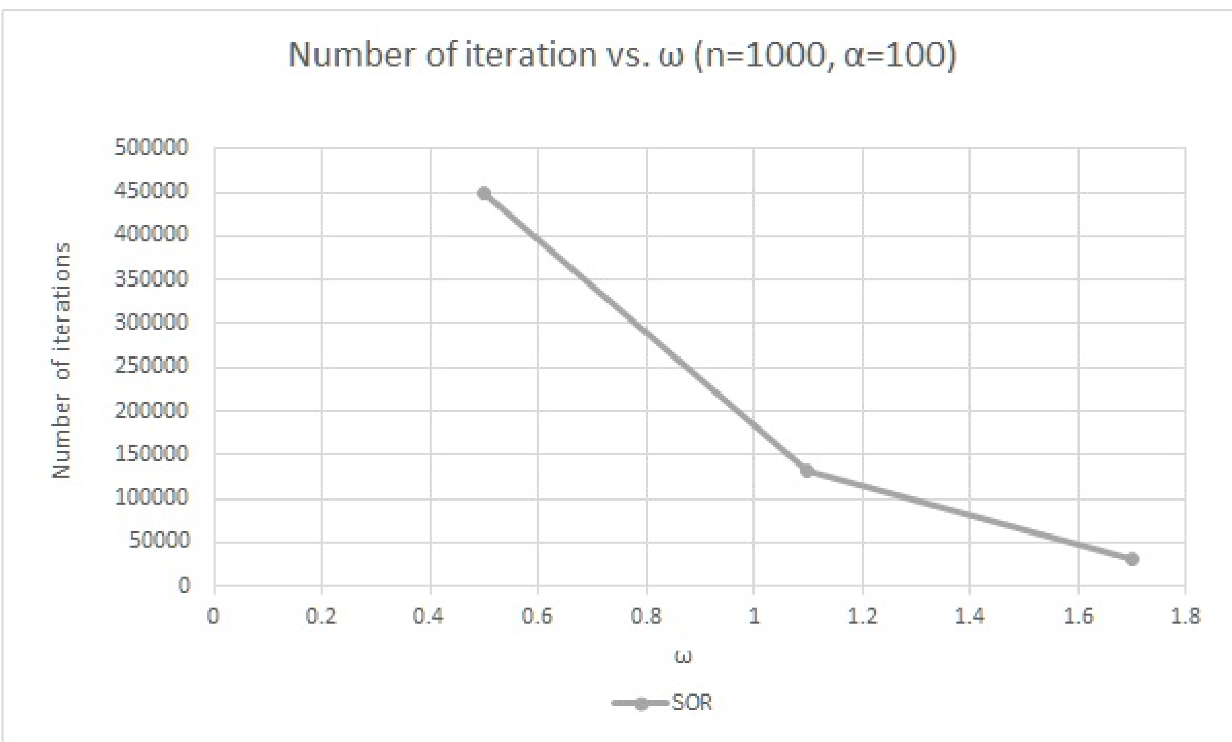

Figure 4: Number of iteration vs. ω ($n = 1000, \alpha = 100$)

- According to the figure 1 and 2, we can easily tell that in both cases, the number of iterations of all three methods are decreasing along with the increasing of α. If we take a look at our matrix A, diagonal entries will be larger with larger α, so that it can help converge more quickly. (Diagonal dominant)

- However, with larger matrix size n, it can increase the computational scale. Also, because the diagonal entries include a h^2, then it will cause the diagonal entries smaller with larger matrix size n. ($h = \frac{1}{n}$). Therefore, we can see in figure 3, when matrix size increase 10 times from $n = 100$ to $n = 1000$, the number of iterations of three methods increase $77 - 79$ times.

- According to figure 4, by applying different ω value to successive over relaxation, a higher ω value can lead to a smaller number of iterations. SOR takes the form of a weighted average between the previous iterate and the computed Gauss-Seidel iterate successively for each component. If $\omega = 1$, the SOR method simplifies to the Gauss-Seidel method. However, by doing some research, I figured out that ω might fails to converge if it is outside the interval $(0, 2)$. To find the best ω, we need to do some heuristic estimate.