Astrophysical Dynamics

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*Partially derived from *Galactic* Dynamics by J. Binney and S. Tremaine and notes by M. Weinberg

Contents

1 Introduction

Relation to 507, 508, 509.

Last in series of courses that are not astrophysics class *per se*. They provide some physics foundations for astrophysics. In this case, the course gives a basic introduction to dynamics as a physical discipline because it is important in astrophysical situations.

Examples include:

- Celestial mechanics
- Evolution of planetary orbits
- Planet formation
- Open clusters
- Globular clusters
- Galactic structure
- Galaxy formation
- Galaxy clusters
- Large Scale structure of the Universe

I don't think that is a subtle fact that most of our knowledge of astronomy relies on an understanding of gravitational dynamics. Consider: measuring the mass of the Earth using a calculator:

$$a = \frac{2d}{t^2} = \frac{GM_\oplus}{R_\oplus^2}$$

Note application to cosmology: dark matter, dark energy etc, are all based on dynamical arguments.

Note on personal interest: most of my research is on dynamics via N-body simulations, so there will be more of a N-body slant here than if someone else teaches the class. However, a fundamental understanding of dynamics is necessary to properly interpret the complex simulations.

I have a dream speech?

Hence, focus on analytical results and fundamental principles here.

Note contrast with hydro: dynamics is somewhat unique to astronomy. Very few situation where there is dissipationless, collisionless motion as we have in astronomy.

In fact let me recall how I introduced hydro.

If we introduce $f(\vec{x}, \vec{v}, t)$, which is the probability density of finding one particle at a position and velocity after averaging over all the motions of the other particles. Conservation of probability, assuming a large number of particles in an isolated system, and much algebraic manipulation gives the following: (see Binney & Tremaine, ch. 8 (ch. 7.2 new edition)

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{v}} = \int \mathrm{d}\vec{X}_2 \frac{\partial \phi(\vec{x}_1, \vec{x}_2)}{\partial \vec{x}_2} \cdot \frac{\partial E_2(\vec{X}_1, \vec{X}_2)}{\partial \vec{v}_1},$$

where E_2 is the excess probability of finding particle 1 at (6D coordinate) $\vec{X}_1$ and particle 2 at $\vec{X}_2$.

In hydro we assumed that the collision term dominated. Mass, momentum and energy conservation in collisions gave us the form that f should take. The collision term turned into a pressure force, and we treated the $\nabla\Phi$ term very cavalierly. Here the opposite is true: a star has to travel $10^7 Mpc$ before it collides with another star, and understanding the trajectories of individual bodies in the gravitational potential is crucial to understanding the dynamics.

Relation to other branches of physics:

1) Celestial mechanics: small number of bodies

2) Be careful when making analogies to hydro: I will make them for intuition, but only after having the correct dynamical result.

3) stat. mech can also be confusing particularly since there is no maximum entropy state for a gravitating system.

Book recommendations.

Prereqs: I'll assume that people have had classical mechanics at the upper undergraduate level, but I'll review it a little.

Course outline:

The outline of the course is based around answering a number of questions about the motion of objects mutually interacting with each other via gravitation.

1. Given a gravitational potential, how does a star (or planet) move. Of course, if we are talking about a galaxy or globular cluster, we are

rarely interested in a single orbit, we are interested in the statistics, the distributions in space and velocity. However, we shall see that these distributions sometimes need to be specified in terms of properties of the individual orbits. Of course, N-body simulations are all about doing orbits, and you aren't going to do the system correctly if you can't correctly calculate individual orbits.

$$\Phi(x) \Rightarrow x(t)$$

2. The second question is in the opposite direction: given a distribution of stellar positions and velocities: what is the gravitational field.

$$\rho(x) \Rightarrow \Phi(x)$$

3. Next we have to put these together: how do we put together a distribution of stars that generate a potential, which will make the stars move in such a way that their positions continue to generate the potential in which they move? That is, a mutually consistent model of a steady state stellar system. This will be the crux of the course.

$$x(t) \Rightarrow \rho(x) \Rightarrow \Phi(x) \Rightarrow x(t)$$

4. Finally, how to these systems evolve. Both close to equilibrium (stability) and from equilibrium to equilibrium.

To start we will have to make an assumption: that the orbits of stars in a system depend on the gravitational potential as a whole, and not on the interaction with individual stars. We will see in a while that this is indeed the case for typical stellar systems.

2 Orbits

We might say "We know Newton's laws; let's code them up and let 'er rip!". Two problems: one I mentioned before, one gets very little understanding with this approach. Two: it's still impossible, and requires lots of approximations.

In order to make progress, we need to understand the underlying structure. An example of using brains rather than brawn that you should be familiar with is the symmetries that lead to conservation laws. Emmy Nöther provided the mathematical foundation for this idea, but it should be intuitive.

Consider Newton's laws:

$$m\ddot{x} = f(x)$$

What symmetry of this leads to energy conservation? (Add g(t) if they have trouble.)

Where does linear momentum conservation come from?

Let's consider what we can learn from Energy conservation. For a one dimensional motion we have

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mv^2$$

and

$$\Phi(x) = -\int_{x_0}^{x} f(x)dx$$

Note the arbitrary constant.

And it can be easily shown that $E = T + \Phi$ is conserved. Multiply Newton's eq. by $\dot{x}$. $\dot{x}\ddot{x}$ can be identified as $\frac{dT}{dt}$ and $\dot{x}f(x)$ is $\frac{-d\Phi}{dt}$ by the Liebnitz rule, and so we have $\frac{d(T+\Phi)}{dt}$ is 0, or E is conserved.

Example of 1D motion in a potential well. Note how conservation of energy can tell us:

1. physical extent of the orbit

2. given a position, what the velocity at that position is.

Now let's look at the system a little differently: the equation of motion is a second order non-linear differential equation, which can be rewritten as a coupled set of first order differential equations:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= f(x) \end{aligned}$$

Note that rather than focusing on x(t) and v(t), we will consider the solution within the co-ordinates x and v. Note: we are considering v as a variable independent of x.

Example: simple harmonic oscillator.

Note the symmetry between x and v.

Note how energy conservation defines a line in a 2-D phase space; no two surfaces intersect.

2.1 Orbits in spherical potentials

Orbits in spherical potentials are easy to consider and lead to some important concepts.

The equation of motion in a spherical potential is:

$$\ddot{\mathbf{r}} = F(r)\hat{\mathbf{e}}_r.$$

Crossing through by $\mathbf{r}$, we show that the angular momentum vector, $\mathbf{L} \equiv \mathbf{r}\times\mathbf{v}$ is conserved.

But also symmetry implies L conserved.

The equations of motion in the plane are

$$\begin{aligned} \ddot{r} - r\dot{\psi}^2 &= F(r) \\ 2\dot{r}\dot{\psi} + r\ddot{\psi} &= 0 \Rightarrow r^2\dot{\psi} = L. \end{aligned}$$

Eliminating ψ, we have a one dimensional equation of motion:

$$\ddot{r} - L^2/r^3 = F(r).$$

From this we define an effective potential

$$\Phi_{\text{eff}} \equiv \Phi(r) + \frac{L^2}{2r^2}.$$

Note that this is a thoroughly anharmonic oscillator.

The energy of the oscillator is:

$$\frac{1}{2}\dot{r}^2 + \Phi_{\text{eff}} = \frac{1}{2}\dot{r}^2 + \Phi + \frac{L^2}{2r^2} \equiv E.$$

As in the 1D case, we learn all about the motion from just studying the Energy equation.

2.2 2-D motion

Nearly all systems of interest will have more than one degree of freedom, however, even 2-D motion is not generally solvable.

In 2D we have

$$\ddot{\vec{x}} = \vec{f}(\vec{x})$$

where $\vec{x} = (x_1, x_2), \vec{f} = (f_1(\vec{x}), f_2(\vec{x}))$ and this can be generalized to more dimensions.

Contrary to the 1D case, only a subclass of f are *conservative*, that is with $\vec{f} = -\nabla\Phi(x)$. For these systems (which will be the ones we consider)

$$\begin{aligned} \ddot{\vec{x}} &= -\nabla\Phi(\vec{x}) \\ \dot{\vec{x}} \cdot \ddot{\vec{x}} &= -\dot{\vec{x}} \cdot \nabla\Phi(\vec{x}) \\ \frac{d}{dt}\frac{\dot{x}^2}{2} &= -\frac{d}{dt}\Phi(\vec{x}) \end{aligned}$$

So energy is conserved. As in the one D case the orbits are trapped in a domain $\vec{x}$ such that $\Phi(\vec{x}) < E$. If the domain is finite, then the particle is trapped in a potential well.

Again, we want to look at this geometrically: it is now a set of 4 coupled ODEs in 4 independent variables. The 2d spatial projection is the trajectory or orbit.

Note

- orbits intersect, but not the phase space curves (why?)
- Conservation of energy is one constraint that restricts the orbit to a subspace. Sometimes (often?) there are other conservation laws which will further restrict the phase space.

1 dof: 2-d space
2 dof: 4-d phase space
3 dof: 6-d phase space
3N DOF: 6N-d phase space
But conserved quantities reduce the dimensionality of the motion.
1DOF + energy: 1D motion
2DOF + energy: 3D motion.
In 2DOF there may be another constraint which will further restrict motion.

Example: anisotropic harmonic oscillator.

$$V(x_1, x_2) = (x_1^2 + qx_2^2)/2$$

The motion is separable, and there are two energy-like constraints. Each new conservation law (*integral of the motion*) reduces the dimensionality. Again, each law corresponds to a symmetry (Nöther's Theorem).

2.3 Central force law

We will be frequently interested in force laws of the form

$$\vec{f} = f(r)\hat{e}_r$$

This is obviously a conservative system. Also note: $\vec{L}$ is conserved. (By what symmetry?). But $\vec{L}$ is a vector: there are 3 conserved quantities. So we have the dimensionality of the system is: 6 - 1(energy) - 3(L) = 2. That is, the central force problem is reducible to 1 DOF, and is therefore completely solvable.

On to N particle systems: a generalization of the above.

3 Aside: numerical solutions of ordinary differential equations

3.1 Euler Method

Consider again:

$$\frac{dy}{dx} = f(x, y); y(a) = y_0.$$

and we would like to know $y(b)$.

The simplest algorithm is the Euler method:

$$y_{n+1} = y_n + hf(x_n, y_n).$$

Advantages:

- Simple
- Self starting

Disadvantages:

- Low order
- Unstable

3.2 Leapfrog

Consider a class of problems:

$$\begin{aligned} \frac{dx}{dt} &= f(v) \\ \frac{dv}{dt} &= g(x) \end{aligned}$$

Leapfrog can be used:

$$\begin{aligned} v_{n+1/2} &= v_{n-1/2} + hg(x_n) \\ x_{n+1} &= x_n + hf(v_{n+1/2}) \end{aligned}$$

- Second order
- Not self starting (but an Euler 1/2 step can be used to start)

Now let's relabel:

$$\begin{aligned} v_{n+1} &= v_n + hg(x_n) \\ x_{n+1} &= x_n + hf(v_{n+1}) \end{aligned}$$

and compare with first order Euler:

$$\begin{aligned} v_{n+1} &= v_n + hg(x_n) \\ x_{n+1} &= x_n + hf(v_n) \end{aligned}$$

The difference is subtle. Should it make a big difference? If it does why?

Note that leapfrog is typically implemented as:

$$\begin{aligned} v_{n+1/2} &= v_n + 0.5hg(x_n) \\ x_{n+1} &= x_n + hf(v_{n+1/2}) \\ v_{n+1} &= v_{n+1/2} + 0.5hg(x_{n+1}) \end{aligned}$$

3.3 Runge-Kutta

The workhorse.

Fourth order:

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\ k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \\ k_4 &= hf(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5) \end{aligned}$$

Going to higher order is generally not justified.

Error can be estimated by half-stepping. This gives an overhead of 1.375.

Use $\Delta \sim h^5$ to adjust stepsize.

3.4 Bulirsch-Stoer

For high-order, accurate integrations:

- Richardson extrapolation

  Allows high order with few function evaluations.

  Let $Int[h]$ be a second order method of integrating a differential equation, and let I_t be the true solution. Then

  $$Int[h_1] = I_t + Ch_1^2 + \ldots$$

  $$Int[h_2] = I_t + Ch_2^2 + \ldots$$

  If $h_1 = \frac{1}{2}h_2$, then $4Int[h_1] - Int[h_2]$ will elimitate the h^2 term, giving a higher order estimate of I_t.

- Rational function extrapolation

  Allows large extrapolations without worries about poles.

- Use method with even error function: modified midpoint method,

$$
\begin{aligned}
z_1 &= y_n + \frac{h}{2} f(x_n, y_n) \\
z_2 &= z_1 + h f(x_{n+1/2}, z_1) \\
y_{n+1} &= \frac{1}{2}[z_2 + z_1 + \frac{h}{2} f(x_{n+1}, z_2)]
\end{aligned}
$$

Allows function approximation in h^2.

Extrapolator can give error estimate.

Adjust stepsize to keep number of extrapolations reasonable.

4 Review of Hamiltonian dynamics

Lagrange: **Arbitrary choice of coordinate system.**

Recall why we are doing this: symmetries will give us a more fundamental understanding of the system that simply integrating systems of ODEs does not.

4.1 Lagrangian Mechanics

The principle of virtual work is that in equilibrium,

$$\mathbf{F}_i \cdot \delta \mathbf{q}_i = 0,$$

where $\mathbf{F}_i$ is the applied force and $\delta \mathbf{q}_i$ is a displacement consistent with constraints. Hence Φ must be an extremum for equilibrium. This is how we can find the equilibrium configuration in statics even in a situation where we have complicated coordinate systems.

A similar principle can be used for a dynamical system, but we need to find a function to take the place of Φ. Any dynamical state can be an equilibrium if we add a force, $-\dot{\mathbf{p}}$ so

$$(\mathbf{F}_i \cdot \delta \mathbf{q}_i - \dot{\mathbf{p}}_i \cdot \delta \mathbf{q}_i) = 0.$$

and each point in the tractectory is an equilibrium between the real and fictious forces.

Integrating over a finite time, we have

$$\int_{t1}^{t2} \left[-\delta\Phi + \mathbf{p}_i \cdot \frac{\mathrm{d}(\delta\mathbf{q}_i)}{\mathrm{d}t} \right] \mathrm{d}t - [\mathbf{p}_i \cdot \delta\mathbf{q}_i]_{t_1}^{t2}.$$

Where I have integrated the second term by parts. The integrated term vanishes since we assume that the tractory starts and stops on the end points, so the displacements are constrainted to be 0 there.

Now we need a function T_i such that

$$\delta T_i = \mathbf{p}_i \cdot \delta\mathbf{v}_i = \mathbf{p}_i \cdot \frac{\mathrm{d}(\delta\mathbf{q}_i)}{\mathrm{d}t}.$$

This is obviously $T_i = \frac{1}{2} m_i \mathbf{v}_i^2$, so

$$\delta \int_{t1}^{t2} (T - \Phi) \mathrm{d}t = \delta \int_{t1}^{t2} L \mathrm{d}t = 0,$$

where $L = T - \Phi$ is the **Lagrangian**.

L plays a rôle in dynamics similar to that of Φ in statics:

Note that L is a function of $\mathbf{q}$ and $\dot{\mathbf{q}}$.

4.2 Hamilton's equations

$\Rightarrow$ we cannot have velocity dependent coordinates in Lagrangian mechanics. This is a problem for, e.g., the local standard of rest of the Sun about the Galaxy where the coordinates depend on angular momentum.

Can we come up with new coordinates to use as dynamical variables? *Legendre transformations* are the way we do this in thermodynamics, changing from U to enthalpy to Helmholtz free energy, etc. This is done by adding a perfect differential to L. This is the same as adding a constant since the value of the differential in the action integral is determined by the value at the end points.

Since the δq vanish at the end-points, we have

$$\int_{t1}^{t2} \left[\delta L - \frac{\mathrm{d}(\mathbf{p} \cdot \delta\mathbf{q})}{\mathrm{d}t} \right] \mathrm{d}t = 0.$$

for any function $\mathbf{p}$.

$$\int_{t1}^{t2} \left[\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} - \mathbf{p} \right) \cdot \delta\dot{\mathbf{q}} + \left(\frac{\partial L}{\partial \mathbf{q}} - \dot{\mathbf{p}} \right) \cdot \delta\mathbf{q} \right] \mathrm{d}t = 0.$$

Choose $\mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$ and note that this gives $\dot{\mathbf{p}} = \partial L/\partial \mathbf{q}$. Now define

$$H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p} \cdot \dot{\mathbf{q}} - L$$

so that

$$\mathrm{d}H = \dot{\mathbf{q}} \cdot \mathrm{d}\mathbf{p} - \frac{\partial L}{\partial \mathbf{q}} \cdot \mathrm{d}\mathbf{q} - \frac{\partial L}{\partial t}\mathrm{d}t.$$

this implies

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = \frac{\partial L}{\partial \mathbf{q}} = -\frac{\partial H}{\partial \mathbf{q}},$$

which are Hamilton's equations of motion.

The variational principle is now

$$\delta \int (\mathbf{p} \cdot \dot{\mathbf{q}} - H)\mathrm{d}t = 0.$$

The dynamical variables are now $\mathbf{p}$ and $\mathbf{q}$.

4.3 Canonical transformations

The power of the Hamiltonian is the ability to describe the motion in any coordinate system, but retain the same form of the equations of motion.

So if we have $H(\mathbf{q}, \mathbf{p})$ and $H'(\mathbf{Q}, \mathbf{P})$, the Hamiltonian in the new coordinate system, then

$$\delta \int (\mathbf{p} \cdot \dot{\mathbf{q}} - H)dt = \delta \int (\mathbf{P} \cdot \dot{\mathbf{Q}} - H')dt.$$

The variation is of the path, holding the end points fixed, so we can add any total differential to the integrand, and the variation will be the same. Therefore we have the following:

$$\mathbf{p} \cdot \dot{\mathbf{q}} - H = \mathbf{P} \cdot \dot{\mathbf{Q}} - H' + \frac{\mathrm{d}}{\mathrm{d}t}F.$$

If $F = F(\mathbf{q}, \mathbf{Q})$ we see that in order for the above to hold, we must have (noting that $dF/dt = \partial F/\partial q\dot{q} + \partial F/\partial Q\dot{Q} + \partial F/\partial t$):

$$\mathbf{p} = \frac{\partial F}{\partial \mathbf{q}}, \quad \mathbf{P} = -\frac{\partial F}{\partial \mathbf{Q}}, \quad H' = H + \frac{\partial F}{\partial t}.$$

Likewise, if we consider $F = F(\mathbf{q}, \mathbf{P})$ then we can add $\mathbf{Q} \cdot \dot{\mathbf{P}}$ to both sides, set $S = F + \mathbf{Q} \cdot \mathbf{P}$ and get the requirement that

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}$$

and

$$\mathbf{Q} \equiv \frac{\partial S}{\partial \mathbf{P}}.$$

The transformation $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \mathbf{Q})$ is called a *canonical transformation* and S the *generating function* of the transformation.

4.4 Poisson brackets and symplectic structure

To make it obvious that Hamilton's equations are coordinate-free, lets define the **Poisson bracket** $\{A, B\}$ of any two functions $A(\mathbf{q}, \mathbf{p})$, $B(\mathbf{q}, \mathbf{p})$ on phase space by

$$\{A, B\} \equiv \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}}. \tag{1}$$

It is straightforward to verify that the coordinates $(\mathbf{q}, \mathbf{p})$ satisfy the **canonical commutation relations**

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \quad \text{and} \quad \{q_i, p_j\} = \delta_{ij}.$$

This is easy to verify since q_i and p_i are all independent variables.

If we write $(z_i \equiv x_i,\ z_{3+i} \equiv v_i \ \ i = 1, 2, 3)$, and define the **symplectic matrix c** by

$$c_{\alpha\beta} \equiv \{z_\alpha, z_\beta\} = \begin{cases} \pm 1 & \text{for } \alpha = \beta \mp 3,\ 1 \le \alpha, \beta \le 6; \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

we have

$$\{A, B\} = \sum_{\alpha,\beta=1}^{6} c_{\alpha\beta} \frac{\partial A}{\partial z_\alpha} \frac{\partial B}{\partial z_\beta}. \tag{3}$$

Any set of 6 phase-space coordinates $\{Z_\alpha,\ \alpha = 1, \ldots, 6\}$ is called a set of **canonical coordinates** if $\{Z_\alpha, Z_\beta\} = c_{\alpha\beta}$. Let $\{Z_\alpha\}$ be such a set; then

with (3) and the chain rule we have

$$\{A,B\} = \sum_{\alpha,\beta=1}^{6} c_{\alpha\beta}\frac{\partial A}{\partial z_\alpha}\frac{\partial B}{\partial z_\beta} = \sum_{\kappa\lambda}\Big(\sum_{\alpha\beta} c_{\alpha\beta}\frac{\partial Z_\kappa}{\partial z_\alpha}\frac{\partial Z_\lambda}{\partial z_\beta}\Big)\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda} \quad (4)$$

$$= \sum_{\kappa\lambda}\{Z_\kappa, Z_\lambda\}\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda} = \sum_{\kappa\lambda} c_{\kappa\lambda}\frac{\partial A}{\partial Z_\kappa}\frac{\partial B}{\partial Z_\lambda}. \quad (5)$$

Thus the derivatives involved in the definition (1) of the Poisson bracket can be taken with respect to any set of canonical coordinates, just as the vector formula $\nabla \cdot \mathbf{a} = \sum_i (\partial a_i/\partial x_i)$ is valid in any Cartesian coordinate system. It is conventional to denote the first 3 coordinates W_i by q_i and the last 3 by $p_i = W_{3+i}$.

Note that if we can express the equations of motion in terms of Poisson brackets, then these equations of motion will be valid *independent of the coordinate system*.

Hamilton's eqns may be written

$$\dot{q}_i = \{q_i, H\} \quad ; \quad \dot{p}_i = \{p_i, H\}. \quad (6)$$

The rate of change of an arbitrary canonical coordinate Z_α along an orbit is

$$\dot{Z}_\alpha = \sum_{\beta=1}^{6}\frac{\partial Z_\alpha}{\partial z_\beta}\dot{z}_\beta,$$

where, as usual, $\mathbf{z} \equiv (\mathbf{q}, \mathbf{p})$. With Hamilton's equations (6) and equation (3) this becomes

$$\dot{Z}_\alpha = \sum_{\beta=1}^{6}\frac{\partial Z_\alpha}{\partial z_\beta}\{z_\beta, H\} = \sum_{\beta\gamma\delta}\frac{\partial Z_\alpha}{\partial z_\beta}c_{\gamma\delta}\frac{\partial z_\beta}{\partial z_\gamma}\frac{\partial H}{\partial z_\delta} = \sum_{\gamma\delta} c_{\gamma\delta}\frac{\partial Z_\alpha}{\partial z_\gamma}\frac{\partial H}{\partial z_\delta} \quad (7)$$

$$= \{Z_\alpha, H\}. \quad (8)$$

Thus Hamilton's equations (6) are valid in *any* canonical coordinate system.

The Hamiltonian is therefore very much like the stream function in fluid mechanics. The system follows surfaces of constant H, and the Poisson bracket gives the direction along the surface of the flow.

4.5 Example: Hamiltonian Formulation of Hill's equations

(Taken from Quinn et al 2010)

The Lagrangian for Hill's equations in the orbital plane is

$$\mathcal{L} = \frac{1}{2}\left[(\dot{x} - \Omega y)^2 + (\dot{y} + \Omega x)^2\right] + \frac{1}{2}\Omega^2(2x^2 - y^2) - \Phi(x, y), \tag{9}$$

(Heggie 2001) where x and y are, respectively, the distances perpendicular to and along the direction of rotation from the center of a frame in circular motion with angular speed Ω. Φ is the potential due to other forces, e.g., interactions with other particles. In all that follows, we will neglect the motion in the z direction since it is trivial to integrate in the standard way.

Lagrange's equations ,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0,$$

give the standard Hill's equations of motion,

$$\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x = -\frac{\partial \Phi}{\partial x} \tag{10}$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial \Phi}{\partial y}. \tag{11}$$

In this form, the presence of the velocity-dependent terms requires a modification to the leapfrog method such that a predicted velocity is used in the estimate of the final acceleration. This maintains second order, but is obviously not time reversible and destroys the symplectic nature of leapfrog.

Now we derive the Hamiltonian form of the equations of motion. From their definitions, the canonical momenta are

$$p_x \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} - \Omega y \tag{12}$$

$$p_y \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y} + \Omega x \tag{13}$$

and the Hamiltonian is

$$H = p \cdot \dot{q} - \mathcal{L} \tag{14}$$

$$H(x, y, p_x, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \Omega(y p_x - x p_y) - \frac{1}{2}\Omega^2(2x^2 - y^2) + \Phi(x, y). \tag{15}$$

Now consider a new set of canonical coordinates, (X, Y, P_x, P_y), derived from the generating function

$$S_2(x, y, P_x, P_y) = xP_x + yP_y - \Omega xy. \tag{16}$$

The rules of canonical transformations then give

$$p_x = \frac{\partial S_2}{\partial x} = P_x - \Omega y; \quad p_y = \frac{\partial S_2}{\partial y} = P_y - \Omega x, \tag{17}$$

and

$$X = \frac{\partial S_2}{\partial P_x} = x; \quad Y = \frac{\partial S_2}{\partial P_y} = y. \tag{18}$$

In terms of the original positions and velocities, these new canonical coordinates are x, y, $P_x = \dot{x}$, and $P_y = \dot{y} + 2\Omega x$. The Hamiltonian in these coordinates is

$$H(x, y, P_x, P_y) = \frac{P_x^2}{2} + \frac{P_y^2}{2} - 2\Omega x P_y + \frac{\Omega^2 x^2}{2} + \Phi(x, y), \tag{19}$$

a somewhat simpler form than equation (15).

Hamilton's equations of motion are therefore

$$\dot{x} = P_x \tag{20}$$

$$\dot{y} = P_y - 2\Omega x \tag{21}$$

$$\dot{P}_x = 2\Omega P_y - \Omega^2 x - \frac{\partial \Phi}{\partial x} \tag{22}$$

$$\dot{P}_y = -\frac{\partial \Phi}{\partial y}. \tag{23}$$

From these equations, it is obvious that P_y is constant in the absence of perturbing forces. This is equivalent to the conservation of angular momentum, and leads to the conserved quantity, $\sum_i P_{y_i}$, in a many-particle system. In particular, $\sum_i P_{y_i}$ is conserved in a collision between particles (Wisdom & Tremaine 1988), which will be useful when calculating collision outcomes (see below). However, for periodic boundary conditions, P_y will change as a particle crosses the boundary in x because of the shear across the box. Nevertheless the time-averaged total P_y should be constant for a system that does not have a net motion in the x direction (Wisdom & Tremaine88). Also from the equations of motion in this form it is clear why Hill's equations are easy to integrate numerically. If the guiding center of the motion is at $x = 0$, then $P_y = 0$ for all time in the absence of perturbations, and the motion reduces to a harmonic oscillator with frequency Ω.

4.6 Hamilton-Jacobi equation

Jacobi had a clever idea: let's choose the momentum coordinates to be the integrals of the motion.

If that's the case then Hamilton's equations show us that:

$$\dot{P} = -\frac{\partial H'}{\partial Q} = 0.$$

(since P is a constant) Hence $H'(P)$ is a function of P only. Furthermore,

$$\dot{Q} = -\frac{\partial H'}{\partial P} = \omega(P) = constant,$$

so the solution is trivial: P =constant, $Q(t) = \omega t + Q_0$. That is, Q increases linearly with time.

We can transform from our regular coordinates to these new variables using a generating function of the type $S(q, P)$. In this case $p = \frac{\partial S}{\partial q}$, so we can plug this into the Hamiltonian to get:

$$H\left(q, \frac{\partial S}{\partial q}\right) = H'(P) = E = constant.$$

The simplicity of the E.O.M. are offset by the fact that the *Hamilton-Jacobi equation* is an implicit partial differential equation for S. Once we solve for S, we can get the corresponding coordinates from $Q = \partial S/\partial P$.

Since these coordinates increase linearly without bound, it would therefore make sense to choose "angle-like" variables. If the orbit is bound, the Cartesian coordinates x_i cannot increase without limit as the Q_i do. From this we infer that the x_i are periodic functions of the Q_i. In other words if we scale the Q_i correctly, we can expand $\mathbf{x}$ in a Fourier series

$$\mathbf{x}(\mathbf{Q}, \mathbf{P}) = \sum_{\mathbf{n}} \mathbf{X_n}(\mathbf{P}) e^{i\mathbf{n}\cdot\mathbf{Q}} = \sum_{\mathbf{n}} \mathbf{X_n}(\mathbf{P}) e^{i\mathbf{n}\cdot\omega t},$$

where the sum is over all vectors $\mathbf{n}$ with integer components, and the Q_i have been scaled so that $\mathbf{x}$ returns to its original value after Q_i has increased by 2π. The 3-surface of fixed $\mathbf{P}$ and varying $\mathbf{Q}$ constitutes a 3-torus.

This is Quasi-periodic or multiply periodic motion.

Significance of Q: draw harmonic oscillator.

4.7 Actions

How can we choose these magic variables? We want:

$$2\pi = \oint dQ = \oint dq \cdot \frac{\partial Q}{\partial q}$$

for some coordinate Q. Let's use the fact that $Q = \frac{\partial S}{\partial P}$ to get:

$$2\pi = \oint dq \cdot \frac{\partial^2 S}{\partial q \partial P} = \frac{\partial}{\partial P} \oint dq \cdot p.$$

This is satisfied if we choose:

$$P = \frac{1}{2\pi} \oint dq \cdot p = \frac{1}{2\pi}(\text{Phase space Area})$$

This is obviously an integral of the orbit, and is usually called the *action*. It is usually designed I or J.

This solution of the H-J equation is fundamental to understanding orbits, their classification, and evolution, and the construction of galactic models. In particular, the restriction into a subspace of phase space by the action restricts the orbit in physical space as well.

5 Examples of Action Finding

5.1 Existence of Solutions

There is no guarantee that we can solve the H-J equation for a particular configuration.

If we can, the orbit is completely specified by the 3 actions (at most 1 per degree of freedom). This is referred to as regular motion. The actions restrict the motion to a 6-3 dimensional subspace. The solution is an n-torus. In practice the actions can only be found if the H-J equation is separable.

If we can't solve the H-J equation (more precisely, a solution doesn't exist), then the orbits will be irregular. They will fill all space allowed by energy conservation, and have the property that 2 nearby orbits diverge exponentially. (The expansion in angles shown above can't be performed.)

E.g. Henon-Heiles (1964) potential:

$$\Phi(x, y) = \frac{1}{2}(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3)$$

5.1.1 Free particle

The Hamiltonian of the particle in free space is

$$H = \frac{p^2}{2}.$$

i.e., the kinetic energy, but written in terms of p (not v!)

In this case, the H-J equation is:

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = E.$$

Note that this would be, in general, a non-linear partial differential equation, so a solution could be quite difficult. However in this case it's easy to simplify:

$$\frac{\partial S}{\partial q} = \sqrt{2E} = \text{constant} \equiv P$$

Note that we can pick the new momentum to be a constant of motion, or any function of the constant. Here we pick $\sqrt{2E}$ or just the velocity. So $S = qv$ and we have

$$p = \frac{\partial S}{\partial q} = v$$

$$Q = \frac{\partial S}{\partial P} = x$$

and we have rectilinear motion. (Although this was rather difficult.)

5.1.2 Harmonic Oscillator

The Hamiltonian is:

$$H = \frac{\omega^2 x^2}{2} + \frac{p^2}{2}$$

The Hamilton-Jacobi equation is therefore

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{x^2}{2} = E \equiv \frac{\omega^2 I^2}{2}$$

We are choosing the new (constant) momentum to be the turning point of the orbit: $P = I = \sqrt{2E}/\omega$. Note that we are free to choose any function

of E (the constant of the motion.) I choose this for convenience below. The equation for the generating function is:

$$\frac{\partial S}{\partial q} = \omega\sqrt{I^2 - x^2}$$

which has the solution:

$$S(x, I) = \omega \int^x dx\sqrt{I^2 - x^2}$$

The lower limit can be arbitrary: S(x,I) is only defined up to a constant. The rules for canonical transformations gives us:

$$Q = \frac{\partial S}{\partial I} = \omega \int^x dx \frac{I}{\sqrt{I^2 - x^2}} = \omega I \arcsin(x/I) + Q_0.$$

or

$$x = I \sin((Q - Q_0)/\omega I).$$

Note that Q has unusual units for a coordinate. However, we would like to choose a momentum whose action goes through 2π in one cycle of the oscillator's motion. That is, we want $J = 1/2\pi \oint dxp$. Using the solution above, we have

$$J = \frac{1}{2\pi} \oint dx \frac{\partial S}{\partial x} = \frac{1}{2\pi} \oint dx\omega\sqrt{I^2 - x^2} = \frac{\omega I^2}{2\pi} \int_0^{2\pi} d\xi cos^2(\xi) = \frac{\omega I^2}{2}.$$

where I have used $\xi = Q/\omega I$ for the last equality. The action is therefore $\omega I^2/2$. The canonical "angle" Q (different from above) is $Q = \partial S/\partial J$ or $Q = \arcsin(x/I) + Q_0$ or $x = I sin(Q - Q_0)$.

Note that:

$$E = \frac{\omega^2 I^2}{2}$$

so if the action is $J = \omega I^2/2$, then

$$H' = \omega J.$$

Actions as Adiabatic invariants: it can be shown that as the parameters of the Hamiltonian are changed slowly, is the *actions* that remain constant. In the case of the Harmonic oscillator, as we change, e.g. ω, the ratio E/ω remains invariant while E changes.

5.1.3 Galactic Disk

For homework.

5.2 Orbit classification for the pendulum

Librating orbits, circulating orbits, marginal orbits.

5.3 Box and Loop orbits

5.3.1 Non-axisymmetric potentials

Consider the 2-D potential

$$\Phi(x,y) = \frac{1}{2}v_0^2 \ln(R_c^2 + x^2 + y^2/q^2).$$

Two different families of orbits are present: **loop orbits** and **box orbits**.

Orbits in both families are constrained by another integral.

5.3.2 Stäckel potentials

Finding the right coordinate system helps to find the integrals.

We need a coordinate system (u, v) such that:

1. u = constant and v = constant are boundaries of orbits.
2. $H(u, v, p_u, p_v) \times f(u, v)$ breaks into a sum $F_u(u, p_u) + F_v(v, p_v)$.

Consider the coordinates defined by

$$x = \Delta \sinh u \cos v, \quad y = \Delta \cosh u \sin v.$$

u is a radial coordinate, v is an angular coordinate.

The Lagrangian in these coordinates is

$$\mathcal{L} = \frac{1}{2}\Delta^2(\sinh^2 u + \cos^2 v)(\dot{u}^2 + \dot{v}^2) - \Phi.$$

The momenta are therefore

$$\begin{aligned} p_u &\equiv \frac{\partial \mathcal{L}}{\partial \dot{u}} = \Delta^2(\sinh^2 u + \cos^2 v)\dot{u}, \\ p_v &\equiv \frac{\partial \mathcal{L}}{\partial \dot{v}} = \Delta^2(\sinh^2 u + \cos^2 v)\dot{v}, \end{aligned}$$

and the Hamiltonian is

$$\begin{aligned} H(u,v,p_u,p_v) &= p_u\dot{u} + p_v\dot{v} - \mathcal{L} \\ &= \frac{1}{2}\Delta^2(\sinh^2 u + \cos^2 v)(\dot{u}^2 + \dot{v}^2) + \Phi \\ &= \frac{p_u^2 + p_v^2}{2\Delta^2(\sinh^2 u + \cos^2 v)} + \Phi. \end{aligned}$$

It is clear that the Hamiltonian will only separate if

$$\Phi(u,v) = \frac{U(u) - V(v)}{\sinh^2 u + \cos^2 v}.$$

Then $(\sinh^2 u + \cos^2 v) \times (H = E)$ breaks into

$$E\sinh^2 u - \frac{p_u^2}{2\Delta^2} - U(u) \equiv I_2 = \frac{p_v^2}{2\Delta^2} - V(v) - E\cos^2 v,$$

where I_2 is a constant.

The momenta are therefore

$$\begin{aligned} p_u &= \pm\sqrt{2}\Delta\sinh u\sqrt{E - U_{\text{eff}}}, \quad \text{where} \quad U_{\text{eff}} \equiv \frac{I_2 + U(u)}{\sinh^2 u}, \\ p_v &= \pm\sqrt{2}\Delta\cos v\sqrt{E - V_{\text{eff}}}, \quad \text{where} \quad V_{\text{eff}} \equiv \frac{I_2 + V(v)}{\cos^2 v}. \end{aligned}$$

Now look at the potential of a **perfect ellipsoid** which corresponds to a density of

$$\rho(\mathbf{x}) = \frac{\text{constant}}{1 + m^2}, \quad \text{where} \quad m^2(\mathbf{x}) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

For the case $a > b = c$ we have

$$\begin{aligned} U(u) &= -W\sinh u\arctan\left(\frac{\Delta\sinh u}{c}\right) \\ V(v) &= W\cos v\text{arctanh}\left(\frac{\Delta\cos v}{c}\right) \end{aligned}$$

For $I_2 > 0$ we have loop orbits and for $I_2 < 0$ we have box orbits.

For $I_2 > 0$ there is a $u_{\min}$ and a $u_{\max}$ which are bounding ellipses for the orbit. For $I_2 < 0$, u can go to zero and v is constrained to lie between $v_{\min}$ and $v_{\max}$.

5.4 Surfaces of Section and approximate integrals

Plots from BT on near-axisymmetric potentials.

5.5 Kepler Motion

Example for how near-integralness can be handled.

5.5.1 Delaunay variables

The H-J equation for Kepler motion is

$$\begin{aligned} E &= \frac{1}{2}|\nabla S|^2 - \frac{GM}{r} \\ &= \frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial S}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{GM}{r} \end{aligned}$$

We write $S(\mathbf{x}) = S_r(r) + S_\theta(\theta) + S_\phi(\phi)$ and solve by separation of variables. We find

$$\text{constant} \equiv L_z^2 = \left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = p_\phi^2 \tag{24}$$

$$L^2 - \frac{L_z^2}{\sin^2\theta} = \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = p_\theta^2 \tag{25}$$

$$2E + 2\frac{GM}{r} - \frac{L^2}{r^2} = \left(\frac{\partial S_r}{\partial r}\right)^2 = p_r^2. \tag{26}$$

Each of these 3 equations is a relation of the form $p_i(q_i)$. The orbital torus is the 3-surface generated by varying the phase-space coordinates through all values compatible with these equations. Since ϕ doesn't occur in (24), it can take any value while p_ϕ is restricted to the single value L_z. Equations (25) and (26) restrict θ and r to ranges and allow non-zero variation of p_θ and p_r. The orbit's inclination is just $i = \frac{\pi}{2} - \theta_{\min}$ and from equation (25) we have that $\sin\theta_{\min} = L_z/L$, so

$$i = \arccos(L_z/L).$$

The actions are

$$J_\phi = \frac{1}{2\pi}\oint p_\phi(\phi)\,\mathrm{d}\phi = L_z$$

$$
\begin{aligned}
J_\theta &= \frac{1}{2\pi}\oint p_\theta(\theta)\,\mathrm{d}\theta \\
&= \frac{2}{\pi}\int_{\theta_{\min}}^{\pi/2}\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}\,\mathrm{d}\theta = L - L_z \\
J_r &= \frac{1}{2\pi}\oint\sqrt{2E+\frac{2GM}{r}-\frac{L^2}{r^2}}\,\mathrm{d}r \\
&= \frac{GM}{\sqrt{2|E|}} - L.
\end{aligned}
$$

The last equation can be written

$$E = H = -\frac{G^2M^2}{2(J_r+J_\theta+J_\phi)^2},$$

from which it follows that $\omega_r = \omega_\theta = \omega_\phi = -2H/(J_r+J_\theta+J_\phi)$. It is this unusual double degeneracy that causes all Kepler orbits to close, and gives rise to specially complex behavior when we perturb H_K.

A new way to think about closed orbits: the orbit is closed because the radial frequency is equal to the ϕ frequency.

Also new way to think of orbit confined to a plane: the "up-down" frequency is the same as the ϕ frequency.

Three of the 5 constants of motion of a Kepler orbit may be taken to be its three actions. The other two are constant by virtue of the system's degeneracy: this makes two differences in angles constant, for example $w_1 \equiv w_\phi - w_\theta$ and $w_2 \equiv w_\theta - w_r$. It is useful to make a canonical transformation to a set which includes these new angles. These new angles will have frequencies of

$$\omega_1 = 0; \quad \omega_2 = 0.$$

The generating function of this transformation is

$$S' = (w_\phi - w_\theta)J_1 + (w_\theta - w_r)J_2 + w_r J_3.$$

Differentiating S' w.r.t. the old angles we discover the connection between the new and old actions:

$$
\begin{array}{lll}
J_\phi &= J_1 & \\
J_\theta &= J_2 - J_1 \quad \Rightarrow & J_2 = J_\theta + J_\phi = L \\
J_r &= J_3 - J_2 \quad \Rightarrow & J_3 = J_r + J_\theta + J_\phi.
\end{array}
$$

Replacing the old actions by the new in our formulæ, we have

$$\begin{aligned} H &= -\frac{G^2M^2}{2J_3^2} = -\frac{GM}{2a} \quad \Rightarrow \quad J_3 = \sqrt{GMa}, \\ e &= \sqrt{1-\frac{L^2}{GMa}} = \sqrt{1-J_2^2/J_3^2}, \\ i &= \arccos(J_1/J_2). \end{aligned}$$

Now w_3 is just N, the mean anomaly. It remains only to relate w_1 and w_2 to Ω and ω (which they actually equal). We have $w_i = (\partial S/\partial J_i)$. Now replacing E, L and L_z in our expressions for $\partial S_\phi/\partial\phi$ etc. we find

$$S = \int J_1 \mathrm{d}\phi - \int \sqrt{J_2^2 - \frac{J_1^2}{\sin^2\theta}}\,\mathrm{d}\theta + \int \sqrt{-\frac{G^2M^2}{J_3^2} + 2\frac{GM}{r} - \frac{J_2^2}{r^2}}\,\mathrm{d}r. \qquad (27)$$

(We take the negative square root in S_θ because as planet passes through the *ascending* node, θ is decreasing and so $p_\theta < 0$.) Differentiating we get

$$\begin{aligned} w_1 = \frac{\partial S}{\partial J_1} &= \phi + J_1 \int \frac{\mathrm{d}\theta}{\sin^2\theta\sqrt{J_2^2 - J_1^2\csc^2\theta}} \\ &= \phi + \int \frac{\mathrm{d}\theta}{\sin\theta\sqrt{\sin^2\theta\sec^2 i - 1}} \\ &= \phi + \int \frac{\cot i \csc^2\theta \mathrm{d}\theta}{\sqrt{1-\cot^2 i\cot^2\theta}} \\ &= \phi - u, \end{aligned}$$

where

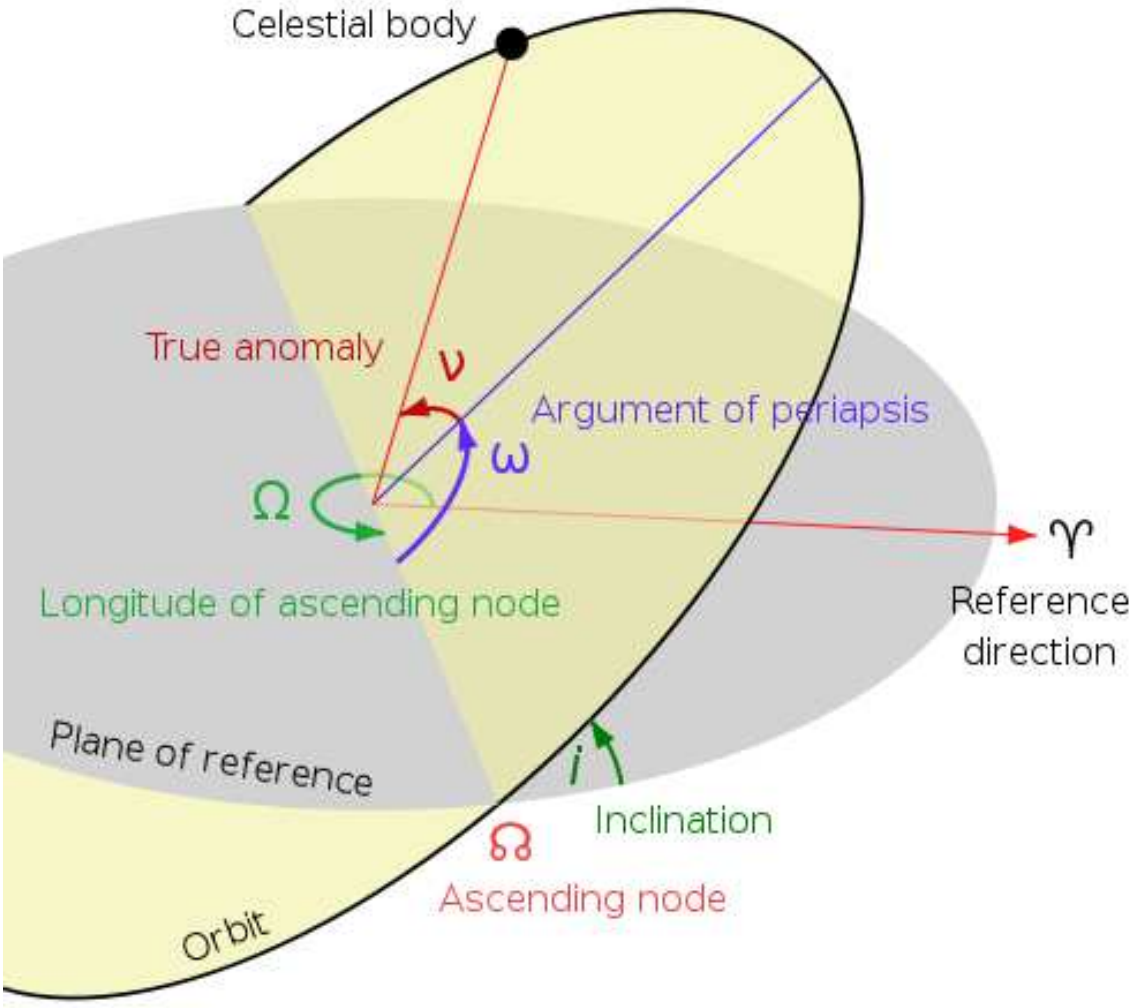
$$\sin u \equiv \cot i \cot\theta.$$

A figure demonstrates that the new variable u is actually $\phi - \Omega$ and thus that $w_1 = \Omega$:

By differentiating (27) show that $w_3 = G^2M^2t/J_3^3$. Obtain this result by another route. Show that w_3 is, in fact, the mean anomaly.

Show that the integrals obtained by differentiating (27) w.r.t. J_2 evaluate to $\arcsin(\cos\theta/\sin i)$ and φ, where φ is the angle in the orbital plane between the direction of pericenter and the position of the planet. Hence conclude that $w_2 = \omega$, the argument of perihelion.

The variables (w_i, J_i) are called Delaunay variables. Let the true Hamiltonian of a planet be written $H = H_K + \epsilon H_1$, where H_K is the Kepler


Celestial body
True anomaly
ν
Argument of periapsis
ω
Ω
♈
Reference direction
Longitude of ascending node
Plane of reference
i
Inclination
☊
Ascending node
Orbit

Hamiltonian defining the Delaunay variables and $\epsilon \ll 1$, then the equations of motion of the Delaunay variables are

$$\begin{aligned} \dot{J}_1 = -\epsilon\frac{\partial H_1}{\partial w_1} \quad ; \quad \dot{J}_2 &= -\epsilon\frac{\partial H_1}{\partial w_2} \quad ; \quad \dot{J}_3 = -\epsilon\frac{\partial H_1}{\partial w_3} \\ \dot{w}_1 = \epsilon\frac{\partial H_1}{\partial J_1} \quad ; \quad \dot{w}_2 &= \epsilon\frac{\partial H_1}{\partial J_2} \quad ; \quad \dot{w}_3 = -\frac{2H_K}{J_3} + \epsilon\frac{\partial H_1}{\partial J_3} \end{aligned}$$

Thus the rates of change of five of these variables are of order ϵ, and to zeroth-order, the sixth increases linearly in time.

5.6 The disturbing function

The only snag with this elegant scheme is the tedium of expressing H_1 as a function of Delaunay variables. Since the Newtonian gravitational force is not velocity-dependent, H_1 is a function of the planet's spatial coordinates. Thus it is a function on three-dimensional real space. Unfortunately, Delaunay variables are coordinates for six-dimensional phase space, so we have to treat H_1 as a function in this bigger space. Since $\mathbf{x}$ is a periodic function of $\mathbf{w}$, H_1 can be expanded in a Fourier series:

$$H_1(\mathbf{x}(\mathbf{J}, \mathbf{w}; t)) = \sum_{\mathbf{n}} h_{\mathbf{n}}(\mathbf{J}; t) e^{\mathbf{i}\mathbf{n}\cdot\mathbf{w}},$$

where $\mathbf{n}$ runs over vectors with integer components.

Let's see how this works out in a simple case. Consider the disturbance of an asteroid m_a by Jupiter.

The other two terms in the potential on the right can be considered to constitute a perturbing potential. So in the Hamiltonian formulation of this problem the zeroth-order Hamiltonian is $H_K = \frac{1}{2}\dot{r}_a^2 - G(m_s + m_a)/r_a$ and the perturbing Hamiltonian is

$$H_1 = -Gm_J\Big(\frac{1}{|\mathbf{r}_a - \mathbf{r}_J|} - \frac{\mathbf{r}_a \cdot \mathbf{r}_J}{r_J^3}\Big). \tag{28}$$

Minus H_1 is called the *disturbing function*.

5.7 Perturbation Theory

Consider first the simple case in which both Jupiter and the asteroid are on circular orbits in the ecliptic. Then

$$H_1 = -\frac{Gm_J}{a_J}\Big((1+\alpha^2-2\alpha\cos\varphi)^{-1/2} - \alpha\cos\varphi\Big), \tag{29}$$

where $\alpha \equiv a_a/a_J$ and φ is the angle between $\mathbf{r}_a$ and $\mathbf{r}_J$. The radical in (29) is often expanded in a Fourier series to give

$$H_1 = -\frac{Gm_J}{a_J}\left(\sum_{m=-\infty}^{\infty}\frac{1}{2}b_{1/2}^{(m)}\cos m\varphi - \alpha\cos\varphi\right).$$

The functions $b_{1/2}^{(m)}(\alpha)$ are known as *Laplace coefficients* and are tabulated, e.g., in Brouwer & Clements.

$w_1 = \Omega$ is redundant for orbits confined to the ecliptic and may be set to zero. I will also assume pure circular motion for simplicity. Then $\varphi = w_3 - w_3'$, where the primes denote Jupiter's coordinates. The natural thing to do now is to solve for the evolution of the unprimed coordinates with the primed coordinates replaced by their unperturbed values, $w_3' = \nu_J t$, where $\nu_J \equiv 2\pi/T_J$ is Jupiter's angular frequency. For example,

$$\dot{J}_3 = \frac{Gm_J}{a_J}\frac{\partial}{\partial w_3}\left(\sum_m b_{1/2}^{(m)}\cos\left(m(w_3-\nu_J t)\right) - \alpha\cos(w_3-\nu_J t)\right)$$

There is clearly no difficulty integrating these coupled equations. The solutions will show the J_i to wiggle around without going anywhere much.

Now imagine replacing φ with $\phi - \nu_J t$, substituting for η and η' and ϕ in terms of w_3 and $\nu_J t$, and expanding the right side in a Fourier series in w_3 and t. The products of circular functions will clearly generate all sorts of sum and difference frequencies, so H_1 will be of the form

$$H_1 = -\frac{Gm_J}{a_J}\sum_{\mathbf{n}} h_{\mathbf{n}}(\alpha)e^{\mathrm{i}(N\nu_J t + n_3 w_3)}. \tag{30}$$

The equations of motion of J_3 and w_3 are now

$$\begin{aligned}\dot{J}_3 &= \frac{Gm_J}{a_J}\sum_{\mathbf{n}}\mathrm{i}n_3 h_{\mathbf{n}}(\alpha)e^{\mathrm{i}(N\nu_J t+n_3 w_3)}\\ \dot{w}_3 &= -\frac{Gm_J}{a_J}\sum_{\mathbf{n}}\frac{\partial h_{\mathbf{n}}}{\partial\alpha}\frac{\partial\alpha}{\partial J_3}e^{\mathrm{i}(N\nu_J t+n_3 w_3)}.\end{aligned} \tag{31}$$

The n_i are expected to range over all positive and negative integers, and in the unperturbed motion w_2 is constant and $w_3 = \nu_a t + \phi_0$, where ϕ_0 is the asteroid's phase with respect to Jupiter. So for an asteroid with $\nu_a = 3\nu_J$ the right side of (31) will contain 'long-period terms' such as those with $\mathbf{n}$ of the form $\mathbf{n} = (3, n_2, -1)$—that is, it will contain terms in which the exponent is constant in the unperturbed case. These terms cause disaster when we integrate up (31) with the right side evaluated along the unperturbed orbit:

$$J_3 = \frac{Gm_J}{a_J} \sum_{\mathbf{n}} \frac{n_3 h_{\mathbf{n}}}{N\nu_J + n_3\nu_a} e^{\mathrm{i}t(N\nu_J + n_3\nu_a)} e^{\mathrm{i}n_3\phi_0} + \text{constant}.$$

The local spot of bother caused by the near vanishing of some of the denominators $n_1\nu_J + n_3\nu_a$ is called the 'small divisor problem'. Despite the efforts of the greatest mathematicians up to and including H. Poincaré, (Poincaré showed that because of the small divisors, the expansion is, in general, divergent) it was only mastered, and then but partially, in the 1950-60s by Kolmogorov, Arnold & Moser (the 'KAM' theorem). This work showed that the quasi-periodic trajectories of an integrable problem usually remain quasi-periodic under the influence of "sufficiently small" perturbations to the Hamiltonian. The exceptions are trajectories where the ratios of characteristic frequencies of the original problem are sufficiently well approximated by rational numbers – i.e. the theorem fails near resonances. Since rational numbers are inextricably mixed with irrationals along the real number line, regions of quasi-periodic and chaotic behavior are similarly intertwined. Furthermore, the tori which are "destroyed" form a finite set which grows with the strength of the perturbation. Arnold showed that the Solar System could be described by quasiperiodic orbits if the masses, eccentricities and inclinations were sufficiently small. Unfortunately, the real Solar System does not satisfy Arnold's requirements.

5.7.1 Resonances and Pendulum equations

The physical origin of the small denominator problem is that the forces described by long-period terms act in one sense for long enough to cause the orbit to deviate significantly from its unperturbed form. So these terms which cause grief are actually the interesting terms. The basic idea of celestial mechanics is to neglect the short period terms on the grounds that they average out to zero, and to concentrate on the long-period terms.

Suppose we are interested in an asteroid along whose orbit $N_1\nu_J + N_3\nu_a \simeq 0$. To keep things simple let's set $J_2 = w_2 = 0$ and concentrate on the evolution of (J_3, w_3). We define a new variable

$$\psi \equiv N_1\nu_J t + N_3 w_3$$

and discard all terms in the sum (30) for H_1 except the zero-frequency term (the *secular* term) and the terms (at positive and negative frequency) that comprise $\cos(\psi)$:[1]

$$H_1 \simeq h_0 + h_{\rm res}\cos\psi.$$

This discarding of the short-period terms is called 'averaging the Hamiltonian'. If we are to use ψ as a coordinate, we need to know what its conjugate momentum J_ψ is. The generating function for the transformation $(w_3, J_3) \leftrightarrow (\psi, J_\psi)$ is

$$S(w_3, J_\psi) = (N_1\nu_J t + N_3 w_3)J_\psi \quad \Rightarrow \quad \begin{aligned} \psi &= \frac{\partial S}{\partial J_\psi} = N_1\nu_J t + N_3 w_3, \\ J_3 &= \frac{\partial S}{\partial w_3} = N_3 J_\psi. \end{aligned}$$

Since the generating function contains explicit t-dependence in the new variables, the Hamiltonian is

$$\begin{aligned} H_1(J_\psi, \psi) &= H_1(J_3, w_3, t) + \frac{\partial S}{\partial t} \\ &= h_0 + N_1\nu_J J_\psi + h_{\rm res}\cos\psi. \end{aligned}$$

Neither the unperturbed Hamiltonian $H_0(J_3) = H_0(N_3 J_\psi)$ nor the new H_1 contains explicit t-dependence. So the motion is confined to surfaces of constant

$$\begin{aligned} H(J_\psi, \psi) &= H_0 + h_0 + N_1\nu_J J_\psi + h_{\rm res}\cos\psi \\ &= \alpha(J_\psi) - \beta(J_\psi)\cos\psi. \end{aligned}$$

We assume that $\beta > 0$—if it isn't we define $\psi' = \psi + \pi$ and then $\beta' = -\beta$ will be positive. Note that $\beta/|\alpha| \simeq m_J a_a / m_s a_J$ is small.

For some value J_0 of J_ψ near the asteroid's action the unperturbed orbit is perfectly resonant, i.e. $0 = \dot\psi = \partial H_0/\partial J_\psi = {\rm d}\alpha/{\rm d}J_\psi$. So Taylor expanding α in powers of $\Delta \equiv J_\psi - J_0$ we have

$$H(\Delta, \psi) = A + \frac{1}{2}B\Delta^2 - \beta\cos\psi,$$

[1] We can arrange for the phase of the cosine to be zero by choosing our origin of time intelligently.

where A and B are constants. Since β is small, $\beta(J_\psi) \simeq \beta(J_0)$ to a sufficient approximation and the equations of motion are approximately

$$\begin{array}{ll} \dot{\psi} & = B\Delta \\ \dot{\Delta} & = -\beta \sin\psi \end{array} \qquad \Rightarrow \quad \ddot{\psi} = -B\beta \sin\psi.$$

This is the equation of motion of a pendulum. Two qualitatively different motions are possible: either the pendulum circulates in a constant sense because it has enough energy to carry it over top dead center, or it lacks this critical energy and swings to and fro. Quantitatively it circulates when

$$\text{K.E. at bottom} = \frac{1}{2}\dot{\psi}^2 > 2B\beta = \text{P.E. difference between top and bottom.}$$

When ψ is oscillating around $\psi = 0$, one says that the asteroid is **librating** in w_3. There are asteroids trapped in this way for $N_1/N_2 = 1$ (**Trojan asteroids**) and 3/2 (**Hilda asteroids**). Other important resonances are marked by **Kirkwood gaps**.

Resonance Protection An asteroid on a librating resonant orbit will never have a close encounter with with Jupiter. For example the Hilda asteroids have $3\lambda_J - \varpi - 2\lambda_a \approx 0$. Therefore when ever a Hilda is in conjunction with Jupiter ($\lambda_J = \lambda_a$), it is at perihelion, and far from Jupiter.

Order of resonance The ratio N_1/N_3 is frequently expressed as $(p+q)/q$, where q is the order of the resonance. The leading term in H_1 is proportional to e^q.

Each stable resonant orbit is surrounded by a family of trapped librating orbits—the stable orbit is that on which $\psi = 0$ and the librating orbits are those on which the pendulum swings to and fro. Chaos ensues when two or more families of neighboring resonances claim the allegiance of the same orbits, that is, when resonances 'overlap'. Whether or not resonance overlap occurs depends on (i) the magnitude of the $h_{\mathbf{n}}$, (which determine the width of the resonances), and (ii) the nearness of the resonances.

The ring systems of Saturn and Uranus are profoundly affected by the flattenings of the planets, which depress ω_1 below zero and raise ω_2 above zero.

Model the effect of the planet's flattening by adding to the regular monopole potential $-GM/r$ a quadrupole term $\alpha GM(3z^2 - r^2)/r^5$. Verify these statements about ω_1 and ω_2 by showing that the radial and vertical epicycle frequencies are given by

$$\kappa_r^2 = \Big(1 - 3\frac{\alpha^2}{r^2}\Big)\frac{GM}{r^3} \quad ; \quad \kappa_z^2 = \Big(1 + 9\frac{\alpha^2}{r^2}\Big)\frac{GM}{r^3}.$$

Around Saturn the resonances are sufficiently well separated for order to be the rule. Around Uranus many important resonances overlap and chaos is widespread.

5.7.2 Numerical integration and Mappings

Despite having an elegant formalism for working with perturbations, the theory gets very messy, and estimating the strength of competing resonances can be quite tedious. Therefore, Numerical integrations have recently come to the forefront to investigate the structure of phase space. A numerical integration can be thought of as a transformation

$$(\mathbf{q}_0, \mathbf{p}_0) \rightarrow (\mathbf{q}_\tau, \mathbf{p}_\tau),$$

and this transformation is repeated. In order to preserve the structure of phase space, this transformation must be canonical or symplectic: $(\mathbf{q}_0, \mathbf{p}_0)$ are related to $(\mathbf{q}_\tau, \mathbf{p}_\tau)$ by an appropriate generating function.

Now consider our Hamiltonian of

$$H = H_K + H_1.$$

Hamilton's equations are

$$\dot{\mathbf{z}} = \{\mathbf{z}, H_K + H_1\},$$

where $\mathbf{z}$ is $(\mathbf{q}, \mathbf{p})$. Introducing the operators

$$K \equiv \{, H_K\} \quad \text{and} \quad P \equiv \{, H_1\},$$

the formal solution of the above is

$$\mathbf{z}(\tau) = \exp[\tau(K + P)]\mathbf{z}(0).$$

We can approximate the true evolution operator, $\exp[\tau(K+P)]$ as a product of terms like $\exp(\tau K)$ and $\exp(\tau P)$ via the Baker-Campbell-Hausdorff (BCH) identity:

$$\exp K \exp P = \exp\left(K + P + \frac{1}{2}[K,P] + \frac{1}{12}[K-P,[K,P]] + \langle \text{more terms} \rangle\right).$$

Consider

$$\exp(\tau K)\exp(\tau P) = \exp[\tau(K+P) + \langle \text{error} \rangle].$$

This says that by stepping the system under H_K, then stepping the system under H_1 we are evolving the system under the Hamiltonian

$$H_K + H_1 + H_{\text{err}}.$$

In this case, H_{err} is of order τ, so this is a first order integration scheme. Considering more terms in the BCH identity will give you higher order integrators. Since H_K is integrable, if we can approximate H_1 with something that is also integrable, we can take large steps in τ and still preserve the structure of phase space. Hence we have a tool to calculate surfaces of section using an *Area preserving map*.

In particular, consider the case where

$$H_K = \frac{p^2}{2}$$

and

$$H_1 = \Phi(x).$$

Note how $\exp(\tau K) \equiv x_\tau = x_0 + \tau p$ and $\exp(\tau P) \equiv p_\tau = p_0 - \tau \frac{\partial \Phi}{\partial x}$. Hence we have derived the basics of Leap Frog. In more detail, the operator combination $\exp(\tau K/2)\exp(\tau P)\exp(\tau K/2)$ eliminates the next order error term.

6 The epicycle approximation

Expand Φ_{eff} in a Taylor series about its minimum:

$$\Phi_{\text{eff}} = \text{const} + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2 + \cdots,$$

where

$$x \equiv R - R_g, \quad \kappa^2 \equiv \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\right|_{(R_g,0)}, \quad \nu^2 \equiv \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2}\right|_{(R_g,0)}.$$

The equations of motion decouple and we have two integrals:

$$x = X\cos(\kappa t + \phi_0) \qquad z = Z\cos(\nu t + \zeta)$$
$$E_R \equiv \frac{1}{2}[v_R^2 + \kappa^2(R - R_g)^2] \qquad E_z \equiv \frac{1}{2}[v_z^2 + \nu^2 z^2].$$

Now compare the **epicycle frequency**, κ, with the angular frequency, Ω.

$$\Omega^2 \equiv \frac{v_c^2}{R^2} = \frac{1}{R}\frac{\partial\Phi}{\partial R} = \frac{L_z^2}{R^4},$$
$$\kappa^2 = \frac{\partial(R\Omega^2)}{\partial R} + \frac{3L_z^2}{R^4} = R\frac{\partial\Omega^2}{\partial R} + 4\Omega^2.$$

Since Ω always decreases, but never faster than Keplerian,

$$\Omega \le \kappa \le 2\Omega.$$

The epicycle approximation also makes a prediction for the ϕ-motion since $L_z = R^2\dot{\phi}$ is conserved. Let

$$y \equiv R_g[\phi - (\phi_0 + \Omega t)]$$

be the displacement in the ϕ direction from the "guiding center". If we expand L_z to first order in displacements from the guiding center, we obtain

$$\phi = \phi_0 + \Omega t - \frac{2\Omega X}{\kappa R_g}\sin(\kappa t + \phi_0).$$

Therefore

$$y = -Y\sin(\kappa t + \phi_0) \quad \text{where} \quad \frac{Y}{X} = \frac{2\Omega}{\kappa} \equiv \gamma \ge 1.$$

$\Rightarrow$ The epicycles are elongated tangentially.

Now eliminate $(R - R_g)$ from E_R. Let $\mathbf{v} \equiv \mathbf{v}' - \mathbf{v}_{LSR}$ be the velocity w.r.t. the Local Standard of Rest. Then

$$\begin{aligned} v_\phi(R_0) &\equiv v'_\phi(R_0) - v_c(R_0) = R_0(\dot{\phi} - \Omega_0) \\ &= R_0(\dot{\phi} - \Omega_g + \Omega_g - \Omega_0) \\ &\simeq R_0\left[(\dot{\phi} - \Omega_g) - \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0} x\right]. \end{aligned}$$

Replacing $\dot{\phi}$ by $\Omega_g(1-2x/R_g)$ this becomes

$$\begin{aligned} v_\phi &\simeq -R_0 x\left[\frac{2\Omega_g}{R_g}+\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0}\right] \simeq -x\left[2\Omega_0+R_0\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0}\right] \\ &\equiv 2Bx. \end{aligned}$$

So $E_R \simeq \frac{1}{2}[v_R^2+\gamma^2 v_\phi^2]$ where $\gamma = \frac{1}{2}\kappa/B = 2\Omega_g/\kappa$. So E_R looks like a kinetic energy of peculiar motion.

7 Potentials

- Most of the mass of stellar systems is in point-like objects.

  (Stars! How surprising!)

- Calculating the potential by summing individual contributions is impractical, since it involves a sum over N terms where N is large.

  However, the fact we have many stars means that we can make the approximation that:

$\Rightarrow$ stellar systems can be represented by smooth density fields generating smooth force fields.

The smooth density will be everywhere proportional to the local star density. I will talk about the validity of this approximation later, when we consider the orbits of stars and evolution of systems.

The next several lectures will be spent discussing forces and the details of their calculation.

7.1 Poisson's equation

The relationship between forces and the underlying density distribution is described by Poisson's equation, which can be derived from the Newtonian force low as follows:

To calculate the force on a unit mass at at a point $\mathbf{x}$, we sum over all the contributions from each element $\delta^3\mathbf{x}'$:

$$\delta\mathbf{F}(\mathbf{x}) = G\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\delta m(\mathbf{x}') = G\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3}\rho(\mathbf{x}')\delta^3\mathbf{x}'.$$

This gives

$$\mathbf{F}(\mathbf{x}) = G \int \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \rho(\mathbf{x}') \mathrm{d}^3\mathbf{x}'.$$

This could be the end of things, but this is not convenient to calculate.

If we introduce the gravitational potential which we define as:

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \mathrm{d}^3\mathbf{x}',$$

and note that

$$\nabla_{\mathbf{x}} \left(\frac{1}{|\mathbf{x}' - \mathbf{x}|} \right) = \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3},$$

(so $\nabla_{\mathbf{x}} \Phi$ looks like $\mathbf{F}$) we have

$$\mathbf{F}(\mathbf{x}) = \nabla \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \mathrm{d}^3\mathbf{x}' = -\nabla\Phi.$$

Deriving the force from a potential field has several advantages:

- It constrains the force field to be conservative. (The work required to get a mass from one position to another is independent of the path, or $\int \mathbf{F}\dot{\mathrm{d}}\mathbf{x}$ is path independent.) Also note that we can't choose an arbitrary $\mathbf{F}$.
- The scalar field, Φ is easier to visualize than a vector field.
- A scalar field is often easier to calculate than a vector field.

  (1/3 the work).

The divergence of the force is:

$$\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) = G \int \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) \rho(\mathbf{x}') \mathrm{d}^3\mathbf{x}'.$$

But (from the product rule)

$$\nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \right) = -\frac{3}{|\mathbf{x}' - \mathbf{x}|^3} + \frac{3(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^5}.$$

which is 0 for $\mathbf{x}' \neq \mathbf{x}$. So we can restrict the volume of integration to an arbitrarily small sphere (of radius h) about $\mathbf{x}$. We can take $\rho(\mathbf{x}')$ out of the integral and we have

$$\begin{aligned}
\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) &= G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}|\leq h} \nabla_{\mathbf{x}} \cdot \left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \right) \mathrm{d}^3\mathbf{x}' \\
&= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}|\leq h} \nabla_{\mathbf{x}'} \cdot \left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^3} \right) \mathrm{d}^3\mathbf{x}' \\
&= -G\rho(\mathbf{x}) \int_{|\mathbf{x}'-\mathbf{x}|=h} \frac{(\mathbf{x}'-\mathbf{x}) \cdot \mathrm{d}^2\mathbf{S}'}{|\mathbf{x}'-\mathbf{x}|^3},
\end{aligned}$$

where we have replaced a divergence with respect to $\mathbf{x}$ with a divergence with respect to $\mathbf{x}'$, and used the divergence theorem to replace a volume integral with a integral over the enclosing surface. Now on the surface, we have $|\mathbf{x}'-\mathbf{x}| = h$ and $\mathrm{d}^2\mathbf{S}' = (\mathbf{x}'-\mathbf{x})h\mathrm{d}^2\Omega$. So

$$\nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{x}) = -G\rho(\mathbf{x}) \int \mathrm{d}^2\Omega = -4\pi G\rho(\mathbf{x}).$$

Substituting Φ for $\nabla \cdot \mathbf{F}$ we have **Poisson's equation**:

$$\nabla^2\Phi = 4\pi G\rho.$$

That was a bit of work, but I wanted to go through the details so that I could point out the following:

Note that the derivation of this equation stems from the facts that 1) the gravitational force is conservative (so we can define a Φ) and 2) the central $1/r^2$ nature of the force between two mass elements, so that the divergence of the force is zero in a vacuum. So, if we wished to calculate the potential of point masses that do not attract each other with $1/r^2$ forces (say Plummer spheres), Poisson's equation would NOT hold if ρ is proportional to the mass density of the objects. Rather $\rho(x)$ must be the mass density at $\mathbf{x}$ assuming all the particles' mass is smoothed out appropriately to account for non-$1/r^2$ forces. Possibly a trivial point, but one that has been missed by some clever people.

Integrating Poisson's equation over an arbitrary volume gives

$$4\pi G \int \rho \mathrm{d}^3\mathbf{x} = 4\pi GM = \int \nabla^2\Phi \mathrm{d}^3\mathbf{x} = \int \nabla\Phi \cdot \mathrm{d}^2\mathbf{S}.$$

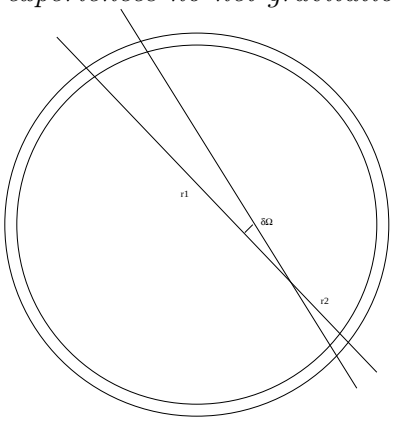
where the divergence theorem is used. This is **Gauss's theorem**: *the integral of the normal component of* $\nabla\Phi$ *over any closed surface equals* $4\pi G$ *times the mass contained within that surface.*

Again, this depends on $F \sim 1/r^2$. In E&M, one uses this to actually test the $1/r^2$ dependence.

7.2 Spherical Systems

This might be better demonstrated with Gauss' theorem.

Newton's first theorem: *A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.*



Consider the section of the shell enclosed by a cone centered at an arbitrary interior point. The mass contained δm_1 is proportional to r_1^2 and the force on the point is proportional to $\delta m_1/r_1^2$. Therefore,

$$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}.$$

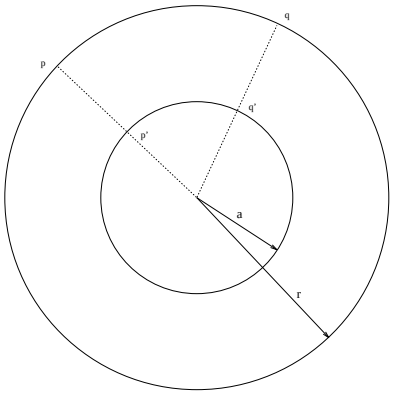
Summing over all solid angles, there is no net force on an interior point due to the spherical shell.

Also note that since $\nabla\Phi = -\mathbf{F} = 0$, Φ is a constant within the shell. Using the definition of Φ at the center of the sphere we have

$$\Phi = \frac{-GM}{R}$$

within the sphere.

Newton's second theorem: *The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.*



Compare the potential at point p due to a patch at point q' on a shell of radius a of mass M:

$$\delta\Phi = -\frac{GM}{|\mathbf{p}-\mathbf{q}'|}\frac{\delta\mathbf{\Omega}}{4\pi}$$

with the potential at point p' due to a patch at point q on a shell of radius r of mass M:

$$\delta\Phi' = -\frac{GM}{|\mathbf{p}'-\mathbf{q}|}\frac{\delta\mathbf{\Omega}}{4\pi}.$$

By symmetry $\delta\Phi = \delta\Phi'$, and by summing over q and q', $\Phi = \Phi'$. But we just calculated that $\Phi' = -GM/r$, so $\Phi = -GM/r$ which is exactly the potential if all the mass of the shell was concentrated at the center.

A similar theorem can be proved for Homeoids namely: a homeodial shell produces isopotential surfaces which are spheroids confocal with the shell, and a constant interior potential.

Since the potentials add linearly, we can easily calculate the potential at any point in a spherical density distribution by separately calculating contributions from the interior and exterior parts:

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2\mathrm{d}r' + \int_r^\infty \rho(r')r'\mathrm{d}r'\right].$$

The first term is the interior mass taken to be at the center, and the second is a sum over the potentials due to exterior shells.

The force at a point r is entirely due to the mass inside r:

$$\mathbf{F}(r) = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\hat{\mathbf{e}}_r = -\frac{GM(r)}{r^2},$$

where $M(r)$ is the mass interior to r.

A quantity of interest is velocity of a test particle on a circular orbit, or the **circular speed**, v_c. Setting the centripetal acceleration equal to the force we get

$$v_c^2 = r\frac{\mathrm{d}\Phi}{\mathrm{d}r} = r|\mathbf{F}| = \frac{GM(r)}{r}.$$

So the circular speed is a measure of the mass interior to r. Now we have something we can use: if you tell me what v_c is as a function of r for a galaxy, and I can assume it is spherical, I can tell you what the mass is as a function of r. (Not the case for a non-spherical distribution.)

Another quantity is the **escape speed**, v_e defined by

$$v_e(r) = \sqrt{2|\Phi(r)|}.$$

This definition comes from setting the kinetic energy of a star equal to the absolute value of its potential energy. That is, stars with positive total energy are not bound to the system. In order for a star to escape from from the gravitational field represented by Φ, it is necessary that its speed be greater than v_e. This can be used to get the local Φ of the galaxy.

Some Simple Spherical Systems

Point mass:

$$\Phi(r) = -\frac{GM}{r} \quad ; \quad v_c(r) = \sqrt{\frac{GM}{r}} \quad ; \quad v_e(r) = \sqrt{\frac{2GM}{r}}.$$

Whenever the circular speed declines as $r^{1/2}$ it is referred to as **Keplerian**. It usually implies that there is no significant mass at that radius.

Homogeneous sphere:

$$M = \frac{4}{3}\pi r^3\rho \quad ; \quad v_c = \sqrt{\frac{4\pi G\rho}{3}}r.$$

The equation of motion for a particle in such a body is

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3}r,$$

which is the equation of motion for a harmonic oscillator with period

$$T = \sqrt{\frac{3\pi}{G\rho}}.$$

Independent of r, if a particle is started at r, it will reach the center in a time

$$t_{dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}},$$

known as the **dynamical time**. Although this result is only true for a homogeneous sphere, it is common practice to use this definition with any system of density ρ.

By integrating the density, we can get the potential:

$$\Phi = \begin{cases} -2\pi G\rho(a^2 - \frac{1}{3}r^2), & r < a \\ -\frac{4\pi G\rho a^3}{3r}, & r > a. \end{cases}$$

One would expect the center of a galaxy to have a potential of this type if there is no cusp in the central density. (implying a linear rise in v_c.

Some spherical potentials in common use include variations of the 2 power law profile

$$\rho(r) = \frac{\rho_0}{(r/a)^\alpha(1 + r/a)^{\beta-\alpha}}$$

where the inner power law slope is $-\alpha$ and the outer power law slope is $-\beta$. In particular the Hernquist profile:

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^3}$$

is used for Elliptical galaxies and galactic nuclei, while the NFW profile

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^2}$$

is used for dark matter halos.

7.3 Flattened Systems

Following are a few simple models that can be used to illustrate the dynamics of axisymetric galaxies.

Logarithmic potentials:

If we wish to model real galaxies, we need a potential where the circular speed is a constant at large radii. If $v_c = v_0$, a constant, then $\mathrm{d}\Phi/\mathrm{d}R \propto 1/R$, therefore $\Phi \propto v_0^2 \ln R + C$. So consider

$$\Phi = \frac{1}{2}v_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q_\Phi^2}\right) + \text{constant}.$$

where $q_\Phi \leq 1$ for oblate potentials. Poisson's equation gives:

$$\rho = \left(\frac{v_0^2}{4\pi G q_\Phi^2}\right) \frac{(2q_\Phi^2+1)R_c^2 + R^2 + 2(1-\frac{1}{2}q_\Phi^{-2})z^2}{(r_c^2 + R^2 + z^2 q_\Phi^{-2})^2}.$$

The density asymptotes to R^{-2} or z^{-2}. Note that this implies an infinite mass. This potential also gives a drastic example of a general phenomenon: the density distribution is much flatter than the potential distribution. In this case, the density can even go negative if $q_\Phi < 1/\sqrt{2}$. (giving unphysical Φ.)

Poisson's equation for thin disks:

In an axisymetric system Poisson's equation is

$$\frac{1}{R}\frac{\partial}{\partial R} R\left(\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G\rho.$$

Note that if the density is concentrated in the plane, both ρ and the second derivative w.r.t. z will get very large while the second derivative w.r.t R remains well behaved. For a thin disk, therefore, Poisson's equation simplifies to

$$\frac{\partial^2 \Phi(R,z)}{\partial z^2} = 4\pi G\rho(R,z).$$

So in the thin disk approximation one can first determine the potential in the plane of the disk $\Phi(R,0)$, and then at each radius solve for the vertical structure.

For useful models of the Milky Way see Dehnen and Binney 1998.

7.4 Multipole Expansions

Now lets talk about how to solve Poisson's equation for a more general density distribution. Poisson's equation is a linear equation, so if we find a complete set of orthogonal functions ψ_n such that

$$\nabla^2 \psi_n = \lambda_n \psi_n,$$

where orthogonal means

$$\int \psi_n^* \psi_m w(\mathbf{x}) \mathrm{d}^3\mathbf{x} = \delta_{nm},$$

where $w(\mathbf{x})$ is some weight function. we have an eigenvalue problem and the potential for any arbitrary mass distribution is easily found. The density is decomposed into a sum over the eigenfunctions:

$$\rho(\mathbf{x}) = \sum_n C_n \psi_n(\mathbf{x}),$$

where

$$C_n = \int \psi_n^* \rho \mathrm{d}^3\mathbf{x}.$$

The potential is now just

$$\Phi(\mathbf{x}) = 4\pi G \sum_n \frac{C_n}{\lambda_n} \psi_n(\mathbf{x}).$$

and we are done.

For example, in Cartesian coordinates,

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{i}(k_x x + k_y y + k_z z)}$$

are eigenfunctions of the Laplacian, with eigenvalues $-|\mathbf{k}|^2$. So that if we decompose the density field into its Fourier modes, $\rho(\mathbf{k})$, then the potential is just

$$\Phi = \sum_{\mathbf{k}} -\frac{4\pi G \rho(\mathbf{k}) e^{\mathrm{i}(k_x x + k_y y + k_z z)}}{k^2}.$$

Now this procedure is usually only practical if the density distribution in question can be reasonably approximated by a small number of eigenfunctions. For the case of a nearly spherical mass distribution, we certainly have a hope of this if we look at Poisson's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = \lambda \psi.$$

Using the method of separation of variables we can split this up into three equations, one in each of the independent variables. To do this we assume ψ will be a product of eigenfunctions of the individual equations.

$$\psi(r, \theta, \phi) = R(r) P(\theta) Q(\phi).$$

Substituting this in, we have

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda r^2 \sin^2\theta + \frac{\sin\theta}{P}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}P}{\mathrm{d}\theta}\right) = -\frac{1}{Q}\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2}.$$

The left side of the equation does not depend on ϕ, and the right side does not depend on r or θ, so both sides must be equal to a constant which we will, with foresight, guess as m^2. So

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}\phi^2} = -m^2 Q,$$

whose eigenfunctions are easily seen as

$$Q(\phi) = \sum_m e^{\mathrm{i}m\phi}.$$

Where m must be an integer for single valued functions.

Rewriting the other side of the equation, we have

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \lambda r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{P\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right).$$

Again, we can set both sides equal to a constant, $l(l+1)$. In terms of $x \equiv \cos\theta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right] - \frac{m^2}{1-x^2}P = -l(l+1)P,$$

the eigenfunctions of which are associated Legendre functions, $P_l^{|m|}$, with eigenvalues, $l(l+1)$. The combination of Legendre functions and circular functions are referred to as spherical harmonics, $Y_l^m(\theta,\phi)$, where

$$\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi Y_l^{m*}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}.$$

The eigenfunctions of the radial equation are spherical Bessel functions, $j_l(kr)$ with eigenvalues $-k^2$.

In practice, the radial eigenfunctions are rarely used since they do a very poor job of representing the radial density distribution of a galaxy, but since many galaxies are nearly spherical, the spherical harmonics can be of considerable practical use.

However, we can expand the radial part in any set of functions for which we can solve the radial part of Poisson's equation.

A more practical way to proceed is to consider the potential of a spherical shell. In this case, everywhere except on the shell, we have Laplace's equation: $\nabla^2\Phi = 0$. The solution to this equation is very similar to Poisson's equation except that the radial equation is now:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - l(l+1)R = 0.$$

The simplest solutions of which are

$$R(r) = Ar^l \quad \text{and} \quad Br^{-(l+1)}.$$

Note that spherical Bessel functions reduce to these in the limit of large k.

Now we expect the potential for a thin shell to be finite in its interior so it should be comprised of terms that look like:

$$\Phi_{\mathrm{int}}(r,\theta,\phi) = Ar^l Y_l^m(\theta,\phi).$$

On the other hand externally, the field should fall to zero. So

$$\Phi_{\mathrm{ext}}(r,\theta,\phi) = Dr^{-(l+1)} Y_l^m(\theta,\phi).$$

The interior and the exterior solutions can be matched at the shell by requiring that the potential be continuous and matching the gradient of the potential using Gauss's theorem on a small volume around the shell. We can therefore find the potential at any point by summing the exterior potentials of all shells interior to that point, and summing the interior potentials of all shells exterior to that point:

$$\begin{aligned}\Phi(r,\theta,\phi) &= -4\pi G\sum_{l,m}\frac{Y_l^m(\theta,\phi)}{2l+1}\left[\frac{1}{r^{(l+1)}}\int_0^r \rho_{lm}(r')r'^{(l+2)}\mathrm{d}r'\right.\\ &\qquad \left. + r^l\int_0^\infty \rho_{lm}(r')\frac{\mathrm{d}r'}{r'^{(l-1)}}\right].\end{aligned}$$

Note in particular the exterior potential: the monopole declines as $1/r$, the dipole ($l = 1$) as $1/r^2$ and the quadrapole ($l = 2$) as $1/r^3$. If we expand about the center of mass, the dipole will be zero, and the correction to the monopole potential gets very small with increasing r. If the quadrapole is very large, we can have super-keplerian falloff.

7.5 Designer basis functions

See Saha, 1993, MNRAS 262, 1062.

We are not forced to use orthogonal functions. If we choose any linearly independent ψ_n, then we can decompose the potential as

$$\Phi(\mathbf{x}) = \sum_n C_n \psi_n(\mathbf{x}),$$

where the C_n satisfy

$$\sum_n M_{mn} C_n = 4\pi \int \psi_m^*(\mathbf{x}) \rho \mathrm{d}^3\mathbf{x},$$

where

$$M_{mn} = \int \psi_m^* \nabla^2 \psi_n \mathrm{d}^3\mathbf{x}.$$

The M_{mn} need only be calculated once for any set of ψ_n.

A reasonable scheme is choose

$$\psi_{klm} = F_{kl}(r) Y_{lm}(\theta, \phi).$$

Now let

$$F_{kl} = W_l(r) U_k(r)$$

Where $W_l(r)$ is a reasonable approximation to the radial dependence, and $U_k(r) = (u(r))^k$.

One example would be:

$$W_l = \frac{r^l}{(1+r)^{2l+1}}, \quad u = \frac{r-1}{r+1}.$$

7.5.1 Sturm-Liouville solvers

See Weinberg, M. 1999, Ap J. 117, 629

Given an arbitrary radial density-potential pair one can construct:

$$\Phi(r, \theta, \phi) = \Phi_0(r) u(r) Y_{lm}(\theta, \phi)$$

and

$$\rho(r, \theta, \phi) = \rho_0(r) u(r) Y_{lm}(\theta, \phi)$$

Where Φ_0 and ρ_0 are a known potential density pair.

When you plug these into the radial part of Poisson's equation, you now have an eigenvalue problem for $u(r)$, the Sturm-Liouville equation. One can then use SLE solvers to get the eigenfunctions of this equation, the first of which is just a constant.

7.6 Tree codes

Lets go back to our original definition of Φ:

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \mathrm{d}^3\mathbf{x}',$$

One can Taylor expand this about $x' = 0$ and get:

$$\Phi(\mathbf{x}) = -G \int d^3x' \left[\frac{\rho(x')}{|x|} + \frac{x'\rho(x') \cdot x}{|x|^3} + \frac{(3x_i'x_j' - |x'|^2\delta_{ij})\rho(x')x_ix_j}{2|x|^5} + \cdots \right]$$

Note that

$$\int \rho(x') = M$$

etc.

This expansion converges quickly for $x >> x'$, since the error scales as $(x'/x)^{l+1}$. Also note that these are the cartesian forms of the Y_{lm}.

This leads to the tree-code idea: if a collection is far away compute the potential from multipoles. If it is close by, break it into pieces.

8 Equilibrium Models

To complete the loop, we need to follow an ensemble of orbits. The first question is what happens to an ensemble of orbits with certain initial conditions under time evolution. For this we have:

8.1 Liouville's theorem

Liouville's theorem tells us how a volume of phase space changes with time. In fact, it is an incompressible fluid. This can be shown following the treatment I introduced for Numerical Mappings. That is, Hamilton's Equations,

$$\dot{\mathbf{z}} = \{\mathbf{z}, H\}$$

define a flow in phase space, and a flow operator K^t maps a phase space point forward in time $\mathbf{z}(0) \rightarrow \mathbf{z}(t)$. We wish to show for a phase space volume Γ if

$$K^t\langle \text{volume of } \Gamma \rangle = \langle \text{volume of } \Gamma \rangle.$$

This is simple to prove. Consider K^t as $t \to 0$. We have

$$K^t(\mathbf{z}) = \mathbf{z} + t\{\mathbf{z}, H\} + O(t^2)$$

It is fairly intuitive that if $\nabla \cdot \{\mathbf{z}, H\} = 0$ then the flow preserves volume. Explicitly, we have the volume of Γ:

$$V(t) = \int_{\Gamma(0)} d\mathbf{z} \left| \frac{\partial K^t(\mathbf{z})}{\partial \mathbf{z}} \right|.$$

The Jacobian of the transformation can be evaluated from the expansion:

$$\frac{\partial K^t(\mathbf{z})}{\partial \mathbf{z}} = \mathbf{I} + \frac{\partial \{\mathbf{z}, H\}}{\partial \mathbf{z}} t + O(t^2)$$

Therefore:

$$V(t) = \int_{\Gamma(0)} d\mathbf{z}(1 + t\nabla \cdot \{\mathbf{z}, H\} + O(t^2))$$

and

$$\frac{dV(t)}{dt} = \int_{\Gamma(0)} d\mathbf{z} \nabla \cdot \{\mathbf{z}, H\}.$$

But from the definition of the Poisson Bracket we have:

$$\nabla \cdot \{\mathbf{z}, H\} = \frac{\partial}{\partial q} \cdot \left(\frac{\partial H}{\partial p} \right) - \frac{\partial}{\partial p} \cdot \left(\frac{\partial H}{\partial q} \right) = 0!$$

Hence phase space volume is conserved (Liouville's theorem). Corollaries:

- Phase trajectories don't cross.
- A boundary in phase space always encloses the same particles as in the initial conditions.

8.2 Liouville equation

We would now like to be able to apply Liouville's theorem to an ensemble of N points.

We start with a distribution function of points in phase space: $f(q, p)$ is defined such that the number of points in a volume dV is $dN = f(q, p)dqdp$. We can use our operator machinery to find out how dN evolves with time:

$$K^t dN = K^t f(q, p, t) K^t dV = f(q, p, t)dV + \{f, H\}tdV$$

so

$$\frac{dN}{dt} = \frac{\partial f(p,q,t)}{\partial t} dV + \{f, H\} dV$$

or

$$\frac{df(q,p)}{dt} = \frac{\partial f(q,p,t)}{\partial t} + \{f, H\}.$$

If particles are neither created or destroyed, then we have $df/dt = 0$ which is *Liouville's equation*. Further note that if f is explicitly time independent (say, f describes an equilibrium distribution) then we have

$$\{f, H\} = 0.$$

Note that this equation also holds for any integral of motion.

8.3 Boltzmann equation

Going from a system of N interacting particles to the description of one particle interacting with all others.

Liouville's equation now takes the form

$$\frac{\partial f^{[N]}}{\partial t} + \left\{ f^{[N]}, H \right\} = 0$$

where $f^{[N]}$ referrs to an N particle distribution function of $6-N$ dimensional phase space.

Assume particle motions are uncorrelated.

Integrate Liouville's equation over all coordinates with indices $(2, 3, 4, \ldots)$ to get an espression that only $f^1 \equiv f$. (See BT. ch 8, ch 7.2 in new edition).

Get collisionless Boltzmann equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0, \quad \text{or} \quad \frac{\partial f}{\partial t} + \{f, H\} = 0,$$

where H is the single particle Hamiltonian with the potential given by the sum of interactions of all other particles.

Let's rewrite it in a from familiar from hydro:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0.$$

Note that this f is NOT the actual star density, but a probability density. The star density in phase space does not obey a CBE like equation, in particular, it does not behave like an incompressible fluid. Observable quantities can be found by integrating f over suitable volumes. ("coarse-graining")

8.4 Moment Equations

Now let us integrate the CBE over all velocities:

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{32}$$

We can define

$$\nu \equiv \int f \mathrm{d}^3\mathbf{v} \quad \text{and} \quad \overline{v}_i \equiv \frac{1}{\nu} \int f v_i \mathrm{d}^3\mathbf{v},$$

which allows (32) to be written

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu \overline{v}_i)}{\partial x_i} = 0. \tag{33}$$

This is just the continuity equation in real space. Now take the first velocity moment of the CBE:

$$\frac{\partial}{\partial t} \int f v_j \mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{34}$$

We can use the divergence theorem to manipulate the last term

$$\int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = -\int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\mathbf{v} = -\int \delta_{ij} f \mathrm{d}^3\mathbf{v} = -\delta_{ij} \nu,$$

which can be substituted into (34) giving

$$\frac{\partial (\nu \overline{v_j})}{\partial t} + \frac{\partial (\nu \overline{v_i v_j})}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0,$$

where

$$\overline{v_i v_j} \equiv \frac{1}{\nu} \int v_i v_j f \mathrm{d}^3\mathbf{v}.$$

This is an equation of momentum conservation, very much like Euler's equation.

It is helpful to define

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v}_i \overline{v}_j.$$

At each point $\mathbf{x}$ the symmetric tensor σ^2 defines an ellipsoid whose principal axes run parallel to σ^2's eigenvectors and whose semi-axes are proportional to the square roots of σ^2's eigenvalues. This is called the **velocity ellipsoid** at $\mathbf{x}$.

8.4.1 Jeans equations for spherical systems

For a spherical system, Jeans equations are derived by starting with the Hamiltonian in spherical coordinates,

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + \Phi(\mathbf{r}),$$

and evaluating the Poisson bracket using the canonical coordinates $r, \theta, \phi, p_r, p_\theta, p_\phi$, and integrating over $dp_r dp_\theta dp_\phi$. Note that $p_r = v_r$, $p_\theta = r v_\theta$ and $p_\phi = r \sin\theta v_\phi$.

Consider observing a galaxy that is rotationally invariant. Then

$$\overline{v_\theta^2} = \overline{v_\phi^2}.$$

The Jeans equation for a spherical system in steady state is

$$\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r} + \frac{\nu}{r}\left[2\overline{v_r^2} - \left(\overline{v_\theta^2} + \overline{v_\phi^2}\right)\right] = -\nu\frac{\mathrm{d}\Phi}{\mathrm{d}r}. \tag{35}$$

We can define

$$\beta \equiv 1 - \frac{\overline{v_\theta^2}}{\overline{v_r^2}},$$

and (35) becomes

$$\frac{1}{\nu}\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r} + 2\frac{\beta\overline{v_r^2}}{r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}.$$

Given that $(\mathrm{d}\Phi/\mathrm{d}r) = GM(r)/r^2$, if we were able to measure $\overline{v_r^2}$, β, and the luminosity density ν, we could derive the mass $M(r)$ interior to r from

$$M(r) = -\frac{r\overline{v_r^2}}{G}\left(\frac{\mathrm{d}\ln\nu}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln\overline{v_r^2}}{\mathrm{d}\ln r} + 2\beta\right).$$

Comparing this with $L(r)$ might give us evidence for a massive black hole at the center of a galaxy with an active nucleus. However, we can only measure $I(R)$, the surface brightness, and the line-of-sight velocity dispersion, $\sigma_p(R)$ at a projected radius, R. If $\beta = 0$ then we have

$$I(R) = 2\int_R^\infty \frac{\nu r \mathrm{d}r}{\sqrt{r^2 - R^2}} \tag{36}$$

and

$$I(R)\sigma_p^2(R) = 2\int_R^\infty \frac{\nu\overline{v_r^2}r\mathrm{d}r}{\sqrt{r^2-R^2}}.$$

These are Abel integral equations which can be solved for $\nu(r)$ and $\overline{v_r^2}$. See plots by Sargent et.al. (1978).

However, if we *assume* a constant mass to light ratio, then σ_p is related to β, etc. by

$$I(R)\sigma_p^2(R) = 2\int_R^\infty \left(1-\beta\frac{R^2}{r^2}\right)\frac{\nu\overline{v_r^2}r\mathrm{d}r}{\sqrt{r^2-R^2}}.$$

If we use this to eliminate β from (35), we obtain:

$$I\sigma_p^2 - G\Upsilon R^2\int_R^\infty \frac{\nu L(r)\mathrm{d}r}{r^2\sqrt{r^2-R^2}} = R^2\int_R^\infty \left[2\nu\overline{v_r^2} + \frac{R^2}{r}\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r}\right]\frac{r\mathrm{d}r}{\sqrt{r^2-R^2}}.$$

The left side is completely determined by the observations. (ν can be gotten from (36).) We therefore have a linear integro-differential equation for $\nu\overline{v_r^2}$, which can be solved (Binney & Mamon, 1982).

The problem with the Jeans equations is that we have too many variables (10) and too few equations (4). Taking higher moments of the CBE just makes matters worse.

8.4.2 Jeans equations for axisymmetric systems

Asymmetric drift: empirically $v_a \equiv v_c - \overline{v}_\phi \sim \overline{v_R^2}/120km/s$. Where does this come from?

The Jeans equations for an axisymmetric system in steady state are

$$\begin{aligned}
\frac{\partial(\nu\overline{v_R^2})}{\partial R} + \frac{\partial\nu\overline{v_Rv_z}}{\partial z} + \nu\left(\frac{\overline{v_R^2}-\overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R}\right) &= 0, \\
\frac{\partial(\nu\overline{v_Rv_\phi}}{\partial R} + \frac{\partial(\nu\overline{v_\phi v_z})}{\partial z} + \frac{2\nu}{R}\overline{v_\phi v_R} &= 0, \qquad (37)\\
\frac{\partial(\nu\overline{v_Rv_z})}{\partial R} + \frac{\partial(\nu\overline{v_z^2})}{\partial z} + \frac{\nu\overline{v_Rv_z}}{R} + \nu\frac{\partial\Phi}{\partial z} &= 0.
\end{aligned}$$

At $z=0$, the radial equation can be written for the midplane:

$$\frac{R}{\nu}\frac{\partial(\nu\overline{v_R^2})}{\partial R} + R\frac{\partial\overline{v_Rv_z}}{\partial z} + \overline{v_R^2} - \overline{v_\phi^2} + R\frac{\partial\Phi}{\partial R} = 0,$$

where $\partial\nu/\partial z = 0$ at the midplane.

Now define an azimuthal velocity dispersion

$$\sigma_\phi^2 = \overline{(v_\phi - \overline{v}_\phi)^2} = \overline{v_\phi^2} - \overline{v}_\phi^2.$$

Note that $R\partial\Phi/\partial R = v_c^2$, and we have

$$\sigma_\phi^2 - \overline{v_R^2} - \frac{R}{\nu}\frac{\partial(\nu\overline{v_R^2})}{\partial R} - R\frac{\partial(\overline{v_R v_z})}{\partial z} = v_c^2 - v_\phi^2.$$

Note that from BT's epicycle approximation discussion: $\sigma_\phi^2 \sim 0.45\overline{v_R^2}$. This non-intuitive result is because we are looking at a particular position, not a particular orbit. v_ϕ from inner guiding centers are slowed down, while v_ϕ from outer guiding centers are sped up. The observation of a lag is therefore due to a decrease in the density with radius.

8.5 Virial equation

Just as we took velocity moments of the CBE to obtain the Jeans equations, we can now take spatial moments of the CBE. If we multiply Jeans equation by x_k and integrate over space we obtain

$$\int x_k \frac{\partial(\nu\overline{v}_j)}{\partial t}\mathrm{d}^3\mathbf{x} = -\int x_k \frac{\partial(\nu\overline{v_i v_j})}{\partial x_i}\mathrm{d}^3\mathbf{x} - \int \nu x_k \frac{\partial\Phi}{\partial x_j}\mathrm{d}^3\mathbf{x}. \tag{38}$$

The second term on the right hand side can be identified with something known as the **Chandrasekhar potential energy tensor, W**.

$$W_{jk} = -\int \rho(\mathbf{x}) x_j \frac{\partial\Phi}{\partial x_k}\mathrm{d}^3\mathbf{x},$$

It can be shown that (for Φ being generated by gravity):

$$W_{jk} = -\frac{1}{2}G\int\int \rho(\mathbf{x})\rho(\mathbf{x}')\frac{(x_j' - x_j)(x_k' - x_k)}{|\mathbf{x} - \mathbf{x}'|^3}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x}.$$

From this we can see that

$$\begin{aligned} W &= \mathrm{trace}(\mathbf{W}) = -\frac{1}{2}G\int\rho(\mathbf{x})\int\frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\mathrm{d}^3\mathbf{x}'\mathrm{d}^3\mathbf{x} \\ &= \frac{1}{2}\int\rho(\mathbf{x})\Phi(x)\mathrm{d}^3\mathbf{x}. \end{aligned}$$

So we see that W is just the gravitational potential energy.

If we now introduce the **moment of inertia tensor I**

$$I_{jk} \equiv \int \nu x_j x_k \mathrm{d}^3\mathbf{x}.$$

and a **kinetic energy tensor K** by

$$K_{jk} \equiv \frac{1}{2}\int \nu \overline{v_i v_j} \mathrm{d}^3\mathbf{x},$$

and as As with the velocity moments we can split up **K** into ordered and random parts:

$$K_{jk} = T_{jk} + \frac{1}{2}\Pi_{jk},$$

where

$$T_{jk} \equiv \frac{1}{2}\int \nu \overline{v}_j \overline{v}_k \mathrm{d}^3\mathbf{x} \quad ; \quad \Pi_{jk} \equiv \int \nu \sigma_{jk}^2 \mathrm{d}^3\mathbf{x}, \tag{39}$$

the above can be transformed into the the **tensor virial theorem**

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_{jk} = 2T_{jk} + \Pi_{jk} + W_{jk}. \tag{40}$$

Zwicky's derivation: (Ap. J. 1937, 86, 217)

$$m_i \frac{d^2\vec{x}_i}{dt^2} = \vec{F}_i$$

where $\vec{F}_i$ is the total forces on galaxy i.

Scalar multiplication with $\vec{x}_i$ gives:

$$\frac{1}{2}\frac{d^2}{dt^2}(m_i x_i^2) = \vec{x}_i \cdot \vec{F}_i + m_i \left(\frac{d\vec{x}_i}{dt}\right)^2$$

Summing over all cluster members gives the virial theorem. For Newtonian gravity only, the virial can be converted into the potential energy of the system.

If the system is in steady state, $\ddot{\mathbf{I}} = 0$ and we have the **scalar virial theorem**

$$2K + W = 0. \tag{41}$$

Here, $K \equiv \frac{1}{2}\sum m_i v_i^2$, and $W \equiv \frac{1}{2}\sum m_i \Phi_i$.

That last definition is somewhat non-trivial to get from $\vec{x}_i \cdot \vec{F}_i$:

$$\sum \vec{x}_i \cdot \vec{F}_i = \sum_i \sum_j \frac{G m_j m_i \vec{x}_i \cdot (\vec{x}_j - \vec{x}_i)}{|\vec{x}_i - \vec{x}_j|^3}$$

but the indices are dummies so:

$$\sum_i \vec{x}_i \cdot \vec{F}_i = \sum_i \sum_j \frac{G m_j m_i \vec{x}_j \cdot (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3}$$

Adding the above and dividing by 2 gives:

$$\sum_i \vec{x}_i \cdot \vec{F}_i = \sum_i \sum_j \frac{G m_j m_i (\vec{x}_j - \vec{x}_i) \cdot (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} = \frac{1}{2} \sum_i m_i \Phi(\vec{x}_i).$$

The arguement that $\ddot{\mathbf{I}}$ is on average zero usually starts from considering the time average of a single orbit. $\frac{1}{2}\frac{dI}{dt} = m\vec{v} \cdot \vec{x}$ (the "virial"). If the orbit is bound, then both v and x are bounded and any average time derivative will go to zero for long times.

If E is the energy of the system, we have

$$E = K + W = -K = \frac{1}{2} W.$$

We can therefore estimate the mass of a system in equilibrium as

$$M = \frac{2v^2 r_g}{G}$$

Zwicky used this to estimate the mass of the Coma cluster to be $4.5 \times 10^{13} M\odot$. This sounds small, but he estmated the galaxies to contain 8.5×10^7 stars, and hence the mass to lumisity would be 500, compared to about 3 for the local stellar population. Hence *dark matter*.

8.5.1 Spherical collapse

(Ignore p. 214 of BT!) (page 361 in the 2nd edition).

In the standard picture of galaxy formation, the material that formed our galaxy started out in expansion then turned around and collapsed. Question: what is the size of the galaxy as a function of the turn around radius?

At turnaround, $E = W_t = -1/2GM/r_t$. At virialization, $E = 1/2W = -1/4GM/r_v$. Or $r_v = 1/2 r_t$!

8.6 Jeans' theorems

The CBE states that f is constant along orbits. Therefore f is an integral. Since any function of integrals is an integral, any function of integrals solves the CBE, hence we have

Jeans Theorem Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion, and any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation.

First part is true since f *is* an integral (from Liouville's theorem). More explicitly, assume f depends on a set of independent variables, y_i. Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 = \sum_i \frac{\partial f}{\partial y_i}\frac{\mathrm{d}y}{\mathrm{d}t} = \sum_i \frac{\partial f}{\partial y_i}\{y_i, H\}.$$

Since the y_i are independent, this can only be true of $\{y_i, H\}$ for all the y_i. That is, the y_i must be constants of the motion.

Second part is true because

$$\frac{\mathrm{d}}{\mathrm{d}t} f[I_1(\mathbf{x},\mathbf{v}),\ldots,I_n(\mathbf{x},\mathbf{v})] = \sum_{m=1}^{n} \frac{\partial f}{\partial I_m}\frac{\mathrm{d}I_m}{\mathrm{d}t} = 0.$$

This is useful for toy models. What about real galaxies? More useful is

Strong Jeans Theorem The DF of a steady-state galaxy in which almost all orbits are regular with incommensurable frequencies may be presumed to be a function only of three independent isolating integrals.

The proof is non-trivial, but it basically involves using the fact that if the frequencies are incommensurable, then orbits will completely and uniformly cover their tori. Note the "may be presumed". A more complicated f could give correct results, but you get the same results using a simpler f.

The parts of the proof are:

1. The regular orbits are tori in phase space.

2. The density on a given torus is constant since the motion in angle is linear

3. So we can build a model by the linear combination of these tori.

4. The variables, (labels) we use to distinguish one torus from another are the Actions, of which there are 3, and which are integrals of motion.

An example where this doesn't work: the Solar System, because of the commensurability of frequencies in the Kepler problem.

We will now be looking for self-consistent solutions for f. So as well as satisfying the CBE, (which it does by being a function of the integrals) but also Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho = 4\pi G \int f \mathrm{d}^3\mathbf{v}.$$

It will be convenient to define the relative potential and the relative energy by

$$\Psi \equiv -\Phi + \Phi_0 \quad \text{and} \quad \mathcal{E} \equiv -E + \Phi_0 = \Psi - \frac{1}{2}v^2.$$

Φ_0 is chosen to be the value of Φ at the edge of the galaxy (where $f = 0$).

8.7 Distribution functions $f(E)$

Definitions of Ψ and $\mathcal{E}$. Make drawings.

$f(E)$ and isotropic velocity distribution.

Poisson equation as differential equation for Ψ given f.

If $f = f(\mathcal{E}) = f(\Psi - \frac{1}{2}v^2)$ then the velocity dispersion in the radial direction is given by

$$\overline{v_r^2} = \frac{1}{\rho}\int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi v_r^2 f[\Psi - \frac{1}{2}(v_r^2 + v_\theta^2 + v_\phi^2)].$$

Likewise we have for v_θ^2:

$$\overline{v_\theta^2} = \frac{1}{\rho}\int \mathrm{d}v_r \mathrm{d}v_\theta \mathrm{d}v_\phi v_\theta^2 f[\Psi - \frac{1}{2}(v_r^2 + v_\theta^2 + v_\phi^2)].$$

Note that these only differ by the labeling of the variables. We can therefore conclude that $v_r^2 = v_\theta^2 = v_\phi^2$. In other words, the velocity dispersion tensor

is everywhere isotropic. Using spatial spherical symmetry as well, Poisson's equation becomes

$$\begin{aligned}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) &= -16\pi^2 G\int_0^{\sqrt{2\Psi}} f(\Psi-\frac{1}{2}v^2)v^2\mathrm{d}v \\ &= -16\pi^2 G\int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi-\mathcal{E})}\mathrm{d}\mathcal{E}\end{aligned}$$

One can either assume f and solve for Ψ or assume Ψ and solve for f. We'll do the former first.

8.7.1 Polytropes

A simple form for f is

$$f(\mathcal{E}) = \begin{cases} F\mathcal{E}^{n-3/2}, & \mathcal{E} > 0; \\ 0, & \mathcal{E} \le 0, \end{cases}$$

with F and n constants. Then

$$\rho = 4\pi\int_0^{\infty} f(\Psi-\frac{1}{2}v^2)v^2\mathrm{d}v = 4\pi F\int_0^{\sqrt{2\Psi}}(\Psi-\frac{1}{2}v^2)^{n-3/2}v^2\mathrm{d}v.$$

Substituting $v^2 = 2\Psi\cos^2\theta$, this becomes

$$\rho = c_n\Psi^n \quad (\Psi > 0),$$

where

$$c_n \equiv (2\pi)^{3/2}\frac{(n-\frac{3}{2})!F}{n!}.$$

c_n is finite for $n > \frac{1}{2}$. Putting this into Poisson's equation, we have

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) + 4\pi G c_n\Psi^n = 0.$$

If we rescale the variables according to

$$s \equiv \frac{r}{b}, \quad \psi \equiv \frac{\psi}{\psi(0)},$$

where

$$b \equiv (4\pi G\Psi(0)^{n-1}c_n)^{-1/2},$$

we have the Lane-Emden equation

$$\frac{1}{s^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\frac{\mathrm{d}\psi}{\mathrm{d}s}\right) = \begin{cases} -\psi^n, & \psi > 0; \\ 0, \psi \leq 0. \end{cases} \tag{42}$$

This equation first arose form modeling self-gravitating gas spheres in the context of stellar structure. The adiabatic index γ is related to n by $\gamma = 1+\frac{1}{n}$.

For $n = 5$, the solution to (42) is

$$\psi = \frac{1}{\sqrt{1+\frac{1}{3}s^2}}$$

since

$$\frac{1}{s^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\frac{\mathrm{d}\psi}{\mathrm{d}s}\right) = -\frac{1}{(1+\frac{1}{3}s^2)^{5/2}} = -\psi^5.$$

Plummer sphere ($n = 5$).

$$\rho = \frac{C}{(r_c^2+r^2)^{5/2}}$$

Note that

$$\Psi = \rho^{1/5} \propto \frac{1}{(r_c^2+r^2)^{1/2}}$$

Note that it is a particularly simple form for both the density and the potential. There is a constant density core and an asymptotic density of r^{-5}.

8.7.2 isothermal sphere

In the limit $n \to \infty$ the polytropes asymptote to the isothermal sphere ($\gamma = 1$). The DF is

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}}e^{\mathcal{E}/\sigma^2}.$$

where ρ_1 and σ are constants. The mean-square speed of starts is

$$\overline{v^2} = \frac{\int_0^\infty \exp\left(\frac{\Psi-\frac{1}{2}v^2}{\sigma^2}\right)v^4\mathrm{d}v}{\int_0^\infty \exp\left(\frac{\Psi-\frac{1}{2}v^2}{\sigma^2}\right)v^2\mathrm{d}v} = 2\sigma^2\frac{\int_0^\infty e^{-x^2}x^4\mathrm{d}x}{\int_0^\infty e^{-x^2}\mathrm{d}x} = 3\sigma^2.$$

The parameter σ therefore sets the systems "temperature" according to $\sigma^2 = k_BT/m$. Note that the distribution of velocities is everywhere Maxwellian.

Integrating over velocities, we find

$$\rho = \rho_1 e^{\Psi/\sigma^2},$$

and Poisson's equation can be written

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right) = -4\pi G \rho_1 r^2 e^{\Psi/\sigma^2}.$$

This can be put into dimensionless form by defining the **King radius** by

$$r_0 \equiv \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}}$$

and scaling $\tilde{\rho} \equiv \rho/\rho_0$ and $\tilde{r} \equiv r/r_0$. We then have

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r}^2 \frac{\mathrm{d}\ln\tilde{\rho}}{\mathrm{d}\tilde{r}}\right) = -9\tilde{r}^2\tilde{\rho},$$

which can be solved numerically. For large r the solution asymptotes to

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2},$$

known as the **singular isothermal sphere**. This provides a very good fit to dark halos since the circular velocity is constant. Unfortunately, it has infinite density in the center, and an infinite total mass.

8.7.3 King model

We can fix up the isothermal sphere by truncating it at finite radius. A natural way to do this in terms of the DF is to "lower" f:

$$f(\mathcal{E}) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2}\left(e^{\mathcal{E}/\sigma^2} - 1\right) & \mathcal{E} > 0; \\ 0 & \mathcal{E} \leq 0. \end{cases}$$

This defines **King models** (actually a subset of **Michie models**, which we will discuss later) or **lowered isothermal spheres**. The spatial structure can be determined by integrating over $\mathbf{v}$, then numerically integrating starting from $r = 0$ with $\frac{\mathrm{d}\Psi}{\mathrm{d}r} = 0$ and a given $\Psi(0)$. At some point the density will drop to zero. This is the **tidal radius**. The **concentration** is defined by

$$c \equiv \log_{10}(r_t/r_0).$$

King models can also be paramaratized by $\Psi(0)/\sigma^2$.

8.7.4 Eddington inversion

Given a $\rho(r)$ (and hence, a $\Psi(r)$), can we find an f that generates it? Since Ψ is a monotonic function of r we can write the density as

$$\frac{1}{\sqrt{8}\pi}\rho(\Psi) = 2\int_0^\Psi f(\mathcal{E})\sqrt{\Psi-\mathcal{E}}\mathrm{d}\mathcal{E}.$$

If we differentiate both sides, we have an Abel integral equation, which has the solution

$$\begin{aligned} f(\mathcal{E}) &= \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int_0^\mathcal{E}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}} \\ &= \frac{1}{\sqrt{8}\pi^2}\left[\int_0^\mathcal{E}\frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\right)_{\Psi=0}\right] \end{aligned}$$

In order for the DF to be physical, the integral has to be a monotonically increasing function of $\mathcal{E}$.

8.8 Anisotropic distribution functions

Why L^2 and not L.

If we allow f to also depend on the integral L, we obtain models in which $\overline{v_\theta^2} = \overline{v_\phi^2} \neq \overline{v_r^2}$. A one-parameter family of DFs that generate the same density profile can be obtained by a modification of Eddington's inversion.

Suppose f depends on $\mathcal{E}$ and L only through

$$Q \equiv \mathcal{E} - \frac{L^2}{2r_a^2} = \Psi - \frac{1}{2}v^2\left(1+\frac{r^2}{r_a^2}\sin^2\eta\right),$$

where r_a is a constant and η is the polar angle in velocity space. The integral over velocity is now

$$\rho(r) = 2\pi\int_0^\pi \sin\eta\mathrm{d}\eta\int_0^\Psi f(Q)\frac{\sqrt{2(\Psi-Q)}\mathrm{d}Q}{[1+(r/r_a)^2\sin^2\eta]^{3/2}}.$$

The integral over η can be done giving the relation

$$\sqrt{32}\pi\int_0^\Psi f(Q)\sqrt{\Psi-Q}\mathrm{d}Q = \left(1+\frac{r^2}{r_a^2}\right)\rho(r) \equiv \rho_Q.$$

Now the relation between $f(Q)$ and $\rho_Q(\Psi)$ is the same as that of $F(\mathcal{E})$ and $\rho(\Psi)$ in Eddington's relation. Hence, we can solve for $f(Q)$. For any density distribution $\rho(r)$, we now have a family of DF's which differ by the parameter r_a, an *anisotropy radius*. This are known as Osipkov-Merritt models.

The figure shows models of this family made from a Jaffe profile

$$\rho = \frac{M}{4\pi r_J^3}\frac{r_J^4}{r^2(r+r_J)^2} \quad \Rightarrow \quad \Phi = \frac{GM}{r_J}\ln\left(\frac{r}{r+r_J}\right).$$

Note the velocity dispersion profile is more centrally peaked for more radial velocity dispersion tensors.

8.8.1 Michie models

The more general form of King models are the Michie models with the DF

$$f(\mathcal{E}, L) = \begin{cases} \rho_1(2\pi\sigma^2)^{-3/2}e^{-L^2/(2r_a^2\sigma^2)}[e^{\mathcal{E}/\sigma^2}-1], & \mathcal{E} > 0, \\ 0, & \mathcal{E} \leq 0. \end{cases}$$

This reduces to the King models as $r_a \to \infty$. In both this model and the Osipkov-Merritt models, the velocity dispersion is isotropic in the center, and changes to radial at about r_a.

8.9 Real galaxies

Kuijken and Dubinski (1995, Ap. J. 277, 1341).

Use $E_z \equiv \Psi(R,z) - \Psi(R,0) + \frac{1}{2}v_z^2$ as third integral.

Widrow and Dubinski (2005, Ap. J. 631, 838).

Exponential disk, Hernquist bulge, NFW halo and central supermassive black hole; all with distribution functions. However, halo and bulge DFs are isotropic, in conflict with expectations from cosmological N-body simulations.

8.10 Numerically constructing 3D DFs

8.10.1 N-body simulations as models of stellar systems

Typically $N_{simulation} \neq N_{real}$ by a large factor. So

$$\ddot{\mathbf{x}}_i = \sum_{j\neq i}^{N} -\nabla\phi(\mathbf{x}_i, \mathbf{x}_j)$$

where ϕ is the gravitational potential between two point masses are NOT the equations of motion we should be solving. The difference in N between our model and the actual system should significantly affect the model. Instead we should be solving the CBE:

$$\frac{\partial f}{\partial t} + \{f, H\} = 0; \quad \int f \mathrm{d}^6 z = 1.$$

Now the CBE is a first order non-linear partial differential equation. Such equations (as a hyperbolic second order equation) can be solved by *the method of characteristics*. The characteristics of a partial differential equation is the path in the independent variables along which information propagates. (The most intuitive example of this is the wave equation.) But for the CBE, the characteristics are defined by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = v; \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla\Phi.$$

These are the equations of motion of a particle the potential Φ. Furthermore, we know that f is constant along the characteristics. Therefore each particle can be considered as carrying a piece of f as it follows its trajectory.

The only difficulty is in evaluating Φ. In terms of the distribution function,

$$\Phi(\mathbf{x}) = -GM \int \mathrm{d}^6 z' \frac{f(\mathbf{z}')}{|\mathbf{x} - \mathbf{x}'|}.$$

This integral can be done by Monte Carlo integration. This is based on the result that for any reasonable function $g(\mathbf{z})$,

$$\int \mathrm{d}^6 \mathbf{z} g(\mathbf{z}) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{g(\mathbf{z}_i)}{f_s(\mathbf{z}_i)},$$

where the points $\mathbf{z}_i$ are randomly chosen with sampling probability density, $f_s(\mathbf{z})$. Applying this to the Poisson integral,

$$\Phi(\mathbf{x}) \simeq -\frac{GM}{N} \sum_{i=1}^{N} \frac{f(\mathbf{z}_i)/f_s(\mathbf{z}_i)}{|\mathbf{x} - \mathbf{x}_i|}.$$

So in a conventional N-body simulation, $f_s(\mathbf{z}) = f(\mathbf{z})$, so the particle density represents the underlying phase space density. In this case all the $f(\mathbf{z})$ are equal and one is tempted to imagine that sampling points are real stars, but we really are trying to integrate a partial differential equation, the CBE.

8.10.2 Softening

Actually, the Poisson integral is not well suited to Monte Carlo integration because of the singularity at $\mathbf{x} = \mathbf{x}'$. This causes very large scatter in the estimation of Φ when a sampling point falls particularly close to the singularity. This scatter produces a fluctuation in the potental, $\delta\Phi$, which has two effects.

The change in a particle's energy along its orbit is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial x_i}\dot{x}_i + \frac{\partial E}{\partial v_i}\dot{v}_i + \frac{\partial E}{\partial t}.$$

But remember that Hamilton's equations tell us $\dot{x}_i = \frac{\partial H}{\partial v_i}$ and $\dot{v}_i = -\frac{\partial H}{\mathbf{x}_i}$, so the only term that remains is $\frac{\partial E}{\partial t}$, and the explicit time depence of E is the $\delta\Phi$ from the fluctuations in the potental. So we have that a particles energy is changing as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial \Phi}{\partial t}.$$

The fluctuations in Φ due to discrete sampling will break the conservation of energy of a particle, (likewise, it will break other constants of the motion). This random walk in energy is the same as the two-body relaxation discussed in the beginning of the section on orbits.

If massive and less massive particles are present, the less massive particles will typically recoil from an encounter with more velocity than the massive particle. Hence mass segregation occurs, with the massive particles falling deeper into the potential well.

Both these effects can be lessened by introducing a softening into the inter-particle potential. Note that *all* N-body techniques suffer from two body relaxation, although for some (particularly the FFT and multipole methods) the softening is introduced by the numerical technique, and the effect of two-body encounters are less apparent.

The problem of noise in estimating the potential is exacerbated because of the nature of gravity. Any fluctuations from a smooth density distribution will tend to grow.

8.10.3 Applicability of the CBE

Any orbit integration will involve errors. So the value of f which we measure at some calculated position, $\mathbf{z}$, should really be associated with a true posi-

tion, $\mathbf{z}_t$. If there are large gradients of f then our estimation of f is grossly in error.

Indeed we expect large gradients in f if the phase space has significant chaotic structure, which by both intuition and experience is indeed the case.

Note that using the coarse grain distribution function will not help because it obeys no known equation of motion.

This problem cannot be answered completely satisfactorily, but we have a couple of assurances.

The first is that we usually only interested in moments of f, which are in general insensitive to the fine structure of f.

If we integrate the equations of motion using a symplectic method, we have the reassurance that we are integrating a Hamiltonian system that closely approximates the true one. In fact, Quinlan and Tremaine have shown that it is possible to find an orbit in the true Hamiltonian that closely "shadows" an orbit that is numerically integrated. The "shadow" orbit has only slightly different initial conditions. This indicates that our N-body system can be shadowed by orbits in the underlying collisionless system.

Also see Valluri et al 2010 for evidence that N-body models can accurately preserve the actions of orbits as the potential is adiabatically changed.

8.10.4 Other numerical methods

There are other techniques for generating models from either a DF or density distribution.

If we have f, we can estimate how to populate orbits in energy. Surfaces of constant energy are approximately planes in $\mathbf{J}$ space, and we can move orbits around on these planes and only marginally effect the density distribution.

If we have ρ, we can integrate orbits. If we build up a library of orbits, we can construct a model from this library by requiring that

$$\rho(\mathbf{x}_\beta) = \sum_1^N m_\alpha P_{\alpha\beta}/\Delta_\beta,$$

where $P_{\alpha\beta}$ is the percentage of time that an orbit α spends in the volume Δ_β centered at $\mathbf{x}_\beta$. This problem can be solved as linear programming problem or using a maximum entropy technique.

8.10.5 Made to Measure models

See Syer & Tremaine 1996.

Any observable must be derived from the distribution function from

$$Y_j = \int K_j(z) f(z) d^6 z$$

For an N-body system, where each particle is given a weight, w_i, this becomes

$$y_j(t) = \sum^N w_i K_j(z_i(t)).$$

If we define a goodness of fit parameter

$$\Delta_j \equiv y_j(t)/Y_j - 1$$

then we can create a differential equation (the "force of change") for the weights

$$\frac{dw_i(t)}{dt} = -\epsilon w_i(t) \left[\mu \frac{\partial S}{\partial w_i} - \sum_j \frac{K_j(z_i(t))}{Y_j} \Delta_j(t) \right]$$

to find a model that satisfies the constraints.

8.10.6 Torus construction

See McGill & Binney, 1990.

Suppose you have a potential for which you know the actions and which could be a (poor) approximation for the potential in question. One can find the actual actions using a generating function

$$S(\theta^T, J) = \theta^T \cdot J + \sum S_n(J) \sin(n \cdot \theta^T)$$

One can find the $S_n(J)$ by enforcing $H = const$ on the target torus. This can be done with a minimization.

9 Choice of equilibrium

Remember for collisional systems: form of collision term sets equilibrium distribution. (That is, conservation of momentum and energy.)

Maximum entropy?

$$S \equiv -N \int \mathrm{d}^6 z f \ln f$$

System that maximizes entropy is isothermal sphere!

Can always increase entropy by shrinking core and expanding envelope.

Assume a core-envelope system with M_1, r_1 and M_2, r_2 with $M_2 << M_1$. In this case

$$E_1 = \frac{1}{2}W = -\frac{1}{2}\frac{GM_1^2}{r_1}$$

and

$$E_2 \sim -\frac{GM_1 M_2}{r_2}.$$

Now let's shrink the core by ϵr_1, allowing it to assume a new viral equilibrium. The energy will **decrease** by

$$\Delta E \sim \frac{\epsilon G M_1^2}{r_1}.$$

If this energy goes into the envelope, then its new radius is

$$r_2' \sim -\frac{GM_1 M_2}{E_2 + \Delta E}$$

which can get very large. The velocity dispersion is

$$\sigma_2' \sim \sqrt{\frac{GM_1}{r_2'}},$$

so the phase space volume is

$$\mathcal{V} \sim \sigma_2^3 r_2^3 \sim (GM_1 r_2')^{3/2}.$$

The entropy of the envelope is

$$S \sim N_2 \ln(\mathcal{V}) \sim \frac{3}{2} N_2 \ln(r_2')$$

which can get arbitrarily large. Meanwhile the entropy of the core has changed by only a small amount. Therefore arbitrarily large entropies can be achieved by rearranging the mass while conserving energy.

9.1 Gravi-thermal catastrophy

If we define a temperature as

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}k_B T,$$

where m is the mass of a single star, and k_B is the Boltzmann constant. If we define a mass weighted mean temperature $\overline{T}$, then the total Kinetic Energy of the system of stars is

$$K = \frac{3}{2}Nk_B\overline{T}.$$

But the virial theorem tells us the total energy is $E = -K$, so

$$E = -\frac{3}{2}Nk_B\overline{T},$$

and the heat capacity is

$$C \equiv \frac{\mathrm{d}E}{\mathrm{d}\overline{T}} = -\frac{3}{2}Nk_B$$

which is *negative*. Consider the self gravitating system in contact with a heat bath at the same temperature . If a small amount of heat is transfered to the bath, the temperature of the system will *increase* so there will be a temperature gradient causing more heat to flow from the system to the bath, ad infinitum. On the other hand, if a small bit of heat goes from the bath to the system, the temperature will eventually fall to zero. How fast does this happen?

9.2 Relaxation time

Consider an infinite homogeneous system of identical stars with mass m, and number density n. In the smooth potential approximation, a test star moving through this system will experience no forces. In actuality it get pulled this way and that as it passes individual stars. If the star's velocity is approximately constant, we can easily estimate the impulse it recieves from passing a given star. If the star's velocity is $\mathbf{v}$ and the impact parameter is b then the impulse is

$$|\delta\mathbf{v}_\perp| \simeq \frac{2Gm}{bv}$$

This is approximately the force at the closest approach, Gm/b^2 times the time spent there, b/v. Since the background stars are distributed randomly, the impulses will add incoherently so that their sum after a time t is:

$$\begin{aligned} |\Delta \mathbf{v}_\perp|_t^2 &= \sum |\delta \mathbf{v}_\perp|^2 = \frac{4G^2m^2t}{v^2} \int \frac{nv2\pi b \mathrm{d}b}{b^2} \\ &= \frac{8\pi G^2 m^2 n t}{v} \int \frac{\mathrm{d}b}{b} \end{aligned}$$

where the number of interactions in a time t is $n2\pi bv$ The integral at small b diverges because our approximation breaks down there. The test star's trajectory deviates significantly from a straight line when the impact parameter becomes smaller than $b_{\min} \equiv Gm/v^2$. It is possible to make a more exact calculation, but we're just looking for an order of magnitude estimate, so we can just truncate the integration at $b_{\min}$.

The integral also diverges at large b. This is due to the nature of gravity: that masses from all distances contribute equally to the gravitational force. But clearly b cannot be greater than the charactaristic size of the system, R.

We therefore define the **Coulomb logarithm** by

$$\ln \Lambda \equiv \ln(b_{\max}/b_{\min}) \simeq \frac{Rv^2}{Gm},$$

we conclude that $|\Delta \mathbf{v}_\perp|_t^2 \simeq v^2$ after a time

$$t_{\mathrm{relax}} = \frac{v^3}{8\pi (Gm)^2 n \ln \Lambda}.$$

According to the virial theorem

$$v^2 \simeq \frac{GmN}{R}, \quad \text{where} \quad N \equiv \frac{4}{3}\pi n R^3.$$

So the ratio of the relaxation time to the crossing time, $t_{\mathrm{cross}} \equiv R/v$ is

$$\frac{t_{\mathrm{relax}}}{t_{\mathrm{cross}}} = \frac{v^3 N \frac{4}{3}\pi R^3}{8\pi (v^2 R)^2 \ln \Lambda} \frac{v}{R} = \frac{N}{6 \ln \Lambda}.$$

Note the significance of the logarithm for N-body simulations: far collisions are just as important as near ones, hence increasing softening only logarithmically reduces the relaxation time.

Table of crossing times and relaxation times for astronomical objects.

Object	N	$t_{\rm cross}$	$t_{\rm relax}$	Age
Open Cluster	10^2	10^6 yr	10^7 yr	10^8 yr
Globular Cluster	10^5	10^5 yr	10^9 yr	10^{10} yr
Galactic Nucleus	10^8	10^4 yr	10^{11} yr	10^{10} yr
Galaxy	10^{11}	10^8 yr	10^{17} yr	10^{10} yr
Galaxy Cluster	10^3	10^9 yr	10^{11} yr	10^{10} yr

Note that for galaxies, we can reduce the number of particles easily and not effect relaxation. For globular clusters we have to use particles as stars.

9.3 Choice of equilibrium

How do we set equilibrium? Initial conditions.

If this is the case, how come D.F.s are so close to isothermal (e.g. King models)?

Evolving f: phase mixing and evolving potential: violent relaxation. See Lynden-Bell, 1967, MNRAS 136, 101. Using the time depend scalar virial theorem, he shows that if the variation in galaxy size is of the same order as the size, the relaxation time (calculated in the way we did for two body relaxation, $t_r = (v^2/(d\phi/dt)^2)^{1/2}$) is of order 1/8th(!!!) the oscillation period.

Reason for cosmo simulations.

Initial conditions for cosmo simulations:

CDM:

$$f = \rho(\vec{x})\delta(\vec{v} - \vec{\Psi}(\vec{x}))$$

HDM:

$$f = \rho(\vec{x})\exp((\vec{v} - \vec{\Psi}(\vec{x}))^2/2\sigma^2)$$

These simulations result in the NFW (1997 MNRAS) profile:

$$\frac{\rho(r)}{\rho_{crit}} = \frac{\delta_c}{(r/r_s)(1 + r/r_s)^2}$$

10 Stability

10.1 Jeans Instability

We first study the perturbations in an infinite homogeneous self-gravitating system. The problem here is that there is no equilibrium configuration for

such a system. As known, since Newton, gravity requires everything to collapse on itself (e.g. Einstein's introduction of the cosmological constant). The problem can be noted by looking at Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$. In a homogeneous system, Φ must be a constant by translational symmetry, but if that is the case, $\nabla^2\Phi$ must be zero and the only self consistent solution is $\rho = 0$ everywhere. Therefore, we perpetrate what Binney and Tremaine call the **Jeans swindle** by assuming that Poisson's equation only describes the relation between the perturbed potential and density, and the unperturbed potential is assumed zero everywhere. This can be justified as long as the scale for changes in Φ in the equilibrium system is much larger than the scale of the perturbations.

Note that in an expanding universe, the above does not hold: there is an equilibrium configuration. However, because of the expansion, the growth rates of perturbations are slower than what is calculated below.

The other place where this does not hold is the Einstein static Universe.

10.1.1 Linearized equations of motion

As with most stability analysis we proceed by linearizing the equations of motion by dividing variables into an equilibrium part and a perturbed part which is assumed small. Hence for a collisionless system,

$$f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{x}, \mathbf{v}) + \epsilon f_1(\mathbf{x}, \mathbf{v}, t),$$
$$\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \epsilon\Phi_1(\mathbf{x}, t).$$

Plugging these into the CBE and Poisson's equation, and dropping all terms proportional to ϵ^2, we have the **linearized collisionless Boltzmann equation**

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} - \nabla\Phi_0\cdot\frac{\partial f_1}{\partial \mathbf{v}} - \nabla\Phi_1\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0,$$

and

$$\nabla^2\Phi_1 = 4\pi G\int f_1 \mathrm{d}^3\mathbf{v}.$$

Similarly for a self gravitating barytropic fluid ($p = p(\rho)$), the continuity equation becomes

$$\frac{\partial \rho_1}{\partial t} + \nabla\cdot(\rho_0\mathbf{v}_1) + \nabla\cdot(\rho_1\mathbf{v}_0) = 0.$$

Euler's equation becomes

$$\frac{\partial v_1}{\partial t} + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_0 = \frac{\rho_1}{\rho_0^2}\nabla p_0 - \frac{1}{\rho_0}\nabla p_1 - \nabla\Phi_1,$$

and the equation of state is

$$p_1 = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 \rho_1 \equiv v_s^2 \rho_1.$$

Here we have introduced the sound speed defined by

$$v_s^2 \equiv \left(\frac{\mathrm{d}p(\rho)}{\mathrm{d}\rho}\right)_{\rho_0}.$$

10.1.2 Jeans Instability for a fluid

If we assume ρ_0 is constant and $\mathbf{v}_0 = 0$ and swindle Φ_0 to be zero, we have

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 &= 0, \\
\frac{\partial \mathbf{v}_1}{\partial t} &= -\frac{1}{\rho_0}\nabla p_1 - \nabla\Phi, \\
\nabla^2\Phi_1 &= 4\pi G\rho_1, \\
p_1 &= v_s^2 \rho_1.
\end{aligned}$$

Taking the time derivative of the conservation equation and the divergence of the Euler equation, and eliminating in favor of ρ_1 give us

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 - 4\pi G\rho_0\rho_1 = 0.$$

This is a wave equation, so we expect solutions of the form:

$$\rho_1 = Ce^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}.$$

This satisfies the equation provided the dispersion relation holds:

$$\omega^2 = v_s^2 k^2 - 4\pi G\rho_0.$$

For large k (small wavelength), this reduces to just sound waves, and things are stable, but for some k

$$k^2 < k_J^2 \equiv \frac{4\pi G\rho_0}{v_s^2}$$

the system will be unstable. We define the **Jeans length** to be $\lambda_J = 2\pi/k_J$ or

$$\lambda_J^2 = \frac{\pi v_s^2}{G\rho_0}.$$

Note that this length can be determined by comparing the sound crossing time of a ball of gas (v_s) with the timescale for gravitational collapse, $\sqrt{G\rho_0}$.

The **Jeans Mass** is the mass contained within a sphere of diameter λ_J:

$$M_J = \frac{4\pi}{3}\rho_0(\frac{1}{2}\lambda_J)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi v_s^2}{G\rho_0}\right)^{3/2}.$$

Of note for star formation note what happens as the system starts to collapse. The density gets larger, and, if the system is isothermal, the Jeans mass gets smaller ($M_J \propto \rho^{-1/2}$). Hence smaller regions can become independently unstable as the cloud collapses. On the other hand, if the material is adiabatic, then $M_J \propto \rho^{1+\frac{3}{2}(\gamma-2)}$, and the Jeans mass grows if $\gamma > 4/3$.

10.1.3 Jeans Instability for a stellar system

The CBE and Poisson's equation also admit wave solutions, $f_1 = f_a(\mathbf{v})\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$, $\Phi_1 = \Phi_a \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$, giving the relations:

$$(\mathbf{k}\cdot\mathbf{v} - \omega)f_a - \Phi_a\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0$$
$$-k^2\Phi_a = 4\pi G\int f_a \mathrm{d}^3\mathbf{v}$$

Combining these gives a dispersion relation

$$1 + \frac{4\pi G}{k^2}\int \frac{\mathbf{k}\cdot\partial f_0/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega}\mathrm{d}^3\mathbf{v} = 0.$$

The stability is determined by the dependence of f_0 on $\mathbf{v}$. If we assume a Maxwellian,

$$f_0(\mathbf{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}e^{\frac{1}{2}v^2/\sigma^2}.$$

If we choose a coordinate system such that v_x lies along the $\mathbf{k}$ axis then the dispersion relation becomes

$$1 - \frac{2\sqrt{2\pi}G\rho_0}{k\sigma^3}\int_{-\infty}^{\infty}\frac{v_x e^{-\frac{1}{2}v_x^2/\sigma^2}}{kv_x - \omega}\mathrm{d}v_x = 0.$$

The point that divides stability from instablility is where $\omega = 0$, and this where the integral can be evaluated:

$$k_J^2 = \frac{4\pi G\rho_0}{\sigma^2}.$$

Note that this resembles the Jeans length of a fluid with the substitution of velocity dispersion for v_s.

Features of collisionless systems:

- Transition between stable and unstable happens at the same k as fluid systems.
- There are no sound waves: $Im(\omega)$ is 0 only for $k = k_J$.
- All solutions with $k > k_J$ have $Im(\omega) < 0$: waves are damped by Landau damping.

10.2 Stability of Rotating Sheet

A uniformly rotating sheet is a case where the background potential can be self-consistently solved.

The perturbation equations are

$$\frac{\partial \Sigma_1}{\partial t} + \Sigma_0 \nabla \cdot \vec{v}_1 = 0$$

$$\frac{\partial \vec{v}_1}{\partial t} = -\frac{v_s^2}{\Sigma_0}\nabla\Sigma_1 - \nabla\Phi_1 - 2\vec{\Omega} \times \vec{v}_1$$

$$\nabla^2\Phi_1 = 4\pi G\Sigma_1\delta(z)$$

Assume a solution of the form

$$\Phi_1 = \Phi_k \exp(i(kx - \omega t) - |kz|)$$

Poisson's equation gives:

$$\Phi_1 = -\frac{2\pi G\Sigma_k}{|k|} \exp(i(kx - \omega t) - |kz|)$$

and the perturbation equations become:

$$-i\omega\Sigma_k = -ik\Sigma_0 v_{kx}$$

$$-i\omega v_{kx} = -\frac{v_s^2 ik\Sigma_k}{\Sigma_0} + \frac{2\pi G i\Sigma_k k}{|k|} + 2\Omega v_{ky}$$
$$-i\omega v_{ky} = -2\Omega v_{kx}$$

Combining these give us the dispersion relation

$$\omega^2 = 4\Omega^2 - 2\pi G\Sigma_0|k| + k^2 v_s^2$$

Sheet is unstable if $\omega^2 < 0$.

This will happen for

$$\frac{v_s\Omega}{G\Sigma_0} < \frac{\pi}{2}$$

If we move from a uniform sheet to a piece of a gaseous disk, the appropriate frequency is the epicycle frequence (= 2Ω for uniform disk) and we have the *Toomre criterion*:

$$Q \equiv \frac{v_s\kappa}{\pi G\Sigma_0} < 1$$

for instability.

The physical basis for the criterion is a combination of Jeans stability at small wavelengths,

$$\lambda < \frac{v_s^2}{G\Sigma_0},$$

and centrifugal stability at large wavelengths,

$$\lambda > \frac{G\Sigma_0}{\Omega^2}.$$

Note differences from Jeans instability:

- For Jeans there is always a wavelength beyond which the system is unstable. For the disk, if v_s and Ω are sufficiently large there may be NO unstable region.
- At small wavelengths, the growth is suppressed by the pressure; at large wevelengths the growth is supressed by the shear.
- Even if $v_s = 0$ there is a wavelength beyond which the instability doesn't grow:

$$\lambda_{crit} = \frac{2\pi}{k_{crit}} \equiv \frac{4\pi^2 G\Sigma}{\kappa^2}$$

- The maximum growth rate occurs at roughly $\lambda_{crit}/2$.
- If the rotation is 0, then we do have a Jeans like criteria:

$$k_J = \frac{2\pi G\Sigma_0}{v_s^2}$$

  where $k < k_J$ is unstable.

This is only for *local* modes. From simulations we know that even for disks with $1 < Q < 2$ we have can have more global modes go unstable.

10.3 Radial Orbit Instability

The cosmological picture of galaxy formation implies that orbits should be very radial, how do we get non-radial orbits? The radial orbit instability is a possible mechanism.

Consider a purely radial orbit: it will appear as a straight wire. As in the Kepler problem, the wire is not precessing because of a match in frequencies, in this case $\Omega_r = 2\Omega_\phi$.

Now define ψ_a as the azimuth of an apocenter. For slightly non-zero L, we have

$$\frac{\mathrm{d}\psi_a}{\mathrm{d}t} = \frac{\Delta\psi - \pi}{T_r} = \Omega_\phi - \frac{1}{2}\Omega_r,$$

which will be the precession rate of the "wire". For a non-singular potential (certainly not Kepler), we can Taylor expand $\Delta\psi$ in L:

$$\Delta\psi = \pi + p(E)L + \mathcal{O}(L^2).$$

So the nearly radial orbits in a galaxy can be considered as slowly precessing "spines on a sea urchin".

If there is a non-radial perturbation, then the precession rates will be changed by:

$$\frac{\mathrm{d}^2\psi_a}{\mathrm{d}t^2} \sim \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p(E)}{T_r(E)}L\right) \sim \frac{p(E)}{T_r(E)}N$$

where N is the torque from the perturbation. Compare with the rigid body equation of motion:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} = \frac{N}{T},$$

so we can identify a moment of inertia of an orbit: $I = T_r(E)/p(E)$.

Now suppose there is a slight overdensity of wires at a given direction, this will provide a torque that will tend to attract more wires in that direction, but the random motion will smear out the perturbation. Since the random velocities are proportional to L, we expect a criterion of a typical L:

$$L^2 > G\rho r^2 I$$

for stability.

10.4 Instabilities in Barred Potentials

(Sellwood & Wilkinson 1993 for a review of bars)

Consider nearly circular orbits in a rotating frame, where x is the radial displacement, and y is the tangential displacement. The equations of motion are:

$$\ddot{x} - 2\Omega\dot{y} + 2R\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r}x = -\frac{\partial\Phi}{\partial x} \quad (43)$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{\partial\Phi}{\partial y}. \quad (44)$$

Recall that solutions with $\Phi = 0$ are in the form

$$x = Ae^{i\kappa t+\phi_0}$$

and

$$y = iA\frac{2\Omega}{\kappa}e^{i\kappa t+\phi_0}$$

where κ is the epicyclic frequency.

Now introduce a perturbation of the form

$$\Phi = P(r)e^{im(\theta-\Omega_p t)}$$

where m is the azimuthal wavenumber of the perturbation, and Ω_p is the pattern speed.

The equations become:

$$\ddot{x} - 2\Omega\dot{y} + 2R\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r}x = -\frac{\partial P}{\partial r}e^{im(\theta+[\Omega-\Omega_p]t)} \quad (45)$$

$$\ddot{y} + 2\Omega\dot{x} = -\frac{im}{R}Pe^{im(\theta+[\Omega-\Omega_p]t)}. \quad (46)$$

(note that $y = R\theta$.)

Define the angular fequency that the guiding center overtakes the perturbation as $\omega \equiv m(\Omega - \Omega_p)$. Then the solution is

$$x = \frac{-1}{\kappa^2 - \omega^2}\left[\frac{2m\Omega}{R\omega}P + \frac{\partial P}{\partial r}\right]e^{i(m\theta+\omega t)}$$

and

$$y = \frac{-1}{\kappa^2 - \omega^2}\left[\frac{2m\Omega}{\omega}\frac{\partial P}{\partial r} + \frac{4\Omega^2 - \kappa^2 + \omega^2}{\omega^2}\frac{m}{R}P\right]e^{i(m\theta+\omega t)}.$$

As in the Solar System case, the interesting points are where the denominators are 0. Here there are two separate cases:

- Corotation: $\omega = 0$. A star will maintain fixed phase with respect to the perturbation
- Linblad resonances: $\omega = \pm\kappa$. A star will encounter the pertubation at the same point in its epicycle.

10.5 Instabilities in Comoving coordinates

We would like write the equations in comoving coordinates $\vec{x}$ and "peculiar" velocity $\vec{v}$ defined by

$$\begin{aligned}\vec{x} &= \vec{r}/a(t)\\ \dot{\vec{r}} &= \dot{(a\vec{x})} = \dot{a}\vec{x} + a\dot{\vec{x}} = H\vec{r} + \vec{v}\end{aligned}$$

In these coordinates, the Poisson equation is:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \nabla\cdot(\rho\vec{v}) = 0$$

Note the extra factor due to expansion.

Euler's equation is:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} + 2H\vec{v} + \frac{\ddot{a}}{a}\vec{x} = -\frac{1}{a^2}\left(\frac{\nabla p}{\rho} + \nabla\Phi\right).$$

Note a couple of extra terms:

1. The "Hubble drag" term: in the absence of forces, the comoving peculiar velocity decays with time.

2. The cosmological accleration term: the overall rate of expansion is changing.

Doing our standard linearization, we get:

$$\frac{\partial \rho_1}{\partial t} + 3H\rho_1 + \rho_0 \nabla \cdot \vec{v} = 0$$

and

$$\frac{\partial \vec{v}}{\partial t} + 2H\vec{v} = -\frac{1}{a^2}\left(\frac{\nabla p_1}{\rho_0} + \nabla \Phi_1\right).$$

Note that the cosmological accleration term is a zeroth order quantity that is balanced by the gradient in the zeroth order potential (actually the 0,0 component of the Freidmann equations).

In accordance with standard use, we introduce the quantity

$$\delta \equiv \frac{\rho_1 - \rho_0}{\rho_0} = \frac{\rho_1}{\rho_0} - 1$$

In terms of this quantity, the continuity equation is now:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \vec{v} = 0$$

Now we do the usual procedure to eliminate v between these equations, and we will also assume that δ has a wave form: $\delta_k \exp(i\vec{k} \cdot \vec{x})$. This gives:

$$\ddot{\delta}_k + 2H\dot{\delta}_k = \delta_k \left(4\pi G\rho_0 - \frac{v_s^2 k^2}{a^2}\right)$$

Note that because of explicit time dependence in H and ρ_0, we don't have a solution which is an exponential in time. Instead we have, for a matter dominated universe and small sound speed,

$$\delta_k \propto t^{2/3} \quad \text{and} \quad \delta_k \propto t^{-1}$$

For total radiation domination

$$p = \frac{\rho c^2}{3}$$

This gives the linearized equation for δ in comoving coordinates

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = \delta\left(\frac{32\pi}{3}G\rho_0 - \frac{v_s^2 k^2}{a^2}\right)$$

where the sound speed in this limit is

$$v_s^2 = \frac{c^2}{3}$$

10.5.1 Solutions for $\delta(t)$

Consider $\lambda > \lambda_J(t)$. This is almost the horizon size for radiation domination.

In both radiation and matter domination with $\Omega = 1$ we have $\rho_0 \propto 1/t^2$.

$$\begin{aligned} \text{matter domination } (a \propto t^{2/3}): &\qquad 4\pi G\rho_0 = \tfrac{2}{3t^2} \\ \text{radiation domination } (a \propto t^{1/2}): &\qquad \tfrac{32\pi G}{3}\rho_0 = \tfrac{1}{t^2} \end{aligned}$$

This is a second-order linear differential equation for $\delta(\vec{x}, t)$.

There are two solutions: a growing and decaying mode.

Try a power law solution $\delta \propto t^n$

Since for matter $H = 2/3t$, we have

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2} = 0.$$

The power law solution gives:

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$$

with solutions of $n = -1$ and $n = 2/3$.

For matter domination

$$\begin{aligned} \delta(\vec{x}, t) &= A(\vec{x})D_+(t) + B(\vec{x})D_-(t) \\ &= A(\vec{x})t^{2/3} + B(\vec{x})t^{-1} \end{aligned}$$

For radiation domination

$$\delta(\vec{x}, t) = A(\vec{x})t^1 + B(\vec{x})t^{-1}$$

Hence for the growing modes

$$\delta \propto \begin{cases} a(t)^2 = (1+z)^{-2} & \text{radiation domination} \\ a(t) = (1+z)^{-1} & \text{matter domination} \end{cases}$$

If we assume that density fluctuations appear at some very early time, then at a much later time we care only about the growing mode, which we will just call $D(t)$. Thus,

$$\delta(\vec{x}, t) = \delta(\vec{x}, t_i) \frac{D(t)}{D(t_i)}$$

As long as $\delta \ll 1$, density contrasts simply "grow in place" in comoving coordinates. More complicated analytic expressions exist for $\Omega < 1$.

If matter dominated and $\Lambda = 0$ an approximate expression is

$$D(z) = \left[1 + \frac{2.5\Omega_0 z}{1 + 1.5\Omega_0}\right]^{-1}$$

For large z, $\delta \propto (1+z)^{-1}$ as before.

For small z, growth ceases and one gets **gravitational freezeout**.

During the radiation dominated epoch perturbations do not grow since $\lambda_J \sim cH^{-1}$

Not true for collisionless matter, e.g. dark matter, since it feels no pressure now as before

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G \rho_m \delta$$

but

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = 8\pi G(\rho_m + \rho_r)$$

This leads to the solution

$$\delta \propto \frac{\rho_m}{\rho_r} + \frac{2}{3} = \frac{1 + z_{eq}}{1+z} + \frac{2}{3}$$

so during radiation domination no growth occurs.

Following perturbations bigger than the horizon involves complicated General Relativity calculations.

The result is that perturbations larger than the horizon grow like perturbations larger the Jeans length within the horizon.

10.5.2 Spectrum of Fluctuations

It is often convenient to work with Fourier components of δ:

$$\begin{aligned} \delta(\vec{x}) &= \frac{V}{(2\pi)^3}\int \delta_{\vec{k}}(\vec{k})e^{-i\vec{k}\cdot\vec{x}}d^3k \\ \delta_{\vec{k}}(\vec{k}) &= \frac{1}{V}\int \delta(\vec{x})e^{i\vec{k}\cdot\vec{x}}d^3x \end{aligned}$$

Each $\delta_{\vec{k}} = A_{\vec{k}}e^{i\theta_{\vec{k}}}$ is a complex number, with A_k the Fourier amplitude.

If the phases $\theta_{\vec{k}}$ are uncorrelated, then the field $\delta(\vec{x})$ is **Gaussian**.

This implies that the 1-point probability distribution function of δ is Gaussian,

$$P(\delta) = (2\pi\sigma^2)^{-1/2}e^{-\delta^2/2\sigma^2}$$

where the variance

$$\sigma^2 \equiv \langle\delta^2\rangle = \int_0^\infty 4\pi k^2 P(k)dk$$

Here $P(k)$ is the **power spectrum**

$$P(k) \equiv \langle A_{\vec{k}}A^*_{\vec{k}}\rangle = \langle|\delta_k|^2\rangle$$

which tells the RMS fluctuation as a function of scale (whether or not the field is Gaussian).

The probability distribution of δ_k is

$$P(\delta_k) = (2\pi P(k))^{-3/2}\, e^{-|\delta_k|^2/2P(k)}$$

In linear theory, each Fourier mode evolves independently, $\delta_{\vec{k}}(t) \propto D(t)$.

The spatial scale is $\lambda = 2\pi/k$, so even if small scale modes have become nonlinear, $k^3\delta_{\vec{k}} \gtrsim 1$, large scale modes may still follow linear theory.

On scales in the linear regime, the shape of the power spectrum is preserved, and its amplitude grows $\propto D^2(t)$.

Inflation produces a Gaussian field with power law power spectra

$$P(k) = \langle|\delta_k|^2\rangle \propto k^n$$

with $n \approx 1$.

This is called the **Harrison–Zel'dovich** or scale invariant spectrum. **The transfer function** Modes of short wavelength have their amplitudes

reduced relative to long wavelength modes owing to dissipative and pressure effects.

This is effect is encapsulated in the **transfer function**

$$T_k \equiv \frac{\delta_k(z=0)D(z_i)}{\delta_k(z_i)D(0)}$$

z_i is arbitrary as long it is before any scale of interest enters the horizon.

$$P(k,t) = P_i(k,t_i)T_k^2(k,t)$$

There are two main effects that alter the shape of the **primordial power spectrum**

1. **Jeans mass effects.** Prior to z_{eq} perturbations inside the horizon do not grow.

   After z_{eq} perturbations in collisionless dark matter on all scales grow.

   One of the critical scales for the power spectrum is the comoving horizon size, d_H, at matter-radiation equality

   $$d_{eq} = 16.0\left(\Omega h^2\right)^{-1} \text{ Mpc}$$

   Lets see how the power spectrum is affected for adiabatic fluctuations.

   If radiation dominated

   $$d_H \propto a$$

   Perturbations with $kd_{eq} \ll 1$ always grow as

   $$\delta \propto a^2 \propto d_H^2.$$

   Perturbations with larger k, i.e. smaller λ, enter the horizon when

   $$d_H \sim 1/k$$

   and are frozen until z_{eq} when they grow again.

   The missing growth factor is just the square in the change in d_H during that period $\propto k^{-2}$.

   Hence

   $$T_k \approx \begin{cases} 1 & (kd_{eq} \ll 1) \\ (kd_{eq})^{-2} & (kd_{eq} \gg 1) \end{cases}$$

2. Damping effects

   Free streaming

   Silk damping

10.5.3 Non-linear collapse

See G.D. 9.2.1

Consider a spherical top-hat perturbation; that is, δ is constant and non-zero within some r of the origin and zero everywhere else. δ grows with time, but obviously from conservation of mass, the comoving size will get smaller.

One can solve for the evolution of the size of the perturbation by considering it as an isolated sphere in physical (non-comoving coordinates). Birkhoff's theorem allows you to do this. Because the sphere is uniform density, the dynamical time is the same everywhere, so it stays uniform denisity. The surface of the sphere follows the equation of a cycloid. In parametric form this is

$$r = A(1 - \cos\eta)$$
$$t = B(\eta - \sin\eta).$$

From these you can calculate t_{max} (turn-around) at $\eta = \pi$ and collapse at $\eta = 2\pi$.

Doing the calculation gives

$$t_{max} \sim t_i 1.095 \delta_i^{-3/2}$$

for an $\Omega = 1$ universe. Hence for a given δ_i at t_i we can calculate when it will collapse, or conversely, given an initial time and a time of collapse we can calculate what δ will collapse:

$$\delta_c(t_i, t) = \left(\frac{2 \times 1.095 t_i}{t}\right)^{2/3} = 1.686 \left(\frac{t_i}{t}\right)^{2/3}$$

Hence, given a density perturbation field at one epoch, we can calculate for some arbitrary epoch, which parts of that field collapsed into bound (virialized) objects.

Given the statistics of a random field (see BBKS 1986), both number of bound objects and their mass function can be calculated. This is the basis for *Press-Schechter Theory*. Essentially this theory involves smoothing the density field on a given mass scale and calculating the number of separate regions or fraction of mass that is above δ_c. Press-Schechter theory is able to calculate:

- bottom-up structure formation: Since σ_k grows with k smaller (lower mass) regions collapse earlier.

- Mass function: by smoothing on different scales at a given epoch, one can calculate the number density as a function of mass.
- Merger rates: one can calculate the conditional probability of a region that is currently collapsed on one scale being collapsed at a smaller scale at an earlier epoch.

10.6 Disk stability and spiral structure

The limitations of the standard theory of spiral structure are illustrated in the following example.

Imagine trying to solve the Schrödinger equation for the hydrogen atom using linear analysis.

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\psi}{\mathrm{d}r}\right) - \left(\frac{l(l+1)}{r^2} - \frac{q}{r}\right)\psi + \omega\psi = 0.$$

We assume $\psi \propto e^{ikr}$ and obtain a dispersion relation

$$k^2 = \omega - \frac{l(l+1)}{r^2} + \frac{q}{r}.$$

For $\omega < 0$ there is a range over which waves oscillate, and outside of which waves decay. One then packs an integral number of oscillations between these two limits and gets a quantum condition.

The reason this does not work well is that the interesting regime is for one or two oscillations, and the linear approximation is no good.

10.7 Conventional theory of spiral structure

When we perturb the Poisson-bracket form of the CBE for a Hamiltonian, $H = H_0 + \delta\Phi$ and linearize, we obtain

$$\frac{\partial \delta f}{\partial t} + \{\delta f, H_0\} = \{\delta\Phi, f_0\}.$$

If we expand in the actions and angles of H_0, we have

$$\delta f(\mathbf{J}, \theta) = \sum_{\mathbf{n}} \delta f_{\mathbf{n}}(\mathbf{J}) \exp(i\mathbf{n}\cdot\theta),$$

$$\delta\Phi(\mathbf{J},\theta) = \sum_{\mathbf{n}} \delta\Phi_{\mathbf{n}}(\mathbf{J}) \exp(i\mathbf{n}\cdot\theta),$$

If we assume $\delta f = \delta\overline{f} e^{-i\nu t}$ and $\delta\Phi = \overline{\Phi} e^{-i\nu t}$, and plug these into the linearized CBE, we get

$$\delta\overline{f}_{\mathbf{n}} = \frac{\mathbf{n}\cdot\partial f_0/\partial\mathbf{J}}{\mathbf{n}\cdot\omega - \nu}\delta\overline{\Phi}_{\mathbf{n}}.$$

where

$$\omega = \frac{\partial H_0}{\partial \mathbf{J}}.$$

Poisson's equation therefore tells us the normal modes satisfy

$$(4\pi G)^{-1}\nabla^2 \sum_{\mathbf{n}} \delta\overline{\Phi}_{\mathbf{n}} e^{i\mathbf{n}\cdot\theta} = \delta\overline{\rho} = \sum_{\mathbf{n}} \int \frac{\mathbf{n}\cdot\partial f_0/\partial\mathbf{J}}{\mathbf{n}\cdot\omega - \nu}\delta\overline{\Phi}_{\mathbf{n}} e^{i\mathbf{n}\cdot\theta} \mathrm{d}\mathbf{v}.$$

Now we guess that the normal mode of a disk will be of the form $\epsilon e^{i(kR+m\phi)}$. This is called the "tight winding approximation" since it is implicitly assumed that the modes of interest have $2\pi m \ll kR$. Next we need to get actions and angles in terms of $\mathbf{x}$ and $\mathbf{v}$, so we use the epicycle approximation. This gives

$$R = R_g + a\cos\theta_r, \quad \phi = \theta_a + \frac{\gamma a}{R_g}\sin\theta_r, \quad \text{where} \quad a \equiv \sqrt{\frac{2J_r}{\kappa}}.$$

The representation of $\delta\overline{\Phi}$ in action-angles is therefore

$$\begin{aligned}\delta\overline{\Phi} &= \epsilon e^{i(kR+m\phi)} = \epsilon e^{ikR_g}\exp(ika\cos\theta_r)e^{im\theta_a}\exp\left(i\frac{m\gamma a}{R_g}\sin\theta_r\right) \\ &= \epsilon\sum_{l=-\infty}^{\infty}\exp[i(kR_g + l\alpha)]J_l(\mathcal{K}a)e^{i(l\theta_r+m\theta_a)},\end{aligned}$$

where

$$\alpha(J_a) \equiv \arctan\left(\frac{m\gamma}{kR_g}\right) \quad \text{and} \quad \mathcal{K} \equiv \sqrt{k^2 + \frac{m^2\gamma^2}{R_g^2}},$$

and J_l is a Bessel function. So we've accomplished the expansion of $\overline{\Phi}$ in action angle coordinates.

$$\delta\overline{\Phi}_{(l,m)} = \epsilon\exp[i(kR_g + l\alpha)]J_l(\mathcal{K}a).$$

The linearized CBE now gives us

$$\delta\overline{f}_{\mathbf{n}}(\mathbf{J}) = \epsilon \exp[i(kR_g + l\alpha)] J_l(\mathcal{K}a) \frac{\mathbf{n}\cdot\partial f_0/\partial\mathbf{J}}{\mathbf{n}\cdot\omega - \nu} \quad [\mathbf{n} \equiv (l,m)].$$

In order to evaluate $\delta\Sigma$, we need to integrate $\delta\overline{f}$ over all velocities, and make an assumption for f_0. If we assume Schwarzschild's DF in the form $f_0(\mathbf{J}) = (\gamma\Sigma_0/2\pi\sigma^2)e^{-\kappa J_r/\sigma^2}$ and do the messy integrals, (Heavy use of Gradshteyn and Ryzhik) we have

$$\delta\Sigma \simeq \frac{\mathcal{K}^2\Sigma_0\delta\Phi}{\kappa^2(1-s^2)}\mathcal{F}(s,\chi),$$

where

$$s \equiv \frac{\nu - m\Omega}{\kappa}, \quad \chi \equiv \frac{\mathcal{K}^2\sigma^2}{\kappa^2}, \quad \mathcal{F}(s,\chi) \equiv 2(1-s^2)\frac{e^{-\chi}}{\chi}\sum_{l=1}^{\infty}\frac{I_l(\chi)}{1-s^2/l^2},$$

where $I_l(\chi)$ is a modified Bessel function. This is to be compared with Poisson's equation for a disk:

$$\delta\Sigma = -\frac{|k|\delta\Phi}{2\pi G}.$$

This implies that the normal modes must satisfy (after we approxmate $\mathcal{K}$ with $|k|$)

$$\frac{|k|}{k_{\rm crit}}\mathcal{F}(s,\chi) = 1 - s^2 = 1 - \frac{(\nu - m\Omega)^2}{\kappa^2}, \quad \text{where} \quad k_{\rm crit} \equiv \frac{\kappa^2}{2\pi G\Sigma_0}.$$

This is the Lin-Shu-Kalnajs dispersion relation for tightly wound spirals.

- Waves are trapped between radii at which $\kappa = \pm(\nu - m\Omega)$. These are the inner and outer **Linblad resonances**.
- The dispersion relation for leading and trailing waves are identical.
- The function $k\mathcal{F}$ behaves qualitatively like $k - \epsilon k^2$, where $\epsilon > 0$. This implies two allowed values of k for any value of s, long and short-wave branches.
- At the **corotation resonance** where $\nu = m\Omega_p = m\Omega$, it can happen that there is no solution for k. The two branches merge at the edge of this forbidden zone.
- The **group velocity**, $v_g = \partial\omega/\partial k$ tells how the waves will propagate. Wave packets will also reflect off the resonances.

10.8 Swing amplification

If the parameter

$$X \equiv \frac{k_{crit} R}{m}$$

becomes small, then swing amplification can happen. This is not a phenomenon captured by the WKB approximation. For this case an unwinding spiral arm matches the epicycle frequency of the stars, creating a significant amplification of an initial leading spiral density wave.

11 Encounters of stellar systems

If stellar systems are dissipationless, why do mergers happen?

Although energy is conserved, ordered motion (from the orbit) can be converted into internal velocity dispersion of the colliders. Hence stellar systems are inelastic.

Note that because of the impulse approximation ($|\Delta v| \propto 1/v$), slow encounters are more disruptive than fast encounters.

11.1 Dynamical Friction

Chandrasekhar (1943) derived the force that a moving particle experiences when moving through a uniform sea of smaller particles is:

$$\frac{dv_M}{dt} = -4\pi G^2 M m_a \ln\Lambda \int \mathrm{d}^3\mathbf{v}_a f(\mathbf{v}_a) \frac{\mathbf{v}_M - \mathbf{v}_a}{|\mathbf{v}_M - \mathbf{v}_a|^3}$$

Where $f(\mathbf{v}_a)$ is the distribution function of background stars of mass m_a, and $M >> m_a$ is the mass of the subject body. This formula is derived in the diffusion limit, i.e. assuming a large number of small, uncorrelated scatterings. It can be thought of as the gravitational force due to the wake behind the moving object.

For an isotropic DF, this simplifies to:

$$\frac{dv_M}{dt} = -\frac{4\pi G^2 M \mathbf{v}_M}{v_M^3} \ln\Lambda \rho(< v_M)$$

where M is the mass of the object, v is its speed, $\ln\Lambda$ is the Coulomb logarithm and $\rho(< v_M)$ is the density of stars with speed less than v_M (obtained from the obvious integral over the distribution function).

The slow velocity limit is:

$$\frac{dv_M}{dt} = -\frac{16\pi^2}{3} G^2 M m_a \ln\Lambda f(0) \mathbf{v}_M,$$

similar to Stokes drag.

The large velocity limit is:

$$\frac{dv_M}{dt} = -\frac{4\pi G^2 M \mathbf{v}_M}{v_M^3} \ln\Lambda\rho$$

Note:

- The deceleration is greater for larger masses (force is proportional to M^2: the wake is larger.
- It is independent of the masses of the smaller objects, just the density: there could be a spectrum of masses, and the formula still holds.
- The deceleration is smaller for higher velocity dispersions: the wake is weakened.
- One can also think of this as equipartition: in equilibrium all objects should have equal kinetic energy.

Although this equation has been used widely, are some serious difficulties with this equation.

1. The Coulomb logarithm implies that the drag force comes from all scales. The assumption of a homogeneous medium is clearly violated.
2. This does not consider the self gravity of the wake.
3. It violates Poincare's recurrence theorem: there are no secular energy changes in 1st or second order perturbation theory. (Illustration with recurring orbit).
4. If the drag term is proportional to the local density, then a satellite orbiting completely outside a galaxy should experience no friction.
5. If a bar is modeld as a pair of masses, the slowing time is incredibly short compared to the age of a galaxy.

   Also consider Type I migration in a protoplanetary disk.

The answer to all these issues is that in a finite stellar system, dynamical friction occurs through interactions with resonances (Tremaine and Weinberg 1984). Hence tidal drag can occur because some orbital resonances lie within the galaxy. The recurrence theorem breaks down at resonances. Furthermore as the object slows, the resonances change, sweeping through phase space.

Analytic work has been done on this with linear response theory: the subject body is treated as an external potential, which generates a response in the system, and the potential of the response is used to calculate the force on the subject body.

11.2 Tidal Shocks

Example of a high speed encounter where

- The kinetic energy of the relative motion is larger than the kinetic energy of the internal motion.
- The majority of the stars will barely move with respect to their system center. We can use the impulse approximation.

We will also only be concerned about accelerations relative to the system center.

11.2.1 Distant-tide approximation

If the size of the subject system is smaller than the typical impact parameter (or the scale lengths within the perturbing object), then the perturbing potential will vary smoothly, and the force can be written as

$$-\frac{\partial\Phi(\mathbf{x})}{\partial x_j} = -\left.\frac{\partial\Phi}{\partial x_j}\right|_{x=0} - \left.\frac{\partial^2\Phi}{\partial x_j\partial x_k}\right|_{x=0} x_k + \mathcal{O}(x^2).$$

In this case, accelerations with respect to the system center are

$$\dot{\mathbf{v}}_j = -\left.\frac{\partial^2\Phi}{\partial x_j\partial x_k}\right|_{x=0} x_k$$

This tide can "pinch" globular clusters as they pass through the disk of the galaxy. In this case, the cluster is small compared to the scale height of the disk, so the relative accleration is

$$\dot{v}_z = -\left(\frac{\partial^2\Phi_d}{\partial Z^2}\right)_{cm} z$$

where the tide is evaluated at the cluster center. Poisson's equation tells us:

$$\frac{\mathrm{d}^2\Phi_d}{\mathrm{d}Z^2} = 4\pi G\rho_d,$$

so the relative acceleration is

$$\dot{v}_z = -4\pi G\rho_d(R, Z_{cm})z.$$

For a high speed passage, a star doesn't move in the cluster during the encounter, so the impulse is

$$\Delta v_z = \int dt\dot{v}_z = -4\pi Gz \int dt\rho_d(R, Z_{cm}(t)).$$

Now $Z_{cm} \sim V_z t$, so we can instead integrate over Z_{cm}:

$$\Delta v_z = -\frac{4\pi Gz}{|V_z|}\int \mathrm{d}Z_{cm}\rho_d = -\frac{4\pi G\Sigma_d(R)z}{|V_z|}.$$

The energy gained by a star in this passage is therefore

$$\Delta E = \frac{1}{2}\langle(\Delta v)^2\langle = \frac{8\pi G^2\Sigma^2}{V_z^2}\langle z^2\rangle.$$

Expressing the mean z in terms of the semi-major axis of a stellar orbit in the cluster, a, we have:

$$\Delta E = \frac{14\pi^2 G^2\Sigma_d^2 a^2}{3V_z^2}.$$

If the cluster passes through the disk twice per orbital period, T_ψ, and we use the Keplerian value for energy, $E = \frac{1}{2}GM_{gc}/a$,

$$t_d \sim \frac{1}{2}T_\psi\frac{|E|}{\Delta E} = 0.005\frac{M_{gc}V_z^2 T_\psi}{G\Sigma_d^2 a^3}.$$

This works out to be < 10 Gyr for $a > 30$ pc for solar neighborhood values. It will be shorter closer in since the disk is denser and the orbital period is shorter. Hence cluster erosion by tidal shocks is significant.

11.3 Tides

Now let's consider the case of a static tide. This would be appropriate for a satellite orbiting in a nearly circular orbit. The orbit of the stars in the satellite can be treated in the restricted three-body problem. In this case there is a constant of motion, the Jacobi integral:

$$E_J = \frac{1}{2}v^2 + \Phi(\mathbf{x}) - \frac{1}{2}|\Omega \times \mathbf{x}|^2$$

$$E_J = \frac{1}{2}v^2 + \Phi_{eff}(\mathbf{x})$$

Contours of constant Φ_{eff} are zero velocity surfaces for stars with $E_J = \Phi_{eff}$. The largest zero velocity surface that surrounds the satellite is the tidal or Roche surface.

The radius of this surface (Jacobi radius or Hill radius or Roche radius) is approximately the distance to the L_3 Lagrange point. This can be found from

$$\left(\frac{\partial \Phi_{eff}}{\partial x}\right)_{x_s - x_{r_J}} = 0.$$

Now

$$\Phi_{eff} = -G\left[\frac{M}{|\mathbf{x} - \mathbf{x}_M|} + \frac{m}{|\mathbf{x} - \mathbf{x}_m|} + \frac{M+m}{2R_0^3}(x^2 + y^2)\right],$$

where M is the mass of the large body, m is the mass of the satellite, and R_0 is their separation. Taking the derivative gives

$$0 = \frac{M}{(R_0 - r_J)^2} - \frac{m}{r_J^2} - \frac{M+m}{R_0^3}\left(\frac{MR_0}{M+m} - r_J\right).$$

This needs to be solved for r_J numerically. However for $m << M$, we can Taylor expand $(R_0 - r_J)^{-2}$, and we get

$$0 = \frac{M}{R_0^2}\left(1 + \frac{2r_J}{R_0} + \cdots\right) - \frac{m}{r_J^2} - \frac{M}{R_0^2} + \frac{M+m}{R_0^3}r_J \sim \frac{3Mr_J}{R_0^3} - \frac{m}{r_J^2}.$$

This is easily solved for

$$r_J = \left(\frac{m}{3M}\right)^{1/3} R_0.$$

One gets the same result in the Hill approximation.

One way to think of the tidal forces is as a limit on the density of a satellite. If the mean density of the satellite is

$$\bar{\rho} = m/(\frac{4}{3}\pi r_J^3)$$

and the the mean density of the main galaxy is

$$\bar{\rho}_g = M/(\frac{4}{3}\pi r_J^3),$$

Then the hills radius is equivalent to

$$\bar{\rho} = 3\bar{\rho}_g,$$

that is, a satellite will be stripped until its mean density matches the mean density of the galaxy interior to its orbital radius.

Also, as a satellite sinks, it will lose mass and then be destroyed according to the mean density. Furthermore, despite the heating of dynamical friction a sinking satellite will not decrease the density of a galaxy.