

1.0 Planets with Atmospheres – a brief overview

Currently, observational characterization of planetary atmospheres is similar to what had been learned about the Earth's atmosphere by the early 1960s, after rocket and satellite measurements. From telescope observations and planetary missions we have (mostly) determined the main atmospheric constituents and altitude profiles of temperature. Some books divide atmospheres into 4 groups:

N₂ atmospheres (Earth, Titan, Triton, Pluto),
CO₂ atmospheres (Venus, Mars),
H₂-rich giants (Jupiter, Saturn, Uranus, Neptune), and
Extremely tenuous atmospheres (Io; bodies with exospheres, e.g., Europa).

However, a more meaningful categorization is chemical character, specifically redox. We can think of two groups:

1) Reducing atmospheres

Relatively rich in gases that are *reducing agents* (prone to donate electrons in chemical reactions), typically hydrogen-bearing gases, such as hydrogen itself (H₂), methane (CH₄) or possibly ammonia (NH₃).

- giant planets
- Titan
- Early Earth (and at some point early Mars and Venus).

They tend to have *hydrocarbon aerosols* (fine, suspended particles) in hazes.

2) Oxidizing atmospheres

- Earth
- Mars
- Venus (upper atmosphere).

They tend to have *sulfate aerosols* (if volcanic sulfur is in the atmosphere), e.g., clouds on Venus, the *Junge layer* @20-25 km in Earth's stratosphere.

There are also tenuous atmospheres (surface pressure in bars):

i) **N₂-rich atmospheres above surfaces covered in N₂ ice**: Triton ($\sim(14-20)\times 10^{-6}$), the largest moon of Neptune, and Pluto ($\sim 10\times 10^{-6}$);

ii) **O₂-rich exospheres above moons covered in water ice**: Jupiter's moons Europa ($\sim 10^{-12}$ - 10^{-13} O₂), Ganymede ($\sim 10^{-12}$ O₂) and Callisto ($\sim 10^{-9}$ bar of O₂ with some CO₂), and Saturn's moons Rhea ($\sim 10^{-12}$ O₂ and CO₂) and Dione ($\sim 10^{-12}$ O₂);

iii) **Volcanogenic atmospheres**: SO₂ on Io ($\sim 10^{-7}$ - 10^{-9}), a moon of Jupiter, and a water vapor-CO₂ exosphere on Enceladus, a moon of Saturn.

iv) **Exospheres above rocky surfaces**: Mercury ($\sim 10^{-15}$), Moon (3×10^{-15}) – ballistic atmospheres with atoms hopping across the surface

Physical class: “Clausius-Clayperon” or “collapsed” atmospheres – where the mean surface pressure is just in vapor-solid equilibrium at the prevailing temperature (e.g., Pluto, Triton, Mars(?))

For this course, we are mainly interested in the terrestrial planets (Earth, Venus, Mars), Titan, the giant planets, and exoplanets (rocky and giant).

Table 1: Reducing atmospheres. Gases are given as volume mixing ratios (Sources: Lodders (2010); Atreya et al. (2003); Niemann et al. (2010)).

Species	Jupiter	Saturn	Uranus	Neptune	Titan
H ₂	86.4 ± 2.6 %	88 ± 2 %	82.5 ± 3.3 %	80 ± 3.2 %	0.099%
He	13.6 ± 0.3 %	12 ± 2 %	15.2 ± 3.3 %	19.0 ± 3.2 %	
CH ₄	0.181 ± 0.34 %	0.47 ± 0.2 %	~2.3 %	~1-2%	5.65% (1.48% in stratosphere)
¹³ CH ₄	1.9 ± 0.1 × 10 ⁻⁵	5.1 ± 0.2 × 10 ⁻⁵			0.062%
N ₂					94.2%
NH ₃	6.1 ± 2.8 × 10 ⁻⁴	(1.6 ± 1.1) × 10 ⁻⁴	<1 × 10 ⁻⁷	<6 × 10 ⁻⁷	
H ₂ O	6.5 ± 2.9 × 10 ⁻⁴	0.2-2 × 10 ⁻⁸	5-12 × 10 ⁻⁹	1.5-3.5 × 10 ⁻⁹	< 10 ⁻⁸
H ₂ S	6.7 ± 0.4 × 10 ⁻⁵	≤0.4 × 10 ⁻⁶	<8 × 10 ⁻⁷	<3 × 10 ⁻⁶	-
Ne	2.3 ± 0.25 × 10 ⁻⁵				<2 × 10 ⁻⁵ ²⁰ Ne
Ar	1.0 ± 0.4 × 10 ⁻⁵				3.4 × 10 ⁻⁵ ⁴⁰ Ar 2.1 × 10 ⁻⁷ ³⁶ Ar
C ₂ H ₆	5.8 ± 1.5 × 10 ⁻⁶	7.0 ± 1.5 × 10 ⁻⁶	1-20 × 10 ⁻⁹	1.5 × 10 ⁻⁶	≤1 × 10 ⁻⁵
PH ₃	1.1 ± 0.4 × 10 ⁻⁶	4.5 ± 1.4 × 10 ⁻⁶			-
C ₂ H ₂	1.1 ± 0.3 × 10 ⁻⁷	3.0 ± 0.2 × 10 ⁻⁷	~1.0 × 10 ⁻⁸	6.0 ^{+14.0} _{-4.0} × 10 ⁻⁹	≤2 × 10 ⁻⁶
CO ₂	5-35 × 10 ⁻⁹	≤3 × 10 ⁻¹⁰	4.0 ± 0.5 × 10 ⁻¹¹	5 × 10 ⁻¹⁰	10 ⁻⁸ (above stratospheric condensation level)
CO	1.6 ± 0.3 × 10 ⁻⁹	1.4 ± 0.7 × 10 ⁻⁹	<4 × 10 ⁻⁸	6.5 ± 3.5 × 10 ⁻⁶	4.7 × 10 ⁻⁵
D/H	2.3-2.6 × 10 ⁻⁵	1.7 ^{+0.75} _{-0.45} × 10 ⁻⁵			1.35 ± 0.3 in H ₂ 1.32 ^{+0.15} _{-0.11} in CH ₄
¹³ C/ ¹² C	0.0108 ± 0.0005	0.011			0.01099 ± 0.00017 in CH ₄

Table 2: The composition and properties of oxidizing atmospheres (noting that the lower atmosphere of Venus could be considered somewhat reducing). (Sources: Lodders and Fegley (1998); Pollack (1991); Krasnopolsky and Lefevre (2013); Mahaffy et al. (2013); Franz et al. (2017).

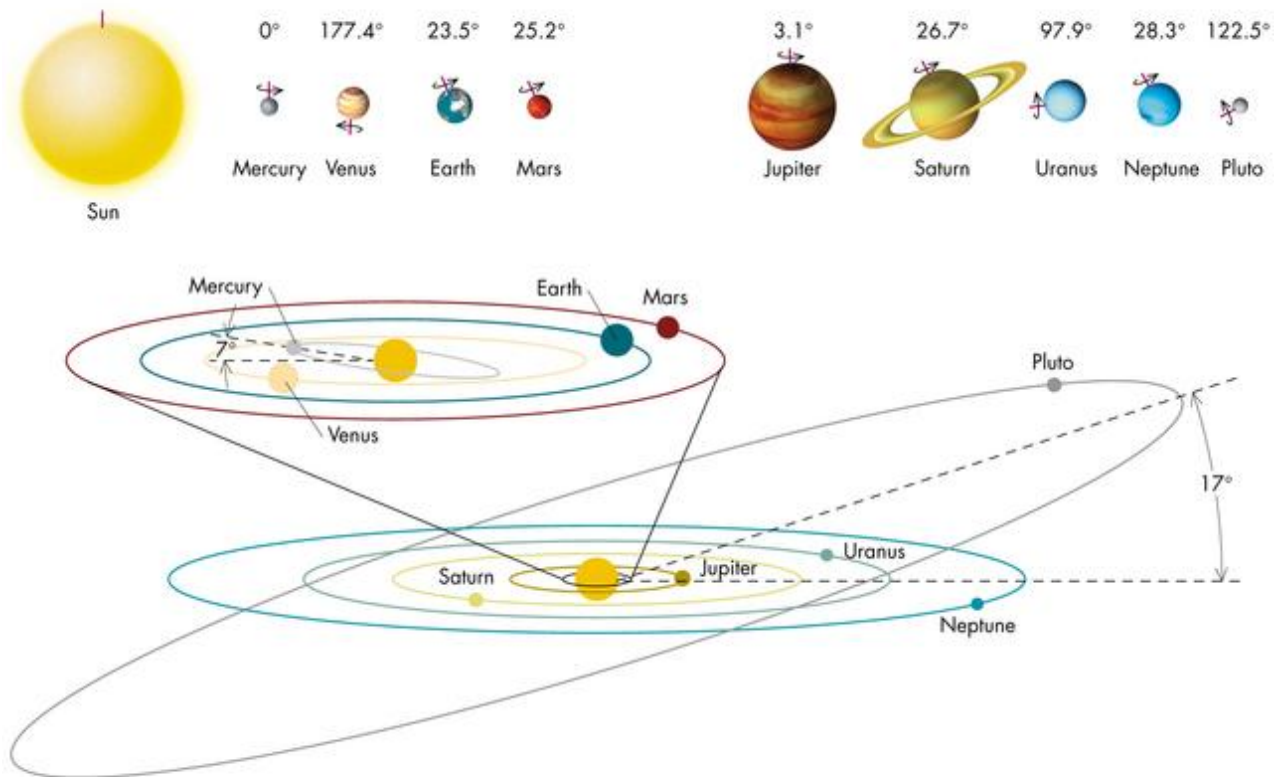
Note: The composition of Mars' atmosphere is from *Curiosity Rover* results (Franz et al., 2017).

Parameter	Venus	Earth	Mars
Mean surface pressure (bar)	95.6	1.0	0.006
Mean surface temperature (K)	735	288	218
Mass relative to Earth (5.97×10^{24} kg)	0.815	1.0	$0.107 \approx 1/9$
Mean radius relative to Earth (6371 km).	0.950	1.0	$0.532 \approx 1/2$
Key gases in atmosphere (by volume)	CO ₂ 96.5% N ₂ 3.5% SO ₂ [¶] •150±30 ppm (22-42 km) •25±150 ppm (12-22 km) ⁴⁰ Ar 70±25 ppm ³⁶⁺³⁸ Ar 75±35 ppm H ₂ O [¶] 30±15 ppm (5-45 km) CO [¶] •45±10 ppm (cloud top) •17±1 ppm (12 km) He 12 (+24/-8) ppm Ne 7±3 ppm H ₂ [¶] 2.5±1 ppm (50-60 km) HCl 0.4 ppm (70 km) ⁸⁴ Kr 50±25 ppb [¶] Altitude-dependent	N ₂ * 78.084% O ₂ * 20.946% H ₂ O 0.1 ppm - 4% (varies) Ar 9340 ppm CO ₂ * [§] •~280 ppm (pre-industrial) • 396.5 ppm (year 2013) Ne 18.18 ppm ⁴ He 5.24 ppm CH ₄ * 1.8 ppm Kr 1.14 ppm H ₂ * 0.55 ppm N ₂ O* ~320 ppb CO* 125 ppb *Biologically cycled and influenced §Increasing ~2 ppm/yr	CO ₂ 94.9±1.6% N ₂ 2.79±0.05% Ar 2.08±0.02% O ₂ 0.174±0.006 % CO 747±3 ppm H ₂ O ~0.03% (varies) He 10 ppm H ₂ 15±5 ppm Ne 2.5 ppm Kr 0.3 ppm O ₃ 0-80 ppb H ₂ O ₂ 0-40 ppb SO ₂ <0.3 ppb [†] *Indicates lack of volcanic outgassing

What we don't know:

- Exoplanet atmospheric compositions.
- The degree to which any of these are influenced by life and whether particular structures of terrestrial biochemistry and thus metabolic waste gases (CH₄, O₂, N₂O, N₂, (CH₃)₂S, COS, CS₂) or microbial food (H₂, CO) are outcomes pre-determined by a geochemical context and the universality of the Periodic Table (e- orbitals/ affinities, redox, etc).

SOLAR SYSTEM OVERVIEW



Upper sketch shows the angle (from the normal to the orbital plane) of the rotation axes of the planets, the **obliquity**. Substantial differences in obliquity affects the seasonal distribution of sunlight quite drastically. (Orbits & bodies are not to scale).

What do we mean by 'north'? Two conventions (the second is the official one):

- 1) *NASA right-hand rule* (as illustrated above): If curled fingers represent the direction of rotation, the thumb points north. Using this definition, Venus has an obliquity of 177.4°, is upside down, rotating in a conventional counter-clockwise direction when viewed from above its North Pole.
- 2) *International Astronomical Union (IAU)*: the North Pole of a planet points upwards from the plane defined by its counter-clockwise orbital motion. Thus, Venus and Uranus have obliquities of 2.6° and 82.1°, respectively, and both spin "backwards" (to the west) when viewed from above the orbital plane

When we look at the orbital planes of the planets, we see two anomalies:

- **Mercury** has an orbital tilt that may be due to a late collision with a large body during planetary formation.
- **Pluto** has large eccentricity and orbital tilt; it's a captured Kuiper Belt Object. KBOs are icy, comet-like objects that lie between 35 to ~50 AU. there are probably about 2×10^8 KBOs with radii ~10 km.

[Planetary orbital parameters are affected by other planets in a chaotic way. We cannot predict even the Earth's orbital parameters *exactly* beyond ~60 m.y., only statistically (work by Jacques Laskar).]

Giant Planets Overview

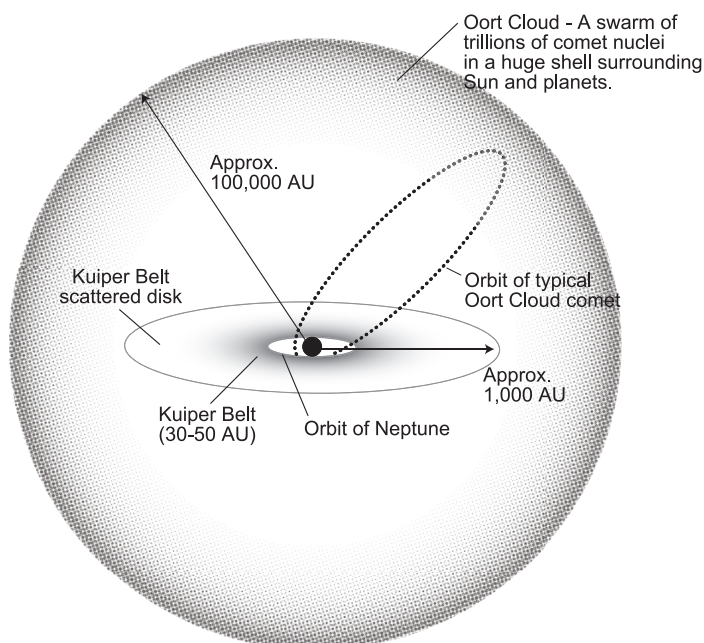
Giant planets consist mostly of H_2 and He but are enhanced in ice-forming elements from Jupiter to Saturn to Uranus/Neptune. The term *ice-forming elements* means O, C and N, which are responsible for a great deal of ice in the universe including those of H_2O , CH_4 , NH_3 , CO_2 , CO and gas hydrates.

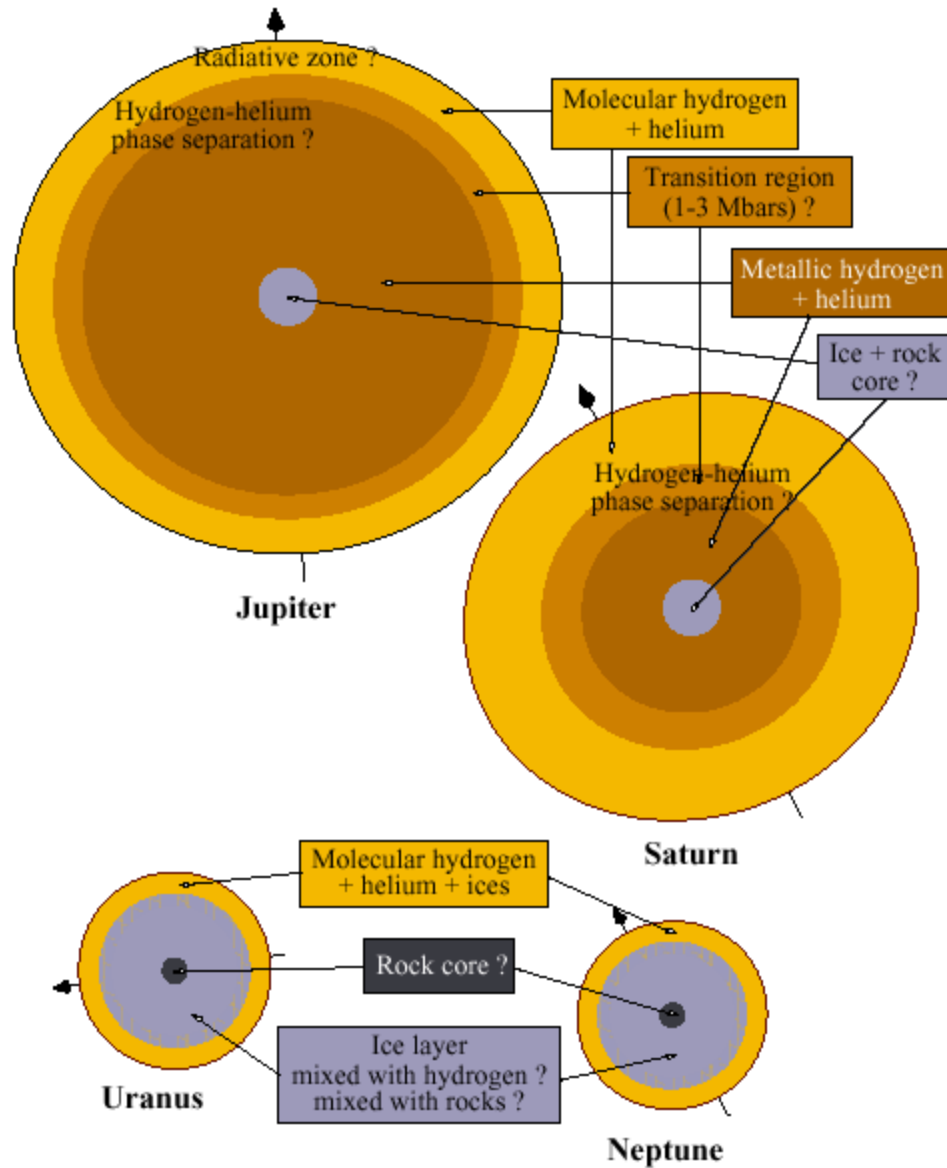
Indeed, after H and He the next most abundant elements are O, C and N.

(C abundance is 0.6 times that of O, N abundance is 0.3 times that of C.)

The C/H ratio steadily increases from Jupiter to Neptune, for example. In gas giant planet interiors, hydrogen becomes metallic at a transition pressure of 1-3 Mbar.

Q) What kind of atomic structure would you expect for metallic hydrogen?





▲ Schematic interiors of giant planets and ice giants. Inside Jupiter and Saturn, hydrogen becomes metallic at a transition pressure of 1-3 Mbar.

1.1 Atmospheric Structure: Hydrostatics

The vertical structure of atmospheric pressure and density determine where physical and chemical processes take place. The 3-D distribution of temperature determines the distribution of pressure, which drives the winds. Therefore, we need to be able to connect distributions of temperature, pressure, and density. To do so, we use:

- 1) an equation of state, relating these three variables (P , T , ρ) to each other
- 2) the hydrostatic equation, relating the pressure to mass per unit area of overlying air.

Q) An application is planetary descent probes, e.g., Mars landers, Jupiter's *Galileo* probe, Titan's *Huygens* probe. These measure deceleration from air resistance. This is used to back out the temperature profile, i.e., T versus altitude. How might that work?

1.1.1 Atmospheric temperature structure: An overview

How the temperature varies with altitude is one of the most basic features of planetary atmospheres -- the *atmospheric structure*. The Earth's atmosphere serves to introduce the terminology for describing vertical regions of the atmosphere. The average temperature structure of Earth's atmosphere consists of five layers of increasing altitude:

1. the *troposphere* from the surface to the *tropopause*, which lies at an average height of 12 km but varies in altitude from ~8 km at the poles to ~17 km in the tropics
2. the *stratosphere* from the tropopause to the *stratopause* at ~50 km altitude (where the air pressure is about 100 Pa (1 mbar))
3. the *mesosphere* from the stratopause to the *mesopause*, at ~85 km altitude (where the air pressure is about 1-0.1 Pa (0.01-0.001 mbar))
4. the *thermosphere* from the mesopause to the *thermopause* (also known as the *exobase*) at about ~500 km.
5. the *exosphere*, which lies above the thermopause and joins interplanetary space.

Inspection of Fig. 1.1 shows that each new layer above the troposphere begins where the temperature undergoes an inflection. Consequently, the terminology developed for Earth's atmosphere depends upon the presence of the ozone layer. Ozone absorption of ultraviolet light causes temperature to increase with height above the troposphere and this defines the stratosphere. Other planets, such as Mars and Venus, do not have ozone layers, so the nomenclature for atmospheric layers breaks down somewhat.

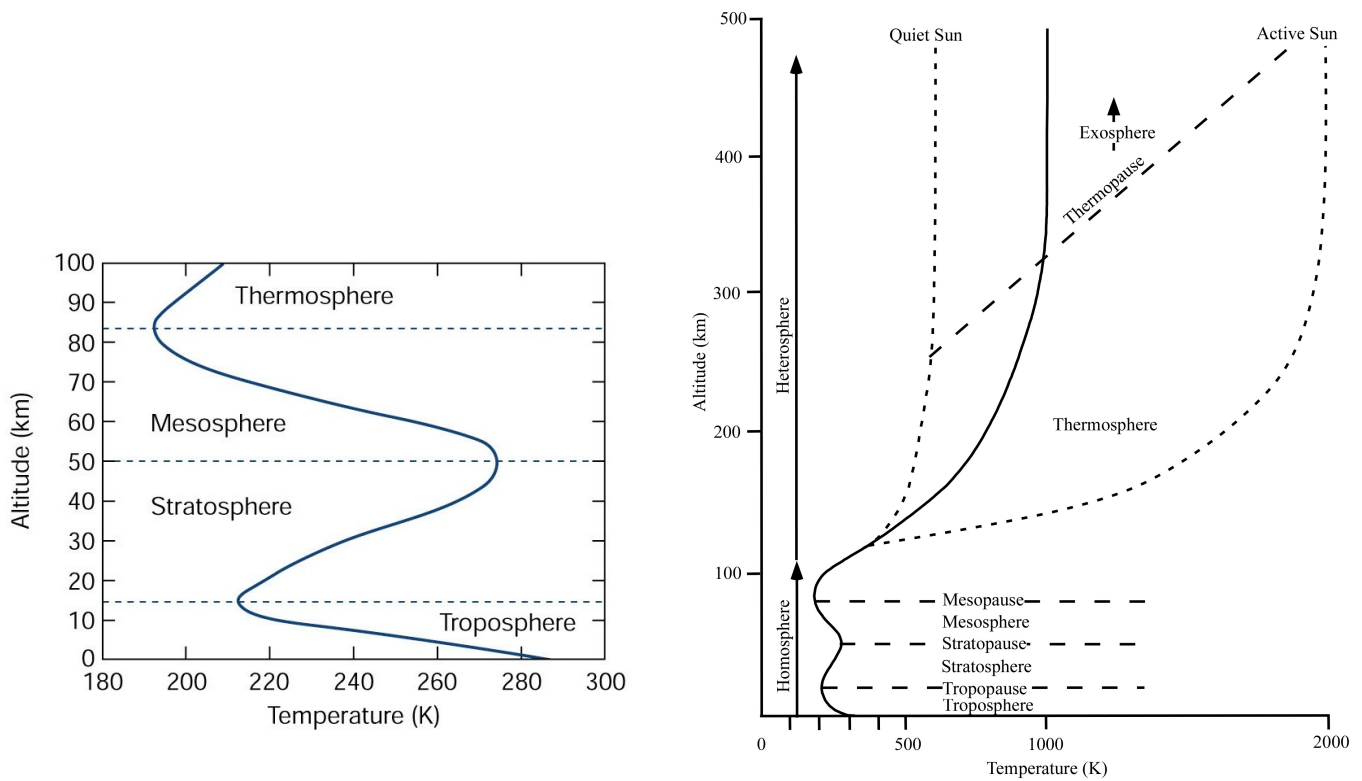


Fig. 1.1: Standard atmospheric structure of Earth.

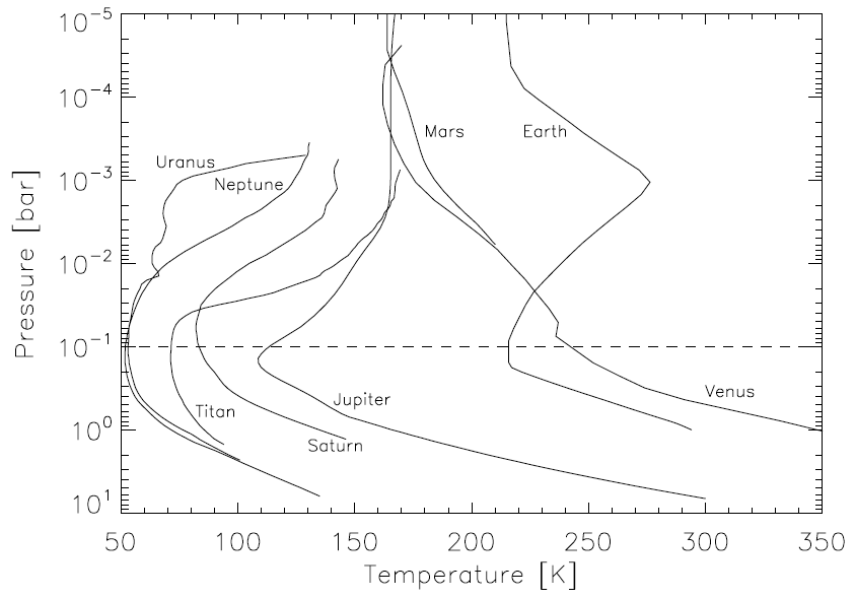


Fig. 1.2: Atmospheric structures of giant and terrestrial planets. Why are there these structures?

What we don't know: 1) Why thermospheres on Jupiter, Saturn, Uranus, and Neptune are hot. Solar energy absorption is too small. Suggestions are deposition of energy by waves. 2) The range of exoplanet atmospheric structures.

1.1.2.1 The ideal gas equation

The equation of state relevant to all atmospheric situations that we will study, is the ideal gas law,

$$p = nkT = \underbrace{(n\bar{m})}_{\rho} \left(\frac{k}{\bar{m}} \right) T \Rightarrow p = \rho \bar{R} T \quad (1.1)$$

Here k is Boltzmann's constant, \bar{m} is the mean mass of molecules in the atmosphere, n is the number density of molecules, mass density $\rho = n\bar{m}$, and gas constant $\bar{R} = k / \bar{m}$. Of course, the ideal gas law fails very deep in the atmospheres of the giant planets. Note that the ideal gas constant R (8.314 J mol^{-1}) is related to atmospheric gas constant via the relation $\bar{R} = R / \bar{M} = k / \bar{m}$, where we introduce \bar{M} , the mean molar mass of the atmosphere (kg mol^{-1}), which is related to the mean molecular mass of individual molecules by $\bar{M} = N_A \bar{m}$, where N_A is Avogadro's number, $6.022 \times 10^{23} \text{ molecules mol}^{-1}$. A useful reference value for mass of molecules is the mass of the hydrogen atom $m = 1.66 \times 10^{-27} \text{ kg}$

What defines the mean molar mass? Planetary atmospheres consist of mixtures of gases. Mixtures of ideal gases obey Dalton's Law for the partial pressures and number densities of the individual constituent gases p_i , n_i ,

$$p_i = kn_i T \quad (1.2)$$

Thus, we have

$$\rho = \sum_i m_i n_i, \quad n = \sum_i n_i, \quad \bar{m} = \frac{\sum_i m_i n_i}{n} = \frac{\bar{M}}{N_A} \quad (1.3)$$

where m_i is the mass of the i^{th} molecule.

For example, in the Earth's atmosphere we have a mean molar mass, \bar{M} , of dry air of 28.97 g mol^{-1} and $\bar{R} = R / \bar{M} = 287 \text{ J K}^{-1} \text{ kg}^{-1}$. On Mars, if the atmosphere were pure CO_2 , we would expect $\bar{M} = 44 \text{ g mol}^{-1}$, but mass spectrometry measurements by the Viking landers in the 1970s and Curiosity rover in 2013 showed that nitrogen and argon make up the balance so that $\bar{M} = 43.49 \text{ g mol}^{-1}$, and $\bar{R} = R / \bar{M} = 191.2 \text{ J kg}^{-1} \text{ K}^{-1}$.

1) A basic form of the gas law is:

$$PV = nRT \quad (\text{B1})$$

Here, n = number of moles in volume V at pressure P and temperature T . $R = 8.314 \text{ J K}^{-1} \text{ mol}^{-1}$ = **universal gas constant**. This ideal gas law incorporates **Boyle's Law** ($PV = \text{const.}$), **Charles Law** ($V/T = \text{const.}$), and **Avogadro's hypothesis** ($n = PV/RT$, i.e. samples at the same P & T contain same number of moles).

2) Rather than moles, we can use the total number of molecules, N . The number of molecules per mole is Avogadro's number, N_A , so $N = nN_A$. The equation of state is

$$PV = \frac{N}{N_A} RT \quad (\text{B2})$$

$$PV = NkT$$

Here we have introduced $k = R/N_A$ = Boltzmann's constant = $1.381 \times 10^{-23} \text{ J K}^{-1}$.

3) Another way of expressing the gas law is to use molecules per unit volume:

$$P = nkT \quad (\text{B3})$$

where n is the number density of molecules ($=N/V$), the number of molecules per unit volume (molecules m^{-3} , although frequently molecules cm^{-3} is used in atmospheric science, which requires other consistent units).

4) A fourth way is to use mass density rather than number density. Density = mass/volume = $\rho = M/V = N\bar{m}/V$, where \bar{m} = mean molecular mass and N = number of molecules in volume V . Hence

$$P = \frac{\rho kT}{\bar{m}} \quad (\text{B4})$$

5) A fifth way is to take the above eqn, substitute for $k = R/N_A$ and note that $N_A \bar{m}$ = molar mass, \bar{M} . So we have:

$$P = \frac{\rho RT}{\bar{M}} \quad (\text{B5})$$

$$P = \frac{\rho RT}{\bar{M}} = \rho \bar{R} T$$

Note that \bar{M} sometimes goes by the name of "mean molecular weight. Molar mass is a much more sensible name. Frequently the "gas constant" \bar{R} is used, which is not to be confused with the universal gas constant, R . The gas constant \bar{R} is per unit mass [$\text{J K}^{-1} \text{ kg}^{-1}$] and depends on gas composition.

EXAMPLE:

(a) What is the density of air on the surface of Mars? The average pressure is 6.1 mbar, the molecular gas constant is $191 \text{ J K}^{-1} \text{ kg}^{-1}$, and the average temperature is -55 C .

(b) How does this compare with the density on the surface of the Earth? On Earth, the average pressure is 1013 mbar, the average temperature is 288 K, and the molar mass of dry air is 28.97 gm.

1.1.2.2 The Composition of an Atmosphere*Mixing Ratio*

The amount of a gas is usually given as a *volume mixing ratio*, f_i , which is the mole fraction. By Avogadro's hypothesis, the mole fraction is simply the ratio of the number of molecules N_i , of a gas to the total number of molecules N , so

$$\text{volume mixing ratio, } f_i = \frac{N_i}{N} = \frac{p_i}{p} = \frac{V_i}{V} = \frac{n_i}{n} \quad (1.4)$$

The other relations in eq. (1.4) follow from the ideal gas law (eq. (B3)) and *Dalton's Law*. Here, V_i is the volume occupied by molecules of the gas in total sample volume, V . We can also define a mass mixing ratio, μ_i , which is ratio of the mass of a particular gas to the total mass of the gas mixture sample, or

$$\text{mass mixing ratio, } \mu_i = \frac{N_i m_i}{N \bar{m}} = \frac{m_i}{\bar{m}} \frac{p_i}{p} = \frac{m_i}{\bar{m}} f_i \quad (1.5)$$

Here, we have used eq. (1.4) to relate the mass-mixing ratio to the volume mixing ratio.

Virtual Temperature [notes only – we will skip in class]

Atmospheric compositions are not fixed; they can vary because of chemical reactions or condensation and evaporation. For example, the mass-mixing ratio of water vapor in Earth's atmosphere varies from about 0.1 ppmv (in Antarctica) to about 4% (wet tropical regions). As a consequence, the gas constant \bar{R} varies. For condensable gases, such as water vapor in the atmosphere of Earth or ammonia in Jupiter's atmosphere, it is convenient to define the virtual temperature. This is a *fictitious temperature that allows the ideal gas law to be satisfied using the gas constant for the dry gas alone*. This requires that virtual temperature T_v satisfies

$$T_v = \frac{\bar{R}}{\bar{R}_d} T = \frac{T}{[1 - (e/p)(1 - \epsilon)]}, \quad \epsilon \equiv m_c / \bar{m}_d \quad (1.6)$$

where m_c is the molecular mass of the condensable gas, \bar{m}_d is the mean molecular mass for the dry gas (combination of atmospheric gases excluding the condensable species), \bar{R}_d is the gas constant for the dry gas, and e is the vapor pressure of the condensable species.

Derivation of the above is by Dalton's law of partial pressures $p = p_d + e$. Thus we can use the ideal gas law to get:

$$\rho = \frac{(p - e)}{\bar{R}_d T} + \frac{e}{R_c T}, \quad R_c = R / M_c$$

$$\rho = \frac{p}{\bar{R}_d T} - \frac{e}{\bar{R}_d T} + \frac{e}{R_c T}$$

$$\rho = \frac{p}{\bar{R}_d T} \left[1 - \left(\frac{e}{p} \right) \left(1 - \frac{\bar{R}_d}{R_c} \right) \right]$$

The last equation may be written $p = \bar{R}_d \rho T_v$.

Since $\bar{R}_d = R / \bar{M}_d = k / \bar{m}_d$ and $\bar{R}_c = R / M_c = k / m_c$, then $\bar{R}_d / R_c = m_c / \bar{m}_d$.

The usefulness of the definition (1.6) is that the equation of state for the gas including the condensable can be expressed in the form

$$p = \bar{R}_d \rho T_v \quad (1.7)$$

that is, the gas constant remains the same in all cases, provided we replace the temperature with the virtual temperature T_v , by using (1.6). (For precise computations, it may be necessary to include the mass of condensate in clouds in the definition of virtual temperature).

- Since water vapor has a molecular mass that is lighter than the mean molecular mass of the dry gas of Earth's atmosphere (a mixture mainly of O₂ and N₂), the density of moist air is **less** than the density of dry air at the same temperature and pressure.
- For Jupiter, the condensable gas ammonia is heavier than the mean molecular mass of the dry gas (a mixture of molecular hydrogen and helium), so the density of "moist" air is greater than that of the "dry" gas at the same pressure and temperature. Consequently, virtual temperature due to water vapor is slightly **greater** than actual temperature for Earth, and virtual temperature accounting for variable ammonia concentration is slightly **less** than actual temperature in the atmosphere of Jupiter.

1.1.2.3 The hydrostatic equation (p.28 Ingersoll; p. 96 Pierrehumbert)

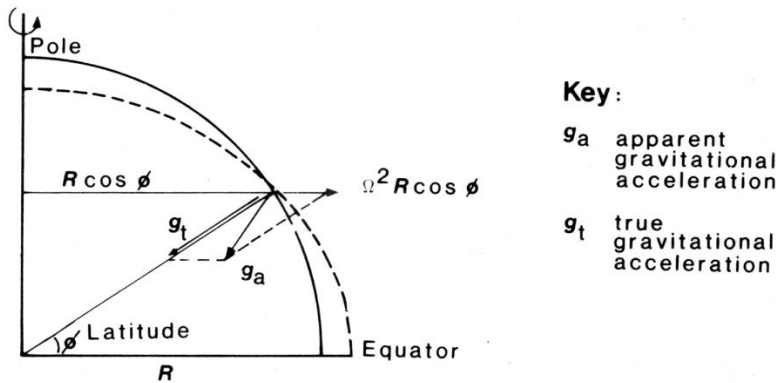
If air settles under gravity, high pressure lower down in the atmosphere pushes upwards against the weight of overlying air. This balance of forces is **hydrostatic equilibrium**, an approximation that is valid over a horizontal scale larger than a few kilometers on Earth.

The equation is derived by considering a volume of air of unit surface area and the change in its pressure ΔP over height Δz . The column with unit area cross-section and small height Δz has a mass $\rho \Delta z$, so that the pressure change going up height Δz is $\Delta p = -(\rho \Delta z)g = \text{force}/(\text{unit area})$, where g is gravitational acceleration.

In differential and integral form, this equation is

$$\frac{\partial p}{\partial z} = -g\rho, \quad p(z) = \int_z^\infty g\rho dz \quad (1.8)$$

Here g obviously depends on height, but will also depend on latitude because a planet rotates. With an angular velocity, Ω , the centrifugal force is given by $\Omega^2 R$, where R is the radial distance from the axis of rotation. At the equator, R is just the radius of the planet.



The centrifugal force at latitude ϕ is $\Omega^2 R \cos \phi$. The component of this force upward in local vertical is $\Omega^2 R \cos^2 \phi$. Hence the first order correction to g is to subtract this component. The acceleration due to gravity is:

$$g(r, \Phi) \approx \frac{GM}{r^2} \left[1 - \frac{\Omega^2 r^3 \cos^2 \Phi}{GM} + \dots \right] \approx g_0 \left(\frac{r_0}{r_0 + z} \right)^2 \quad (1.9)$$

Here Ω is the planetary angular rotation rate

e.g., $\Omega = 2\pi/(\text{sidereal rotational period}) = 2\pi/(88642.663) = 7.088 \times 10^{-5} \text{ s}^{-1}$ for Mars

$\Omega = 2\pi/(35727.3) = 1.76 \times 10^{-5} \text{ s}^{-1}$ for Jupiter

M is the mass of the planet below radius r and G is the universal gravitational constant.

In the second approximate expression, r_0 is a reference radius, usually a mean solid surface or cloud-top radius and z is the geometric height above that radius.

In reality, the Earth and other planets bulge at the equator (because even the rocky ones are fluid over geological timescales) so that the local g-vector is towards the center of the planet. Thus with the oblate planet, there is no apparent g offset from the local vertical.

Does the latitudinal correction to 'g' matter?

The correction term in the middle expression of (1.9) accounts for the centrifugal acceleration at a given latitude due to planetary angular rotation rate, Ω . For the rocky planets, this correction is a few parts per thousand, and it produces the first order correction to sphericity of gravitational potential surfaces. Higher order corrections also appear due to the departure from spherical symmetry of the mass distribution due mainly, but not entirely, to rotation.

- **Rocky planets:** the last approximation of (1.9) is acceptable for most atmospheric applications.
- **Giant planets:** the rotational correction reaches as much as 7%; the planetary shape (of Jupiter, for example) is strongly deformed from spherical, and a more accurate treatment is needed.

1.1.2.4 Atmospheric Scale Height

The **scale height** of an atmosphere is the vertical distance over which pressure or density drops by a factor of $1/e = 1/(2.7183) \sim 1/3$. *The pressure scale height will generally differ slightly from the density scale height* because they are connected via the ideal gas law through temperature, which varies with height, so we should specify which. Only in an isothermal atmosphere does the pressure scale height equal the density scale height.

We can take the hydrostatic equation and the ideal gas equation to derive the scale height of an atmosphere:

$$\frac{\partial p}{\partial z} = -g(z) \underbrace{\frac{p}{R(z)T(z)}}_{\rho} \Rightarrow \frac{\partial p}{p} = -\left(\frac{\partial z}{H}\right) \Rightarrow p = p_s \exp\left(-\int_0^z \left(\frac{dz}{H}\right)\right) \Rightarrow p \approx p_s \exp\left(-\frac{z}{H}\right)_{\text{isothermal}} \quad (1.10)$$

where p_s is the surface pressure. This equation is simplified when we assume that the atmosphere is isothermal, well mixed, and that the variation of gravity with height is small. For most planetary atmospheres, these are not actually bad assumptions. Assuming an isothermal atmosphere at temperature \bar{T} , we get

$$p = p_s \exp\left(-\frac{z}{H}\right), \text{ where } H = \frac{k\bar{T}}{\bar{m}g} = \frac{\bar{R}\bar{T}}{g} \quad (1.11)$$

We can also think of H in terms of an energy distribution. In an isothermal atmosphere (meaning constant temperature with height) the average molecular thermal energy is $\sim kT$. If all this energy were used to lift the molecule to a height H , the gravitational potential energy would be mgH . We would not expect many molecules to have much more energy than kT , so the atmosphere should thin out above a scale height. Thus the height in $mgH \sim kT$ defines an energy level analogous to a Boltzmann distribution.

1.1.2.5 Geopotential height and geometric height

a) Geopotential height. To avoid dealing with the variation of g with height, a transformation is sometimes used that is analogous to the use of virtual temperature that we encountered earlier. *Geopotential height* Z , is the height of a given point in the atmosphere in units proportional to the potential energy of unit mass at this height relative to some reference altitude. Think of it as a “gravity-adjusted height”, which is usually of some pressure level; or rather, geopotential height is used as a function of pressure to make calculations easier. On a weather map, a plot of geopotential height at a single pressure shows ridges and troughs—highs and lows.

The potential energy per unit mass at height z is the integral of potential energy to that height, which is the *geopotential*, given by

$$\Phi(z) = \int_0^z g \, dz \quad \text{and } d\Phi = g \, dz \quad [\text{m}^2\text{s}^{-2} \text{ or J kg}^{-1}] \quad (1.12)$$

Geopotential height, Z , is defined by

$$Z = \frac{\Phi(z)}{g_0} = \frac{1}{g_0} \int_0^z g(z, \phi) \, dz \quad [\text{m}], \text{ where } g_0 = \frac{GM}{r_0^2} \quad (1.13)$$

and can be used as the dependent variable in place of geometric height, where ϕ is latitude and r_0 is a reference radius. Since $z \ll r$, it follows that $g_0 \approx g(z, \phi)$ and $Z \approx z$. For example, Z differs only 1.55% from z at 100 km on Earth. Using the hydrostatic equation in basic form, $dp = -g\rho \, dz = -\rho d\Phi$, we have, with the ideal gas law:

$$dp = -\left(\frac{p}{RT}\right) d\Phi = -\left(\frac{p}{RT}\right) g_0 dZ \Rightarrow \frac{dp}{p} = -\frac{1}{H} dZ, \text{ so}$$

$$p(Z) = p_0 \exp\left(-\int_0^Z g \rho \, dZ\right) = \exp\left(-\int_0^Z \frac{dZ}{H}\right) \quad (1.14)$$

where the scale height is

$$H(Z) = \frac{\bar{R}T(Z)}{g_0} = \frac{\bar{R}_d T_v(Z)}{g_0} \quad (1.15)$$

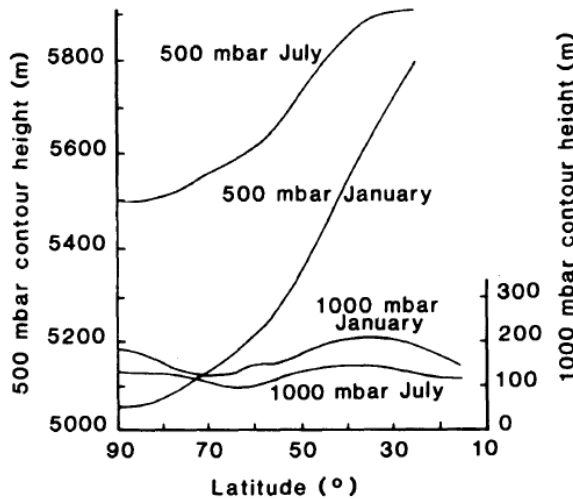
The difference in geopotential height between two pressure surfaces is called **thickness**:

$$\Delta Z = Z(p) - Z(p_0) = \int_p^{p_0} H(p) \frac{dp}{p} = \bar{H} \ln\left(\frac{p_0}{p}\right) \propto \bar{T} \quad (1.16)$$

where \bar{H} is the log (ln)-pressure average scale height between p and p_0 , it is proportional to \bar{T} , the log-pressure mean temperature.

Eq. (1.16) is called the **hyposometric equation**. The practical utility of this transformation is that one can construct a distribution of heights of isobaric surfaces given the temperature distribution (measured with remote sensing, for example) and the height of one isobaric surface that is measured or assumed. This approach directly relates the temperature distribution to its influence on pressure variations, which, in turn, force winds in an atmosphere. (Lateral gradients of temperature cause lateral gradients of

pressure to develop at altitude. For example, the 500-mbar surface in Earth's mid-troposphere (~5-6 km altitude) tends to slope downwards from warm equator to cold high latitudes. Greater height of a pressure surface is associated with greater pressure below because pressure increases with depth, so there is generally higher pressure towards the tropics in midlatitudes. This situation causes strong eastward winds in (winter) northern midlatitudes because when account is taken of Earth's rotation, winds tend to flow such that low pressure is to the left of the flow in the northern hemisphere).



← Monthly & zonally averaged (i.e., around lines of latitude) height contours (above mean sea-level (MSL)) of 500 mbar and 100 mbar surfaces in the northern hemisphere. When the 1000 mbar contour is 100-200 m, the MSL pressure is ~1008-1016 mbar. (Source: McIveen (1992), p. 83).

Q) How many scale heights to the 500 mb level on Earth? (Hint: eq. 1.11). Do mountains reach that high?

Q) Where is broad plateau of high pressure

(associated with dry, calm weather).

In hydrostatic equilibrium, p is a unique function of z for a given distribution of T / \bar{m} from eq. (1.10). So pressure or any function of pressure can be used an alternative vertical coordinate to geometric height z . GCMs (Global Climate Models) can use pressure or normalized pressure as a vertical variable with height as a dependent variable. Also, in some dynamical studies, a common coordinate is log-pressure. This approximates height and is defined by

$$\xi = -H_{\text{ref}} \ln \left(\frac{p}{p_{\text{ref}}} \right), \text{ where } H_{\text{ref}} = \frac{kT_{\text{ref}}}{gm_{\text{ref}}} \quad (1.17)$$

Here H_{ref} is a reference scale height, p_{ref} is a reference pressure, and T_{ref} and m_{ref} are some global mean reference values

1.1.2.6 The Surface Pressure and Mass of an Atmosphere

The surface pressure, p_s , integrates the mass, M_c , of a column of air per unit area because, by definition, $p_s = \text{weight}/(\text{unit area}) = M_c g$. So (ignoring lateral forces in spherical geometry), the mass of a column of air per unit area is given by

$$M_c = \frac{p_s}{g} \quad (1.18)$$

Examples: Column mass on Mars, Titan, Earth