A RANDOM MATRIX APPROACH TO THE EXISTENCE OF TRACIAL STATES ON UNITAL STABLY FINITE EXACT C^*-ALGEBRAS

PUI KUEN LEUNG

Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{A}_+$ denote the set of all positive elements of $\mathcal{A}$. A *state* is an additive linear functional $\rho : \mathcal{A}_+ \to [0, \infty]$ satisfying $\rho(\mathbf{1}) = 1$. We say a state ρ is *tracial* if it satisfies $\rho(xx^*) = \rho(x^*x)$ for all $x \in \mathcal{A}$. Tracial states are important objects of study in the theory of operator algebras as they are intricately linked with the structural properties of C^*-algebras, shedding light on the interaction between algebraic and topological aspects of C^*-algebras. In particular, they have found use in the classification of C^*-algebras and especially von Neumann algebras. In this project, we aim to give a self contained overview of a random matrix approach to showing the existence of tracial states on unital exact stably-finite C^*-algebras. This approach first appeared in the paper [2] and compared to the more traditional approach making use of ultraproduct techniques relating to AW^*-algebras, has the advantage of using relatively elementary tools. As such we will only need to assume the basic theory of C^*-algebras.

1. Exact C^*-Algebras

1.1. **Spatial tensor product.** Given C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, we denote by $\mathcal{A}_1 \odot \mathcal{A}_2$ their algebraic tensor product which has a natural structure as a $*$-algebra with multiplication given by $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$ and involution given by $(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*$. Given representations π_1 and π_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$ on some Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, we have a representation $\pi_1 \otimes \pi_2$ of $\mathcal{A}_1 \odot \mathcal{A}_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ given by $(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$. We can then define $\|\cdot\|_{\min}$ on $A \odot B$ by

$$\left\| \sum_{i=1}^n a_i \otimes b_i \right\|_{\min} := \sup \left\{ \left\| (\pi_1 \otimes \pi_2) \left(\sum_{i=1}^n a_i \otimes b_i \right) \right\| : \pi_i \text{ is a representation of } \mathcal{A}_i \text{ for } i = 1, 2 \right\}.$$

It can be easily seen that $\|\cdot\|_{\min}$ is always finite since the π_i are contractions, and it can be easily verified that indeed it defines a C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$. In fact, it is called the minimal C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$ as it turns out to be the smallest C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$. The *spatial tensor product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is then the completion of $\mathcal{A}_1 \odot \mathcal{A}_2$ with respect to this norm, which we denote by $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

1.2. **Exact C^*-algebras.** Given any C^*-algebra $\mathcal{B}$ and a closed (two-sided) ideal $\mathcal{I}$ in $\mathcal{B}$, the first isomorphism theorem gives an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{I} \longrightarrow 0.$$

We then say a C^*-algebra $\mathcal{A}$ is *exact* if for every such pair $(\mathcal{B}, \mathcal{I})$, tensoring the above sequence with $\mathcal{A}$ preserves exactness, i.e. if

$$0 \longrightarrow \mathcal{A} \otimes_{\min} \mathcal{I} \longrightarrow \mathcal{A} \otimes_{\min} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\min} \mathcal{B}/\mathcal{I} \longrightarrow 0$$

is an exact sequence for every C^*-algebra $\mathcal{B}$ and closed ideal $\mathcal{I}$ in $\mathcal{B}$. The most important examples of exact C^*-algebras are probably the class of *nuclear* C^*-algebras. These are C^*-algebras $\mathcal{A}$ where for each C^*-algebra $\mathcal{B}$ there is a unique C^*-norm on $\mathcal{A} \odot \mathcal{B}$ (i.e. the minimal and maximal C^*-norms on the algebraic tensor product coincide), and they include all commutative C^*-algebras and all finite-dimensional C^*-algebras.

2. Operator Topologies

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$. There are several distinct topologies on $\mathcal{B}(\mathcal{H})$ which will be of interest to us. Apart from the familiar norm, weak, strong operator, weak operator topologies, we shall also consider:

- the σ-strong topology – this is the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the family of seminorms $\{\rho_T : T \in \mathcal{K}(\mathcal{H})\}$ where $\rho_T(S) = \|ST\|$ and $\mathcal{K}(\mathcal{H})$ is the space of compact operators on $\mathcal{H}$,
- the σ-strong* topology – this is the locally convex topology generated by the seminorms $\{\rho_T, \lambda_T : T \in \mathcal{K}(\mathcal{H})\}$ where ρ_T is as above and $\lambda_T(S) = \|TS\|$, and
- the σ-weak topology – the locally convex topology generated by the seminorms $\{\omega_T : T \in \mathcal{L}_1(\mathcal{H})\}$ where $\omega_T(S) = |\operatorname{Tr}(ST)|$ and $\mathcal{L}_1(\mathcal{H})$ is the Banach space of trace class operators on $\mathcal{H}$.

These topologies are related by the inclusions

$$\begin{array}{ccccccc} \text{norm} & \supseteq & \sigma\text{-strong}^* & \supseteq & \sigma\text{-strong} & \supseteq & \sigma\text{-weak} \\ & & & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\ & & & & \text{SOT} & \supseteq & \text{WOT} \end{array}$$

It can be shown that every continuous linear functional on the Banach space $\mathcal{L}_1(\mathcal{H})$ has the form $T \mapsto \operatorname{Tr}(ST)$ for some $S \in \mathcal{B}(\mathcal{H})$. Hence we have $\mathcal{B}(\mathcal{H}) = (\mathcal{L}_1(\mathcal{H}))^*$ and we see that the σ-weak topology on $\mathcal{B}(\mathcal{H})$ is really just the weak* topology. We now consider some density theorems relating to these topologies. Let $\mathcal{A}$ be a C^*-algebra. We denote by π_U the representation of $\mathcal{A}$ formed by taking the direct sum of the GNS representations of all states on $\mathcal{A}$, called the *universal representation* of $\mathcal{A}$. Then $\pi_U(\mathcal{A})$ is a $*$-subalgebra of bounded operators on some Hilbert space $\mathcal{H}_U$, and its closure $\overline{\pi_U(\mathcal{A})}^{\text{WOT}}$ in the weak operator topology is a von Neumann algebra which we call the *enveloping von Neumann algebra* of $\mathcal{A}$. The **Sherman–Takeda theorem** says that the bidual $\mathcal{A}^{**}$ of a C^*-algebra $\mathcal{A}$ is isometrically isomorphic to $\overline{\pi_U(\mathcal{A})}^{\text{WOT}}$ as Banach spaces. The **von Neumann bicommutant theorem** says that the weak operator closure of a unital $*$-subalgebra $\mathcal{M}$ of bounded operators on a Hilbert space is equal to its bicommutant $(\mathcal{M}')'$. Since π_U is faithful and $\mathcal{S} \subseteq (\mathcal{S}')'$ holds for any set of operators $\mathcal{S}$, we have

$$\mathcal{A} \simeq \pi_U(\mathcal{A}) \subseteq (\pi_U(\mathcal{A})')' = \overline{\pi_U(\mathcal{A})}^{\text{WOT}}$$

and therefore we obtain a unital embedding of $\mathcal{A}$ into the enveloping von Neumann algebra which may be identified with the canonical embedding $i : \mathcal{A} \hookrightarrow \mathcal{A}^{**}$. From now on, we shall use $\mathcal{A}^{**}$ to refer to both the bidual and the enveloping von Neumann algebra. The following can be viewed as an analogue of the Goldstine theorem[1] for the σ-strong topology.

Theorem 2.1 (Kaplansky density)**.** *Let $\mathcal{A}$ be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The unit ball of $\mathcal{A}$ is σ-strong dense in the unit ball of $(\mathcal{A}')'$.*

We also have the following.

Lemma 2.2. *Let $\mathcal{A}$ be a C^*-algebra. The restriction of the σ-weak topology on $\mathcal{A}^{**}$ to $\mathcal{A}$ coincides with the weak topology on $\mathcal{A}$.*

Proof. This follows from the fact that the σ-weak topology on $\mathcal{A}^{**}$ is just the weak* topology, and that the weak topology on a Banach space coincides with the relative weak* topology from the bidual. $\square$

[1]Recall that Goldstine theorem says that the image of the unit ball of a Banach space under the canonical embedding is weak* dense in the unit ball of the bidual.

3. Comparison Theory for Projections

Let $\mathcal{A}$ be a C^*-algebra. Recall that an element $x \in \mathcal{A}$ is a *projection* if $x = x^* = x^2$. Let p, q be projections in $\mathcal{A}$. We say p and q are *orthogonal* if $pq = 0$. We say p is a *subprojection* of q if $p \leq q$. We say p and q are *Murray-von Neumann equivalent* in $\mathcal{A}$, written $p \sim q$, if there is a $u \in \mathcal{A}$ with $u^*u = p$ and $uu^* = q$. Then we say $p \prec q$ if p is equivalent to a subprojection of q. A projection p in $\mathcal{A}$ is *finite* if $p \sim p' \leq p$ implies $p' = p$, *infinite* if it is not finite, and *properly infinite* if there exist projections p_1 and p_2 such that $p \sim p_1 \leq p$, $p \sim p_2 \leq p$, and $p_1 \perp p_2$. A unital C^*-algebra $\mathcal{A}$ is finite (resp. infinite, properly infinite) if the unit $\mathbf{1}$ is finite (resp. infinite, properly infinite). One of the useful properties of properly infinite projections in von Neumann algebras is that they can be "halved" in the following sense.

Proposition 3.1. *Let $\mathcal{M}$ be a von Neumann algebra and $p \in \mathcal{M}$ be a properly infinite projection. Then there exist projections $p_1, p_2 \in \mathcal{M}$ such that $p = p_1 + p_2$, $p_1 \perp p_2$, and $p_1 \sim p_2 \sim p$.*

3.1. Matrix algebras.

Let us now consider matrix algebras of $\mathcal{A}$. By the Gelfand–Naimark theorem, we can without loss of generality take $\mathcal{A}$ to be a concrete C^*-algebra of bounded operators on a Hilbert space $\mathcal{H}$. For $n \in \mathbb{N}$, we consider the matrix algebra $M_n(\mathcal{A})$ whose elements are the $n \times n$ matrices with entries in $\mathcal{A}$. The algebra $M_n(\mathcal{A})$ becomes a $*$-algebra when equipped with the involution $(a_{ij})^* = (a_{ji}^*)$, and each element can be viewed in a natural way as a linear operator on $\mathcal{H}^n$ and so we have an induced operator norm on $M_n(\mathcal{A})$ from $\mathcal{B}(\mathcal{H}^n)$, giving it the structure of a C^*-algebra. For the sake of convenience, we shall use tensor product notation in matrix algebras. Denote by $e_{ij} \in M_n(\mathbb{C})$ the matrix with 1 in the (i, j)th entry and zeroes elsewhere. Then for $a \in \mathcal{A}$, we write $a \otimes e_{ij}$ for the element of $M_n(\mathcal{A})$ with a in the (i, j)th entry and zeroes elsewhere. For an arbitrary matrix $Y \in M_n(\mathbb{C})$ we then define $a \otimes Y \in M_n(\mathcal{A})$ by linear combinations of terms of the form $a \otimes e_{ij}$. We then say $\mathcal{A}$ is *stably finite* if $M_n(\mathcal{A})$ is finite for every $n \in \mathbb{N}$. Examples of stably finite C^*-algebras include all *residually finite dimensional* C^*-algebras, namely C^*-algebras $\mathcal{A}$ in which for any non-zero $a \in \mathcal{A}$ there is a finite dimensional representation π of $\mathcal{A}$ with $\pi(a) \neq 0$. In particular, all commutative C^*-algebras are stably finite.

3.2. Comparison theory for projections in matrix algebras.

We can consider a version of Murray-von Neumann equivalence of projections in matrix algebras by allowing comparison of matrices of different dimensions. Consider

$$M_\infty(\mathcal{A}) = \bigcup_{n \in \mathbb{N}} M_n(\mathcal{A})$$

where an element $x \in M_n(\mathcal{A})$ may be identified as an element in $M_{n+k}(\mathcal{A})$ for any $k \in \mathbb{N}$ by adding rows and columns of zeros. Then we say projections $p, q \in M_\infty(\mathcal{A})$ are Murray-von Neumann equivalent, and we write $p \sim q$, if they are Murray-von Neumann equivalent as elements of $M_n(\mathcal{A})$ for some n, i.e. there exists $u \in M_n(\mathcal{A})$ such that $u^*u = p$ and $uu^* = q$. Denote by $V(\mathcal{A})$ the set of equivalence classes $[p]$ of projections p in $M_\infty(\mathcal{A})$ with respect to Murray von Neumann equivalence. We endow $V(\mathcal{A})$ with an order structure by declaring $[p] \leq [q]$ if $p \prec q$ in $M_n(\mathcal{A})$ for some n, and an addition operation $\oplus$ by taking

$$[p] \oplus [q] := [\mathrm{diag}(p, q)] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

Let p, q be projections in a C^*-algebra $\mathcal{A}$. Below we state a result characterising when $\tau(p)$ is strictly dominated by $\tau(q)$ for any lower semicontinuous trace τ on $\mathcal{A}$, in terms of their comparison in $V(\mathcal{A}^{**})$.

Proposition 3.2 (Corollary 9.8 in [2])**.** *Let $\mathcal{A}$ be a C^*-algebra, and let $p, q \in \mathcal{A}$ be projections. The following are equivalent.*

(1) *There exists* $\epsilon > 0$ *such that* $\tau(p) \leq (1-\epsilon)\tau(q)$ *for any lower semicontinuous trace* τ *on* $\mathcal{A}$.
(2) *There exists* $k \in \mathbb{N}$ *such that* $k[p] \leq (k-1)[q]$ *in* $V(\mathcal{A}^{**})$.

The proof of the above is mostly topological in nature and makes no use of concepts beyond those of elementary properties of the operator topologies introduced earlier, but we omit it due to space constraint. As a consequence, we have the following as a corollary.

Corollary 3.3. *Let* $\mathcal{A}$ *be a* C^**-algebra, and let* $p, q \in \mathcal{A}$ *be projections. The following are equivalent.*
(1) *There exists* $\epsilon > 0$ *such that* $\tau(p) \leq (1-\epsilon)\tau(q)$ *for any lower semicontinuous trace* τ *on* $\mathcal{A}$.
(2) *There exists* $\epsilon > 0$ *and* $c_1, \ldots, c_r \in q\mathcal{A}p$ *such that*

$$\sum_{i=1}^{r} c_i^* c_i = p \quad \text{and} \quad \sum_{i=1}^{r} c_i c_i^* \leq (1-\epsilon)q.$$

Proof. (2) $\Rightarrow$ (1) is obvious by noting that for any trace τ we have

$$\tau(p) = \sum_{i=1}^{r} \tau(c_i^* c_i) = \sum_{i=1}^{r} \tau(c_i c_i^*) \leq (1-\epsilon)\tau(q).$$

Assume now that (1) holds, so that $p, q \in \mathcal{A}$ are projections and $\epsilon > 0$ is such that $\tau(q) \leq (1-\epsilon)\tau(p)$ for any lower semicontinuous trace τ on $\mathcal{A}$. By Proposition 3.2, there exists $k \in \mathbb{N}$ such that $k[p] \leq (k-1)[q]$ in $V(\mathcal{A}^{**})$, so we have

$$p \otimes \mathbf{1}_k \prec q \otimes \left(\sum_{i=1}^{k-1} e_{ii} \right) \quad \text{in } M_k(\mathcal{A}^{**}).$$

That is, there exists $u \in M_k(\mathcal{A}^{**})$ such that $u^*u = p \otimes \mathbf{1}_k$ and $uu^* \leq q \otimes \left(\sum_{i=1}^{k-1} e_{ii} \right)$. We have

$$kp = \sum_{i=1}^{k} (u^*u)_{ii} = \sum_{i=1}^{k} \sum_{j=1}^{k} u_{ij}^* u_{ij} \quad \text{and} \quad (k-1)q \geq \sum_{i=1}^{k} (uu^*)_{ii} = \sum_{i=1}^{k} \sum_{j=1}^{k} u_{ij} u_{ij}^*.$$

Hence fixing $a_1, \ldots, a_{k^2}$ to be an enumeration of $\{\frac{1}{\sqrt{k}} u_{ij} : 1 \leq i, j \leq k\}$, we have

$$\sum_{i=1}^{k^2} a_i^* a_i = \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} u_{ij}^* u_{ij} = p \quad \text{and} \quad \sum_{i=1}^{k^2} a_i a_i^* = \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} u_{ij} u_{ij}^* \leq \frac{k-1}{k} q.$$

We note that $a_i \in q\mathcal{A}^{**}p$ and $a_i a_i^* \in (q\mathcal{A}^{**}q)_+$ for all $1 \leq i \leq k^2$. By the Kaplansky density theorem, we have that $q\mathcal{A}p$ and $(q\mathcal{A}q)_+$ are σ-strong dense (and hence σ-weak dense) in $q\mathcal{A}^{**}p$ and $(q\mathcal{A}^{**}q)_+$ respectively. Hence there exists a net $(b_\alpha^{(1)}, \ldots, b_\alpha^{(k^2)}, g_\alpha)_{\alpha \in A}$ in $\left[\oplus_{i=1}^{k^2} q\mathcal{A}p \right] \oplus (q\mathcal{A}q)$ such that $g_\alpha \geq 0$ and

$$\lim_\alpha b_\alpha^{(i)} = a_i \quad \text{and} \quad \lim_\alpha g_\alpha = \frac{k-1}{k} q - \sum_{i=1}^{k^2} a_i a_i^* \quad \sigma\text{-weakly}.$$

It follows that

$$\lim_\alpha \sum_{i=1}^{k^2} (b_\alpha^{(i)})^* b_\alpha^{(i)} = p \quad \text{and} \quad \lim_\alpha \left(g_\alpha + \sum_{i=1}^{k^2} b_\alpha^{(i)} (b_\alpha^{(i)})^* \right) = \frac{k-1}{k} q \quad \sigma\text{-weakly}.$$

Consider now the convex cone

$$K := \left\{ \left(\sum_{i=1}^{r} b_i^* b_i, g + \sum_{i=1}^{r} b_i b_i^* \right) : r \in \mathbb{N},\ b_1, \ldots, b_r \in q\mathcal{A}p,\ g \in (q\mathcal{A}q)_+ \right\} \subseteq \mathcal{A} \oplus \mathcal{A}.$$

Since the restriction of the σ-weak topology on $\mathcal{A}^{**}$ to $\mathcal{A}$ is the weak topology (Lemma 2.2), it follows that

$$\left(p, \frac{k-1}{k}q\right) \in \overline{K}^w = \overline{K}$$

where $\overline{K}^w$ and $\overline{K}$ denote respectively the the weak and norm closures of K (which coincide by Mazurs' theorem). Choose now $\delta \in (0,1)$ such that

$$(1-\delta)^{-1}\left(\frac{k-1}{k} + \delta\right) < 1.$$

Note that this is always possible since $(1-\delta)^{-1}\left(\frac{k-1}{k}+\delta\right) \to \frac{k-1}{k} < 1$ as $\delta \to 0$. Since $\left(q, \frac{k-1}{k}p\right) \in \overline{K}$, there exists $b_1, \ldots, b_r \in q\mathcal{A}p$ and $g \in (q\mathcal{A}q)_+$ such that

$$\left\| p - \sum_{i=1}^r b_i^* b_i \right\| < \delta \quad \text{and} \quad \left\| \frac{k-1}{k}q - \left(g + \sum_{i=1}^r b_i b_i^*\right) \right\| < \delta.$$

Since $p = \mathbf{1}_{p\mathcal{A}p}$, we have $\|\mathbf{1}_{p\mathcal{A}p} - \sum_{i=1}^r b_i^* b_i\| < 1$ and thus we see that $\mathbf{1}_{p\mathcal{A}p} - (\mathbf{1}_{p\mathcal{A}p} - \sum_{i=1}^r b_i^* b_i) = \sum_{i=1}^r b_i^* b_i$ is invertible in the C^*-algebra $p\mathcal{A}p$, with inverse $h \in (p\mathcal{A}p)_+$. Then

$$(1-\delta)p \leq \sum_{i=1}^r b_i^* b_i \leq (1+\delta)p \implies (1+\delta)^{-1}p \leq h \leq (1-\delta)^{-1}p.$$

Taking $c_i := b_i h^{1/2}$, we see that $\sum_{i=1}^r c_i^* c_i = \mathbf{1}_{p\mathcal{A}p} = p$ and moreover

$$\sum_{i=1}^r c_i c_i^* = \sum_{i=1}^r b_i h b_i^* \leq (1-\delta)^{-1} \sum_{i=1}^r b_i b_i^* \leq (1-\delta)^{-1}\left(g + \sum_{i=1}^r b_i b_i^*\right) \leq (1-\delta)^{-1}\left(\frac{k-1}{k} + \delta\right) p.$$

Since $(1-\delta)^{-1}\left(\frac{k-1}{k}+\delta\right) < 1$, we may write this as $1-\epsilon$ for some $\epsilon > 0$, and we have the desired conclusion (2). $\square$

4. Random Matrices

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say a random variable $Y^{(n)} : \Omega \to M_n(\mathbb{C})$ is a *Gaussian random matrix* with variance σ^2 if the real and imaginary parts of the entries $\{\mathrm{Re}(Y_{ij}^{(n)}), \mathrm{Im}(Y_{ij}^{(n)}) : 1 \leq i,j \leq n\}$ are independent and identically distributed with Gaussian distribution[2] with mean zero and variance σ^2. For each $n \in \mathbb{N}$, let $Y_1^{(n)}, \ldots, Y_r^{(n)}$ be independent $n \times n$ Gaussian random matrices with variance $\frac{1}{n}$. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $c_1, \ldots, c_r \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be bounded operators satisfying

$$\sum_{i=1}^r c_i^* c_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \quad \text{and} \quad \sum_{i=1}^r c_i c_i^* \leq \mathbf{1}_{\mathcal{B}(\mathcal{K})} \quad \text{for some } c \geq 1. \tag{4.1}$$

Put $S_n := \sum_{i=1}^r c_i \otimes Y_i^{(n)}$. We are interested in an asymptotic lower bound as $n \to \infty$ for the spectrum of the random matrix $S_n^* S_n$, which we note is positive and hence has spectrum contained in $[0, \infty)$.

Theorem 4.1 (Theorem 8.7 in [2])**.** *Let $\mathcal{A}$ be the unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\{c_i^* c_j : 1 \leq i,j \leq r\}$. If $\mathcal{A}$ is exact, then*

$$\liminf_{n\to\infty} \min\{\sigma_n(S_n(\omega)^* S_n(\omega))\} \geq \left(\sqrt{c}-1\right)^2 \quad \textit{for almost every } \omega \in \Omega,$$

where σ_n denotes the spectrum in $M_n(\mathcal{A})$.

[2]We say that a random variable $X : \Omega \to \mathbb{R}$ is Gaussian distributed with mean μ and variance σ^2 if $\mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) = (2\pi\sigma^{-2})^{-1/2} \int_A e^{-(x-\mu)^2/2\sigma^2}\, dx$ for all Borel sets $A \in \mathcal{B}(\mathbb{R})$.

A full proof of the above would take up dozens of pages and is thus infeasible to include in this project. The rough idea is as follows. First, under the slightly stronger assumption that

$$\sum_{i=1}^{r} c_i^* c_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})} \quad \text{and} \quad \sum_{i=1}^{r} c_i c_i^* = \mathbf{1}_{\mathcal{B}(\mathcal{K})} \quad \text{for some } c \geq 1, \tag{4.2}$$

we establish the bound

$$\mathbb{E}\left[\exp\left(-tS_n^*S_n\right)\right] \leq \exp\left(-(\sqrt{c}-1)^2 t + (c+1)^2 \frac{t^2}{n}\right) \mathbf{1}_{M_n(\mathcal{H})} \quad \text{for all } 0 \leq t \leq \frac{n}{2c}$$

through combinatorial techniques. A bit of work involving an application of the Borel-Cantelli lemma will then allow us to establish the result of Theorem 4.1 when $\mathcal{H}$ is finite-dimensional and under assumption (4.2). The exactness assumption for $\mathcal{A}$ comes in when passing from finite-dimensional $\mathcal{H}$ to infinite-dimensional. More precisely, we make use of the characterisation that a unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is exact if and only if there exists a net $(M_{n_\alpha}(\mathbb{C}))_{\alpha \in A}$ of matrix algebras such that the inclusion map $i : \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ admits for each $\alpha \in A$ a factorisation

$$\mathcal{A} \xrightarrow{\varphi_\alpha} M_{n_\alpha}(\mathbb{C}) \xrightarrow{\psi_\alpha} \mathcal{B}(\mathcal{H})$$

where φ_α and ψ_α are unital completely positive[3] maps satisfying $\lim_\alpha \|\psi_\alpha \circ \varphi_\alpha(x) - x\| = 0$ for all $x \in \mathcal{A}$. This allows us to establish Theorem 4.1 for infinite-dimensional $\mathcal{H}$, but still under the stronger assumption (4.2). Relaxing to the less restrictive assumption of (4.1) is then done by a dilation argument. That is, given $c_1, \ldots, c_r \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying the less restrictive assumption (4.1), we construct larger Hilbert spaces $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ and $\tilde{\mathcal{K}} \supseteq \mathcal{K}$ and operators $\tilde{c}_1, \ldots, \tilde{c}_s \subseteq \mathcal{B}(\tilde{\mathcal{H}}, \tilde{\mathcal{K}})$ where $s \geq r$, such that

$$\tilde{c}_i|_{\mathcal{H}} = \begin{cases} c_i & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r+1 \leq i \leq s, \end{cases}$$

and the $\tilde{c}_i$ satisfy the stronger assumption (4.2) for $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}$. We can then apply the established version of the theorem to the operators $\tilde{c}_1, \ldots, \tilde{c}_s$, yielding the desired conclusion for $c_1, \ldots, c_r$.

5. Proof of Main Result

We now embark on proving the main result, namely that if $\mathcal{A}$ is a unital exact stably finite C^*-algebra then it must admit a tracial state.

Theorem 5.1. *Let $\mathcal{A}$ be an exact C^*-algebra, and let $p, q \in \mathcal{A}$ be projections. If there exists $\epsilon > 0$ such that $\tau(p) \leq (1-\epsilon)\tau(p)$ for any lower semicontinuous trace τ on $\mathcal{A}$, then we have*

$$p \otimes \mathbf{1}_n \prec q \otimes \mathbf{1}_n \quad \text{in } M_n(\mathcal{A}) \text{ for some } n \in \mathbb{N}.$$

Proof. Let $p, q \in \mathcal{A}$ and $\epsilon > 0$ be quantities satisfying the stated assumption. By Corollary 3.3, there exist $c_1, \ldots, c_r \in q\mathcal{A}p$ such that

$$\sum_{i=1}^{r} c_i^* c_i = cp \quad \text{and} \quad \sum_{i=1}^{r} c_i c_i^* \leq q,$$

where $c := (1-\epsilon)^{-1/2} > 1$. By the Gelfand-Naimark theorem, we may without loss of generality assume that $\mathcal{A}$ is a concrete C^*-algebra of bounded operators on some Hilbert space $\mathcal{H}$. Hence we

[3]A linear map $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is *positive* if it maps positive elements to positive elements. We then say ϕ is *completely positive* if $\phi \otimes \mathbf{1}_{M_n(\mathbb{C})} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is a positive map for all $n \in \mathbb{N}$.

may consider $c_1, \dots, c_r$ as elements of $\mathcal{B}(p\mathcal{H}, q\mathcal{H})$, and we have

$$\sum_{i=1}^{r} c_i^* c_i = c\mathbf{1}_{\mathcal{B}(q\mathcal{H})} \quad \text{and} \quad \sum_{i=1}^{r} c_i c_i^* \leq \mathbf{1}_{\mathcal{B}(q\mathcal{H})}.$$

The algebra $\mathcal{F}$ generated by $\{c_i^* c_j : 1 \leq i, j \leq r\}$ is exact, being a subalgebra of the exact algebra $p\mathcal{A}p$. Hence for $n \in \mathbb{N}$, choosing $Y_1^{(n)}, \dots, Y_r^{(n)}$ to be independent $n \times n$ Gaussian random matrices with variance $\frac{1}{n}$, we have by Theorem 4.1 that

$$\liminf_{n\to\infty} \min \{\sigma_n(S_n^*(\omega)S_n(\omega))\} \geq \left(\sqrt{c} - 1\right)^2 \quad \text{for almost every } \omega \in \Omega,$$

where $S_n = \sum_{i=1}^{r} c_i \otimes Y_i^{(n)}$. In particular, there exists $\omega \in \Omega$ and $n \in \mathbb{N}$ such that $S_n(\omega)^* S_n(\omega)$ is invertible in $M_n(\mathcal{F})$ and hence invertible in $M_n(p\mathcal{A}p)$. Let us define

$$u := S_n(\omega)[S_n(\omega)^* S_n(\omega)]^{-1/2}.$$

Then $u \in M_n(q\mathcal{A}p)$ and we note that $u^*u \in M_n(\mathcal{B}(p\mathcal{H}))$ and $uu^* \in M_n(\mathcal{B}(q\mathcal{H}))$. Since $u^*u = \mathbf{1}_{\mathcal{B}(p\mathcal{H})} \otimes \mathbf{1}_n = p \otimes \mathbf{1}_n$ is a projection in $M_n(\mathcal{B}(p\mathcal{H}))$, we have that uu^* is a projection in $M_n(\mathcal{B}(q\mathcal{H}))$ and it satisfies

$$uu^* \leq \mathbf{1}_{\mathcal{B}(q\mathcal{H})} \otimes \mathbf{1}_n = q \otimes \mathbf{1}_n.$$

That is, we have $p \otimes \mathbf{1}_n \prec q \otimes \mathbf{1}_n$ in $M_n(\mathcal{A})$ as desired. $\square$

Theorem 5.2. *Let $\mathcal{A}$ be a unital exact C^*-algebra. If $\mathcal{A}$ has no tracial states, then there exists $n \in \mathbb{N}$ and projections $p, q \in M_n(\mathcal{A})$ such that*

$$p \perp q \quad \textit{and} \quad p \sim q \sim \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n.$$

Proof. Assume that $\mathcal{A}$ is a unital exact C^*-algebra with no tracial states. Consider the projections $p', q' \in M_2(\mathcal{A})$ defined by

$$p' = \begin{pmatrix} \mathbf{1}_{\mathcal{A}} & 0 \\ 0 & \mathbf{1}_{\mathcal{A}} \end{pmatrix} \quad \text{and} \quad q' = \begin{pmatrix} \mathbf{1}_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathcal{A}$ has no tracial states and it embeds unitally into $\mathcal{A}^{**}$, neither does $\mathcal{A}^{**}$. As von Neumann algebras either admit a tracial state or is properly infinite, we see that $\mathcal{A}^{**}$ is properly infinite. Hence by Proposition 3.1 we have

$$[\mathbf{1}_{\mathcal{A}}] = [\mathbf{1}_{\mathcal{A}}] \oplus [\mathbf{1}_{\mathcal{A}}] \oplus [\mathbf{1}_{\mathcal{A}}] \oplus [\mathbf{1}_{\mathcal{A}}] \quad \text{in } V(\mathcal{A}^{**}),$$

and so we have $[q'] = [p'] \oplus [p']$ in $V(M_2(\mathcal{A}^{**}))$. By Proposition 3.2 it follows that there exists $\epsilon > 0$ such that $\tau(p) \leq (1 - \epsilon)\tau(q)$ for any lower semicontinuous trace τ on $\mathcal{A}$, and by Theorem 5.1 there exists $n \in \mathbb{N}$ such that

$$p' \otimes \mathbf{1}_n \prec q' \otimes \mathbf{1}_n \quad \text{in } M_{2n}(\mathcal{A}).$$

That is, there exists $u \in M_{2n}(\mathcal{A})$ such that

$$u^*u = \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n \end{pmatrix} \quad \text{and} \quad uu^* \leq \begin{pmatrix} \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.1}$$

It follows from the inequality in (5.1) that u must have the form

$$u = \begin{pmatrix} u_{11} & u_{12} \\ 0 & 0 \end{pmatrix} \quad \text{for some } u_{11}, u_{12} \in M_n(\mathcal{A}).$$

The equality in (5.1) then implies

$$u_{11}^* u_{11} = u_{12}^* u_{12} = \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n \quad \text{and} \quad u_{11}^* u_{12} = 0.$$

Taking $p := u_{11} u_{11}^*$ and $q := u_{12} u_{12}^*$, we see that p, q are projections in $M_n(\mathcal{A})$ which are orthogonal and satisfy $p \sim q \sim \mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n$. $\square$

The main result now follows immediately. Indeed, if $\mathcal{A}$ is a unital exact stably finite C^*-algebra which admits no tracial states, then Theorem 5.2 implies that there exists $n \in \mathbb{N}$ and projections $p, q \in M_n(\mathcal{A})$ such that the unit $\mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n$ is equivalent to two projections which are orthogonal. In particular, this implies $\mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_n$ is not finite, contradicting the assumption that $M_n(\mathcal{A})$ is finite for every n.

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