A_∞-ABSOLUTE CONTINUITY OF ELLIPTIC MEASURES IN 1-SIDED CHORD-ARC DOMAINS

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1. Introduction

Let Ω be a 1-sided chord-arc domain in $\mathbb{R}^n$ and let $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω. Recall that there exists a unique collection of Borel probability measures $\omega_L = \{\omega_L^X\}_{X\in\Omega}$ called the elliptic measure, such that for any $g \in H^{1/2}(\partial\Omega, \sigma) \cap C(\partial\Omega)$, the function

$$u_g(X) := \int_{\partial\Omega} g(y)\, d\omega_L^X(y)$$

determines the unique weak solution to $Lu = 0$ with $\operatorname{Tr} u = g$. An interesting question to ask is when does absolute continuity of ω_L with respect to the surface measure hold? We will specifically consider a quantitative version of absolute continuity, where the Radon-Nikodym derivative satisfies a reverse Hölder inequality, or equivalently, where the Hardy-Littlewood maximal function is of weak type (p, p) for some exponent p. This will be made precise soon. The aim in this project is to show, following [CHMT20], that a sufficient condition for this quantitative absolute continuity of the elliptic measure to hold is that every bounded weak solution satisfies a certain Carleson measure condition. In fact, this is also a necessary condition as was also shown in [CHMT20], but the proof of which requires Green functions and is beyond the scope of this project.

2. Setting up

To simplify the notation, for $x \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$ let us write $\Delta(x, r) := B(x, r) \cap \partial\Omega$ for the surface ball with centre x and radius r. We now introduce a set of notations that will be used throughout the article.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ where $n \geq 2$, be a 1-sided chord-arc domain (1-sided CAD) and $L = -\operatorname{div} A\nabla$ be a uniformly elliptic operator on Ω.

(i) For every $X \in \Omega$, we write $\delta(X) := \operatorname{dist}(X, \partial\Omega)$ for the distance from X to the boundary.
(ii) We write ϵ for the Corkscrew constant. That is, ϵ is a positive constant such that for every surface ball $\Delta = \Delta(x, r)$, there exists a point $X_\Delta \in \Delta$ which we recall is called the corkscrew point, such that $B(X_\Delta, \epsilon r) \subset \Omega$.
(iii) We denote by C_{AR} the Ahlfors regular constant of $\partial\Omega$. That is, $C_{\mathrm{AR}} > 0$ is such that $C_{\mathrm{AR}}^{-1} r^{n-1} \leq \sigma(\Delta(x, r)) \leq C_{\mathrm{AR}} r^{n-1}$ for all $x \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$, where $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$ is the restriction of the $(n-1)$-dimensional Hausdorff measure to $\partial\Omega$.
(iv) We write Λ for the ellipticity constant of L. That is, $\Lambda > 0$ is such that $A(X)\xi \cdot \xi \geq \Lambda^{-1}|\xi|^2$ and $|A(X)\xi \cdot \zeta| \geq \Lambda|\xi||\zeta|$ for all $\xi, \zeta \in \mathbb{R}^n$ and $X \in \Omega$.

The constants ϵ and C_{AR} as well as the Harnack chain constants will be collectively called the **1-sided CAD constants**. Let us say $\omega_L \ll \sigma$ if there exists $X \in \Omega$ such that $\omega_L^X \ll \sigma$. By the Harnack inequality, ω_L^X and ω_L^Y are mutually absolutely continuous for all $X, Y \in \Omega$, hence $\omega_L^X \ll \sigma$ holding for one $X \in \Omega$ is equivalent to it holding for all $X \in \Omega$. We now give the precise definition of the aforementioned quantitative version of absolute continuity we will work with.

Definition 2.2. We say that $\omega_L \in A_\infty(\partial\Omega)$ if there exist $\alpha, \beta \in (0,1)$ such that if $\Delta_0 = B_0 \cap \partial\Omega$ is a surface ball, then for every surface ball $\Delta = B \cap \partial\Omega$ with $B \subset B_0$ and for every Borel set $F \subset \Delta$, we have

$$\frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)} \le \alpha \implies \frac{\sigma(F)}{\sigma(\Delta)} \le \beta.$$

As remarked in the introduction, the condition that $\omega_L \in A_\infty(\partial\Omega)$ is related to properties of the Hardy-Littlewood maximal function and the Radon-Nikodym derivative. We make this connection precise now. Let us recall that the Hardy-Littlewood maximal function M_σ is defined for a locally integrable function f by

$$M_\sigma f(x) := \sup \fint_\Delta |f(y)|\, d\sigma(y),$$

where the supremum is taken over all surface balls Δ centred at x.

Proposition 2.3. The following are equivalent.

(i) $\omega_L \in A_\infty(\partial\Omega)$.

(ii) There exists $p \in [1,\infty)$ such that for every surface ball Δ_0 and every surface ball Δ such that $\Delta \subset \Delta_0$, the Hardy-Littlewood maximal function M_σ is of weak type $(L^p(\Delta, d\omega_L^{X_{\Delta_0}}), L^p(\Delta, d\omega_L^{X_{\Delta_0}}))$.

(iii) $\omega_L \ll \sigma$ and there exists $r \in (1,\infty)$ such that for every surface ball Δ_0 and every surface ball Δ such that $\Delta \subset \Delta_0$, the Radon-Nikodym derivative $k_L^{X_{\Delta_0}} := d\omega_L^{X_{\Delta_0}}/d\sigma$ satisfies the reverse Hölder inequality

$$\left(\fint_\Delta |k_L^{X_{\Delta_0}}(y)|^r\, d\sigma(y)\right)^{1/r} \le C \fint_\Delta k_L^{X_{\Delta_0}}(y)\, d\sigma(y).$$

In the case that condition (ii) above holds with $p > 1$, the weak type (p,p) property can be replaced by simply L^p boundedness. Hence (ii) can be equivalently stated as

(ii') Either M_σ is of weak type $(L^1(\Delta, d\omega_L^{X_{\Delta_0}}), L^1(\Delta, d\omega_L^{X_{\Delta_0}}))$ for all surface balls Δ_0 and Δ such that $\Delta \subset \Delta_0$, or there exists $p \in (1,\infty)$ such that M_σ is bounded on $L^p(\Delta, d\omega_L^{X_{\Delta_0}})$ for all surface balls Δ_0 and Δ such that $\Delta \subset \Delta_0$.

See Appendix A for the discussion on this and the proof of Proposition 2.3. To conclude this section, we give here the statement of the main result which we aim to prove.

Theorem 2.4. Let $\Omega \subset \mathbb{R}^n$ be 1-sided chord-arc domain and L be a uniformly elliptic operator on Ω. If there exists $C > 0$ such that every bounded weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ to $Lu = 0$ satisfies the Carleson measure condition

$$\sup_{x\in\partial\Omega,\, r>0} \frac{1}{r^n} \int_{B(x,r)\cap\Omega} |\nabla u(X)|^2 \delta(X)\, dX \le C\|u\|^2_{L^\infty(\Omega)},$$

then $\omega_L \in A_\infty(\partial\Omega)$.

3. Dyadic decomposition of the boundary

A crucial tool we shall need for our analysis is a "dyadic decomposition" of the boundary. Similar to the dyadic cubes in $\mathbb{R}^n$, at each scale $k \in \mathbb{Z}$ we have a partition of $\partial\Omega$ into "cubes" with analogous properties to that of dyadic cubes in $\mathbb{R}^n$.

3.1. **Dyadic decomposition.** The following gives the existence of a dyadic decomposition of $\partial\Omega$. We omit the proof; interested readers are directed to Section 2 of [DS91] for a construction of these sets.

Lemma 3.1. There exist constants $a > 0$, $\eta > 0$ and C, depending only on n and C_{AR}, such that for every $k \in \mathbb{Z}$ there exists a collection of Borel sets $\mathbb{D}_k := \{Q_j^k : j \in \mathcal{J}_k\}$ called **dyadic cubes** such that

(i) For every $k \in \mathbb{Z}$, $\partial\Omega = \bigcup_j Q_j^k$.
(ii) If $m \geq k$, then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.
(iii) If $k \in \mathbb{Z}$ and $j \in \mathcal{J}_k$, then for each $m < k$ there is a unique $i \in \mathcal{J}_m$ such that $Q_j^k \subset Q_i^m$. The dyadic cube Q_i^m is called a parent of Q_j^k.
(iv) $\operatorname{diam}(Q_j^k) \leq C2^{-k}$.
(v) Each dyadic cube Q_j^k contains a surface ball $\Delta(x_j^k, a_0 2^{-k})$ for some $x_j^k \in \partial\Omega$.
(vi) For every $\tau \in (0, a_0)$, $k \in \mathbb{Z}$, and $j \in \mathcal{J}_k$, it holds that

$$\sigma\left(\left\{x \in Q_j^k : \operatorname{dist}(x, \partial\Omega \backslash Q_j^k) \leq \tau 2^{-k}\right\}\right) \leq C\tau^n \sigma(Q_j^k).$$

Let us write $\mathbb{D}(\partial\Omega) := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k$ for the collection of all dyadic cubes at any scale. For a dyadic cube $Q \in \mathbb{D}(\partial\Omega)$, we write $k(Q) := k$ whenever $Q \in \mathbb{D}_k$, and we define the **length** of Q to be $\ell(Q) := 2^{-k(Q)}$. We also write Q^* for the **parent** of Q, that is, Q^* is the unique dyadic cube containing Q such that $\ell(Q^*) = 2\ell(Q)$. We note that conditions (iv) and (v) above together imply that $\operatorname{diam}(Q) \approx \ell(Q)$ and moreover, the existence of a point $x_Q \in \partial\Omega$ and a surface ball $\Delta(x_Q, r_Q)$ such that

$$c\ell(Q) \leq r_Q \leq \ell(Q) \quad \text{and} \quad \Delta(x_Q, 2r_Q) \subset Q \subset \Delta(x_Q, Cr_Q) \tag{3.1}$$

for some uniform constants $c > 0$ and $C > 1$. Let us write

$$B_Q := B(x_Q, r_Q), \quad \widetilde{B}_Q := B(x_Q, Cr_Q), \quad \Delta_Q := \Delta(x_Q, r_Q), \quad \widetilde{\Delta}_Q := \Delta(x_Q, Cr_Q). \tag{3.2}$$

The point x_Q is called the **centre** of the cube Q. By means of the surface ball Δ_Q, we can define the corkscrew point relative to a dyadic cube Q by $X_Q := x_{\Delta_Q}$ which has the property that $\delta(X_Q) \approx \operatorname{dist}(X_Q, Q) \approx \operatorname{diam}(Q)$.

3.2. **Discretized Carleson regions and sawtooths.** Given a dyadic cube $Q \in \mathbb{D}(\partial\Omega)$, we define a corresponding discretized Carleson region by $\mathbb{D}_Q := \{Q' \in \mathbb{D}(\partial\Omega) : Q' \subset Q\}$. For a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}(\partial\Omega)$, we define the global discretized sawtooth relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D}(\partial\Omega) \backslash \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j},$$

and if in addition we have a fixed cube $Q \in \mathbb{D}(\partial\Omega)$, we define the associated local discretized sawtooth relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \backslash \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q.$$

3.3. **Carleson boxes and sawtooths.** We will also require a "geometric" version of the above. We follow the construction in Section 3 of [HMUT14]. Let $\mathcal{W} = \{W_i\}$ denote the Whitney decomposition of Ω, which we recall is the collection $\{W_i\}$ of maximal dyadic cubes in $\mathbb{R}^n$ satisfying $40W_i \subset \Omega$. Recall also that $\mathcal{W}$ forms a partition of Ω and we have

$$4\operatorname{diam}(I) \leq \operatorname{dist}(4I, \partial\Omega) \leq \operatorname{dist}(I, \partial\Omega) \leq 40\operatorname{diam}(I) \quad \text{for all } I \in \mathcal{W},$$

and $\operatorname{diam}(I) \approx \operatorname{diam}(J)$ whenever the boundaries of I and J touch. For $I \in \mathcal{W}$, we write $\ell(I)$ for the length of I and $k_I := k$ if $\ell(I) = 2^{-k}$. For $Q \in \mathbb{D}(\partial\Omega)$, define

$$\mathcal{W}_Q := \{I \in \mathcal{W}(\Omega) : k(Q) - m_0 \leq k_I \leq k(Q) + 1 \text{ and } \operatorname{dist}(I, Q) \leq C_0 2^{-k(Q)}\},$$

for some constants m_0 and C_0 depending only on the Corkscrew constant ϵ and the constants from Lemma 3.1, chosen such that:

- $X_Q \in I$ for some $I \in \mathcal{W}_Q$, and
- for each dyadic child $R \subset Q$, we have $X_R \in J$ for some $J \in \mathcal{W}_Q$.

In particular, the collection $\mathcal{W}_Q$ is non-empty for every $Q \in \mathbb{D}(\partial\Omega)$. Moreover, by choosing C_0 large enough, we may assume that there always exists $I \in \mathcal{W}_Q$ such that $k(Q) - 1 \le k_I \le k(Q)$, that

$$\mathcal{W}_{Q_1} \cap \mathcal{W}_{Q_2} \neq \emptyset \quad \text{whenever } 1 \le \frac{\ell(Q_2)}{\ell(Q_1)} \le 2 \text{ and } \operatorname{dist}(Q_1, Q_2) \le 1000\ell(Q_2),$$

and that $C_0 \ge 1000\sqrt{n}$. Given $A \subset \Omega$, let us write $X \longrightarrow_A Y$ if the interior of A contains all the balls in a Harnack chain in Ω which connects X to Y and if for any point Z contained in any ball in the Harnack chain, it holds that

$$\operatorname{dist}(Z, \partial\Omega) \approx \operatorname{dist}(Z, \Omega \backslash A) \tag{3.3}$$

with uniform control of the implicit constants. We give an augmentation $\mathcal{W}_Q^*$ of $\mathcal{W}_Q$ to take advantage of the Harnack chain condition. For each $I \in \mathcal{W}_Q$, form a Harnack chain $H(I)$ from the centre $X(I)$ to the Corkscrew point X_Q relative to Q. Let $\mathcal{W}(I)$ be the collection of all Whitney cubes which intersects at least one ball in $H(I)$, and set

$$\mathcal{W}_Q^* := \bigcup_{I \in \mathcal{W}_Q} \mathcal{W}(I).$$

For $\lambda \in (0, 1)$ to be chosen, define the fattening $I^* := (1 + \lambda)I$ and let

$$U_Q := \bigcup_{I \in W_Q^*} I^*.$$

It is easy to see that $\mathcal{W}_Q \subset \mathcal{W}_Q^* \subset \mathcal{W}$, $X_Q \in U_Q$, and $X_R \in U_Q$ for each dyadic child $R \subset Q$. Moreover, one can see by construction that there exist constants k^* and K_0 such that for all $I \in \mathcal{W}_Q^*$, it holds that

- $k(Q) - k^* \le k_I \le k(Q) + k^*$,
- $X(I) \longrightarrow_{U_Q} X_Q$, and
- $\operatorname{dist}(I, Q) \le K_0 \ell(Q)$,

where k^*, K_0, and the implicit constants in (3.3) depend only on λ and the constants which m_0 and C_0 depend on. Now choose $\lambda \in (0, 1)$ small enough so that for any $I, J \in \mathcal{W}$, we have $\operatorname{dist}(I^*, J^*) \approx \operatorname{dist}(I, J)$ and $\operatorname{int}(I^*) \cap \operatorname{int}(J^*) \neq \emptyset \iff \partial I \cap \partial J \neq \emptyset$. We may also assume that there exists $\tau \in (1/2, 1)$ such that $\tau J \cap I^* = \emptyset$ whenever $I, J \in \mathcal{W}$ are distinct.

Definition 3.2. For $Q \in \mathbb{D}(\partial\Omega)$, the **Carleson box** associated to Q is defined by

$$T_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \right).$$

For a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}(\partial\Omega)$, the **global sawtooth relative to** $\mathcal{F}$ is defined by

$$\Omega_{\mathcal{F}} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F}}} U_{Q'} \right),$$

and if in addition we have a fixed cube $Q \in \mathbb{D}(\partial\Omega)$ we define the associated **local sawtooth relative to** $\mathcal{F}$ by

$$\Omega_{\mathcal{F},Q} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'} \right).$$

Remark. From (3.1), one sees that for every $Q \in \mathbb{D}(\partial\Omega)$ there are constants $\kappa_1 \in (0,1)$ and $\kappa_0 > 1$ depending only on the Corkscrew constant ϵ and the constants from Lemma 3.1 such that

$$\kappa_1 B_Q \cap \Omega \subset T_Q \subset \kappa_0 B_Q \cap \Omega. \tag{3.4}$$

Let $\mathcal{F} = \{Q_j\} \subset \mathbb{D}(\partial\Omega)$ be a disjoint family of dyadic cubes in $\partial\Omega$. For every $\rho > 0$, we further define another family $\mathcal{F}(\rho) \subset \mathbb{D}(\partial\Omega)$ by adding to $\mathcal{F}$ all cubes $Q \in \mathbb{D}(\partial\Omega)$ such that $\ell(Q) \leq \rho$ and taking $\mathcal{F}(\rho)$ to be the corresponding collection of maximal cubes. It is easy to see that $\mathbb{D}_{\mathcal{F}(\rho)}$ is precisely the union of all cubes $Q \in \mathbb{D}_{\mathcal{F}}$ such that $\ell(Q) > \rho$. If we take $Q \in \mathbb{D}(\partial\Omega)$, $\delta \in (0,1)$, and $\mathcal{F}_0 = \emptyset$, then $\mathcal{F}_0(\delta\ell(Q)) = \{Q' \in \mathbb{D}(\partial\Omega) : \delta\ell(Q)/2 < \ell(Q') \leq \delta\ell(Q)\}$, so we have $\mathbb{D}_{\mathcal{F}_0(\delta\ell(Q)),Q} = \{Q' \in \mathbb{D}_Q : \ell(Q') > \delta\ell(Q)\}$. Define $U_{Q,\delta} := \Omega_{\mathcal{F}_0(\delta\ell(Q)),Q}$. We note that each $U_{Q,\delta}$ is a Whitney region relative to Q whose distance to $\partial\Omega$ is on the order of $\delta\ell(Q)$. It will be useful to note the following.

Lemma 3.3. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. Then the sets $\{U_{Q,\delta}\}_{Q\in\mathbb{D}_{Q_0}}$ have bounded overlap with constant depending on δ.

Proof. Assume $X \in U_{Q,\delta} \cap U_{Q',\delta}$ where $Q, Q' \in \mathbb{D}(\partial\Omega)$. Then $\ell(Q) \approx_\delta \delta(X) \approx_\delta \ell(Q')$ and $\operatorname{dist}(Q,Q') \leq \operatorname{dist}(Q,X) + \operatorname{dist}(X,Q') \lesssim_\delta \ell(Q) + \ell(Q') \approx_\delta \ell(Q)$. The conclusion then follows once we note that the number of pairs of cubes Q and Q' satisfying $\ell(Q) \approx_\delta \ell(Q')$ and $\operatorname{dist}(Q,Q') \lesssim_\delta \ell(Q)$ is finite (depending on δ.) ◻

4. Further regularity of weak solutions

We give here some further results relating to weak solutions of the equation $Lu = 0$ in Ω. The following lemmas may be found in the work in progress paper [HMT14]. However, we note that Lemma 4.1 is well known in the case of the harmonic measure as it was shown in [Bou87], while Lemma 4.2(i) is an immediate consequence of the Hölder continuity of weak solutions near the boundary.

Lemma 4.1. There exist constants $c \in (0,1)$ and $C > 1$ depending only on C_{AR} and Λ, such that for every ball $B := B(x,r)$ centred on $\partial\Omega$,

$$\omega_L^Y(\Delta(x,r)) \geq C^{-1} \quad \text{for all } Y \in B(x,cr) \cap \Omega.$$

In particular, for every surface ball Δ we have $\omega_L^{X_\Delta}(\Delta) \geq C^{-1}$.

Lemma 4.2. There exist constants $C > 0$ and $0 < \gamma \leq 1$ depending only on n, Λ, and the 1-sided CAD constants, such that for every ball $B := B(x,r)$, the following hold:

(i) If $u \in W^{1,2}_{\mathrm{loc}}(B \cap \Omega) \cap C(\overline{B_0 \cap \Omega})$ is a nonnegative weak solution to $Lu = 0$ such that $u = 0$ on $B \cap \partial\Omega$, then

$$u(X) \leq C\left(\frac{|X-x|}{r}\right)^\gamma \sup_{Y\in\overline{B\cap\Omega}} u(Y), \quad \text{for all } X \in \frac{1}{2}B \cap \Omega.$$

(ii) If $X \in \Omega \backslash 4B$, then $\omega_L^X(2B \cap \partial\Omega) \leq C\omega_L^X(B \cap \partial\Omega)$.

(iii) For every Borel $F \subset B \cap \partial\Omega$ and every $X \in \Omega \backslash 2\kappa_0 B$ where κ_0 is the constant from (3.4),

$$C^{-1}\omega_L^{X_{B\cap\partial\Omega}}(F) \leq \frac{\omega_L^X(F)}{\omega_L^X(B\cap\partial\Omega)} \leq C\omega_L^{X_{B\cap\partial\Omega}}(F).$$

5. Good covers

Definition 5.1. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. Let $\epsilon_0 \in (0,1)$ and $F \subset Q_0$ be Borel. A **good ϵ_0-cover** of F with respect to a regular Borel measure μ on Q_0 is a collection $\{O_\ell\}_{\ell=1}^k$ of Borel subsets of Q_0 together with pairwise disjoint families $\mathcal{F}_\ell = \{Q_i^\ell\} \subset \mathbb{D}_{Q_0}$ such that

(i) $F \subset O_k \subset O_{k-1} \subset \cdots \subset O_1 \subset Q_0$,

(ii) $\mathcal{O}_\ell = \bigcup_{Q_i^\ell \in \mathcal{F}_\ell} Q_i^\ell$ for every $1 \le \ell \le k$, and
(iii) $\mu(\mathcal{O}_\ell \cap Q_i^{\ell-1}) \le \epsilon_0 \mu(Q_i^{\ell-1})$ for every $Q_i^{\ell-1} \in \mathcal{F}_{\ell-1}$ and $2 \le \ell \le k$.

The following shows that one can iterate in some sense the property (iii).

Lemma 5.2. If $\{\mathcal{O}_\ell\}_{\ell=1}^k$ is a good ϵ_0-cover of F with respect to μ, then for every $1 \le \ell \le k$, it holds that

$$\mu(\mathcal{O}_\ell \cap Q_i^m) \le \epsilon_0^{\ell-m} \mu(Q_i^m) \quad \text{for all } Q_i^m \in \mathcal{F}_m,\ 1 \le m \le \ell. \tag{5.1}$$

Proof. Let $1 \le \ell \le k$. We show the result by induction on m. If $m = \ell$ then (5.1) trivially holds since $Q_i^\ell \subset \mathcal{O}_\ell$. If $m = \ell - 1$ then this is just (iii) in the definition of a good ϵ_0-cover. Assume now that (5.1) holds for some $2 \le m \le \ell$, and we aim to show that it holds for $m-1$. We first claim that for every $Q_i^{m-1} \in \mathcal{F}_{m-1}$,

$$\mathcal{O}_\ell \cap Q_i^{m-1} \subset \bigcup_{Q_j^m \in \mathcal{F}_m,\, Q_j^m \subsetneq Q_j^{m-1}} \mathcal{O}_\ell \cap Q_j^m.$$

Indeed, let $x \in \mathcal{O}_\ell \cap Q_i^{m-1}$. Since $\mathcal{O}_\ell \cap Q_i^{m-1} \subset \mathcal{O}_m$ and $\mathcal{F}_m$ forms a partition of $\mathcal{O}_m$, there exists a unique $Q_j^m \in \mathcal{F}_m$ such that $x \in Q_j^m$. Hence $Q_i^{m-1} \cap Q_j^m \ne \emptyset$ and we have either $Q_i^{m-1} \subset Q_j^m$ or $Q_j^m \subsetneq Q_i^{m-1}$. If $Q_i^{m-1} \subset Q_j^m$ then $\mu(Q_i^{m-1}) = \mu(\mathcal{O}_m \cap Q_i^{m-1}) \le \epsilon_0 \mu(Q_i^{m-1})$, which is a contradiction. Hence we must have $Q_j^m \subsetneq Q_i^{m-1}$, and this proves the claim. Then using the claim and applying the induction hypothesis, we have

$$\mu(\mathcal{O}_\ell \cap Q_i^{m-1}) \le \sum_{Q_j^m \in \mathcal{F}_m,\, Q_j^m \subsetneq Q_j^{m-1}} \mu(\mathcal{O}_\ell \cap Q_j^m) \le \epsilon_0^{\ell-m} \sum_{Q_j^m \in \mathcal{F}_m,\, Q_j^m \subsetneq Q_j^{m-1}} \mu(Q_j^m) \le \epsilon_0^{\ell-m} \mu(\mathcal{O}_m \cap Q_i^{m-1}) \le \epsilon_0^{\ell-m} \mu(Q_i^{m-1})$$

which proves the lemma. $\square$

If $\{\mathcal{O}_\ell\}_{\ell=1}^k$ is a good ϵ_0-cover of F with respect to μ, we say the length of this cover is k. Let $Q_0 \in \mathbb{D}(\partial\Omega)$ and let us say μ is **dyadically doubling** on Q_0 if there exists $C_\mu \ge 1$ such that $\mu(Q^*) \le C_\mu \mu(Q)$ for every $Q \in \mathbb{D}_{Q_0} \setminus \{Q_0\}$, where Q^* is the parent of Q.

Lemma 5.3. Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. Assume that μ is a dyadically doubling regular Borel measure on Q_0. Then for every $\epsilon_0 \in (0, e^{-1}]$, if F is a Borel subset of Q_0 with $\mu(F) \le \alpha\mu(Q_0)$ for some $\alpha \in (0, \epsilon_0^2/(2C_\mu^2)]$, then F has a good ϵ_0-cover with respect to μ of length $k_0 = k_0(\alpha, \epsilon_0)$ with $k_0 \approx \log \alpha^{-1} / \log \epsilon_0^{-1}$.

Proof. Let ϵ_0, F, and α as above and let $a := C_\mu/\epsilon_0 > 1$. Since $0 < \alpha < \epsilon_0^2/(2C_\mu^2) = a^{-2}/2$, there is a unique $k_0 = k_0(\alpha, \epsilon_0) \in \mathbb{N}$ such that $a^{-k_0-1} < 2\alpha \le a^{-k_0}$. Taking the logarithm of both sides yields

$$\frac{1}{3(1 + \log C_\mu)} \frac{\log \alpha^{-1}}{\log \epsilon_0^{-1}} \le k_0 \le \frac{\log \alpha^{-1}}{\log \epsilon_0^{-1}}.$$

The outer regularity of μ implies there exists a Borel set $U \subset Q_0$ such that U is open in $\partial\Omega$ and $\mu(U \setminus F) < \alpha\mu(Q_0)$. Let $\widetilde{F} := U \cap Q_0$ and define the level sets

$$\Omega_k := \left\{ x \in Q_0 : M_{\mu,Q_0}^d(\chi_{\widetilde{F}})(x) > a^{-k} \right\}, \quad \text{for all } 1 \le k \le k_0,$$

where

$$M_{\mu,Q_0}^d f(x) := \sup_{Q \ni x,\, Q \in \mathbb{D}_{Q_0}} \fint_Q f(y)\, d\mu(y), \quad f \in L^1_{\text{loc}}(Q_0, d\mu)$$

is the dyadic maximal function with respect to μ. Clearly $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_{k_0} \subset Q_0$. We claim also that $\widetilde{F} \subset \Omega_1$. Indeed, let $x \in \widetilde{F}$. Since U is open in $\partial\Omega$, there exists $r_x > 0$ such that $B_x \cap \partial\Omega \subset U$, where $B_x := B(x, r_x)$. Let $Q_x \in \mathbb{D}(\partial\Omega)$ be a dyadic cube such that $x \in Q_x$, $\ell(Q_x) < \ell(Q_0)$, and $\text{diam}(Q_x) < r_x$.

Since $x \in \widetilde{F} \cap Q_x \subset Q_x \cap Q_0$ and $\ell(Q_x) < \ell(Q_0)$, it follows that $Q_x \in \mathbb{D}_{Q_0}$. Also since $\operatorname{diam}(Q_x) < r_x$ we see that $Q_x \subset B_x \cap \partial\Omega \subset U$, so $Q_x \subset U \cap Q_0 = \widetilde{F}$ and hence

$$M^d_{\mu,Q_0}(\chi_{\widetilde{F}})(x) \geq \frac{\mu(\widetilde{F} \cap Q_x)}{\mu(Q_x)} = 1 > a^{-1},$$

which implies $x \in \Omega_1$. We see from this that $\Omega_k \neq \emptyset$ for any k. By the definition of k_0, for every $1 \leq k \leq k_0$ we see that

$$\mu(\widetilde{F}) \leq \mu(U) \leq \mu(U \backslash F) + \mu(F) < 2\alpha\mu(Q_0) \leq a^{-k_0}\mu(Q_0) \leq a^{-k}\mu(Q_0).$$

For every $1 \leq k \leq k_0$, let $\mathcal{F}_k = \{Q_i^k\} \subset \mathbb{D}_{Q_0} \backslash \{Q_0\}$ be a maximal pairwise disjoint collection of dyadic cubes which satisfies

$$\mu(\widetilde{F} \cap Q_i^k) > a^{-k}\mu(Q_i^k). \tag{5.2}$$

We note that $\mathcal{F}_k \neq \emptyset$ for all k since $\Omega_k \neq \emptyset$, and also that $\Omega_k = \bigcup_{Q_i^k \in \mathcal{F}_k} Q_i^k$. Hence if we set $O_k := \Omega_{k_0-k+1}$, we note that $\{O_k\}_{k=1}^{k_0}$ satisfies (i) and (ii) in the definition of a good ϵ_0-cover of F. It remains to show that it satisfies (iii), to which end it suffices to show that $\mu(\Omega_k \cap Q_j^{k+1}) \leq \epsilon_0\mu(Q_j^{k+1})$ for every $Q_j^{k+1} \in \mathcal{F}_{k+1}$. First, we note that as μ is dyadically doubling, we have

$$\frac{\mu(\widetilde{F} \cap Q_i^k)}{\mu(Q_i^k)} \leq C_\mu \frac{\mu(\widetilde{F} \cap (Q_i^k)^*)}{\mu((Q_i^k)^*)} \leq C_\mu a^{-k},$$

where the second inequality is due to the maximality of $\mathcal{F}_k$. Next, we note that if $Q_i^k \cap Q_j^{k+1} \neq \emptyset$, then it must hold that $Q_i^k \subset Q_j^{k+1}$ as otherwise if $Q_j^{k+1} \subsetneq Q_i^k$ then (5.2) yields $a^{-k}\mu(Q_i^k) < \mu(\widetilde{F} \cap Q_i^k) \leq a^{-k-1}\mu(Q_i^k)$ where the last inequality follows from the maximality of $\mathcal{F}_{k+1}$, giving a contradiction as $a > 1$. Hence

$$\begin{aligned}
\mu(\Omega_k \cap Q_j^{k+1}) = \sum_{Q_i^k : Q_i^k \subset Q_j^{k+1}} \mu(Q_i^k \cap Q_j^{k+1}) &= \sum_{Q_i^k : Q_i^k \subset Q_j^{k+1}} \mu(Q_i^k) \\
&< a^k \sum_{Q_i^k : Q_i^k \subset Q_j^{k+1}} \mu(\widetilde{F} \cap Q_i^k) \\
&\leq a^k \mu(\widetilde{F} \cap Q_j^{k+1}) \leq a^{-1}C_\mu \mu(Q_j^{k+1}) = \epsilon_0 \mu(Q_j^{k+1}).
\end{aligned}$$

This completes the proof of the lemma. □

In particular, if $\mu(F) = 0$, then α can be taken arbitrarily small and thus the above lemma implies that there exists a good ϵ_0-cover of F of arbitrary length.

6. A conical square function estimate

Given $Q_0 \in \mathbb{D}(\partial\Omega)$ and $\eta \in (0, 1)$, define the modified non-tangential cone

$$\Gamma^n_{Q_0}(x) := \bigcup_{Q \in \mathbb{D}_{Q_0},\, Q \ni x} U_{Q,\eta^3}, \quad \text{where we recall that } U_{Q,\eta^3} = \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q,\, \ell(Q') > \delta\ell(Q)} U_{Q'} \right).$$

The aim of this section is to prove following important estimate which the main result relies on.

Proposition 6.1. There exists a constant $\eta \in (0, 1)$ depending only on n, Λ, and the 1-sided CAD constants, as well as constants $\alpha_0 \in (0, 1)$ and $C_\eta \geq 1$ depending on the same constants and additionally on η, such that for every triple (Q_0, α, F) where $Q_0 \in \mathbb{D}(\partial\Omega)$, $\alpha \in (0, \alpha_0)$, and $F \subset Q_0$ is a Borel set satisfying $\omega_L^{X_{Q_0}}(F) \leq \alpha \omega_L^{X_{Q_0}}(Q_0)$, there exists a Borel set $S \subset Q_0$ such that the bounded weak solution $u(X) = \omega_L^X(S)$ satisfies

$$S_{Q_0}^\eta u(x) := \left(\int_{\Gamma_{Q_0}^\eta(x)} |\nabla u(Y)|^2 \delta(Y)^{2-n} \, dY \right)^{1/2} \geq C_\eta^{-1} \left(\log \alpha^{-1}\right)^{1/2}, \quad \text{for all } x \in F.$$

The proof of the above will build upon a series of lemmas. Let $k_* \geq 1$ be a positive integer to be chosen and let $\eta := 2^{-k_*} < 1$. For $Q \in \mathbb{D}(\partial\Omega)$, we define $\widetilde{Q} \in \mathbb{D}_Q$ to be the unique dyadic cube such that $x_Q \in \widetilde{Q}$ and $\ell(\widetilde{Q}) = \eta \ell(Q)$.

Lemma 6.2. There exists constants $C > 0$ and $\gamma \in (0, 1]$ depending only on n, Λ, and the 1-sided CAD constants, such that

$$\omega_L^{X_{\widetilde{Q}}}(\partial\Omega \backslash Q) = \omega_L^{X_{\widetilde{Q}}}(\partial\Omega) - \omega_L^{X_{\widetilde{Q}}}(Q) \leq 1 - \omega_L^{X_{\widetilde{Q}}}(Q) \leq C\eta^\gamma.$$

Proof. Let $\varphi(X) = \varphi_0(r_Q^{-1}(X - r_Q))$, where $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi_{B(0,1)} \leq \varphi_0 \leq \chi_{B(0,2)}$. Then $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \varphi \leq 1$, $\operatorname{supp}(\varphi) \subset 2B_Q$, and $\varphi \equiv 1$ in B_Q. Since $\Delta(x_Q, 2r_Q) \subset Q$, we have

$$v(X_{\widetilde{Q}}) := \int_{\partial\Omega} \varphi(y) \, d\omega_L^{X_{\widetilde{Q}}}(y) \leq \omega_L^{X_{\widetilde{Q}}}(Q). \tag{6.1}$$

The function $v \in W_{\mathrm{loc}}^{1,2}(\Omega) \cap C(\overline{\Omega})$ is a weak solution to $Lu = 0$ with $\operatorname{Tr} u = \varphi|_{\partial\Omega} \equiv 1$, so $\widetilde{v} = 1 - v \in W_{\mathrm{loc}}^{1,2}(\Omega) \cap C(\overline{\Omega})$ is a weak solution to $Lu = 0$ with $\operatorname{Tr} u = 0$. Then (6.1) and Lemma 4.2(i) give the existence of the required C and γ such that

$$1 - \omega_L^{X_{\widetilde{Q}}}(Q) \leq 1 - v(X_{\widetilde{Q}}) = \widetilde{v}(X_{\widetilde{Q}}) \leq C \left(\frac{|X_{\widetilde{Q}} - x_Q|}{r_Q} \right)^\gamma \|\widetilde{v}\|_{L^\infty(\Omega)} \leq C\eta^\gamma,$$

where the last inequality follows from $|X_{\widetilde{Q}} - x_Q| \leq |X_{\widetilde{Q}} - x_{\widetilde{Q}}| + |x_{\widetilde{Q}} - x_Q| \lesssim \ell(\widetilde{Q}) = \eta\ell(Q)$ and the fact that $r_Q \approx \ell(Q)$. □

Lemma 6.3. There exist constants $c_0 \in (0, 1)$ depending only on C_{AR} and Λ, and δ depending only on C_{AR}, such that for every $\eta < \delta$, we have $c_0 \leq \omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}) \leq 1 - c_0$.

Proof. The first estimate $c_0 \leq \omega_L^{X_{\widetilde{Q}}}(\widetilde{Q})$ follows from Lemma 4.1 and the Harnack inequality. Next, we claim that if η is sufficiently small then there exists $\widetilde{Q}' \in \mathbb{D}(\partial\Omega)$ such that $\ell(\widetilde{Q}') = \ell(\widetilde{Q})$, $\widetilde{Q}' \cap \widetilde{Q} = \emptyset$, and $\operatorname{dist}(\widetilde{Q}', \widetilde{Q}) \lesssim \ell(\widetilde{Q})$. Indeed, let $\widetilde{Q}_j \in \mathbb{D}(\partial\Omega)$ denote the unique dyadic cube satisfying $\widetilde{Q} \subset \widetilde{Q}_j$ and $\ell(\widetilde{Q}_j) = 2^j \widetilde{Q}$. Then by Ahlfors regularity of $\partial\Omega$, we have $\sigma(\widetilde{Q}_j) \gtrsim \ell(\widetilde{Q}_j)^{n-1} = 2^{j(n-1)}\ell(\widetilde{Q})^{n-1} > \sigma(\widetilde{Q})$ whenever j is chosen to be large enough so that $2^{j(n-1)} > 2C_{\mathrm{AR}}$. Fixing this choice of j, we then have $\widetilde{Q} \subsetneq \widetilde{Q}_j$ and the desired $\widetilde{Q}'$ satisfying $\ell(\widetilde{Q}') = \ell(\widetilde{Q})$, $\widetilde{Q}' \cap \widetilde{Q} = \emptyset$, and $\operatorname{dist}(\widetilde{Q}', \widetilde{Q}) \lesssim \ell(\widetilde{Q})$, can then be picked from $\mathbb{D}_{\widetilde{Q}_j}$. Thus, by the Harnack inequality and Lemma 4.1, we have $\omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}') \gtrsim \omega_L^{X_{\widetilde{Q}'}}(\widetilde{Q}') \geq C^{-1}$ for some $C > 1$, and so

$$\omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}) = \omega_L^{X_{\widetilde{Q}}}(\partial\Omega) - \omega_L^{X_{\widetilde{Q}}}(\partial\Omega \backslash \widetilde{Q}) \leq 1 - \omega_L^{X_{\widetilde{Q}}}(\widetilde{Q}') \leq 1 - C^{-1}.$$

□

Now fix $Q_0 \in \mathbb{D}(\partial\Omega)$ and choose η small enough so that Lemma 6.3 holds. By Lemma 4.2 and the Harnack inequality, $\omega := \omega_L^{X_{Q_0}}$ is a dyadically doubling regular Borel measure on $\partial\Omega$, with dyadic doubling constant C_ω dependent only on n, Λ, and the 1-sided CAD constants. Let $\epsilon_0 \in (0, e^{-1})$ and $\alpha \in (0, \epsilon_0^2/(2C_\omega^2))$ to be chosen, and let $F \subset Q_0$ be a Borel set with $\omega(F) \leq \alpha\omega(Q_0)$. Then Lemma

5.3 yields the existence of a good ϵ_0-cover $\{O_\ell\}_{\ell=1}^k$ of F with respect to ω, with length $k \approx \frac{\log \alpha^{-1}}{\log \epsilon_0^{-1}}$. For $1 \leq \ell \leq k$, let $\mathcal{F}_\ell = \{Q_i^\ell\} \subset \mathbb{D}_{Q_0} \backslash \{Q_0\}$ be the collection of disjoint cubes such that $O_\ell = \bigcup_{Q_i^\ell \in \mathcal{F}_\ell} Q_i^\ell$. Define $\widetilde{O}_\ell := \bigcup_{Q_i^\ell \in \mathcal{F}_\ell} \widetilde{Q}_i^\ell$, and $S := \bigcup_{j=2}^k \widetilde{O}_{j-1} \backslash O_j$. Then S is a union of pairwise disjoint sets, so

$$\chi_S(y) = \sum_{j=2}^k \chi_{\widetilde{O}_{j-1} \backslash O_j}(y) \quad \text{for all } y \in \partial\Omega.$$

For $y \in F$ and $1 \leq \ell \leq k$, let $Q_i^\ell(y)$ denote the unique cube in $\mathcal{F}_\ell$ containing y, and let $P_i^\ell(y)$ be the unique cube in $\mathbb{D}_{Q_i^\ell(y)}$ containing y such that $\ell(P_i^\ell(y)) = \eta \ell(Q_i^\ell(y))$. Moreover, define $\widetilde{P}_i^\ell(y)$ to be the unique cube in $\mathbb{D}_{P_i^\ell(y)}$ containing the centre $x_{P_i^\ell(y)}$ of $P_i^\ell(y)$ and such that $\ell(\widetilde{P}_i^\ell(y)) = \eta \ell(P_i^\ell(y))$. Set $u(X) := \omega_L^X(S)$, so that

$$u(X) = \int_{\partial\Omega} \chi_S(y)\, d\omega_L^X(dy) = \sum_{j=2}^k \omega_L^X(\widetilde{O}_{j-1} \backslash O_j). \tag{6.2}$$

Recall that $X_{\widetilde{Q}_i^\ell(y)}$ and $X_{\widetilde{P}_i^\ell(y)}$ denote corkscrew points relative to the dyadic cubes $\widetilde{Q}_i^\ell(y)$ and $\widetilde{P}_i^\ell(y)$ respectively. The following estimate is the last missing tool before we can embark on the proof of Proposition 6.1.

Lemma 6.4. There exists $\delta' > 0$ depending only on n, Λ, and the 1-sided CAD constants, such that if η and ϵ_0 are less than δ', then it holds that

$$\left| u(X_{\widetilde{Q}_i^\ell(y)}) - u(X_{\widetilde{P}_i^\ell(y)}) \right| \geq \frac{1}{2} c_0 \quad \text{for all } y \in F \text{ and } 1 \leq \ell \leq k-1,$$

where c_0 is the constant from Lemma 6.3.

Proof. We will estimate $u(X_{\widetilde{Q}_i^\ell(y)})$ and $u(X_{\widetilde{P}_i^\ell(y)})$ separately. Let $y \in F$ and write Q_i^ℓ and P_i^ℓ for $Q_i^\ell(y)$ and $P_i^\ell(y)$ respectively.

Step 1: estimate for $u(X_{\widetilde{Q}_i^\ell})$. By Lemma 6.2 and (6.2), we have

$$u(X_{\widetilde{Q}_i^\ell}) = \omega_L^{X_{\widetilde{Q}_i^\ell}}(S) \leq \omega_L^{X_{\widetilde{Q}_i^\ell}}(\partial\Omega \backslash Q_i^\ell) + \omega_L^{X_{\widetilde{Q}_i^\ell}}(S \cap Q_i^\ell) \leq C\eta^\gamma + I, \tag{6.3}$$

where $I := \omega_L^{X_{\widetilde{Q}_i^\ell}}(S \cap Q_i^\ell)$. For every $1 \leq \ell \leq k-1$ and $2 \leq j \leq \ell$, we have $Q_i^\ell \subset O_\ell \subset O_j$, so $Q_i^\ell \cap (\widetilde{O}_{j-1} \backslash O_j) = \emptyset$ and hence by (6.2) we have

$$I = \sum_{j=2}^k \omega_L^{X_{\widetilde{Q}_i^\ell}}(Q_i^\ell \cap (\widetilde{O}_{j-1} \backslash O_j)) = \sum_{j=\ell+1}^k \omega_L^{X_{\widetilde{Q}_i^\ell}}(Q_i^\ell \cap (\widetilde{O}_{j-1} \backslash O_j)) = I_1 + I_2,$$

where $I_1 := \sum_{j=\ell+2}^k \omega_L^{X_{\widetilde{Q}_i^\ell}}(Q_i^\ell \cap (\widetilde{O}_{j-1} \backslash O_j))$ and $I_2 := \omega_L^{X_{\widetilde{Q}_i^\ell}}(Q_i^\ell \cap (\widetilde{O}_\ell \backslash O_{\ell+1}))$ with the notation that if $\ell = k-1$ then $I_1 = 0$. We claim that $I_1 \leq C_\eta \epsilon_0$. This is trivially true if $\ell = k-1$, so assume $1 \leq \ell \leq k-2$. Then by the Harnack inequality to move from $X_{\widetilde{Q}_i^\ell}$ to $X_{Q_i^\ell}$ with constant dependent on η, we have

$$\begin{aligned} I_1 &\leq C_\eta \sum_{j=\ell+2}^k \omega_L^{X_{Q_i^\ell}}(Q_i^\ell \cap (\widetilde{O}_{j-1} \backslash O_j)) && \text{by Harnack inequality,} \\ &\leq \frac{C_\eta}{\omega(Q_i^\ell)} \sum_{j=\ell+2}^k \omega(Q_i^\ell \cap (\widetilde{O}_{j-1} \backslash O_j)) && \text{by Lemma 4.2(ii) and (iii),} \end{aligned}$$

$$\leq \frac{C_\eta}{\omega(Q_i^\ell)} \sum_{j=\ell+2}^{k} \omega(Q_i^\ell \cap \mathcal{O}_{j-1}) \qquad \text{since } \widetilde{\mathcal{O}}_{j-1} \backslash \mathcal{O}_j \subset \mathcal{O}_{j-1},$$

$$\leq C_\eta \sum_{j=\ell+2}^{k} \epsilon_0^{j-1-\ell} \leq C_\eta \epsilon_0 \qquad \text{by Lemma 5.2,}$$

which shows the claim. Next, we note that since $Q_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell$, then Lemma 6.3 yields

$$I_2 = \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \backslash \mathcal{O}_{\ell+1}) \leq \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell) \leq 1 - c_0.$$

Hence if η and ϵ_0 are chosen small enough so that $C\eta^\gamma < c_0/8$ and $C_\eta \epsilon_0 < c_0/8$, then using equation (6.3), that $I = I_1 + I_2$, and the bounds we obtained from I_1 and I_2, we have

$$u(X_{\widetilde{Q}_i^\ell}) \leq C\eta^\gamma + C_\eta \epsilon_0 + 1 - c_0 \leq 1 - \frac{3}{4} c_0.$$

Next, we attempt to obtain a lower bound for $u(X_{\widetilde{Q}_i^\ell})$. To this end we note that

$$\begin{aligned} u(X_{\widetilde{Q}_i^\ell}) &= \omega_L^{X_{\widetilde{Q}_i^\ell}}(S) \geq \omega_L^{X_{\widetilde{Q}_i^\ell}}(Q_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1})) \\ &= \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \backslash \mathcal{O}_{\ell+1}) = \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell) - \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \geq c_0 - \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}), \end{aligned}$$

where to obtain the first equality on the second line we used again that $Q_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell$ and to obtain the last inequality we used Lemma 6.3. Then using the Harnack inequality to move from $X_{\widetilde{Q}_i^\ell}$ to $X_{Q_i^\ell}$ with constant depending on η, we get

$$\begin{aligned} \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) &\leq C_\eta \omega_L^{X_{Q_i^\ell}}(Q_i^\ell \cap \mathcal{O}_{\ell+1}) && \text{by Harnack inequality and since } \widetilde{Q}_i^\ell \subset Q_i^\ell, \\ &\leq C_\eta \frac{\omega(Q_i^\ell \cap \mathcal{O}_{\ell+1})}{\omega(Q_i^\ell)} && \text{by Lemma 4.2(ii) and (iii),} \\ &\leq C_\eta \epsilon_0 && \text{by Lemma 5.2.} \end{aligned} \tag{6.4}$$

Hence choosing ϵ_0 even smaller if necessary, we may assume $C_\eta \epsilon_0 < c_0/4$ and obtain

$$u(X_{\widetilde{Q}_i^\ell}) \geq c_0 - C_\eta \epsilon_0 \geq \frac{3}{4} c_0. \tag{6.5}$$

Step 2: estimate for $u(X_{\widetilde{P}_i^\ell})$. Consider two cases.

Case 1: $P_i^\ell \cap \widetilde{Q}_i^\ell = \emptyset$. Then in the exact same fashion as above, we can show that $u(X_{\widetilde{P}_i^\ell}) \leq C\eta^\gamma + \widehat{I}_1 + \widehat{I}_2$, where

$$\widehat{I}_1 := \sum_{j=\ell+2}^{k} \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \backslash \mathcal{O}_j)) \quad \text{and} \quad \widehat{I}_2 := \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1})).$$

Using the Harnack inequality to move from $X_{\widetilde{P}_i^\ell}$ to $X_{\widetilde{Q}_i^\ell}$ and the fact that $\widetilde{P}_i^\ell \subset Q_i^\ell$, we have

$$\widehat{I}_1 \leq C_\eta \sum_{j=\ell+2}^{k} \omega_L^{X_{\widetilde{Q}_i^\ell}}(Q_i^\ell \cap (\widetilde{\mathcal{O}}_{j-1} \backslash \mathcal{O}_j)) = C_\eta I_1 \leq C_\eta \epsilon_0.$$

Moreover, we have $P_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \backslash \mathcal{O}_{\ell+1}) = (P_i^\ell \cap \widetilde{Q}_i^\ell) \backslash \mathcal{O}_{\ell+1} \subset P_i^\ell \cap \widetilde{Q}_i^\ell = \emptyset$, so $\widehat{I}_2 = 0$. Hence if η and ϵ_0 are chosen small enough so that $C\eta^\gamma < c_0/8$ and $C_\eta \epsilon_0 < c_0/8$, then it holds that

$$u(X_{\widetilde{P}_i^\ell}) \leq C\eta^\gamma + C_\eta \epsilon_0 \leq \frac{1}{4} c_0.$$

Hence $|u(X_{\widetilde{Q}_i^\ell}) - u(X_{\widetilde{P}_i^\ell})| \geq u(X_{\widetilde{Q}_i^\ell}) - u(X_{\widetilde{P}_i^\ell}) \geq \frac{3}{4}c_0 - \frac{1}{4}c_0 = \frac{1}{2}c_0$.

Case 2: $P_i^\ell \cap \widetilde{Q}_i^\ell \neq \emptyset$. In this case, since $\ell(P_i^\ell) = \ell(\widetilde{Q}_i^\ell)$ we must have $P_i^\ell = \widetilde{Q}_i^\ell$. Hence $P_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell \cap \widetilde{\mathcal{O}}_\ell = \widetilde{Q}_i^\ell = P_i^\ell$, so by Lemma 6.2 we have

$$
\begin{aligned}
u(X_{\widetilde{P}_i^\ell}) = \omega_L^{X_{\widetilde{P}_i^\ell}}(S) &\geq \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap (\widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1})) = \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \widetilde{\mathcal{O}}_\ell \setminus \mathcal{O}_{\ell+1}) = \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \setminus \mathcal{O}_{\ell+1}) \\
&= \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell) - \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \mathcal{O}_{\ell+1}) \geq 1 - C\eta^\gamma - \omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \mathcal{O}_{\ell+1}).
\end{aligned}
$$

Using the Harnack inequality to move from $X_{\widetilde{P}_i^\ell}$ to $X_{\widetilde{Q}_i^\ell}$ with constant depending on η, and (6.4), we have

$$
\omega_L^{X_{\widetilde{P}_i^\ell}}(P_i^\ell \cap \mathcal{O}_{\ell+1}) = \omega_L^{X_{\widetilde{P}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \leq C_\eta \omega_L^{X_{\widetilde{Q}_i^\ell}}(\widetilde{Q}_i^\ell \cap \mathcal{O}_{\ell+1}) \leq C_\eta \epsilon_0.
$$

Hence if η and ϵ_0 are chosen small enough so that $C\eta^\gamma < c_0/8$ and $C_\eta\epsilon_0 < c_0/8$, then combining the two equations above we have the lower bound

$$
u(X_{\widetilde{P}_i^\ell}) \geq 1 - C\eta^\gamma - C_\eta\epsilon_0 \geq 1 - \frac{1}{4}c_0.
$$

Thus, $|u(X_{\widetilde{Q}_i^\ell}) - u(X_{\widetilde{P}_i^\ell})| \geq u(X_{\widetilde{P}_i^\ell}) - u(X_{\widetilde{Q}_i^\ell}) \geq 1 - \frac{1}{4}c_0 - \left(1 - \frac{3}{4}c_0\right) = \frac{1}{2}c_0$. □

We give now the proof of Proposition 6.1.

Proof of Proposition 6.1. Choose η and ϵ_0 smaller if necessary, so that Lemma 6.4 holds. Fix $y \in F$ and $1 \leq \ell \leq k-1$, and write Q_i^ℓ and P_i^ℓ for $Q_i^\ell(y)$ and $P_i^\ell(y)$ respectively. By construction of $U_{\widetilde{Q}_i^\ell}$ and $U_{\widetilde{P}_i^\ell}$, we have $X_{\widetilde{Q}_i^\ell(y)} \in U_{\widetilde{Q}_i^\ell}$ and $X_{\widetilde{P}_i^\ell(y)} \in U_{\widetilde{P}_i^\ell}$, so there exist $I_{\widetilde{Q}_i^\ell} \in \mathcal{W}^*_{\widetilde{Q}_i^\ell}$ and $I_{\widetilde{P}_i^\ell} \in \mathcal{W}^*_{\widetilde{P}_i^\ell}$ such that $X_{\widetilde{Q}_i^\ell(y)} \in I^*_{\widetilde{Q}_i^\ell}$ and $X_{\widetilde{P}_i^\ell(y)} \in I^*_{\widetilde{P}_i^\ell}$. Moreover, we note that $\ell(\widetilde{Q}_i^\ell) = \eta\ell(Q_i^\ell)$ and $\ell(\widetilde{P}_i^\ell) = \eta^2\ell(Q_i^\ell)$, so $\ell(\widetilde{Q}_i^\ell) > \ell(\widetilde{P}_i^\ell) > \eta^3\ell(Q_i^\ell)$ since $\eta \in (0,1)$. Also, since $\widetilde{Q}_i^\ell \subset Q_i^\ell$ and $\widetilde{P}_i^\ell \subset P_i^\ell \subset Q_i^\ell$, it follows that $I^*_{\widetilde{Q}_i^\ell}$ and $I_{\widetilde{P}_i^\ell}$ are contained in $U_{Q_i^\ell,\eta^3}$. Thus, by Lemma 6.4 we obtain

$$
\begin{aligned}
\frac{1}{2}c_0 &\leq \left|u(X_{\widetilde{Q}_i^\ell}) - u_{U_{Q_i^\ell,\eta^3}}\right| + \left|u_{U_{Q_i^\ell,\eta^3}} - u(X_{\widetilde{P}_i^\ell})\right| \\
&\lesssim \left(\fint_{I^*_{\widetilde{Q}_i^\ell}} |u(Y) - u_{U_{Q_i^\ell,\eta^3}}|^2\,dY\right)^{1/2} + \left(\fint_{I^*_{\widetilde{P}_i^\ell}} |u(Y) - u_{U_{Q_i^\ell,\eta^3}}|^2\,dY\right)^{1/2} \\
&\leq \left(|I^*_{\widetilde{Q}_i^\ell}|^{-1} \fint_{U_{Q_i^\ell,\eta^3}} |u(Y) - u_{U_{Q_i^\ell,\eta^3}}|^2\,dY\right)^{1/2} + \left(|I^*_{\widetilde{P}_i^\ell}|^{-1} \fint_{U_{Q_i^\ell,\eta^3}} |u(Y) - u_{U_{Q_i^\ell,\eta^3}}|^2\,dY\right)^{1/2} \\
&\leq C_\eta \left(\ell(Q_i^\ell)^{-n} \int_{U_{Q_i^\ell,\eta^3}} |u(Y) - u_{U_{Q_i^\ell,\eta^3}}|^2\,dY\right)^{1/2} \\
&\leq C_\eta \left(\ell(Q_i^\ell)^{-n} \int_{U_{Q_i^\ell,\eta^3}} |\nabla u(Y)|^2\,dY\right)^{1/2} \\
&\leq C_\eta \left(\int_{U_{Q_i^\ell,\eta^3}} |\nabla u(Y)|^2 \delta(Y)^{2-n}\,dY\right)^{1/2}
\end{aligned}
$$

where the second line follows from Moser's estimate, the third from $I^*_{\widetilde{Q}_i^\ell} \subset U_{Q_i^\ell,\eta^3}$ and $I^*_{\widetilde{P}_i^\ell} \subset U_{Q_i^\ell,\eta^3}$, the fourth from $|I^*_{\widetilde{Q}_i^\ell}| \approx_\eta \ell(Q_i^\ell)^n \approx |I^*_{\widetilde{P}_i^\ell}|$, the fifth from the Poincaré inequality, and the last from the observation that $\delta(Y) \approx_\eta \ell(Q_i^\ell)$ whenever $Y \in U_{Q_i^\ell,\eta^3}$. Finally, by Lemma 5.3 we have if α is chosen to be less than $\epsilon^2/(2C_\mu^2)$, then

$$\frac{c_0^2}{4}\frac{\log\alpha^{-1}}{\log\epsilon_0^{-1}} \approx \frac{c_0^2}{4}(k-1) \le C_\eta \sum_{\ell=1}^{k-1} \int_{U_{Q_i^\ell,\eta^3}} |\nabla u(Y)|^2 \delta(Y)^{2-n}\, dY \le C_\eta \int_{\Gamma^\eta_{Q_0}(x)} |\nabla u(Y)|^2 \delta(Y)^{2-n}\, dY,$$

where the last inequality follows once we recall from Lemma 3.3 that the sets $\{U_{Q,\eta^3}\}_{Q\in\mathbb{D}_{Q_0}}$ have bounded overlap with constant depending on η only. □

7. Proof of main result

Finally, we prove the main result Theorem 2.4. Let us first recall that Theorem 2.4 says that $\omega_L \in A_\infty(\partial\Omega)$ if there exists $C > 0$ such that every bounded weak solution $u \in W^{1,2}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ to $Lu = 0$ satisfies the Carleson measure condition

$$\sup_{x\in\partial\Omega,\, r>0} \frac{1}{r^n} \int_{B(x,r)\cap\Omega} |\nabla u(X)|^2 \delta(X)\, dX \le C\|u\|^2_{L^\infty(\Omega)}.$$

Building upon the conical square function estimate in Proposition 6.1, we now give the proof of the above.

Proof of Theorem 2.4. Assume that every bounded weak solution satisfies the Carleson measure condition above. To show that $\omega_L \in A_\infty(\partial\Omega)$, we need to show the existence of $\widetilde{\alpha}, \beta \in (0,1)$ such that for every surface balls $\Delta_0 = B_0 \cap \partial\Omega$ and $\Delta := B \cap \partial\Omega$ with $B \subset B_0$, and for every Borel set $F \subset \Delta$, we have

$$\frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)} \le \widetilde{\alpha} \implies \frac{\sigma(F)}{\sigma(\Delta)} \le \beta. \tag{7.1}$$

In fact, we will show a slightly stronger version where $\beta \in (0,1)$ may be taken to be arbitrary.

Step 1: a dyadic version of (7.1). Let us first show that a dyadic version of (7.1) holds. That is, we show that if $\beta \in (0,1)$ then there exists $\widetilde{\alpha} \in (0,1)$ such that if $Q^0 \in \mathbb{D}(\partial\Omega)$ then for every dyadic cube $Q_0 \subset Q^0$ and every Borel set $F \subset Q_0$, it holds that

$$\frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(Q_0)} \le \widetilde{\alpha} \implies \frac{\sigma(F)}{\sigma(Q_0)} \le \beta. \tag{7.2}$$

Fix $\beta \in (0,1)$ and $Q^0 \in \mathbb{D}(\partial\Omega)$. Also fix a dyadic cube $Q_0 \subset Q^0$ and choose a Borel set $F \subset Q_0$ such that $\omega_L^{X_{Q_0}}(F) \le \alpha\omega_L^{X_{Q_0}}(Q_0)$ for some $\alpha \in (0,1)$ to be chosen. Define $u(X) := \omega_L^X(S)$ and note that $u \in W^{1,2}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to $Lu = 0$ satisfying $\|u\|_{L^\infty(\Omega)} \le 1$. By Proposition 6.1, if α is chosen small enough so that $\alpha < \alpha_0$, then it holds that

$$\begin{aligned}
C_\eta^{-2}\log\alpha^{-1}\sigma(F) &\le \int_F |S^\eta_{Q_0}u(x)|^2\, d\sigma(x) \\
&\le \int_F \int_{\Gamma^\eta_{Q_0}(x)} |\nabla u(Y)|^2 \delta(Y)^{2-n}\, dY\, d\sigma(x) \\
&= \int_{2\kappa_0 B_Q\cap\Omega} |\nabla u(Y)|^2 \delta(Y)^{2-n} \left(\int_{Q_0} \chi_{\Gamma^\eta_{Q_0}(x)}(Y)\, d\sigma(x)\right) dY,
\end{aligned}$$

where on the last line we used Fubini's theorem and we recall that $\kappa_0 > 1$ is the constant from (3.4) which satisfies $\Gamma^\eta_{Q_0}(x) \subset T_{Q_0} \subset 2\kappa_0 B_Q \cap \Omega$. Now let $Y \in 2\kappa_0 B_Q \cap \Omega$ and let $\widehat{y} \in \partial\Omega$ be a point on the boundary such that $|Y - \widehat{y}| = \delta(Y)$. We claim that

$$\{x \in Q_0 : Y \in \Gamma^\eta_{Q_0}(x)\} \subset \Delta(\widehat{y}, C\eta^{-3}\delta(Y)).$$

Indeed, let $x \in Q_0$ be such that $\Gamma^\eta_{Q_0}(x)$ contains Y. Then by definition of $\Gamma^\eta_{Q_0}(x)$ there exists $Q \in \mathbb{D}_{Q_0}$ such that $x \in Q$ and $Y \in U_{Q,\eta^3}$, so that there is $Q' \in \mathbb{D}_Q$ with $\ell(Q') > \eta^3\ell(Q)$ such that $Y \in U_{Q'}$. Thus, $\delta(Y) \approx \operatorname{dist}(Y, Q') \approx \ell(Q')$, so

$$\begin{aligned}
|x - \hat{y}| &\le \operatorname{dist}(x, Q') + \operatorname{dist}(Q', Y) + \delta(Y) \\
&\le \operatorname{diam}(Q) + \operatorname{dist}(Q', Y) + \delta(Y) \\
&\lesssim \ell(Q) + \delta(Y) && \text{since } \operatorname{diam}(Q) \approx \ell(Q) \text{ and } \operatorname{dist}(Q', Y) \approx \delta(Y), \\
&\le C\eta^{-3}\delta(Y) && \text{since } \ell(Q') > \eta^3\ell(Q) \text{ and } \ell(Q') \approx \delta(Y),
\end{aligned}$$

which shows the claim. Thus,

$$\int_{Q_0} \chi_{\Gamma^\eta_{Q_0}(x)}(Y)\, d\sigma(x) \le \sigma(\Delta(\widehat{y}, C\eta^{-3}\delta(Y))) \lesssim \eta^{-3(n-1)}\delta(Y)^{n-1}$$

where the last estimate follows from the Ahlfors regularity of $\partial\Omega$. Hence

$$C_\eta^{-2} \log \alpha^{-1}\sigma(F) \lesssim \eta^{-3(n-1)} \int_{2\kappa_0 B_Q \cap \Omega} |\nabla u(Y)|^2 \delta(Y)\, dY \lesssim \eta^{-3(n-1)}\sigma(2\kappa_0 B_Q \cap \partial\Omega)$$

where the last estimate follows from the Carleson measure condition which u satisfies and that $\|u\|_{|L^\infty(\Omega)} \le 1$. Hence by the Ahlfors regularity of $\partial\Omega$, we have $C_\eta^{-2}\log\alpha^{-1}\sigma(F) \le C\eta^{-3(n-1)}\sigma(Q_0)$ for some $C > 0$ and it follows that whenever $\alpha \in (0, \min\{\alpha_0, e^{-CC_\eta^2\eta^{-3(n-1)}\beta^{-1}}\})$ we have $\sigma(F)/\sigma(Q_0) \le \beta$. Fixing such a choice of α and recalling that F is chosen so that $\omega_L^{X_{Q_0}}(F)/\omega_L^{X_{Q_0}}(Q_0) \le \alpha$, we see that

$$\frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q_0)} \le \alpha \implies \frac{\sigma(F)}{\sigma(Q_0)} \le \beta. \tag{7.3}$$

Now by Lemma 4.2(ii) and (iii), the Harnack inequality, and Lemma 4.1, we have

$$\frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q_0)} \approx \frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(\Delta_0)} \lesssim \omega_L^{X_{Q_0}}(F) \approx \omega_L^{X_{\Delta_0}}(F) \lesssim \frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(\Delta_0)} \approx \frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(Q_0)},$$

so there exists $C' > 0$ such that

$$\frac{\omega_L^{X_{Q_0}}(F)}{\omega_L^{X_{Q_0}}(Q_0)} \le C' \frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(Q_0)}.$$

Combining this with (7.3) we see that

$$\frac{\omega_L^{X_{Q^0}}(F)}{\omega_L^{X_{Q^0}}(Q_0)} \le \widetilde{\alpha} \implies \frac{\sigma(F)}{\sigma(Q_0)} \le \beta, \quad \text{where } \widetilde{\alpha} := \alpha/C'.$$

This shows the desired dyadic version of the A_∞ property.

Step 2: conclude. To show the ball version (7.1), we prove its contrapositive. Fix a surface ball $\Delta_0 := B_0 \cap \partial\Omega$ where $B_0 = B(x_0, r_0)$, and let $\Delta = B \cap \partial\Omega$ where $B = B(x, r)$ satisfies $B \subset B_0$. Let $F \subset \Delta$ be a Borel set such that $\sigma(F)/\sigma(\Delta) > \beta$. Define the family $\mathcal{F} := \{Q \in \mathbb{D}(\partial\Omega) : Q \cap \Delta \ne \emptyset,\ r/(4C) < \ell(Q) \le r/(2C)\}$, where $C > 1$ is the uniform constant in (3.2). We note that $\mathcal{F}$ is a disjoint family of dyadic cubes since there exists only one dyadic number in the interval $(r/(4C), r/(2C)]$, and that $\Delta \subset \bigcup_{Q\in\mathcal{F}} Q \subset 2\Delta$. Now we claim that there exists a constant $\widetilde{C} > 1$ and $Q_0 \in \mathcal{F}$ such that

$$\frac{\sigma(F\cap Q_0)}{\sigma(Q_0)} > \frac{\beta}{\widetilde{C}}.$$

Indeed, if we assume the contrary then we have $\sigma(F\cap Q) \leq c^{-1}\beta\sigma(Q)$ for all $Q\in\mathcal{F}$ and $c > 1$, so that

$$\sigma(F) = \sum_{Q\in\mathcal{F}} \sigma(F\cap Q) \leq c^{-1}\beta \sum_{Q\in\mathcal{F}} \sigma(Q) \leq c^{-1}\beta\sigma(2\Delta) \leq \beta\sigma(\Delta)$$

where the last inequality can be made to hold by choosing c large enough depending on the Ahlfors regular constant so that $c^{-1}\sigma(2\Delta) \leq \sigma(\Delta)$, and this contradicts $\sigma(F) > \beta\sigma(\Delta)$. In particular, we see that $\widetilde{C}$ depends only on the Ahlfors regular constant. Let $Q^0 \in \mathbb{D}(\partial\Omega)$ be the unique dyadic cube such that $Q_0 \subset Q^0$ and $r_0/2 < \ell(Q^0) \leq r_0$. By the contrapositive of (7.2), there exists $\widetilde{\alpha} \in (0,1)$ such that

$$\frac{\omega_L^{X_{Q^0}}(F\cap Q_0)}{\omega_L^{X_{Q^0}}(Q_0)} > \widetilde{\alpha}.$$

By Lemma 4.2 and the Harnack inequality, we have

$$\frac{\omega_L^{X_{Q^0}}(F\cap Q_0)}{\omega_L^{X_{Q^0}}(Q_0)} \approx \frac{\omega_L^{X_{\Delta_0}}(F\cap Q_0)}{\omega_L^{X_{\Delta_0}}(Q_0)} \approx \frac{\omega_L^{X_{\Delta_0}}(F\cap Q_0)}{\omega_L^{X_{\Delta_0}}(\Delta)} \leq \frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)},$$

so we have shown that

$$\frac{\sigma(F)}{\sigma(\Delta)} > \beta \implies \frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)} > \widetilde{\alpha},$$

which is the contrapositive of the desired result. □

Appendix A. Muckenhoupt weights

Let $\Omega \subset \mathbb{R}^n$ be a domain. We state here several results pertaining to the theory of Muckenhoupt weights, before proving Proposition 2.3 building upon these results. All results are taken from [SM95] where the theory of Muckenhoupt weights with respect to the Lebesgue measure on $\mathbb{R}^n$ is discussed. However, with minor modifications to the proofs, these results remain valid when replacing the Lebesgue measure on $\mathbb{R}^n$ with the surface measure σ on $\partial\Omega$, and we will restate each result in this setting. Fix a subset $E \subset \partial\Omega$.

Definition A.1. Let $\omega \in L^1_{\mathrm{loc}}(E, d\sigma)$. For $p \in (1,\infty)$, we say ω belongs to the A_p class of Muckenhoupt weights and write $\omega \in A_p(E)$ if there exists $C > 0$ such that the inequality

$$\fint_\Delta \omega(y)\, d\sigma(y) \left(\fint_\Delta \omega(y)^{-p'/p}\, d\sigma(y)\right)^{p/p'} \leq C$$

holds for all surface balls $\Delta \subset E$, where $1/p + 1/p' = 1$. In the case $p = 1$, we say $\omega \in A_1(E)$ if there exists $C > 0$ such that for all surface balls $\Delta \subset E$,

$$\fint_\Delta \omega(y)\, d\sigma(y) \leq C\omega(x) \quad \text{for a.e. } x \in \Delta.$$

Definition A.2. Let $\omega \in L^1_{\mathrm{loc}}(E, d\sigma)$. We say $\omega \in A_\infty(E)$ if there exist $\alpha, \beta \in (0,1)$ such that for all surface balls $\Delta \subset E$ and all Borel sets $F \subset \Delta$,

$$\frac{\mu(F)}{\mu(\Delta)} \leq \alpha \implies \frac{\sigma(F)}{\sigma(\Delta)} \leq \beta,$$

where $d\mu = \omega\, d\sigma$.

Remark. In [SM95], the A_∞ class of weights is first defined in Chapter V §1.7 where "there exist $\alpha, \beta \in (0,1)$" is replaced by "for any $\alpha \in (0,1)$ there exists $\beta \in (0,1)$". At first glance this appears to be a stronger condition than Definition A.2 above, but it was later shown in §3.1 that the two definitions are actually equivalent.

Proposition A.3 (Proposition 2 in Chapter V of [SM95])**.** Let μ be a Borel measure on E and let $p \in [1,\infty)$. Then the maximal operator M_σ is of weak type $(L^p(E, d\mu), L^p(E, d\mu))$ if and only if $\mu \ll \sigma|_E$ and $\omega := d\mu/d\sigma \in A_p(E)$.

Theorem A.4 (Theorem 3 in Chapter V of [SM95])**.** Let $\omega \in L^1_{\mathrm{loc}}(E, d\sigma)$ be nonnegative. The following are equivalent.

(i) $\omega \in A_\infty(E)$.
(ii) $\omega \in A_p(E)$ for some $p \in [1,\infty)$.
(iii) There exists $C > 0$ and $r \in (1,\infty)$ such that ω satisfies the reverse Hölder continuity

$$\left(\fint_\Delta |\omega(y)|^r \, d\sigma(y) \right)^{1/r} \le C \fint_\Delta \omega(y) \, d\sigma(y)$$

for all surface balls $\Delta \subset E$.

Remark. Examining the proof in [SM95], we see that in fact one can take $r = p'$.

We now prove Proposition 2.3. Let us recall its statement first.

Proposition. The following are equivalent.

(i) $\omega_L \in A_\infty(\partial\Omega)$.
(ii) There exists $p \in [1,\infty)$ such that for every surface ball Δ_0 and every surface ball Δ such that $\Delta \subset \Delta_0$, the Hardy-Littlewood maximal function M_σ is of weak type $(L^p(\Delta, d\omega_L^{X_{\Delta_0}}), L^p(\Delta, d\omega_L^{X_{\Delta_0}}))$.
(iii) $\omega_L \ll \sigma$ and there exists $r \in (1,\infty)$ such that for every surface ball Δ_0 and every surface ball Δ such that $\Delta \subset \Delta_0$, the Radon-Nikodym derivative $k_L^{X_{\Delta_0}} := d\omega_L^{X_{\Delta_0}}/d\sigma$ satisfies the reverse Hölder inequality

$$\left(\fint_\Delta |k_L^{X_{\Delta_0}}(y)|^r \, d\sigma(y) \right)^{1/r} \le C \fint_\Delta k_L^{X_{\Delta_0}}(y) \, d\sigma(y).$$

Proof. (i) $\Rightarrow$ (ii): Assume that $\omega_L \in A_\infty(\partial\Omega)$, so that there exist $\alpha, \beta \in (0,1)$ such that if Δ_0 and Δ are surface balls such that $\Delta \subset \Delta_0$, then for every Borel set $F \subset \Delta$ we have

$$\frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)} \le \alpha \implies \frac{\sigma(F)}{\sigma(\Delta)} \le \beta.$$

Then for every surface ball Δ_0 and every surface ball Δ such that $\Delta \subset \Delta_0$, we have $\omega_L^{X_{\Delta_0}} \in A_\infty(\Delta)$. Hence by Theorem A.4, $\omega \in A_p(\Delta)$ for some $p \in [1,\infty)$. By Proposition A.3, this implies the maximal operator M_σ is of weak type $(L^p(\Delta, d\omega_L^{X_{\Delta_0}}), L^p(\Delta, d\omega_L^{X_{\Delta_0}}))$. As α and β are independent of Δ_0 and Δ, so is p. Hence (ii) holds.

(ii) $\Rightarrow$ (iii): Let Δ_0 be a surface ball. If (ii) holds, then by Proposition A.3 we have $\omega_L^{X_{\Delta_0}} \ll \sigma|_\Delta$ for every surface ball $\Delta \subset \Delta_0$. As Δ_0 is arbitrary, the mutual absolute continuity of the family of measures $\{\omega_L^X : X \in \Omega\}$ implies $\omega_L^{X_{\Delta_0}} \ll \sigma$ and hence we have $\omega_L \ll \sigma$. Also by Proposition A.3, we have $k_L^{X_{\Delta_0}} := d\omega_L^{X_{\Delta_0}}/d\sigma \in A_p(\Delta)$ for every surface ball $\Delta \subset \Delta_0$. By Theorem A.4, there exists $C > 0$ and

$r \in (1, \infty)$ such that ω satisfies the reverse Hölder inequality

$$\left(\fint_{\Delta} |k_L^{X_{\Delta_0}}(y)|^r \, d\sigma(y) \right)^{1/r} \leq C \fint_{\Delta} k_L^{X_{\Delta_0}}(y) \, d\sigma(y)$$

for all surface balls $\Delta \subset \Delta_0$.

(iii) $\Rightarrow$ (i): Assume that (iii) holds. Let Δ_0 and Δ be surface balls such that $\Delta \subset \Delta_0$. To show that $\omega_L \in A_\infty(\partial\Omega)$, we need to show that $\omega_L^{\Delta_0} \in A_\infty(\Delta)$. But this is an immediate consequence of Theorem A.4. □

We make the final note that in (ii) above, if $p > 1$ then one can replace the weak type (p, p) property with L^p boundedness. Indeed, from Chapter V §3 in [SM95], if $\omega \in A_p$ for some $p > 1$ then there is a $p_1 < p$ such that $\omega \in A_{p_1}$. Combining with Proposition A.3, we see that M_σ is of weak type (p, p) implies it is of weak type (p_1, p_1) and therefore by the Marcinkiewicz interpolation theorem (using the obvious L^∞-boundedness of M_σ for the upper exponent), M_σ is bounded on L^p.

References

[Bou87] J. Bourgain. On the hausdorff dimension of harmonic measure in higher dimension. *Inventiones Mathematicae*, 87(3):477–483, 1987.

[CHMT20] Juan Cavero, Steve Hofmann, José María Martell, and Tatiana Toro. Perturbations of elliptic operators in 1-sided chord-arc domains. part ii: Non-symmetric operators and carleson measure estimates. *Transactions of the American Mathematical Society*, 373(11):7901–7935, 2020.

[DS91] Guy David and Stephen Semmes. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond lipschitz graphs. *Astérisque*, 193, 1991.

[HMT14] Steve Hofmann, José María Martell, and Tatiana Toro. General divergence form elliptic operators on domains with adr boundaries, and on 1-sided nta domains. *work in progress*, 2014.

[HMUT14] Steve Hofmann, José María Martell, and Ignacio Uriarte-Tuero. Uniform rectifiability and harmonic measure, ii: Poisson kernels in L^p imply uniform rectifiability. *Duke Mathematical Journal*, 163(8), 2014.

[SM95] Elias M. Stein and Timothy S. Murphy. *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*. Princeton University Press, 1995.