Q1.1

$$\hat{y}_i = b_0 + b_1 z_{i1} + b_2 z_{i2}$$

$$SSE = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2$$

$$SSE = \sum_{i=1}^{N} \left[ y_i - (b_0 + b_1 z_{i1} + b_2 z_{i2}) \right]^2$$

Q1.2

let $e_i = y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}$

$$\frac{\partial SSE}{\partial b_0} = -2 \sum_{i=1}^{N} e_i = 0$$

$$\frac{\partial SSE}{\partial b_1} = -2 \sum_{i=1}^{N} e_i z_{i1} = 0$$

$$\frac{\partial SSE}{\partial b_2} = -2 \sum_{i=1}^{N} e_i z_{i2} = 0$$

Q1.3

$$\sum_{i=1}^{N} e_i = 0 \Rightarrow \frac{1}{N} \sum_{i=1}^{N} e_i = 0$$

so average error is zero.

$$\sum_{i=1}^{N} e_i z_{i1} = 0$$

$$\sum_{i=1}^{N} e_i z_{i2} = 0$$

which means the vector of residuals $e$ is orthogonal to $z_1$ and $z_2$ in $\mathbb{R}^N$.

Q1.4

From $\sum e_i = 0$ :

$$\sum_{i=1}^{N} [y_i - b_0 - b_1 z_{i1} - b_2 z_{i2}] = 0$$

$$\sum y_i - N b_0 - b_1 \sum z_{i1} - b_2 \sum z_{i2} = 0$$

but $\sum z_{i1} = 0$ and $\sum z_{i2} = 0$ because $z_{ij}$ are centered.

so: $\sum y_i - N b_0 = 0$

$$b_0^* = \frac{1}{N} \sum_{i=1}^{1} y_i = \bar{y}$$

$$\bar{y}_i = b_1 z_{i1} + b_2 z_{i2} + \text{error}$$

Q1.5

From $\sum e_i z_{i1} = 0$ and $\sum e_i z_{i2} = 0$ with $b_0$ eliminated:

$$\sum \tilde{y}_i z_{i1} = b_1 \sum z_{i1}^2 + b_2 \sum z_{i1} z_{i2}$$

$$\sum \tilde{y}_i z_{i2} = b_1 \sum z_{i1} z_{i2} + b_2 \sum z_{i2}^2$$

$$\begin{bmatrix} \sum z_{i1}^2 & \sum z_{i1} z_{i2} \\ \sum z_{i1} z_{i2} & \sum z_{i2}^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \sum \tilde{y}_i z_{i1} \\ \sum \tilde{y}_i z_{i2} \end{bmatrix}$$

So $A = \begin{bmatrix} \sum z_{i1}^2 & \sum z_{i1} z_{i2} \\ \sum z_{i1} z_{i2} & \sum z_{i2}^2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, $C = \begin{bmatrix} \sum \tilde{y}_i z_{i1} \\ \sum \tilde{y}_i z_{i2} \end{bmatrix}$

$$Ab = C$$

Q1.6

$$\frac{1}{N} = A = \begin{bmatrix} \frac{1}{N} \sum z_{i1}^2 & \frac{1}{N} \sum z_{i1} z_{i2} \\ \frac{1}{N} \sum z_{i1} z_{i2} & \frac{1}{N} \sum z_{i2}^2 \end{bmatrix} = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix}$$

$$\frac{1}{N} C = \begin{bmatrix} \frac{1}{N} \sum \tilde{y}_i z_{i1} \\ \frac{1}{N} \sum \tilde{y}_i z_{i2} \end{bmatrix} = \begin{bmatrix} Cov(y, X_1) \\ Cov(y, X_2) \end{bmatrix}$$

Normal equation: $\begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} Cov(y, X_1) \\ Cov(y, X_2) \end{bmatrix}$