Theorem 1. $\mathrm{Fib}(n)$ *is the closest integer to* $\phi^n/\sqrt{5}$*, where* $\phi = (1+\sqrt{5})/2$

证明. 令 $\psi = (1-\sqrt{5})/2$, $f(n) = (\phi^n - \psi^n)/\sqrt{5}$

证明 $\mathrm{Fib}(n) = f(n)$

$$\frac{1}{\phi} = \frac{2}{1+\sqrt{5}} = \frac{2(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{2} = -\psi$$

$$\frac{1}{\psi} = \frac{2}{1-\sqrt{5}} = \frac{2(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} = -\frac{1+\sqrt{5}}{2} = -\phi$$

$$\phi + \psi = 1$$

$$\begin{aligned}
f(0) &= 0 \\
f(1) &= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = 1 \\
f(n-1) + f(n-2) &= \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\phi^{n-1} + \phi^{n-2}}{\sqrt{5}} - \frac{\psi^{n-1} + \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\phi^{n-1}(1 + 1/\phi)}{\sqrt{5}} - \frac{\psi^{n-1}(1 + 1/\psi)}{\sqrt{5}} \\
&= \frac{\phi^{n-1}(1 - \psi)}{\sqrt{5}} - \frac{\psi^{n-1}(1 - \phi)}{\sqrt{5}} \\
&= \frac{\phi^n - \psi^n}{\sqrt{5}} \\
&= f(n)
\end{aligned}$$

$$f(n) = \mathrm{Fib}(n)$$

证明 $\mathrm{Fib}(n)$ 是最接近 $\phi^n/\sqrt{5}$ 的整数，即证 $\left|f(n) - \frac{\phi^n}{\sqrt{5}}\right| < \frac{1}{2}$

$$\left|f(n) - \frac{\phi^n}{\sqrt{5}}\right| = \left|\frac{\psi^n}{\sqrt{5}}\right| = \frac{|\psi^n|}{\sqrt{5}}$$

$$\frac{|\psi^n|}{\sqrt{5}} < \frac{1}{2}$$

$$|\psi^n| < \frac{\sqrt{5}}{2}$$

$$|\psi| \approx 0.618 < 1$$

$$|\psi|^n < 1 < \frac{\sqrt{5}}{2}$$

$\square$