Problem Set 5 (Lectures 9-10)

Problem 1 (40 points: Averaging Type I/Type II errors in Simple Hypothesis Testing Problems) Consider the testing problem:

$$\mathbf{H}_0 : \theta_0 \quad \text{vs.} \quad \mathbf{H}_1 : \theta_1.$$

Let $f(x|\theta_0)$ denote the likelihood under the null, and $f(x|\theta_1)$ the likelihood under the alternative.

In class, we argued that—under the 0-1 loss—a test ϕ for the simple hypothesis testing problem is characterized by its profile of Type I/Type II errors:

$$(R(\phi, \theta_0), R(\phi, \theta_1))' \in \mathbb{R}^2.$$

Suppose we want to pick a test ϕ that minimizes the *average* rate of error. That is, let $\pi_0 \in (0, 1)$ and consider the problem

$$\inf_{\phi} \ \pi_0 R(\phi, \theta_0) + (1 - \pi_0) R(\phi, \theta_1).$$

Show that the test that minimizes the average rate of error is the likelihood ratio test (with critical value $\pi_0/1 - \pi_0$).

Problem 2 (40 points: Minimax Tests for Simple Hypothesis Testing Problems) Consider the testing problem

$$\mathbf{H}_0 : x \sim \mathcal{N}(0, 1) \quad \text{vs.} \quad \mathbf{H}_1 : x \sim \mathcal{N}(2 * (1.64), 1).$$

A test for this problem is *minimax* if it solves the problem

$$\inf_{\phi} \sup\{R(\phi, \theta_0), R(\phi, \theta_1).\}$$

What is the minimax test for this problem?

Problem 3 (20 points: Generalized Likelihood Ratio) Consider the composite testing problem

$$\mathbf{H}_0 : \theta \in \Theta_0 \quad \text{vs.} \quad \mathbf{H}_1 : \theta \in \Theta_1.$$

A test for this problem is *minimax* if it solves the problem

$$\inf_{\phi} \sup\{R(\phi,\theta_0), R(\phi,\theta_1).\}$$

Casella and Berger (Statistical Inference, 2nd Edition, p. 375) define the Generalized Likelihood Ratio as the test that rejects the null hypothesis whenever:

$$\frac{\sup_{\theta\in\theta_0} f(x|\theta)}{\sup_{\theta\in\Theta} f(x|\theta)} \leq c,$$

where $c \in [0,1]$. Is this test equivalent to the test that rejects the null hypothesis whenever

$$\frac{\sup_{\theta\in\Theta_1} f(x|\theta)}{\sup_{\theta\in\Theta_0} f(x|\theta_0)} \geq 1/c?$$