Problem Set 1 (Lectures 1 and 2)

1. (The σ-algebra generated by a collection of sets) Let Ω be an arbitrary non-empty set and let A be a collection of elements of 2^{Ω}. Define

$$F^*(A) \equiv \{\mathcal{F} \mid \mathcal{F} \text{ is a } \sigma\text{-algebra of } \Omega \text{ containing } A\}$$

   i) Show that $F^*(A)$ is non-empty.
   ii) Let $\sigma(A)$ denote the intersection over all the σ-algebras contained in $F^*(A)$. Show that $\sigma(A)$ is a σ-algebra.

   Optional: It turns out that there is no other σ-algebra $\mathcal{F}$ such that $A \subseteq \mathcal{F}$ and $\mathcal{F} \subset \sigma(A)$ (you can also show this if you are interested). The set $\sigma(A)$ is the (unique) smallest σ- algebra containing A.

2. Show that σ-algebra is "closed" under countable intersections. That is, if $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ then $\cap_{n=1}^{\infty} F_n \in \mathcal{F}$.

3. (Properties of a probability measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that:

   (a) $\mathbb{P}(F_1 \cup F_2) = \mathbb{P}(F_1) + \mathbb{P}(F_2) - \mathbb{P}(F_1 \cap F_2)$ for any $F_1, F_2 \in \mathcal{F}$
   (b) $\mathbb{P}(\cup_{n\in\mathbb{N}} F_n) \leq \sum_{n\in\mathbb{N}} \mathbb{P}(F_n)$ for any countable collection $\{F_n\}$

   Comment: These are useful properties implied by the definition of probability measure. We will use some of them throughout the course.

4. Proof the following Proposition: If F_X is the c.d.f. of a random variable $X : \Omega \to \mathbb{R}$ then

   (a) F_X is non-decreasing.
   (b) $\lim_{x\uparrow\infty} F_X(x) = 1$
   (c) $\lim_{x\downarrow-\infty} F_X(x) = 0$
   (d) $\lim_{h\to 0^+} F(x+h) = F(x)$

   Comment: Combined with the optional part below, this gives a full characterization of how c.d.f.s for real-valued random variables can look.

   (Optional) Furthermore, if F is a function satisfying 1,2,3,4, then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$ such that F coincides with F_X.

5. (Moments of some common distributons) Use the definitions of expectations provided in class to solve the following problems:

   (a) Show that if X is a Bernoulli random variable with parameter p, then $\mathbb{E}_F[X] = p$ and $\mathbb{E}_F[(X-p)^2] = p(1-p)$

   (b) Show that if X is a Normal Random variable with parameters μ and σ^2 then $\mathbb{E}_F[X] = \mu$ and $\mathbb{E}_F[(X-\mu)^2] = \sigma^2$.[1]

   (c) Show that if X is a Pareto distribution with parameters (x_m, α), then for any $n \geq \alpha$, $\mathbb{E}_F[X^n] = \infty$.

   (d) Show that the moment generating function of a Normal random variable with parameters μ and σ^2 is given by:
$$\mu_X(t) = \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right)$$

   (e) Show that if $X \sim \mathcal{N}(0,1)$, then the random variable $Y : \mathbb{R} \to \mathbb{R}$ given by $\mu + \sigma X$ has the c.d.f. of Normal random variable with parameters (μ, σ^2).

[1] Here you can use the fact that $\int_{-\infty}^{\infty} u(1/\sqrt{2\pi})e^{-u^2/2}du = 0$ and $\int_{-\infty}^{\infty} u^2(1/\sqrt{2\pi})e^{-u^2/2}du = 1$.