Mathematical Foundation of DNN : HW 2

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In this assigment, I implemented every algorith both with torch package and without it(except problem 7). I'll focus on the result of optimization and the comparison both implmentations. Also, I attached the code what I used in this assignment in appendix of this document. Please refer to it.

1

I refer the code given by lecture to implement the SGD with logistic regression. The result of this code is described in figure 1. As figure 1 depicted, both implementation succeeded convergence, since their losses were decreasing and saturated. Since the algorithm without torch and optimizer is too naive, the performace of SGD without SGD is worse than that with torch. To be specific, the result drawn by SGD without torch has larger loss and sizeable variance. I guess the optimizer torch using is more sophisticated and effective.

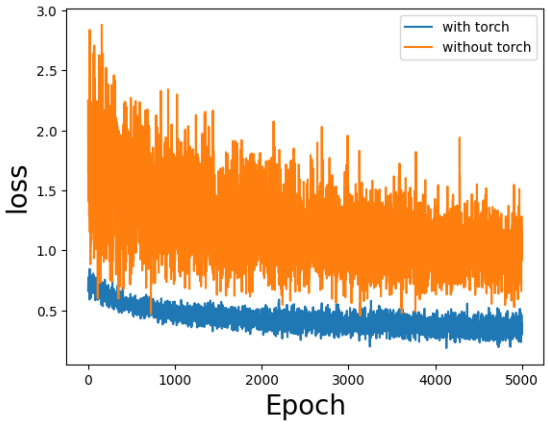


Figure 1: The comparison of the algorithm of SGD both with torch and without it.

2

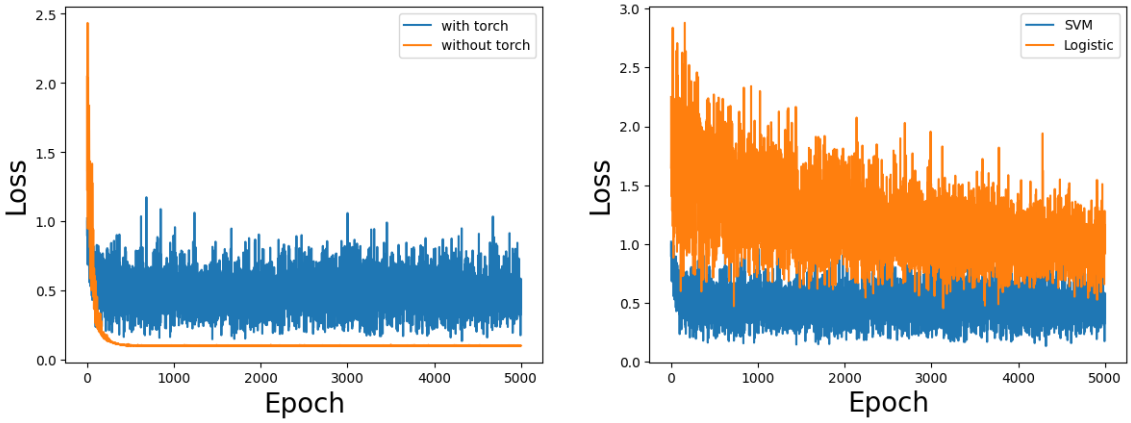
Simiar to problem 1, both algorithm were successful to converge. Surprisingly, in this problem, my own SGD algorithm outperform the implementation with torch. I conjecture that this outperformace is intrinsic algorithmic issues in torch or the hyperparameter tuning. At the first, I suspect that the SGD in torch hit the non-differentiable point of ReLU function. However, when I counted the number of hit that SGD meet the non-differential point within its training, it result in zero, that is, SGD never meet non-differentiable point in its training process. Even if the implementation with torch cannot show good performance, it still better than logistic regression.

3

As one can check from figure 3, the given data is not linearly separable, that is, there is no line that separate both red and blue dot into two sectors. However, after transform the X using φ, this data can be linearly separable. To calculate the decision boundary, I used SVM which is more superior than logistic regression.

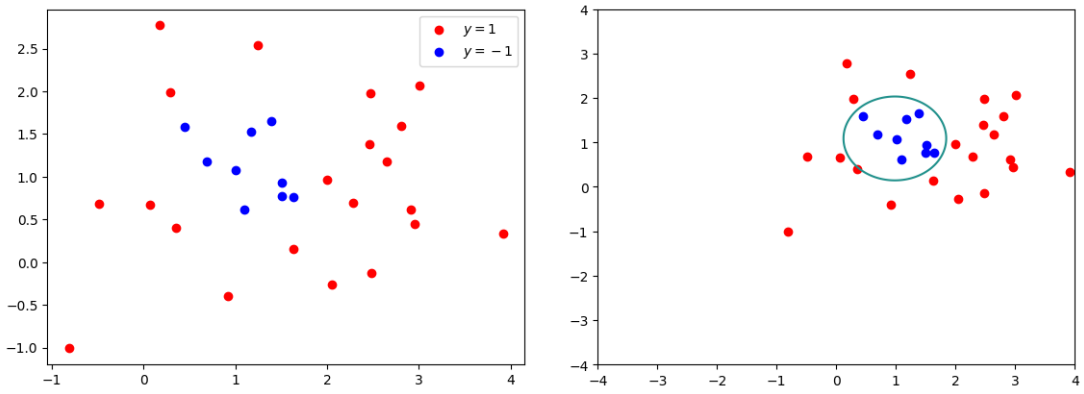
4

CLAIM : $y = -\log(x)$ function is strict convex function. To prove the **CLAIM** above, I proved the following **LEMMA**.



(a) (b)

Figure 2: (a) The comparison of the algorithm of SGD both with torch and without it. (b) The comparision between logistic regression and SVM.



(a) (b)

Figure 3: (a) The scatter plot of given data with its label. (b) The scatter plot with decision boundary(green) obtained by SVM.

LEMMA : The differentiable function f is strictly convex *if and only if* its derivate f' is strictly increasing.
proof
$\Longrightarrow$: $a, b \in \mathbb{R}(a < b)$.Let$x_1, x_2, x_3 \in \mathbb{R}$ be chosen s.t. $a < x_1 < x_2 < x_3 < b$.
From Chordal Slope Lemma, which is famous lemma applicable to convex function,

$$\frac{f(x_1) - f(a)}{x_1 - a} < \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(x_3) - f(x_2)}{x_3 - x_2} < \frac{f(b) - f(x_3)}{b - x_3} \tag{1}$$

$$\frac{f(x_2) - f(a)}{x_2 - a} < \frac{f(b) - f(x_2)}{b - x_2} \tag{2}$$

Taking limit $x_1 \to a, x_3 \to b$ on equation 1,

$$f'(a) \leq \frac{f(x_2) - f(a)}{x_2 - a} < \frac{f(b) - f(x_2)}{b - x_2} \leq f'(b) \tag{3}$$

$\Longleftarrow$: Let $x_1, x_2, x_3 \in \mathbb{R}$ s.t. $x_1 < x_2 < x_3$. From Mean Value Theorem,

$$\exists a \text{ s.t } \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(a) \tag{4}$$

$$\exists b \text{ s.t } \frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(b) \tag{5}$$

Note that $x_1 < a < x_2 < b < x_3$. Since f' is strictly increasing, $f'(a) < f'(b)$, which implies

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{6}$$

The result of equation 6 is equivalent that f is strictly convex according to Chordal Slope Lemma.■

proof of CLAIM From the Lemma prove above, it is enough to show that the derivative of $y = -\log(x)$ is strictly increasing.

$$\frac{d^2}{dx^2}\left(-\log(x)\right) = \frac{1}{x^2} > 0 \text{ for } x > 0 \tag{7}$$

Since the second derivative of $f(x) = -\log(x)$ is positive definite within the region that logarithm is defined. This implies that f' is strictly increasing.■

From the claim, one can show that $D_{KL} > 0$ for different probability mass function p, q. Furthermore, $D_{KL} = 0$ *if and only if* $p = q$.

$$-D_{KL}(p||q) = -\sum_i p_i \log \frac{p_i}{q_i} = \mathbb{E}_p\left[\log \frac{q_i}{p_i}\right] < \log \mathbb{E}_p\left[\frac{q_i}{p_i}\right] = \log q_i \leq \log 1 = 0 \tag{8}$$

If $p = q$, then,

$$D_{KL}(p||q) = \sum_i p_i \log \frac{p_i}{q_i} = \sum_i p_i \log 1 = 0 \tag{9}$$

Since if $p \neq q$, we can apply the formulation made in equation 8, D_{KL} never be zero.

5

6

This problem can be solved easily by considering the each element of the gradient vector. Taking $\odot$ into account, considering elementwisely is more efficient. Thus, it is suffient to show that the jth component of left hand side is equal to that of right hand side one. Please note that the second equality of ∇_b, ∇_a are trivial by refering that multiplying diagonal matrix to a vector is identical to multipyling diagonal entry to each vector element.

$$\begin{aligned}
\left[\nabla_u f_\theta(x)\right]_j &= \frac{\partial}{\partial u_j} f_\theta(x) \\
&= \frac{\partial}{\partial u_j} \sum_i u_i \sigma(a_i x + b_i) \\
&= \sum_i \delta_{ij} \sigma(a_i x + b_i) \\
&= \sigma(a_j x + b_j)
\end{aligned}$$

$$\begin{aligned}
\left[\nabla_b f_\theta(x)\right]_j &= \frac{\partial}{\partial b_j} \sum_i u_i \sigma(a_i x + b_i) \\
&= \sum_i u_i \sigma'(a_i x + b_i) \left(\frac{\partial}{\partial b_j}(a_i x + b_i)\right) \\
&= \sum_i u_i \sigma'(a_i x + b_i) \delta_{ij} \\
&= u_j \sigma'(a_j x + b_j)
\end{aligned}$$

$$\begin{aligned}
\left[\nabla_a f_\theta(x)\right]_j &= \frac{\partial}{\partial a_j} \sum_i u_i \sigma(a_i x + b_i) \\
&= \sum_i u_i \sigma'(a_i x + b_i) \left(\frac{\partial}{\partial a_j}(a_i x + b_i)\right) \\
&= \sum_i u_i \sigma'(a_i x + b_i) \delta_{ij} x_i \\
&= u_j \sigma'(a_j x + b_j) x_j
\end{aligned}$$

7

I used aformentioned code to implement SGD via SVM. The figure 7 shows the results of the following code. As the figure depicted, training is successful. To be specific, the blue one, which learnd from only a epoch does not approximate the true function, well. However, as epoch increases, our model becomes well representation of the true function.

```
import numpy as np
import matplotlib.pyplot as plt

def f_true(x) :
    return (x-2)*np.cos(x*4)

def sigmoid(x) :
    return 1 / (1 + np.exp(-x))

def sigmoid_prime(x) :
    return sigmoid(x) * (1 - sigmoid(x))

K = 10000
alpha = 0.007
N, p = 30, 50
np.random.seed(0)
a0 = np.random.normal(loc = 0.0, scale = 4.0, size = p)
b0 = np.random.normal(loc = 0.0, scale = 4.0, size = p)
u0 = np.random.normal(loc = 0, scale = 0.05, size = p)
theta = np.concatenate((a0,b0,u0))
losses =[]
tmp_losses = []

X = np.random.normal(loc = 0.0, scale = 1.0, size = N)
Y = f_true(X)

def f_th(theta, x) :
    return np.sum(theta[2*p : 3*p] * sigmoid(theta[0 : p] * np.reshape(x,(-1,1)) + theta[p : 2
                                                    *p]), axis=1)

def diff_f_th(theta, x) : # from problem 6, I extensively deploy the elementwise product of
                                                    numpy array.
    a = theta[:p]
    b = theta[p:2*p]
    u = theta[2*p:]
    return np.append(np.append(np.diag(sigmoid_prime(a*x+b))@u*x, np.diag(sigmoid_prime(a*x+b)
                                                    )@u), sigmoid(a*x+b))

xx = np.linspace(-2,2,1024)
plt.plot(X,f_true(X),'rx',label='Data points')
plt.plot(xx,f_true(xx),'r',label='True Fn')

for k in range(K) :
    idx = np.random.randint(0,N-1)
    input, label = X[idx], Y[idx]
    loss = (f_th(theta,input) - label)**2/2 #MSE
    tmp_losses.append(loss)
    #SGD
    theta = theta - alpha*(f_th(theta,input) - label) * diff_f_th(theta,input)

    if (k+1)%2000 == 0 :
        plt.plot(xx,f_th(theta, xx),label=f'Learned Fn after {k+1} iterations')

    if (k+1)%p == 0: #for each epoch
        losses.append(np.mean(tmp_losses)) # store the mean loss of each epoch
        tmp_losses = []

plt.legend()
plt.show()
plt.savefig('plot.png')
```

A

```
import numpy as np
import torch
import torch.nn as nn
```

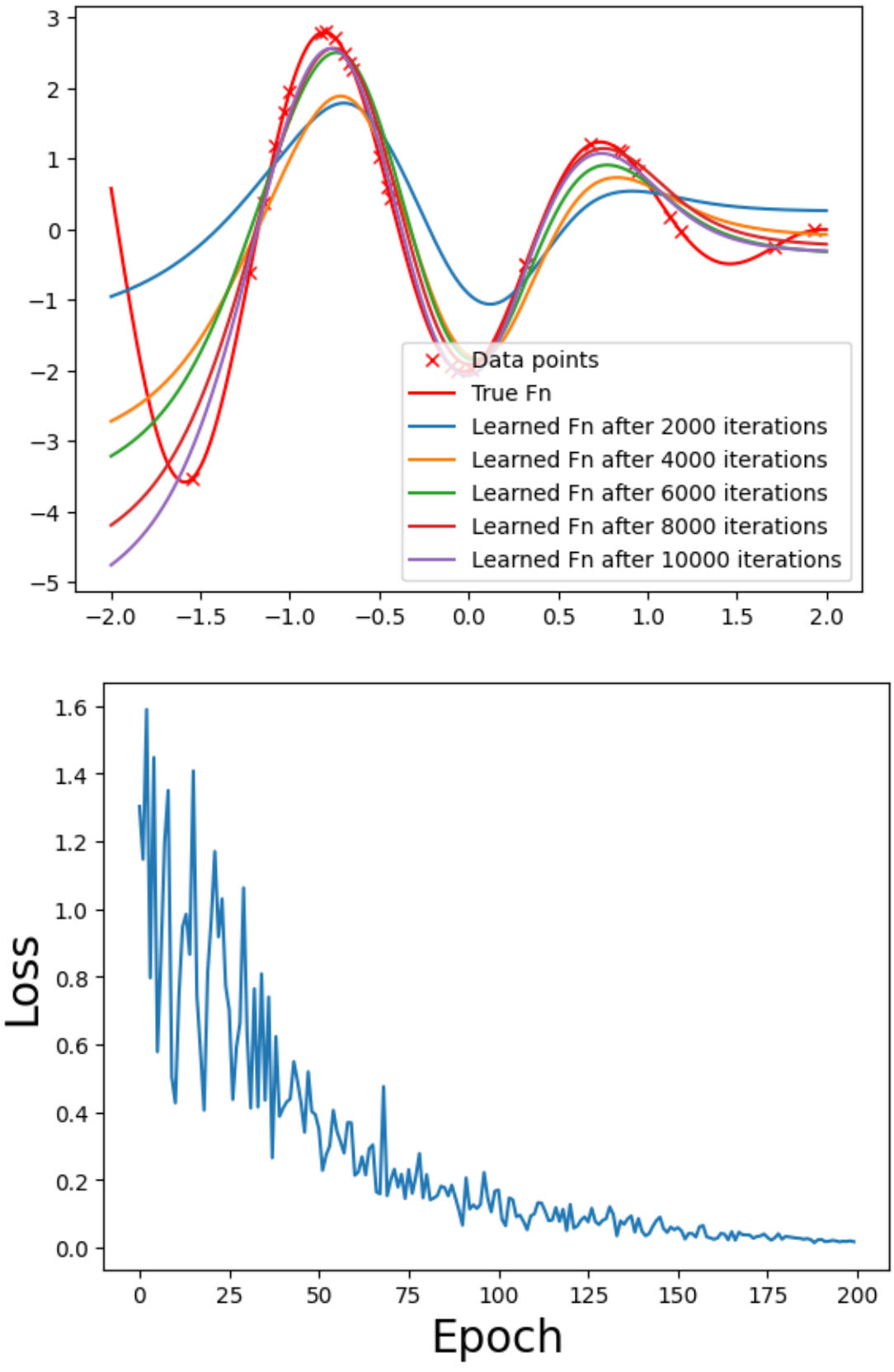


Figure 4: (a) The result of SGD training. One can notice that as SGD learned more, the f_θ became more similar to true function. (b) This figure shows that the SGD train was success.

```
from random import randint
import matplotlib.pyplot as plt
# Hyperparameters setup and Prepare Dataset
```

```
N,p = 30,20
np.random.seed(0)
X = np.random.randn(N,p)
Y = 2*np.random.randint(2,size = N)-1
lr = 1e-4
epoch = 5000
class LR(nn.Module):
    def __init__(self,input_dim = p):
        super().__init__()
        self.linear = nn.Linear(input_dim, 1, bias = True)

    def forward(self, x):
        return self.linear(x)

model = LR()

def logistic_loss(output, target):
    return -torch.nn.functional.logsigmoid(target*output)

loss_function = logistic_loss
optimizer = torch.optim.SGD(model.parameters(), lr=lr)
losses = []
for _ in range(epoch) :
    tmp_losses = []
    for _ in range(N):
        # Sampling
        idx = randint(0, N-1)
        target, label = X[idx], Y[idx]

        optimizer.zero_grad()

        train_loss = loss_function(model(torch.FloatTensor(target)), label)
        tmp_losses.append(train_loss)
        train_loss.backward() # calculate graident via backpropagation
        optimizer.step() # update the parameteres
    losses.append(np.mean([x.detach().numpy() for x in tmp_losses]))
```

B

```
import numpy as np
from random import randint
import matplotlib.pyplot as plt
N,p = 30,20
np.random.seed(0)
X = np.random.randn(N,p)
Y = 2*np.random.randint(2,size = N)-1
lr = 1e-4
epoch = 5000
class LR():
    def __init__(self,input_dim = p):
        self.weight = np.random.uniform(0,1,input_dim+1) # last row is bias term
    def output(self,x):
        x = np.append(x,1) # add dummy column that is one
        return x.T@self.weight
    def update_weight(self,grad,lr = 1e-4):
        self.weight = self.weight - lr * grad

model = LR()

def logistic_loss(output, target):
    return np.log(1+np.exp(-target*output))

loss_function = logistic_loss
losses = []

for _ in range(epoch) :
    tmp_losses = []
    for _ in range(p):
        # Sampling
        idx = randint(0, N-1)
        target, label = X[idx], Y[idx]

        train_loss = loss_function(model.output(target), label)
```

```
        tmp_losses.append(train_loss)
        grad = (1-train_loss)*label*np.append(target,label)
        model.update_weight(grad,lr = lr)
    losses.append(np.mean([x for x in tmp_losses]))
```

C

```
import numpy as np
import torch
import torch.nn as nn
from random import randint
import matplotlib.pyplot as plt
# Hyperparameters setup and Prepare Dataset
N,p = 30,20
np.random.seed(0)
X = np.random.randn(N,p)
Y = 2*np.random.randint(2,size = N)-1
lr = 1e-3
epoch = 500
regularizer = 0.1 # lambda
class SVM(nn.Module):
    def __init__(self,input_dim = p):
        super().__init__()
        self.linear = nn.Linear(input_dim, 1, bias = True)

    def forward(self, x):
        return self.linear(x)

model = SVM()
optimizer = torch.optim.SGD(model.parameters(), lr=lr)
svm_losses = []
cnt = 0
for _ in range(epoch) :
    tmp_losses = []
    for _ in range(p):
        # Sampling
        idx = randint(0, N-1)
        target, label = X[idx], Y[idx]

        optimizer.zero_grad()

        train_loss = torch.clamp(1-label*model(torch.FloatTensor(target)),min = 0) +
                                                    regularizer*model.linear.weight@model.
                                                    linear.weight.T
        train_loss.requires_grad_(True)
        tmp_losses.append(train_loss)
        train_loss.backward() # calculate graident via backpropagation
        optimizer.step() # update the parameteres
    svm_losses.append(np.mean([x.detach().numpy() for x in tmp_losses]))
```

D

```
import numpy as np
from random import randint
import matplotlib.pyplot as plt

# Hyperparameters setup and Prepare Dataset
N,p = 30,20
np.random.seed(0)
X = np.random.randn(N,p)
Y = 2*np.random.randint(2,size = N)-1
lr = 1e-3
epoch = 5000
regularizer = 0.1 # lambda

class SVM():
    def __init__(self,input_dim = p):
        self.weight = np.random.uniform(0,1,input_dim+1) # last row is bias term
        self.regularizer =regularizer
    def output(self,x):
```

```
        return x.T@self.weight
    def update_weight(self,grad,lr = 1e-4):
        self.weight = self.weight - lr * grad

    def loss_function(self, x, y):
        x = np.append(x,y)
        return np.max([0, 1-y*self.output(x)]) + self.weight@self.weight.T *regularizer

    def loss_prime(self, x,y):
        x = np.append(x,y)
        if y*self.output(x) <1 :
            return -y*x +2*self.regularizer*self.weight
        else:
            return 2*self.regularizer*self.weight

model = SVM()
svm_losses = []
cnt = 0

for _ in range(epoch):
    tmp_loss = []
    for _ in range(p):
        idx = randint(0,N-1)
        input,label = X[idx], Y[idx]
        z = model.output(np.append(input,label))
        if z==1:
            cnt+=1
        train_loss = model.loss_function(input, label)
        tmp_loss.append(train_loss)
        grad = model.loss_prime(input,label)

        model.update_weight(grad=grad, lr = lr)
    svm_losses.append(np.mean(tmp_loss))

plt.plot(range(epoch), np.abs(svm_losses))
plt.xlabel("Epoch",fontsize = 20)
plt.ylabel("Loss",fontsize = 20)
plt.show()
print(cnt)
```

E

```
for i in range(N):
    if y[i] == 1:
        plt.scatter(X[0,i],X[1,i],color = "red")
    if y[i] == -1:
        plt.scatter(X[0,i],X[1,i],color = "blue")
plt.legend(["$y = 1$","$y = -1$"])
plt.show()

def trans(x):
    return [1,x[0],x[0]**2, x[1],x[1]**2]
X_trans = np.asarray([trans(x) for x in X.T]).T # (5,N)

import numpy as np
from random import randint
import matplotlib.pyplot as plt

N,p = 30,5
lr = 1e-3
epoch = 500
regularizer = 0.1 # lambda

class SVM():
    def __init__(self,input_dim = p):
        self.weight = np.random.uniform(0,1,input_dim+1) # last row is bias term
        self.regularizer =regularizer
    def output(self,x):
        return x.T@self.weight
    def update_weight(self,grad,lr = 1e-4):
        self.weight = self.weight - lr * grad

    def loss_function(self, x, y):
```

```
        x = np.append(x,y)
        return np.max([0, 1-y*self.output(x)]) + self.weight@self.weight.T *regularizer

    def loss_prime(self, x,y):
        x = np.append(x,y)
        if y*self.output(x) <1 :
            return -y*x +2*self.regularizer*self.weight
        else:
            return 2*self.regularizer*self.weight

model = SVM()
svm_losses = []
cnt = 0

for _ in range(epoch):
    tmp_loss = []
    for _ in range(p):
        idx = randint(0,N-1)
        input,label = X_trans.T[idx], y[idx]
        z = model.output(np.append(input,label))
        train_loss = model.loss_function(input, label)
        tmp_loss.append(train_loss)
        grad = model.loss_prime(input,label)

        model.update_weight(grad=grad, lr = lr)
    svm_losses.append(np.mean(tmp_loss))

w = model.weight
xx = np.linspace(-4,4,1024)
yy = np.linspace(-4,4,1024)
xx,yy = np.meshgrid(xx,yy)
Z = w[0] + (w[1]*xx + w[2]*xx**2) +(w[3]*yy + w[4]*yy**2)
plt.contour(xx,yy,Z,0)
for i in range(N):
    if y[i] == 1:
        plt.scatter(X[0,i],X[1,i],color = "red")
    if y[i] == -1:
        plt.scatter(X[0,i],X[1,i],color = "blue")
plt.show()
```