Introduction to Computer Vision : HW 1

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1 Camera Calibration

a

From the result of SVD,

$$\mathbf{AV} = \mathbf{U\Sigma} \tag{1}$$

Since $\mathbf{V} \in \mathbf{O(n)}$, the column vector of $\mathbf{V}$, denoting $\mathbf{v_i}$, can form the basis of $\mathbb{R}^n$. That is, $\forall \mathbf{x} \in \mathbf{R}^n, \mathbf{x} = \sum_i a_i \mathbf{v_i},$ for $a_i \in \mathbb{R}$. This results in $\mathbf{Ap} = \sum_i a_i \mathbf{Av_i} = \sum_i a_i \sigma_i \mathbf{u_i}$. Thus,

$$\|\mathbf{Av}\|^2 = \sum_i a_i^2 \sigma_i^2 \tag{2}$$

It is a convention to make the diagonal entries of $\mathbf{\Sigma}$, σ_i, be ordered, that is, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$. Thus, to minimize equation 2, $a_1 = a_2 = \cdots = a_{r-1} = 0$ and $a_r = 1$, since $\mathbf{p}$ is normalized. This implies $\mathbf{p} = \mathbf{v_n}$.(note that $r = n$ since we assumed over-determined system.)

b

c

Before determining $\mathbf{P}$, I proved that $\mathbf{p} = \mathbf{A}^\dagger \mathbf{b}$. Consider SVD of $\mathbf{A} = \mathbf{U\Sigma V^T} \in \mathbb{R}^{m\times n}$, where $\mathbf{U} \in O(m), \mathbf{\Sigma} \in \mathbf{R}^{m\times n}, \mathbf{V} \in O(n)$.

$$\|\mathbf{Ap} - \mathbf{b}\|^2 = \|\mathbf{U\Sigma V^T p} - \mathbf{b}\|^2 = \|\mathbf{\Sigma V^T p} - \mathbf{U^T b}\| (\because \mathbf{U} \in \mathbf{O(m)}) \tag{3}$$

Let $\mathbf{V^T p} = \mathbf{q}, \mathbf{U^T b} = \mathbf{r}$ and rewrite the problem as follow.

$$\text{Find } \arg\min_{\mathbf{q}} \|\mathbf{\Sigma q} - \mathbf{r}\|^2 \tag{4}$$

This problem can be solved as follow.

$$\|\mathbf{\Sigma q} - \mathbf{r}\|^2 = \sum_{i=1}^{r} (\sigma_i q_i - r_i)^2 \text{ where } r = \min\{m, n\} \tag{5}$$

We can uniquely select the $q_i = r_i/\sigma_i$, for $i = 1, \cdots, r$, or $\mathbf{q} = \mathbf{\Sigma^{-1} r}$, to minimize this expression. This implies $\mathbf{p} = \mathbf{V\Sigma^{-1}U^T b}$. Thus, proof is done, noting that $\mathbf{A}^\dagger = (\mathbf{A^T A})^{-1}\mathbf{A^T} = (\mathbf{V\Sigma^T \Sigma V^T})^{-1}\mathbf{\Sigma^T U^T} = \mathbf{V\Sigma^{-1}U^T}$