Suppose the sum of a set of integers is the result of adding together all the integers in that set. Given a non-empty set S consisting of n integers, What is the number of non-empty subsets of S which have an even sum?

Case 1: S consists only of even integers. Then with certainty we can say that for all subsets s of S, the sum of s is even. Thus, the number of non-empty subsets of S with even sum is equal to $2^n - 1$.

Case 2: S consists of even and odd integers. We know that all subsets of S formed exclusively by even integers have even sum. Additionally, we know that all subsets of S with even sum formed exclusively by odd integers must have even cardinality.

Let O be the set of all odd integers in S. Finding the number of even-sized subsets of the set O is equivalent to summing the number of ways one can choose 2 elements from O, and the number of ways to choose 4 elements, and then 6 elements, and so forth.

Let x be the number of elements in O, then the number of even-sized subsets of O is equal to the sum of all $2k$ combinations of O, where $k \in \mathbb{Z}$, $0 \leq k \leq \lfloor \frac{x}{2} \rfloor$, which we write as

$$\sum_{k=0}^{\lfloor \frac{x}{2} \rfloor} \binom{x}{2k} = \binom{x}{0} + \binom{x}{2} + \cdots + \binom{x}{2k} \tag{1}$$

If we observe the binomial expansion of $(1-1)^x$, we notice that

$$\begin{aligned}
(1-1)^x &= \sum_{k=0}^{x} \binom{x}{k} 1^{x-k} (-1)^k \\
0 &= \sum_{k=0}^{x} \binom{x}{k} (-1)^k \\
0 &= \binom{x}{0} - \binom{x}{1} + \binom{x}{2} - \binom{x}{3} + \cdots \\
\binom{x}{1} + \binom{x}{3} + \cdots &= \binom{x}{0} + \binom{x}{2} + \cdots \\
\sum_{k=0}^{\lfloor \frac{x}{2} \rfloor} \binom{x}{2k+1} &= \sum_{k=0}^{\lfloor \frac{x}{2} \rfloor} \binom{x}{2k}
\end{aligned}$$

$\sum_{k=0}^{\lfloor \frac{x}{2} \rfloor} \binom{x}{2k+1}$ is equal to the number of odd-sized subsets of O. Thus, the number of odd-sized subsets of O is equal to the number of even-sized subsets of O. Let α be the number of odd-sized subsets of O, and let β be the number of even-sized subsets of O. We know that $\alpha + \beta$ must equal the total number of subsets in O. Thus, we have the following equations

$$\alpha - \beta = 0 \tag{2}$$

$$\alpha + \beta = 2^x \tag{3}$$

thus

$$\begin{aligned} 2\alpha &= 2^x \\ \alpha &= 2^{x-1} \end{aligned}$$

We can pair any even-sized subsets of O with any subset of S containing only even integers to produce a subset of S with even sum. Let y be the number of even integers in S. Then the following expression will give us the number of non-empty subsets of S with even sum

$$2^{x-1} \cdot 2^y - 1 \tag{4}$$

But, $x + y$ must equal n. Therefore

$$\begin{aligned} 2^{x-1} \cdot 2^y &= 2^{x+y-1} - 1 \\ &= 2^{n-1} - 1 \end{aligned}$$

Thus, the number of non-empty subsets of S with even sum equals $2^{n-1} - 1$