GRE Mathematics Subject Test GR1768 Solutions

2nd edition

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Preface

Thank you for checking out my solutions to the GRE mathematics subject test GR1768. If you haven't received the questions for the GR1768 in the mail, you can find them at `GR1768`.

This document was designed so that students can look at any solution independently and understand it. As a result, explanations are more or less self-contained, though some theorems are simply referenced.

There is a `glossary` which contains most of the theorems that I used in the solutions. In particular, I made sure to include theorems from Calculus and below, because it has probably been a while for most of you. A few students have told me that they studied the glossary to prepare for the subject test. Though I think knowing what is in the glossary is important, I would like to note that it was never intended to be an exhaustive list of the material you need to know for the exam.

A few words of acknowledgement are in order. A whole bunch of websites helped me while I was writing this. In particular, `mathematicsgre.com`, `math.stackexchange.com`, and `wikipedia.org` are great resources and I recommend them. Even more importantly, I would like to acknowledge and thank the readers that emailed me about errors. The mistakes have been fixed, and it's all thanks to you! Of course, I am responsible for any of the mistakes remaining.

While preparing for the GRE math subject test, I recommend old exams as your primary resources. Past performance is the best indicator of future performance, so try your best to take old tests using actual exam conditions. In particular, be mindful of the time constraint, because it makes the exam substantially more difficult. The techniques used in these solutions should be viable given the time pressure, but please try to get a feel for how long it takes you to complete problems and when it's best to guess or skip a question.

Your solutions will not require proofs, and it's unnecessary to show a lot of detail. In this sense, most solutions here do not mimic how one ought to solve problems during the exam. There are some proofs because I thought that understanding the ideas behind some of the calculations was important. Also, this text has way more detail than you need because the solutions are intended to be both didactic and a reintroduction to concepts from lower division courses. Keep this in mind. I recommend that you explicitly think about what you would do to solve a problem as you read the corresponding solution.

If you're in need of further GRE preparation material, there a few resources available. You are welcome to check out one of my two practice test at `rambotutoring.com/GREpractice.pdf`. The book *Cracking the GRE Mathematics Subject Test* is good. The problems are a bit easier than what would be ideal, but the questions are similar to those on the GRE mathematics subject test. There are also chapters within *Cracking the GRE Mathematics Subject Test*, which reintroduce concepts that you'll need for the GRE. I also recommend the material from a workshop that my friend Charlie ran while he attended UCLA. I posted the questions from the workshop and their solutions at `rambotutoring.com/math-gre/`. His problems are substantially more challenging than actual GRE math subject test questions, which makes it a good resource for students looking to achieve high scores. The REA book *GRE Mathematics* is also good—mainly due to its difficulty—but the authors weren't able to emulate the actual GRE test as closely as the other sources I cited.

To take care of a bit of shop work:

- My apologies for any errors. Alas, one of the problems with free documents is that it's not feasible to hire editors. And I must admit that I'm not the most careful writer. I welcome your help: If you find errors or have questions feel free to email me at `charles.tutoring@gmail.com`. Feedback is greatly appreciated. The most up-to-date version of this document can be found at `rambotutoring.com/GR1768-solutions.pdf`.

- If you find these solutions helpful, you might be interested in *GRE Mathematics Subject Test Solutions: Exams GR1268, GR0568, and GR9768*. The GR1268's questions are identical to the GR1768's. The GR0568 and GR9768 are the other two most recent official tests. The GR0568, in particular, is good to work through because the questions are about as hard as the GR1768's. The booklet is on sale at `amazon.com`.

- For details about my tutoring business, check out my website `rambotutoring.com`.

- Please write a review for *GRE Mathematics Subject Test Solutions: Exams GR1268, GR0568, and GR9768*. Feedback is very important.

Good luck on the GRE!

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GRE mathematics subject test GR1768 solutions

Question 1.

When we plug 0 into the expression for x, we obtain the ratio $0/0$ which is an indeterminate form. According to *L'Hôspital's rule*, when a ratio is in the $0/0$ indeterminate form, the limit is equal to the limit of the ratio of the derivatives of the top and bottom. So,

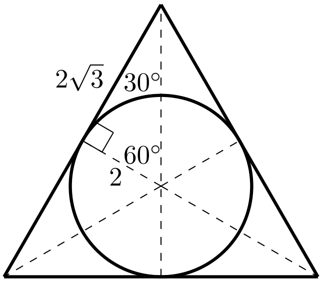
$$\lim_{x\to 0} \frac{\cos(3x)-1}{x^2} \overset{LH}{=} \lim_{x\to 0} \frac{-3\sin(3x)}{2x}$$

Notice that the new limit is also in the $0/0$ indeterminate form, which means that we can apply L'Hôspital's rule again:

$$\begin{aligned} \lim_{x\to 0} \frac{-3\sin(3x)}{2x} &\overset{LH}{=} \lim_{x\to 0} \frac{-9\cos(3x)}{2} \\ &= \frac{-9\cos(0)}{2} \\ &= -\frac{9}{2}. \end{aligned}$$

Hence, we select (E). See *sine and cosine values in quadrant I* in the glossary for a table of sine and cosine values at popular radian measures. ■

Question 2.



Since the circle is inscribed within an equilateral triangle, it is tangent to the triangle at all points of intersection. This implies that the radius of the circle is perpendicular to the triangle at intersection points. With this in mind, we divide the equilateral triangle into six congruent right triangles.

Each of the right triangles is a $30° - 60° - 90°$ special right triangle. This is because the hypotenuse of each right triangle bisects an interior angle of the equilateral triangle. As a result, one interior angle of each right triangle must have measure $60°/2 = 30°$, which implies the other acute angle has measure $90° - 30° = 60°$.

We will find the area of each of our right triangles. Let the radial length of 2 be the length of each triangle's height. Then the base of each right triangle has length

$$2\tan(60°) = 2\sqrt{3}.$$

Hence, each right triangle has area

$$\frac{2(2\sqrt{3})}{2} = 2\sqrt{3}.$$

Because all six right triangles are congruent, the total area of the equilateral triangle is

$$6(2\sqrt{3}) = 12\sqrt{3}.$$

The correct answer must be (C). ■

Question 3.

Let $u = \log x$, which implies $du = dx/x$. Hence,

$$\begin{aligned}
\int_{e^{-3}}^{e^{-2}} \frac{1}{x \log x}\, dx &= \int_{x=e^{-3}}^{e^{-2}} \frac{du}{u} \\
&= \int_{u=\log e^{-3}}^{\log e^{-2}} \frac{du}{u} \\
&= \int_{-3}^{-2} \frac{du}{u} \\
&= \log|u| \Big|_{-3}^{-2} \\
&= \log 2 - \log 3 \\
&= \log \frac{2}{3}.
\end{aligned}$$

We conclude that the answer is (D). A list of *logarithm properties* is located in the glossary. ■

Question 4.

This reduces to an equivalent problem with *bases*, since every subspace has a basis and the number of elements in a basis is its dimension. Consider a basis for $V \cap W$ and extend it to a basis of V. Let the set $\mathcal{B}_1$ be this basis for V. Extend $\mathcal{B}_1 \cap W$ to a basis of W, and call it $\mathcal{B}_2$. Extend $\mathcal{B}_1 \cup \mathcal{B}_2$ to a basis of X, and call it $\mathcal{B}$. We need to find what the cardinality of $\mathcal{B}_1 \cap \mathcal{B}_2$ cannot be. Using the *inclusion-exclusion principle* on $\mathcal{B}_1$ and $\mathcal{B}_2$,

$$|\mathcal{B}_1 \cup \mathcal{B}_2| = |\mathcal{B}_1| + |\mathcal{B}_2| - |\mathcal{B}_1 \cap \mathcal{B}_2|.$$

Since $\mathcal{B}_1 \cup \mathcal{B}_2 \subseteq \mathcal{B}$,

$$|\mathcal{B}_1| + |\mathcal{B}_2| - |\mathcal{B}_1 \cap \mathcal{B}_2| \leq |\mathcal{B}|.$$

Because $\dim(V) = \dim(W) = 4$ and $\dim(X) = 7$, we have

$$4 + 4 - |\mathcal{B}_1 \cap \mathcal{B}_2| \leq 7 \quad \text{implies} \quad |\mathcal{B}_1 \cap \mathcal{B}_2| \geq 1.$$

Therefore, $V \cap W$ cannot have dimension 0, and we pick (A). ■

Question 5.

We want to find $P(A) = 1 - P(A^c)$, where A is defined to be the event that "one integer is *not* the square of the other" and A^c is the complement of this event, i.e. A^c is the event that "one integer *is* the square of the other". Suppose the first coordinate of each ordered pair corresponds to Sofia's number and the second corresponds to Tess's. Then $A^c = \{(1,1),(2,4),(4,2),(3,9),(9,3)\}$ has five elements, and the sample space

$$\{(1,1),(1,2),\ldots,(1,10),\ldots,(10,10)\}$$

has a hundred elements. Thus, the probability that neither number selected is the square of the other must be

$$P(A) = 1 - P(A^c) = 1 - 0.05 = 0.95.$$

We fill in bubble (E) and continue. ■

Question 6.

The function $f(x) := x^6$ increases monotonically on the interval $[0, \infty)$. This can be proven, for example, by taking the derivative. Because

$$f(2^{1/2}) = 8, \quad f(3^{1/3}) = 9, \quad \text{and} \quad f(6^{1/6}) = 6,$$

we conclude

$$6^{1/6} < 2^{1/2} < 3^{1/3}.$$

So, the answer is (C). ■

Question 7.

Because f increases on the closed interval $[0, 2]$, and decreases on the closed interval $[2, 4]$, the value $f(2)$ must be an absolute maximum.

Using the *Fundamental theorem of Calculus*, we know

$$\int_0^4 f'(x)\,dx = f(4) - f(0).$$

Since the definite integral of f' from 0 to 4 measures the net signed area between f' and the x-axis over the interval, we conclude that the left side of the equation is positive, because the positive area between f' and the x-axis over the interval $(0, 2)$ has greater magnitude than the negative area between f' and the x-axis over the interval $(2, 4)$. It follows that $f(0) < f(4)$. Therefore,

$$f(0) < f(4) < f(2)$$

and we select (C). ■

Question 8.

The nonzero integers under multiplication are not a *group* because the multiplicative inverses of some integers are not integers, e.g. $1/3$ is the multiplicative inverse of 3. Fill in the bubble for (B). ■

Question 9.

Recall the geometric interpretation of the first and second derivatives. The value $g'(x)$ tells us the slope of g at x, and $g''(x)$ tells us the concavity of g at x. As a result, g is flat at $x = 0$, concave up at $x = -1$, and concave down on the open interval $(0, 2)$. Let's go through our options. Choice (B) fails because the graph is concave up on a subset of $(0, 2)$, (C) doesn't work because the graph isn't flat at $x = 0$, (D) can't be it because the graph is concave up on a subset of $(0, 2)$, and we remove (E) from consideration because the graph is concave down at $x = -1$. The only available option is (A). ■

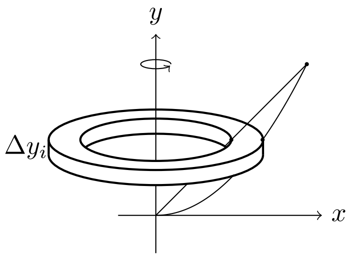
Question 10.

We will solve this problem using algebra:

$$\begin{aligned}
& \sqrt{(x+3)^2 + (y-2)^2} &&= \sqrt{(x-3)^2 + y^2} \\
\Rightarrow\quad & (x+3)^2 + (y-2)^2 &&= (x-3)^2 + y^2 \\
\Rightarrow\quad & x^2 + 6x + 9 + y^2 - 4y + 4 &&= x^2 - 6x + 9 + y^2 \\
\Rightarrow\quad & 12x - 4y &&= -4.
\end{aligned}$$

The answer is (A), because this is the equation of a line. ■

Question 11.



We will find a formula in terms of y_i for the volume of a thin washer ΔV_i, and then formulate the total volume as the limit of a Riemann sum of ΔV_i's.

Each slice of volume is the area of its annular cross-section multiplied by a small vertical length Δy_i. The outer radius of the annular cross-section lies on the curve $y = x^2$ and inner radius on the line $y = x$. Hence, at the y-value y_i the outer and inner radii are $\sqrt{y_i}$ and y_i, respectively. Thus,

$$\Delta V_i = \pi\Big((\sqrt{y_i})^2 - (y_i)^2\Big)\Delta y_i = \pi\Big(y_i - (y_i)^2\Big)\Delta y_i.$$

The minimum and maximum y-values, within the region bounded by $y = x$ and $y = x^2$, are 0 and 1, respectively. Consider the partition $P = \{1, 1/2, 1/3, \ldots, 1/(n+1)\}$ of the interval $[0, 1]$. Suppose $y_i = 1/(n-i+1)$ and $\Delta y_i = 1/(n-i+1) - 1/(n-i+2)$. Notice that $\sum_{i=1}^n \Delta V_i \approx V$. Furthermore, as $n \to \infty$, we have $\sum_{i=1}^n \Delta V_i \to V$ and

$$\sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} \pi\left(y_i - (y_i)^2\right)\Delta y_i \longrightarrow \int_0^1 \pi\Big(y - y^2\Big)dy.$$

Hence, the volume obtained from rotating the region bounded by $y = x$ and $y = x^2$ about the y-axis is

$$V = \int_0^1 \pi\left(y - y^2\right)\, dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

We select (B) and continue. Note that we went through the reasoning behind the *washer method*. ■

Question 12.

Any group of prime order must be cyclic. This follows from *Lagrange's theorem* which says the order of a subgroup must divide the order of the entire group. So, if a group has prime order, then any nonidentity element must generate the entire group because it generates a subgroup of order other then 1.

It follows that all groups of prime order are isomorphic to all other groups of the same prime order. This is due to the fact that all prime ordered groups are cyclic and all cyclic groups of the same order are isomorphic.

It is easy enough to find groups that are non-isomorphic but of the same order for the composite values of n. For example, there are two groups, up to isomorphism, of order 9; they are $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Ergo, (B) must be the correct answer. ■

Question 13.

Due to an *integration property*, if $f'(x) \geq -1$ for all x, then

$$\int_0^3 f'(x)\ dx \geq \int_0^3 -1\ dx = -3.$$

Because of the *Fundamental theorem of Calculus*,

$$\int_0^3 f'(x)\ dx = f(3) - f(0) = 5 - f(0).$$

Hence, $5 - f(0) \geq -3$. Solving for $f(0)$ yields $f(0) \leq 8$. We conclude that option (D) is correct. ■

Question 14.

Since there is no area under a point,

$$\int_c^c g(t)\ dt = 0.$$

Due to the given equation, it follows that $3c^5 + 96 = 0$. After a little algebra, we find that $c = -2$. Pick option (B). ■

Question 15.

The function f must be one-to-one. Suppose otherwise. Then there are unique elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$. But this implies $(g \circ f)(s_1) = (g \circ f)(s_2)$, which is a contradiction of the assumption that $g \circ f$ is one-to-one.

During an the exam, this would be a good time to select (A) and continue, but let's find counterexamples for the others. Consider

$$f = \{(1,2)\} \quad \text{and} \quad g = \{(2,4),(3,4)\},$$

where

$$S = \{1\}, \quad T = \{2,3\}, \quad \text{and} \quad U = \{4,5\}.$$

The composite function $g \circ f = \{(1,4)\}$ is clearly one-to-one. However, the function f is not onto, because 3 is not in the image of f; g is not one-to-one because $g(2) = g(3)$; the function g is not onto because 5 is not in its range; and the image of $g \circ f$ does not contain 5 which means $g \circ f$ cannot be onto. ■

Question 16.

The first step is to clearly formulate this scenario as a conditional statement:

If either A or B, then C.

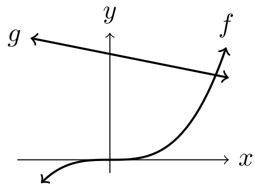
Note: most mathematicians consider *or* to be inclusive, so X *or* Y being true implies X could be true, Y could be true, or both X and Y could be true. The *either* in front makes the *or* exclusive.

Since the negation of *either A or B* is *A and B, or not A and not B*, the contrapositive of our conditional statement is

If not C, then $\big($A and B$\big)$ or $\big($not A and not B$\big)$.

In other words, when C is false, the statements A and B must have the same truth value. Thus, given that C is false, A being false implies that B is false. We choose (B). ■

Question 17.



Let's reformulate the first option as a graphing problem. Suppose

$$f(x) := x^3 \quad \text{and} \quad g(x) := 10 - x.$$

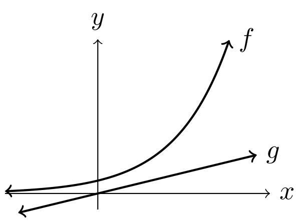
The number of solutions to the equation in (A) is the same as the number of intersections of the graphs of f and g. So, after we draw a rough sketch of f and g, we see that the equation in (A) has one real solution.

Consider (B). We know

$$x^2 + 5x - 7 = x + 8 \quad \text{if and only if} \quad x^2 + 4x - 15 = 0.$$

The *discriminant* of the latter quadratic equation is $4^2 - 4(1)(-15) = 76 > 0$. This implies that there are two real solutions.

Algebra shows that the one solution of equation (C) is $x = -2/5$.



The number of real solutions of equation (D) is the same as the number of intersections of the graphs of

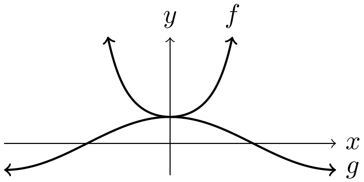
$$f(x) := e^x \quad \text{and} \quad g(x) := x.$$

A sketch shows that there are no intersections, so equation (D) has no real solution.

Since this is tougher to graph, a few remarks are in order. Notice that $f(x) > 0$ and $g(x) < 0$ for $x < 0$, which means there is no hope of the graphs intersecting for negative values of x. To prove there is no intersection for $x \geq 0$, notice

$$f(0) = e^0 = 1 \quad \text{and} \quad g(0) = 0,$$

while $f'(x) = e^x > 1$ and $g'(x) = 1$ for $x > 0$. In other words, f is bigger at $x = 0$ and grows faster for $x > 0$, so there is no chance of f and g intersecting for positive values of x.



Examine (E). Notice

$$\sec x = e^{-x^2} \quad \text{if and only if} \quad e^{x^2} = \cos x.$$

So, the number of solutions of equation (E) is the same as the number of intersections of the graphs of

$$f(x) := e^{x^2} \quad \text{and} \quad g(x) := \cos x.$$

Because $f'(x) = 2xe^{x^2}$,

$$f'(x) < 0 \text{ for } x < 0 \quad \text{and} \quad f'(x) > 0 \text{ for } x > 0.$$

It follows that $f(0) = 1$ is the absolute minimum. Since $g(0) = \cos 0 = 1$, we conclude that the graphs intersect at $x = 0$. Since the range of g is the interval $[-1, 1]$, there are no other intersections. Note: a list of *sine and cosine values in quadrant I* is given in the glossary.

Therefore, equation (B) has the greatest number of solutions. Fill in the bubble and move on! ■

Question 18.
The function

$$f'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{d}{dx}\left(x^n\right) = \sum_{n=1}^{\infty} x^{n-1}.$$

We need one of the *summation formulas* from Calculus:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}, \quad \text{when} \quad |r| < 1.$$

In our case, $a = 1$ and $r = x$, so

$$f'(x) = \frac{1}{1-x}.$$

We select (A) and proceed to the next question. ■

Question 19.
For any complex number z,

$$z = |z|e^{i\theta} = |z|\left(\cos\theta + i\sin\theta\right),$$

where $|z|$ is the modulus of z and θ is some number in the interval $(-\pi, \pi]$. Using the identity, we see

$$\bar{z} = |z|\left(\cos\theta - i\sin\theta\right) = |z|\left(\cos(-\theta) + i\sin(-\theta)\right) = |z|e^{-i\theta},$$

because $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$. Furthermore, notice that $z \to 0$ is equivalent to saying $|z| \to 0$. We are ready to compute our limit:

$$\begin{aligned}
\lim_{z\to 0} \frac{(\bar{z})^2}{z^2} &= \lim_{|z|\to 0} \frac{\left(|z|e^{-i\theta}\right)^2}{\left(|z|e^{i\theta}\right)^2} \\
&= \lim_{|z|\to 0} \frac{e^{-2i\theta}}{e^{2i\theta}} \\
&= e^{-4i\theta}.
\end{aligned}$$

Since θ could be any value in the interval $(-\pi, \pi]$, the limit does not exist and we pick option (E). ■

Question 20.

The limit is in the 0/0 indeterminate form, because the numerator is $g\big(g(0)\big) - g(e) = g(e) - g(e) = 0$ and the denominator is 0 when $x = 0$. So, we will use *L'Hôspital's rule*. We continue as follows:

$$\lim_{x\to 0} \frac{g(g(x)) - g(e)}{x} \overset{LH}{=} \lim_{x\to 0} \frac{g'(g(x))g'(x) - 0}{1} = g'(g(0))g'(0).$$

Since $g'(x) = 2e^{2x+1}$, we have

$$g'(g(0))g'(0) = g'(e) \cdot 2e = 2e^{2e+1} \cdot 2e = 4e^{2e+2}.$$

Thus, the correct answer must be (E). ■

Question 21.

We first prove that $\sqrt{1+t^2}\sin^3 t\cos^3 t$ is odd:

$$\begin{aligned}\sqrt{1+(-t)^2}\sin^3(-t)\cos^3(-t) &= \sqrt{1+t^2}(-\sin t)^3\cos^3 t \\ &= -\sqrt{1+t^2}\sin^3 t\cos^3 t.\end{aligned}$$

If f is an odd function, then $\int_{-a}^{a} f(t)\ dt = 0$ for all real numbers a for which the integral makes sense. This is because the signed area to the left of the origin has the same magnitude, but the opposite sign, as the area to the right of the origin.

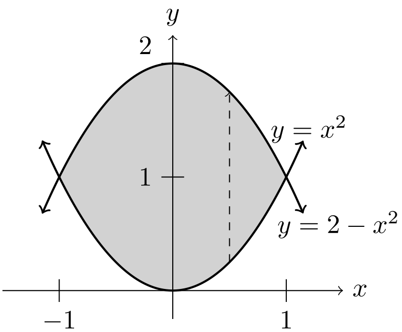
With this and our basic *integration properties* in mind, we compute the definite integral:

$$\begin{aligned}&\int_{-\pi/4}^{\pi/4} \left(\cos t + \sqrt{1+t^2}\sin^3 t\cos^3 t\right)\ dt \\ &\qquad = \int_{-\pi/4}^{\pi/4} \cos t\ dt + \int_{-\pi/4}^{\pi/4} \sqrt{1+t^2}\sin^3 t\cos^3 t\ dt \\ &\qquad = \int_{-\pi/4}^{\pi/4} \cos t\ dt + 0 \\ &\qquad = \sin t\Big|_{-\pi/4}^{\pi/4} \\ &\qquad = \sin(\pi/4) - \sin(-\pi/4) \\ &\qquad = \sqrt{2}.\end{aligned}$$

Choose (B) and move on. Note: there is a list of *sine and cosine values in quadrant I* located in the glossary. ■

Question 22.

The first step is to draw the base of the solid.



We observe that the planes $z = y + 3$ and $z = 0$ never intersect within the region above. It follows that the volume is

$$\begin{aligned}
V &= \int_{-1}^{1} \int_{x^2}^{2-x^2} \int_{0}^{y+3} dz dy dx \\
&= \int_{-1}^{1} \int_{x^2}^{2-x^2} y + 3 \; dy dx \\
&= \int_{-1}^{1} \frac{(2-x^2)^2}{2} + 3(2-x^2) - \frac{x^4}{2} - 3x^2 \; dx \\
&= \int_{-1}^{1} \frac{x^4}{2} - 5x^2 + 8 - \frac{x^4}{2} - 3x^2 \; dx \\
&= \int_{-1}^{1} 8 - 8x^2 \; dx \\
&= \frac{32}{3}.
\end{aligned}$$

The correct answer is (C). ■

Question 23.

Consider the following addition and multiplication tables for S:

$$\begin{array}{c|ccccc} + & 0 & 2 & 4 & 6 & 8 \\ \hline 0 & 0 & 2 & 4 & 6 & 8 \\ 2 & 2 & 4 & 6 & 8 & 0 \\ 4 & 4 & 6 & 8 & 0 & 2 \\ 6 & 6 & 8 & 0 & 2 & 4 \\ 8 & 8 & 0 & 2 & 4 & 6 \end{array} \quad \text{and} \quad \begin{array}{c|ccccc} \cdot & 0 & 2 & 4 & 6 & 8 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 8 & 2 & 6 \\ 4 & 0 & 8 & 6 & 4 & 2 \\ 6 & 0 & 2 & 4 & 6 & 8 \\ 8 & 0 & 6 & 2 & 8 & 4. \end{array}$$

It is easy to see that S is closed under $+$ and $\cdot$, 0 is the identity element under $+$, 6 is the identity element under $\cdot$, and both operations are commutative. Ergo, statement (D) is false. ∎

Question 24.

Let's reformulate the problem using matrices. Define

$$A := \begin{pmatrix} 1 & 3 & 2 & 2 \\ 1 & 4 & 1 & 0 \\ 3 & 5 & 10 & 14 \\ 2 & 5 & 5 & 6 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x} := \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}.$$

Then our system can be written as $A\boldsymbol{x} = \mathbf{0}$.

Statement (A) is true. For any homogeneous system, where the dimension makes sense, $\boldsymbol{x} = \mathbf{0}$ is a solution.

Statement (C) is true. Suppose $\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} = \boldsymbol{c}$ are solutions to $A\boldsymbol{x} = \mathbf{0}$. Then

$$A(\boldsymbol{b} + \boldsymbol{c}) = A\boldsymbol{b} + A\boldsymbol{c} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

A quick computation shows that (D) is true:

$$\begin{array}{ccccccccc} -5 & + & 3(1) & + & 2(1) & + & 2(0) & = & 0 \\ -5 & + & 4(1) & + & 1 & & & = & 0 \\ 3(-5) & + & 5(1) & + & 10(1) & + & 14(0) & = & 0 \\ 2(-5) & + & 5(1) & + & 5(1) & + & 6(0) & = & 0. \end{array}$$

Statement (B) follows from (D). If there is a vector $\boldsymbol{b} \neq \mathbf{0}$ such that $A\boldsymbol{b} = \mathbf{0}$, then for each scalar k in $\mathbb{R}$ the vector $k\boldsymbol{b}$ is a solution because

$$A(k\boldsymbol{b}) = kA\boldsymbol{b} = k \cdot \mathbf{0} = \mathbf{0}.$$

By the process of elimination, (E) must be false so we select it. We don't recommend that you find another solution to the system during the exam because it's somewhat time-consuming and it's very easy to make a mistake. That said, $\boldsymbol{x} = (-8, 2, 0, 1)$ is also a solution and it is not a scalar multiple of $(-5, 1, 1, 0)$. ■

Question 25.

This question is primarily testing your memory. Recall that $(c, h(c))$ is an *inflection point* of h if and only if h'' switches signs at c.

The value of h'' is positive when h' is increasing, and h'' is negative when h' is decreasing. Since h' goes from decreasing to increasing within the open interval $(-2, -1)$, there is a c in the interval such that $(c, h(c))$ is an inflection point. Fill in the bubble for (A)! ■

Question 26.

Since

$$4 \cdot 3 \equiv 12 \equiv 1 \pmod{11} \quad \text{and} \quad 6 \cdot 2 \equiv 12 \equiv 1 \pmod{11},$$

the multiplicative inverses of 3 and 2 are 4 and 6, respectively. So,

$$\begin{aligned} 3x &\equiv 5 \pmod{11} &\quad \text{and} \quad 2y &\equiv 7 \pmod{11} \\ \Rightarrow 12x &\equiv 20 \pmod{11} &\quad \text{and} \quad 12y &\equiv 42 \pmod{11} \\ \Rightarrow \quad x &\equiv 9 \pmod{11} &\quad \text{and} \quad y &\equiv 9 \pmod{11}. \end{aligned}$$

Thus,

$$\begin{aligned} x + y &\equiv 9 + 9 \\ &\equiv 18 \\ &\equiv 7 \pmod{11}. \end{aligned}$$

The correct answer must be (D). ■

Question 27.

Recall the following identity. Suppose $z \neq 0$ is a complex number. Then

$$z = |z|e^{i\theta} = |z|\left(\cos\theta + i\sin\theta\right),$$

where $|z|$ is the modulus of z and θ is some number in the interval $(-\pi, \pi]$.

Our goal is to write $z = 1 + i$ into the form $|z|e^{i\theta}$, use exponent rules to raise z to the 10th power, and then use the expression with trigonometric functions to write our result in standard form.

Let's find the modulus of z:

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Therefore,

$$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right).$$

It follows that $\cos\theta = \sqrt{2}/2$ and $\sin\theta = \sqrt{2}/2$. From what we know about the unit circle, or by inspection of the list of *sine and cosine values in quadrant I* in the glossary, we can conclude that $\theta = \pi/4$. The rest of the problem is a simple computation:

$$\begin{aligned}
(1+i)^{10} &= \left(\sqrt{2}e^{\pi i/4}\right)^{10} \\
&= 2^5 e^{5\pi i/2} \\
&= 32\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) \\
&= 32\left(0 + 1i\right) \\
&= 32i.
\end{aligned}$$

The solution must be (D). ■

Question 28.

We know

$$y - 4 = 3(x-1) \quad \text{implies} \quad y = 3x+1.$$

Because f is tangent to $y = 3x + 1$ at $x = 1$, $f(1) = 3(1) + 1 = 4$ and $f'(1) = 3$. This allows us to jettison (A) as an option.

The *inverse function theorem* says

> Suppose f has a continuous non-zero derivative in some connected open neighborhood of $x = a$. Further, assume the graph of f within this neighborhood contains the point (a, b). Then
>
> $$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

We note that injectivity of the differentiable function f implies f' is non-zero in a connected neighborhood of $x = 1$. It follows that $(f^{-1})'(4) = 1/3$. We remove (B) from consideration.

With the *derivative rules* from Calculus in mind, the last three options are simple computations.

Using the product rule,

$$\begin{aligned}(fg)'(1) &= f(1)g'(1) + f'(1)g(1) \\ &= 4\left(\frac{1}{2\sqrt{1}}\right) + 3\left(\sqrt{1}\right) \\ &= 2 + 3 \\ &= 5.\end{aligned}$$

So we eliminate (C).

Option (D) is false, so we select it. The untruth of (D) is proven using the chain rule:

$$\begin{aligned}(g \circ f)'(1) &= g'(f(1)) \cdot f'(1) \\ &= g'(4) \cdot 3 \\ &= \left(\frac{1}{2\sqrt{4}}\right) \cdot 3 \\ &= 3/4 \\ &\neq 1/2.\end{aligned}$$

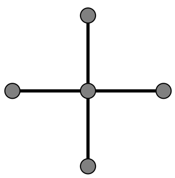
The last one, (E), is true. We have

$$\begin{aligned}(g \circ f)(1) &= g\left(f(1)\right) \\ &= g(4) \\ &= \sqrt{4} \\ &= 2.\end{aligned}$$

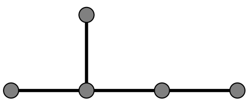
■

Question 29.

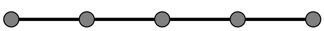
Let's go through the cases. It is clear that we can connect at most four edges to a vertex of our tree.



The next possibility is that one of the vertices has three edges connected to it.

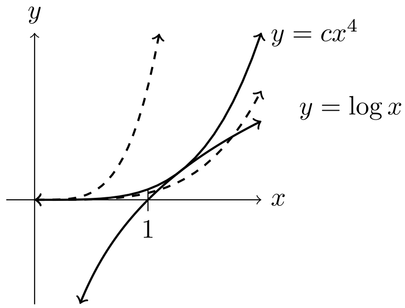


The last case we need to consider is when there are no more than two edges connect to each vertex.



Ergo, there are three non-isomorphic trees with five vertices, so (C) is the correct answer. ■

Question 30.



As can be seen above, if the graph of $y = cx^4$ and $y = \log x$ intersect at exactly one point, then the graphs are tangent at their intersection. Being tangent at a location implies that they have the same slope, which means the derivatives of the two functions are equal. So,

$$\frac{1}{x} = 4cx^3 \quad \text{implies} \quad x = \sqrt[4]{\frac{1}{4c}}.$$

Hence,

$$\begin{aligned}
\log x &= cx^4 \\
\Rightarrow \quad \log\left(\sqrt[4]{\frac{1}{4c}}\right) &= c \cdot \frac{1}{4c} \\
\Rightarrow \quad \frac{1}{4}\log\left(\frac{1}{4c}\right) &= \frac{1}{4} \\
\Rightarrow \quad \log\left(\frac{1}{4c}\right) &= 1 \\
\Rightarrow \quad \frac{1}{4c} &= e \\
\Rightarrow \quad c &= \frac{1}{4e}.
\end{aligned}$$

And we have proven that the answer is (A). A list of *logarithm properties* is located in the glossary. ■

Question 31.

To find the eigenvalues, we need to find the characteristic polynomial

$$p(\lambda) = \det \begin{pmatrix} 3-\lambda & 5 & 3 \\ 1 & 7-\lambda & 3 \\ 1 & 2 & 8-\lambda \end{pmatrix} = -(\lambda-2)(\lambda-5)(\lambda-11).$$

The eigenvalues are the solutions of $p(\lambda) = 0$. Thus, we select (C), because 2 and 5 are eigenvalues but not 3. ■

Question 32.

Recall the *Fundamental theorem of Calculus*:

> Suppose f is continuous on the closed interval $[a, b]$. Then
> $$\int_a^b f(t)\ dt = F(b) - F(a),$$
> where $F'(t) = f(t)$.

Supposed $F(t)$ is an antiderivative of e^{t^2}. Then

$$\begin{aligned} \frac{d}{dx}\int_{x^3}^{x^4} e^{t^2}\ dt &= \frac{d}{dx}\left(F(x^4) - F(x^3)\right) \\ &= 4x^3F'(x^4) - 3x^2F'(x^3) \\ &= 4x^3e^{x^8} - 3x^2e^{x^6} \\ &= x^2e^{x^6}\left(4xe^{x^8-x^6} - 3\right). \end{aligned}$$

Select (E). A list of *derivative rules* is located in the glossary. ■

Question 33.

We will rewrite the ratio as a product and use the product rule, which is one of the *derivative rules* in the glossary. Let

$$f(x) := \frac{x-1}{e^x} = (x-1)e^{-x}.$$

So,

$$f'(x) = 1e^{-x} - (x-1)e^{-x} \qquad f''(x) = -1e^{-x} + (x-2)e^{-x}$$
$$= (2-x)e^{-x} \qquad\qquad = (x-3)e^{-x},$$
$$= -(x-2)e^{-x},$$

and

$$\begin{aligned} f'''(x) &= e^{-x} - (x-3)e^{-x} \\ &= (4-x)e^{-x} \\ &= -(x-4)e^{-x}. \end{aligned}$$

It appears that

$$f^{(n)}(x) = (-1)^n(x-n-1)e^{-x}.$$

To prove this, we would use induction on n. However, proofs aren't necessary for the GRE and this result seems sufficiently self-evident. We conclude

$$f^{(19)}(x) = -(x-20)e^{-x} = (20-x)e^{-x},$$

and we fill in the bubble for (C). ■

Question 34.

Let's compute $\det(A)$. There is a theorem that says the determinant of an upper or lower triangular matrix is the product of the entries on the main diagonal. Therefore,

$$\det(A) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

We can immediately exclude (E) as a possibility. Furthermore, since a matrix is invertible if and only if the determinant is nonzero, A is invertible and we jettison (A). We remove (D) as a possibility because there is a sequence of elementary row operations that can be used to transform A into the identity; one algorithm to find the inverse is to consider $(A|I)$ and perform elementary row operations until you find $(I|A^{-1})$ so (D) is also equivalent to asking whether A is invertible.

Let's turn our attention to (B). It is not too hard to see that the proposition $A\boldsymbol{x} = \boldsymbol{x}$ implies $\boldsymbol{x} = \mathbf{0}$ is equivalent to the claim that 1 is *not* an eigenvalue of A. We will show that 1 is an eigenvalue by proving that it is a zero of the characteristic polynomial. Since

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 0 & 2-\lambda & 3 & 4 & 5 \\ 0 & 0 & 3-\lambda & 4 & 5 \\ 0 & 0 & 0 & 4-\lambda & 5 \\ 0 & 0 & 0 & 0 & 5-\lambda \end{pmatrix},$$

we know

$$p(1) = \det \begin{pmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = 0.$$

Select (B) as your answer.

Since we are not being timed (now) we will show that (C) is valid. The last row of A^2 is

$$(0\,0\,0\,0\,5) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = (0\,0\,0\,0\,25)\,.$$

■

Question 35.

We would like to find the point on the plane $2x + y + 3z = 3$ that is closest to the origin. This is equivalent to finding the point (x, y, x) on the plane that minimizes $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Since square roots increase monotonically, the point that minimizes d will also minimize $f(x, y, z) := x^2 + y^2 + z^2$. As such, for simplicity, we will minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) := 2x + y + 3z = 3$. Via the *method of Lagrange multipliers*, we know relative extrema occur when

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z),$$

for some λ. It follows that

$$2x = 2\lambda, \quad 2y = \lambda, \quad \text{and} \quad 2z = 3\lambda.$$

A bit of algebra shows that $y = x/2$, and $z = 3x/2$. Thus,

$$\begin{aligned} 3 &= g\left(x, \frac{x}{2}, \frac{3x}{2}\right) \\ &= 2x + \left(\frac{x}{2}\right) + 3\left(\frac{3x}{2}\right) \\ &= 7x. \end{aligned}$$

Hence, $x = 3/7$, $y = 3/14$, and $z = 9/14$. The minimum is a relative extremum, because there is a disk on the plane containing the closest point such that every other point in the disk is farther away from the origin. Since $(3/7, 3/14, 9/14)$ is the only relative extremum, it must minimize f. Thus, (B) is correct. ■

Question 36.

Option (A) need not be true. Consider $S = \{0, 1\}$. There is no continuous function from the closed interval $[0, 1]$ to S.

Option (B) is not the solution. Consider $S = (0, 1)$. Since 0 is a limit point of S, there is no open neighborhood U of 0 such that $U \cap S = \varnothing$.

Option (C) must be true, which means this is the correct answer. Let

$$W := \{v \in S : \exists \text{ an open } V \subseteq \mathbb{R} \text{ s.t. } v \in V \subseteq S\}.$$

If W is empty, then we are done because the empty set is open. Suppose not and consider v in W. By definition, there is an open neighborhood V of v such that $V \subseteq S$. For u in V, we have u in W, because there exists an open subset of u contained within S, namely V. It follows that each point of W has an open neighborhood contained within W. Thus, W is open.

Option (D) is not always true. Consider $S = [0, 1]$. It is not too tough to see that

$$\{w \notin S : \exists \text{ an open } W \subseteq \mathbb{R} \text{ s.t. } w \in W,\ W \cap S = \varnothing\} = (-\infty, 0) \cup (1, \infty)$$

is open.

Option (E) is out. When S is open, it cannot be recreated via the intersections of closed subsets, because the intersection of closed subsets is always closed. ■

Question 37.

Statement I is false. Consider the case where $P : \mathbb{R} \to \mathbb{R}$ such that $P : x \mapsto 0$. Then $P^2 : x \mapsto 0$. It follows that $P^2 = P$, but P is not invertible.

Statement II is true. Consider $\boldsymbol{v}$ in V. Then $\boldsymbol{v} = (\boldsymbol{v} - P\boldsymbol{v}) + P\boldsymbol{v}$. Since

$$P(\boldsymbol{v} - P\boldsymbol{v}) = P\boldsymbol{v} - P^2\boldsymbol{v} = P\boldsymbol{v} - P\boldsymbol{v} = \boldsymbol{0},$$

so we can write every element of V as the sum of a vector in the null space of P and a vector in the range of P. Furthermore, the vectors in the range of P are invariant under P because $P(P\boldsymbol{v}) = P^2\boldsymbol{v} = P\boldsymbol{v}$, which implies vectors in the range are eigenvectors with an eigenvalue of 1. Clearly, vectors in the null space of P are eigenvectors with an eigenvalue of 0. We can conclude that P is diagonalizable because there exists a basis of eigenvectors, namely a basis for the null space of P union a basis for the range of P.

Statement III is false. There do exist linear transformations such that $P^2 = P$, but P is not the identity or the zero transformation. For example, if

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{then} \quad P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We select (C). ■

Question 38.

We know that the sum of the interior angles of an n-gon is $180°(n-2)$. This implies that the sum of the interior angles of our 10-gon is $180°(8) = 1440°$. Since our polygon is convex, all interior angles must have measure less than $180°$. We will consider the degenerate case, where there are m angles of measure $90°$ and n angles of measure 180°. We know $m+n=10$ because there are ten interior angles. So, we will solve

$$\begin{cases} 90^\circ m + 180^\circ n &= 1440^\circ \\ m+n &= 10. \end{cases}$$

This yields $m=4$ and $n=6$. Since our obtuse angles must have measure less than 180° and our acute angles must have measure less than 90°, this is an impossibility. However, if we make one of our 90° angles obtuse, then we can decrease the measures of the 180° angles and the other 90° angles by a small amount. This yields the optimal number of acute angles in our 10-gon. Thus, there can be at most three acute angles. And we have proven (C) is correct. ■

Question 39.

The solution is (D). We input `n=88`. The algorithm sets `i=1`. Since `i=1` is less than `n=88`, we enter the first while loop. This changes `i` to `2`, and sets `k=n=88`. Because `k=88` is greater than or equal to `i=2`, we enter the second while loop which reduces `k` by `1` each iteration, until `k=i=2`. At this time `i=2` is printed, and we go back to the beginning of the first while loop. This process continues up to `i=88`, at which time the criterion for the second while loop will pass for the last time which will result in `88` being printed. Then we go back to the first while loop which increases `i` to `89`. Because `i=89` is not less than `k=88`, we will not enter the second while loop. As a result, no more numbers will be printed. ■

Question 40.

Statement III is true, and the others are false. To disprove I and II, consider

$$f(x) := 1, \quad g(x) := 2, \quad \text{and} \quad h(x) := 1 + x.$$

We can disprove commutativity of $\circ$ by considering

$$(f \circ g)(x) = 1 \quad \text{and} \quad (g \circ f)(x) = 2.$$

Furthermore, we disprove that $\circ$ is distributive on the left via considering

$$f \circ (g+h) = f(3+x) = 1 \quad \text{and} \quad (f \circ g)(x) + (f \circ h)(x) = 1+1 = 2.$$

The truth of statement III follows directly from the definition of function addition. By definition,

$$(g+h)(x) := g(x) + h(x).$$

Replace the quantity x with $f(x)$ within this definition and the result follows. Fill in (C) and continue. ■

Question 41.

To find an equation of this plane, we need a point on the plane (which we have), and a vector normal to the plane.

To find a normal vector, we construct a vector parallel to ℓ. Solving the system

$$\begin{cases} x+y+z &= 3 \\ x-y+z &= 5 \end{cases},$$

shows that any point of the form $(4-t, -1, t)$, for t a real number, lies on ℓ. Any vector with tip and tail on ℓ will be parallel, so we select two values of t and find the vector between them. Letting $t = 0$ yields the point $(4, -1, 0)$. Letting $t = 1$ yields the point $(3, -1, 1)$. It follows that the vector

$$\boldsymbol{u} := \big(4-3, -1-(-1), 0-1\big) = \big(1, 0, -1\big)$$

is parallel to ℓ, and therefore perpendicular to the plane.

We are ready to construct the plane. Suppose (x, y, x) is a point on it. Then the vector

$$\boldsymbol{v} := \big(x-0, y-0, z-0\big) = \big(x, y, z\big)$$

lies on the plane. Let's compute the dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \big(1, 0, -1\big) \cdot \big(x, y, z\big) = x - z.$$

Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. Hence, an equation for the plane is $x - z = 0$. The correct answer must be (A). ■

Question 42.

All of the propositions listed are true. This metric induces the discrete topology on $\mathbb{Z}^+$, and within this topology, every set is both open and closed. This forces all functions with domain $\mathbb{Z}^+$ to be continuous, because the inverse image of an open set will, no doubt, be open (since all sets in the domain are open). However, let's go through our options supposing that we don't know about the discrete topology.

Proposition I: For each n in $\mathbb{Z}^+$, the open ball centered at n of radius $1/2$ is contained within $\{n\}$. It follows that every point of $\{n\}$ is an interior point, which proves that $\{n\}$ is open.

Proposition II: Consider an arbitrary set $A \subseteq \mathbb{Z}^+$. We will prove it's closed by showing that the complement $\mathbb{Z}^+ \setminus A$ is open. Because of proposition I, we know all singleton sets are open. Furthermore, the union of open sets is open and

$$\bigcup_{n \in \mathbb{Z}^+ \setminus A} \{n\} = \mathbb{Z}^+ \setminus A.$$

It follows that the complement of A is open, so A is closed.

Proposition III: Recall the ε-δ definition of continuity for real-valued functions: A function $f : X \to \mathbb{R}$ is continuous at c if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad d_X(x, c) < \delta,$$

where d_X is the metric on X. This criterion clearly holds for any real-valued function with domain $\mathbb{Z}^+$ given our metric; simply let $\delta = 1/2$, and it follows vacuously.

Hence, we select (E). ■

Question 43.

We need to find the second derivative of y with respect to x. It would be a mess to remove the parameter. Instead, recall the following two Calculus formulas for the *slope and concavity of curves with parametric equations*:

> Suppose $x = f(t)$ and $y = g(t)$ describe a curve. Then the slope and concavity of the curve at the point corresponding to t, respectively, are
>
> $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d^2y/dtdx}{dx/dt}.$$

It is not hard to see that $dy/dt = 12t^3 + 12t^2$ and $dx/dt = 2t + 2$, so

$$\frac{dy}{dx} = \frac{12t^3 + 12t^2}{2t + 2} = 6t^2.$$

Then $d^2y/dtdx = 12t$. Hence,

$$\frac{d^2y}{dx^2} = \frac{12t}{2t + 2} = \frac{6t}{t + 1}.$$

Our next task is to find the t corresponding to the point $(8, 80)$. Since $x(t) = t^2 + 2t = 8$, $t = -4$ or $t = 2$. Of these two values of t, it is not a difficult computation to conclude that $t = 2$ is the only one that satisfies $y(t) = 3t^4 + 4t^3 = 80$. Hence,

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{6(2)}{2 + 1} = 4.$$

Fill in the bubble for (A). ■

Question 44.

Let's find the general solution to our differential equation. To do this, we will separate variables:

$$y' + xy = x \quad \text{implies} \quad \frac{y'}{y-1} = -x.$$

Then we integrate both sides with respect to x. On the left side, we have

$$\int \frac{y'}{y-1}\, dx = \int \frac{dy}{y-1} = \log|y-1|.$$

We omit the C, because only one is necessary per equation. On the right side, we have

$$\int -x\, dx = -\frac{x^2}{2} + C.$$

Hence,

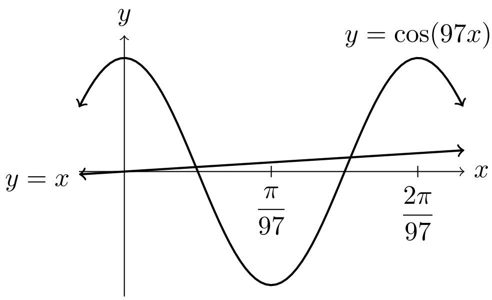
$$\begin{aligned} \log|y-1| &= -\frac{x^2}{2} + C \\ \Rightarrow \qquad y - 1 &= \pm e^{-x^2/2 + C} \\ \Rightarrow \qquad y &= \pm e^{C} e^{-x^2/2} + 1 \\ &= K e^{-x^2/2} + 1, \end{aligned}$$

where $K = \pm e^{C}$. Since

$$\lim_{x \to -\infty} y(x) = \lim_{x \to -\infty} 1 + K e^{-x^2/2} = 1,$$

regardless of K, we need not bother finding it. We conclude that the answer is (B). ∎

Question 45.



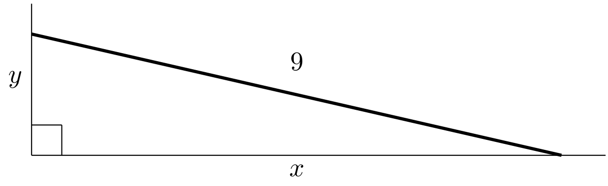
During the first quarter of each period cosine goes from 1 to 0, and in the fourth quarter cosine goes from 0 to 1. When $0 \le x \le 1$, this implies that the graphs of $y = \cos(97x)$ and $y = x$ intersect once in the first and fourth quarter of each period. The graph above illustrates this for the period within the interval $[0, 2\pi/97]$. When $x > 1$, $y = x$ will never intersect $y = \cos(97x)$.

As a result, we can find the number of intersections by computing the number of periods of $y = \cos(97x)$ within the interval $[0, 1]$. The period of $y = \cos(97x)$ is $2\pi/97$. It follows that there are

$$\frac{1}{2\pi/97} = \frac{97}{2\pi} \approx 15.4$$

periods between $x = 0$ and $x = 1$. Since $15.25 < 15.4 < 15.75$, $y = \cos(97x)$ and $y = x$ intersect $2(15) + 1 = 31$ times within our interval. Thus, the correct answer is (C). ■

Question 46.



Let the variables be as shown. We omit units until our conclusion. We know $dx/dt = 2$ because x is increasing at a rate of 2. Our goal is to find the magnitude of $dy/dt|_{y=3}$. Using the Pythagorean theorem, we know $x^2 + y^2 = 81$. Via implicit differentiation,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

and solving for dy/dt yields

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}.$$

The Pythagorean theorem tells us $x = \sqrt{81 - 9} = 6\sqrt{2}$ when $y = 3$. Hence,

$$\left.\frac{dy}{dt}\right|_{y=3} = -\frac{6\sqrt{2}}{3} \cdot 2 = -4\sqrt{2}.$$

We conclude that the ladder is sliding down the wall at a rate of $4\sqrt{2}$ meters per second. Pick (C). ■

Question 47.

We will first deal with continuity. We know $\lim_{x\to a} f(x) = L$ if and only if for each sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to a$ as $n \to \infty$, we have $\lim_{n\to\infty} f(x_n) = L$. Suppose a is a real number, and f is continuous at a. The claim that f is continuous at a is equivalent to saying $\lim_{x\to a} f(x) = f(a)$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of rational numbers such that $r_n \to a$ as $n \to \infty$. Then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{n\to\infty} f(r_n)\\ &= \lim_{n\to\infty} 3r_n^2\\ &= 3a^2.\end{aligned}$$

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of irrational numbers such that $y_n \to a$ as $n \to \infty$. Then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{n\to\infty} f(y_n)\\ &= \lim_{n\to\infty} -5y_n^2\\ &= -5a^2.\end{aligned}$$

These two results must be equal, if f is continuous at a. It follows that $3a^2 = -5a^2$, which implies $a = 0$. We conclude that f is only continuous at 0.

Differentiability is a stronger claim than continuity. As a result, f is either differentiable nowhere or only at 0. We will use the derivative definition to determine differentiability at 0:

$$\begin{aligned} f'(0) &:= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h\to 0} \frac{f(h)}{h} \\ &= 0, \end{aligned}$$

because

$$\frac{f(h)}{h} = \begin{cases} 3h & \text{if } h \in \mathbb{Q} \setminus \{0\} \\ -5h & \text{if } \quad h \notin \mathbb{Q} \end{cases} \to 0 \quad \text{as} \quad h \to 0.$$

We select (B). ■

Question 48.

Recall the definition of the directional derivative:

> Suppose all first order partial derivatives of f exist at the point P. Then the directional derivative of f in the direction of $\boldsymbol{u} \neq \boldsymbol{0}$ is
>
> $$D_{\boldsymbol{u}} f|_P := \nabla f|_P \bullet \frac{\boldsymbol{u}}{|\boldsymbol{u}|},$$
>
> where $\bullet$ denotes the dot product.

It is not too tough to show that $\nabla g = \big(6xy, 3x^2, 1\big)$, which implies $\nabla g|_{(0,0,\pi)} = \big(0, 0, 1\big)$. Furthermore, if $\boldsymbol{u} = \big(1, 2, 3\big)$, then

$$\frac{\boldsymbol{u}}{|\boldsymbol{u}|} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right).$$

Hence,

$$D_{\boldsymbol{u}}g|_{(0,0,\pi)} = \big(0,0,1\big) \cdot \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right) = \frac{3}{\sqrt{14}}.$$

Tortuously, we still need to approximate the directional derivative. It is clear

$$\frac{3}{\sqrt{16}} < \frac{3}{\sqrt{14}} < \frac{3}{\sqrt{9}} \quad \text{implies} \quad 0.75 < \frac{3}{\sqrt{14}} < 1.$$

The only value within this range is 0.8, so we select (B). ■

Question 49.

Every element of the symmetric group of n elements can be written as the product of disjoint cycles. Furthermore, the order of the product of disjoint cycles is the least common multiple of the orders of the cycles. This is because we must raise the product to the smallest number that has every cycle's order as a factor to obtain the minimal number which reduces the product to the identity element.

To illustrate this we will provide two examples. The product $(1,2,3)(4,5)$ has order 6 because the first cycle in the product has order 3 and the second has order 2, so the product must be raised to the least common multiple of these numbers, i.e. 6, to reduce it to the identity element. In contrast, $(1,2)(3,4)(5)$ has order 2, because the orders of the cycles are 2, 2, and 1, and the least common multiple of these numbers is 2. Note that disjoint cycles will permute unique elements, so the sum of the orders of the cycles must be 5.

Our task, therefore, is to find integers m_i that maximize the least common multiple of the m_i, given that $\sum_i m_i = 5$. Let's go through the cases: lcm(5)=5, lcm(1,4)=4, lcm(2,3)=6, lcm(1,1,3)=3, lcm(1,2,2)=2, lcm(1,1,1,2)=2, and lcm(1,1,1,1,1)=1. Hence, the least common multiple is maximized at a value of 6. We conclude that the solution is (B). ■

Question 50.

Only I and III are ideals.

I: It is not too tough to see that $U + V$ remains a subring. It is also an ideal: Suppose r is in R, u is in U, and v is in V. By definition of an ideal, ru is in U and rv is in V. It follows that $r(u + v) = ru + rv$ is in $U + V$. This proves that $U + V$ is a left ideal. The argument to prove that $U + V$ is a right ideal is nearly identical.

II: For u_1 and u_2 in U and v_1 and v_2 in V, we cannot guarantee that $u_1v_1 + u_2v_2$ is in $U \cdot V$. Hence, it is not generally true that $U \cdot V$ is a subring. Since all ideals are subrings, $U \cdot V$ need not be an ideal. For example, consider the ring $\mathbb{R}[x, y]$ which is defined to be set of polynomials with variables x and y. Then $U = V = \{xp + yq : p, q \in \mathbb{R}[x, y]\}$ is an ideal of $\mathbb{R}[x, y]$. However, $U \cdot V = \{(xp_1 + yq_1) \cdot (xp_2 + yq_2) : p_1, p_2, q_1, q_2 \in \mathbb{R}[x, y]\}$ is not a subring and therefore not an ideal. Notice that x^2 and y^2 are in $U \cdot V$, but $x^2 + y^2$ is not in $U \cdot V$, because it cannot be factored into the product of polynomials of the form $xp + yq$ where p and q are in $\mathbb{R}[x, y]$.

III: The intersection of two subrings is always a subring, so we are safe in that respect. If w is in $U \cap V$, then w is in U and V. Because U and V are ideals, rw and wr are in U and V, which implies they are in $U \cap V$.

We select (D). $\blacksquare$

Question 51.

We can immediately see that the second column adds nothing to the column space, because it is -1 times the first column. Hence, we need only find the column space of

$$\begin{pmatrix} 1 & 2 & -3 \\ -1 & -3 & 2 \\ 2 & 5 & -5 \end{pmatrix}.$$

We will row reduce the matrix to reduced row echelon form to find a basis for the column space. This yields

$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

form a basis for the column space, since there are pivots in the first and second column of the reduced row echelon form of our matrix.

Let's go through our candidate bases because we can exclude a few. We know the dimension of our column space is two, which means (B) is out. The basis in option (C) is not orthogonal, so we eliminate it as a possibility. The basis in option (D) is not normalized, which allows us to exclude it from consideration. We can safely conclude that either (A) or (E) is correct.

Option (E) looks like a more viable basis than (A), so we will modify our basis to try to make it look like (D). Notice that there is no entry in the last row of the second vector in (E), and all of its entries are positive. Let's build a vector in the column space with these properties:

$$5\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

It is not hard to see that our constructed vector is orthogonal to

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

As a result, we will normalize the above vector and the one we constructed. This yields

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

We have found an orthonormal basis, and it is the same as (E). ■

Question 52.

The first professor to receive their assignments could be assigned two classes in ${}_{20}C_2 = 20 \cdot 19/2$ different ways. Similarly, the second professor to be assigned courses could be assigned in ${}_{18}C_2 = 18 \cdot 17/2$ ways. Generally, the n-th professor could be assigned two classes in ${}_{(22-2n)}C_2 = (22-2n)(21-2n)/2$ different ways. It follows that there are

$$\left(\frac{20 \cdot 19}{2}\right)\left(\frac{18 \cdot 17}{2}\right) \cdots \left(\frac{4 \cdot 3}{2}\right)\left(\frac{2 \cdot 1}{2}\right) = \frac{20!}{2^{10}}$$

ways for the professors to be assigned classes. We conclude that (A) is correct. ■

Question 53.

This is an application of the *Fundamental theorem of Calculus*, several *integration properties*, and the product rule which is one of our *derivative rules*. We have

$$g(x) = \int_0^x f(y)(y-x)\ dy = \int_0^x yf(y)\ dy - x\int_0^x f(y)\ dy.$$

It follows that

$$g'(x) = xf(x) - \int_0^x f(y)\ dy - xf(x) = -\int_0^x f(y)\ dy.$$

So,

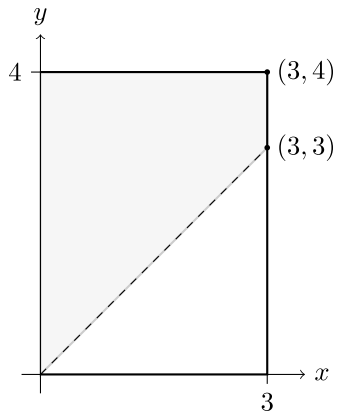
$$g''(x) = -f(x) \quad \text{and} \quad g'''(x) = -f'(x).$$

We conclude that f need only be continuously differentiable once, and we select (A). ■

Question 54.

Since all points (x, y) in $[0,3] \times [0,4]$ are equally as likely, the probability that $x < y$ is the ratio of the areas of $\{(x,y) \in [0,3] \times [0,4] :$

$x < y\}$ and $[0,3] \times [0,4]$.



Hence,

$$P(x < y) = \frac{3(4+1)/2}{3(4)} = \frac{15}{24} = \frac{5}{8}.$$

Fill in (C) and continue! ■

Question 55.

We have

$$\begin{aligned}
\int_0^\infty \frac{e^{ax} - e^{bx}}{(1+e^{ax})(1+e^{bx})}\ dx &= \lim_{t\to\infty} \int_0^t \frac{1 + e^{ax} - (1+e^{bx})}{(1+e^{ax})(1+e^{bx})}\ dx \\
&= \lim_{t\to\infty} \int_0^t \frac{1+e^{ax}}{(1+e^{ax})(1+e^{bx})} \\
&\qquad - \frac{1+e^{bx}}{(1+e^{ax})(1+e^{bx})}\ dx \\
&= \lim_{t\to\infty} \left(\int_0^t \frac{dx}{1+e^{bx}} - \int_0^t \frac{dx}{1+e^{ax}} \right)
\end{aligned}$$

Let $u := e^{bx}$ which implies $du = be^{bx}\ dx = bu\ dx$, and let $v := e^{ax}$

which implies $dv = ae^{ax}\ dx = av\ dx$. It follows that

$$\begin{aligned}
\lim_{t\to\infty} &\left(\int_0^t \frac{dx}{1+e^{bx}} - \int_0^t \frac{dx}{1+e^{ax}} \right) \\
&= \lim_{t\to\infty} \left(\frac{1}{b} \int_{x=0}^{t} \frac{du}{u(u+1)} - \frac{1}{a} \int_{x=0}^{t} \frac{dv}{v(v+1)} \right) \\
&= \lim_{t\to\infty} \left(\frac{1}{b} \int_{u=1}^{e^{bt}} \frac{du}{u(u+1)} - \frac{1}{a} \int_{v=1}^{e^{at}} \frac{dv}{v(v+1)} \right)
\end{aligned}$$

We need to break up the rational expressions in both integrands. Using *partial fraction decomposition*, we know that there are values of A and B such that

$$\frac{1}{w(w+1)} = \frac{A}{w} + \frac{B}{w+1}.$$

Multiplying both sides by $w(w+1)$ yields

$$1 = A(w+1) + Bw.$$

By equating coefficients or by selecting arbitrary values of w, we see $A = 1$ and $B = -1$. So,

$$\begin{aligned}
\lim_{t\to\infty} &\left(\frac{1}{b} \int_{u=1}^{e^{bt}} \frac{du}{u(u+1)} - \frac{1}{a} \int_{v=1}^{e^{at}} \frac{dv}{v(v+1)} \right) \\
&= \lim_{t\to\infty} \left(\frac{1}{b} \int_1^{e^{bt}} \frac{1}{u} - \frac{1}{u+1}\ du - \frac{1}{a} \int_1^{e^{at}} \frac{1}{v} - \frac{1}{v+1}\ dv \right) \\
&= \lim_{t\to\infty} \left(\frac{1}{b} \Big[\log u - \log(u+1) \Big]_1^{e^{bt}} - \frac{1}{a} \Big[\log v - \log(v+1) \Big]_1^{e^{at}} \right) \\
&= \lim_{t\to\infty} \left(\frac{1}{b} \log\left(\frac{e^{bt}}{e^{bt}+1} \right) - \frac{1}{b} \log\left(\frac{1}{2} \right) - \frac{1}{a} \log\left(\frac{e^{at}}{e^{at}+1} \right) \right. \\
&\qquad \left. + \frac{1}{a} \log\left(\frac{1}{2} \right) \right) \\
&= \frac{1}{b} \log 1 + \frac{1}{b} \log 2 - \frac{1}{a} \log 1 - \frac{1}{a} \log 2 \\
&= \frac{a-b}{ab} \log 2.
\end{aligned}$$

The answer must be (E).

Note: a list of *logarithm properties* is located in the glossary. Also, if you feel that the above computation is too intensive for the GRE, the following two suggestions may be of utility: (1) You can choose values for a and b and compute the easier corresponding integrals. (2) The two integrals are nearly identical, so you can compute

$$\int \frac{dw}{w(w+1)}$$

and then insert the result where needed. ■

Question 56.

Statement I is true. Because $\log 1 = 0 < 2\sqrt{1} = 2$ and

$$\frac{d}{dx}\big(\log x\big) = \frac{1}{x} \leq \frac{d}{dx}\big(2\sqrt{x}\big) = \frac{1}{\sqrt{x}}$$

for $x \geq 1$, we can conclude $\log x \leq 2\sqrt{x}$ for $x \geq 1$, because $2\sqrt{x}$ starts out larger and grows faster.

Statement II is false. One of the *summation formulas* from Calculus states that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since the sum is equal to a polynomial of degree three with a positive leading coefficient, it will always overtake Cn^2, regardless of C, for sufficiently large n.

Statement III is true. Recall that the *Maclaurin series formula* for $f(x) := \sin x$ is

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

Furthermore, since $f(0) = 0$ and $f''(0) = 0$,

$$x = 0 + x + 0x^2 = \sum_{k=0}^{2} \frac{f^{(k)}(0)x^k}{k!}$$

qualifies as a second degree Taylor polynomial centered at 0. *Taylor's theorem* provides an error bound for this approximation. Let I be the open interval with endpoints x and 0. Then

$$\begin{aligned}\left|\sin x - x\right| &\leq \sup_{t\in I}\left|f'''(t)\right|\frac{|x|^3}{3!} \\ &= \sup_{t\in I}\left|-\sin t\right|\frac{|x|^3}{6} \\ &\leq \frac{|x|^3}{6}.\end{aligned}$$

Hence, the solution is (D). ■

Question 57.

Statement I is true. We have $\lim_{n\to\infty} x_n = 0$ because

$$0 < x_n < \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} 0 = 0 \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Statement II is false. Consider

$$x_n := \frac{1}{n+1} \quad \text{and} \quad f(x) := \frac{1}{x}.$$

The function f is continuous and real-valued on the open interval $(0,1)$, but

$$f(x_n) = n+1 \to \infty \quad \text{as} \quad n \to \infty.$$

Because $\mathbb{R}$ is complete, a sequence converges whenever it is Cauchy. Since $\{f(x_n)\}_{n=1}^{\infty}$ diverges, it cannot be Cauchy.

Statement III is true. Recall the definition of *uniform continuity*:

> Consider the metric spaces (X,ρ) and (Y,σ). A function $f : X \to Y$ is uniformly continuous on $U \subseteq X$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that
>
> $$\sigma\left(f(x_1), f(x_2)\right) < \varepsilon \quad \text{whenever} \quad \rho(x_1, x_2) < \delta,$$
>
> for all x_1 and x_2 in U.

Within the context of this problem, this means that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that, for x and y in the open interval $(0, 1)$,

$$|g(x) - g(y)| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta.$$

From statement I, we know that x_n converges. So, for all $\eta > 0$ there exists a real number N such that

$$|x_m - x_n| < \eta \quad \text{whenever} \quad m, n > N.$$

When we pick an ε, let η equal the corresponding δ. Hence, for all $\varepsilon > 0$ there exists an N such that

$$|g(x_m) - g(x_n)| < \varepsilon \quad \text{whenever} \quad m, n > N.$$

Thus, $\{g(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. All Cauchy sequences converge within the set of real numbers. The conclusion follows.

We are ready to select (C) and move on. ■

Question 58.

The formula for the *arc length* of a parametric equation whose graph is contained within $\mathbb{R}^2$ is assumed knowledge. Our parametric equation is in $\mathbb{R}^3$, which makes finding its arc length more difficult.

Suppose that ΔL_i is the change in arc length from $\theta = \theta_{i-1}$ to θ_i. Let us find a formula for a small change of arc length ΔL_i in terms of the other variables, and then formulate the total length as the limit of a Riemann sum of ΔL_i's.

Let $\Delta x_i = x(\theta_i) - x(\theta_{i-1})$, $\Delta y_i = y(\theta_i) - y(\theta_{i-1})$, and $\Delta z_i = z(\theta_i) - z(\theta_{i-1})$. Because of the Euclidian metric, ΔL_i equals the square root of the sum of the squares of changes in each variable, i.e.

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}.$$

If follows that

$$\begin{aligned}\Delta L_i &= \sqrt{\left(\frac{\Delta x_i}{\Delta \theta_i} \cdot \Delta \theta_i\right)^2 + \left(\frac{\Delta y_i}{\Delta \theta_i} \cdot \Delta \theta_i\right)^2 + \left(\frac{\Delta z_i}{\Delta \theta_i} \cdot \Delta \theta_i\right)^2} \\ &= \sqrt{\left(\frac{\Delta x_i}{\Delta \theta_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta \theta_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta \theta_i}\right)^2}\ \Delta \theta_i,\end{aligned}$$

where $\Delta\theta_i = \theta_i - \theta_{i-1}$.

Suppose θ goes from α to β. Consider the partition of the interval $\{\theta_i\}_{i=0}^{n}$ such that $\theta_i = (\beta - \alpha)i/n$. Notice $\sum_{i=1}^{n} \Delta L_i \approx L$, where L is the length of the arc from $\theta = \alpha$ to $\theta = \beta$. Furthermore, as $n \to \infty$, we have $\sum_{i=1}^{n} \Delta L_i \to L$ and

$$\begin{aligned}\sum_{i=1}^{n} \Delta L_i &= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta\theta_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta\theta_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta\theta_i}\right)^2}\ \Delta\theta_i \\ &\longrightarrow \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta.\end{aligned}$$

We conclude that

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta.$$

We are ready to find a formula for $L(\theta)$, which is defined to be the arc length from θ to the point $(5, 0, 0)$. Let's consider the bounds of integration. Clearly, the lower bound of integration is the variable θ. Since $(5\cos\theta, 5\sin\theta, \theta) = (5, 0, 0)$ implies $\theta = 0$, the upper bound is $\theta = 0$. We introduce the dummy variable t within the integrand to avoid confusion between the bounds of integration, and the variable of integration. Hence,

$$\begin{aligned}L(\theta) &= \int_{\theta}^{0} \sqrt{(-5\sin t)^2 + (5\cos t)^2 + (1)^2}\ dt \\ &= \int_{\theta}^{0} \sqrt{25\sin^2 t + 25\cos^2 t + 1}\ dt \\ &= \int_{\theta}^{0} \sqrt{25 + 1}\ dt \\ &= \int_{\theta}^{0} \sqrt{26}\ dt \\ &= -\theta\sqrt{26}.\end{aligned}$$

So, $L(\theta_0) = 26$ implies $\theta_0 = -\sqrt{26}$. It follows that

$$x(\theta_0) = 5\cos\left(-\sqrt{26}\right), \quad y(\theta_0) = 5\sin\left(-\sqrt{26}\right), \quad \text{and} \quad z(\theta_0) = -\sqrt{26}.$$

Therefore,

$$\begin{aligned}
D(\theta_0) &= D\left(-\sqrt{26}\right) \\
&= \sqrt{\left(5\cos\left(-\sqrt{26}\right)-0\right)^2+\left(5\sin\left(-\sqrt{26}\right)-0\right)^2+\left(-\sqrt{26}-0\right)^2} \\
&= \sqrt{25\cos^2\left(-\sqrt{26}\right)+25\sin^2\left(-\sqrt{26}\right)+26} \\
&= \sqrt{25+26} \\
&= \sqrt{51}.
\end{aligned}$$

We select (B). A list of *Pythagorean identities* is given in the glossary. ■

Question 59.

We will go through our options. Some *determinant properties* are given in the glossary.

Option (A) is enough to conclude that A is invertible, because

$$\det(-A) = (-1)^3 \det(A) = -\det(A) \neq 0$$

implies $\det(A) \neq 0$.

Option (B) implies that A is invertible. This is due to the fact that

$$\det(A^k) = \big(\det(A)\big)^k \neq 0,$$

and by taking the k-th root of both sides, we see $\det(A) \neq 0$.

Option (C) is enough to conclude that A is invertible. We note the vector $\boldsymbol{v}$ being in the null space of A implies that either $\boldsymbol{v}$ is an eigenvector with an eigenvalue of 0 or $\boldsymbol{v} = \mathbf{0}$, so if A has no eigenvectors or all of its eigenvectors have eigenvalues other than 0, A is invertible. Suppose $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ; if no such $\boldsymbol{v}$ exists, we are done. We want to show that λ cannot equal 0. The vector $\boldsymbol{v}$ is also an eigenvector of $I-A$ with eigenvalue $1-\lambda$, because

$$(I-A)\boldsymbol{v} = I\boldsymbol{v} - A\boldsymbol{v} = \boldsymbol{v} - \lambda\boldsymbol{v} = (1-\lambda)\boldsymbol{v}.$$

It follows that

$$(I - A)^k \boldsymbol{v} = (1 - \lambda)^k \boldsymbol{v} = \mathbf{0}.$$

Since the eigenvector $\boldsymbol{v} \neq \mathbf{0}$, we have

$$(1 - \lambda)^k = 0 \quad \text{implies} \quad \lambda = 1 \neq 0.$$

We conclude A is invertible.

Option (D) is enough to conclude that A is invertible, due to the *rank nullity theorem*. It says

> Suppose V is a finite dimensional vector space and let $T : V \to W$ be a linear map. Then
>
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

For us, $\text{nullity}(T)+\text{rank}(T) = 3$. Since $\{A\boldsymbol{v} : \boldsymbol{v} \in \mathbb{R}^3\} = \text{range}(A) = \mathbb{R}^3$, we know that the rank is three. This implies that the nullity is zero, which is equivalent to A being invertible.

By the process of elimination, the solution must be (E). Let's construct a counterexample. Let

$$\boldsymbol{v}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Further, suppose

$$A := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that $\boldsymbol{v}_1,\ \boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are linearly independent, and $A\boldsymbol{v}_i \neq \mathbf{0}$ for each i. However, A is not invertible. ■

Question 60.

The correct answer is (D), which says $\lim_{|x|\to\infty} |f(x)| = \infty$. We will consider each direction.

Suppose for any large $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(1)| \geq \varepsilon \quad \text{whenever} \quad |x - 1| \geq \delta.$$

This implies $\lim_{|x|\to\infty} |f(x)| = \infty$, because we are guaranteed that we can make the distance between $f(x)$ and $f(1)$ arbitrarily large via selecting any x sufficiently far from 1.

The other direction is a little less clear. But let's prove this way works too. Suppose $\lim_{|x|\to\infty} |f(x)| = \infty$. Then for all $\eta > 0$, there exists an $N > 0$ such that

$$|f(x)| \geq \eta \quad \text{whenever} \quad |x| \geq N.$$

Pick an $\varepsilon > 0$. Let $\eta = \varepsilon + |f(1)|$. There exists an $N > 0$ such that $|f(x)| \geq \varepsilon + |f(1)|$ whenever $|x| \geq N$. Due to the triangle inequality,

$$|f(x) - f(1)| + |f(1)| \geq |f(x)| \geq \varepsilon + |f(1)|.$$

It follows that $|f(x) - f(1)| \geq \varepsilon$.

If $|x| \geq N$, then

$$x \geq N \quad \text{or} \quad x \leq -N.$$

This is equivalent to

$$x - 1 \geq N - 1 \quad \text{or} \quad x - 1 \leq -(N + 1).$$

Since

$$x - 1 \geq N + 1 \quad \text{implies} \quad x - 1 \geq N - 1$$

we have $|x| \geq N$ whenever $|x - 1| \geq N + 1$. So, there exists a $\delta > 0$ such that

$$|f(x) - f(1)| \geq \varepsilon \quad \text{whenever} \quad |x - 1| \geq \delta,$$

specifically $\delta = N + 1$.

Note: On the actual GRE proofs are not required, and can even be counter-productive because they tend to take up more time. Instead, we advise drawing pictures that illustrate the phenomenon above. We omitted such pictures from this text because it would make the presentation highly heterodox, and explaining the meaning and choice of our arbitrary notation and pictures would require a lot of space. ■

Question 61.

Let's solve this mixing problem via the usual differential equation techniques. We omit units during calculations for convenience. Suppose that y is the amount of salt in the tank after t minutes. We are given that $y(0) = 3$. Because the rate of change of salt with respect to time is the rate salt comes into the tank minus the rate it goes out,

$$\frac{dy}{dt} = 4(0.02) - 4\left(\frac{y}{100}\right) = \frac{2-y}{25}.$$

This implies

$$\begin{aligned}
\frac{1}{y-2}\frac{dy}{dt} &= -\frac{1}{25} \\
\Rightarrow \quad \int \frac{dy}{y-2} &= -\frac{1}{25}\int dt \\
\Rightarrow \quad \log|y-2| &= -\frac{t}{25} + C \\
\Rightarrow \quad y - 2 &= \pm e^{C} \cdot e^{-t/25} \\
\Rightarrow \quad y &= 2 + Ke^{-t/25},
\end{aligned}$$

where $K = \pm e^{C}$. It follows that

$$y(0) = 2 + Ke^{0} = 2 + K = 3 \quad \text{implies} \quad K = 1.$$

Thus,

$$y(100) = 2 + e^{-4}.$$

Option (E) is correct. ■

Question 62.

Let's go through why S is neither open nor closed. Every open ball in $[0,1] \times [0,1]$ contains points in $\mathbb{Q} \times \mathbb{Q}$. As such, S has no interior points, which implies it is not open. The set S does not contain all of its limit points so it cannot be closed, e.g. the point $(1/2, 1/2)$ is a limit point of S not contained within S.

Since S is not closed, it is not *compact*. This is due to the *Heine-Borel theorem* which says a subset of $\mathbb{R}^n$, for any natural number n, is compact if and only if it is closed and bounded.

The set S is connected. For any a and b in $[0,1] \setminus \mathbb{Q}$, S contains the paths $\{(a,t) : 0 \leq t \leq 1\}$ and $\{(t,b) : 0 \leq t \leq 1\}$. As a result, it is not difficult to find a path between two arbitrary points in S. It follows that S is path connected, which implies that it is connected because path connectivity is a stronger claim. Obviously, if S is connected, we can safely rule out it being completely disconnected. It is time to select (C). ■

Question 63.

The answer is (E). It is easy enough to falsify the others with counterexamples.

For (A), (B), (C), and (D), consider $A = (1,2)$ and $B = (-2,-1)$. Then

- $\sup(A)\sup(B) = -2$,
- $\sup(A)\inf(B) = -4$,
- $\max\{\sup(A)\sup(B), \inf(A)\inf(B)\} = \max\{-2,-2\} = -2$,
- $\max\{\sup(A)\sup(B), \sup(A)\inf(B)\} = \max\{-2,-4\} = -2$,

but $\sup(A \cdot B) = -1$. ■

Question 64.

We will use the *divergence theorem*. It says:

> Suppose the closed surface S with outward orientation is the boundary of a solid E, and $\boldsymbol{F}$ is a vector field with continuous first order partial derivatives. Then the *flux* of $\boldsymbol{F}$ through S is
>
> $$\oiint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \operatorname{div}(\boldsymbol{F})\ dV,$$
>
> where
>
> $$\operatorname{div}(\boldsymbol{F}) := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \boldsymbol{F}.$$

Let S be the surface described by $z = \sqrt{1 - x^2 - y^2}$. Unfortunately, S is not a closed surface. To avoid a direction computation of the flux through S, we consider the flux though the closed surface $S \cup T$, where

$$T := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1,\ z = 0\}.$$

Since T is contained within the plane $z = 0$, it will be easier to compute the flux of $\boldsymbol{F}$ through T directly than it would be to compute the flux through S directly.

It is not too tough to see that $S \cup T$ forms the boundary of

$$E := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}.$$

As a result, the divergence theorem tells us

$$\oiint_S \boldsymbol{F} \cdot d\boldsymbol{S} + \oiint_T \boldsymbol{F} \cdot d\boldsymbol{S} = \oiint_{S \cup T} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \operatorname{div}(\boldsymbol{F})\ dV.$$

From here, the calculation is fairly straightforward. Notice that $z = \sqrt{1 - x^2 - y^2}$ is the upper half of a sphere of radius one, which implies that its volume is

$$\frac{1}{2}\left(\frac{4}{3}\pi(1)^3\right) = \frac{2\pi}{3}.$$

Then the flux of $\boldsymbol{F}$ through $S \cup T$ is

$$\begin{aligned}\iiint\limits_E \operatorname{div}(\boldsymbol{F})\ dV &= \iiint\limits_E \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\ dV \\ &= \iiint\limits_E 3\ dV \\ &= 3\left(\frac{2\pi}{3}\right) \\ &= 2\pi.\end{aligned}$$

To find the flux of $\boldsymbol{F}$ through S, we compute the flux though T and subtract this result from the flux through $S \cup T$. Recall

$$\oiint\limits_T \boldsymbol{F} \cdot d\boldsymbol{S} = \oiint\limits_T \boldsymbol{F} \cdot \boldsymbol{n}\ dS,$$

where $\boldsymbol{n}$ is normal to T. Since T has outward orientation relative to E, and T lies on the plane $z = 0$, $\boldsymbol{n} = \big(0, 0, -1\big)$. Hence,

$$\begin{aligned}\oiint\limits_T \boldsymbol{F} \bullet \boldsymbol{n}\ dS &= \oiint\limits_T \big(x, y, 0\big) \bullet \big(0, 0, -1\big)\ dS \\ &= \oiint\limits_T 0\ dS \\ &= 0.\end{aligned}$$

We conclude

$$\oiint\limits_S \boldsymbol{F} \cdot d\boldsymbol{S} = 2\pi - 0 = 2\pi.$$

Fill in the bubble for (E). ■

Question 65.

Recall the *necessary and sufficient condition for a function to be analytic*:

> The function $f(z) = u(x, y) + iv(x, y)$ is analytic if and only if
> $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It follows that

$$g_y(x, y) = e^x \sin y \quad \text{and} \quad g_x(x, y) = -e^x \cos y.$$

Using "partial integration" (i.e. treating the variables that we are not integrating with respect to as constants during integration), we can see that

$$\begin{aligned} g(x, y) &= \int g_y(x, y)\ dy \\ &= \int e^x \sin y\ dy \\ &= -e^x \cos y + h_1(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} g(x, y) &= \int g_x(x, y)\ dx \\ &= \int -e^x \cos y\ dx \\ &= -e^x \cos y + h_2(y). \end{aligned}$$

It follows that $h_1(x) = h_2(y)$ is constant, because there are no terms of only y in the first integral and no terms of only x in the second. Hence,

$$g(3, 2) - g(1, 2) = -e^3 \cos 2 + e^1 \cos 2 = (e - e^3) \cos 2.$$

The solution must be (E). ■

Question 66.

Recall that m is a unit of $\mathbb{Z}_n$ if and only if the greatest common factor of m and n is 1. Since 17 is a prime number, every non-zero element of $\mathbb{Z}_{17}$ is a unit. It follows that the order of $\mathbb{Z}_{17}^{\times}$ is 16.

Lagrange's theorem says that the order of a subgroup must divide the order of the entire group. It follows that our contenders, 5, 8, and 16, could only generate cyclic subgroups of order 1, 2, 4, 8, or 16. We can immediately exclude 1 from being the order of any of our subgroups because it is clear that none of our contenders could be the multiplicative identity.

Let's consider 5. We will compute 5 raised to each of the possible orders:

$$\begin{aligned} 5^2 &\equiv 25 \\ &\equiv 8 \pmod{17}, \end{aligned} \qquad \begin{aligned} 5^4 &\equiv 8^2 \\ &\equiv 64 \\ &\equiv 13 \pmod{17}, \end{aligned} \qquad \begin{aligned} 5^8 &\equiv 13^2 \\ &\equiv 169 \\ &\equiv 16 \pmod{17}, \end{aligned}$$

and

$$\begin{aligned} 5^{16} &\equiv 16^2 \\ &\equiv 256 \\ &\equiv 1 \pmod{17}. \end{aligned}$$

We can conclude that 5 generates a cyclic subgroup of order 16, which implies it generates $\mathbb{Z}_{17}^{\times}$.

We have also proven that 8 and 16 are not generators, because

$$5^2 \equiv 8 \pmod{17} \quad \text{implies} \quad 5^{16} \equiv 8^8 \equiv 1 \pmod{17},$$

which means 8 generates a cyclic subgroup of order 8. Similarly,

$$5^8 \equiv 16 \pmod{17} \quad \text{implies} \quad 5^{16} \equiv 16^2 \equiv 1 \pmod{17},$$

which means that 16 generates of cyclic subgroup of order 2. Thus, we select (B). ■

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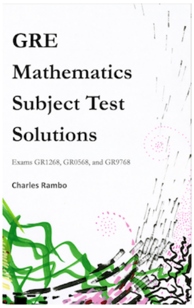

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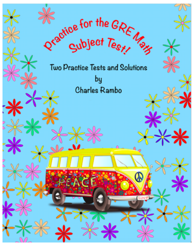

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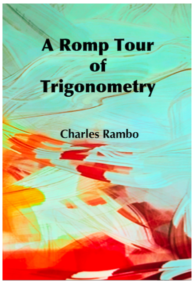
Trigonometry Book



You can read a sample of my trigonometry book `here`. The full book could be useful to you as a GSI (or TA) once you're in graduate school. The title is *A Romp Tour of Trigonometry*. It's \$17.50. You can buy it at `amazon.com`.

Glossary

Antiderivatives Useful antiderivatives.

- $\displaystyle\int u^n \; du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
- $\displaystyle\int e^u \; du = e^u + C$
- $\displaystyle\int \frac{du}{u} = \log|u| + C$
- $\displaystyle\int \sin u \; du = -\cos u + C$
- $\displaystyle\int \cos u \; du = \sin u + C$
- $\displaystyle\int \tan u \; du = -\log|\cos u| + C$
- $\displaystyle\int \frac{du}{1+u^2} = \text{Arctan } u + C$

Arc length

- The arc length of the curve from $x = a$ to $x = b$ described by $y = f(x)$ is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dt}\right)^2} \; dx.$$

- The arc length of the curve from $t = a$ to $t = b$ described by $(x, y) = (f(t), g(t))$ is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The arc length of the curve from $\theta = \alpha$ to $\theta = \beta$ described by the polar equation $r = f(\theta)$ is

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Basis The set $\mathcal{B}$ is a basis of a vector space V over a field $\mathbb{F}$ if and only if the following hold.

- The set $\mathcal{B}$ is nonempty.
- Every element in V can be written as a linear combination of elements in $\mathcal{B}$.
- The elements of $\mathcal{B}$ are linearly independent.

Compact Consider the set X under some topology. A collection $\mathcal{U}$ of open sets is said to be an "open cover" of X if and only if

$$X \subseteq \bigcup_{U \in \mathcal{U}} U.$$

The set X is compact if and only if every open cover $\mathcal{U}$ has a finite subcover $\{U_1, U_2, \ldots, U_n\} \subseteq \mathcal{U}$ such that

$$X \subseteq U_1 \cup U_2 \cup \ldots \cup U_n.$$

Derivative rules Suppose that f and g are differentiable on some domain D. Assume c and n are constants.

- Constant rule:

$$\frac{d}{dx}(c) = 0.$$

- Constant multiple rule:

$$(c \cdot f)'(x) = c \cdot f'(x).$$

- Power rule:
$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

- Sum and difference rules:

$$(f \pm g)'(x) = f'(x) \pm g'(x).$$

- Product rule:

$$(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x).$$

- Quotient Rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2},$$

  where $g(x) \neq 0$.

- Chain rule:

$$(f \circ g)'(x) = f'\left(g(x)\right) g'(x).$$

Derivatives Useful derivatives.

- $\dfrac{d}{dx}u^n = nu^{n-1}u'$
- $\dfrac{d}{dx}e^u = u'e^u$
- $\dfrac{d}{dx}\log|u| = \dfrac{u'}{u}$
- $\dfrac{d}{dx}\sin u = u'\cos u$
- $\dfrac{d}{dx}\cos u = -u'\sin u$
- $\dfrac{d}{dx}\tan u = u'\sec^2 u$
- $\dfrac{d}{dx}\text{Arctan } u = \dfrac{u'}{1+u^2}$

Determinant properties Suppose A and B are $n \times n$ matrices.

- The matrix A is invertible if and only if $\det(A) \neq 0$.
- If A^{-1} exists, $\det(A^{-1}) = 1/\det(A)$.
- For k in $\mathbb{R}$, $\det(kA) = k^n \det(A)$.
- The value of $\det(AB) = \det(A)\det(B)$.
- For k an integer, $\det(A^k) = (\det(A))^k$.

Directional derivative Suppose all first order partial derivatives of f exist at the point P. Then the directional derivative of f in the direction of $\boldsymbol{u} \neq \boldsymbol{0}$ is

$$D_{\boldsymbol{u}} f\big|_P := \nabla f\big|_P \bullet \frac{\boldsymbol{u}}{|\boldsymbol{u}|},$$

where $\bullet$ denotes the dot product.

Discriminant Consider a quadratic function $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $n \neq 0$. The discriminant is $\Delta := b^2 - 4ac$. Furthermore,

- If $\Delta > 0$, f has two real zeros.
- If $\Delta = 0$, f has one real zero of multiplicity two.
- If $\Delta < 0$, f has two complex zeros.

Divergence theorem Suppose the closed surface S has outward orientation and is the boundary of a solid E, and $\boldsymbol{F}$ is a vector field with continuous first order partial derivatives. Then the flux of $\boldsymbol{F}$ through S is

$$\oiint\limits_S \boldsymbol{F} \bullet d\boldsymbol{S} = \iiint\limits_E \operatorname{div}(\boldsymbol{F})\ dV,$$

where

$$\operatorname{div}(\boldsymbol{F}) := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \bullet \boldsymbol{F}.$$

First derivative test Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on the open interval (a, b) and differentiable on $(a, b) \setminus \{c\}$, where $a < c < b$.

- If $f'(x) > 0$ for x in (a, c) and $f'(x) < 0$ for x in (c, b), then $f(c)$ is a relative maximum.
- If $f'(x) < 0$ for x in (a, c) and $f'(x) > 0$ for x in (c, b), then $f(c)$ is a relative minimum.

In other words, if f' switches from positive to negative at c then $f(c)$ is a relative maximum, and if f' switches from negative to positive at c then $f(c)$ is a relative minimum.

Flux Let $\boldsymbol{F}$ be a vector field. Denote the flux of $\boldsymbol{F}$ through the surface S by Φ. We can calculate Φ by means of the limit of a Riemann sum. Consider the change in flux $\Delta\Phi$ which is the result of the flow of $\boldsymbol{F}$ through a small amount of the surface, say $\Delta\boldsymbol{S} := \boldsymbol{n}\Delta S$. We assume ΔS is small enough that it is approximately flat and $\boldsymbol{n}$ is a unit normal vector of ΔS. Then

$$\Delta\Phi \approx \boldsymbol{F} \cdot \Delta\boldsymbol{S} := \boldsymbol{F} \cdot \boldsymbol{n}\ \Delta S.$$

If θ is the acute angle between $\boldsymbol{F}$ and $\boldsymbol{n}$, then

$$\boldsymbol{F} \cdot \boldsymbol{n}\ \Delta S = |\boldsymbol{F}|\,|\boldsymbol{n}| \cos\theta\ \Delta S = |\boldsymbol{F}| \cos\theta\ \Delta S.$$

As our ΔS's become smaller and smaller,

$$\sum \Delta\Phi = \sum_{\Delta\boldsymbol{S}} \boldsymbol{F} \cdot \Delta\boldsymbol{S} = \sum_{\Delta\boldsymbol{S}} \boldsymbol{F} \cdot \boldsymbol{n}\ \Delta S \to \Phi$$

and

$$\sum_{\Delta\boldsymbol{S}} \boldsymbol{F} \cdot \Delta\boldsymbol{S} = \sum_{\Delta\boldsymbol{S}} \boldsymbol{F} \cdot \boldsymbol{n}\ \Delta S \to \oiint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \oiint_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS.$$

So,

$$\Phi = \oiint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \oiint_S \boldsymbol{F} \cdot \boldsymbol{n}\ dS.$$

Note that in $\mathbb{R}^3$, at any point on a surface S there are always two unit normal vectors. The choice of which unit normal vector is used to describe S is called the "orientation" of S.

Let S be a subset of $\mathbb{R}^3$, and say $\boldsymbol{F}$ is a function of (x, y, z). Suppose $\boldsymbol{r}(u, v)$ is a parameterization of S, where (u, v) is in the set R. It can be shown that $d\boldsymbol{S} = \boldsymbol{r}_u \times \boldsymbol{r}_v \ dudv$, which implies

$$\oiint_S \boldsymbol{F}(x, y, z) \bullet d\boldsymbol{S} = \iint_R \boldsymbol{F}(\boldsymbol{r}) \bullet (\boldsymbol{r}_u \times \boldsymbol{r}_v) \ dudv.$$

When S is orientated upward and described by the equation $z = f(x, y)$, where (x, y) is in R, a popular parametrization of S is $\boldsymbol{r}(x, y) = \big(x, y, f(x, y)\big)$. This implies

$$\boldsymbol{r}_x \times \boldsymbol{r}_y = \big(- f_x(x, y), -f_y(x, y), 1\big).$$

Hence,

$$\oiint_S \boldsymbol{F}(x, y, z) \bullet d\boldsymbol{S}$$
$$= \iint_R \boldsymbol{F}\big(x, y, f(x, y)\big) \bullet \big(- f_x(x, y), -f_y(x, y), 1\big) \ dA.$$

Fundamental counting principle Suppose there are n_1 ways for an event to occur, and n_2 ways for another independent event to occur. Then there are

$$n_1 \cdot n_2$$

ways for the two events to occur. More generally, if there are n_i ways for the i-th independent event to occur, where $i = 1, 2, \ldots, m$, then there are

$$n_1 \cdot n_2 \cdot \ldots \cdot n_m$$

ways for the consecutive occurrence of the m events to occur.

Fundamental theorem of Calculus Suppose f is continuous on the closed interval $[a, b]$. Then

$$\int_a^b f(x) \ dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

Fundamental theorem of finitely generated abelian groups Let G be a finitely generated abelian group. Then it is isomorphic to an expression of the form

$$\mathbb{Z}^k \times \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \ldots \mathbb{Z}_{p_n^{\alpha_m}},$$

where $k, \alpha_1, \alpha_2, \ldots, \alpha_m$ are whole numbers and $p_1, p_2, \ldots, p_m$ are primes which are not necessarily distinct. Alternatively, G is isomorphic to an expression of the form

$$\mathbb{Z}^k \times \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_n},$$

where $k, r_1, r_2, \ldots, r_n$ are whole numbers and r_i divides r_{i+1} for all $i = 1, 2, \ldots, n-1$. The values of k and each r_i are uniquely determined by G.

Group The set G together with a binary operation $\cdot$ is a group if and only if the following properties of G and $\cdot$ hold:

- Closed: a and b in G implies $a \cdot b$ in G.
- Associative: for all a, b, and c in G, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Contains the identity element: there is an element e such that $e \cdot a = a \cdot e = a$ for all a in G.
- Contains inverse elements: for all a in G there is a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

`http://en.wikipedia.org/wiki/Group_(mathematics)`

Heine-Borel theorem For all positive integers n, a set in $\mathbb{R}^n$ is closed and bounded if and only if it is compact.

Ideal A left ideal I of a ring R is a subring such that for all r in R we have $rI \subseteq I$. The subring I is a right ideal if for all r in R, we have $Ir \subseteq I$. If both criteria are met, then we say that I is a two-sided ideal or simply an ideal.

Inclusion-exclusion principle For finite sets $U_1, U_2, \ldots, U_n$,

$$\left|\bigcup_{k=1}^{n} U_k\right| = \sum_{k=1}^{n} |U_k| - \sum_{1\le k<\ell\le n} |U_k \cap U_\ell| + \sum_{1\le k<\ell<m\le n} |U_k \cap U_\ell \cap U_m| - \ldots + (-1)^{n-1} |U_1 \cap U_2 \cap \cdots \cap U_n| .$$

http://en.wikipedia.org/wiki/Inclusion-exclusion_principle

Inflection point Suppose f is a twice differentiable real-valued function on the set $(a, b) \setminus \{c\}$, where $a < c < b$. Then a point $(c, f(c))$ is an inflection point of the graph of f if and only if there is some open interval (a, b), such that $f''(x) < 0$ for x in (a, c) and $f''(x) > 0$ for x in (c, b), or $f''(x) > 0$ for x in (a, c) and $f''(x) < 0$ for x in (c, b). In other words, $(c, f(c))$ is an inflection point if and only if f'' switches signs at c.

Integration properties Suppose f and g are integrable real-valued functions over the closed interval $[a, b]$. Let α and β be in $\mathbb{R}$, and let c be in $[a, b]$. Then

- $\displaystyle\int_a^b \alpha f(x) + \beta g(x)\ dx = \alpha \int_a^b f(x)\ dx + \beta \int_a^b g(x)\ dx$
- $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
- $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$
- $\displaystyle\int_c^c f(x)\ dx = 0$
- $\displaystyle\int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$, whenever $f(x) \le g(x)$ for x in $[a, b]$

Inverse function theorem Suppose f is one-to-one and has a continuous derivative f' within some connected open neighborhood of $x = a$. Further, assume the graph of f within this neighborhood contains the point (a, b). Then

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

L'Hôspital's rule Let f and g be functions differentiable on $(a, b) \setminus \{c\}$, and $g(x) \neq 0$ for all x in $(a, b) \setminus \{c\}$, where c is in (a, b). Assume $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Lagrange's theorem Suppose G is a finite group and H is a subgroup of G. Then the order of H divides the order of G.

Logarithm properties The GRE assumes log is base e <u>not</u> base 10.

- $\displaystyle\int \frac{du}{u} = \log |u| + C$
- $\log x = y \iff e^y = x$
- $\log(e^x) = x$ and $e^{\log x} = x$
- $\log 1 = 0$
- $\log e = 1$
- $\log(xy) = \log x + \log y$
- $\log \left(\dfrac{x}{y} \right) = \log x - \log y$
- $\log x^y = y \log x$

Maclaurin series formula Suppose the n-th derivative of f exists and is continuous. The Maclaurin polynomial of degree n for f is

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

If f is infinitely differentiable, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Well known Maclaurin series include:

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$
- $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$
- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, where $-1 < x < 1$

Method of Lagrange multipliers Suppose $f(x,y,z)$ and $g(x,y,z)$ have continuous first order partial derivatives, and there is a constant k such that $g(x,y,z) = k$. Then relative extrema of f occur at points (x,y,z) that satisfy

$$\begin{aligned} f_x(x,y,x) &= \lambda g_x(x,y,z), \qquad & f_y(x,y,z) &= \lambda g_y(x,y,z), \\ &\text{and} \quad f_z(x,y,z) = & \lambda g_z(x,y,z) & \end{aligned}$$

for some λ in $\mathbb{R}$.

Necessary and sufficient condition for a function to be analytic The function $f(x+iy) = u(x,y) + iv(x,y)$ is analytic if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Partial fraction decomposition Suppose we have a rational expression of the form $p(x)/q(x)$ where p and q are polynomials with no common factors, the degree of q is larger than the degree of p, and $q \neq 0$. The objective of partial fraction decomposition is to write our rational expression as the sum of more simple rational expressions. What follows is a nonexhaustive list of rules:

Factor of q	Term(s) of partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^m$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_m}{(ax+b)^m}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^m$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$,

where $a, b, c, A, B, A_i,$ and B_i are real numbers, $a \neq 0$, and m is a natural number. For further explanation, we suggest any quality Calculus text, e.g. *Calculus* by Stewart.

Pythagorean identities Suppose θ is in $\mathbb{R}$. Then

$$\cos^2\theta+\sin^2\theta = 1, \quad 1+\tan^2\theta = \sec^2\theta, \quad \text{and} \quad 1+\cot^2\theta = \csc^2\theta.$$

Rank nullity theorem Suppose V is a finite dimensional vector space and let $T : V \to W$ be a linear map. Then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Ring A set R is a ring if and only if it is an abelian (commutative) group under $+$ and the following properties of R and $\cdot$ hold

- Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, and c in R.
- Distributive on the right: $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, and c in R.
- Distributive on the left: $(b + c) \cdot a = b \cdot a + c \cdot a$ for all a, b, and c in R.

`http://en.wikipedia.org/wiki/Ring_(mathematics)`

Sine and cosine values in quadrant I To convert the radian measures in the first row to degrees, simply multiply $180°/\pi$.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

Slope and concavity of curves with parametric equations Suppose $x = f(t)$ and $y = g(t)$ are twice differentiable real-valued functions and t is a real number. At the point corresponding to t, the slope of the curve described by $\{(x(t), y(t)) \in \mathbb{R}^2 : t \text{ real}\}$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

when $dx/dt \neq 0$. Furthermore, at the point corresponding to t, the concavity of the curve $\{(x(t), y(t)) \in \mathbb{R}^2 : t \text{ real}\}$ is

$$\frac{d^2y}{dx^2} = \frac{d^2y/dtdx}{dx/dt},$$

where $dx/dt \neq 0$.

Summation formulas

- $\displaystyle\sum_{k=1}^{n} a_k + b_k = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\displaystyle\sum_{k=1}^{n} c \cdot a_k = c\sum_{k=1}^{n} a_k$
- $\displaystyle\sum_{k=1}^{n} 1 = n$
- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
- $\displaystyle\sum_{k=1}^{n} a_k = \frac{n(a_1+a_n)}{2}$, where $\sum a_k$ is an arithmetic series
- $\displaystyle\sum_{k=1}^{n} a_1 r^{k-1} = \frac{a_1(1-r^n)}{1-r}$, where $r \neq 1$
- $\displaystyle\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}$, where $|r| < 1$

Taylor's theorem Let f be a real-valued function defined on some set which contains the interval $[a, b]$. Suppose $f^{(n)}$ is continuous on $[a, b]$ and $f^{(n+1)}$ exists on the open interval (a, b), where n is a positive integer. Then for each x and c in $[a, b]$ there is a z between x and c such that

$$f(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} + \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Hence, f can be approximated by the polynomial

$$\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

and the error is less than or equal to the Lagrange error bound of

$$\frac{\sup_{z\in I} |f^{(n+1)}(z)|}{(n+1)!}|x-c|^{n+1},$$

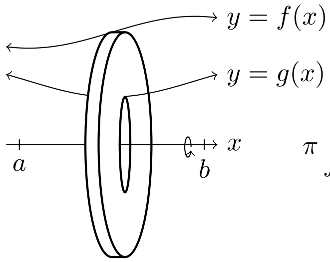
where I is the open interval with endpoints x and c.

Uniform continuity Consider the metric spaces (X, ρ) and (Y, σ). A function $f : X \to Y$ is uniformly continuous on $U \subseteq X$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sigma\left(f(x_1), f(x_2)\right) < \varepsilon \quad \text{whenever} \quad \rho(x_1, x_2) < \delta,$$

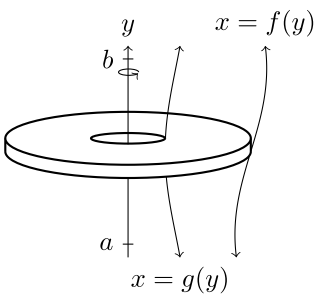
for all x_1 and x_2 in U.

Washer method Consider the region bound between $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ for x in the interval $[a, b]$. Then the volume of the solid from $x = a$ to $x = b$ generated by revolving the region about the x-axis is



$$\pi \int_a^b \big(f(x)\big)^2 - \big(g(x)\big)^2 \, dx$$

Consider the region bound between $x = f(y)$ and $x = g(y)$, where $f(y) \geq g(y)$ for y in the interval $[a, b]$. Then the volume of the solid from $y = a$ to $y = b$ generated by revolving the region about the y-axis is



$$\pi \int_a^b \big(f(y)\big)^2 - \big(g(y)\big)^2 \, dy$$