Suppose $T \in L(V)$. Prove that if $U_1, ..., U_m$ are subspaces of V invariant under T, then $U_1+, ..., +U_m$ is invariant under T.

Proof. Let $v \in U_1+, ..., +U_m$. Then Tv can be written as $T(\alpha_1 u_1 + ..., +\alpha_m u_m)$, where $u_j \in U_j$ and $\alpha_j \in \mathbb{F}$. Then $Tv = T(\alpha_1 u_1) + ... + T(\alpha_m u_m) = \beta_1 u_1 + ... + \beta_m u_m \in U_1+, ..., +U_m$. Therefore $U_1+, ..., +U_m$ is invariant under T. □

Suppose $T \in L(V)$. Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.

Proof. Let $v \in V_1 \cap ... \cap V_m$, where $V_1, ..., V_m$ are subspaces invariant under T. Then $v \in V_j$ for some $V_j \in V_1, ..., V_m$. Then v is invariant under T.

□

Suppose $S, T \in L(V)$ are such that $ST = TS$. Prove that $null(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbb{F}$.

Proof. Let $v \in null(T - \lambda I)$. Then $Tv = \lambda v$ (definition of eigenvector). Then $STv = TSv \implies S\lambda v = TSv \implies \lambda(Sv) = T(Sv)$. Therefore $Sv \in null(T - \lambda I)$, so $null(T - \lambda I)$ is invariant under S.

□

Suppose $T \in L(V)$ and dim range $T = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

Proof. stuff

□

Suppose $T \in L(V)$ is invertible and $\lambda \in \mathbb{F}$ 0. Prove that λ is an eigenvalue of T iff $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Proof. Let $v \in V$ be an eigenvector of T with eigenvalue λ. Then $Tv = \lambda b$. Apply T^{-1} to both sides. $T^{-1}Tv = T^{-1}\lambda v \implies Iv = \lambda T^{-1}v \implies \frac{1}{\lambda}v = T^{-1}v$. Therefore $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}. □