Decomposition Methods in Demography

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ISBN 90 5170 727 4
NUR 740

Cover illustration: “The creation of the universe” mural of the master painter Daniel Ponce Montuy. Cover design: PuntSpatie, Amsterdam / Vladimir Canudas Romo, Groningen.

Rijks*universiteit* Groningen

Decomposition Methods in Demography

Proefschrift

ter verkrijging van het doctoraat in de
Ruimtelijke Wetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. F. Zwarts,
in het openbaar te verdedigen op
maandag 30 juni 2003
om 16:00 uur

door

Vladimir Canudas Romo

geboren op 22 november 1972
te Parijs, Frankrijk

Promotores: Prof. dr. J.W. Vaupel
Prof. dr. ir. F.J. Willekens

Beoordelingscommissie: Prof. dr G. Caselli
Prof. dr G. Feichtinger
Prof. dr. J. Oosterhaven

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Preface

The adventure of decomposition methods is the most challenging responsibility that I have encountered so far. I accepted this formidable task because, I believe, it is the key idea of all the methods founded in the core of demography. Decomposition theory is based on the simple principle of separating demographic measures into components that contribute to an understanding of the phenomena under study. Most of the decompositions presented here refer to changes over time in demographic variables. For this reason, this research contributes to the understanding of population dynamics. The principle of decomposition methods is not only applicable to demography, however. All sciences, in one or other way, use decomposition as a research tool. As such, other social and natural sciences could benefit from the results presented in this book.

My background in actuarial science and population studies provided me with the skills to learn, develop and enjoy the decompositions presented here. The formulas are simple but they explain the fundamental relations existing among demographic variables. An understanding of differential equations and algebra and an interest in population phenomena are useful in understanding the power of decomposition.

I had special interest in looking at the developments of different decomposition methods and finding out which questions these methods could help answer. From reviewing previous methods, insight was gained into the desired properties of decomposition methods and progress in decomposition theory was made.

Stimulated by the early work of James Vaupel on decomposition methods, I began studying the research carried out in this field. As I became familiar with the state of the art in decomposition analysis, I was inspired with many ideas for future developments of Vaupel's formulas. Advice from Frans Willekens in other areas of demography helped me further develop my work. I am grateful to both James Vaupel and Frans Willekens for their roles as "Doktorvaters". Their guidance and friendship was very valuable during the process of this Ph.D. I am particularly grateful to James Vaupel because his dedication and enthusiasm for formal demography were very inspiring for me. I particularly enjoyed those demographic lessons on the ferry from Rostock to Denmark. I am particularly grateful to Frans Willekens because his suggestions and encouragement allowed me to expand my research to a broader area of study. I was delighted when the meetings were held at his house and included the wonderful hospitality of Maria Willekens.

This Ph.D. was carried out as a collaborative work between two institutions. I came to the Max Planck Institute for Demographic Research in Rostock, Germany, as the first Sergio Camposortega Cruz Ph.D. Fellow in 1999. As a member of this institution, I had the opportunity to meet many well-established demographers, as well as the promising future generation of demographers. Exchanges with all these people who are dedicated to demography were very stimulating for my work. My appreciation of support goes to Gunnar, Sara, Jan, Vladka, Roland, Michaela, Dirk, Lena, Annette B., Rasmus, Iliana, Hans Peter, Jonathan, Ines, Konstantin, Edlira, Nika, Alexia, Heiner, Gunde, Rainer, Ulrike, Patricio, Elizabetha, Francesco, Nadege, Jutta, and many others.

During my time in Rostock, I also had the chance to participate in courses of the International Max Planck Research School for Demography, first as a student and later as a lecturer. Concerning these courses, I would like to thank the students for their patience and for their feed-back. The courses of this School provided me with the opportunity to visit my second host institution, the University of Groningen. During my Ph.D. studies I had several extended visits to the Population Research Centre (PRC) in Groningen, which also provided me with a scholarship for my final year. I enjoyed the stimulating international environment of the PRC. My gratitude for support goes to Stiny, Karen, Mamun, Moury, Sarbani, Hideko, José, Maaike, Luisa, Ajay, Zula, Elda, Marloes, Andrea, Susan, Vanessa, Rob, Miriam, Bettina, Thomas, Inge, among many others. Mrs. Marijke Steenstra, my Dutch host, also deserves special thanks for letting me feel at home.

Both institutions were very welcoming and I enjoyed the homely environment, especially among friendly colleagues who contributed with their energetic spirit to create perfect places for developing new ideas. I have reached this far, in academic terms and in distance, thanks to the education received in my former universities in México the UNAM and FLACSO. Thank you to all of you, dear Demographers. Finally, I wish to thank the Consejo Nacional de Ciencia y Tecnología of México for supporting me with a complementary scholarship.

I am grateful to Annette Erlangsen, Jenae Tharaldson, Gina Rozario and Sara Grainger for their English editing of early versions of the book and to Rozenberg Publishers for publishing the book. My appreciation also goes to the master painter Daniel Ponce Montuy for allowing me to use his art on the cover of the book.

Before closing this preface I would like to acknowledge all those who were at all times in my thoughts and helped me through the difficult times when days were too short and work too intense. All my admiration goes to my friends, some far some near - I keep on learning from your delight for life. All my love goes to my family who has taught me how to keep smiling in hard times. In those hard cold days, when even the demographic spirit was giving up, a warm hug from a loved one reminded me my compromise with life, Thanks Annettita.

Part I

Introduction

1 Demography

1.1 Demography

The word demography is derived from the Greek words “demos” meaning people, and “graphia” meaning the science of. Demography is the study of human populations, and it investigates such aspects as size, structure, distribution over space, socioeconomic characteristics, households and families, migration, labor force, and vital processes of populations. (For further elaboration see the Encyclopedia of the Social and Behavioral Sciences (2001).)

Researchers from many different backgrounds identify themselves as demographers. An attempt to synthesize the perspectives of these different disciplines into a definition of demography could be summed up as a science that analyzes populations based on the methods and techniques of mathematics and statistics, and on theories from the social and biological sciences.

The focal points of demography are changes in size, growth rates, and the composition of populations. The discipline is not limited to examining aggregated processes or individual-level behavior, but it also looks at interactions between the micro and macro level of analysis.

In order to accurately explain population phenomena, demographers usually follow the principle of separating demographic variables into parts that explain the different components of the phenomena. This aids in specifically analyzing the effects that each component has on the whole dynamic. The present book encompasses the study of how to properly separate demographic variables' changes over time. This methodology is known as decomposition technique and corresponds to the field of demography generally referred to as formal demography.

Like most other sciences, demography may be defined narrowly or broadly. In the narrowest sense, formal demography constitutes the core of demography.

Demographic measures are described by mathematical formulas. The mathematics used in the formulas and the methodology to explain the relations between different demographic measures is the essential part of formal demography (for more details see Keyfitz (1985)). A vital part of formal demography is the development of new methodologies and applications of mathematics in demographic analysis.

Formal demography provides us with several different methods that facilitate comparisons of populations over time. One of these techniques is the decomposition method, which will be explained here in detail.

1.2 Decomposition Techniques

Rates, indices and average ages of population are demographic variables which are used as convenient summary measures of mortality, fertility, migration, and other population phenomena. For example, Schoen (1970) points out that a mortality measure or index should have four properties. It should be unique, respect the proportional difference between the corresponding elements in the age-specific schedules, reflect the nature of the underlying mortality function and not be affected by confounding factors. Nevertheless, these desirable properties are seldom fully fulfilled by an index. On the one hand, indices have the advantage of being easily interpreted, but on the other hand, they usually suffer from the disadvantage of being single-figured. In other words, all the demographic phenomena has to be learned from this single value. This ambiguity becomes more evident when comparing measures over time or by population, sex, age or other categories.

Confounding influences on population data may not be solely due to age, but also to other compositional traits such as sex, race-ethnicity, urban-rural residence, marital status, socioeconomic status, and many other characteristics. Populations often differ greatly in these traits and any given population can undergo important compositional changes over time.

We examine some questions on changes which might be affected by confounding factors: Why is the Mexican crude death rate declining over time? Is it due to a decline in mortality or to a change in the population's age structure? Why are the crude birth rates in Denmark, the Netherlands and Sweden declining? Is it due to a decline in fertility among married and unmarried, or to the fact that compositions from both groups are changing over time? We could also examine the overall life expectancy of selected European countries. The question then would be whether the change over time in life expectancy is because of a decline (or increase) in life expectancy in each country or because of change over time in the population composition of the respective countries.

In order to obtain summary measures which take account of compositional effects (age, marital status, nationality or other characteristics), demographers have devised a number of techniques. In this book, we focus on how decomposition methods can be used to analyze the problems of confounding compositional effects.

In general, to decompose means to separate something into its constituent parts or elements or into simpler compounds. The decomposition methods used in demography also follow this separation principle by dividing demographic variables into specific components.

Decomposition methods are used when comparing demographic variables that belong to different populations, or when comparing variables of the same population over time. To answer our research questions, here, we mainly apply decomposition methods to study changes in demographic variables over time.

1.3 Research Question

This book focuses on a comparison of the different decomposition methods used in demography. The main purpose of this research is to answer the following question:

Which are the components of the change over time of demographic variables?

The answer to this question can be found in Part III, where a decomposition method presented by Vaupel (1992) is further developed. This method separates the changes over time of demographic variables into two components and it is simply referred in the rest of the book as the *direct versus compositional decomposition.* This decomposition method has similarities to previous techniques of separating changes over time. The most relevant of these techniques used for studying demographic variables are reviewed in Part II. Taking the similarities between existing methods in this field and direct vs. compositional decomposition into account led to new queries about decomposition methods, which can be summed up as follows:

What are the advantages and disadvantages of direct vs. compositional decomposition relative to the previous methods?

The answer to this question is found in Part IV wherein direct vs. compositional decomposition and the previous methods are compared.

The following is an overview of the parts of the book.

1.4 Organization of the Book

The book is divided into four parts:

- Part I consists of two chapters, including the present *Introduction.* Chapter 1 provides an orientation to this book as well as to the field of study that utilizes decomposition methods. Chapter 2 describes the special notation used throughout the book. The mathematical calculations and proofs used throughout the text are not difficult and do not require any mathematical knowledge beyond elementary arithmetic and calculus.

- Part II, on *Decomposition Methods*, is a literature review of existing decomposition methods, and it is divided into three chapters. The first chapter, Chapter 3 presents some of the early developments in decomposition methods. Chapter 4 goes on to describe contributions to the methodological development of particular demographic fields: mortality, fertility and population growth. Chapter 5 lists alternative decomposition methods used in demography.

- Part III, which presents a method for *Decomposing Change Into Direct Versus Compositional Components*, is the main part of this book and includes four chapters. Chapter 6 presents a decomposition method for the change over time of demographic functions developed by Vaupel (1992). Also this chapter includes applications presented by Vaupel and Canudas Romo (2002) and decompositions suggested by Vaupel and Canudas Romo (2003). The remaining three chapters include some of this author's contributions to this area by extending Vaupel's method. Chapter 7 outlines an age, category and cause of death decomposition. This chapter shows how to calculate the contribution of change in any given age group, category or cause of death, to the total change of demographic measures. Chapter 8, on *Multidimensional Decompositions,* includes the decomposition of change over time for a demographic variable when numerous compositional components are present. The last chapter in Part III, *Estimation of Derivatives in Demography*, is concerned with the estimation procedures of derivatives and intensities over time for demographic functions involved in the analysis.

- Part IV, *Evaluation of the Decomposition Methods*, is devoted to comparing direct vs. compositional decomposition presented in Part III with the previous decomposition methods presented in the preceding literature review provided in Part II. This part of the book contains two chapters. The former, Chapter 10, is divided into five main sections corresponding to sections and chapters in Part II. The first section contrasts direct vs. compositional decomposition with earlier methods. Next there is an examination of the decompositions in life expectancy, crude birth rate and population growth association with direct vs. compositional decomposition. Finally, some relationships between the alternative methods and direct vs. compositional decomposition are shown. Chapter 11 comprising general conclusions brings this book to a close. The sections of this chapter consist of an introduction, a section on desired properties of decomposition methods and concluding remarks. This part of the book concludes the project by showing how and when direct vs. compositional decomposition takes precedence.

2

Notation

2.1 Introduction

In this chapter we introduce some general expressions that have facilitated the development of many formulas found in this book. All notations are listed in the index of notations at the end of the book.

Demographic averages are introduced first, followed by an explanation of the reasoning behind the focus on averages. Next is a description of the operators used to measure changes over time and finally, the issue of data sources and an illustration of how the results from the estimations in the tables are presented.

2.2 Demographic Averages

The decompositions of change over time demonstrated here, mainly focus on averages, also known as expected values or expectations. Let the expectation operator at time t be denoted as $\bar{v}(t)$ the mean value of the function $v(x,t)$ over variable x, can be expressed as follows:

$$
\begin{aligned}
E(v) \equiv \bar{v}(t) &= \frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx},\ x\ continuous, \\
&= \frac{\sum_x v_x(t)w_x(t)}{\sum_x w_x(t)},\ x\ discrete, \qquad (2.1)
\end{aligned}
$$

where $v(x,t)$ is some demographic function and $w(x,t)$ is some weighting function. For example, if $v(x,t)$ is substituted with the age-specific death rates and $w(x,t)$ with the age-specific population size we get a $\bar{v}(t)$ which is equal to the crude death rate. Equation (2.7) shows this substitution in detail.

Variable x in (2.1) can be continuous or discrete. This is denoted respectively as $v(x,t)$, or in the discrete case as a subindex, $v_x(t)$. In the applications presented in this book, x can denote age, region of residence or marital status, although other applications are also of interest. The variable t, which here denotes time exclusively, is always continuous.

In the text, the variables x, $v(x,t)$ and $w(x,t)$ are used for general formulas and are substituted with known demographic variables depending on the application. For example, in several applications the weighting function $w(x,t)$ equals $N(a,t)$ the age-specific population size at age a and time t. Here it is implicit that variable x is changed by a, representing age. (See Encyclopedia of the Social and Behavioral Sciences (2001) for a discussion on this basic but subtle quantity.)

If the demographic function $w_x(t)$ is aggregated over all the possible values of the discrete variable x then this is denoted with a dot in place of the variable x, $w.(t) = \sum_x w_x(t)$. For example, the total population at time t denoted $N.(t)$ is equal to the addition over all ages of the single age population sizes, denoted $N_a(t)$ for age a to $a+1$ at time t. This is expressed as $N.(t) = \sum_a N_a(t)$. In the case of two discrete variables, x and z, adding all values of one of the variables, we denote this with a dot instead of with the variable. When adding over z we use $w_{x\cdot}(t) = \sum_z w_{xz}(t)$, and adding over x we get $w_{\cdot z}(t) = \sum_x w_{xz}(t)$. So when adding over both x and z we obtain $w_{\cdot\cdot}(t) = \sum_x \sum_z w_{xz}(t)$.

For the continuous case the variables are simply omitted, i.e., integrating $w(x,t)$ over all values of x we obtain $w(t) = \int_0^\infty w(x,t)dx$ and $w(x,t) = \int_0^\infty w(x,z,t)dz$. The new variable z is useful when two compositional components, for example, age and country of residence, are included in the same decomposition.

The reason we focus on averages is that many demographic measures can be expressed as averages. Demographic formulas that can be generalized by a mathematical expectation, or average, are presented in the next section.

2.3 Demographic Measures: Ratios, Proportions, Rates, and Probabilities

As pointed out by Palmore and Gardner (1983), the most commonly used measures in demography are ratios, proportions, rates, percentages and probabilities. This section shows that all these measures can be considered as averages.

A *ratio* is a single number that expresses the relative size of two numbers. A commonly used ratio in demography is the ratio of male to female births, the sex ratio at birth. The result of dividing a number $v.(t) = \sum_x v_x(t)$ by another number $w.(t) = \sum_x w_x(t)$ is the ratio of v to w,

$$\frac{v.(t)}{w.(t)} = \frac{\sum_x v_x(t)}{\sum_x w_x(t)}. \tag{2.2}$$

This ratio can be seen as an average of ratios by carrying out an arithmetic manipulation of dividing and multiplying the numerator by the same term,

$$\frac{v.(t)}{w.(t)} = \frac{\sum_x \left(\frac{v_x(t)}{w_x(t)}\right) w_x(t)}{\sum_x w_x(t)} = \overline{\left(\frac{v}{w}\right)}(t). \tag{2.3}$$

The numerator and denominator of a ratio may or may not be subsets of the same variable. By using the average of ratios (2.3) the nominator and denominator are related. The sex ratio at birth, for example, is equal to the average of sex ratios by age of the mother.

Proportions are a special type of ratio where the denominator includes the numerator. A well-known demographic variable is the proportion of the total population belonging to an age group, for example the proportion below age a. It is possible to express proportions as averages by using an indicator function $I_a(x)$. This function has values of 1 when a condition is satisfied, x below a for example, or 0 otherwise. By using this function it is possible to include only the desired elements in the numerator,

$$\frac{v.(t)}{w.(t)} = \frac{\sum_x v_x(t)}{\sum_x w_x(t)} = \frac{\sum_x I_a(x) w_x(t)}{\sum_x w_x(t)} = \bar{I}_a(t). \tag{2.4}$$

The right hand side of equation (2.4) shows that the proportion of the variable $v.(t)$ over $w.(t)$ is equal to the average of the indicator weighted by $w_x(t)$. The proportion of the total population below age 15, for example, is equal to the average indicator $\bar{I}_a$ with $a = 15$ or less, and weighted by the age-specific population size at age a and time t, $N_a(t)$,

$$\frac{N_{0-15}(t)}{N.(t)} = \frac{\sum_{a=0}^{15} N_a(t)}{\sum_{a=0}^{\omega} N_a(t)} = \frac{\sum_{a=0}^{\omega} I_{0-15}(a) N_a(t)}{\sum_{a=0}^{\omega} N_a(t)} = \bar{I}_{0-15}(t). \tag{2.5}$$

Percentages are a special type of proportion in which the ratio is multiplied by a constant, normally 100, so that the result is expressed per hundred.

A *rate* refers to the occurrence of events over a given time interval, where the denominator is the duration of persons exposed to risk during this time interval. The duration of persons "exposed to risk" is specified by using the concept of "person-years lived". The crude death rate (CDR) for year t, for example, is sometimes denoted by $d(t)$ and calculated as the total number of deaths during the year t, $D.(t)$, over the number of person-years lived during the period, $N.(t)$. The number of deaths at age a, $D_a(t)$, is equal to the product of the age-specific death rates $m_a(t)$ at age a and time t, and the population size at that age, $N_a(t)$; $D_a(t) = m_a(t) N_a(t)$,

$$d(t) = \frac{D.(t)}{N.(t)} = \frac{\sum_{a=0}^{\omega} D_a(t)}{\sum_{a=0}^{\omega} N_a(t)} = \frac{\sum_{a=0}^{\omega} m_a(t) N_a(t)}{\sum_{a=0}^{\omega} N_a(t)} = \bar{m}(t), \tag{2.6}$$

where ω is the highest age attained. The population size functions as an estimate of the person-years lived.

In the continuous case when the age interval gets shorter the death rates can be changed to the "force of mortality", denoted $\mu(a,t) = \lim_{\Delta a \to 0} m_{a,a+\Delta a}(t)$. The continuous expression of the CDR is the average of the force of mortality weighted by the age-specific population size, $N(a,t)$,

$$d(t) = \frac{D(t)}{N(t)} = \frac{\int_0^{\omega} D(a,t)da}{\int_0^{\omega} N(a,t)da} = \frac{\int_0^{\omega} \mu(a,t) N(a,t)da}{\int_0^{\omega} N(a,t)da} = \bar{\mu}(t). \tag{2.7}$$

Due to the fact that demographers come from many academic disciplines and for various historical reasons, the actual rates used by demographers are not always consistent with the rates described here. The crude birth rate (CBR), for example, is defined as the total number of births over the total population size. Demographers working in the field of fertility mainly use female age-specific fertility rates. In our research we follow this convention and refer to female rates as age-specific fertility rates. Births occurring during the year t, denoted as $B(t)$, are the product of age-specific fertility rates, $b\,(a,t)$, and the female population size denoted at age a and time t as $N_f\,(a,t)$; $B(a,t) = b(a,t)N_f(a,t)$. The crude birth rate contains the female population size in the numerator as weights, while the total population size is the denominator,

$$CBR(t) = \frac{B(t)}{N(t)} = \frac{\int_0^\omega B\,(a,t)\,da}{\int_0^\omega N\,(a,t)\,da} = \frac{\int_0^\omega b\,(a,t)\,N_f\,(a,t)\,da}{\int_0^\omega N\,(a,t)\,da}. \tag{2.8}$$

It is possible to transform this rate into an average by carrying out an arithmetic manipulation as in equations (2.3) and (2.4). The CBR is modified to an average of the product of the age-specific fertility rates and the proportion of females,

$$CBR(t) = \frac{\int_0^\omega b\,(a,t)\,c_f(a,t)N\,(a,t)\,da}{\int_0^\omega N\,(a,t)\,da} = \overline{bc_f}(t), \tag{2.9}$$

where $c_f(a,t) = \frac{N_f(a,t)}{N(a,t)}$ is the proportion of females in the population at age a and time t.

Normally, ratios, proportions and percentages are used for analyzing the composition of a set of certain events or a population. Rates, in contrast, are used to study the dynamics of change. A *probability* is similar to a rate except for one important difference: a probability's denominator consists of all persons in a given population at the beginning of the observation period. The probability of dying in the year 2005 in a population closed to migration given that one survives to age 50 in January 2005 is equal to the deaths that occur during the year in this age group divided by the population of age 50 and above present on January 1, 2005. Arithmetic manipulations, similar to those for rates, can also be carried out to study a probability as an average.

As can be seen, averages have a central position within mathematical expressions of demographic measures. Demographic rates and other measures are influenced by the population composition. The study of changes over time, as described above in terms of averages, leads us to the field of decomposition methods. The mathematical formula resultants of derivation over time for demographic variables have different elements which represent the components responsible for the change. The following section presents the notation for the operators needed to study changes over time for demographic variables.

2.4 Operators

Let a dot over a variable denote the derivative with respect to time, t,

$$\dot{v}(t) \equiv \dot{v}(x,t) = \frac{\partial}{\partial t}v(x,t). \tag{2.10}$$

The change over time in the crude death rate, as seen in equation (2.7), is expressed following (2.10) as $\dot{d}(t) = \frac{\partial}{\partial t}d(t)$.

An acute accent is used to denote the relative derivative or intensity with respect to time, t

$$\acute{v}(t) \equiv \acute{v}(x,t) = \frac{\frac{\partial}{\partial t} v(x,t)}{v(x,t)}. \tag{2.11}$$

In this way, the relative change in the crude birth rate over time, as seen in equation (2.8), is then denoted as $C\acute{B}R(t) = \frac{\frac{\partial}{\partial t} CBR(t)}{CBR(t)}$. The use of the acute accent, which reduces the clutter in many demographic formulas, originated from Vaupel (1992) and is used in Vaupel and Canudas Romo (2000). Note that for simplicity we often omit the arguments x and t. Derivation is how change over time is measured and the notation presented here has proven to be very useful in developing direct vs. compositional decomposition formulas.

The covariance operator is widely used in statistical analysis but is rare in formal demography. The results and applications presented by Vaupel (1992), and Vaupel and Canudas Romo (2002), also found in Part III, suggest that this and various other covariances are of general significance to the understanding of population dynamics. In this context, the covariance measures the extent to which a demographic variable rises and falls with another demographic measure. For example, the covariance between age and the population growth rate is used by Preston, Himes and Eggers (1989) to analyze population aging. The covariance operator is defined as the average of the product of the deviations of two variables from their respective means,

$$C(v,u) = E\left[(v-\bar{v})(u-\bar{u})\right]$$

$$= \frac{\int_0^\infty [v(x,t)-\bar{v}(t)]\,[u(x,t)-\bar{u}(t)]\,w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{2.12}$$

The term $[v(x,t)-\bar{v}(t)]$ corresponds to the distance between the variable $v(x,t)$ and its mean $\bar{v}(t)$, and it is similar to the interpretation for $[u(x,t)-\bar{u}(t)]$. Another expression for the covariance which is used in this book is the difference between the expectation of a product and the product of expectations

$$C(v,u) = E(vu) - E(v)\,E(u) = \overline{vu} - \bar{v}\bar{u}$$

$$= \frac{\int_0^\infty v(x,y)u(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx} - \frac{\int_0^\infty v(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}\frac{\int_0^\infty u(x,y)w(x,y)dx}{\int_0^\infty w(x,y)dx}. \tag{2.13}$$

For example, $C(\mu,r)$ is the covariance between the force of mortality $\mu(a,t)$ and the age-specific growth rates $r(a,t)$, weighted by the population size $N(a,t)$. The basic idea is that the covariance will be positive if the force of mortality tends to be higher (or lower) than average at those ages when the age-specific growth rate tends to be higher (or lower) than average.

Note that equation (2.13) implies that the expectation of a product can be decomposed as

$$\overline{uv} = \bar{u}\bar{v} + C(u,v), \tag{2.14}$$

which is a useful result in many contexts.

We also need to define a special notation for the operators of difference and relative difference. Parallel to the dot notation for the derivation operator let the difference operator between $t+h$ and t be denoted as

$$\Delta v(t) \equiv v(t+h) - v(t), \tag{2.15}$$

and the relative difference operator as

$$\grave{v}(t) \equiv \frac{\Delta v(t)}{v(t)} \equiv \frac{v(t+h) - v(t)}{v(t)}. \tag{2.16}$$

2.5 Data and Applications

The applications presented throughout this book come from freely available data. This section describes some of these data sources. The applications of decomposition formulas are organized in tables or figures which show the results. Following the presentation of the data sources is a subsection explaining how to read these tables.

2.5.1 Data Sources

The data sources used in the applications are listed below in the order of their appearance in the text. There is also a brief description of the databases and the tables which we used for the illustrations.

The United Nations Demographic Yearbook (2001): is a comprehensive collection of international demographic statistics, prepared by the Statistics Division of the United Nations. It contains historical demographic statistics from 1948 to 1997 and presents time series of population size by age, sex and urban/rural residence, natality, mortality and nuptiality as well as selected derived measures concerning these components of population change for a 50-year period. Applications in the book using this data source are employed in Tables 3.1, 6.1, 6.3, 6.4, 7.1, 9.1 and 10.1. Figure 7.1 is also based on this data source.

The Human Mortality Database (2002) was created to provide detailed mortality and population data for researchers interested in the history of human longevity. The main goal of the database is to document the longevity revolution of the modern era and to facilitate research into its causes and consequences. There is open international access to these data. At present the database contains detailed data for a collection of 17 countries. However, this database is limited to countries where death registration and census data are virtually complete, since this type of information is required by the uniform method followed in this database. As a result, the countries included here are relatively wealthy and for the most part highly industrialized. The book employs this data source in Tables 3.2, 3.3, 3.4, 3.5, 4.1, 4.7, 4.8, 4.10, 4.11, 5.5, 6.2, 6.7, 8.2, 8.3, 8.5, 10.2 and 10.3. Figures 6.1, 6.2, 6.3, 8.1, 8.2 and 9.1 are among the figures that are based on this data source.

The Berkeley Mortality Database (2001) has been replaced by a bigger and better project, i.e., the Human Mortality Database. All except one item are available from the Human Mortality Database. This item is the Japanese cause-of-death data and is used in Tables 4.2, 7.4, 7.5 and 8.4. Figure 7.5 is based on this data source.

The Eurostat Database (2000) is accessed through the New Cronos interface which allows one to navigate in the database to find and download the data of interest. The database collects information on an annual basis from 36 countries comprising EU countries, EFTA countries, and other east and central European countries. Data are collected by the respective National Statistical Institutes and depend on the registration systems used in each country. Eurostat collects raw numbers, not indicators directly from the countries. Time series for the EU and EFTA countries begin from 1950 and continue through 1999, but some demographic data by age for the period 1960-1990 has not been fully processed. This data source is used in Tables 4.3, 4.4, 6.5, 6.6, 8.6 and 10.4.

The International Data Base (IDB) is a computerized data bank containing statistical tables of demographic and socio-economic data for 227 countries and areas of the world. The IDB provides quick access to specialized information, with emphasis on demographic measures, for individual countries or selected groups of countries. The IDB combines data from country sources (especially censuses and surveys) with estimates and projections to provide information dating back as far as 1950 and as far ahead as 2050. Because the IDB is maintained as a research tool in response to sponsor requirements, the amount of information available for each country may vary. The database covers the national population, and selected data by urban/rural residence, and by age and sex. Tables 4.9, 7.2, 7.3, 8.1 and 10.5 employ this data from the U.S. Census Bureau (2001).

The National Retrospective Demographic Survey (1998) or EDER in Spanish, was carried out in Mexico in 1998. The data pertains to Mexican residents in 1998. This retrospective survey with duration data is representative of the entire population of the country. A total of 2,496 persons were interviewed in the EDER survey. Three cohorts of Mexicans are interviewed: those born in 1936-1938, 1951-1953, and 1966-1968. For each person, individual events are related to calendar time, from the year of birth to the moment of the survey in 1998. The event-history data of each individual can be divided into five data sets: migration history, labor history, education history, family history, and fertility history. Chapter 5 includes four tables which used this database: Tables 5.1, 5.2, 5.3 and 5.4.

The data used in Figures 7.2, 7.3 and 7.4 were supplied by Professor Virgilio Partida of the National Population Board from Mexico, Conapo (2002). A data source which comes from a table in the article by Islam et al. (1998) is used in Tables 4.5 and 4.6 in this book. Historical data for Japan in Table 4.11 is based on Japan Statistical Association (2002).

2.5.2 Tables of Applications

In general the configuration of the tables follows the same presentation as illustrated by Table 2.1. has five parts. There are five features, the first of which is the table number and caption. top heading has the number and name of the Table. At the beginning of the book is compilation of all tables used in this study.

In the actual table, the first row is for the heading of the columns, separated from the results by a line. For example, sometimes there is need to know information about different years, different regions, or the contribution of different components, in which case each column contains the respective information.

The next section of the table is the observed values of the variable under study. For example, the information of the demographic variable in the years under study is provided

Table 2.1: Example of how the results are presented in the tables.

Heading for the columns of the table
Observed demographic changes
Estimated components of the change

Source of the data and notes.

here. Furthermore, the observed change in the period under study is stated here. Again there is another line separating this information from the next part of the table.

The next section provides estimations of the change in the demographic variable and of the components of this change. This section is crucial for our work since it contains the results of the methods explained here. Sources and special notes accompany the table.

Part II

Decomposition Methods

3

Standardization and Decomposition Techniques

3.1 Introduction

This chapter presents the first decomposition method in demography and some of the generalizations that have been suggested to improve it. We begin with a section on *Methods of Standardization* which shows the origins of the decomposition methods. The next section presents the first decomposition method and applies it to study the changes in the Mexican crude death rate, (CDR). The decomposition allocates the change into two components that explain the change over time in the CDR.

The section that follows is entitled *Further Decomposition Research* which includes four subsections, one for each of the generalizations of the decomposition techniques. In these four subsections we study the change over time of the CDR of a selected group of European countries. This European CDR involves not only the structure of the population of one nation but that of many countries. The decompositions of this section decompose the change into terms that account for these different structures of the population.

3.2 Methods of Standardization

Chronologically, the precursors of the decomposition methods are the techniques of standardization. *Indirect standardization* is mentioned for the first time in 1786 by Tetens (1786), and *direct standardization* in 1844 by Neison (1844), as cited by Hoem (1987). In the early days of its usage, standardization was known as the "method of standards" and was applied to various rates in order to compare regional and occupational mortality.

Demographers use the technique of standardization to eliminate compositional effects from overall rates of some phenomena in two or more populations. The technique of *direct standardization* assumes a particular population as standard and recomputes the overall rates in both populations by replacing their compositions by the compositional schedule of the standard population. *Indirect standardization* assumes a particular age schedule of risk as standard and recomputes the overall rates in the other populations by replacing their age schedules. The arguments for and against *direct* and *indirect standardization* has been settled in favor of the latter. Presumably this preference is due to less weighting to groups with smaller exposures in *indirect methodology* (see Hoem (1987) for more detail on this comparison).

Because of its importance in demographic analysis, methods of standardization have been extensively described in most demography textbooks, see for instance, Keyfitz (1977), Elandt-Johnson and Johnson (1980), Pressat (1980), Smith (1992), Newell (1994), Leridon and Laurent (1997), Brown (1997), Caselli et al. (2001) and Preston et al. (2001).

Regardless of their importance and wide usage in almost all fields of demography, standardized rates have always been considered a bit problematic. They are viewed as unreliable due to their dependence on an arbitrary standard. This problem was the impetus behind further research on methods of comparison.

Beginning with the classic paper by Kitagawa (1955) another field of research, namely the decomposition of the difference between the crude rates of two populations, was developed. The next section reviews this method.

3.3 First Decomposition Method

Let $\bar{v}(t+h)$ and $\bar{v}(t)$ be the values for the average under study at time t and $t+h$, as defined in equation (2.1). Kitagawa's proposed decomposition is based on the difference between these two measures $\bar{v}(t+h) - \bar{v}(t)$, also expressed as $\Delta\bar{v}(t)$:

$$\bar{v}(t+h) - \bar{v}(t) = \Delta\bar{v}(t) = \sum_x v_x(t+h)\frac{w_x(t+h)}{w.(t+h)} - \sum_x v_x(t)\frac{w_x(t)}{w.(t)},$$

where the weights $w_x(t+h)$ are normalized (weights for all ages that summed up add to one) by dividing the total value of the weights $w.(t) = \sum_x w_x(t)$.

The decomposition focuses on the additive contribution of two components that sum up to the difference in their overall rates. The first term is the difference between the population rates over time, $v_x(t+h) - v_x(t)$. The second component is the difference in weights or population structure, $\frac{w_x(t+h)}{w.(t+h)} - \frac{w_x(t)}{w.(t)}$. The decomposition is written as

$$\begin{aligned}\Delta\bar{v}(t) \quad &= \quad \sum_x \left(\frac{\frac{w_x(t+h)}{w.(t+h)} + \frac{w_x(t)}{w.(t)}}{2}\right)[v_x(t+h) - v_x(t)] \qquad (3.1)\\ &+ \sum_x \frac{v_x(t+h) + v_x(t)}{2}\left(\frac{w_x(t+h)}{w.(t+h)} - \frac{w_x(t)}{w.(t)}\right). \qquad (3.2)\end{aligned}$$

In (3.1) the change of the average $\bar{v}(t)$ is due to changes in the variables $v_x(t)$ at every value of x. We refer to it as $\Delta\ v_x$. The weight of (3.1) is the average of the two normalized weights.

The other term, in (3.2), is the change in averages due to changes in the normalized weights, denoted as $\Delta\ \frac{w_x}{w.}$, with the average of the demographic function $v_x(t)$ as weights.

Note that the weights in both (3.1) and (3.2) are arithmetic averages. Kitagawa (1955) states that "changes in rates and composition are seldom independent, rather a change in one is likely to affect the other. It may be argued, therefore, that since both were changing during the period, a logical set of weights for summarizing changes in specific rates, for example, would be the average composition of the population during the period."

As an example, we look at the Mexican crude death rate (CDR). As shown in equation (2.6), the CDR is the average of the age-specific death rates $m_a(t)$, at age a and time t, weighted by the population size $N_a(t)$, at age a and time t. The change over time of the CDR can be decomposed according to Kitagawa's proposal as

$$\Delta d(t) = \sum_a m_a(t+h)\frac{N_a(t+h)}{N.(t+h)} - \sum_a m_a(t)\frac{N_a(t)}{N.(t)}$$

$$= \sum_a \left(\frac{\frac{N_a(t+h)}{N.(t+h)} + \frac{N_a(t)}{N.(t)}}{2}\right)[m_a(t+h) - m_a(t)] \tag{3.3}$$

$$+\sum_a \frac{m_a(t+h) + m_a(t)}{2}\left(\frac{N_a(t+h)}{N.(t+h)} - \frac{N_a(t)}{N.(t)}\right). \tag{3.4}$$

Table 3.1 presents the Mexican CDR and the decomposition of its annual change over time from 1965 to 1975, from 1975 to 1985 and from 1985 to 1995. In the table the components of change are denoted as $\Delta\ m_a$ and $\Delta\ \frac{N_a}{N.}$, referring to the changes in the age-specific death rates and in age structure, respectively. The Mexican CDR declined continuously during the

Table 3.1: Crude death rate, $d(t)$, per thousand, and Kitagawa's decomposition of the annual change over time in 1965-1975, 1975-1985 and 1985-1995 for Mexico.

t	1965	1975	1985
$d(t)$	9.514	7.456	5.532
$d(t+10)$	7.456	5.532	4.755
$\Delta d(t)$	-0.206	-0.192	-0.078
$\Delta\ m_a$	-0.161	-0.159	-0.116
$\Delta\ \frac{N_a}{N.}$	-0.045	-0.033	0.038
$\Delta d(t) = \Delta\ m_a + \Delta\ \frac{N_a}{N.}$	-0.206	-0.192	-0.078

Source: Author's calculations, based on the United Nations Data Base (2001).

decades under study. These changes are mainly attributable to changes in the age-specific death rates. The change in the CDR due to the age structure moves from negative values in the first two decades to opposing the decline in the crude death rate in the last period. The third row, of the observed change, $\Delta d(t)$, and the last row, of the estimated change, are exactly identical.

Kitagawa's decomposition (3.1 and 3.2) is an arithmetic manipulation of a difference. Therefore, this decomposition is not unique since other terms could have been incorporated into the formula. But it is accepted as a very good approximation of the share of change of the elements of a demographic average.

Two remarks made by Kitagawa are relevant for the following sections. First, the effects of factors do not imply causal relationships. Second, the total effect is not necessarily better explained by increasing the number of factors. This last statement is also reviewed in the next section when more confounding variables are involved, and other techniques of decomposition are needed to deal with the averages.

3.4 Further Decomposition Research

Kitagawa's formula has successfully been used in cases of one confounding factor by separating the difference in demographic variables into two components. When more confounding factors are present, it is necessary to expand equations (3.1) and (3.2) further. Four methods are shown here; three have further developed Kitagawa's formulation and the last is a similar methodology applied in economics and recently in demography. The first three methods are by Cho and Retherford, Kim and Strobino and Das Gupta, respectively, while the fourth section presents the work of Oosterhaven and Van der Linden.

3.4.1 Cho and Retherford's Decomposition

Consider data cross-classified by three factors, letting $v_{xz}(t)$ and $w_{xz}(t)$ be demographic functions and x, z and t be the factors. The decomposition of the change over t (populations, time or another variable) should then account for the change in the functions $v_{xz}(t)$ and $w_{xz}(t)$, as well as the influence of the factors x and z on these functions.

For example, we can study the change over time in the average crude death rate of selected European countries. Let the age structure of the selected countries be the x factor and the population distribution between the countries be the z factor. A decomposition of this change should take into account the influence of the x factor (age structure) and the z factor (population distribution).

Cho and Retherford (1973) examine averages of averages and apply Kitagawa's formula twice. Their study interest is double averages defined as the average over x of the averages $\bar{v}_x(t)$, denoted as $\tilde{\bar{v}}(t)$,

$$\tilde{\bar{v}}(t) = \sum_x \bar{v}_x(t) \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}, \tag{3.5}$$

where $w_{x\cdot}(t) = \sum_z w_{xz}(t)$ and $w_{\cdot\cdot}(t) = \sum_x w_{x\cdot}(t)$, and where the average over z, $\bar{v}_x(t)$, is as already defined in (2.1) as

$$\bar{v}_x(t) = \sum_z v_{xz}(t) \frac{w_{xz}(t)}{w_{x\cdot}(t)}. \tag{3.6}$$

Following Kitagawa's formula, the difference of the averages over time (or other characteristics) is

$$\tilde{\bar{v}}(t+h) - \tilde{\bar{v}}(t) =$$

$$\sum_x \frac{\bar{v}_x(t+h) + \bar{v}_x(t)}{2} \left(\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} - \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)} \right) \tag{3.7}$$

$$+ \sum_x \left(\frac{\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} + \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}}{2} \right) \left[\bar{v}_x(t+h) - \bar{v}_x(t) \right]. \tag{3.8}$$

The difference $\bar{v}_x(t+h) - \bar{v}_x(t)$, in the second term in equation (3.8), is further decomposed by applying Kitagawa's formula a second time

$$\bar{v}_x(t+h) - \bar{v}_x(t) =$$

$$\sum_z \left(\frac{\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)} + \frac{w_{xz}(t)}{w_{x\cdot}(t)}}{2} \right) \left[v_{xz}(t+h) - v_{xz}(t) \right] \tag{3.9}$$

$$+ \sum_z \frac{v_{xz}(t+h) + v_{xz}(t)}{2} \left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)} - \frac{w_{xz}(t)}{w_{x\cdot}(t)} \right). \tag{3.10}$$

The final decomposition is the change in the normalized weight $w_{xz}(t)$ due to the effect of factor x, seen in the term (3.7) denoted as $\Delta \frac{w_{x\cdot}}{w_{\cdot\cdot}}$. The other two components come from substituting (3.9) and (3.10) in (3.8). The term (3.9) corresponds to the changes in variable $v_{xz}(t)$ due to x and z, Δv_{xz}. The last element (3.10) is the change in the normalized weight $w_{xz}(t)$ due to factor z, $\Delta \frac{w_{xz}}{w_{x\cdot}}$. It has to be noted that the weight in the term (3.8) is shared by the two elements (3.9) and (3.10) while (3.7) only has one weight. This implies that Cho and Retherford's decomposition depends on the order in which the variables x and z are elected.

As an example, we look at the crude death rate in selected European countries. The aim is to know what the contribution to the total change of the changes in age structure and population distribution between countries is. The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

Let the age- and country-specific death rates be denoted as $m_{ac}(t)$, where the sub-indexes a and c correspond to age and country, respectively. The population size is defined over age and country as $N_{ac}(t)$. The overall CDR of these selected countries is denoted as $\bar{d}_E(t)$, and is an average of crude death rates:

$$\bar{d}_E(t) = \sum_c d_c(t) \frac{N_{\cdot c}(t)}{N_{\cdot\cdot}(t)}, \tag{3.11}$$

where the CDR of every country $d_c(t)$ is as in (2.6) defined as

$$d_c(t) = \sum_a m_{ac}(t) \frac{N_{ac}(t)}{N_{\cdot c}(t)}.$$

Table 3.2 presents the overall CDR of the selected European countries, $\bar{d}_E(t)$. The rate is calculated as the average of the countries' crude death rates weighted by the population size as in (3.11). In the table, Cho and Retherford's decomposition of the change in the CDR is shown for the periods 1960 to 1970, from 1975 to 1985 and from 1992 to 1996. There are three components in the decomposition: changes in rates, changes in age structure within countries and change produced by changes in the population distribution between countries. During all

Table 3.2: Crude death rate, $\bar{d}_E(t)$, per thousand, and Cho and Retherford's decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996. The crude death rate is the average of crude death rates of selected European countries.

t	1960	1975	1992
$\bar{d}_E(t)$	10.720	10.592	10.917
$\bar{d}_E(t+h)$	10.892	10.965	11.596
$\Delta\bar{d}_E(t)$	0.0172	0.0374	0.1697
$\Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}}$	0.1104	0.1433	0.1310
$\Delta\frac{N_{ac}}{N_{a\cdot}}$	0.0001	0.0032	-0.0078
Δm_{ac}	-0.0933	-0.1091	0.0465
$\Delta\bar{d}_E(t) = \Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}} + \Delta\frac{N_{ac}}{N_{a\cdot}} + \Delta m_{ac}$	0.0172	0.0374	0.1697

Source: Author's calculations from Cho and Retherford formulation, based on the Human Mortality Database (2002). For the years 1960 and 1975 eleven-year periods were used (1960-1970 and 1975-1985), $h = 10$. For the year 1992 a five-year period was used (1992-1996), $h = 4$. The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

the examined periods, the average CDR of the European countries is increasing. The change is mainly explained by changes in the age- and country-specific death rates, Δm_{ac}, and the effect of changes in the age structure within the countries, $\Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}}$. The term for the change in the structure of population distribution between the countries, $\Delta\frac{N_{ac}}{N_{a\cdot}}$, is only notorious in the last period from 1992 to 1996.

The components in the first two periods follow the same pattern, slightly increasing for the period 1975-1985. The increase in the CDR is due to an ageing process of the age structure of the population, seen in the high positive values of the component $\Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}}$. This increase is countered by a decrease in the age- and country-specific death rates, seen in the component Δm_{ac}. In other words, there are improvements in survivorship, but these can not catch up with the ageing process of the studied populations.

The major increase in the average crude death rate was found between 1992 and 1996, with an annual increase of 0.17 per thousand. In this column it is also possible to see the different mortality trends of the countries included in the average. The east European countries included here, particularly Russia, experienced a very dramatic increase in their mortality levels in the 1990s. This can be seen in the contribution of the Δm_{ac} which changed from a negative value in 1960-1970 and 1975-1985 to a positive value in 1992-1996. That is, the increase in some age- and country-specific death rates was greater than the decline in other ages and countries' death rates. The age structure term remains the most important factor accounting for more than 70% of the total change.

As mentioned by Kitagawa, the total change is not necessarily better explained by increasing the number of factors. This was the case for the component denoting the structure of the population distribution between the countries, $\Delta\frac{N_{ac}}{N_{a\cdot}}$. This term increased its contribution in 1975-85 and 1992-1996 but never surpassed its 9% share in the total change of the CDR. Decomposition analysis, therefore, is not like regression analysis, in which the addition of each independent variable to the formula increasingly explains the variations in the dependent variable.

It is important to note that choosing first the crude death rate of countries and then averaging over countries is one option for constructing the overall CDR. It is also possible to look first at the age-specific death rates $m_a(t)$ of all the selected countries

$$m_a(t) = \sum_c m_{ac}(t)\frac{N_{ac}(t)}{N_{\cdot c}(t)},$$

and the resultant overall CDR is an average similar to (2.7). Slightly different results in the decomposition terms would be obtained if the order of the characteristics (age and country) is inverted.

This dependency on the arbitrary selection of the factors in the decomposition inspired Kim and Strobino (1984) to develop another method. They suggested an alternative decomposition method when data involves three factors, arguing that in certain cases it would be more meaningful to treat the three factors according to some hierarchy. Their method is explained in the following subsection.

3.4.2 Kim and Strobino's Decomposition

A rate for a population, as defined in Chapter 2, is the number of events over the total duration involving persons exposed to risk. For example, the crude death rate in (2.7) is equal to the deaths occurring during a period over the total population at risk of dying during the period.

In the case when two compositional variables are involved, this rate is expressed as $\tilde{\bar{v}}(t) = \frac{\sum_{x,z} O_{xz}(t)}{\sum_{x,z} w_{xz}(t)}$, where $O_{xz}(t)$ are the occurrences of the event under study (deaths, births, etc.). Also, in Chapter 2 we showed that these measures can be seen as weighted averages by undertaking the adequate arithmetic manipulation. Kim and Strobino transform the rate into a product of three terms, one for the rates, denoted v_{xz}, a second for the normalized weights over the z factor, denoted $\frac{w_{xz}}{w_{x\cdot}}$, and a third for a normalized weight over the x factor, denoted $\frac{w_{x\cdot}}{w_{\cdot\cdot}}$,

$$\tilde{\bar{v}}(t) = \sum_{x,z}\frac{O_{xz}(t)}{w_{\cdot\cdot}(t)} = \sum_{x,z}\frac{O_{xz}(t)}{w_{xz}(t)}\frac{w_{xz}(t)}{w_{x\cdot}(t)}\frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)} = \sum_{x,z} v_{xz}(t)\frac{w_{xz}(t)}{w_{x\cdot}(t)}\frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}. \tag{3.12}$$

The hierarchy among the factors comes from the inclusion of the weight $w_{x\cdot}(t)$ in (3.12). Different components are obtained if instead of $w_{x\cdot}(t)$ we include $w_{\cdot z}(t)$. The decomposition proposed by Cho and Retherford depends on the order of the factors x and z because each term has a different proportion of weights. In contrast, Kim and Strobino suggest separating the different components in an equal distribution among all factors involved. More recently, Gray (1991) has suggested a generalization of Kim and Strobino's decomposition when more variables are involved (please refer to the section on *The Delta Method*). There are three components of this decomposition and they involve differences between them as illustrated in (3.12). The first is for the difference between rates and it is denoted as $\Delta v_{xz} = v_{xz}(t+h) - v_{xz}(t)$. The second is the difference in normalized weights over the z factor, denoted by $\Delta \frac{w_{xz}}{w_{x\cdot}} = \frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)} - \frac{w_{xz}(t)}{w_{x\cdot}(t)}$. The third is the difference in normalized weight over the x factor, $\Delta \frac{w_{x\cdot}}{w_{\cdot\cdot}} = \frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} - \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}$. Each of these differences is weighted by a function of the other two terms,

$$\tilde{\bar{v}}(t+h) - \tilde{\bar{v}}(t) =$$

$$\sum_{x,z} \frac{\left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)} + \frac{w_{xz}(t)}{w_{x\cdot}(t)}\right)\left(\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} + \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}\right) + \left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)}\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} + \frac{w_{xz}(t)}{w_{x\cdot}(t)}\frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}\right)}{6} \Delta v_{xz}$$
$$+ \sum_{x,z} \frac{\left(\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} + \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}\right)[v(t+h)+v(t)] + \left(\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)}v(t+h) + \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}v(t)\right)}{6} \Delta \frac{w_{xz}}{w_{x\cdot}}$$
$$+ \sum_{x,z} \frac{\left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)} + \frac{w_{xz}(t)}{w_{x\cdot}(t)}\right)[v(t+h)+v(t)] + \left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)}v(t+h) + \frac{w_{xz}(t)}{w_{x\cdot}(t)}v(t)\right)}{6} \Delta \frac{w_{x\cdot}}{w_{\cdot\cdot}}. \tag{3.13}$$

Here, it has to be noted that the differences Δv_{xz}, $\Delta \frac{w_{x\cdot}}{w_{\cdot\cdot}}$ and $\Delta \frac{w_{xz}}{w_{x\cdot}}$ in Kim and Strobino's formula are exactly the same as those found in the formula by Cho and Retherford. The two methods then differ in the selected weights for the differences.

Table 3.3 presents the CDR of selected European countries using the same data as in Table 3.2. Analogous to Table 3.2 where Cho and Retherford's formula was applied, Table 3.3 presents the decomposition by using Kim and Strobino's formula. The differences Δv_{xz}, $\Delta \frac{w_{x\cdot}}{w_{\cdot\cdot}}$ and $\Delta \frac{w_{xz}}{w_{x\cdot}}$ have a parallel notation for our example of Δm_{ac}, $\Delta \frac{N_{a\cdot}}{N_{\cdot\cdot}}$ and $\Delta \frac{N_{ac}}{N_{a\cdot}}$ respectively. The results are exactly the same as those in Table 3.2. This can be explained by noting that the principle of separation is the same in both methods. This principle involves the identification of weights for each of the differences involved in the decomposition. Furthermore, both disaggregations of the averages are achieved by arithmetic manipulation of the differences.

An alternative to this arithmetic manipulation is the decomposition proposed by Das Gupta. Das Gupta (1978), (1989), (1993) and (1994), has extensively contributed to this field of analysis. In his work, the idea of separating differences of rates was extended by multiple factors: by comparing several of the methods mentioned above, by applying them in a different order, and by suggesting new methods. The main contribution of Das Gupta's work is presented in the following subsection.

Table 3.3: Crude death rate, $\bar{d}_E(t)$, per thousand, and Kim and Strobino's decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996. The crude death rate is the average of crude death rates of selected European countries.

t	1960	1975	1992
$\bar{d}_E(t)$	10.720	10.592	10.917
$\bar{d}_E(t+h)$	10.892	10.965	11.596
$\Delta\bar{d}_E(t)$	0.0172	0.0374	0.1697
$\Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}}$	0.1105	0.1434	0.1305
$\Delta\frac{N_{ac}}{N_{a\cdot}}$	0.0000	0.0032	-0.0076
Δm_{ac}	-0.0933	-0.1092	0.0467
$\Delta\bar{d}_E(t) = \Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}} + \Delta\frac{N_{ac}}{N_{a\cdot}} + \Delta m_{ac}$	0.0172	0.0374	0.1697

Source: Author's calculations from formula (3.13), based on the Human Mortality Database (2002). For the years 1960 and 1975 eleven-year periods were used (1960-1970 and 1975-1985), $h = 10$. For the year 1992 a five-year period was used (1992-1996), $h = 4$. The countries included are: Austria, Bulgaria, Finland, France, east and west Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

3.4.3 Das Gupta's Decomposition

The first step is to look at the average of rates $v_{xz}(t)$ weighted by the normalized rates over both factors x and z, $\frac{w_{xz}(t)}{w_{\cdot\cdot}(t)}$. From the left hand side of (3.12) we obtain this average as

$$\tilde{\bar{v}}(t) = \sum_{x,z} \frac{O_{xz}(t)}{w_{\cdot\cdot}(t)} = \sum_{x,z} \frac{O_{xz}(t)}{w_{xz}(t)} \frac{w_{xz}(t)}{w_{\cdot\cdot}(t)} = \sum_{x,z} v_{xz}(t) \frac{w_{xz}(t)}{w_{\cdot\cdot}(t)}. \tag{3.14}$$

Then Kitagawa's formula is applied, obtaining two terms as in (3.7) and (3.8). This last term involves the differences in rates or change in the function $v_{xz}(t)$ denoted as Δv_{xz}.

The difference of weights in (3.7) is further decomposed using Kitagawa's formulation again. Before doing this calculation, Das Gupta notes that the normalized weight $\frac{w_{xz}}{w_{\cdot\cdot}}$ can be expressed as two terms

$$\frac{w_{xz}(t)}{w_{\cdot\cdot}(t)} = \left(\frac{w_{xz}(t)}{w_{x\cdot}(t)} \frac{w_{\cdot z}(t)}{w_{\cdot\cdot}(t)}\right)^{1/2} \left(\frac{w_{xz}(t)}{w_{\cdot z}(t)} \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}\right)^{1/2}, \tag{3.15}$$

where $(w)^{1/2}$ is the square root of the variable in parentheses. Both terms on the right hand side of (3.15) are the product of two normalized weights. The weights $\frac{w_{xz}(t)}{w_{x\cdot}(t)}$ and $\frac{w_{\cdot z}(t)}{w_{\cdot\cdot}(t)}$ are the two normalized weights for z and $\frac{w_{xz}(t)}{w_{\cdot z}(t)}$and $\frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)}$ are normalized for x. Das Gupta separates the difference of weights $\frac{w_{xz}(t+h)}{w_{..}(t+h)} - \frac{w_{xz}(t)}{w_{..}(t)}$ by applying (3.15), which yields

$$\begin{aligned}
\Delta\frac{w_{xz}}{w_{..}} &= \frac{w_{xz}(t+h)}{w_{..}(t+h)} - \frac{w_{xz}(t)}{w_{..}(t)} \\
&= \left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)}\frac{w_{\cdot z}(t+h)}{w_{..}(t+h)}\right)^{1/2}\left(\frac{w_{xz}(t+h)}{w_{\cdot z}(t+h)}\frac{w_{x\cdot}(t+h)}{w_{..}(t+h)}\right)^{1/2} \\
&\quad - \left(\frac{w_{xz}(t)}{w_{x\cdot}(t)}\frac{w_{\cdot z}(t)}{w_{..}(t)}\right)^{1/2}\left(\frac{w_{xz}(t)}{w_{\cdot z}(t)}\frac{w_{x\cdot}(t)}{w_{..}(t)}\right)^{1/2}. \qquad (3.16)
\end{aligned}$$

Let the difference of the two normalized weights for z,

$$\Delta\frac{w_{xz}}{w_{x\cdot}}\frac{w_{\cdot z}}{w_{..}} = \left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)}\frac{w_{\cdot z}(t+h)}{w_{..}(t+h)}\right)^{1/2} - \left(\frac{w_{xz}(t)}{w_{x\cdot}(t)}\frac{w_{\cdot z}(t)}{w_{..}(t)}\right)^{1/2},$$

account for the change in the average due to the z factor. While the difference in the normalized weights for x accounts for the change in the average of interest due to the x factor,

$$\Delta\frac{w_{xz}}{w_{\cdot z}}\frac{w_{x\cdot}}{w_{..}} = \left(\frac{w_{xz}(t+h)}{w_{\cdot z}(t+h)}\frac{w_{x\cdot}(t+h)}{w_{..}(t+h)}\right)^{1/2} - \left(\frac{w_{xz}(t)}{w_{\cdot z}(t)}\frac{w_{x\cdot}(t)}{w_{..}(t)}\right)^{1/2}.$$

Applying Kitagawa's formula again to (3.16) these two differences are obtained

$$\begin{aligned}
\Delta\frac{w_{xz}}{w_{..}} &= \frac{\left(\frac{w_{xz}(t+h)}{w_{\cdot z}(t+h)}\frac{w_{x\cdot}(t+h)}{w_{..}(t+h)}\right)^{1/2} + \left(\frac{w_{xz}(t)}{w_{\cdot z}(t)}\frac{w_{x\cdot}(t)}{w_{..}(t)}\right)^{1/2}}{2}\Delta\frac{w_{xz}}{w_{x\cdot}}\frac{w_{\cdot z}}{w_{..}} \qquad (3.17) \\
&\quad + \frac{\left(\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)}\frac{w_{\cdot z}(t+h)}{w_{..}(t+h)}\right)^{1/2} + \left(\frac{w_{xz}(t)}{w_{x\cdot}(t)}\frac{w_{\cdot z}(t)}{w_{..}(t)}\right)^{1/2}}{2}\Delta\frac{w_{xz}}{w_{\cdot z}}\frac{w_{x\cdot}}{w_{..}}. \qquad (3.18)
\end{aligned}$$

This decomposition also has three elements. The first two elements come from substituting (3.17) and (3.18) in the term (3.7) of the first employment of Kitagawa's formula. First the change in the normalized weights $w_{xz}(t)$ due to z, (3.17) and expressed as $\Delta\frac{w_{xz}}{w_{x\cdot}}\frac{w_{\cdot z}}{w_{..}}$. Second the change in the normalized weights $w_{xz}(t)$ due to x, (3.18) and expressed as $\Delta\frac{w_{xz}}{w_{\cdot z}}\frac{w_{x\cdot}}{w_{..}}$. The third element is the differences in the function $v_{xz}(t)$ denoted as Δv_{xz}.

Table 3.4 presents the CDR of a selection of European countries using Das Gupta's decomposition. This decomposition also gives similar results as those in Tables 3.2 and 3.3. The similarity arises from the fact that Das Gupta's decomposition employs Kitagawa's formula in order to separate the change in the average. It could also be that all the methods correctly allocate the contribution of the components to the total change. In the last section of this chapter for illustration purposes we have included a decomposition used in economics which helps clarify some of the questions arising from the sections above.

3.4.4 Structural Decomposition Analysis

The close ties between economics and demography are well known, and they have mutually benefited from advances in both sciences. Much of the methodology found in demography is

Table 3.4: Crude death rate, $\bar{d}_E(t)$, per thousand, and Das Gupta's decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996. The crude death rate is the average of crude death rates of selected European countries.

t	1960	1975	1992
$\bar{d}(t)$	10.720	10.592	10.917
$\bar{d}(t+h)$	10.892	10.965	11.596
$\Delta\bar{d}(t)$	0.0172	0.0374	0.1697
$\Delta\frac{N_{ac}}{N_{\cdot c}}\frac{N_{a\cdot}}{N_{\cdot\cdot}}$	0.1115	0.1459	0.1286
$\Delta\frac{N_{ac}}{N_{a\cdot}}\frac{N_{\cdot c}}{N_{\cdot\cdot}}$	-0.0010	0.0006	-0.0061
Δm_{ac}	-0.0933	-0.1092	0.0472
$\Delta\bar{d}(t) = \Delta\frac{N_{ac}}{N_{\cdot c}}\frac{N_{a\cdot}}{N_{\cdot\cdot}} + \Delta\frac{N_{ac}}{N_{a\cdot}}\frac{N_{\cdot c}}{N_{\cdot\cdot}} + \Delta m_{ac}$	0.0172	0.0374	0.1697

Source: Author's calculations from formula (3.18), based on the Human Mortality Database (2002). For the years 1960 and 1975 eleven-year periods were used (1960-1970 and 1975-1985), $h = 10$. For the year 1992 a five-year period was used (1992-1996), $h = 4$. The countries included are: Austria, Bulgaria, Finland, France, east and west Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

also used in economics. The case of decomposition techniques is no exception to this methodological overlap (see Rose and Casler (1996) for a review of the literature on decomposition methodology in economics).

In economics the decomposition methods date back to Leontief who, in 1953, analyzed the change in the structure of production (cited by Dietzenbacher and Los (1998)). The first attempts in decomposition methodology were simultaneously undertaken in demography by Kitagawa (1955). Kitagawa's proposal is analogous to the average of the two *polar* decompositions, introduced by Oosterhaven and Van der Linden (1997) and explained here below. Even though Kitagawa (1955) uses population growth rates, fertility rates and mortality rates, one can identify the analogy to the formulations mentioned by Oosterhaven and Van der Linden (1997). This approach, suggested by Oosterhaven and Van der Linden (1997) for economic variables, has been further applied by Oosterhaven and Hoen (1998), Dietzenbacher and Los (1998, 2000) and Dietzenbacher et al. (2000b).

Polar decomposition, as named by Dietzenbacher and Los (1998), is also concerned with the three differences mentioned earlier, i.e., Δv_{xz}, $\Delta\frac{w_{xz}}{w_{x\cdot}}$ and $\Delta\frac{w_{x\cdot}}{w_{\cdot\cdot}}$, for rates and normalized weights, respectively. The idea behind *polar* decomposition is to let the weights move from one time t, the first pole, to another time $t+h$, or the second pole. In the case of equation (3.12) they suggest the following:

$$\tilde{\bar{v}}(t+h) - \tilde{\bar{v}}(t) = \sum_{x,z}\Delta\left[v_{xz}\right]\frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)}\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} +$$

$$\sum_{x,z} v_{xz}(t)\Delta\left[\frac{w_{xz}}{w_{x\cdot}}\right]\frac{w_{x\cdot}(t+h)}{w_{\cdot\cdot}(t+h)} + \sum_{x,z} v_{xz}(t)\frac{w_{xz}(t)}{w_{x\cdot}(t)}\Delta\left[\frac{w_{x\cdot}}{w_{\cdot\cdot}}\right], \tag{3.19}$$

where for Δv_{xz} the weights are at time $t+h$, consecutively the weights for $\Delta \frac{w_{xz}}{w_{x\cdot}}$ are at both times, and for the last difference the weights are at time t. But the opposite could be constructed starting with time t and finishing with time $t+h$,

$$\tilde{\bar{v}}(t+h) - \tilde{\bar{v}}(t) = \sum_{x,z} \Delta \left[v_{xz} \right] \frac{w_{xz}(t)}{w_{x\cdot}(t)} \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)} +$$

$$\sum_{x,z} v_{xz}(t+h) \Delta \left[\frac{w_{xz}}{w_{x\cdot}} \right] \frac{w_{x\cdot}(t)}{w_{\cdot\cdot}(t)} + \sum_{x,z} v_{xz}(t+h) \frac{w_{xz}(t+h)}{w_{x\cdot}(t+h)} \Delta \left[\frac{w_{x\cdot}}{w_{\cdot\cdot}} \right]. \qquad (3.20)$$

Since neither (3.19) nor (3.20) is preferable, the optimal solution is their average.

Table 3.5 presents the crude death rate of selected European countries by using Oosterhaven and Van der Linden's *polar* decomposition. Here again, results similar to those in the other

Table 3.5: Crude death rate, $\bar{d}_E(t)$, per thousand, and Oosterhaven and Van der Linden's decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996. The crude death rate is the average of crude death rates of selected European countries.

t	1960	1975	1992
$\bar{d}_E(t)$	10.720	10.592	10.917
$\bar{d}_E(t+h)$	10.892	10.965	11.596
$\Delta\bar{d}_E(t)$	0.0172	0.0374	0.1697
$\Delta\frac{N_{ac}}{N_{\cdot c}}$	0.1105	0.1434	0.1296
$\Delta\frac{N_{\cdot c}}{N_{\cdot\cdot}}$	0.0000	0.0031	-0.0071
Δm_{ac}	-0.0933	-0.1092	0.0472
$\Delta\bar{d}_E(t) = \Delta\frac{N_{ac}}{N_{\cdot c}} + \Delta\frac{N_{\cdot c}}{N_{\cdot\cdot}} + \Delta m_{ac}$	0.0172	0.0374	0.1697

Source: Author's calculations from formulas (3.19) and (3.20), based on the Human Mortality Database (2002). For the years 1960 and 1975 eleven-year periods were used (1960-1970 and 1975-1985), $h = 10$. For the year 1992 a five-year period was used (1992-1996), $h = 4$. The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

tables of this chapter, are found. In all the decompositions used in this chapter we obtain the same contributions of the components. If there are variations among the different methods these are only found in the third digit after the decimal point of values per thousand.

Further decompositions were carried out by Dietzenbacher and Los (1998) by exchanging the times of the weights for the years (t or $t+h$) in (3.19) and (3.20) obtaining a total of n factorial (equal to the product of $n(n-1)(n-2)...(3)2$) options of decompositions for n factors. The results of all these decompositions do not differ from each other. As the average of all these decompositions does not provide more information than the average of *polar* decompositions, the latter is preferred. Dietzenbacher and Los (1998) suggest including the standard deviation of those decompositions together with the final average. Recently, analogous decompositions were suggested by Andreev et al. (2002) with applications to demography.

Structural decomposition is the last decomposition presented in this chapter. Because it rounds the debate on allocating the change in a demographic variable into additive components. we included this methodology at the end, followed by some concluding comments.

3.5 Conclusion

In this chapter we introduced the origins of decomposition methods and some of the early formulations. Kitagawa's (1955) technique works well when the demographic variables involved only depend on two variables x and t, for example age a and time t. Consequently the variables in the average are $v_a(t)$ and the normalized weighting function $\frac{w_a(t)}{w_\cdot(t)}$. When the studied average involves more variables, like in the average in equation (3.12) for example, then a more complex formulation is needed. The methodologies of Cho and Retherford (1973), Kim and Strobino (1984) and Das Gupta (1978) have been used in demography as a solution in the event of such a situation.

The decompositions formulated by Cho and Retherford, Kim and Strobino and Das Gupta are based on Kitagawa's formula involving arithmetic manipulation of the difference of averages. As a consequence the components of these decompositions include similar differences for rates and normalized weights, namely Δv_{xz}, $\Delta \frac{w_{xz}}{w_{x\cdot}}$ and $\Delta \frac{w_{x\cdot}}{w_{\cdot\cdot}}$, and therefore the results are also similar. As a further option we have also looked here at the structural decomposition used in economics. Oosterhaven and Van der Linden (1997) present the *polar* decomposition and Dietzenbacher and Los (1998) many other arithmetic separations of a change with results that do not vary from each other.

All these methods agree on the differences that are incorporated into the components of the decomposition. We can conclude that the contribution of all these methods pertains to the effort to find the correct set of weights that should multiply these differences. They are also the base of new developments in decomposition methodology which have inspired much of the work presented later.

A second conclusion refers to the main focus of this research. Our interest is to add to the body of knowledge regarding the effect of changes in the components on the total aggregated index, and also to understand the relations between the different components of the demographic average. To achieve both objectives, in Part III of the book we have used, instead of differences, a continuous approach of change over time of demographic averages.

Other studies have also applied decomposition techniques to analyze mortality, fertility, migration or population growth. Although, these studies are also of interest here, they focus on population phenomena rather than on methodology. However many of these decompositions are exclusive to the themes under study. In the next chapter we look at other applications of decomposition methods.

4

Applications of Decomposition Methods

4.1 Introduction

Many of the contributions which improved decomposition methods were developed through the study of particular areas of demography such as mortality, fertility and population growth. This chapter examines some of the studies that contributed to these decompositions. Several illustrative examples are presented here, which later are compared with direct vs. compositional decomposition presented in Part III.

As shown in the previous chapter, some authors analyze mortality by decomposing the difference between crude death rates into several components. Others instead look at the trends of mortality by examining the change in life expectancy and its decomposition over time or between populations or sexes. Among those that have contributed to these studies are Keyfitz (1985), Mitra (1978), a United Nations report (1982), Pollard (1982; 1988), Andreev (1982), Arriaga (1984), Pressat (1985), Vaupel (1986), Goldman and Lord (1986) and Valkovics (in Wunsch (2002)).

Demographic variables such as the total fertility rate, the crude birth rate, the mean parity and other measures of fertility have also been decomposed. The different components of these decompositions explain the change in fertility patterns. Among the authors that have studied the elements of the fertility change are Anderson (1975), Bongaarts (1978), Bongaarts and Potter (1983), Hobcraft and Little (1984), Cutright and Smith (1988), Nathanson and Kim (1989), Pullum et al. (1989), Zeng et al. (1991), Bongaarts (1993), Gertler and Molyneaux (1994), Smith et al. (1996) and Kohler and Ortega (2002).

Among the authors who have carried out research on the population growth rate and its components are Keyfitz (1985), Bennett and Horiuchi (1981), Preston and Coale (1982), Arthur and Vaupel (1984), Kim (1986), Horiuchi and Preston (1988), Kephart (1988), Caselli and Vallin (1990), Horiuchi (1995), Otani (1997) and Vaupel and Canudas Romo (2000).

Many applications of the decomposition methods have influenced the development of new methods of separating components. In the following sections we look at three areas of demography where these extensions of decomposition techniques have been developed.

4.2 Decomposition of Mortality Measures

Measures of the mortality level of a population such as the crude death rate, average age at death, or other age-aggregated indexes are affected by changes in the age structure of the population. In order to avoid confounding results due to the age structure when comparing over time, studies have analyzed mortality by applying decompositions of the kind shown in Chapter 3. These methods are generally characterized as the analysis of the difference of two age-aggregated indexes into several components.

Other attempts to describe the changes in mortality are, for example, found in the work of Keyfitz (1989). He applies a three-dimensional chart to account for past birth and cohort survivorship variation. Some studies have examined the changes in mortality based on parametric models. For example, Horiuchi and Wilmoth (1998) present a period and cohort decomposition of the gamma-Gompertz-Makeham models to demonstrate the force of mortality. Here the interest is focused on changes over time of demographic variables.

Other studies have analyzed the trends of mortality by examining the differences in life expectancy over time. For example, a United Nations report (1982) uses Kitagawa's approach to study the contribution by age group of the difference in life expectancy. The following section describes some of the studies of changes over time in life expectancy.

4.2.1 Decomposing Life Expectancy

This section starts with the definition of life expectancy and other lifetable functions. Then, four methods are shown together with their corresponding contributors and formulations. Next Arriaga's decomposition of the change in life expectancy and Pollard's cause-specific decomposition are applied.

Letting the radix of the lifetable be equal to one, $\ell(0,t) = 1$, life expectancy at birth at time t can be expressed as

$$e^o(0,t) = \int_0^\omega \ell(a,t)da, \tag{4.1}$$

where $\ell(a,t)$ is the lifetable probability at time t of surviving from birth to age a, and ω is the highest age attained. The lifetable probability of surviving from birth to age a is also a function of the sum of the force of mortality function between those ages

$$\ell(a,t) = e^{-\int_0^a \mu(x,t)dx}, \tag{4.2}$$

as before, $\mu(a,t)$ is the force of mortality, or hazard, at age a and time t. Because the radix of the lifetable is equal to one the integrated force of mortality, or cumulative hazard, $-\int_0^a \mu(x,t)dx$ acts as the growth rate of $\ell(0,t)$ from age 0 to age a.

Another term should be defined is $\rho(a,t)$, the rate of progress in reducing mortality rates, $\rho(a,t) = -\acute{\mu}(a,t)$, where the acute accent is the relative derivative as specified in Section 2.4.

The person-years lived between two ages x_a and x_{a+1} can be calculated as

$$L(x_a, x_{a+1}, t) = \int_{x_a}^{x_{a+1}} \ell(a,t)da, \tag{4.3}$$

and the person-years above a certain age a as

$$T(a,t) = \sum_{x_a=a}^{\omega} L(x_a, x_{a+1}, t). \tag{4.4}$$

The work of Keyfitz (1985) on the change over time in life expectancy corresponds to equal proportional changes in mortality at all ages. Keyfitz assumes constant improvements in mortality at all ages, $\rho(a,t) = \rho(t)$ for all a. The relative change in life expectancy is then

$$\acute{e}^o(0,t) = \rho(t)\mathcal{H}(t), \tag{4.5}$$

where $\mathcal{H}(t)$ denotes the entropy of the survival function

$$\mathcal{H}(t) = -\frac{\int_0^\omega \ell(a,t) \ln [\ell(a,t)]\, da}{\int_0^\omega \ell(a,t)da}. \tag{4.6}$$

The entropy function $\mathcal{H}(t)$ is a measure of the concavity of the survival curve. For example, if in a population deaths are concentrated in a narrow range of ages then $\mathcal{H}(t) = 0$. If age-specific death rates are constant at all ages then $\mathcal{H}(t) = 1$. Equation (4.5) shows how the logarithm of the survival curve $\ln [\ell(a,t)]$ contains information about the effects of changes in death rates on life expectancy.

Mitra (1978), Goldman and Lord (1986), Vaupel (1986) and Vaupel and Canudas Romo (2003) further developed this approach. A generalization of (4.5) by Vaupel and Canudas Romo (2003) is presented in Part III of this book.

Pollard (1982) expands the exponential function in life expectancy through the powers of the force of mortality in equation (4.2): $\int_0^a \mu(x,t)dx$. Let the times be t and $t+h$. Pollard partitions the changes in the expectation of life as a result of mortality changes by age contributions as

$$\begin{aligned} e^o(0,t+h) - e^o(0,t) &= \int_0^\omega [\ell(a,t+h) - \ell(a,t)]\, da \\ &= \int_0^{x_1} [\mu(a,t) - \mu(a,t+h)]\, w(a)da + \\ &\quad ... + \int_{x_n}^{\omega} [\mu(a,t) - \mu(a,t+h)]\, w(a)da \end{aligned} \tag{4.7}$$

where 0, x_1,..., x_n, and ω are ages and the weights are defined as

$$w(a) = \frac{1}{2}\left[\ell\,(a,t)\, e^o\,(a,t+h) + \ell\,(a,t+h)\, e^o\,(a,t)\right]. \tag{4.8}$$

Pollard's equation (4.7) is useful to analyze the contribution of mortality improvements in the various age groups.

In the 1980s, Andreev (1982) (cited by Andreev et al. (2002)), Arriaga (1984) and Pressat (1985) independently developed a decomposition of the difference in life expectancies. We present them here as Arriaga's decomposition. The change in life expectancy is divided into direct, indirect and combined effects, by age categories as follows:

$$\begin{aligned} e^o(0,t+h) - e^o(0,t) &= [\Delta_D(0,x_1) + \Delta_I(0,x_1)] + ... \\ &\quad + [\Delta_D(x_n,\omega) + \Delta_I(x_n,\omega)], \end{aligned} \tag{4.9}$$

where the direct component from age x_a to x_{a+1} is defined as

$$\Delta_D(x_a,x_{a+1}) = \frac{\ell(x_a,t)}{\ell(0,t)} \left(\frac{L(x_a,x_{a+1},t+h)}{\ell(x_a,t+h)} - \frac{L(x_a,x_{a+1},t)}{\ell(x_a,t)} \right), \tag{4.10}$$

and the indirect, which also includes an interaction component, is

$$\Delta_I(x_a,x_{a+1}) = \frac{T(x_{a+1},t+h)}{\ell(0,t)} \left(\frac{\ell(x_a,t)}{\ell(x_a,t+h)} - \frac{\ell(x_{a+1},t)}{\ell(x_{a+1},t+h)} \right). \tag{4.11}$$

The direct effect (4.10) in the ages x_a to x_{a+1} is the change in life expectancy as a consequence of change in the number of years lived within that particular age group. The additional number of years is due to an alteration of mortality between ages x_a and x_{a+1}. Changes in mortality in each age group produce a different number of survivors at the end of the age interval. The indirect effect, here shown together with the interaction component, describes the changes in life expectancy due to the exposure of the additional survivors to new mortality conditions. Pollard (1988) shows that his formulation is analogous to Arriaga's.

Table 4.1 shows the application of equation (4.9) to the annual change in life expectancy at birth for the Swedish population from 1900 to 1905, 1950 to 1955 and 1995 to 2000. Three

Table 4.1: Life expectancy at birth, $e^o(0,t)$, and Arriaga's decomposition of the annual change over time from 1900 to 1905, from 1950 to 1955 and from 1995 to 2000, in Sweden.

t	1900	1950	1995
$e^o(0,t)$	52.239	71.130	78.784
$e^o(0,t+5)$	54.527	72.586	79.740
$\Delta e^o(0,t)$	0.458	0.291	0.191
Δ_D	0.008	0.010	0.007
Δ_I	0.450	0.282	0.184
$\Delta e^o(0,t) = \Delta_D + \Delta_I$	0.458	0.291	0.191

Source: Author's calculations based on equation (4.9). Lifetable data is derived from the Human Mortality Database (2002). Lifetable values from the years 1900 and 1905, 1950 and 1955, 1995 and 2000, were used to obtain the results.

six-year periods were chosen for the calculations, at the beginning of the twentieth century, 1900 - 1905, in the middle of the century, 1950 - 1955, and at the end of the century, 1995 - 2000. During all periods, gains in life expectancy (measured in years) were achieved. The greatest increases occurred in 1900-1905, followed by 1950-1955 while the most recent years registered the smallest increase. The main contributor to the changes is the component that combines the indirect and the interaction effects with minor participation of the direct effect.

Silber (1992) adapts the Gini concentration ratio to study age distribution of the causes of death. Three components are obtained: the inequality of age at death in each cause of death (the "within causes of death" element), the inequality of the average ages of the different causes of death (the "between causes of death" element), and lastly the overlapping between ages at death in the different causes of death. Similarly, Valkovics (in Wunsch (2002), pp. 79-94) suggests a decomposition of the male-female difference in life expectancies into the different causes of death categories. One last decomposition by cause of death is Pollard's proposed equation

$$e^o(0, t+h) - e^o(0, t) = \sum_{i=1}^{n} \int_0^{\omega} \left[\mu_i(a, t) - \mu_i(a, t+h)\right] w(a) da, \tag{4.12}$$

where $\mu_i(a, t)$ is the force of mortality at age a, time t and cause of death i, and $w(a)$ is as defined in equation (4.8).

Table 4.2 shows the decomposition obtained by combining the two formulas proposed by Pollard (4.7) and (4.12), by age and causes of death respectively. The decomposition is applied to Japan for the years 1980 and 1990. We divided the grouped data into five age groups

Table 4.2: Age and cause of death decomposition for the annual change over time in life expectancy, in percentages, for Japan for the period 1980-1990, following Pollard's decomposition methods.

Causes of death	$\Delta e^o(0, 1980) = 2.880$					
$\mu_i(a, t)$	0 - 1	1 - 9	10 - 50	50 - 69	70 +	*All ages*
Malignant neoplasm	-0.001	0.014	0.077	0.053	0.005	0.148
Heart disease	0.002	0.000	0.033	0.119	0.270	0.424
Cerebrovascular disease	0.001	0.001	0.072	0.348	0.800	1.222
Infectious diseases	0.025	0.010	0.024	0.027	-0.036	0.050
Violent deaths	0.011	0.052	0.092	0.013	0.021	0.190
Stomach, liver and kidney disorders	-0.002	0.001	0.055	0.045	0.042	0.142
Senility without psychosis	0.000	0.000	0.000	0.003	0.260	0.263
Other causes	0.181	0.011	0.051	0.053	0.074	0.370
All causes of death	0.218	0.090	0.406	0.660	1.435	2.809

Source: Author's calculations based on equation (4.7) and (4.12), based on the Berkeley Mortality Database (2001).

and eight causes of death. The cause of death which contributes the most to the increase in life expectancy is cerebrovascular disease while infectious diseases are the most influential inhibitors in life expectancy in the oldest age group. The increase in life expectancy was mainly due to changes in the life expectancy of the population aged seventy and above whereas children between 1 and 9 experienced the smallest change.

The contributions to the increase in the Japanese life expectancy by the different causes of death can be divided into four types. First we have death decreases concentrated mainly in the very young ages in the category of "other causes of death"; then are the improvements located in the age group 1 to 50 years, as seen in violent deaths; another category is concentrated in the age group 10 years and above, as seen in malignant neoplasm, and stomach, liver and kidney disorders. Another category consists of the causes of death which show great improvements in old age groups, among which are heart disease, cerebrovascular disease, and senility without psychosis.

4.2.2 Conclusion

Life expectancy is among the most frequently used demographic measures. Still there is a ongoing debate regarding misleading indications of life expectancy (Bongaarts and Feeney (2002) and Vaupel (2002)). In the absence of a better measure of the current conditions of mortality, life expectancy will continue to play a central role in demography.

The above section has covered some of the studies which have contributed to the decomposition in life expectancy. All these methodologies share the common interest of separating the changes in life expectancy into components of mortality variation, age at death distribution and cause of death distribution.

Criteria about the choice of methodology for our analysis and the priority of a particular decomposition method over the others are provided in Part IV. We also compare the methods of this section with direct vs. compositional decomposition.

4.3 Decomposition of Fertility Measures

This section includes decompositions associated with fertility measures. We start with a discussion on the crude birth rate and how to decompose this measure's change over time. Then we look at changes over time in the total fertility rate.

Crude rates are influenced by the age structure of a population, due to the pervasive association between age and demographic rates.

In Chapter 3, on *Standardization and Decomposition Techniques,* the crude death rate was decomposed in order to separate the confounding effects of the age structure. The decompositions of mortality measures shown in Chapter 3 have also been applied to the study of components of fertility change. Nathanson and Kim (1989), for example, used the decomposition proposed by Kim and Strobino (1984) to study adolescent fertility. Smith et al. (1996) used Das Gupta's (1978) separation techniques to study the fertility of unmarried women.

The crude birth rate (CBR), as expressed in equation (2.8), has the same problem as the crude death rate of confounding effects due to the age structure. To take account of this the CBR has also been decomposed. Zeng et al. (1991) explained the increase in the Chinese CBR between 1984 and 1987 by applying a decomposition. The CBR, as defined in equation (2.8), was modified by Zeng and colleagues to include only the fertility of married women (because in China most births occur in marriage). Let the proportion of married women at age a be denoted as $\pi_{ma}(t)$, and the marital fertility rate as $b_{ma}(t)$, the CBR of married women $CBR_m(t)$ is

$$CBR_m(t) = \frac{\sum_{a=0}^{\omega} b_{ma}(t)\pi_{ma}(t)N_{f,a}(t)}{\sum_{a=0}^{\omega} N_a(t)} = \sum_{a=0}^{\omega} b_{ma}(t)\pi_{ma}(t)\pi_{fa}(t), \tag{4.13}$$

where $\pi_{fa}(t)$ is the proportion of women at age a in the total population, $\pi_{fa}(t) = \frac{N_{fa}(t)}{\sum_{a=0}^{\omega} N_a(t)}$. By using a variant of *polar* decomposition shown in Section 3.4.4, Zeng and colleagues decompose the change in $CBR_m(t)$ into three main effects. The change in $CBR_m(t)$ is due to changes in marital fertility rates Δb_{ma}, changes in the proportion of married women $\Delta\pi_{ma}$, and changes in the proportion of women in the total population $\Delta\pi_{fa}$,

$$CBR_m(t+h) - CBR_m(t) \approx \sum_{a=0}^{\omega} \Delta\left[b_{ma}\right]\pi_{ma}(t+h)\pi_{fa}(t+h)$$

$$+\sum_{a=0}^{\omega} b_{ma}(t+h)\Delta\left[\pi_{ma}\right]\pi_{fa}(t+h) + \sum_{a=0}^{\omega} b_{ma}(t+h)\pi_{ma}(t+h)\Delta\left[\pi_{fa}\right]. \tag{4.14}$$

equation (4.14) is an approximation because the terms of interactions between the differences Δb_{ma}, $\Delta\pi_{ma}$ and $\Delta\pi_{fa}$ are ignored. The conclusion of the research by Zeng et al. (1991) is that the age structure, $\Delta\pi_{fa}$, and the declining age at marriage, seen in $\Delta\pi_{ma}$, are the two main contributors to the increase in the crude birth rate in China. Table 4.3 presents the CBR of married women and the decomposition of the annual change over time, as suggested by Zeng and colleagues, for Denmark, the Netherlands and Sweden from 1992 to 1997.

Table 4.3: Crude birth rate of married women and Zeng's decomposition of the annual change over time for Denmark, the Netherlands and Sweden from 1992 to 1997.

	Denmark	Netherlands	Sweden
$CBR_m(1992)$	0.702	1.134	0.716
$CBR_m(1997)$	0.702	0.997	0.470
ΔCBR_m	0.000	-0.027	-0.049
Δb_{ma}	0.013	0.015	-0.020
$\Delta\pi_{ma}$	-0.013	-0.037	-0.024
$\Delta\pi_{fa}$	0.000	-0.005	0.000
$\Delta CBR_m = \Delta b_{ma} + \Delta\pi_{ma} + \Delta\pi_{fa}$	0.000	-0.027	-0.044

Source: Author's calculations from formula (4.14), based on Eurostat (2000).

The first three rows are for the observed CBR of married women in the selected countries, and the observed change in CBR during this period. The lower part of the table is for the estimated components of the decomposition. Table 4.4 below contains similar calculations but these are applied to the CBR of unmarried women.

In a country like China where most births occur in marriage, equation (4.14) provides good estimates. For European countries it is more appropriate to add a CBR for married women and another for unmarried women in an equation similar to (4.13). This is possible because the total number of births is equal to the births of married and unmarried women. As such, the crude birth rate is equal to the CBR of married women plus the CBR of unmarried women as follows:

$$\begin{aligned} CBR(t) &= \frac{B(t)}{N(t)} = \frac{B_m(t)}{N(t)} + \frac{B_u(t)}{N(t)} \\ &= CBR_m(t) + CBR_u(t), \end{aligned} \tag{4.15}$$

where $B(t)$, as before, denotes the births, and $B_m(t)$ and $B_u(t)$ are births of married and unmarried women, respectively. The expression for the CBR of unmarried women, $CBR_u(t)$, is

$$CBR_u(t) = \frac{\sum_{a=0}^{\omega} b_{ua}(t)\pi_{ua}(t)N_{fa}(t)}{\sum_{a=0}^{\omega} N_a(t)} = \sum_{a=0}^{\omega} b_{ua}(t)\pi_{ua}(t)\pi_{fa}(t), \tag{4.16}$$

where $\pi_{ua}(t)$ corresponds to the proportion of unmarried women at age a.

Table 4.4 presents the CBR for unmarried women and the decomposition of the annual change over time, as suggested by Zeng and colleagues, which is also applied to data from Denmark, the Netherlands and Sweden from 1992 to 1997.

Table 4.4: Crude birth rate of unmarried women and Zeng's decomposition of the annual change over time for Denmark, the Netherlands and Sweden from 1992 to 1997.

	Denmark	Netherlands	Sweden
$CBR_u(1992)$	0.608	0.161	0.701
$CBR_u(1997)$	0.578	0.236	0.553
ΔCBR_u	-0.006	0.015	-0.030
Δb_{ua}	-0.008	0.012	-0.041
$\Delta \pi_{ua}$	0.006	0.007	0.012
$\Delta \pi_{fa}$	-0.003	-0.003	-0.002
$\Delta CBR_u = \Delta b_{ua} + \Delta \pi_{ua} + \Delta \pi_{fa}$	-0.005	0.016	-0.031

Source: Author's calculations from formula (4.14), based on Eurostat (2000).

The decomposition by Zeng et al. (1991) is an approximation. Adding the results of Tables 4.3 and 4.4 gives us the observed components (first three rows of the tables) and the estimated components of the decomposition in the CBR for both married and unmarried women. For example, the total crude birth rate arising from the addition of the results in Tables 4.3 and 4.4 is compiled for the Netherlands in Table 10.4. The Netherlands is the only country which experiences an increase in the CBR for unmarried women. Sweden had the largest decrease for both groups, and the Netherlands had an important decline in marital CBR. Some general dynamics in the fertility change of these European countries should be noted. The proportion of married women decreases while the proportion of unmarried women increases. The Netherlands experienced an increase in both marital and non-marital age-specific fertility rates. In Sweden, both groups, married and unmarried, experienced a decline in fertility. For the three countries and both populations of women, the change in the age structure, $\pi_{fa}(t)$, is responsible for only a minor contribution to the total change, in contrast to Zeng's results (1991), for China, where it was the most important factor.

Anderson (1975) and Anderson et al. (1977) also present a variant of *polar* decomposition for age-specific birth rates. The age-specific fertility rates are separated into three components for each education class in terms of fertility of married women $b_{ema}(t)$, the proportion of married women $\pi_{ema}(t)$, and the proportion of women of level of education e, $\pi_{ea}(t)$,

$$b_{ma}(t) = \sum_{e \in E} b_{ema}(t)\, \pi_{ema}(t)\, \pi_{ea}(t), \tag{4.17}$$

where $e \in E$ indicates that all the education classes in E are considered. The change over time is then calculated for one of the components while the other terms remain constant as in time t,

$$b_{ma}(t+h) - b_{ma}(t) = \sum_{e \in E} \Delta\left[b_{ema}\right] \pi_{ema}(t)\, \pi_{ea}(t)$$

$$+ \sum_{e \in E} b_{ema}(t)\, \Delta\left[\pi_{ema}\right] \pi_{ea}(t) + \sum_{e \in E} b_{ema}(t)\, \pi_{ema}(t)\, \Delta\left[\pi_{ea}\right] + \epsilon_{ma}, \tag{4.18}$$

where the term ϵ_{ma} is simply the residual between observed and estimated change in $b_{ma}(t)$. The result of all the differences in age-specific fertility rates is summarized in the total fertility rate (TFR).

Analogous to the study of change over time in life expectancy, most of the research has been concentrated on the change in the total fertility rate. Similar to life expectancy, the TFR is an indicator of a hypothetical cohort, which does not depend on the structure of the population. The TFR is the average number of children per woman if the age-specific birth rates remain constant over a long period,

$$TFR = \int_{\alpha}^{\beta} b(a,t)da, \tag{4.19}$$

where α and β are the lower and upper limits of childbearing. In the following section we look at the changes over time of the TFR.

4.3.1 Decomposing the Total Fertility Rate

Bongaarts (1978) presented a decomposition called "the proximate determinants of fertility" of the total fertility rate. The TFR at time t is expressed as an identity involving indexes

$$TFR(t) = C_m(t)C_c(t)C_a(t)C_i(t)TF(t), \tag{4.20}$$

where C_m, C_c, C_a, and C_i represent indexes of the proportion of married women, contraception use, induced abortion and postpartum infecundability. The last component is the total fertility TF, which comprises natural fecundability, spontaneous intrauterine mortality and permanent sterility. Hobcraft and Little (1984) tested Bongaarts' decomposition (4.20) using an individual-level analysis, and arrived at an additive decomposition.

Any change in a population's level of fertility is necessarily caused by a change in one or more of the proximate determinants. Bongaarts and Potter (1983) present a decomposition of the change over time of Bongaarts' proposal (4.20). The trend in the TFR is expressed as

$$\frac{TFR(t+h)}{TFR(t)} = \frac{C_m(t+h)C_c(t+h)C_a(t+h)C_i(t+h)TF(t+h)}{C_m(t)C_c(t)C_a(t)C_i(t)TF(t)}, \tag{4.21}$$

which is then observed as a relative difference expressed in additive terms as

$$T\grave{F}R = \grave{C}_m + \grave{C}_c + \grave{C}_a + \grave{C}_i + T\grave{F} + \epsilon, \tag{4.22}$$

where the grave accent denotes the relative difference as defined in Section 2.4. For example, the proportional change of the TFR between year $t+h$ and year t with respect to year t is $T\grave{F}R(t) = \frac{TFR(t+h)-TFR(t)}{TFR(t)}$. On the right hand side of the equation the other components are similarly defined.

Cutright and Smith (1988) also present a decomposition of ratios for the comparison of different subpopulations. Their suggestion for a decomposition is to look at the logarithm of the ratios in equation (4.21).

Table 4.5 presents the results of applying Bongaarts and Potter's equation (4.22) in the analysis of the change over the period 1975 to 1993/94 of the proximate determinants of the TFR, for Bangladesh. Table 4.5 differs from the rest of the tables in the book in terms of

Table 4.5: Total fertility rate as a product of five factors, $TFR(t)$, and Bongaarts and Potter's decomposition of the annual relative change over the period 1975-1993/94 for Bangladesh.

t	1975	1993/94	*Relative change*
TFR	7.360	4.503	-0.388
C_m	0.850	0.761	-0.105
C_c	0.937	0.610	-0.349
C_a	0.604	0.653	0.081
C_i	1.000	0.971	-0.029
TF	15.300	15.300	0.000
ϵ			0.013
$T\grave{F}R = \grave{C}_m + \grave{C}_c + \grave{C}_a + \grave{C}_i + T\grave{F} + \epsilon$			-0.388

Source: Author's calculations from formula (4.22), based on the data from Islam et al. (1998).

information about the proximate determinants of fertility in 1975 and 1993/94 in the first two columns of values. In the third column are the contributions of each proximate to the total change over the period, $\grave{C}_m$, $\grave{C}_c$, $\grave{C}_a$, $\grave{C}_i$, $T\grave{F}$ and ϵ. The change in use of contraception, $\grave{C}_c$, is the major component of the change in Bangladesh's TFR. A second component which constitutes an important contribution is the change in the proportion of married women. In contrast to the decline in TFR is the increase in the proportion of induced abortions.

Based on the proximate determinants in equation (4.20) Gertler and Molyneaux (1994) present a simplified decomposition, which they applied to study the reduction in Indonesian fertility. The TFR is decomposed into three main components: indexes of marriage and contraception, and a residual for the remaining factors, $\epsilon(t)$,

$$TFR(t) = C_m(t)C_c(t)\epsilon(t). \tag{4.23}$$

To study the change over time Gertler and Molyneaux suggest calculating the derivatives of the logarithms of equation (4.23) with respect to time,

$$\frac{\partial \ln(TFR)}{\partial t} = \frac{\partial \ln(C_m)}{\partial t} + \frac{\partial \ln(C_c)}{\partial t} + \frac{\partial \ln(\epsilon)}{\partial t}. \tag{4.24}$$

Equation (4.24) can be further developed to obtain a general formula. When studying change over time of multiplicative models, similar to expressions (4.20) and (4.23), it is generally advisable to look at the relative change. The change is then decomposed into the contribution of components that explain the total change. In general terms, let a variable be the product of components $v = v_1 v_2 ... v_n$. The intensity of this variable is

$$\acute{v} = \acute{v}_1 + \acute{v}_2 + ... + \acute{v}_n. \tag{4.25}$$

As this decomposition is a particular case of the proposed direct vs. compositional decomposition presented in Part III, more details are provided there.

Applying equation (4.25) to (4.20) results in the decomposition of the change over time of the TFR as the addition of relative changes of the indexes,

$$T\acute{F}R = \acute{C}_m + \acute{C}_c + \acute{C}_a + \acute{C}_i + T\acute{F}. \tag{4.26}$$

Table 4.6 presents the results of equation (4.26) by analyzing the change over time of the proximate determinants of the TFR, as given in equation (4.20), for Bangladesh in the period 1975-1993/94. Table 4.6 has the same structure as Table 4.5. Table 4.6 also leads to the conclusion that the major component of change for the TFR in Bangladesh is the contraception index. It is important to distinguish the differences between Tables 4.5 and 4.6, since there is no residual term in the last table. A residual term is the impossibility of allocating change to the components of the decomposition. Equation (4.26) contains an inherent advantage over (4.22). The relative change is further studied in Part III as an element of direct vs. compositional decomposition.

Another study which included a decomposition of fertility measures is the study by Pullum et al. (1989). These authors looked at the cohort TFR^c, or mean parity of a cohort that has completed fertility, defined as

$$TFR^c = \sum_i i\pi_i^c, \tag{4.27}$$

Table 4.6: Total fertility rate as a product of five factors, $TFR(t)$, and Gertler and Molyneaux's decomposition of the relative change over time in the period 1975-1993/94 for Bangladesh.

t	1975	1993/94	*Relative change*
TFR	7.360	4.503	-2.729
C_m	0.850	0.761	-0.614
C_c	0.937	0.610	-2.385
C_a	0.604	0.653	0.433
C_i	1.000	0.971	-0.163
TF	15.300	15.300	0.000
$\acute{TFR} = \acute{C}_m + \acute{C}_c + \acute{C}_a + \acute{C}_i + \acute{TF}$			-2.729

Source: Author's calculations using formula (4.26) described in Chapter 9, based on the data from Islam et al. (1998).

where π_i^c is the proportion of women at parity i. The mean parity was decomposed by Pullum and colleagues by using the Delta method which is examined in Chapter 5 entitled *Alternative Decomposition Methods*.

Kohler and Ortega (2002) looked at the age-specific and parity-specific fertility rates and proposed a decomposition of these measures into two components. The first component is a level term for each parity during the period, which increases or decreases childbearing at all ages. The second component is a schedule of fertility that determines the age pattern of each parity.

4.3.2 Conclusion

Analogous to the study of mortality in which life expectancy is frequently used, the total fertility rate is the fertility measure that is used more often. The question of how to properly adjust the TFR has sparked recent debate. Among researchers involved in the development of the TFR are Bongaarts and Feeney (1998), Kim and Schoen (2000), Van Imhoff and Keilman (2000) and Kohler and Ortega (2002). Future decompositions of the TFR are expected to account for each of the components of this adjusted TFR.

This section contains some of the works that have contributed to the decomposition methods of the TFR and also of the crude birth rate. The decomposition methods of the CBR have a common interest in separating the changes into components of fertility and the composition of the population. For the TFR we have mainly concentrated on those studies that cover the decomposition of relative changes in the proximate determinants of fertility.

In Part IV these methods are compared with direct vs. compositional decomposition.

4.4 Decomposition of Growth Measures

One of the basic aspects in demography is the balancing equation of population change. There are only four possible ways of entering or leaving a population. The population at time t, $N(t)$, is the result of the population at the beginning, time 0, $N(0)$, plus the births occurring between these years, $B(t)$, minus the deaths, $D(t)$, and plus the net migration (immigration minus emigration), $I(t)$. The changes in the size of the population must be attributable to the magnitude of these flows, expressed as follows:

$$N(t) = N(0) + B(t) - D(t) + I(t). \tag{4.28}$$

Expressing equation (4.28) in term of rates yields

$$r(t) = CBR(t) - CDR(t) + i(t). \tag{4.29}$$

The crude growth rate, $r(t) = \acute{N}(t) = \frac{dN(t)/dt}{N(t)}$, is equal to the crude birth rate, $CBR(t)$, minus the crude death rate, $CDR(t)$, plus the crude net migration, $i(t)$. In other words, the change in population size is decomposed into change due to the flows of births, deaths and net migration.

Table 4.7 presents the figures obtained by applying (4.29) in the period 1985-1995 for France, Japan and the USA. Because the change occurs continuously from the initial point of 1985 to the final moment of 1995, we choose to study the change at the mid-point or mid-year, which is 1990. In all the tables of this section the values of the averages at all these years are displayed together with the observed change at the mid-year. Japan has the lowest relative

Table 4.7: The population growth rate, $r(t)$, rates in percentages, and the balancing equation in 1985-1995 for France, Japan and the USA.

	France	*Japan*	*USA*
$N(1990)$	56,570,545	121,907,530	248,940,972
$N(1985)$	55,157,223	119,740,316	236,872,781
$N(1995)$	58,020,081	124,113,968	261,624,012
$r(1990)$	0.506	0.359	0.994
$CBR(1990)$	1.333	1.050	1.589
$CDR(1990)$	0.938	0.664	0.871
$i(1990)$	0.111	-0.027	0.275
$r(1990) = CBR - CDR + i$	0.506	0.359	0.994

Source: Author's calculations from formula (4.29) described in Chapter 9. Data from Human Mortality Database (2002).

increase in population, $r(1990)$, while the highest increase is seen in the USA. Among the components of the balancing equation, the CBR accounts for the greatest rate in all countries. Again the country with the lowest level, of CBR is Japan and USA has the highest CBR level. The second most important component is the crude death rate which tempers the increase in the growth rate, due to the negative sign in (4.29). Here again, we find that Japan is the country with the lowest level of mortality while France has the highest. The growth rate of the Japanese population is also reduced due to emigration. Both the French and the American populations are increasing due to a positive net migration rate.

New research has been developed in the area of decomposition of the growth rate, the start of which can be attributed to Bennett and Horiuchi (1981). Instead of looking at the population growth rate, $r(t)$, they propose looking at the age-specific growth rates, $r(a,t) = \frac{\partial N(a,t)}{N(a,t)\partial t}$. The age-specific growth rate at age a and time t, $r(a,t)$, is the relative change over time in the population size. Another way of calculating this relative change over time is to measure the relative change over age and the relative change over cohort together. The age-specific growth rate can then be expressed as

$$r(a,t) = \nu(a,t) - \mu(a,t), \tag{4.30}$$

where $\nu(a,t)$ is the relative change in the rate as age increases $\nu(a,t) = -\frac{\partial N(a,t)}{N(a,t)\partial a}$, and $\mu(a,t)$ is, as above, the force of mortality or change over cohort in the population size, $\mu(a,t) = -\frac{\partial N(a+c,t+c)}{N(a,t)\partial c}$. In this equation and others of this section, net migration is omitted. It could easily be included as a way of exiting (or entering) the population at any age, that is, treated in the same way as the force of mortality.

Two articles look at this new idea. Preston and Coale (1982) generalized Lotka's (1939) fundamental equations for any population by using equation (4.30). Their base identity is the relation between the populations at age 0 and a, both at time t, so that

$$\begin{aligned} N(a,t) &= N(0,t)e^{-\int_0^a r(x,t)dx}e^{-\int_0^a \mu(x,t)dx} \\ &= B(t)R(a,t)s(a,t), \end{aligned} \tag{4.31}$$

where $R(a,t) = e^{-\int_0^a r(x,t)dx}$ is the accumulated growth rate from age 0 to a, and $s(a,t) = e^{-\int_0^a \mu(x,t)dx}$ is the period survival function from 0 to a.

Arthur and Vaupel (1984) developed a second system of formulas also relating populations at age 0 and a by

$$N(a,t) = N(0,t)e^{-\int_0^a r(0,t-x)dx}e^{-\int_0^a \mu(x,t-a+x)dx}, \tag{4.32}$$

where $\mu(x, t-a+x)$ is the cohort force of mortality, and $r(0, t-x)$ is the growth rate of births at time $t-x$. For calculations of these systems Kim (1986) presented the formulas in discrete time and ages which are reviewed in Chapter 9.

The systems of Preston and Coale, and Arthur and Vaupel are decompositions of the population size. While equation (4.31) uses a period decomposition, (4.32) uses a cohort decomposition. An application of their methods was performed by Otani (1997) who substituted (4.32) and a variant of (4.31) in the difference of age distribution and in the average age of the Japanese population.

Arthur and Vaupel also presented a decomposition of the age-specific growth rates by combining both systems as follows,

$$\begin{aligned} r(a,t) &= r(0,t-a) - \int_0^a \frac{\partial \mu(x,u)}{\partial u}|_{u=t-a+x}dx \\ &= r(0,t-a) - \varphi(a,t). \end{aligned} \tag{4.33}$$

Horiuchi and Preston (1988) interpreted this formula as the legacy of past population dynamics. The age-specific growth rates are decomposed into two terms: the growth rate of the number of births $t-a$ years earlier and the accumulation of changes in the cohort age-specific mortality rates up to age a at time t, denoted by $\varphi(a,t)$.

Table 4.8 presents the age-specific growth rates for the ages 20, 50 and 80 years for France and the decompositions shown in equations (4.30) and (4.33). From Table 4.7 we learned

Table 4.8: Age-specific growth rates, $r(a,t)$, rates in percentages, and decompositions as suggested in the systems of Preston and Coale, and Arthur and Vaupel for France at ages 20, 50 and 80 in 1990.

a	20	50	80
$N(a,1990)$	8,555,162	5,920,990	1,820,266
$N(a,1985)$	8,531,224	6,202,743	1,598,866
$N(a,1995)$	8,579,167	5,652,035	2,072,324
$r(a,1990)$	0.056	-0.930	2.594
$\nu(a,1990)$	0.061	-0.474	8.600
$\mu(a,1990)$	0.005	0.456	6.006
$r(a,1990)=\nu-\mu$	0.056	-0.930	2.594
$r(0,1990-a)$	-1.114	3.349	-2.707
$\varphi(a,1990)$	-1.170	4.278	-5.301
$r(a,1990)=r-\varphi$	0.056	-0.930	2.594

Source: Author's calculations of equations (4.30) and (4.33) as described in Chapter 9. Age groups are of ten years starting from the age indicated. Data based on the Human Mortality Database (2002).

that in 1990 France had a low population growth rate. In Table 4.8 one can see the different levels of age-specific growth rates for the French population. France has an elderly population. We see this in the minor relative increase of the population size of persons aged 20 while the population size of those aged 80 exhibits a very high growth rate. However the use of the relative change in population size as age increases, $\nu(a,1990)$ is still not completely established in demography. We could say that is a measure of the difference in cohort sizes and past mortality. The $\mu(a,1990)$ accounts for attrition due to mortality and migration. The age group 80-90 years shows a very high mortality rate compared to the other age groups. From the second decomposition we see that those aged 20 and 80 display decreasing birth growth rates. In other words, they come from cohorts of babies $B(t-a)$ that were smaller than their neighboring cohort $B(t-a+1)$. The change in cohort mortality is seen in the results of $\varphi(a,1990)$. The age group between 50 and 60 years is from a cohort that experienced cohort survival situations that were worse than successive cohorts. As a result of this the $\varphi(50,1990)$ implies a decline in the age-specific growth rate $r(50,1990)$.

Decompositions (4.30) and (4.33) can be substituted whenever the age-specific growth rates are present. For example, Caselli and Vallin (1990) substitute equation (4.33) in the derivative over time of the age distribution of the population. In Part III it is shown that direct vs. compositional decomposition involves age-specific growth rates, which allows substitutions of equations (4.30) and (4.33) at any time.

4.4.1 Decomposing the Crude Growth Rate

The relation between the crude growth rate and age-specific growth rates is an average of the age-specific rates weighted by the population size $N(a,t)$,

$$\bar{r}(t) = \frac{\int_0^\omega r(a,t)N(a,t)da}{\int_0^\omega N(a,t)da} \tag{4.34}$$

Keyfitz (1985) assumes that the age-specific growth rates are fixed over time, and by examining the derivative over time of equation (4.34) he obtains

$$\dot{\bar{r}}(t) = \frac{\int_0^\omega r^2(a)N(a,t)da}{\int_0^\omega N(a,t)da} - \left(\frac{\int_0^\omega r(a)N(a,t)da}{\int_0^\omega N(a,t)da}\right)^2 = \sigma^2(r). \tag{4.35}$$

Equation (4.35) informs us that the change in the mean rate of increase at time t is equal to the variance among the age group rates of increase. Kephart (1988) uses equation (4.35) to study the heterogeneous character of aggregate rates of spatial units of analysis.

Table 4.9: Population growth rate of the world, $r(t)$, and Keyfitz's estimation of the annual change, around January 1, 1979 and around January 1, 1982.

t	1979	1982
$\bar{r}(t-1.5)$	1.717 %	1.697 %
$\bar{r}(t+1.5)$	1.741 %	1.722 %
$\dot{\bar{r}}(t)$	0.798 *	0.832 *
$\dot{\bar{r}} = \sigma^2(r)$	0.798 *	0.832 *

Source: Author's calculations of formula (4.35) as described in Chapter 9. Data is based on the U.S. Census Bureau (2001). Note: * denotes per 10,000. Growth rates for every country were assumed fixed over intervals of 8 years (1975-1983 and 1978-1986) for the columns of 1979 and 1982, respectively. The growth rates of the world for 1977.5, 1980.5 and 1983.5 were calculated as the averages in formula (4.34). Growth rates were estimated based on data for all the countries of the world for which data were available.

As shown in Table 4.9, it is possible to obtain a good approximation of the change in the growth rate by using Keyfitz's formulation. Nevertheless, the assumption of fixed growth rates for the examined countries is somewhat problematic. For example, the value in the row of $\bar{r}(t+1.5)$ for the column 1979 and in the row $\bar{r}(t-1.5)$ for the column 1982 should have the same estimates of the year 1980.5, but instead are different. In Part III we show a generalization of the formula proposed by Keyfitz for countries' growth rates that change over time.

The total population size comes from the addition of the population sizes at all ages, $N(t) = \int_0^\omega N(a,t)da$. By substituting the population size at age a for the Preston and Coale system and recalling the definition of the population growth rate, $r(t) = \acute{N}(t)$, we obtain a decomposition. By applying the general equation (4.25) Vaupel and Canudas Romo (2000) show that the current population growth rate can be decomposed into three components,

$$r(t) = \acute{B}(t) + \acute{e}_o(t) + R^*(t), \tag{4.36}$$

where $\acute{B}(t)$ is the intensity of births, $\acute{e}_o(t)$ is the intensity of life expectancy and $R^*(t)$ is a residual equal to

$$\begin{aligned} R^*(t) &= \frac{\int_0^\omega \left[\acute{R}(a,t) + \acute{s}(a,t)\right] N(a,t)da}{\int_0^\omega N(a,t)da} - \frac{\int_0^\omega \dot{s}(a,t)da}{\int_0^\omega s(a,t)da} \\ &= \bar{\acute{R}}(t) + \bar{\acute{s}}(t) - \acute{e}_o(t). \end{aligned} \tag{4.37}$$

Equation (4.36) permits a decomposition of the current population growth rate into three components. The first two are the current intensity of change in births and the current intensity of change in period life expectancy (which captures the impact of current mortality change). The third is a residual term which reflects the influence of historical fluctuations. These fluctuations result in a population size and structure that is different from the stationary population size and structure implied by current mortality and birth rates.

Table 4.10 presents the application of equation (4.36) to France, Japan and the USA. These data were also used in Table 4.7. In the period under study, France and Japan both experienced

Table 4.10: The population growth rate, $r(t)$, rates in percentages, and Vaupel and Canudas Romo's decomposition of the growth rates in the period 1985-1995 for France, Japan and the USA.

	France	*Japan*	*USA*
$N(1990)$	56,570,545	121,907,530	248,940,972
$N(1985)$	55,157,223	119,740,316	236,872,781
$N(1995)$	58,020,081	124,113,968	261,624,012
$r(1990)$	0.506	0.359	0.994
$\acute{B}(1990)$	-0.777	-1.450	0.498
$\acute{e}_o(1990)$	0.325	0.263	0.146
$R^*(1990)$	0.958	1.546	0.349
$r(1990) = \acute{B} + \acute{e}_o + R^*$	0.506	0.359	0.994

Source: Author's calculations of formula (4.36) as described in Chapter 9. Data from the Human Mortality Database (2002).

a low fertility rate but the growth rate remained positive. In contrast to the negative intensity of change in birth is the positive change in life expectancy. By applying the Vaupel-Canudas decomposition one learns that new births are being substituted by longer life. For the USA the same conclusion cannot be reached since all three components contribute to an increase in the population size.

By using a variant of equation (4.32), Horiuchi (1995) shows a similar application of the general equation (4.25). Horiuchi also obtains a decomposition of the population growth rate.

The population size at age a and time t is now the product of the total population at time $t-a$, the crude birth rate at time $t-a$ and the cohort survival (mortality and net migration)

$$\begin{aligned} N(a,t) &= N(t-a)\left[\int_{\alpha}^{\beta} \pi_f(x,t-a)b(x,t-a)dx\right] e^{-\int_0^a \mu(x,t-a+x)dx} \\ &= N(t-a)CBR(t-a)s_c(a,t), \end{aligned} \tag{4.38}$$

where $s_c(a,t) = e^{-\int_0^a \mu(x,t-a+x)dx}$ is the cohort survival or probability that a person born in $t-a$ will live to attain age a. As before, $\pi_f(x,t-a)$ is the proportion of women at age x in the total population at time $t-a$. From this equation the age-specific growth rates are calculated as $r(a,t) = \acute{N}(a,t)$. By using the relative change of a product (4.25), a decomposition similar to (4.33) is obtained,

$$r(a,t) = \acute{N}(t-a) + C\acute{B}R(t-a) + \acute{s}_c(a,t). \tag{4.39}$$

As a last step the age-specific growth rates are substituted in the crude growth rate or average growth rate, as seen in equation (4.34). The result is a crude growth rate at time t expressed as the effects of past changes in the population size, fertility, and mortality (migration is also included here),

$$r(t) = \overline{\acute{N}} + \overline{C\acute{B}R} + \overline{\acute{s}_c}, \tag{4.40}$$

where the elements on the right hand side are the average of the relative change of population size, fertility and survivorship. The fertility component is separated into two other components: the female age-distribution and age-specific fertility rates.

In comparison to the decompositions obtained in (4.29) and (4.36) which only required data for the period of interest, equation (4.40) is more demanding. The optimal situation for using equation (4.40) is when countries have enough historical data, not only on mortality but also on fertility, age distribution and migration, such as the Scandinavian countries. Horiuchi notes that his proposal requires data with long time series, but it is also possible to adjust the method to countries with less data available. By using his adjustment and equation (4.40) the population growth rate for France, Japan and the USA, was calculated and compiled in Table 4.11. As previously mentioned, this decomposition is the result of the averages of past changes. Two of these components contribute to the increase in growth rates, while the past change in births neutralizes this increase. The average past change in population size is the major component accounting for the change in France and USA. In Japan, on the other hand, the average change in birth rates is the major component. The average change in cohort survivorship helped to sustain the Japanese growth rate at positive levels.

4.4.2 Conclusion

Several decompositions of the growth rate of the population have been presented in this section. The crude growth rate and the age-specific growth rates are measures of change of the population size. Therefore, it is possible to decompose the change in the growth rates or simply to decompose it into components of the past and present changes in fertility, mortality and migration.

Table 4.11: The population growth rate, $r(t)$, rates in percentages, and Horiuchi's decomposition of the growth rates in the period 1985-1990 for France, Japan and the USA.

	France	*Japan*	*USA*
$N(1990)$	56,570,545	121,907,530	248,940,972
$N(1985)$	55,157,223	119,740,316	236,872,781
$N(1995)$	58,020,081	124,113,968	261,624,012
$r(1990)$	0.506	0.359	0.994
$r(1990)$ *	0.504	0.357	0.994
$\overline{\acute{N}}$	0.547	1.129	1.260
$\overline{C\acute{B}R}$	-0.412	-1.246	-0.668
$\overline{\acute{s}_c}$	0.370	0.474	0.402
$r(1990) = \overline{\acute{N}} + \overline{C\acute{B}R} + \overline{\acute{s}_c}$	0.504	0.357	0.994

Source: Author's calculations of formula (4.40) as described in Chapter 9. Note: * means that the growth rates were calculated as an average of growth rates which is explained in Chapter 9. Data is derived from the Human Mortality Database (2002). Historical data for Japan is based on Japan Statistical Association (2002), and the USA data is from the U.S. Census Bureau (2001).

Formulations by Preston and Coale (1982) and Arthur and Vaupel (1984) have helped us to understand the relationships among the demographic variables. These systems are generalizations of population models that hold for any population, which look at changes over time, age and cohort in a continuous way. Therefore, they are related to the formulations of direct vs. compositional decomposition, in Part III, that have the purpose of accounting for changes over time in a continuous way. Furthermore, the population growth rates are measures of the changes in the age structure, and therefore they are included in one of the components of direct vs. compositional decomposition.

Chapter 9, on estimation procedures, includes a section on population growth rates. Explanations of the estimations of the tables of the present chapter are found in that chapter.

5

Alternative Decomposition Methods

5.1 Introduction

In this chapter, we examine three decomposition methods. These techniques are concerned with parametric models. The total change of the demographic function is decomposed into the contributions of the various factors included in the model.

The first method generally referred to as "Regression Decomposition" (RD) is extensively used in demography. It is based on linear relationships of a dependent variable in terms of independent variables in various subpopulations. As an application here we study the regression models for the achieved number of children for three regions of Mexico: central, north and south. The mean number of children is regressed in each region by four factors with varying regional levels of influence. The reasons for these disparities are studied by comparing the linear models between the different regions. The total change in this regression decomposition exercise is the result of adding the disparities in parameters and independent variables among the different regions.

The second method is called the "Purging Method". Similar to RD, first a relationship model is found. Here, the selected model is a multiplicative (or log-linear) model for cross-classified data. We employ Mexican migration cross-classified by educational achievement and region of origin: central, north and south. The educational level in Mexico differs by region of origin. It is, therefore, important to free the frequencies of the undesirable interaction between region of origin and the distribution of the educational attainment. The confounding factor is eliminated by "purging" the model. In other words, the model is transformed into a model without the undesired factor, which allows comparisons by educational attainment.

The last model, the "Delta Method", is as the two previous models based on a parametric function. The partial derivatives of the defined model with respect to all parameters are calculated. The total change in the defined model is found by addition of the partial derivatives.

Here, assuming that mortality follows a Gompertz trajectory we study the change in mortality explained by the contribution of each of the parameters.

5.2 Regression Decomposition

This section reviews the regression decomposition (RD) method. This methodology's goal is to explain the disparity between linear relationships of the different subpopulations. Two components explain the variation in these linear models. The first effect on the outcome of the dependent variable corresponds to group membership. This component is interpreted as the influence of the factors in the different subpopulations. The second component accounts for the difference in the characteristics of the two groups.

Among the researchers that have contributed to the development of regression decomposition are Coleman et al. (1972), Alwin and Hauser (1975), Sobel (1983), Clogg and Eliason (1986), Firebaugh (1989), Rodgers (1990), Al-Qudsi and Shah (1991), Leibowitz and Klerman (1995), Hanson et al. (1996), Bidani and Ravallion (1997), and Shen (1999).

The decomposition methods examined in Chapter 3 are based on arithmetic manipulations of differences between demographic variables. Similarly, regression decomposition is based on arithmetic manipulations of differences between the estimated relationships of the regressions.

The RD method is explained and illustrated by an example. Coleman et al. (1972) present a regression decomposition that accounts for the difference between two groups based on the relationship of interest. Suppose the two groups are $i = 1, 2$, and in each group there is a linear relation between a dependent variable y_i, and a set of K variables x_{ik}. The size of the groups is specified as n_i and the process generating the linear relation as

$$y_i = a_i + \sum_{k=1}^{K} b_{ik} x_{ik} + \epsilon_i \tag{5.1}$$

where a_i is an intercept, b_{ik} the kth parameter, and the stochastic disturbance ϵ_i has mean zero and variance σ_i^2.

Then the groups' means $\bar{y}_i$ are of the form

$$\bar{y}_i = \alpha_i + \sum_{k=1}^{K} \beta_{ik} \bar{x}_{ik} \tag{5.2}$$

where $\bar{x}_{ik}$ is the mean of the kth explanatory variable in the ith group. The intercept and parameter estimates are denoted by α_i and β_{ik}, respectively.

The disparity between the two groups is studied by comparison of the means, $\bar{y}_2 - \bar{y}_1$. This difference is decomposed as follows:

$$\begin{aligned} \bar{y}_2 - \bar{y}_1 &= (\alpha_2 - \alpha_1) + \sum_{k=1}^{K} \left(\frac{\bar{x}_{1k} + \bar{x}_{2k}}{2} \right) (\beta_{2k} - \beta_{1k}) \\ &\quad + \sum_{k=1}^{K} \left(\frac{\beta_{1k} + \beta_{2k}}{2} \right) (\bar{x}_{2k} - \bar{x}_{1k}) . \end{aligned} \tag{5.3}$$

The components of these expressions refer to two sources of change: a) the difference in parameters, that is, the difference in intercepts α_i, and slopes β_{ik}, which are the first two terms of (5.3), and b) the difference in the mean values of the independent variables $\bar{x}_{ik}$, the third component in (5.3). The component (a) may be interpreted as the influence of group membership on the dependent variable. The second component (b) is the difference in the characteristics between the two groups.

As an application of this technique we look at the total number of children per person for different regions of Mexico. The mean number of children is regressed in each region by four factors, namely educational level, unions in life, age at birth of the first child and the sex of the respondent. These factors are used here as predictors of the achieved number of children. Their level of influence differs from region to region and factor to factor. The reasons for these disparities are examined by applying the *RD* method.

First we look at the outcomes of the regression models for the Mexican regions. These results are based on the National Retrospective Demographic Survey, or EDER (1998) in Spanish, carried out in Mexico in 1998. Table 5.1 includes the regression models for the achieved number of children for the three aforementioned regions of Mexico. Here it should be noted that this example is not meant to be an exhaustive study of Mexican fertility but rather it is an application of the *RD* technique. Again, the mean number of children has been regressed by four factors, namely educational level, unions in life, parental age at birth of the first child and the sex of the respondent. For each region there are two columns containing the coefficients, denoted as β, and means of these factors, $\bar{x}$. The first row is for the intercept α_i. In the analyzed survey the mean number of children is higher among Mexicans from central Mexico than for those from both north and south Mexico, 5.14, 4.50 and 4.60 respectively. Education level is the highest in the north and lowest in the south, as reflected in the average of this factor $\bar{x}$. But this factor negatively influences the achieved number of children, seen in the coefficients of the factor β. The negative influences of education on the mean number of children is more pronounced in the center than in the other two regions.

In a similar way other features come to light. For example, the age at first child is also the highest in the north and the lowest in the south of Mexico. Nevertheless, this factor exhibits only a minor influence on the dependent variable. To understand the disparities between characteristics, $\bar{x}$, and the influence of these characteristics in each region, β, we applied the *RD* method.

Table 5.2 presents the results of applying the *RD* method shown in equation (5.3) with respect to the differential in achieved number of children between Mexicans from the center and those from the north and south of the country. The columns of the difference in parameters $\Delta\beta$ evaluate the influence of the factors in the different regions,

$$\Delta\beta = (\alpha_2 - \alpha_1) + \sum_{k=1}^{K} \left(\frac{\bar{x}_{1k} + \bar{x}_{2k}}{2} \right) (\beta_{2k} - \beta_{1k}).$$

The columns of the change in the average values $\Delta\bar{x}$ represent the proportion of the change that can be explained by the characteristics of the populations in the north or south compared

Table 5.1: Number of children regressed by age at first child, education achievement, sex and the number of unions in life, for Mexicans from the center, north and south of the country.

Variable	Center		North		South	
Number of children	5.14		4.50		4.60	
	β	$\bar{x}$	β	$\bar{x}$	β	$\bar{x}$
Intercept	11.237***		11.366***		9.856***	
	(0.593)		(0.827)		(0.737)	
Education	-1.409***	1.298	-1.261***	1.540	-1.169***	1.225
	(0.083)		(0.123)		(0.107)	
Unions in life	0.354	1.100	0.267	1.110	0.279	1.120
	(0.218)		(0.251)		(0.242)	
Age at first child	-0.170***	22.930	-0.170***	23.450	-0.164***	22.630
	(0.016)		(0.022)		(0.020)	
Sex	-0.490***	1.530	-0.807***	1.540	-0.289	1.530
	(0.168)		(0.241)		(0.207)	
$\bar{\text{R}}$ square	0.287		0.275		0.294	

Source: Based on EDER (1998). The notations β and $\bar{x}$ corresponds to the parameter estimates and the means, respectively. *** p<0.001. The states of the central region are: Aguascalientes, Colima, Distrito Federal, Guanajuato, Hidalgo, Jalisco, México, Michoacán, Morelos, Nayarit, Puebla, Queretaro, San Luis Potosí, Tlaxcala and Zacatecas. The states of the north region are: Baja California, Baja California Sur, Coahuila, Chihuahua, Durango, Nuevo León, Sinaloa, Sonora and Tamauilipas. The states of the south region are: Campeche, Chiapas, Guerrero, Oaxaca, Quintana Roo, Tabasco, Veracruz and Yucatán.

to the center,

$$\Delta\bar{x} = \sum_{k=1}^{K} \left(\frac{\beta_{1k} + \beta_{2k}}{2} \right) (\bar{x}_{2k} - \bar{x}_{1k}) \, .$$

In both, $\Delta\beta$ and $\Delta\bar{x}$ the center is denoted by the subindex 1 and the north or the south by 2. In Table 5.1 both education and age at first child show an inverse influence on the total number of children born. The higher the level of these two variables, the lower the number of children born. These inverse relationships are not equally present in all Mexican regions. Table 5.2 shows the comparison between the northern and central regions and the southern and central regions of Mexico in the columns labeled North-Center and South-Center respectively. A negative value in the table indicates an advantage in favor of Mexicans from the central region, while a positive value indicates advantages for those from the northern or southern regions.

The factors of education and age at first child show negative values of $\Delta\bar{x}$ and T in the columns of the North-Center and positive values in the South-Center. On the other hand, the difference in parameters, $\Delta\beta$, is positive for both North-Center and South-Center. By combining these two results we reach the following conclusion. The central region shows the greatest inverse relationship between the factors of education and age at first child, and the achieved number of children. However, the northern region has achieved levels of the

Table 5.2: Regression decomposition of the differential in achieved number of children between Mexicans from the center and those from the north and south. Number of children is regressed on educational achievement, number of unions in life, age at first child and sex.

Factor	North-Center			South-Center		
	$\Delta\beta$	$\Delta\bar{x}$	T	$\Delta\beta$	$\Delta\bar{x}$	T
Intercept	0.129	0	0.129	-1.381	0	-1.381
Education	0.210	-0.323	-0.113	0.303	0.094	0.397
Unions in life	-0.096	0.003	-0.093	-0.083	0.006	-0.077
Age at first child	0	-0.088	-0.088	0.137	0.050	0.187
Sex	-0.487	-0.006	-0.493	0.308	0	0.308
Total contribution	-0.244	-0.414	-0.658	-0.716	0.150	-0.566

Source: Based on EDER (1998). The notations $\Delta\beta$ and $\Delta\bar{x}$ correspond to the differences in the parameter estimates and the means, respectively. The column T contains the addition of parameters and means differences. A negative entry indicates an advantage in favor of the people from the central region while a positive entry indicates an advantage for people from the north or the south.

characteristics under study seen in the $\Delta\bar{x}$ terms, that surpass the contribution of the greater influence of the factors in the central region, $\Delta\beta < \Delta\bar{x}$. As a result, the total contribution of the factors of education and age at first child is negative in the column T of the North-Center. Medina (2000) presents a thorough study on the Mexican educational level by region. The results shown in Table 5.1 and 5.2 accord with those of Medina's and they indicate the necessity for regional comparisons.

In the last row of Table 5.2 we find the total contribution of the difference in parameters and means. The total contribution of the difference in characteristics, $\Delta\bar{x}$, is the most important factor in the North-Center difference. The northern region has undesirable characteristics that impact on the increase in the number of children. For the South-Center disparities, $\Delta\bar{x}$ is a minor component due to similar characteristics between the southern and central regions. The overall influence of the factors, $\Delta\beta$, is the major component featuring in the South-Center difference.

The RD formula (5.3) described above is not unique. Sobel (1983) used the parameters and independent variables from the second group as weights, and he obtained the following decomposition

$$\begin{aligned} \bar{y}_2 - \bar{y}_1 &= (\alpha_2 - \alpha_1) + \sum_{k=1}^{K} \bar{x}_{2k}(\beta_{2k} - \beta_{1k}) \\ &\quad + \sum_{k=1}^{K} \beta_{2k}\,(\bar{x}_{2k} - \bar{x}_{1k}) - \sum_{k=1}^{K}(\beta_{2k} - \beta_{1k})\,(\bar{x}_{2k} - \bar{x}_{1k})\,. \end{aligned} \tag{5.4}$$

This decomposition then has three components because an interaction term has been added. The interaction component is used to balance the formula and is interpreted as a good (bad) election of the weight if its value is small (large). In some research, decompositions of the type seen in (5.4) are preferred over (5.3). Examples of these are found in the works of Al-Qudsi and Shah (1991) and Hanson et al. (1996). There the parameters and independent variables of one group are considered as weights. Al-Qudsi and Shah (1991) compared the economic progress between foreigners and nationals in Kuwait, taking the Kuwaiti population as the weights. Hanson et al. (1996) studied the trends in child support comparing several periods and taking the values seen in the initial period as the weights.

Another decomposition procedure is proposed by Clogg and Eliason (1986). These authors are concerned with higher-order moments of the independent variable x_i. Let a polynomial regression model of order K be

$$y_i = a_i + \sum_{k=1}^{K} b_{ik} x_i^k + \epsilon_i \tag{5.5}$$

where x_i^k is the kth power of the independent variable x_i. This decomposition is similar to (5.3), but now the means $\bar{x}_{ik}$ are substituted by the sample moments of the independent variable x_i, $\bar{x}_{ik} = \frac{\sum_{j=1}^{n_i} x_{ij}^k}{n_i}$.

The works by Firebaugh (1989) and Rodgers (1990) are focused on the decomposition of changes in aggregate measures of a characteristic of a population. These studies question the possibility of analogous decompositions such as those presented in Chapter 3. The associations are from rate and compositional components, as seen in Chapter 3, into cohort replacement which are a part of and within cohort change. The average cohort replacement is the movement that occurs every year towards younger generations when new birth cohorts are added and old ones disappear. The within cohort change is an alteration of a characteristic of individuals independent of their cohort membership.

Firebaugh (1989) assumes that an aggregate measure $\bar{y}_i$ is linearly related to time, t_i, and to a cohort component C_i. His suggestion for a decomposition of the difference $\bar{y}_2 - \bar{y}_1$ is one which involves the difference between years $t_2 - t_1$ and another denoting the difference between cohorts $C_2 - C_1$. In both cases, the parameters β of the initial time are used. The difference over years corresponds to changes within a cohort while the cohort replacement describes the difference in the growth rates at birth.

Rodgers (1990) questioned the possible interpretation of the components in the decomposition suggested by Firebaugh. Instead, Rodgers suggests searching for variables related to cohort or period effects in order to examine whether they explained the change.

The existent relationships between the independent variables also inspired the development of other decompositions. By using path analysis Alwin and Hauser (1975) divided the linearity expression of a few independent variables into direct and indirect components. The direct component is the effect of the independent variables on the dependent variable. The indirect part describes the effect of the independent variables mediated by other independent variables. Figure 5.1 shows a causal diagram for a linear recursive model. The variable Y is linearly related to X_1, X_2, X_3 and X_4, while the variables X_1, X_2 and X_3 also are interrelated. The corresponding model contains a direct effect of X_1, X_2, X_3 and X_4 on Y depicted in Figure 5.1 by the straight lines. The broken lines correspond to the indirect effect, these are the effects of X_1, X_2 and X_3 on Y which are mediated by the other variables.

We have now introduced the regression decomposition methodology. A common feature of the methods discussed above is that these statistical inferences are used when the information is derived from samples rather than from whole populations. Therefore, together with the

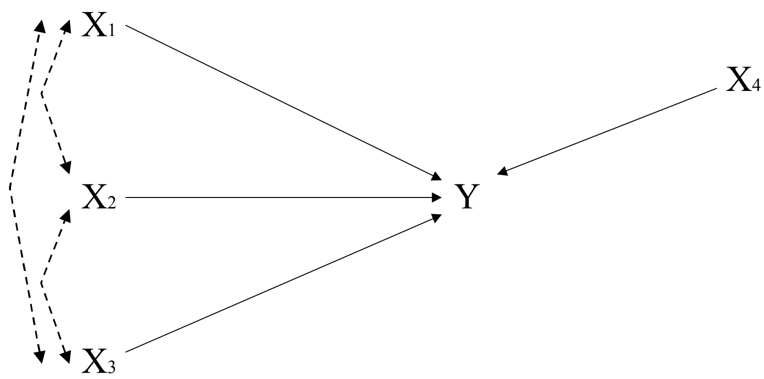


Figure 5.1: Causal diagram for a linear recursive model.

calculations of the regression decomposition a test of statistical hypotheses should also be considered. Since this is beyond the focus of this study, we refer to the study by Sobel (1983), which shows how standard errors and test statistical hypotheses can be computed.

5.3 The Purging Method

Several authors have made contributions to multiplicative models of rates, standardization and components analysis, see Schoen (1970), Teachman (1977), Clogg (1978), Willekens (1982), Willekens and Shah (1984), Clogg and Eliason (1988), Liao (1989) and Xie (1989). In this section the development of this method is presented based on the works of the authors mentioned above.

Schoen (1970) recommended using the geometric mean of the age-specific mortality rates as an average of relative indexes. The author points out the properties which are desirable for a mortality index:

1. Each population should have a real unique number.
2. It should respect proportionality. If mortality is two times higher in the second population, then the index should also be twice as high in the second population.
3. The index should not be affected by confounding components.
4. The index should reflect the nature of the underlying mortality function.

Teachman (1977) further analyzed these properties and showed the relationship between Schoen's index and log-linear models. Log-linear models fulfill all the properties of a mortality index and are useful when making compositional controlled comparisons.

In data, which is cross-classified by different characteristics, we find differences in the rate of occurrence of some event due to group compositions. Let the three-way contingency table be classified by a composition variable C, a group variable G, and a dependent variable D (or occurrence of the event). In the following example concerning Mexico, C is the region of origin, G is educational attainment and D is the dependent variable of Mexican migration.

Let the categories C, G and D be indexed as i, j and k, respectively. In a three-way contingency table the expected frequencies in the cells (i, j, k), are denoted as F_{ijk}, and the cell-specific proportions that fall in the event k can be defined as

$$v_{ij(k)} = \frac{F_{ijk}}{F_{ij\cdot}}, \tag{5.6}$$

where the dot in $F_{ij\cdot}$ corresponds to the summation over the replaced subscript, $F_{ij\cdot} = \sum_k F_{ijk}$. The overall proportion of occurrence in the jth group is given in terms of the frequencies as

$$v_{\cdot j(k)} = \frac{F_{\cdot jk}}{F_{\cdot j\cdot}}. \tag{5.7}$$

Following Goodman's (1970) work, a general multiplicative model in the three-way contingency table CGD may be written as

$$F_{ijk} = \eta\tau_i^C\tau_j^G\tau_k^D\tau_{ij}^{CG}\tau_{ik}^{CD}\tau_{jk}^{GD}\tau_{ijk}^{CGD}, \tag{5.8}$$

where η is the scale factor and the τ parameters denote various kinds of main effects and interactions. The estimates of the proportions of occurrence of the event are obtained by substituting (5.6) or (5.7) by the frequencies obtained from the model parameters in (5.8). Willekens and Shah (1984) show that instead of estimating frequencies and then obtaining proportions it could be possible to calculate directly the model parameter for proportions. They complement their formal proof of the two estimation procedures with applications to migration proportions.

If we assume a multiplicative log-linear model for frequency data of contingency tables, as found in (5.8), the "purged" method suggested by Clogg (1978) can be applied. His suggestion is to remove the confounding effect τ_{ij}^{CG} by looking at the ratios of $F_{ijk}^* = \frac{F_{ijk}}{\tau_{ij}^{CG}}$. The "purged", adjusted or standardized proportions for the confounding effect τ_{ij}^{CG}, are defined as

$$v_{ij(k)}^{**} = \frac{F_{ijk}^*}{F_{ij\cdot}^*}. \tag{5.9}$$

The final adjustments to the "purged" proportions in (5.9) concern rescaling which is carried out to ensure that the sum of the "purged" frequencies is equal to the observed frequencies.

At this juncture we present an application of the purging method. The cross-tabulated data consists of Mexican migration by educational achievement and region of origin. The data is based on the EDER (1998) survey. The observed frequencies of the cross-tabulated data are listed in Table 5.3. The observed proportions of migration by educational attainment in the whole country are 2.04% for persons without any education, 4.11% for persons with some elementary education, 4.23% for persons with secondary education and 3.37% for persons with post-secondary education.

The northern and central regions of Mexico have higher levels of educational attainment than the southern region. It is, therefore, particularly important to free the frequencies of this undesirable interaction. In Table 5.4 the ratios have been adjusted for the confounding influence of the interaction between region of origin and the distribution of the educational attainment. The adjusted ratio for the whole country is now 1.81% for those with no education,

Table 5.3: Observed Mexican migration by education achievement and region of origin.

	No education			Elementary			Secondary			Higher educ.		
	NM	M	T	NM	M	T	NM	M	T	NM	M	T
South	119	1	120	344	5	349	55	4	59	111	3	114
Center	223	6	229	601	34	635	180	4	184	221	8	229
North	43	1	44	269	13	282	105	7	112	127	5	132
Total	385	8	393	1214	52	1266	340	15	355	459	16	475
Ratio %	2.036			4.108			4.226			3.369		

Source: Based on EDER (1998). The notations NM and M correspond to not-migrated and migrated respectively. The states of the Central region are: Aguascalientes, Colima, Distrito Federal, Guanajuato, Hidalgo, Jalisco, México, Michoacán, Morelos, Nayarit, Puebla, Queretaro, San Luis Potosí, Tlaxcala and Zacatecas. The states of the North region are: Baja California, Baja California Sur, Coahuila, Chihuahua, Durango, Nuevo León, Sinaloa, Sonora and Tamauilipas. The states of the South region are: Campeche, Chiapas, Guerrero, Oaxaca, Quintana Roo, Tabasco, Veracruz and Yucatán.

Table 5.4: Mexican migration adjusted by using Clogg's method for purging the confounding influence of the interaction between region of origin and educational achievement.

	No education			Elementary			Secondary			Higher educ.		
	NM	M	T	NM	M	T	NM	M	T	NM	M	T
South	159	1	160	559	8	567	30	2	32	111	3	114
Center	132	4	136	383	22	405	259	6	265	221	8	229
North	95	2	97	280	14	294	54	4	58	127	5	132
Total	386	7	393	1222	44	1266	343	12	355	459	16	475
Ratio %	1.807			3.424			3.251			3.369		

Source: Based on EDER (1998). The notations NM and M are for not-migrated and migrated, respectively.

3.42% for those with elementary education, 3.25% for those with secondary education and 3.37% for those with higher education. As expected we see lower ratios for those with elementary and secondary education, the ratio for the latter group is even below that of the higher educated. The disparities between the educational levels of the different Mexican regions have a confounding effect on migration, which must be controlled. One option for controlling such an interaction is to use the purging method.

Other purging methods can be applied by dividing one or several of the parameter interactions τ from the expected frequencies in equation (5.8), as shown by Clogg and Eliason (1988), Shah (1988), Liao (1989) and Xie (1989).

Liao (1989) proposed a decomposition using purged rates. His model can be extended to several factors by solving a system of simultaneous linear formulas and obtaining the rate differences after the confounding factor has been purged.

The methodology of controlling for confounding factors through purging is mainly used in cases where models of the type seen in (5.8) can be estimated. As mentioned by Willekens

(1982) and Clogg and Eliason (1988), apart from the purged method it is important also to have indicators of the precision of the adjusted rates.

5.4 The Delta Method

The last alternative method here examined is the Delta method. It has been applied by Wilmoth (1988), Pullum, Tedrow and Herting (1989), Foster (1990), Gray (1991) and Pletcher et al. (2000) among others.

The Delta method is based on the chain rule for differentiation of functions of many variables. If v is a function of several variables, $v = f(x_1, x_2, ..., x_n)$, then the total differential of v is given by

$$dv = \sum_i^n \left(\frac{\partial v}{\partial x_i}\right) dx_i, \tag{5.10}$$

where $\frac{\partial v}{\partial x_i}$ is the differential of the function v with respect to the variable x_i, and dx_i is the differential of the variable x_i.

Pullum et al. (1989) used a finite approximation of (5.10),

$$\Delta v = \sum_i^n \left(\frac{\partial v}{\partial x_i}\right) \Delta x_i, \tag{5.11}$$

where Δv and Δx_i are finite differences. A similar link between the delta method and discrete decomposition techniques can be found in the work of Gray (1991).

Pullum and colleagues developed a procedure to allocate the changes of the mean fertility to changes in specific parities or groups of parities. The mean parity of a cohort, presented in equation (4.27) as the cohort TFR^c, can also be defined as a function of the parity progression ratios as

$$TFR^c = P_0 + P_0P_1 + P_0P_1P_2 + ... = \sum_{i=0}^{\infty} \prod_{j=0}^{i} P_j, \tag{5.12}$$

where P_j is the parity progression ratio from parity j to parity $j+1$, that is, the number of women of parity $j+1$ divided by women in parity j. Another way of looking at the P_j is the births of order $j+1$ divided by the births of order j. The total change in (5.12) is obtained by taking the changes in every parity. By using the Delta method we get

$$\Delta TFR^c = \sum_i \left(\frac{\partial TFR^c}{\partial P_i}\right) \Delta P_i. \tag{5.13}$$

For example, for parity 1 the contribution to the total change is

$$\begin{aligned} \left(\frac{\partial TFR^c}{\partial P_1}\right) \Delta P_1 &= P_0 \Delta P_1 + P_0 P_2 \Delta P_1 + ... \\ &= \left[\frac{TFR^c - P_0}{P_1}\right] \Delta P_1. \end{aligned} \tag{5.14}$$

Pullum and colleagues concluded that the impact of the different parities on the total change of the TFR^c diminishes for higher parities.

Another example of the Delta method is found in the work of Wilmoth (1988). He uses the method for analyzing mortality surfaces $\mu(a,t)$. Instead of examining the level of mortality, the author defines three directions of change for the $\mu(a,t)$: age, time and cohort. Using equation (5.10) the decomposition of the change in the $\mu(a,t)$ is

$$d\mu(a,t) = \frac{\partial\mu(a,t)}{\partial a}da + \frac{\partial\mu(a,t)}{\partial t}dt + \frac{\partial\mu(a+c,t+c)}{\partial c}dc, \tag{5.15}$$

where the three terms on the right hand side correspond to the change with respect to age, time and cohort respectively.

Foster (1990) and Pletcher et al. (2000) studied the derivatives of parametric schedules for demographic measures in fertility and mortality, respectively. Parametric schedules for demographic events take into account the strong regularities in vital rates of human populations. The change in the aggregate demographic measures are expressed as vectors of parameters.

As an application of the Delta method we look at the total change in the force of mortality. If mortality follows a shifting Gompertz trajectory then the force of mortality at age a and time t is

$$\mu(a,t) = \alpha(t)e^{\beta(t)a}, \tag{5.16}$$

where $\alpha(t)$ is the base level mortality and $\beta(t)$ the rate of increase with age, both at time t. Then it follows from equation (5.10) that the change in the age-specific mortality curve can be decomposed into

$$\begin{aligned} \Delta\mu(a,t) &= \frac{\partial\alpha e^{\beta a}}{\partial\alpha}\Delta\alpha + \frac{\partial\alpha e^{\beta a}}{\partial\beta}\Delta\beta \\ &= e^{\beta a}\Delta\alpha + \alpha e^{\beta a}a\Delta\beta. \end{aligned} \tag{5.17}$$

The values of α and β can be estimated from the intercept and slope of a regression line fitted to the logarithm of the age-specific death rates from age 30 to 95 years. That is the life span when mortality approximately follows a Gompertz trajectory. Table 5.5 shows the parameters estimate of the Gompertz trajectories for males and females in Hungary and Japan, for the years 1989 and 1999. In both countries, the male base levels of mortality, $\alpha(t)$, are higher than for females, while the values for the female rates of change over age, $\beta(t)$, are higher than the male rates.

The above-mentioned parameters are used in Table 5.5 for calculating the force of mortality at age 75 and the decomposition of the change as formulated in (5.17). Both Hungary and Japan experienced a decline in the force of mortality at age 75. The change is more pronounced in Japan than in Hungary. The parameters seem to influence the dynamic of the force of mortality in these two countries in different directions. In Hungary, we find a reduction in the base level of mortality with an almost equal increase in the rate of change over age. Japan has experienced the opposite change; a minor increase in the base level of mortality and a decline in the rate of change over age. In other words, improvements for those aged 30 have occurred in Hungary, while mortality rates in Japan have mainly changed in the older groups. In both countries females bear higher reductions than males.

Table 5.5: Gompertz trajectories calculated for males and females in Hungary and Japan, for 1989 and 1999. The force of mortality at age 75 is calculated together with the Delta decomposition of the change in the period.

	Hungary				Japan			
	Males		Females		Males		Females	
t	1989	1999	1989	1999	1989	1999	1989	1999
$\alpha(t)$ (%)	0.027	0.023	0.006	0.004	0.003	0.004	0.001	0.001
$\beta(t)$	0.078	0.079	0.092	0.095	0.097	0.095	0.102	0.098
$\mu(75, t)$	0.091	0.089	0.056	0.054	0.049	0.044	0.029	0.023
$\Delta\mu$ (%)	-0.018		-0.021		-0.050		-0.066	
$\frac{\partial\mu}{\partial\alpha}\Delta\alpha$ (%)	-0.124		-0.166		0.032		0.013	
$\frac{\partial\mu}{\partial\beta}\Delta\beta$ %	0.105		0.143		-0.083		-0.081	
$\Delta\mu = \frac{\partial\mu}{\partial\alpha}\Delta\alpha + \frac{\partial\mu}{\partial\beta}\Delta\beta$ (%)	-0.019		-0.023		-0.051		-0.068	

Source: Author's calculations based on equation (5.17). Data derived from the Human Mortality Database (2002).

5.5 Conclusion

Three decomposition methods are presented in this chapter. The regression decomposition and the purging methods are established techniques in demography, while the Delta method is relatively new in the field. These methods are concerned with parametric models. Regression decomposition and the Delta method separate the contribution of the parameters in the total change, while the purging method eliminates the confounding factors.

Much of the demographic information is derived from samples rather than from whole populations. In these cases the statistical inferences of the linear regressions and the log-linear models in the purging method should be accompanied by indicators of the precision of these models or standard errors. Together with the calculations of the regression decomposition and the purging methods, tests of statistical hypotheses should also be considered. Compared with the simple methods presented in Chapters 3 and 4, these standard errors and tests of statistical hypotheses are drawbacks in the present techniques.

The Delta method is more related to the proposed direct vs. compositional decomposition in Part III. The likeness comes from observing both methods cases that are continuously changing. The disadvantage of the Delta method compared to the other two in this chapter is the assumption of an initial relational model.

With this chapter we conclude Part II of the book. We have seen the evolution of a wide range of decomposition methods during the past years. Chapter 3 presented general decompositions based on arithmetic manipulations of a difference in demographic variables. Chapter 4 introduced the decomposition methods of particular demographic measures. In Chapter 5 we illustrated decompositions that are applied in parametric representations of demographic variables. These chapters dedicated to the previous decomposition methods are our framework for presenting direct vs. compositional decomposition. Part III of the book is focused on this decomposition method for changes over time.

Part III

Decomposing Demographic Change Into Direct Versus Compositional Components

6

Decomposing Demographic Averages

6.1 Introduction

This chapter presents direct vs. compositional decomposition developed by Vaupel (1992) and further extensions by this author. This decomposition has been applied to data by Vaupel and Canudas Romo (2002) and the present chapter as well as following chapters include this author's contributions to the field by extending Vaupel's formulation. As already noted we called this method "the direct versus compositional decomposition" because it separates the change over time into these components.

First we introduced the sources that inspired the development of the decomposition. Vaupel's main formula and the interpretation of its components are presented in Section 6.2. This main formula can be extended to any kind of compositional structure, not only to the age structure, as illustrated, by using averages over subpopulations in Section 6.3. In Section 6.4 we examine the derivatives of demographic functions, such as differences and additions. A formula for relative changes is then introduced in Section 6.5 and the study of relative changes of functions is explained in Section 6.6. Finally a new decomposition of life expectancy is displayed in Section 6.7.

In each section where formulas are presented, applications are also included. In the last part of the book, a comparison is made between direct vs. compositional decomposition and the previous methods shown in Part II. To ensure uniformity and comparability, the applications seen here in Part III use the same data as the applications of Part II which is entitled *Decomposition Methods*.

To study population aging, Preston, Himes and Eggers (1989) analyzed the change over time in the average age of the population. The average age of the population is

$$\bar{a}(t) = \frac{\int_0^\omega aN(a,t)da}{\int_0^\omega N(a,t)da}, \tag{6.1}$$

where as before $N(a,t)$ is the population size at age a and time t.

By differentiating this expression with respect to time they found this change equal to the covariance between ages and age-specific growth rates $r(a,t)$,

$$\dot{\bar{a}} = C(a,r), \tag{6.2}$$

where the covariance is as defined in (2.12). The average age of the population increases for aging populations. This can also be seen in equation (6.2) where $\dot{\bar{a}}$ increases when the covariance is positive, that is, when the age-specific growth rates increase with age.

The findings of Preston, Himes and Eggers were extended by Vaupel (1992), and applied by Vaupel and Canudas Romo (2002). In the following section the proof of this generalization is presented together with some illustrations.

6.2 Derivatives of Averages

The object of interest is the change over time of demographic variables. Let $\bar{v}(t)$ be a demographic average as defined in equation (2.1). We analyze the change in an average by studying its derivative with respect to time t. The change in the average, $\dot{\bar{v}}$, is

$$\dot{\bar{v}} = \frac{\partial}{\partial t}\frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{6.3}$$

The decomposition of the change over time of a demographic variable can be simply and memorably expressed as the sum of two components: the average change plus the covariance between the variable of interest and the intensity of the weighting function,

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v,\acute{w}). \tag{6.4}$$

The average change, $\bar{\dot{v}}$, is

$$\bar{\dot{v}} = \frac{\int_0^\infty \left[\frac{\partial}{\partial t}v(x,t)\right] w(x,t)dx}{\int_0^\infty w(x,t)dx}, \tag{6.5}$$

and the covariance is calculated as

$$C(v,\acute{w}) = \frac{\int_0^\infty \left(\left[v(x,t)-\bar{v}(t)\right]\left[\acute{w}(x,t)-\bar{\acute{w}}(t)\right]\right) w(x,t)dx}{\int_0^\infty w(x,t)dx}$$

$$= \frac{\int_0^\infty v(x,t)\acute{w}(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx} - \frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}\frac{\int_0^\infty \acute{w}(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{6.6}$$

The derivation of equation (6.4) follows from substitution of (6.3), (6.5) and (6.6) into the corresponding terms in (6.4). Another proof of the derivation of equation (6.4) can be seen in Vaupel and Canudas Romo (2002).

Equation (6.4) tells us that the change in the average, $\dot{\bar{v}}$, can be decomposed into two terms. The first term, $\bar{\dot{v}}$, the average change, accounts for the change observed in the population produced by a direct change in the characteristic of interest. In the rest of the text we refer to this term as direct change or the level-1 effect of change.

The second component, $C(v, \acute{w})$, the covariance term, is the structural or compositional component of change; it accounts for the changes in population heterogeneity. In the remaining text this compositional term is called compositional component or the level-2 effect of change. Schoen and Kim (1992) also derived a formula, similar to (6.4), in which there is only compositional change and no direct change.

The crude death rate, $\bar{\mu}(t)$ also denoted $d(t)$, defined in equation (2.7) is a demographic average (see Chapter 2 for this discussion). It follows directly from equation (6.4) that the change over time of the variable, $\dot{\bar{\mu}}$, is decomposed into two terms. These are the average changes in the mortality rates, $\bar{\dot{\mu}}$, plus the covariance between mortality rates and growth rates,

$$\dot{\bar{\mu}} = \bar{\dot{\mu}} + C(\mu, r), \tag{6.7}$$

where $r(a,t)$ are the age-specific growth rates $r(a,t) = \acute{N}(a,t)$.

Table 6.1 presents an application of equation (6.7) and it shows decomposition of the change in the crude death rate for the member countries of the North American Free Trade Agreement (NAFTA) from 1985 to 1995.

Equation (6.7) is in continuous form but demographic data are discrete, so we estimated the values in the table using the methods described in Chapter 9. The estimated values are sometimes slightly different from the observed figures. These discrepancies arise when discrete data over a h-year period are used to approximate derivatives and averages at a particular instant. In all the tables of this part of the book there is a line dividing observed and estimated values. Because the change occurs continuously from the initial moment of 1985 to the final moment of 1995, we choose to study the change at the mid-year, here 1990. The values of the average at all these years are displayed together with the observed changes. The lower part of the table corresponds to the estimated data. First, the estimated components are presented followed by the estimated change.

The three countries that are members of the NAFTA experienced a decrease in crude death rates between 1985 and 1995. The United States underwent the highest decrease followed by Mexico and Canada in third position with a minor change. These changes are mainly due to a decrease in the age-specific death rates, or level-1 change. In Canada and Mexico the average change in the mortality rates, $\bar{\dot{\mu}}$, is about the same while the United States experiences a very marked improvement.

The change in the age structure of the three populations is contrary to the level-1 changes. For Canada the level-2 effect is almost as high as the average change in mortality rates and when added to the level-1 effect, to estimate the total change, results in a very small change in the Canadian $\dot{\bar{\mu}}$. Canada's reductions in mortality were balanced out by the aging process of its population.

Table 6.1: Crude death rate, $d(t)$, per thousand, and decomposition of the annual change over time from 1985 to 1995 for Canada, Mexico and United States.

	Canada	Mexico	United States
$d(1990)$	7.150	5.100	8.762
$d(1985)$	7.195	5.532	9.683
$d(1995)$	7.116	4.755	7.941
$\dot{d}(1990)$	-0.008	-0.078	-0.174
$\bar{\dot{\mu}}$	-0.109	-0.116	-0.245
$C(\mu, r)$	0.101	0.039	0.071
$\dot{d} = \bar{\dot{\mu}} + C(\mu, r)$	-0.008	-0.077	-0.174

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

In many situations the demographic average in equation (2.1) is described as the product of two terms

$$\bar{v}(t) = \int_0^\omega v(x,t)c(x,t)dx, \tag{6.8}$$

where $c(x,t)$ denotes the proportion of the total values of the weights that belong to the category x at time t, $c(x,t) = \frac{w(x,t)}{\int_0^\infty w(x,t)dx}$, and therefore $\int_0^\omega c(x,t)dx = 1$. Under these circumstances equation (6.4) changes to

$$\dot{\bar{v}} = \bar{\dot{v}} + \overline{v\acute{c}}. \tag{6.9}$$

This is easily proved by looking at the derivative of $\bar{v}(t)$, which follows the rule of the derivative of a product

$$\dot{\bar{v}} = \int_0^\omega \dot{v}(x,t)c(x,t)dx + \int_0^\omega v(x,t)\dot{c}(x,t)dx = \bar{\dot{v}} + \overline{v\left(\frac{\dot{c}}{c}\right)}. \tag{6.10}$$

As a consequence of equations (6.4) and (6.10) the covariance component $C(v, \acute{w})$ for the level-2 effect of change can also be expressed as

$$C(v, \acute{w}) = \overline{v\acute{c}}. \tag{6.11}$$

6.3 Averages over Subpopulations

The focus so far has been on averages over age. However age heterogeneity is only one of the multitudinous dimensions of population heterogeneity. In this section we present averages over another characteristic, namely population size by country of residence.

Consider a population composed of different subpopulations. Life expectancy at birth at time t for the entire population, $\bar{e}^o(t)$, is defined as the average of life expectancies for the subpopulations

$$\bar{e}^o(t) = \frac{\sum_i e_i^o(t) N_i(t)}{\sum_i N_i(t)}, \tag{6.12}$$

where $N_i(t)$ is the size of the subpopulation i and $e_i^o(t)$ is the subpopulation life expectancy at birth. Here, we use the subpopulation size, $N_i(t)$, as the weights.

The change in $\bar{e}_o$ over time can be decomposed following equation (6.4) as

$$\dot{\bar{e}}^o = \bar{\dot{e}}^o + C(e^o, r), \tag{6.13}$$

where $r_i(t)$ is the population growth rate of the ith subpopulation, $r_i(t) \equiv \acute{N}_i(t)$.

In Table 6.2 equation (6.13) is applied to changes in life expectancy of the chosen European countries. The common life expectancy at birth at time t for specific European countries is calculated as shown in equation (6.12). In Chapter 3 Tables 3.2, 3.3, 3.4 and 3.5 presented the crude death rate of specific European countries. In Table 6.2 we use the same countries to study the life expectancy for the entire population of these countries. The periods of observation are from 1960 to 1970, 1975 to 1985 and from 1992 to 1996, denoted by their mid-years 1965, 1980 and 1994. For the years 1965 and 1980, the common life expectancy

Table 6.2: Life expectancy at birth, $\bar{e}_o(t)$, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996. Life expectancy is calculated as the average over selected European countries.

t	1965	1980	1994
$\bar{e}^o(t)$	70.718	71.747	73.359
$\bar{e}^o(t-5)$	69.856	70.974	73.322
$\bar{e}^o(t+5)$	71.570	72.518	73.201
$\dot{\bar{e}}^o(t)$	0.171	0.154	-0.030
$\bar{\dot{e}}^o$	0.165	0.156	-0.043
$C(e^o, r)$	0.006	-0.002	0.013
$\dot{\bar{e}}^o = \bar{\dot{e}}^o + C(e^o, r)$	0.171	0.154	-0.030

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 ten-year periods were used (1960-1970 and 1975-1985). For the year 1994 a four-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

gained an annual increase of 17% and 15%, respectively. On the other hand, in 1994 we find a decline in life expectancy of 3%. The level-1 effect accounted for most of these changes, i.e.,whether there is an increase or decrease. In 1994, the level-2 effect shows a polarization of the subpopulations' life expectancies and growth rates. Countries which experience the highest population growth rates are also those with the longest life expectancy. The opposite is also true for those countries which have the lowest levels of growth rates and life expectancy. As a result of these relations between life expectancies and growth rates, the covariance term is positive.

In Chapter 4 some methods for decomposing life expectancy were shown. An alternative decomposition of the change in life expectancy analogous to equation (6.4) is shown by Vaupel and Canudas Romo (2003). This decomposition is explained in the last section of this chapter.

6.4 Derivatives of Demographic Differences and Additions

Equation (6.4) yields a simple but powerful result that divides the change over time of a demographic variable into points of main interest for demographers. The use of this general result is not restricted to averages but it could also be used on functions of averages. As an extension of equation (6.4) this section presents the decomposition of the change over time of differences and additions.

An area that has stimulated much interest among demographers has to do with gender differences. A few examples of these are the higher female survivorship in all populations, the higher male Mexican international migration, the higher female Mexican internal migration, the higher male mortality by violent causes of death, and the lower female wages in similar jobs as males.

The change over time of operations among demographic variables can be separated into direct effects and compositional effects. The chief interest is on the difference of demographic measures between the sexes, but other differences are possible. Let $\bar{v}_M(t)$ and $\bar{v}_F(t)$ denote two averages, for males and females respectively, and $\Delta\bar{v}(t)$ denote the difference between them, $\Delta\bar{v}(t) = \bar{v}_F(t) - \bar{v}_M(t)$. The change over time of the difference $\Delta\bar{v}(t)$ can be seen as

$$\dot{\Delta}\bar{v}(t) = \frac{\partial}{\partial t}\left[\bar{v}_F(t) - \bar{v}_M(t)\right] = \frac{\partial}{\partial t}\bar{v}_F(t) - \frac{\partial}{\partial t}\bar{v}_M(t) = \dot{\bar{v}}_F(t) - \dot{\bar{v}}_M(t). \tag{6.14}$$

The desired decomposition is achieved by applying equation (6.4) to both terms $\dot{\bar{v}}_F(t)$ and $\dot{\bar{v}}_M(t)$,

$$\begin{aligned} \dot{\Delta}\bar{v}(t) &= [\bar{\dot{v}}_F - \bar{\dot{v}}_M] + [C_F(v,\acute{w}) - C_M(v,\acute{w})] \\ &= \Delta\bar{\dot{v}} + \Delta C(v,\acute{w}). \end{aligned} \tag{6.15}$$

Table 6.3 presents the decomposition of the change over time of the difference of the Mexican male and female crude death rates. Using equation (6.15) the change in the difference of the crude death rates is explained by level-1 and level-2 effects of change. The periods studied are 1980-1985, 1985-1990 and 1990-1995. During the three studied periods the difference between male and female CDRs narrows. This can be seen in the negative values of the change in the difference, $\dot{\Delta}d(t)$. Between 1980 and 1985 the male-female mortality gap shows the greatest reduction. The level-1 effect is again the main contributor to this change. That is, the average reduction in male mortality is more effective than the female direct change. The two populations seem to have similar changes in age composition resulting in only a small difference due to the level-2 effect of change.

Table 6.3: Difference between the male and female crude death rates, $\Delta d(t)$, per thousand, and decomposition of the annual change over time between 1980-1985, 1985-1990 and 1990-1995 for Mexico.

t	1980		1985		1990		1995
$d_M(t)$	7.407		6.233		5.829		5.405
$d_F(t)$	5.517		4.755		4.394		4.112
$\Delta d(t)$	1.890		1.478		1.435		1.293
$\Delta \dot{d}(t)$		-0.082		-0.009		-0.028	
$\Delta\bar{\dot{\mu}}$		-0.088		-0.018		-0.037	
$\Delta C(\mu, r)$		0.005		0.009		0.009	
$\Delta\dot{d}(t) = \Delta\bar{\dot{\mu}} + \Delta C(\mu, r)$		-0.083		-0.009		-0.028	

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001). Note that $d_M(t)$ and $d_F(t)$ correspond to the male and female crude death rates.

The same formulations for the decomposition of a difference of averages could be developed for an addition of averages. As an example, let the reader be reminded of the expression for the crude birth rate in equation (4.15), in Chapter 4, as the addition of two CBRs,

$$CBR(t) = CBR_m(t) + CBR_u(t),$$

where $CBR_m(t)$ and $CBR_u(t)$ are the married and unmarried CBRs respectively. The change over time of (4.15) is decomposed in a similar way as in (6.15), i.e.,

$$C\dot{B}R(t) = C\dot{B}R_m(t) + C\dot{B}R_u(t). \tag{6.16}$$

In equations (4.13) and (4.16) it is shown that the crude birth rates of married and unmarried women is the product of three terms. For married women, for example, the three terms are the product of fertility of married women $b_{ma}(t)$, the proportion of married women $\pi_{ma}(t)$, and the proportion of women in the total population $\pi_{fa}(t)$. The change over time in this crude death rate is

$$C\dot{B}R_m(t) = \sum_0^\omega \dot{b}_{ma}(t)\pi_{ma}(t)\pi_{fa}(t)$$

$$\sum_0^\omega b_{ma}(t)\dot{\pi}_{ma}(t)\pi_{fa}(t) + \sum_0^\omega b_{ma}(t)\pi_{ma}(t)\dot{\pi}_{fa}(t). \tag{6.17}$$

Equation (6.17) shows the continuous expression for the difference in decomposition presented by Zeng et al. (1991) in (4.14). A similar decomposition is derived for the crude birth rate of unmarried women in (6.16). Here it should be noted that equation (6.17) has the advantage over (4.14) of not including residual terms and it is then an exact decomposition.

The change over time of other demographic functions could be studied. In the next two sections it is shown that the change over time of fractions and products of demographic averages can also be decomposed. The decomposition of these two types of operators is instead based on the relative change.

6.5 Relative Derivatives of Averages

Another formula presented by Vaupel (1992) is the decomposition of a relative change.

The relative change in an average, $\acute{\bar{v}}$, is

$$\acute{\bar{v}} = \frac{\dot{\bar{v}}(t)}{\bar{v}(t)} = \frac{1}{\bar{v}(t)} \frac{\partial}{\partial t} \frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{6.18}$$

The decomposition of the relative change over time of a demographic variable can be simply and memorably expressed as

$$\acute{\bar{v}} = \tilde{\acute{v}} + \left[\tilde{\acute{w}} - \bar{\acute{w}}\right], \tag{6.19}$$

where the tilde denotes an alternative averaging procedure where the weights are proportional to the product of $v(x,t)$ and $w(x,t)$,

$$\tilde{v}(t) = \frac{\int_0^\infty v(x,t)v(x,t)w(x,t)dx}{\int_0^\infty v(x,t)w(x,t)dx}. \tag{6.20}$$

The proof of this formula is directly established by dividing formula (6.4) by $\bar{v}$,

$$\acute{\bar{v}} = \frac{\dot{\bar{v}}}{\bar{v}} = \frac{\bar{\dot{v}}}{\bar{v}} + \frac{C(v, \acute{w})}{\bar{v}},$$

and from the definition of covariance in (2.13) we obtain

$$\acute{\bar{v}} = \tilde{\acute{v}} + \frac{\overline{v\acute{w}} - \bar{v}\bar{\acute{w}}}{\bar{v}} = \tilde{\acute{v}} + \left[\tilde{\acute{w}} - \bar{\acute{w}}\right]. \tag{6.21}$$

This formula separates the relative change into a level-1 and a level-2 change, as formula (6.4) did before.

Equation (6.19) could be applied in all the examples previously shown in this chapter, decomposing the relative change. As an example, we decompose the relative change of the crude death rate, $\bar{\mu}(t)$. This decomposition is obtained by applying equation (6.19) to the CDR,

$$\acute{\bar{\mu}} = \tilde{\acute{\mu}} + \left[r_D - r\right], \tag{6.22}$$

where $r(t)$ is the population growth rate and $r_D(t)$ represents the death-weighted population growth rate (i.e. the population growth rate when age categories are weighted by the number of deaths in that particular age category),

$$r_D(t) = \frac{\int_0^\infty r(a,t)\mu(a,t)N(a,t)da}{\int_0^\infty \mu(a,t)N(a,t)da} = \frac{\int_0^\infty r(a,t)D(a,t)da}{\int_0^\infty D(a,t)da}, \tag{6.23}$$

where $D(a,t)$ are the deaths occurring at age a and time t.

Equation (6.22) indicates that the relative change in the CDR is the result of two components. First, a level-1 change is seen in the average relative change in age-specific death rates.

Table 6.4: Crude death rate, $d(t)$, per thousand, and decomposition of the annual relative change over time in 1965-1975, 1975-1985 and 1985-1995 for Mexico.

t	1970	1980	1990
$\bar{\mu}(t)$	8.486	6.436	5.150
$\bar{\mu}(t-5)$	9.514	7.456	5.532
$\bar{\mu}(t+5)$	7.456	5.532	4.755
$\acute{\bar{\mu}}(t)$	-24.194	-29.774	-15.072
$\tilde{\acute{\mu}}$	-18.992	-24.583	-22.574
r_D	26.662	19.480	26.491
r	31.936	24.697	18.992
$r_D - r$	-5.275	-5.218	7.498
$\acute{\bar{\mu}}(t) = \tilde{\acute{\mu}} + [r_D - r]$	-24.267	-29.801	-15.076

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

Secondly, a compositional change is seen in the difference between the rise in deaths and the population growth.

Table 6.4 includes the Mexican crude death rate and the decomposition of its relative change during the periods 1965-1975, 1975-1985 and 1985-1995, denoted by their mid-years 1970, 1980 and 1990 respectively. Table 3.1, in Chapter 3, showed the Mexican crude death rate and the decomposition of the change over time. The results in the table for 1975-1985 contrast with those indicating relative change in Table 6.4. The relative change is more pronounced in the 1980s than in other periods.

In Table 6.4, the level-1 effect retains its predominance in the relative change. The level-2 effect is the result of the growth rate weighted by the deaths, $r_D(t)$, minus the growth rate of the population, $r(t)$. The growth rates that form the level-2 component are included in Table 6.4. The growth rate of the population decreases over time, falling from 3.1% to 1.8%. The growth rate weighted by the deaths decreases from 1970 to 1980 from 2.6% to 1.9% and later increases to 2.6% in 1990. As a result of the fluctuations in $r_D(t)$ there is a positive level-2 effect of change in 1990 that counters the decrease in the crude death rate. Increase in growth rates at ages with higher levels of mortality $\mu(a,t)$ produced this higher $r_D(t)$ with respect to $r(t)$ in 1990.

6.6 Relative Derivative of Products and Ratios

In Chapter 4 equation (4.25) showed the relative change of a product. If the function $v(t)$ is equal to the product of variables $v_i(t)$, $v(t) = v_1(t)v_2(t)...v_n(t)$, then the relative decomposition of $v(t)$ is the sum of the relative derivatives of $v_i(t)$,

$$\acute{v} = \acute{v}_1 + \acute{v}_2 + ... + \acute{v}_n.$$

This formula is a particular case of the general formula of an average. Let the demographic variable $v(a,t)$ be a product of variables $v(a,t) = v_1(a,t)v_2(a,t)...v_n(a,t)$. Let the demographic function $w(a,t)$ be a product of variables $w(a,t) = w_1(a,t)w_2(a,t)...w_m(a,t)$. The average $\bar{v}(t)$ is now expressed as follows,

$$\begin{aligned}\bar{v}(t) &= \frac{\int_0^\omega v(a,t)\,w(a,t)\,da}{\int_0^\omega w(a,t)\,da} \\ &= \frac{\int_0^\omega v_1(a,t)v_2(a,t)...v_n(a,t)w_1(a,t)w_2(a,t)...w_m(a,t)da}{\int_0^\omega w_1(a,t)w_2(a,t)...w_m(a,t)da}.\end{aligned} \tag{6.24}$$

From equation (6.19), for the decomposition of the relative change, and from equation (4.25), the relative change of a product, we obtain the extension of the decomposition

$$\begin{aligned}\acute{\bar{v}} &= \tilde{\bar{v}}_1 + \tilde{\bar{v}}_2 + ... + \tilde{\bar{v}}_n \\ &\quad + \left[\tilde{\bar{w}}_1 - \bar{\bar{w}}_1\right] + \left[\tilde{\bar{w}}_2 - \bar{\bar{w}}_2\right] ... + \left[\tilde{\bar{w}}_m - \bar{\bar{w}}_m\right].\end{aligned} \tag{6.25}$$

The first n components are the level-1 effects of the n variables $v_1(a,t)v_2(a,t)...v_n(a,t)$. The second group of components are the level-2 effects of the m variables $w_1(a,t)w_2(a,t)...w_m(a,t)$.

Take, for example, the crude birth rate presented in equation (2.8). As mentioned in Chapter 2, the crude birth rate can be seen as an average by using the female ratio, $c_f(a,t) = \frac{N_f(a,t)}{N(a,t)}$,

$$CBR(t) = \frac{\int_0^\omega b(a,t)\,c_f(a,t)\,N(a,t)\,da}{\int_0^\omega N(a,t)\,da}.$$

Applying equation (6.25) to the CBR it is possible to decompose the relative change of the crude birth rate as

$$C\acute{B}R = \tilde{\bar{b}} + \tilde{\bar{c}}_f + [r_B - r]\,, \tag{6.26}$$

where $r(t)$ and $r_B(t)$ are the population growth rates and the births of the weighted population growth rate (i.e. the population growth rate when age categories are weighted by the number of births to women in that particular age category). Hence (6.26) provides a breakdown of the growth rate of the crude birth rate into components related to the growth rate of the age-specific fertility rates, the growth rate of the female ratio and the degree of disproportionate population growth at fertile ages.

Tables 4.3 and 4.4 included the crude birth rate and the decomposition of the change over time for Denmark, the Netherlands and Sweden. Table 6.5 presents the decomposition of the relative change in the CBR seen in (6.26) for these same countries from 1992-1997, with 1995 the mid-year.

Complementary results obtained by looking at Tables 4.3 and 4.4 are further established by observing Table 6.5. The average relative change in birth rates, $\tilde{\bar{b}}$, is negative for all countries. In other words, on average there is a decrease in the age-specific birth rates. This level-1 effect of change shows an enormous reduction and is the main contributor to the relative change in Sweden.

The level-2 change, a result of the difference in growth rates, is the main contributor to the relative change in the CBR for Denmark and the Netherlands. In the latter country, the

Table 6.5: Crude birth rate, $CBR(t)$, in percentage, and decomposition of the annual relative change over time, in percentage, from 1992 to 1997, for Denmark, the Netherlands and Sweden.

	Denmark	Netherlands	Sweden
$CBR(1995)$	1.287	1.258	1.199
$CBR(1992)$	1.310	1.295	1.417
$CBR(1997)$	1.280	1.232	1.023
$\acute{CBR}(1995)$	-0.464	-0.996	-6.548
$\tilde{\acute{b}}$	-0.121	-0.285	-6.238
$\tilde{\acute{c}}_f$	0.087	0.060	0.073
r_B	0.006	-0.212	0.055
r	0.434	0.554	0.408
$r_B - r$	-0.428	-0.766	-0.353
$\acute{CBR} = \tilde{\acute{b}} + \tilde{\acute{c}}_f + [r_B - r]$	-0.462	-0.991	-6.518

Source: Author's calculations described in Chapter 9, based on Eurostat (2000).

growth rate weighted by babies is negative as a result of negative growth rates in the ages where many children were born.

Equation (6.25) can also be extended to include products and ratios of averages. If the average $\bar{v}(t)$ is the product of averages, $\bar{v}(t) = \bar{v}_1(t)\bar{v}_2(t)...\bar{v}_n(t)$, then the decomposition of the relative change is similar to (6.25). The ratios have to be expressed in another way.

Let $\bar{v}_M(t)$ and $\bar{v}_F(t)$ be the averages for males and females, and the ratio of males over females be $\bar{v}(t) = \frac{\bar{v}_M(t)}{\bar{v}_F(t)}$. This measure is decomposed as follows

$$\acute{\bar{v}} = \tilde{\acute{v}}_M - \tilde{\acute{v}}_F(t) + \left[\tilde{\acute{w}}_M - \bar{\acute{w}}_M\right] - \left[\tilde{\acute{w}}_F - \bar{\acute{w}}_F\right]. \tag{6.27}$$

The proof follows from equation (6.25) by changing the ratio for a product $\bar{v}(t) = \bar{v}_M(t)\left[\bar{v}_F(t)\right]^{-1}$ derived with respect to t.

The ratio of the total fertility rate (TFR) of married over unmarried can be calculated as

$$R_{TFR}(t) = \frac{TFR_m(t)}{TFR_u(t)} = \frac{\int_\alpha^\beta b_m(a,t)da}{\int_\alpha^\beta b_u(a,t)da}, \tag{6.28}$$

where $b_m(a,t)$ and $b_u(a,t)$ are, as before, the age-specific fertility rates of married and unmarried women. The decomposition of the relative change can be calculated following (6.27) which yields

$$\acute{R}_{TFR} = T\acute{F}R_m(t) - T\acute{F}R_u(t), \tag{6.29}$$

where the relative change in $T\acute{F}R$ is equal to the average relative change in fertility rates weighted by the fertility rates, for example for the married population we have

$$T\acute{F}R_m(t) = \frac{\int_\alpha^\beta \acute{b}_m(a,t) b_m(a,t) da}{\int_\alpha^\beta b_m(a,t) da}. \tag{6.30}$$

A similar ratio is used by Coale (mentioned by Preston et al. (2001)) to study the historical fertility levels in European populations. Table 6.6 presents the ratio of TFR for those married over those unmarried and the decomposition shown in equation (6.29) for the relative change from 1992 to 1997 applied to data from Denmark, the Netherlands and Sweden. The gap

Table 6.6: Ratio of total fertility rates for married over unmarried women and the decomposition of the annual relative change over time, in percentage, for Denmark, the Netherlands and Sweden from 1992 to 1997.

	Denmark	Netherlands	Sweden
$R_{TFR}(1995)$	2.860	6.938	3.269
$R_{TFR}(1992)$	2.796	7.712	3.237
$R_{TFR}(1997)$	2.925	6.242	3.301
$\acute{R}_{TFR}(1995)$	0.009	-0.042	0.004
$T\acute{F}R_m(t)$	-0.240	-0.558	-0.287
$T\acute{F}R_u(t)$	-0.251	-0.488	-0.293
$\acute{R}_{TFR} = T\acute{F}R_m(t) - T\acute{F}R_u(t)$	0.011	-0.070	0.006

Source: Author's calculations from formula (6.29), based on Eurostat (2000).

between married and unmarried TFR is smaller in the Scandinavian countries than in the Netherlands. All three countries experienced a decline in the average relative change in age-specific fertility rates. For both Denmark and Sweden, the fertility of the unmarried exhibits a more pronounced decrease compared to the married. As a result both have positive relative changes in the ratio while there is a negative change for the Netherlands.

6.7 Decomposition of the Change in Life Expectancy

6.7.1 The Derivative of Life Expectancy

Vaupel and Canudas Romo (2003) have shown that life expectancy can also be decomposed into two components analogous to those in equation (6.4). These terms account for the mortality improvements and the heterogeneity of such improvements at different ages.

The notation of life expectancy introduced in Chapter 4 is used here, together with some new notations. Let $f(a,t)$ denote the probability density function describing the distribution of deaths (i.e., lifespan) in the lifetable population at age a and at time t. This function is equal to the product of force of mortality and the survival function, $f(a,t) = \mu(a,t)\ell(a,t)$.

As already defined, the rate of progress in reducing mortality rates is $\rho(a,t) = -\acute{\mu}(a,t)$. The average improvement in mortality is calculated as

$$\bar{\rho}(t) = \int_0^{\omega} \rho(a,t) f(a,t) da. \tag{6.31}$$

Let $e^o(a,t)$ denote remaining life expectancy at age a and time t:

$$e^o(a,t) = \frac{\int_a^{\omega} \ell(x,t)dx}{\ell(a,t)}, \tag{6.32}$$

the average number of life-years lost as a result of death is

$$e^{\dagger}(t) = \int_0^{\omega} e^o(a,t) f(a,t) da. \tag{6.33}$$

Recalling the definition of covariance in (2.12), the covariance between the rate of improvement in mortality and remaining life expectancy at various ages is

$$C_f(\rho, e^o) = \int_0^{\omega} [\rho(a,t) - \bar{\rho}(t)] \left[e^o(a,t) - e^{\dagger}(t)\right] f(a,t) da. \tag{6.34}$$

Note that the averages and the covariance just introduced have denominators of one because the probability density function describing the distribution of deaths over all ages is equal to one, $\int_0^{\omega} f(a,t)da = 1$.

Vaupel and Canudas Romo's main formula for decomposing the change in life expectancy is

$$\dot{e}^o(0,t) = \bar{\rho}e^{\dagger} + C_f(\rho, e^o). \tag{6.35}$$

The proof follows from substituting equations (6.31), (6.33) and (6.34) into (6.35). Another demonstration is found in Vaupel and Canudas Romo (2003).

The terms in equation (6.35) are on one side of the equation the derivative of life expectancy at birth, $\dot{e}^o(0,t)$, that is the change in life expectancy over time. The right hand side of the formula contains two terms. The first term is the product of the average rate of mortality improvement and the average number of life-years lost. The average rate of mortality improvement $\bar{\rho}(t)$ can be interpreted as the proportion of deaths averted (or lives saved), while $e^{\dagger}(t)$ can be interpreted as the average number of life-years gained per life saved.

The second term, the covariance between rates of mortality improvement and remaining life expectancies, increases or decreases the general effect, depending on whether the covariance is positive or negative. If $\rho(a,t)$ is constant at all ages, then the covariance is zero. Hence, the covariance captures the effect of heterogeneity in $\rho(a,t)$ at different ages. Remaining life expectancy generally declines with age, as shown for three countries in Figure 6.1. At a certain age a^*, the remaining life expectancy is equal to the average number of life-years gained per life saved, that is $e^o(a^*,t) = e^{\dagger}(t)$. The covariance will be positive if before age a^* the age-specific pace of mortality improvement tends to be higher than average and if after age a^* the age-specific pace of mortality improvement tends to be lower than average. Note that ages are weighted according to the distribution of deaths.

Table 6.7: Life expectancy at birth, $e^o(0,t)$, and the decomposition of the annual change from 1990 to 1999 for Japan, Sweden and the United States.

	Japan	*Sweden*	*United States*
$e^o(0,1995)$	79.821	78.551	76.094
$e^o(0,1990)$	78.983	77.591	75.441
$e^o(0,1999)$	80.658	79.512	76.746
$\dot{e}^o(0,1995)$	0.186	0.213	0.145
$\bar{\rho}$ (%)	2.202	1.426	0.537
$e^\dagger$	10.418	10.253	12.399
$\bar{\rho}e^\dagger$	0.229	0.146	0.067
$C_f(\rho, e^o)$	-0.044	0.066	0.078
$\dot{e}^o(0,1995) = \bar{\rho}e^\dagger + C_f(\rho, e^o)$	0.186	0.213	0.145

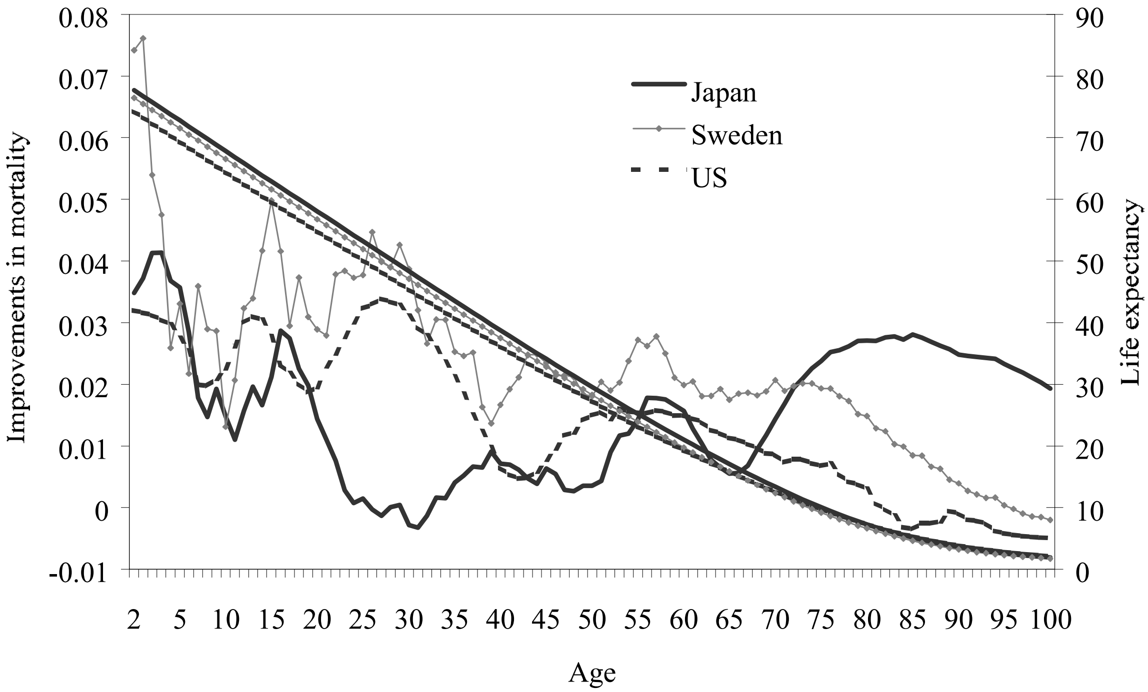
Source: Author's calculations are described in Chapter 9. Lifetable data is derived from the Human Mortality Database (2002). Lifetable values from the years 1990 and 1999 were used to obtain results for the mid-point around January 1995.

Table 6.7 shows the application of equation (6.35) to the annual change in life expectancy at birth for Japan, Sweden and the United States from 1990 to 1999, on January 1, 1995. In the last decade of twentieth century, Japan, Sweden and the United States all experienced increasing levels of life expectancy. Particularly interesting are the female Japanese levels, which are world records. Oeppen and Vaupel (2002) presented a complete list of these world records in the last century and an explanation of how the forecasted limits of life expectancy have been broken. The number of life-years lost as a result of death, $e^\dagger(t)$, varied for each country, with Japan registering 10.41 years, Sweden 10.25 years and the US 12.40 years. These numbers are multiplied by the improvements in mortality, $\bar{\rho}(t)$. Japan exhibits the greatest proportion of prolonged lives saved whereas the US has the lowest proportion. As a consequence, the product $\bar{\rho}(t)e^\dagger(t)$ gives the term of the contribution in life expectancy due to the advance in survivorship. For Japan and the US, this term—which is the main contributor to the increase in life expectancy—is positive.

The second term is the covariance between the improvements in mortality and the remaining life expectancy. Sweden and the United States share similar levels of this component. Nevertheless, for the US this is the most important factor. Figure 6.1 shows the five-year moving average of the improvement in mortality and the remaining life expectancy for ages 2 to 100 for the three countries. In this figure, it is possible to see that the remaining life expectancy generally declines with age. The improvement in mortality underwent numerous fluctuations. The five-year moving average is shown here. The positive covariance between these two measures results from major improvements in mortality in the younger age groups where life expectancy is the highest. Furthermore, in Sweden and the US the curves of improvement in both mortality and remaining life expectancy decline with age.

In Japan the covariance is in opposition to the increase in life expectancy. This is the result of a change in the curve of improvement in mortality around 60 shifting up in the oldest age groups. This covariance term could be divided into positive values before reaching the sixties

Figure 6.1: Five-year moving average of the improvement in mortality and the remaining life expectancy at ages 2 to 100 for Japan, Sweden and the United States in 1995.



and negative values there after. How this age decomposition can be achieved, as well as, a cause of death decomposition is shown in Chapter 7.

6.7.2 The Difference in Male and Female Life Expectancy

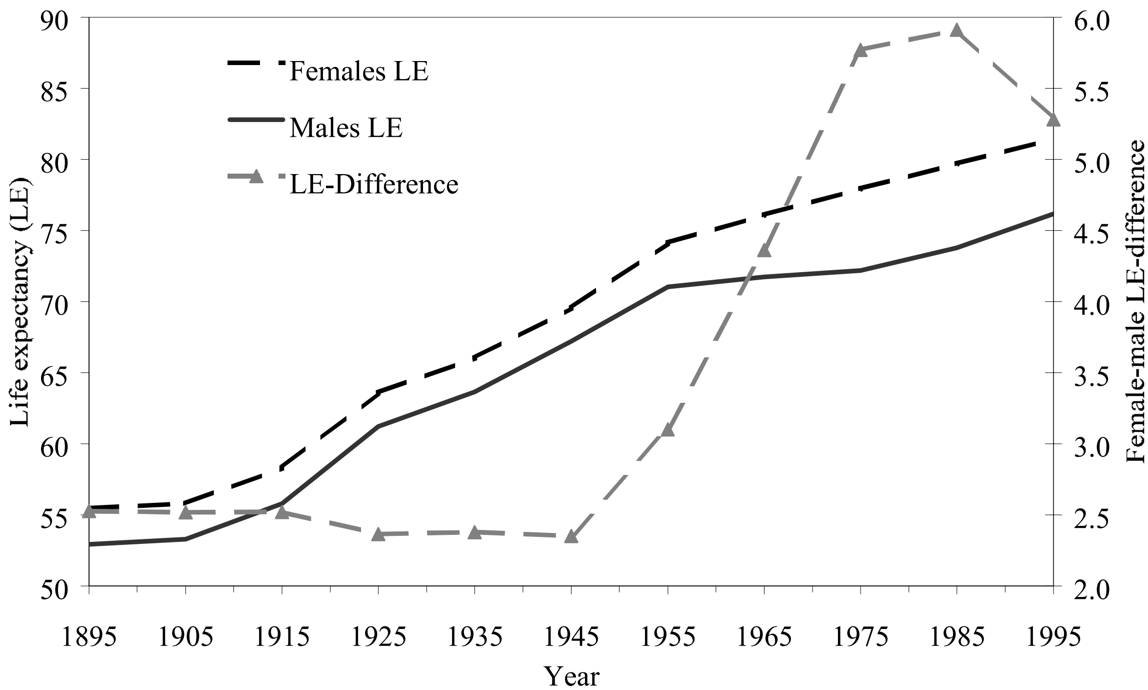
It is well known that a substantial gap exists between the life expectancy of males and females. Retherford (1972), Pollard (1982 and 1988), Van Poppel, Tabeau and Willekens (1996), and Valkovics (in Wunsch (2002)) are some of the studies that look at the mortality differential between the sexes. An interesting question, therefore, concerns how this difference has changed over time.

Let $e^o_F(0,t)$ and $e^o_M(0,t)$ be the life expectancy at birth for females and males respectively. The change over time in the difference between the female-male life expectancies is

$$\frac{\partial}{\partial t}\left[e^o_F(0,t) - e^o_M(0,t)\right] = \dot{e}^o_F - \dot{e}^o_M. \tag{6.36}$$

In equation (6.36) any of the decompositions of life expectancy mentioned in Chapter 4 could be substituted. For an example we look at the decomposition of the change in the difference of the female-male life expectancy in Sweden for every decade between 1895 and 1995. Figure 6.2 has two vertical axes, the left corresponding to life expectancies and the right to the differences, both in years. Until 1945 the difference was around 2.5 years, with an increase after this year until the peak in 1985, followed by a reduction in the gap in the latest years. The decomposition of equation (6.35) is applied to study the difference of the female-male life

Figure 6.2: Life expectancy for Swedish males and females from 1895 to 1995, and the difference in life expectancies.



expectancy in Sweden. For every year an average change and a covariance term is obtained

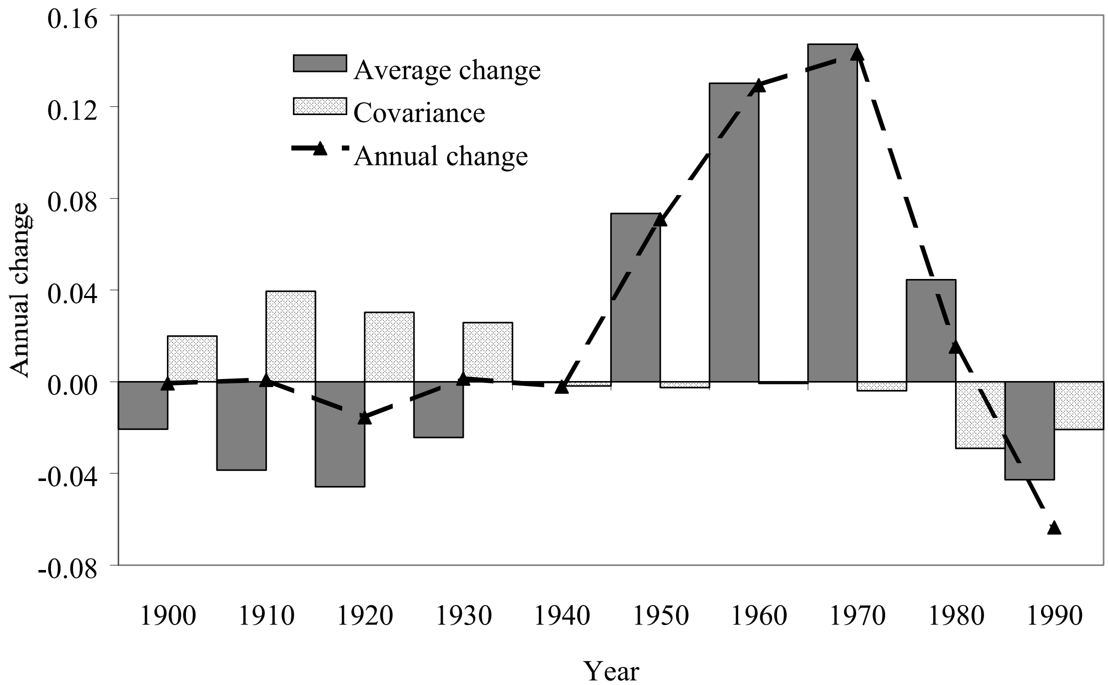
$$\begin{aligned}\frac{\partial}{\partial t}\left[e^o_F(0,t) - e^o_M(0,t)\right] &= \left[\bar{\rho}e^\dagger\right]_F - \left[\bar{\rho}e^\dagger\right]_M + \left[C_f(\rho, e^o)\right]_F - \left[C_f(\rho, e^o)\right]_M \\ &= \left[\bar{\rho}e^\dagger\right]_{F-M} + \left[C_f(\rho, e^o)\right]_{F-M}\,. \end{aligned} \tag{6.37}$$

Figure 6.3 shows the result of applying equation (6.37) to obtain the two components of change over time. Three periods can be noted here: before 1940, from 1940 to 1970, and after 1970. In the period before 1940 the advance in survivorship, $\bar{\rho}e^\dagger$, was better for males than females and the difference was negative. This tendency reverts between 1940-1970 when the females took the lead on advancements in survivorship. During the last period the rate of male survivorship outpaced female survivorship. Similar explanations can be reached for the covariance component.

6.8 Conclusion

In this chapter we present and prove various formulas for decomposing change in a population average into components.

Figure 6.3: Decomposition of the annual change in the Swedish male and female life expectancy difference from 1895 to 1995.



The direct vs. compositional decomposition requires notation that has not become completely standard in demography for three terms, namely the average, derivative and relative derivative, introduced in Chapter 2. Once these notations are adopted we obtain a simple and elegant equation (6.4).

Equation (6.4) provides a straightforward but powerful approach to decomposing direct versus compositional changes for many applications. One component captures the effect of direct change in the characteristic of interest, and the other captures the change that is attributable to a change in the structure or composition of the population.

The decomposition is applied to time derivatives of averages over age and over subpopulations. Several illustrative examples are provided in this chapter. In the examples shown here, two kinds of compositional changes are studied: changes in the age structure of the population and in the size of subpopulations. Many other applications are possible. Such applications will help demographers understand the dynamics of population change.

Equation (6.19) provides a decomposition of the intensity of averages. This alternative way of studying the dynamic of demographic averages is also separated into two components. The first is the average intensity of change of the variable of interest. The second is a difference of averages of the relative change in weights. This second component is analogous to the covariance in equation (6.4) accounting for the change attributed to the change in the structure of the population.

Equation (6.35) provides a new decomposition of the change in life expectancy. The method permits decomposition of the change in life expectancy into the general impact of mortality improvement at all ages and the additional effect of heterogeneity on the age-specific rates of improvement. Vaupel and Canudas Romo (2003) also show that this method permits further decomposition of age-specific and cause-specific effects —for each age category or for each cause of death—of the effects of the pace of mortality improvement, remaining life expectancy, and the frequency of deaths.

The decompositions introduced here, (6.4), (6.19) and (6.35), will lead to interesting demographic insights on population dynamics. The streamlined formulas permit deeper comprehension of the demographic factors that drive changes in demographic variables. Although it is a minor drawback, it is necessary to use approximations when applying them to data, as explained in Chapter 9.

7

Age, Categorical and Cause of Death Decomposition

7.1 Introduction

This chapter shows that direct vs. compositional decomposition can be used to study the contributions of particular ages (levels of the components or categories) to the total change over time of demographic measures. Particularly in the study of life expectancy a cause of death decomposition is shown.

Chapter 4 presented studies analyzing the change in life expectancy over time or between populations. Andreev (1982) (cited by Andreev et al. (2002)), Arriaga (1984), Pressat (1985) and Pollard (1988) decomposed the change in life expectancy into different components. An important part in the analysis of changes in life expectancy is to estimate the contribution of mortality changes for a specific age group to the total difference in life expectancy.

The focus on the contribution in the total change of changes in the different age groups is not exclusive to studies of life expectancy. A similar application can be made with regard to the total fertility rate (TFR). In Chapters 4 and 5 we introduced studies of this kind. The aim is to evaluate the components of change in the TFR, over time or across populations. As in the study of change in life expectancy the analysis focuses on the contributions of the levels to the total change. In the case of the TFR, the area of concern is the contribution of particular parity-specific fertilities to the total difference.

Similar to these studies on TFR and life expectancy we focus here on the crude death rate and the population growth rate of the world. The following are some of the examples of questions which can be asked about the contribution of ages (levels of the components or categories) to both the crude death rate and the population growth rate of the world: How are death rates of different age groups contributing to the change in the Mexican crude death rate? How are different regions of the world contributing to the growth rate of the world's population?

The following section demonstrates a group age decomposition of our main equation (6.4), which is taken to a single age decomposition in the third section. A section on categorical decomposition is then included, where the age decomposition is employed to study categories other than age. The last section in this chapter is a cause of death decomposition of change in life expectancy.

7.2 Age Decomposition

By applying equation (6.4) we obtain two main terms that explain the change of a population average. The change is divided into a term denoting the direct change in the characteristic of interest and a term describing the change in the structure or composition of the population. These two terms can be further decomposed by age.

The age decomposition is accomplished by using the additive property of integrals. In the following text we describe how this property of the integrals is applied to the ages of a given population. The integral from age zero to the last age attained by a person in the population $(0, \omega)$ can be separated into the addition of integrals that account for the different age groups, $(0, x_1)$, (x_1, x_2), (x_2, x_3) and so on until the last age group (x_n, ω). The corresponding equation is

$$\int_0^{\omega} v(a,t)da = \int_0^{x_1} v(a,t)da + \int_{x_1}^{x_2} v(a,t)da + ... + \int_{x_n}^{\omega} v(a,t)da. \tag{7.1}$$

Both terms in equation (6.4), the average change, $\bar{\dot{v}}$, and the covariance component, $C(v, \acute{w})$, are defined over integrals. From the property (7.1) it follows that these components can be separated by age groups. The numerator in the direct component of equation (6.4) is decomposed by age as follows:

$$\begin{aligned} \bar{\dot{v}} &= \frac{\int_0^{\omega} \dot{v}(a,t)w(a,t)da}{\int_0^{\omega} w(a,t)da} \\ &= \frac{\int_0^{x_1} \dot{v}(a,t)w(a,t)da}{\int_0^{\omega} w(a,t)da} + ... + \frac{\int_{x_n}^{\omega} \dot{v}(a,t)w(a,t)da}{\int_0^{\omega} w(a,t)da} \\ &= [\bar{\dot{v}}]_0^{x_1} + ... + [\bar{\dot{v}}]_{x_n}^{\omega}. \end{aligned} \tag{7.2}$$

It should be noted that the denominator has not changed, while the numerator is partitioned over the different age groups. A similar approach is used to separate the compositional component by age,

$$\begin{aligned} C(v, \acute{w}) &= \frac{\int_0^{\omega} [v(a,t) - \bar{v}(t)] \left[\acute{w}(a,t) - \bar{\acute{w}}(t)\right] w(a,t)da}{\int_0^{\omega} w(a,t)da} \\ &= [C(v, \acute{w})]_0^{x_1} + ... + [C(v, \acute{w})]_{x_n}^{\omega}, \end{aligned} \tag{7.3}$$

where the covariance in the age group x_a to x_b is defined as

$$[C(v,\acute{w})]_{x_a}^{x_b} = \frac{\int_{x_a}^{x_b} [v(a,t) - \bar{v}(t)] \left[\acute{w}(a,t) - \bar{\acute{w}}(t)\right] w(a,t)da}{\int_0^{\omega} w(a,t)da}. \tag{7.4}$$

The covariance in the age group x_a to x_b is a component of the covariance over all ages and not a covariance over the smaller age group, the reason being that the averages $\bar{v}(t)$ and $\bar{\acute{w}}(x,t)$ refer to all ages.

The final decomposition formula (6.4) is expressed by using (7.2) and (7.3) which yields:

$$\dot{\bar{v}} = [\bar{\dot{v}} + C(v,\acute{w})]_0^{x_1} + ... + [\bar{\dot{v}} + C(v,\acute{w})]_{x_n}^{\omega} . \tag{7.5}$$

This decomposition allows us to estimate the contribution of each age group to the total change over time of any demographic variable. Another attribute of the age decomposition is the estimation in each age group of the level-1 and level-2 effects.

Table 7.1 shows the age decomposition of the change in the Mexican crude death rate (CDR) from 1965 to 1975, 1975 to 1985 and 1985 to 1995, denoted in the table by the mid-years 1970, 1980 and 1990, respectively. The selected age groups are from 0 to 9 years, 10 to 59 years, and 60 years and above. In the last row, "All ages", the additions of the level-1 and level-2 effects, columns with the legends $[\bar{\dot{\mu}}]_{x_a}^{x_b}$ and $[C(\mu,r)]_{x_a}^{x_b}$ respectively, are found for the three age groups.

Table 7.1: Age decomposition of the annual change over time of the crude death rate, $d(t)$, per thousand, in 1965-1975, 1975-1985 and 1985-1995 for Mexico.

Ages	$[\bar{\dot{\mu}}]_{x_a}^{x_b}$	$[C(\mu,r)]_{x_a}^{x_b}$	$[\bar{\dot{\mu}}]_{x_a}^{x_b} + [C(\mu,r)]_{x_a}^{x_b}$
1970			
$0-9$	-0.151	-0.003	-0.154
$10-59$	-0.034	-0.008	-0.042
60+	0.024	-0.033	-0.009
All ages	-0.161	-0.044	-0.205
1980			
$0-9$	-0.106	-0.013	-0.119
$10-59$	-0.050	-0.017	-0.067
60+	-0.001	-0.002	-0.003
All ages	-0.157	-0.032	-0.189
1990			
$0-9$	-0.046	0.008	-0.038
$10-59$	-0.047	0.002	-0.045
60+	-0.023	0.029	0.006
All ages	-0.116	0.039	-0.077

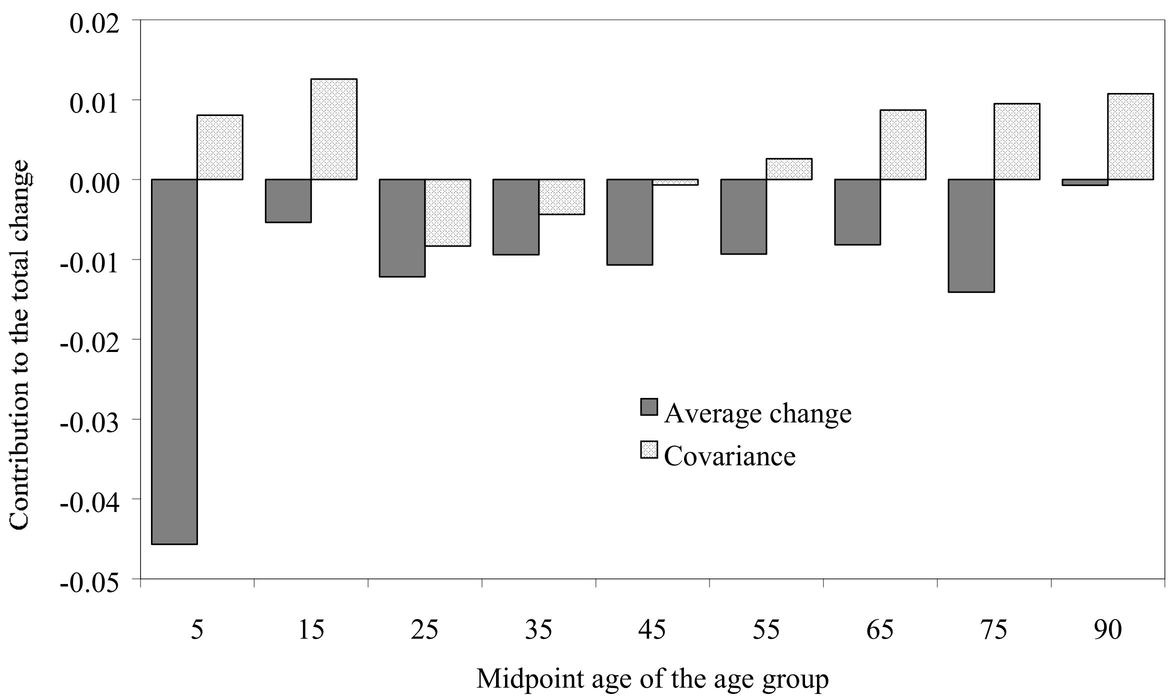
Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

The results of Table 7.1 provide a more detailed picture of the change over time in the Mexican CDR. As can be seen in the negative values of the average change $[\bar{\dot{\mu}}]_{x_a}^{x_b}$, there are improvements in mortality at all ages and periods, except for the age group 60 years and above in 1970. The covariance component, $[C(\mu, r)]_{x_a}^{x_b}$, which accounts for the effect of the change in the structure of the population, opposes the effect of the decreased mortality for all age groups in the year 1990.

In the last column it is shown how in the first two periods the age group 0 to 9 is the major contributor to the decrease in the Mexican CDR, while persons aged 10 to 59 account for the greatest share of the decrease in 1990. In the last period the opposing effect of the compositional component generates an increase in mortality for people aged 60 years and above.

Figure 7.1 shows a bar chart of the components contributing to the change in the Mexican crude death rate from 1985 to 1995. We have analyzed the effects by dividing the population into ten age groups. Again, we find that the major contribution to the decrease in the Mexican

Figure 7.1: Age decomposition of the annual change over time in the Mexican crude death rate from 1985 to 1995.



CDR in 1990 is due to changes in the ages 0 to 9. In contrast, persons aged 80 years and above, in Figure 7.1 indicated by the bar columns at age 90, show the smallest reduction in mortality due to average changes. In this figure the unbalanced contribution by age of the compositional component becomes evident. Among persons less than 20 years or above age 60 we find positive covariance, while negative values are found for the age groups 20 to 59.

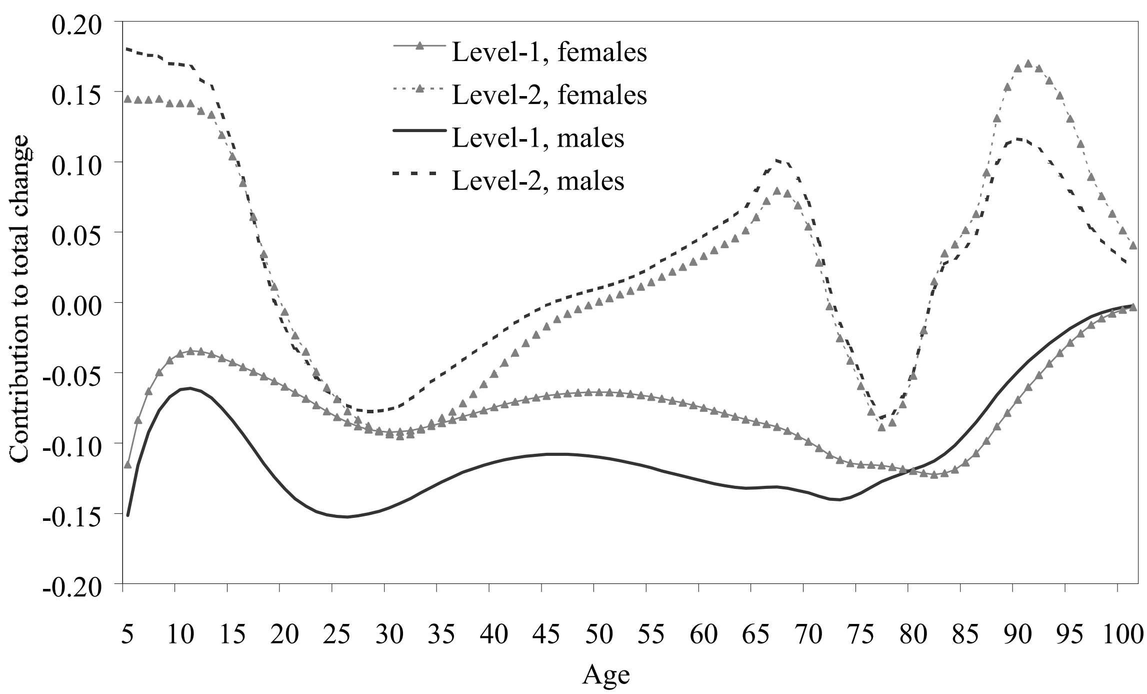
In the next section a single age decomposition is shown. Here, it is shown that the fluctuations over age of the compositional component are due to changes in the age-specific growth rates. These changes are the product of past and present demographic processes.

7.3 Single Age Decomposition

Let the age intervals of the terms in equation (7.5) be as small as one year of age. The direct and compositional components in equation (7.5) can also be calculated for each single age. In this section we examine the reasons for the numerous fluctuations of the compositional component throughout a single-age decomposition.

Figure 7.2 shows a line chart of the values obtained for the direct and the compositional components of the change in the male and female Mexican crude death rates from 1985 to 1995. The values of these components were so large at ages 0, 1 and 2, that the figure is restricted to 5 to 100. The level-1 effects are marked by continuous lines while the level-2 effects are marked by broken lines. Both female components are marked by triangles to distinguish them from their male counterparts. The data used in this section was supplied by Virgilio Partida of the National Population Board of Mexico, Conapo (2002).

Figure 7.2: The direct effect and the compositional effect, in percentages, of the annual change over time in the Mexican male and female crude death rates from 1985 to 1995.



In Figure 7.2 we see that the direct component only has values below zero indicating a decrease in the age-specific death rates at all ages during the analyzed period. The compositional component shows the fluctuations already noted in Figure 7.1. To understand the compositional component it is necessary to look at the elements of this component in equation (7.4).

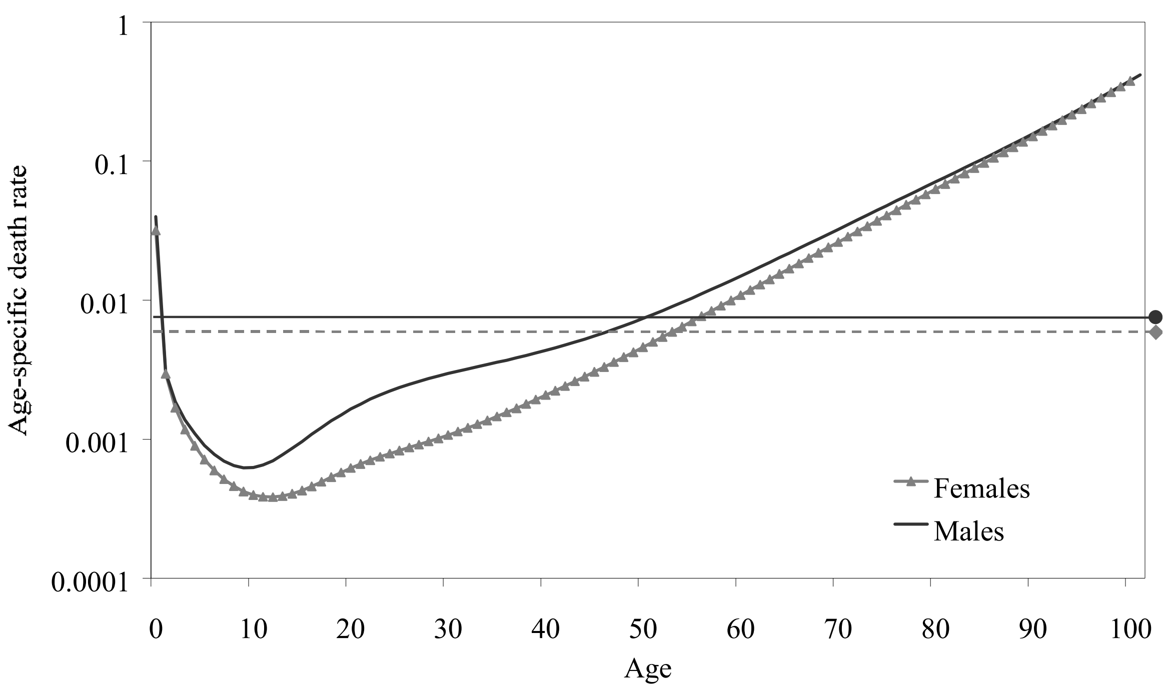
The compositional component of the change over time in the CDR in the single age decomposition is

$$[C(\mu, r)]_{a-0.5}^{a+0.5} \approx [\mu(a,t) - \bar{\mu}(t)]\,[r(a,t) - \bar{r}(t)]\,c(a,t), \tag{7.6}$$

where the term $c(a,t)$ corresponds to the proportion of the population at age a and time t with respect to the total population, $c(a,t) = \frac{N(a,t)}{\int_0^\infty N(a,t)da}$. In other words, the compositional component of a certain age takes the deviation of the age-specific death rate from the crude death rate, $[\mu(a,t) - \bar{\mu}(t)]$, and of the age-specific growth rate from the population total growth rate, $[r(a,t) - \bar{r}(t)]$, as well as the proportion of the total population at that age into account. Fluctuations in the compositional component are due to one of these three factors.

Figure 7.3 shows the age-specific death rates and the Mexican crude death rate for males and females in 1990. The male and female CDRs are indicated by straight lines in the figure, male CDR of 6.233 and female of 4.755 per thousand. After a quite high level of age-specific

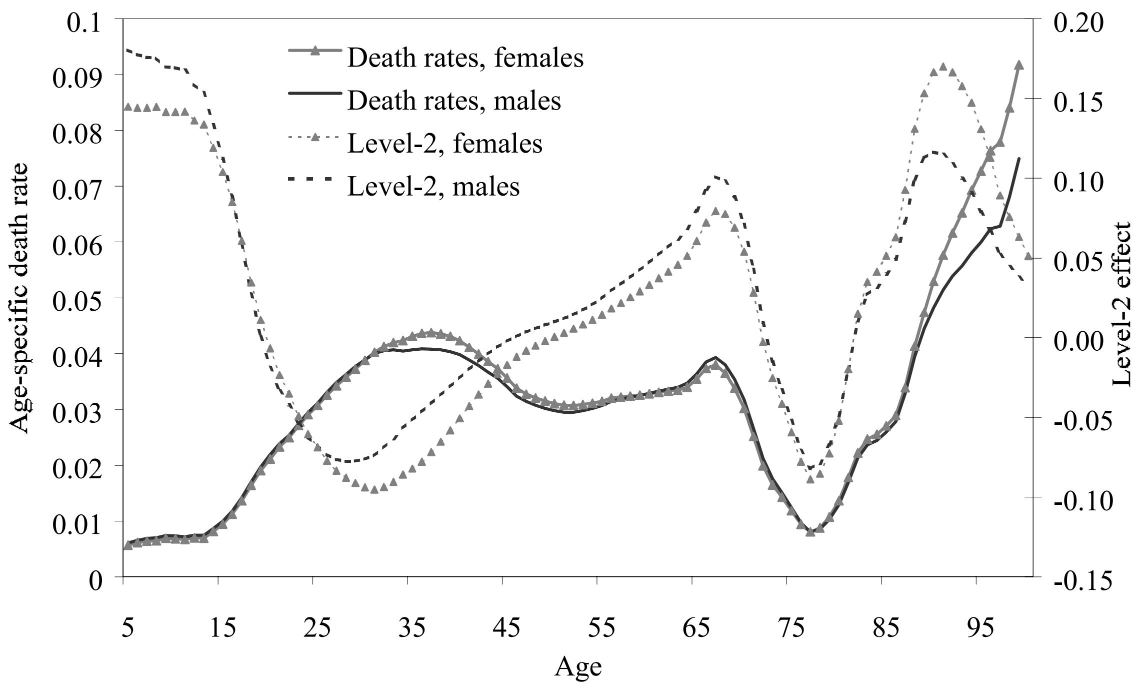
Figure 7.3: The age-specific death rates and the Mexican crude death rate for males and females in 1990.



death rates due to infant mortality, the rates reach their lowest level around age 10. The age-specific death rates for both males and females increase thereafter monotonically. Male age-specific death rates are above the female rates at all ages, with the gap between the sexes most pronounced from age 10 to 60. Around age 50 male and female age-specific death rates have the same values as the crude death rates. This means the difference between the death rates $[\mu(a,t) - \bar{\mu}(t)]$ in equation (7.6) can explain the sign (positive or negative) and magnitude of the compositional component but not the fluctuations we find in Figure 7.2. We reach a similar conclusion for the Mexican population composition, $c(a,t)$, which decreases with age and, therefore, only contributes to the magnitude of the level-2 effect. Further, the product of the difference of death rates and the population composition yields a nearly straight line. We conclude that in order to understand the fluctuations of the compositional component it is necessary to examine the differences between the age-specific growth rates and the population total growth rate, $[r(a,t) - \bar{r}(t)]$.

Figure 7.4 shows the male and female age-specific growth rates together with the level-2 effects depicted in Figure 7.2. At ages below age 50 the compositional components undergo

Figure 7.4: The Mexican male and female age-specific growth rates together with the level-2 effect of the annual change over time in the crude death rate.



exactly the same fluctuations as the age-specific growth rates but in the opposite direction and of higher magnitude. After age 50 the patterns of the fluctuations for both curves follow the same direction, again with higher levels for the compositional component.

In general, the change over time of the population composition does not only depend on the change experienced during the analyzed period, but also on changes of the past. All the information needed to estimate this demographic dynamic is found in a series of age-specific growth rates. Any of the growth rates' decompositions shown in Chapter 4 can be used to examine the components of this change. In other words, much could be learned from substituting the components of the growth rate in the covariance term in equation (6.4). For example, if we recall the decomposition in equation (4.33) for the age-specific growth rates

$$r(a,t) = r(0,t-a) - \varphi(a,t),$$

where $r(0, t-a)$ is the growth rate of the number of births $t-a$ years earlier, and $\varphi(a,t)$ is the cumulation of changes in the cohort age-specific mortality rates up to age a at time t. By substituting equation (4.33) in the decomposition of the crude death rate we obtain

$$\begin{aligned} \dot{\bar{\mu}} &= \bar{\dot{\mu}} + C(\mu, r) \\ &= \bar{\dot{\mu}} + C(\mu, r_0) - C(\mu, \varphi), \end{aligned}$$

where r_0 here denotes $r(0, t-a)$. The CDR can then be decomposed further by age and single age to obtain the contributions of the level-2 effects. These effects are (partly) based on the birth growth rates and the cohort changes in mortality for every age.

When applying the main equation (6.4) to decompose the change of the crude death rate it can be expressed as equation (6.7). The direct component captures mainly the change in the crude death rate due to changes in the age-specific death rates. The compositional component accounts for the change in the crude death rate due to changes in the population structure. By applying the single age decomposition we can see that the compositional component fluctuations are explained by the past and present demographic history within the age-specific growth rates.

7.4 Categorical Decomposition

As shown in the previous sections, equation (6.4) can be further decomposed by age due to the additive property of integrals. It is possible to measure the contribution of a specific age group to the total change over time, as well as the level-1 and level-2 effects of each age group.

Equation (2.1), for a demographic average, has two versions: one for the case of x being continuous and another for the discrete or categorical x. In both cases t was continuous. Age a is a particular case of x being continuous. From this variable we developed the age decomposition. In the same way, it is possible to decompose any category that x stands for. This section presents a categorical decomposition of the change over time in demographic variables.

Let i be the discrete case of x varying from $i = 0, ..., x_n$. Analogous to the property (7.1) of integrals any series of the categories i of the variable $v_i(t)$ can be separated into groups of categories

$$\sum_{i=0}^{x_n} v_i(t) = \sum_{i=0}^{x_1} v_i(t) + \sum_{i=x_2}^{x_3} v_i(t) + ... + \sum_{i=x_{n-1}}^{x_n} v_i(t). \tag{7.7}$$

From equation (6.4), in the case of categorical x, it follows that the direct and compositional components can be further decomposed as

$$\begin{aligned}\dot{\bar{v}} &= \bar{\dot{v}} + C(v,\acute{w}) \\ &= \left[\frac{\sum_{i=0}^{x_1}\dot{v}_i(t)w_i(t)}{\sum_{i=0}^{x_1}w_i(t)}\right]\left[\frac{\sum_{i=0}^{x_1}w_i(t)}{\sum_{i=0}^{x_n}w_i(t)}\right] + ... \\ &\quad + \left[\frac{\sum_{i=x_{n-1}}^{x_n}\dot{v}_i(t)w_i(t)}{\sum_{i=x_{n-1}}^{x_n}w_i(t)}\right]\left[\frac{\sum_{i=x_{n-1}}^{x_n}w_i(t)}{\sum_{i=0}^{x_n}w_i(t)}\right] \\ &\quad + [C(v,\acute{w})]_0^{x_1} + ... + [C(v,\acute{w})]_{x_{n-1}}^{x_n}, \end{aligned} \tag{7.8}$$

where the covariance in the categories x_a to x_b is defined as

$$[C(v,\acute{w})]_{x_a}^{x_b} = \frac{\sum_{i=x_a}^{x_b}\left[v_i(t) - \bar{v}(t)\right]\left[\acute{w}_i(t) - \bar{\acute{w}}(t)\right]w_i(t)}{\sum_{i=0}^{x_n}w_i(t)}. \tag{7.9}$$

The general categorical decomposition is then

$$\dot{\bar{v}} = [\bar{\dot{v}}\pi + C(v,\acute{w})]_0^{x_1} + ... + [\bar{\dot{v}}\pi + C(v,\acute{w})]_{x_{n-1}}^{x_n}, \tag{7.10}$$

where $[\bar{\dot{v}}]_{x_a}^{x_b}$ is now the average change in the categories x_a to x_b and $[\pi]_{x_a}^{x_b}$ the proportion of the population in these categories $[\pi]_{x_a}^{x_b} = \frac{\sum_{i=x_a}^{x_b}w_i(t)}{\sum_{i=0}^{x_n}w_i(t)}$. Note that the term $[\pi]_{x_a}^{x_b}$ is necessary to obtain a direct component for the category x_a to x_b.

Vaupel (1992) demonstrated a decomposition of the change in the world's population growth rate, $\bar{r}(t)$. By applying equation (6.4) the decomposition of the change over time of $\bar{r}(t)$ is equal to the average change in growth rates of all the countries plus the variance in the growth rates

$$\dot{\bar{r}} = \bar{\dot{r}} + \sigma^2(r). \tag{7.11}$$

As shown in Table 7.2, the growth of the world population started to decline around 1980. The pace of this decline was slowed by the variance in growth rates among the world's countries. The average change in country growth rates, $\bar{\dot{r}}$, was negative but the variance term changed this in the late 1970s, yielding an increase in the growth rate of 0.328 per 10,000.

To obtain the contribution of n regions of the world we apply the categorical decomposition demonstrated in equation (7.10) to the decomposition of the population growth rate (7.11). We then get

$$\begin{aligned}\dot{\bar{r}} &= \left[\bar{\dot{r}}\pi + \sigma^2(r)\right]_1 + ... + \left[\bar{\dot{r}}\pi + \sigma^2(r)\right]_n \\ &= \sum_{i=1}^{n}\left[\bar{\dot{r}}\pi + \sigma^2(r)\right]_i, \end{aligned} \tag{7.12}$$

where in region i, the average change in the growth rate is denoted as $[\bar{\dot{r}}]_i$, the proportion of the population in the region is $[\pi]_i$, and $[\sigma^2(r)]_i$ is the variance in growth rates. Table 7.3 shows the regional decomposition of the world's population growth rate from 1975-1980 to 1978-1983, around 1979.

Table 7.2: Population growth rate of the world, $\bar{r}(t)$, and decomposition of the annual change over time around January 1, 1979 and January 1, 1982.

t	1979	1982
$\bar{r}(t-1.5)$	1.722 %	1.732 %
$\bar{r}(t+1.5)$	1.732 %	1.711 %
$\dot{\bar{r}}(t)$	0.328 *	-0.716 *
$\bar{\dot{r}}$	-0.459 *	-1.545 *
$\sigma^2(r)$	0.787 *	0.829 *
$\dot{\bar{r}} = \bar{\dot{r}} + \sigma^2(r)$	0.328 *	-0.716 *

Source: Author's calculations described in Chapter 9, based on the U.S. Census Bureau (2001). Note: * denotes per 10,000. Growth rates were calculated over 5-year intervals (1975-1980, 1978-1983, 1981-1986) in order to estimate growth rates for 1977.5, 1980.5 and 1983.5. Growth rates were estimated based on data for all the countries of the world for which data were available.

The row labeled "World" comprises the addition of the level-1 and level-2 effects over all regions. As seen in the last column, $[\dot{\bar{r}}]_i$, the American and European regions countered the increase in the population growth seen in the African and the Asian regions. For the American and European regions, a similar picture is obtained by looking at the average change in the population growth, $[\bar{\dot{r}}\pi]_i$, while Africa has on average an increasing population growth. As a result of these differences between the continents the direct effect on the world growth is -0.459. The Asian and European regions have variances of the order of 0.280, while very little variation is found in the American region. The total change is an increase in the growth rate of the world.

7.5 Cause of Death Decomposition

This section presents a generalization of the decomposition of the change over time in life expectancy in equation (6.35). Let $\mu_i(a,t)$ be the force of mortality from cause of death i at age a and time t. The chance of surviving cause i, i.e., not dying from cause i, is then $\ell_i(a,t)$. For competing, independent causes of death we have $\ell(a,t) = \ell_1(a,t)...\ell_n(a,t)$. Hence,

$$e^o(0,t) = \int_0^\omega \ell(a,t)da = \int_0^\omega \ell_1(a,t)...\ell_n(a,t)da, \tag{7.13}$$

and the change over time is

$$\begin{aligned}
\dot{e}^o(0,t) &= \int_0^\omega \dot{\ell}_1(a,t)...\ell_n(a,t)da + ... + \int_0^\omega \ell_1(a,t)...\dot{\ell}_n(a,t)da \\
&= \int_0^\omega \acute{\ell}_1(a,t)\ell_1(a,t)...\ell_n(a,t)da + ... + \int_0^\omega \acute{\ell}_n(a,t)\ell_1(a,t)...\ell_n(a,t)da \\
&= \int_0^\omega \acute{\ell}_1(a,t)\ell(a,t)da + ... + \int_0^\omega \acute{\ell}_n(a,t)\ell(a,t)da. \qquad (7.14)
\end{aligned}$$

Table 7.3: Regional decomposition of the annual change over time in the world's population growth rate around 1979, per 10,000.

	Average change in growth rate	*Weight*	*Average weighted change in growth rate*	*Variance in growth rate*	*Change in average growth rate*
i	$[\bar{\dot{r}}]_i$	$[\pi]_i$	$[\bar{\dot{r}}\pi]_i$	$[\sigma^2(r)]_i$	$[\dot{\bar{r}}]_i$
Africa	2.328	0.105	0.244	0.157	0.401
America	-2.741	0.138	-0.378	0.070	-0.307
Asia	-0.062	0.577	-0.036	0.276	0.240
Europe	-1.609	0.180	-0.290	0.284	-0.006
World			-0.459	0.787	0.328

Source: Author's calculations described in Chapter 9, based on the U.S. Census Bureau (2001). Growth rates were estimated over intervals of 5 years (1975-1980, 1978-1983, 1981-1986) for all the countries of the world that had available data. The Asian region includes the Middle East and Oceania, and the American region includes North and South America.

Each of the terms in (7.14) can be reexpressed, by using the fact that $\ell_i(a,t) = e^{-\int_0^a \mu_i(x,t)dx}$ and the remaining life expectancy in equation (6.32), we obtain

$$\int_0^\omega \ell(a,t)\acute{\ell}_i(a,t)da = -\int_0^\omega \ell(a,t)\int_0^a \dot{\mu}_i(x,t)dxda$$
$$= -\int_0^\omega \dot{\mu}_i(a,t)\int_a^\omega \ell(x,t)dxda = -\int_0^\omega \dot{\mu}_i(a,t)\ell(a,t)e^o(a,t)da. \tag{7.15}$$

Thus

$$\dot{e}^o(0,t) = -\sum_{i=1}^n \int_0^\omega \dot{\mu}_i(a,t)\ell(a,t)e^o(a,t)da. \tag{7.16}$$

This equation is the continuous version of the discrete difference equation (4.7) presented by Pollard (1982, 1988). The change in expectation of life of a population between time t and $t+h$ in Pollard's formulation is:

$$e^o(0,t+h) - e^o(0,t) = \sum_{i=1}^n \int_0^\omega [\mu_i(a,t) - \mu_i(a,t+h)]\frac{\ell(a,t+h)e^o(a,t) + \ell(a,t)e^o(a,t+h)}{2}da. \tag{7.17}$$

Let $\rho_i(a,t)$ denote the pace of reduction of mortality from cause i, $\rho_i(a,t) = -\acute{\mu}_i(a,t)$. The proportion of deaths from cause i at age a and time t is $f_i(a,t) = \mu_i(a,t)\ell(a,t)$. It then follows from (7.16) that

$$\dot{e}^o(0,t) = \sum_{i=1}^n \int_0^\omega \rho_i(a,t)e^o(a,t)f_i(a,t)da. \tag{7.18}$$

Note how concise (and elegant) equation (7.18) is compared with (7.17).

Applying the decomposition of the average of a product (2.14) in equation (7.18), we obtain

$$\overline{\rho_i e^o} = \bar{\rho}_i(t) e_i^{\dagger}(t) + C_{f_i}(\rho_i, e^o), \tag{7.19}$$

which yields

$$\dot{e}^o(0,t) = \sum_{i=1}^{n} \left[\bar{\rho}_i(t) e_i^{\dagger}(t) + C_{f_i}(\rho_i, e^o)\right] F_i(t), \tag{7.20}$$

where

$$F_i(t) = \int_0^{\omega} f_i(a,t)da. \tag{7.21}$$

The average pace of reduction of mortality from cause i, $\bar{\rho}_i(t)$, is

$$\bar{\rho}_i(t) = \frac{\int_0^{\omega} \rho_i(a,t) f_i(a,t)da}{\int_0^{\omega} f_i(a,t)da}, \tag{7.22}$$

$e_i^{\dagger}(t)$ is the average number of life-years lost as a result of cause of death i,

$$e_i^{\dagger}(t) = \frac{\int_0^{\omega} e^o(a,t) f_i(a,t)da}{\int_0^{\omega} f_i(a,t)da}, \tag{7.23}$$

and the covariance is between the rate of improvement in mortality from cause of death i and the remaining life expectancy at various ages,

$$C_{f_i}(\rho_i, e^o) = \frac{\int_0^{\omega} [\rho_i(a,t) - \bar{\rho}_i(t)] \left[e^o(a,t) - e_i^{\dagger}(t)\right] f_i(a,t)da}{\int_0^{\omega} f_i(a,t)da}. \tag{7.24}$$

The lifetable distribution of deaths in Japan due to different causes of death in 1980 and 1990 is shown in Table 7.4. This is a distribution of causes of death for a lifetable population in which the proportion of people at each age is determined by lifetable probabilities of survival.

Table 7.5 and Figure 7.5 present the results of applying the decomposition formula in (7.20) to the Japanese data.

Over the decade from 1980 to 1990, Japanese life expectancy rose from 75.91 to 78.80 years, with an estimated annual increase of $\dot{e}^o(0,1985) = 0.288$. Three fifths of this increase in life expectancy at birth can be attributed to a reduction in mortality due to cerebrovascular disease and heart disease. This is indicated in Table 7.5, where $\dot{e}_i^o(0)$ for heart disease is 0.044 and $\dot{e}_i^o(0)$ for cerebrovascular disease is 0.129. The sum, 0.173, accounts for 60% of the total change, $\dot{e}_i^o(0)$, of 0.288.

On average, death rates from malignant neoplasms and infectious diseases increased, yielding negative values of $\bar{\rho}_i$ and negative level-1 changes. As opposed to this, the level-2 changes for these causes of death had positive values, because improvements were made at younger ages with high remaining life expectancy. As a result of the balance between level-1 and level-2

Table 7.4: Causes of death distribution for Japan in 1980 and 1990.

Cause of death	1980 %	1990 %
Heart disease	21.4	23.7
Malignant neoplasm	18.5	21.6
Cerebrovascular disease	24.3	16.1
Infectious diseases	8.6	12.8
Violent deaths	4.6	4.5
Stomach, liver and kidney disorders	4.3	4.3
Senility without psychosis	7.4	5.0
Other causes	10.9	12.0
All causes of death	100.0	100.0

Source: Based on the Berkeley Mortality Database (2001). Heart disease includes hypertensive disease. Other causes of death are those denoted in the Berkeley Mortality Database (2001) as Other Causes, plus congenital malformations and diabetes mellitus. Infectious diseases include pneumonia and bronchitis.

changes, the final column of Table 7.5 shows only positive contributions for all the causes of death.

Similar cause of death decomposition can be applied in other mortality indexes. The crude death rate is the ratio of deaths over population at risk in a particular period. If the number of deaths are divided by cause of death then the CDR can be seen as CDR of different causes of death

$$\bar{\mu} = \bar{\mu}_1 + \bar{\mu}_2 + ... + \bar{\mu}_n, \tag{7.25}$$

where $\bar{\mu}_i$ is the crude death rate of cause of death i,

$$\bar{\mu}_i(t) = \frac{\int_0^\omega \mu_i(a,t)N(a,t)da}{\int_0^\omega N(a,t)da}. \tag{7.26}$$

Equation (6.7) for the decomposition of the CDR then becomes:

$$\dot{\bar{\mu}} = \sum_{i=1}^{n} \left[\bar{\dot{\mu}}_i + C(\mu_i, r)\right], \tag{7.27}$$

where $r(a,t)$ is the age-specific growth rate $r(a,t) = \acute{N}(a,t)$, and the terms $\bar{\dot{\mu}}_i$ and $C(\mu_i, r)$ correspond to the direct and compositional components for cause i.

Another cause of death decomposition is shown in the next chapter for the average age at death. Table 8.4 shows the average age at death in Japan and the decomposition obtained by distribution of deaths and causes of death, between 1980 and 1990.

Table 7.5: Cause of death decomposition for the annual change over time in life expectancy, for Japan from 1980 to 1990.

Cause of death	$\bar{\rho}_i(t)$ %	$e_i^\dagger(t)$	$\bar{\rho}_i(t)e_i^\dagger(t)$	$C_{f_i}(\rho_i, e^o)$	F_i %	$\dot{e}_i^o(0)$
Heart disease	2.058	8.333	0.172	0.022	22.543	0.044
Malignant neoplasm	-0.098	13.276	-0.013	0.088	20.042	0.015
Cerebrovascular disease	6.979	8.594	0.600	0.038	20.226	0.129
Infectious diseases	-0.703	7.942	-0.056	0.104	10.718	0.005
Violent deaths	1.608	23.384	0.376	0.058	4.516	0.020
Stomach, liver and kidney disorders	2.168	11.548	0.250	0.094	4.294	0.015
Senility without psychosis	9.379	4.294	0.403	0.040	6.200	0.027
Other causes	1.432	13.792	0.197	0.137	11.461	0.038
All causes of death	2.675	10.527	0.282	0.007	100.000	0.288

Source: Author's calculations described in Appendix B, based on the Berkeley Mortality Database (2001). The underlying data pertain to five-year age groups. Note that the values of $\bar{\rho}_i(t)$, $e_i^\dagger(t)$, $\bar{\rho}_i(t)e_i^\dagger(t)$ and $C_{f_i}(\rho_i, e^o)$ for all causes of death are complicated functions and not simple sums of the corresponding cause-specific values. The value of $\dot{e}^o(0)$ for all causes of death is slightly different than the sum of the values of $\dot{e}_i^o(0)$ because of approximation errors.

7.6 Conclusion

Age and categorical decompositions are useful extensions of the main decomposition formula (6.4). The contributions of the different ages (or categories) to the total change are complemented with the age and categorical decomposition of the level-1 and level-2 effects.

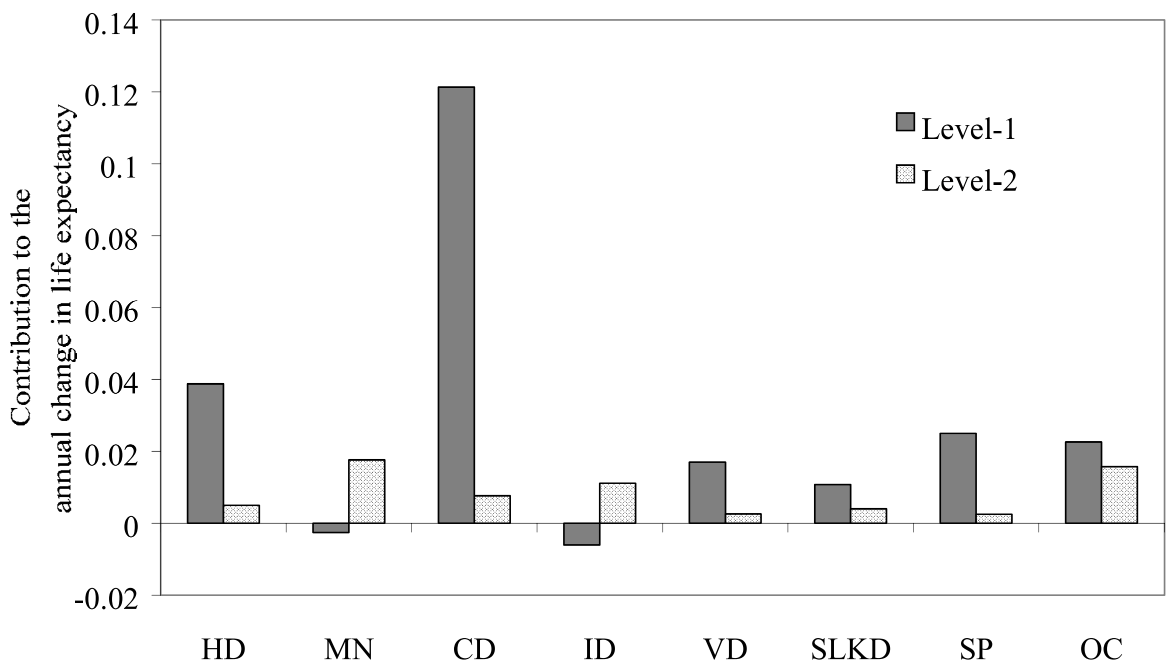
The detailed analysis of the decomposition components also helps the understanding of the changes in the different components. The applications of this chapter illustrate this. When studying the change over time in the CDR, we learned that the past and present demographic history, accounted in the age-specific growth rates, explain the compositional component fluctuations.

In the application of the categorical decomposition, we saw that the total change in the world's population growth rate is based on the changes experienced in the different regions of the world. The regional decomposition allows a detailed analysis of the components responsible for the change in various regions.

As a last decomposition in this chapter a cause of death decomposition is introduced. The change over time in life expectancy is separated into the contributions of the different causes of death. The illustration shows the change over time in the Japanese life expectancy. The direct component comprising cerebrovascular disease and heart disease explained more than half of the total change in life expectancy.

Alternatively it could be possible to combine the cause of death decomposition and the age decomposition to explain the change over time in mortality measures by both categories.

Figure 7.5: Cause of death decomposition of the annual change over time in life expectancy, from 1980 to 1990 for Japan.



Note: The abbreviations correspond to: HD-Heart disease; MN- Malignant neoplasm; CD-Cerebrovascular disease; ID-Infectious diseases; VD-Violent deaths; SLKD-Stomach, liver and kidney disorders; SP-Senility without psychosis; OC-Other causes.

8

Multidimensional Decompositions

8.1 Introduction

In Chapter 6, where the decomposition formula (6.4) was introduced, several applications are shown. These illustrations look at the change over time in demographic variables accounting for the change in the variable of interest and for the compositional effect. Two types of compositions of the population are separately studied, the age structure of the population and the composition of the population by country of residence.

Sometimes several compositions simultaneously affect the change in the average under study. In these situations, it is necessary to have an extension of equation (6.4) to decompose the change over time of the demographic variable. This chapter extends the decomposition (6.4) to the case of numerous dimensions of population heterogeneity contained within the same formula.

In Chapter 3 we presented Kitagawa's decomposition formula which was later extended for situations involving numerous compositions. The methodologies of Cho and Retherford (1973), Kim and Strobino (1984) and Das Gupta (1978) have been used in demography as extensions. A similar approach, suggested by Oosterhaven and Van der Linden (1997) for economic variables is the structural decomposition. These extensions inspired our present extension of equation (6.4).

The total crude death rate, CDR, of selected European countries is calculated as the average of the countries' crude death rates weighted by the population size. There are three components influencing the change of this CDR: changes in the age- and country-specific death rates, changes in the age structure of each country and changes in the population distribution over countries. This chapter shows how to perform an extension of equation (6.4) when demographic variables include numerous structures of the population.

Two ways of solving this problem are shown here. The first method follows Cho and

Retherford's (1973) idea of applying the decomposition formula twice. This part also refers to the work by Kim and Strobino (1984) where some hierarchy among the compositional components is imposed.

The second version is related to the proposals made by Das Gupta (1978) and Clogg's (1978) methodology. Das Gupta's (1978) work is used for separating the weighting function into parts corresponding to each compositional component. Another option for this extension is found by using the Clogg's (1978) methodology of "purging" the undesired factors. In these approaches the different terms account for the different compositional components studied.

8.2 Averages of Averages

8.2.1 Average Age of the Population

Suppose a population consists of a number of subpopulations. Let the average age in each subpopulation i be $\bar{a}_i(t)$,

$$\bar{a}_i(t) = \frac{\sum_{a=0}^{\omega} a n_{ia}(t)}{\sum_{a=0}^{\omega} n_{ia}(t)}, \tag{8.1}$$

where $n_{ia}(t)$ is the age-specific subpopulation size. Let the age-specific growth rates be $r_{ia}(t)$, the total subpopulation size $N_i(t)$, and the overall growth rates $\bar{r}_i(t)$. The average age of the entire population $\tilde{\bar{a}}(t)$ is defined as

$$\tilde{\bar{a}}(t) = \frac{\sum_i \bar{a}_i(t) N_i(t)}{\sum_i N_i(t)}. \tag{8.2}$$

The change over time of the average age of the entire population is due to two reasons. A change in the age structure "within" each subpopulation, and a change "between" the structure of the population given by the subpopulations. Which part of the change in the average age is due to changes "within" the subpopulations, and which part is due to changes "between" the subpopulations?

By applying equation (6.4) to (8.2) the decomposition of the change over time of the average age of the entire population is as follows

$$\dot{\tilde{\bar{a}}}(t) = \tilde{\dot{\bar{a}}} + C_N(\bar{a}, \bar{r}), \tag{8.3}$$

where the two terms on the right-hand side of the formula are weighted by the total subpopulation sizes $N_i(t)$. The first term of the decomposition $\tilde{\dot{\bar{a}}}$ is the average of the changes in the averages. This component can be further decomposed by applying equation (6.4) to each change in the average $\dot{\bar{a}} = C_n(a, r)$, presented in equation (6.2). The final decomposition is

$$\dot{\tilde{\bar{a}}}(t) = \widetilde{C_n(a, r)} + C_N(\bar{a}, \bar{r}). \tag{8.4}$$

The average of the covariance, $\widetilde{C_n(a, r)}$, is the direct effect and corresponds to the change "within" the subpopulations. This component explains the variation experienced in each of the subpopulations. The average of these covariances within each subpopulation is the level-1 effect. Note that the covariances are between ages and age-specific growth rates, $r_{ia}(t)$, and that the covariances use the subpopulation sizes $n_{ia}(t)$ as weighting functions.

The second covariance is the compositional component and corresponds to the change "between" the subpopulations. It captures the variations in the composition of the subpopulations. Note that the covariances are between average ages of the subpopulations, $\bar{a}_i(t)$, and the overall growth rates in the subpopulations, $\bar{r}_i(t)$, and they employ the total subpopulation sizes $N_i(t)$ as weighting functions.

Table 8.1 applies the decomposition in equation (8.4) to the change in the average age of the American population, divided into subpopulations. These subpopulations are defined by age and ethnic group: non-Hispanic white, non-Hispanic black, non-Hispanic Asian and Pacific Islander, American Indian and other non-Hispanic, and Hispanic. The table shows the

Table 8.1: Average age of United States population, $\tilde{\bar{a}}(t)$, calculated as the average over age and race, and decomposition of the annual change over time from 1990 to 2000.

$\tilde{\bar{a}}(1990)$	35.212
$\tilde{\bar{a}}(2000)$	36.491
$\dot{\tilde{\bar{a}}}$	0.128
$C_n\widetilde{(a,r)}$	0.162
$C_N(\bar{a},\bar{r})$	-0.034
$\dot{\tilde{\bar{a}}}(t) = C_n\widetilde{(a,r)} + C_N(\bar{a},\bar{r})$	0.128

Source: Author's calculations described in Chapter 9, based on the U.S. Census Bureau (2001). The population was divided in five subpopulations. These subpopulations are the different ethnic groups in the U.S.: non-Hispanic white, non-Hispanic black, non-Hispanic Asian and Pacific Islander, American Indian and other non-Hispanic, and Hispanic.

well-known aging process of the American population, evident in the positive values of the first term in equation (8.4), the average of the covariances. On average all the ethnic groups show increasing average age. This is the increase in the change "within" the ethnic groups.

On the other hand, the second term of (8.4) focuses on the different growth rates of the ethnic groups and their different average ages. The negative result of the covariance is due to the low average age and high population growth rate of Hispanics and the high average age and low growth rate of non-Hispanic whites. In other words, the lack of correspondence "between" the ethnic groups' average ages and growth rates results in a negative covariance.

This method of studying the heterogeneity "within" and "between" subpopulations can be simplified as the application of equation (6.4) two times. In the next section this extension is applied in the analysis of demographic variables that are not average ages.

8.2.2 Averages of Averages, Generalization

The previous section showed that an important case of averages is when they also involve averages. The development here is similar to the method proposed by Cho and Retherford (1973) to apply Kitagawa's decomposition formula twice. Here, we undertake a similar approach with equation (6.4).

Let x and z be two characteristics that divide the population into subpopulations. For example, in the previous section we had age and ethnic group. If $v(x,z,t)$ is a certain demographic function and $w(x,z,t)$ a weighting function, then the total average is the double integral over both characteristics x and z, expressed as

$$\tilde{\bar{v}}(t) = \frac{\int_0^\omega \bar{v}(x,t)W(x,t)dx}{\int_0^\omega W(x,t)dx}, \tag{8.5}$$

where $W(x,t)$ is the total weight over the characteristic z, $W(x,t) = \int_0^\omega w(x,z,t)dz$, and $\bar{v}(x,t)$ is the average over the characteristic z,

$$\bar{v}(x,t) = \frac{\int_0^\omega v(x,z,t)w(x,z,t)dz}{\int_0^\omega w(x,z,t)dz}. \tag{8.6}$$

It has to be noted that the order in which the indices are elected may not change the resultant overall average. This is proved by noting that $\tilde{\bar{v}}(t) = \bar{\tilde{v}}(t)$. Denoting $\int_x$ and $\int_z$ the integrals over x and z respectively and from the definition of the average of average $\tilde{\bar{v}}(t)$ we have

$$\tilde{\bar{v}}(t) = \frac{\int_0^\omega \bar{v}(x,t)W(x,t)dx}{\int_0^\omega W(x,t)dx} = \frac{\int_x \int_z v(x,z,t)w(x,z,t)dzdx}{\int_x \int_z w(x,z,t)dzdx}$$

and exchanging the integrals we obtain

$$= \frac{\int_z \int_x v(x,z,t)w(x,z,t)dxdz}{\int_z \int_x w(x,z,t)dxdz} = \frac{\int_0^\omega \tilde{v}(z,t)W(z,t)dz}{\int_0^\omega W(z,t)dz} = \bar{\tilde{v}}(t). \tag{8.7}$$

Nevertheless, the components of the decomposition may have different values depending on the selection of $\tilde{\bar{v}}(t)$ or $\bar{\tilde{v}}(t)$. The decomposition of the change over time of equation (8.5) is

$$\dot{\tilde{\bar{v}}}(t) = \tilde{\dot{\bar{v}}} + C_W(\bar{v}, \bar{r}), \tag{8.8}$$

where $\bar{r}(x,t) = \acute{W}(x,t)$ is the intensity with respect to time of the weighting function $W(x,t)$, and $W(x,t)$ is the weight of both terms on the right-hand side, average and covariance. The term $\tilde{\dot{\bar{v}}}$ of equation (8.8) is the average of the key equation (6.4) of Chapter 6, $\dot{v} = \bar{\dot{v}} + C(v, \acute{w})$. Therefore we can further express this result as

$$\dot{\tilde{\bar{v}}}(t) = \tilde{\bar{\dot{v}}} + C_w\widetilde{(v,r)} + C_W(\bar{v}, \bar{r}), \tag{8.9}$$

where $r(x,z,t) = \acute{w}(x,z,t)$ is the intensity with respect to time of the weighting function $w(x,z,t)$. Another decomposition of the same average of averages is applied to the right side of equation (8.7). To find the decomposition of the change in the average $\bar{\tilde{v}}(t)$, when we take first the average over the values of x and then over z, we obtain

$$\dot{\bar{\tilde{v}}}(t) = \bar{\tilde{\dot{v}}} + \overline{C_w(v,r)} + C_W(\tilde{v}, \tilde{r}). \tag{8.10}$$

Kim and Strobino (1984) noted the importance of hierarchy among the compositional components. If there is some factor which might be more important for the dynamics under study it is reasonable to try to capture first its contribution to the total change. Equations (8.9) and (8.10) allow the possibility of reflecting this hierarchy among the compositional components. Choosing among them depends on the relevance of the variables x or z in the study.

By applying equation (8.9) to the formula of the total average age over subpopulations (8.2) and taking into account that $\dot{a} = 0$, equation (8.4) is obtained. The change in the average age of the population can be expressed as an average of covariances plus a covariance of averages

$$\dot{\bar{\tilde{a}}}(t) = C_n\widetilde{(a, r)} + C_N(\bar{a}, \bar{r}). \tag{8.11}$$

Equations (8.9) and (8.10) show the two possibilities of the decomposition for the change over time of $\bar{\tilde{v}}$ and $\tilde{\bar{v}}$, respectively. The average age of the population can only be expressed as one of these averages $\bar{\tilde{a}}$. This is a consequence of the version of $\tilde{\bar{v}}$ applied to the average age of the population which is no longer an average of averages, but is only one average $\tilde{\bar{a}} = \bar{a}$. This follows because the averages over countries are simply equal to the fixed age,

$$\tilde{a}(t) = \frac{\sum_i a n_{ia}(t)}{\sum_i n_{ia}(t)} = a. \tag{8.12}$$

Therefore the decomposition of changes "within" and "between" the subpopulations in the average age of the population is uniquely defined by (8.11).

8.3 Simplifying a Complex Average

8.3.1 Crude Death Rate of a Group of Countries

In the applications of Chapter 6 two types of compositions of the population are separately studied. The composition component is in some examples the age structure of the population, and in other examples the composition of the population is by country of residence.

The previous section presented a method for studying both compositions of the population together. Here we present an alternative procedure. In this section it is shown that it is possible to separate the change over time in a demographic variable into two terms. One term corresponds to the direct effect and the other term accounts for compositional effects, age and country of residence all taken together.

Suppose the aim is to study the crude death rate of a group of countries, for example a selected group of European countries. The CDR of these European countries, denoted as $\bar{d}_E(t)$, can be calculated as an average of the crude death rates of the countries, $d_c(t)$, weighted by the population sizes of each country $N_c(t)$, with $c \in E$ indicating that all selected countries in E are considered, $c = 1, ..., 14$,

$$\bar{d}_E(t) = \frac{\sum_{c\in E} d_c(t) N_c(t)}{\sum_{c\in E} N_c(t)}. \tag{8.13}$$

On the other hand, the CDR for every country, $d_c(t)$, is also an average of the age-specific death rates over age, $m_{ac}(t)$. As in equation (2.7) for country c we have

$$d_c(t) = \frac{\sum_{a=0}^{\omega} m_{ac}(t) N_{ac}(t)}{\sum_{a=0}^{\omega} N_{ac}(t)}, \tag{8.14}$$

where a has 12 age groups 0 to 1, then 1-9, 10-19, and so on until 90-99 and 100 and above. The term $m_{ac}(t)$ is the age- and country-specific death rate and $N_{ac}(t)$ is the population size of country c and age a at time t. Substituting (8.14) in equation (8.13) changes an average of averages to an average. This is obtained by substituting the CDR of every country and keeping in mind that the total population size of each country is equal to the addition of the population sizes at all ages $N_c(t) = \sum_{a=0}^{\omega} N_{ac}(t)$,

$$\bar{d}_E(t) = \frac{\sum_{c \in E} \sum_{a=0}^{\omega} m_{ac}(t) N_{ac}(t)}{\sum_{c \in E} \sum_{a=0}^{\omega} N_{ac}(t)}. \tag{8.15}$$

For every age a and every country c we could define one level of a new variable k in the cross-classified contingency table of ages and countries. The total number of cells in this cross-classification comprises 148 levels, the product of 14 countries by 12 age groups, the product of c and a. The average of averages changes to an average of two single arguments, time t and the variable k that depends on age and country,

$$\bar{d}_E(t) = \frac{\sum_{k=1}^{ac} m_k(t) N_k(t)}{\sum_{k=1}^{ac} N_k(t)}. \tag{8.16}$$

This allows a simpler decomposition of the kind in equation (6.4),

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r), \tag{8.17}$$

where $r(t) = r_k(t)$ is the age- and country-specific growth rate, $r_k(t) = \acute{N}_k(t)$.

Table 8.2 shows the change in the CDR of selected European countries and the decomposition of these changes. The countries are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. The analyzed periods are from 1960 to 1970 denoted in the table as 1965, from 1975 to 1985 denoted as 1980, and from 1992 to 1997 denoted as 1994. Russia was included in the last two columns of the table. The crude death rate of these selected European countries increased in all the studied periods. In the first two columns, 1965 and 1980, there are important changes in survivorship with negative average changes. On average all the countries' mortality rates decreased, seen in the $\bar{\dot{m}}$ term. But there were also significant contributions of the compositional component that opposed the decrease. As a result the CDR of these selected countries increased during the periods. For the last period, 1992 to 1996 in column 1994, both terms contribute to the increase in the crude death rate. In all the studied periods in Table 8.2 the covariance component is the main contributor to the observed change. These covariance accounts for the age structure and the population distribution over countries.

An immediate question the separation of the contribution of the age structure and the population structure over countries.

In the next section the extension of equation (6.4) that accounts for each of the compositional components is shown. Before carrying out further extension we show how it is possible to simplify a complex rate when more than two compositional components are involved.

Table 8.2: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996 for selected European countries.

	1965	1980	1994
$\bar{d}_E(t)$	10.783	10.720	11.187
$\bar{d}_E(t-h/2)$	10.720	10.591	10.916
$\bar{d}_E(t+h/2)$	10.892	10.964	11.594
$\dot{\bar{d}}_E(1994)$	0.017	0.037	0.170
$\dot{\bar{m}}$	-0.090	-0.112	0.036
$C(m,r)$	0.108	0.150	0.133
$\dot{\bar{d}}_E = \dot{\bar{m}} + C(m,r)$	0.018	0.038	0.169

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 eleven-year periods were used (1960-1970 and 1975-1985). For the year 1994 a five-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

8.3.2 Simplifying a Complex Average, Generalization

In general it is possible to change the additions over indexes x_1, x_2, ..., x_m for a single addition. Let x_1 have n_1 categories, x_2 have n_2 categories and so on until x_m with n_m categories. The new single addition over a new index k has a total number of categories equal to the product of $n_1 n_2 ... n_m$. Assuming that the demographic variable $\bar{v}(t)$ can be written as

$$\bar{v}(t) = \frac{\sum_{x_1}\sum_{x_2}...\sum_{x_m} v_{x_1 x_2 ... x_m}(t) w_{x_1 x_2 ... x_m}(t)}{\sum_{x_1}\sum_{x_2}...\sum_{x_m} w_{x_1 x_2 ... x_m}(t)},$$

where $\sum_{x_i}$ is the addition over the different levels of x_i. Define the new variable k as a function of the different levels of x_1, x_2,... and x_m combined. We obtain a simplified version of $\bar{v}(t)$ as

$$\bar{v}(t) = \frac{\sum_{k=1}^{n_1 n_2 ... n_m} v_k(t) w_k(t)}{\sum_{k=1}^{n_1 n_2 ... n_m} w_k(t)}, \tag{8.18}$$

where $\sum_{k=0}^{n_1 n_2 ... n_m}$ indicates that k has $n_1 n_2 ... n_m$ levels.

Equation (6.4) can now be applied without regard to the number of compositional components included. The new result has two components, one for the direct effect of the variable of interest and a second one for all the compositional components taken together.

In the study of the crude death rate it could be possible to include age structure, ethnic group structure, social class structure, education structure and country of residence. By applying equation (8.18) the composition changes to one variable k which includes all the structures: age, ethnic group, social class, education and country of residence.

This procedure of changing the complexity of the multitudinous dimensions of population heterogeneity to a simpler formulation allows a further decomposition of the compositional component. This further decomposition separates the compositional component to allow the distinction between the numerous structures of the population. The next sections show different possibilities for attaining the separation of the compositional component.

8.4 Decomposing the Compositional Component

In equation (6.4) the compositional effect, $C(v, \acute{w})$, accounts for the multitudinous dimensions of population heterogeneity combined. Here an elegant formula for separating the combined compositional components is shown.

Let the reader be reminded of the property of the relative derivative of a product

$$\frac{\frac{\partial}{\partial t}uv}{uv} = \acute{u} + \acute{v}, \tag{8.19}$$

and the covariance property of separating additions

$$C(u_1 + u_2, v) = C(u_1, v) + C(u_2, v), \tag{8.20}$$

which are two properties that help to achieve the decomposition of the compositional components.

Let us assume that there are n compositional components and let the weighting function $w(t)$ be a product of weighting functions $w = w_1 w_2 ... w_n$, where each weighting function $w_i(t)$ accounts for one of the compositional factors. Following the property of intensities (8.19) the intensity of the weighting function is replaced by the addition of the intensities $\acute{w} = \acute{w}_1 + \acute{w}_2 + ... + \acute{w}_n$. Substituting this result in our equation (6.4) and taking into account the property of covariances in (8.20) we have

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}_1 + \acute{w}_2 + ... + \acute{w}_n)$$

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}_1) + C(v, \acute{w}_2) + ... + C(v, \acute{w}_n). \tag{8.21}$$

Again here the first term on the right-hand side of (8.21), the average of the changes, captures the change in the characteristic of interest, this is the direct change. The second component is the covariance term between the underlying variable of interest and the intensity of the weighting function of the compositional factor w_1. This is the structural or compositional component of change due to the compositional factor w_1. In a similar way the third component is the compositional component of change due to compositional factor w_2, and so on until the compositional component of change due to compositional factor w_n.

This formulation is applied in the study of the CDR for the selected European countries. Let us assume that the weighting functions $N_k(t)$ in (8.17) can be separated into a product of age structure weights, $N_a(t)$, and country-structure weights $N_c(t)$, $N_k(t) = N_a(t)N_c(t)$. By applying (8.21) to equation (8.17) the CDR is further decomposed as

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r_a) + C(m, r_c), \tag{8.22}$$

where the growth rates r_a and r_c correspond to the intensities of the age-structure weights $r_a(t) = \acute{N}_a(t)$, and country-structure weights $r_c(t) = \acute{N}_c(t)$, respectively.

The following sections show how to obtain a weighting function $w(a, t)$ as a product of weighting functions $w = w_1 w_2 ... w_n$.

8.5 Separating the Weighting Functions

8.5.1 Two Compositional Factors

Let us look at the case of two compositional factors x and z. The weights, $w_{xz}(t)$, are cross-classified by these two factors. To separate these weights into weights that account independently for the factors x and z, we can follow the technique suggested by Das Gupta (1994).

Let the marginal values of the two-way table be defined as $w_{x\cdot}(t) = \sum_z w_{xz}(t)$ and $w_{\cdot z}(t) = \sum_x w_{xz}(t)$. The trivial case is when each element of the cross-classification can be estimated as the product of the corresponding marginal values of the weighting function, in other words, when $w_{xz}(t) = w_{x\cdot}(t)w_{\cdot z}(t)$. That is when the compositional factors are independent. Because in most cases the elements of the two-way table differ from the product of the marginal values it is necessary to correct for these errors.

For each element $w_{xz}(t)$ the marginal value $w_{x\cdot}(t)$ is corrected by the contribution of the $w_{xz}(t)$ in the corresponding marginal value of $w_{\cdot z}(t)$. This is achieved by the product of the marginal in x and the ratio of $w_{xz}(t)$ over the marginal in z, as $\frac{w_{xz}(t)}{w_{\cdot z}(t)}w_{x\cdot}(t)$. At the same time each $w_{\cdot z}(t)$ is corrected by the contribution of the $w_{xz}(t)$ in the corresponding marginal value of $w_{x\cdot}(t)$. The resultant two elements of the product are

$$w_{xz}(t) = \left[\frac{w_{xz}(t)}{w_{\cdot z}(t)}w_{x\cdot}(t)\right]^{1/2}\left[\frac{w_{xz}(t)}{w_{x\cdot}(t)}w_{\cdot z}(t)\right]^{1/2} = w_x(t)w_z(t), \tag{8.23}$$

where $w_x(t)$ and $w_z(t)$ are the new weighting functions that account for x and z independently, which substitute the weights $w_{xz}(t)$. When each element of the weighting function is divided by the total values of the weighting function $\frac{w_{xz}(t)}{w_{\cdot\cdot}(t)}$, then the weights are normalized. These normalized weights correspond exactly to those suggested by Das Gupta (1994) and they are shown in equation (3.15).

We can now separate the different components of the change in the CDR for the selected European countries. As shown in (8.22) the decomposition of the change in the CDR of the European countries is

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r_a) + C(m, r_c),$$

where the growth rates are calculated from the age and country population size terms obtained from the substitution of $N_{ac}(t)$ in (8.23)

$$N_{ac}(t) = \left[\frac{N_{ac}(t)}{N_{\cdot c}(t)}N_{a\cdot}(t)\right]^{1/2}\left[\frac{N_{ac}(t)}{N_{a\cdot}(t)}N_{\cdot c}(t)\right]^{1/2} = N_a(t)N_c(t),$$

as $r_a(t) = \acute{N}_a(t)$ and $r_c(t) = \acute{N}_c(t)$.

The terms on the right-hand side of (8.22) account for the change in the CDR due to change in the age-specific death rates, changes in the age structure of the countries, and change in the country composition, respectively.

The decomposition of the change in the CDR of the selected European countries from 1960 to 1970, from 1975 to 1985 and from 1992 to 1996 are shown in Table 8.3. The contribution of the two compositional factors is very different. The age structure has the biggest share of

Table 8.3: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996 for selected European countries. Decomposing by direct change and two compositional components: age and country.

	1965	1980	1994
$\bar{d}_E(t)$	10.783	10.720	11.187
$\bar{d}_E(t-h/2)$	10.720	10.591	10.916
$\bar{d}_E(t+h/2)$	10.892	10.965	11.595
$\dot{\bar{d}}_E(t)$	0.017	0.037	0.170
$\bar{\dot{m}}$	-0.090	-0.112	0.036
$C(m,r)$	0.108	0.150	0.133
$C(m,r_a)$	0.110	0.148	0.134
$C(m,r_c)$	-0.002	0.002	-0.001
$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m,r_a) + C(m,r_c)$	0.018	0.038	0.169

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 eleven-year periods were used (1960-1970 and 1975-1985). For the year 1994 a five-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

the compositional component. In 1965 and 1994 the age structure contributes to the increase of the CDR, while the country composition contributes in a minor way to the decrease of the CDR. The addition of these two terms gives the total compositional change of 0.108, 0.150 and 0.133 per thousand seen in Table 8.2.

Equation (8.21) can be used in many other applications. The following section shows the decomposition of average age at death by cause of death.

8.5.2 Decomposing the Average Age at Death

Valkovics (1999) studied the decomposition of the change in life expectancy at birth expressed as the mean age at death,

$$e^o(0,t) = \frac{\int_0^\omega \ell(a,t)da}{\ell(0,t)} = \frac{\int_0^\omega a d(a,t)da}{\int_0^\omega d(a,t)da}, \tag{8.24}$$

where $d(a,t)$ denotes the number of deaths at age a and time t in the lifetable, and $\ell(a,t)$ the survival function at age a and time t. equation (8.24) shows that life expectancy is also an average. Another expression of life expectancy as an average is presented by Vaupel and Canudas Romo (2003).

A similar demographic measure is the average age at death in the population. The number of deaths at any age is equal to the addition of the deaths due to different causes of death. In mathematical notation this is $D_{a.}(t) = \sum_i D_{ai}(t)$ with i the cause of death. A new average can be obtained by exchanging sums for integrals and by inserting the sum over cause of death in equation (8.24). An alternative long averaging proposed in this chapter for the average age at death is

$$\bar{a}_D(t) = \frac{\sum_{a=0}^{\omega} \sum_i aD_{ai}(t)}{\sum_{a=0}^{\omega} \sum_i D_{ai}(t)} = \frac{\sum_{k=1}^{ai} aD_k(t)}{\sum_{k=1}^{ai} D_k(t)}, \tag{8.25}$$

where k depends on the age a and the causes of death i. Applying our main equation (6.4) the change over time in the average age at death is equal to the covariance between age and the intensity of the number of deaths

$$\dot{\bar{a}}_D(t) = C(a, \acute{D}). \tag{8.26}$$

From equation (8.21) the covariance term can be further decomposed into the change due to the distribution of deaths over age and another term for the distribution among the different causes of death

$$\dot{\bar{a}}_D(t) = C(a, \acute{D}_a) + C(a, \acute{D}_i), \tag{8.27}$$

where $\acute{D}_a$ and $\acute{D}_i$ are the intensities of the age structure of deaths and cause of death distribution respectively.

Take as an example the case of Japan. The proportion of deaths due to the different causes of death in the years 1980 and 1990 are shown in Table 7.4. Table 8.4 shows the average age at death in Japan and the decomposition obtained in (8.27) by distribution of deaths and causes of death, between 1980 and 1990. The change over time in the average age at death

Table 8.4: Average age at death, $\bar{a}_D(t)$, and decomposition of the annual change over time for Japan from 1980 to 1990. Decomposing by distribution of deaths over age and by cause of death.

$\bar{a}_D(1985)$	70.492
$\bar{a}_D(1980)$	68.629
$\bar{a}_D(1990)$	72.389
$\dot{\bar{a}}_D(1985)$	0.376
$C(a, \acute{D}_a)$	0.395
$C(a, \acute{D}_i)$	-0.019
$\bar{a}_D(t) = C(a, \acute{D}_a) + C(a, \acute{D}_i)$	0.376

Source: Author's calculations described in Chapter 9, based on the Berkeley Mortality Database (2001).

is mainly due to the first covariance $C(a, \acute{D}_a)$. This term corresponds to the distribution of deaths over age. The positive result in this term is due to a shift of deaths to older ages. The second covariance is between ages and the intensity of the cause of death distribution $C(a, \acute{D}_i)$. This term captures the change in the distribution of cause of death over age. The greatest changes in the distribution of cause of death between the studied years occur at the oldest age groups. By reducing the contribution of some causes of death the contribution of other causes increase. As a consequence for some causes of death the covariance with age is positive while for others is negative. In this application the negative covariance $C(a, \acute{D}_i)$ plays a minor role in the change of the average age at death.

8.5.3 Three Compositional Factors or More

In the case of three compositional factors it could be possible to do a similar general case using Das Gupta's formulations. The case of three compositional factors x, y and z, also follows equation (8.23) by having first the estimation of the elements in the three-way table as the product of marginal values

$$w_{xyz}(t) = w_{x\cdot\cdot}(t)w_{\cdot y\cdot}(t)w_{\cdot\cdot z}(t). \tag{8.28}$$

Following Das Gupta's methodology the marginal value $w_{x\cdot\cdot}(t) = \sum_{y,z} w_{xyz}(t)$ is corrected for the undesired interactions. The cell of the data $w_{xyz}(t)$ is corrected by the interaction marginal value of the other two components $w_{\cdot yz}(t) = \sum_x w_{xyz}(t)$, as $\frac{w_{xyz}(t)}{w_{\cdot yz}(t)}$. But also the interactions including x have to be corrected for other undesired components, $\frac{w_{xy\cdot}(t)}{w_{\cdot y\cdot}(t)}$ and $\frac{w_{x\cdot z}(t)}{w_{\cdot\cdot z}(t)}$. The same applies to the other marginal values $w_{\cdot y\cdot}(t) = \sum_{x,z} w_{xyz}(t)$ and $w_{\cdot\cdot z}(t) = \sum_{x,y} w_{xyz}(t)$. The three new weights are

$$\begin{aligned} w_{xyz}(t) &= \left[\frac{w_{xyz}(t)}{w_{\cdot yz}(t)}\left[\frac{w_{xy\cdot}(t)}{w_{\cdot y\cdot}(t)}\frac{w_{x\cdot z}(t)}{w_{\cdot\cdot z}(t)}\right]^{1/2} w_{x\cdot\cdot}(t)\right]^{1/3} \\ &\quad \left[\frac{w_{xyz}(t)}{w_{x\cdot z}(t)}\left[\frac{w_{xy\cdot}(t)}{w_{x\cdot\cdot}(t)}\frac{w_{\cdot yz}(t)}{w_{\cdot\cdot z}(t)}\right]^{1/2} w_{\cdot y\cdot}(t)\right]^{1/3} \\ &\quad \left[\frac{w_{xyz}(t)}{w_{xy\cdot}(t)}\left[\frac{w_{x\cdot z}(t)}{w_{x\cdot\cdot}(t)}\frac{w_{\cdot yz}(t)}{w_{\cdot y\cdot}(t)}\right]^{1/2} w_{\cdot\cdot z}(t)\right]^{1/3} \\ &= w_x(t)w_y(t)w_z(t). \end{aligned} \tag{8.29}$$

Table 8.5 shows the CDR for the selected European countries and the decomposition of the change over time. The three compositional factors included are age, country and sex and the corresponding decomposition formula is

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r_a) + C(m, r_c) + C(m, r_s), \tag{8.30}$$

where r_a, r_c and r_s correspond to the growth rates of the population accounting only for the compositional effect of the countries, ages and sexes respectively. As in Table 8.3 the main compositional component is the age structure, with the country and sex composition almost canceling each other. It has to be noted that in Tables 8.2, 8.3 and 8.5 the death rates and the population size are over three categories, namely, age, country and sex, $m_{acs}(t)$ and $N_{acs}(t)$. In Tables 8.2 and 8.3 only two of these compositional components are considered while in Table 8.5 all are involved. Therefore, the observed average change and compositional component

Table 8.5: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996 for selected European countries. Decomposing by direct change and three compositional components: age, sex and country.

	1965	1980	1994
$\bar{d}_E(t)$	10.783	10.720	11.187
$\bar{d}_E(t-h/2)$	10.720	10.591	10.916
$\bar{d}_E(t+h/2)$	10.892	10.965	11.595
$\dot{\bar{d}}_E(t)$	0.017	0.037	0.170
$\bar{\dot{m}}$	-0.090	-0.112	0.036
$C(m,r)$	0.108	0.150	0.133
$C(m,r_a)$	0.110	0.148	0.134
$C(m,r_c)$	-0.001	0.001	-0.006
$C(m,r_s)$	-0.001	0.001	0.005
$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m,r_a) + C(m,r_c) + C(m,r_s)$	0.018	0.038	0.169

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 eleven-year periods were used (1960-1970 and 1975-1985). For the year 1994 a five-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

are the same in the three tables. Different results are obtained if the death rates and the population sizes were defined only for two characteristics first and then for three.

Another application of the generalization of the decomposition method is shown in Table 8.6 for the general fertility rate for selected European countries, $GFR_E(t) = \bar{b}_E(t)$. The births of Denmark, France, the Netherlands and Sweden are divided over the total population of females in reproductive ages α and β. The fertility rates are divided by age, marital status (married and unmarried), and by country. Therefore, as done for the CDR of selected European countries now we can decompose the $GFR_E(t)$ into a direct effect of change in the fertility rates and three compositional components. First the $GFR_E(t)$ is simplified into a rate with only one compositional component k which is a function of age, marital status and country,

$$\bar{b}_E(t) = \frac{\sum_{a=\alpha}^{\beta} \sum_s \sum_c B_{asc}(t)}{\sum_{a=\alpha}^{\beta} \sum_s \sum_c N_{fasc}(t)} = \frac{\sum_{k=1}^{asc} b_k(t) N_{fk}(t)}{\sum_{k=1}^{asc} N_{fk}(t)} \tag{8.31}$$

where $N_{fasc}(t)$ and $B_{asc}(t)$ are the female population size and the births from mothers at age a, marital status s and country c. $N_{fk}(t)$ and $b_k(t)$ are the female population size and fertility rates of the compositional component k.

Then the decomposition in equation (6.4) is used to obtain a direct and a compositional component

$$\dot{\bar{b}}_E(t) = \bar{\dot{b}}(t) + C(b,r), \tag{8.32}$$

where $r(t) = r_{fk}(t)$ is the age, marital status and country-specific growth rate, $r_{fk}(t) = \acute{N}_{fk}(t)$.

The compositional component is further decomposed into an age, marital status and country component as shown in this section,

$$\dot{\bar{b}}_E(t) = \bar{\dot{b}}(t) + C(b, r_a) + C(b, r_s) + C(b, r_c), \tag{8.33}$$

where r_a, r_s and r_c correspond to the growth rates of the population accounting only for the compositional effect of the ages, marital status and countries respectively. Table 8.6 presents the decomposition of the general fertility rate for Denmark, France, the Netherlands and Sweden taken together when different compositional components are considered. Similar to Table 8.2 the compositional component explains most of the change in the general fertility rate in Table 8.6. Once this compositional component is separated we obtain half of this component due to change in the marital status composition and the other half due to age composition.

Table 8.6: General fertility rate, $\bar{b}_E(t)$, per hundredth, and decomposition of the annual change over time in 1992-1997 for selected European countries. Decomposing by direct change and three compositional components: age, marital status and country.

Compositional component	1	2	3
$\bar{b}_E(1994)$	5.042	5.042	5.042
$\bar{b}_E(1992)$	5.212	5.212	5.212
$\bar{b}_E(1997)$	4.925	4.925	4.925
$\dot{\bar{b}}_E(t)$	-0.029	-0.029	-0.029
$\bar{\dot{b}}(t)$	0.029	0.029	0.029
$C(b, r)$	-0.057	-0.057	-0.057
$C(b, r_a)$		-0.027	-0.027
$C(b, r_s)$		-0.030	-0.031
$C(b, r_c)$			0.001
$\dot{\bar{b}}_E(t) = \bar{\dot{b}}(t) + C(b, r_a) + C(b, r_s) + C(b, r_c)$	-0.028	-0.028	-0.028

Source: Author's calculations described in Chapter 9, based on Eurostat (2000).

8.5.4 Purging the Weighting Functions

Separating the weighting function into multiplicative components can be done in several ways. The previous section showed how the implementation of Das Gupta's formulations yields one of these solutions. Here, to separate the weights into terms that account independently for each of the compositional components we follow the use of a relationship model. This tool is the basis of the technique of "purging" undesirable compositional effect suggested by Clogg (1978).

In Chapter 5 it is shown that expected frequencies of a contingency table can be substituted by parameters of a general multiplicative model. Let a two-way contingency table be classified by two compositional variables. Let F_{ij} denote the expected frequencies in the cells (i, j) in the two-way contingency table. The general log-linear model in this two-way contingency table is

$$F_{ij} = \eta\tau_i\tau_j\tau_{ij}, \tag{8.34}$$

where η is the scale factor and the τ parameters denote various kinds of main effects and interaction. The parameter τ_{ij} is introduced here to account for the specific information of each cell which is not captured by the main effects.

The tabulated data of the weighting functions $w_{xz}(t)$ classified by x and z are elements of the two-way table. The log-linear model in equation (8.34) applied to the weighting functions gives

$$w_{xz} = \eta\tau_x\tau_z\tau_{xz}, \tag{8.35}$$

where the parameters τ_x and τ_z are new weighting functions that account for x and z independently, and τ_{xz} is an interaction parameter.

The final decomposition is reached by following the property of intensities (8.19), $\acute{w}_{xz} = \acute{\eta} + \acute{\tau}_x + \acute{\tau}_z + \acute{\tau}_{xz}$, and (8.21) to obtain,

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{\eta}) + C(v, \acute{\tau}_x) + C(v, \acute{\tau}_z) + C(v, \acute{\tau}_{xz}). \tag{8.36}$$

Nevertheless, this formulation presents the inconvenience of rapidly increasing the number of compositional parameters when the number of compositional variables increases. For the case of three compositional variables the number of compositional parameters in the saturated model is eight. In general, with n compositional components there are 2^n different parameters. It could be possible to use non-saturated models and thus obtain fewer parameters but the expenses of greater error. Das Gupta (1993) suggests that the interaction terms are negligible and therefore, equation (8.36) simplifies only to covariances with main effects.

For example, by applying equation (8.36) to the change over time in the CDR of the selected European countries five terms are obtained

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, \acute{\eta}) + C(m, \acute{\tau}_a) + C(m, \acute{\tau}_c) + C(m, \acute{\tau}_{ac}). \tag{8.37}$$

In equation (8.37) the first term on the right-hand side accounts for the change in the CDR due to change in the age-specific death rates, while the covariances correspond to the compositional components.

A visual representation of a multidimensional decomposition is shown in the next section.

8.6 Categorical Decomposition in a Two-Way Table

The reason for the high compositional effect in Table 8.2 can be seen through simultaneous analysis of an age and category decomposition. In Chapter 7 age, single age and categorical decompositions were presented. Here we extend this idea to the case of numerous compositional components.

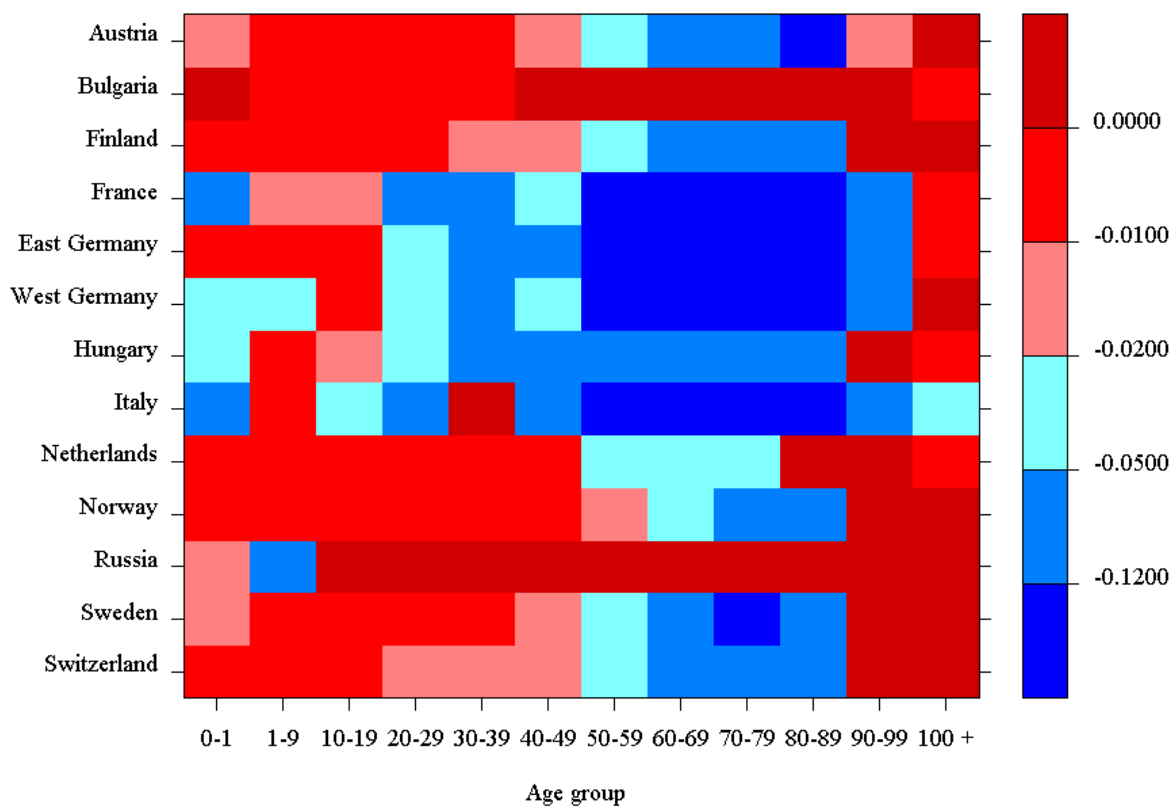
The same formulas used in Chapter 7 for decomposing the direct and compositional component are employed to compile the results of Table 8.2. In this way it is possible to know the contribution of each country and age group to the total change of the CDR. Also it is possible to obtain the contribution of the age and country category in the direct component and compositional component.

The graphic representation of the age and country decomposition is a surface with age groups in the horizontal axis, countries in the vertical axis and levels of these crossings as the surface points. The use of shaded contour maps is implicit in some of Lexis's (1875) original diagrams. Arthur and Vaupel (1984) introduced the phrase "Lexis surface" to describe a surface of demographic rates defined over age and time and we continue to employ that usage here.

The lexis surface is a useful tool that allows a flat representation of surfaces and therefore is perfect for the purpose of showing decomposition over two compositional factors.

Figures 8.1 and 8.2 show the age and country decompositions of the level-1 and level-2 components of change in the CDR, respectively. To visualize these contributions and because some values were extremely low, we show the results multiplied per 100,000. In Figure 8.1 blue

Figure 8.1: Lexis surface of an age and country decomposition of the direct component of the annual change in the crude death rate of selected European countries from 1992 to 1996.

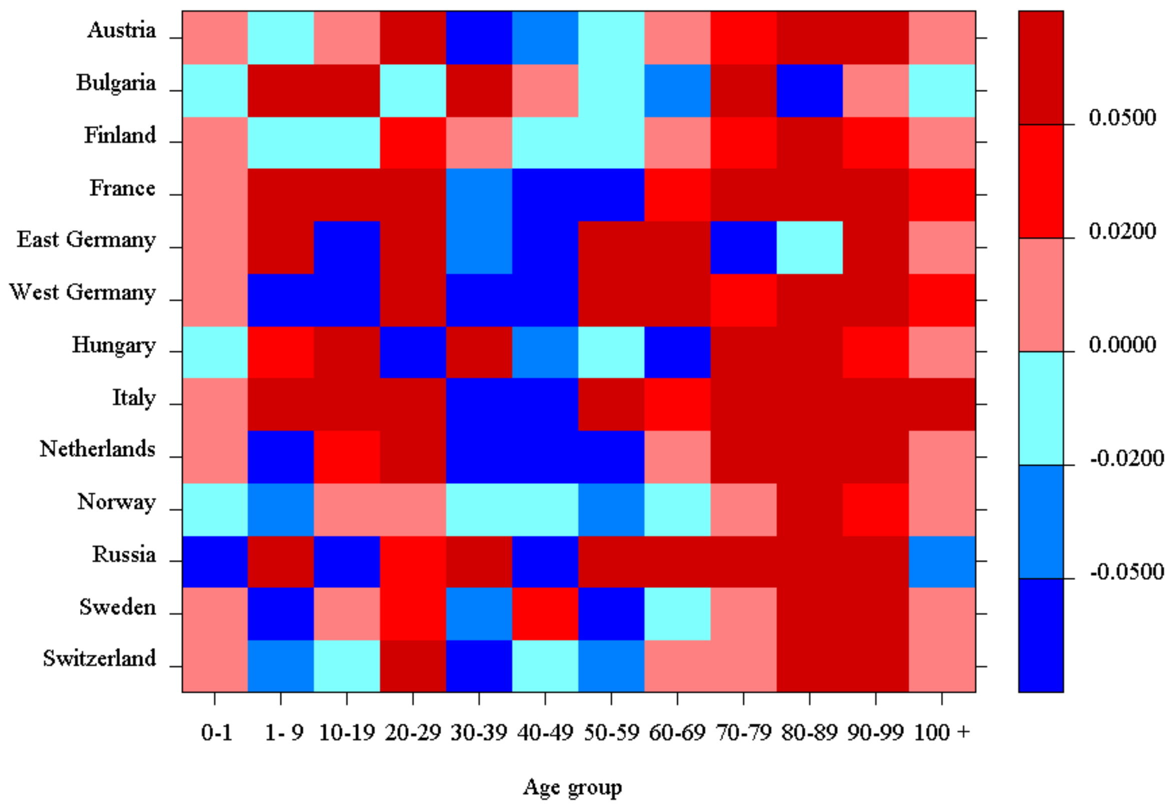


corresponds with negative numbers while red with numbers associated with positive values. It is possible to discern that these European countries have not contributed in the same way to the change in the total CDR. France, Italy, East and West Germany and Hungary are the greatest contributors to the average decline in mortality rates. On the other hand, Bulgaria, Russia and the Netherlands show fewer improvements in survivorship, with a greater number of red boxes.

In Figure 8.1 the improvements previously mentioned are concentrated in the age groups 40 to 99 while young age groups and very old age groups experienced the opposite trend, a decrease in mortality.

In a similar way the compositional component can be decomposed by country and age as shown in Figure 8.2. The scale has changed, but the colors still follow the same pattern. Blue

Figure 8.2: Lexis surface of an age and country decomposition of the compositional component of the annual change in the crude death rate of selected European countries from 1992 to 1996.



indicates a contribution to the decline in the total change in the CDR and red indicates an increase. It is then evident that those age groups that have a compositional component that opposes the decrease in the CDR are those aged 60 and above. The age groups 30 to 60 contribute to the decrease in the total CDR. For the youngest age groups 0 to 1 there are no changes in the compositional components.

8.7 Conclusion

Populations differ by age, sex, race, country of residence, religion, education, and countless other characteristics. Populations often undergo important compositional changes over time that affect comparisons of demographic variables over time. A formula for decomposing changes in a population average into different components is presented. One component captures the effect of direct change in the characteristic of interest, and the others capture the effect of the different compositional effects.

To capture the contribution of a single category to the total change equation (6.4) is further decomposed by age groups and categories. Applications of the newly presented generalization included the decomposition of the change over time in the average age of the United States population, the average age at death in Japan, as well as, the GFR and the CDR for selected countries in Europe. For the last example a categorical decomposition by countries and age groups is also included.

The beginning of this chapter introduced a straightforward solution for the change in averages that contain averages. The solution is to apply equation (6.4) two times. Another solution for the case of combining multitudinous dimensions of population heterogeneity is presented in the rest of the chapter. The proposed solution is an elegant separation of the weighting function into multiplicative components; however, this separation can not be done uniquely. The final two sections present two different ways of separating the weighting function to account independently for the compositional factors.

9

Estimation of the Decomposition Formula

9.1 Introduction

The direct vs. compositional decomposition in equation (6.4) is for continuous changes over time. Nevertheless, demographic data are only available for stocks at a particular point in time or flows over some time period or categorical values for groups. The same accounts for many of the equations presented in Part III. However, these equations can be estimated using the available data and assuming some change over time in the demographic variables under study. The estimations are very precise if these assumptions follow the observed change. However, in some applications the estimations might carry big residual terms, which is a clear sign of wrong assumptions about the change over time of the demographic variables.

This chapter introduces several estimation procedures for the formulas pertaining to direct vs. compositional decomposition. The first section reviews the advantages of visualizing demographic data as vectors and matrices. Rewriting the data in this compact way facilitates the calculations of our main equation. The remaining section is concerned with ways of estimating derivatives over time for demographic measures. Here, we show the estimation procedures used for the applications presented in the book on changes in mortality, fertility and growth rate measures.

9.2 Vector Formulation of Direct vs. Compositional Decomposition

9.2.1 Matrix Notation in Demography

Rewriting demographic data in matrix notation has gradually been established within projection theory since the contribution of Leslie (1945) (in Smith and Keyfitz (1977)). Arranging the data in this way often facilitates computer applications and allows us to establish relations in population dynamics through matrix algebra.

The matrix theory has also been used in another field of demography, namely in the development of multi-state or increment-decrement lifetables. Rogers developed multi-state demography, which simultaneously analyzes the spatial dynamics of a system with several interdependent states that are linked by transitions. By letting M be a matrix, each cell in the matrix m_{ij} represents the transitions from state i to state j. Schoen (1988), Willekens et al. (1993) and Rogers (1995) made further developments in multi-state demography. An example of this state space is shown by Willekens et al. (1993) using four marital statuses and transitions between these states.

The matrix formulation is very important for several areas of demography. Therefore, as a first step, we introduce our main formula in vector terms in this chapter.

9.2.2 The Direct vs. Compositional Decomposition

This subsection introduces the key equation (6.4) in vector terms, considered as a matrix of one by ω-columns. Let $V(t)$ and $W(t)$ be two vectors of ω-entries at time t. The entries of $V(t)$ are $\{v(x,t)\}$ representing the different values of the characteristic x. If x is age a then each cell $v(a,t)$ corresponds to the values of the demographic variable of interest in the age group a to $a+1$ at time t,

$$V(t) = \left[\begin{array}{cccc} v(1,t), & v(2,t), & \cdots, & v(\omega,t) \end{array}\right]. \tag{9.1}$$

Similarly the weighting function $w(x,t)$ can be arranged in $W(t)$

$$W(t) = \left[\begin{array}{cccc} w(1,t), & w(2,t), & \cdots, & w(\omega,t) \end{array}\right]. \tag{9.2}$$

The mathematical expectation shown in equation (2.1) can now be written in matrix notation as

$$\bar{v}(t) = VW^T \left(W1^T\right)^{-1}, \tag{9.3}$$

where W^T is the transpose of W and 1 represents a vector with ones in all the entries. The product $W1^T$ is a fixed number and it can also be considered as a matrix of one by one. The notation $\left(W1^T\right)^{-1}$ denotes the inverse of $W1^T$. This means that the product of the two gives us a value of one.

Equation (6.4) can now be expressed in vector notation following the derivative of a product of vectors in equation (9.3). Willekens (1977) used the matrix differentiation techniques to examine the effects of a change in age-specific rates on the various lifetable functions. This analytical approach is designated as sensitivity analysis. Differentiating a matrix is the procedure of finding partial derivatives of the elements in a matrix function with respect to an argument of the matrix or a scalar. We focus on changes over time by looking at the partial derivatives of the entries in the matrix with respect to time. For example, the change over time of the vector $V(t)$ is

$$\frac{\partial}{\partial t}[V] = \left[\ \tfrac{\partial}{\partial t}v(1,t), \quad \tfrac{\partial}{\partial t}v(2,t), \quad \cdots, \quad \tfrac{\partial}{\partial t}v(\omega,t)\ \right]. \tag{9.4}$$

Similar to the derivative of a product of functions, one can calculate the derivative of a product of vectors. For example, the derivative of the vectors V and W^T is equal to

$$\frac{\partial}{\partial t}\left[VW^T\right] = \frac{\partial}{\partial t}[V]\,W^T + V\frac{\partial}{\partial t}\left[W^T\right]. \tag{9.5}$$

This property is easily generalized to the case of three vectors. It is therefore possible to calculate the derivative of an average as the derivative of the product of the three terms in (9.3),

$$\frac{\partial}{\partial t}\bar{v} = \frac{\partial}{\partial t}\left[VW^T\left(W1^T\right)^{-1}\right]$$

$$= \frac{\partial}{\partial t}[V]\,W^T\left(W1^T\right)^{-1} + V\frac{\partial}{\partial t}\left[W^T\right]\left(W1^T\right)^{-1} + VW^T\frac{\partial}{\partial t}\left[\left(W1^T\right)^{-1}\right]. \tag{9.6}$$

Recalling the notation of the dot over a variable to denote the derivative with respect to time, equation (9.6) can also be written as

$$\dot{\bar{v}} = \dot{V}W^T\left(W1^T\right)^{-1} + V\dot{W}^T\left(W1^T\right)^{-1} - \left(VW^T\left(W1^T\right)^{-2}\right)\left(\dot{W}1^T\right),$$

where the derivative of the last terms is $\frac{\partial}{\partial t}\left[\left(W1^T\right)^{-1}\right] = -\left(W1^T\right)^{-2}\left(\dot{W}1^T\right)$. Let the intensity of the weighting function be

$$\acute{W}(t) = \left[\ \frac{\frac{\partial}{\partial t}w(1,t)}{w(1,t)}, \quad \frac{\frac{\partial}{\partial t}w(2,t)}{w(2,t)}, \quad \cdots, \quad \frac{\frac{\partial}{\partial t}w(\omega,t)}{w(\omega,t)}\ \right]. \tag{9.7}$$

If we rewrite the derivative of a vector as the product of the relative derivative of the vector by the vector, $\dot{W}1^T = \acute{W}W^T$, we obtain the familiar result of equation (6.4),

$$\dot{\bar{v}} = \dot{V}W^T\left(W1^T\right)^{-1} + V\dot{W}^T\left(W1^T\right)^{-1} - \left[VW^T\left(W1^T\right)^{-1}\right]\left[\left(\acute{W}W^T\right)\left(W1^T\right)^{-1}\right] \tag{9.8}$$

or expressed in the notation used in equation (6.4)

$$\begin{aligned}\dot{\bar{v}} &= \bar{\dot{v}} + \overline{v\acute{w}} - (\bar{v})\left(\bar{\acute{w}}\right)\\ &= \bar{\dot{v}} + C(v,\acute{w}).\end{aligned} \tag{9.9}$$

The relevance of this formulation in vector form becomes apparent in cases where we have numerous types of compositions in a population. As shown in Chapter 8, the decomposition formula can be further decomposed into numerous covariance terms. Each of the covariances accounts for one composition of the population. The initial data in a cross-way contingency table can be changed into vectors of the type of $V(t)$ and $W(t)$ as presented in (9.1) and (9.2). For example, if we wish to study the crude death rate of selected European countries we divide the population into age groups and countries. In the applications presented in Chapter 8 we examined 12 age groups and 14 countries. The death rate at age a and for country c is $m_{ac}(t)$, and the matrix $U^*(t)$ of death rates is

$$U^*(t) = \begin{bmatrix} m_{1,1}(t) & m_{1,2}(t) & \cdots & m_{1,12}(t) \\ m_{2,1}(t) & \ddots & & \\ \vdots & & & \\ m_{14,1}(t) & \cdots & & m_{14,12}(t) \end{bmatrix}. \tag{9.10}$$

Similarly the matrix of the population sizes $N^*(t)$ is defined by cells for the population size by age groups and countries $N_{ac}(t)$. Following the rearrangements of matrices into vectors shown by Willekens (1977), we can transform $U^*(t)$ and $N^*(t)$ into the vectors $U(t)$ and $N(t)$. Let the first 12 cells in the vector $U(t)$ be the first row in $U^*(t)$, the next 12 cells of $U(t)$ be the second row of $U^*(t)$, and similarly for the rest of the rows in $U^*(t)$. The new vector $U(t)$ is

$$U(t) = \left[\ \{m_{1,\bullet}(t)\}, \ \{m_{2,\bullet}(t)\}, \ \cdots \ \{m_{14,\bullet}(t)\} \ \right]$$

$$= \left[\ m_{1,1}(t), \ m_{1,2}(t), \ \cdots \ m_{1,12}(t), \ m_{2,1}(t), \ m_{2,2}(t), \ \cdots \ m_{14,1}(t), \ \cdots \ m_{14,12}(t) \ \right].$$

Likewise the population size matrix, $N^*(t)$, can be transformed into the vector $N(t)$. By letting $V(t) = U(t)$ and $W(t) = N(t)$, we can conclude that all the developments shown in this section are valid for these vectors. Equation (9.8) is then expressed as

$$\dot{\bar{d}} = \dot{U}N^T\left(N1^T\right)^{-1} + U\dot{N}^T\left(N1^T\right)^{-1} - \left[UN^T\left(N1^T\right)^{-1}\right]\left[\left(\acute{N}N^T\right)\left(N1^T\right)^{-1}\right], \tag{9.11}$$

which are the explicit calculations required to obtain the decomposition. The only two vectors that correspond to changes are $\dot{U}$ and $\acute{N}$. These changes occur during the entire period under study, from time t to $t+h$. In all the applications shown in the book we have chosen the mid-point as the moment for representing this change, $t + h/2$. The other terms of (9.11) are functions of the vectors $\dot{U}$ and $\acute{N}$ that need to be allocated at time $t + h/2$. Once the four vectors, $\dot{U}(t+h/2)$, $\acute{N}(t+h/2)$, $U(t+h/2)$ and $N(t+h/2)$, are obtained at time $t+h/2$ it is simple a matter of mechanically putting them in the order specified in equation (9.11). This is true for any application that involves equation (9.8).

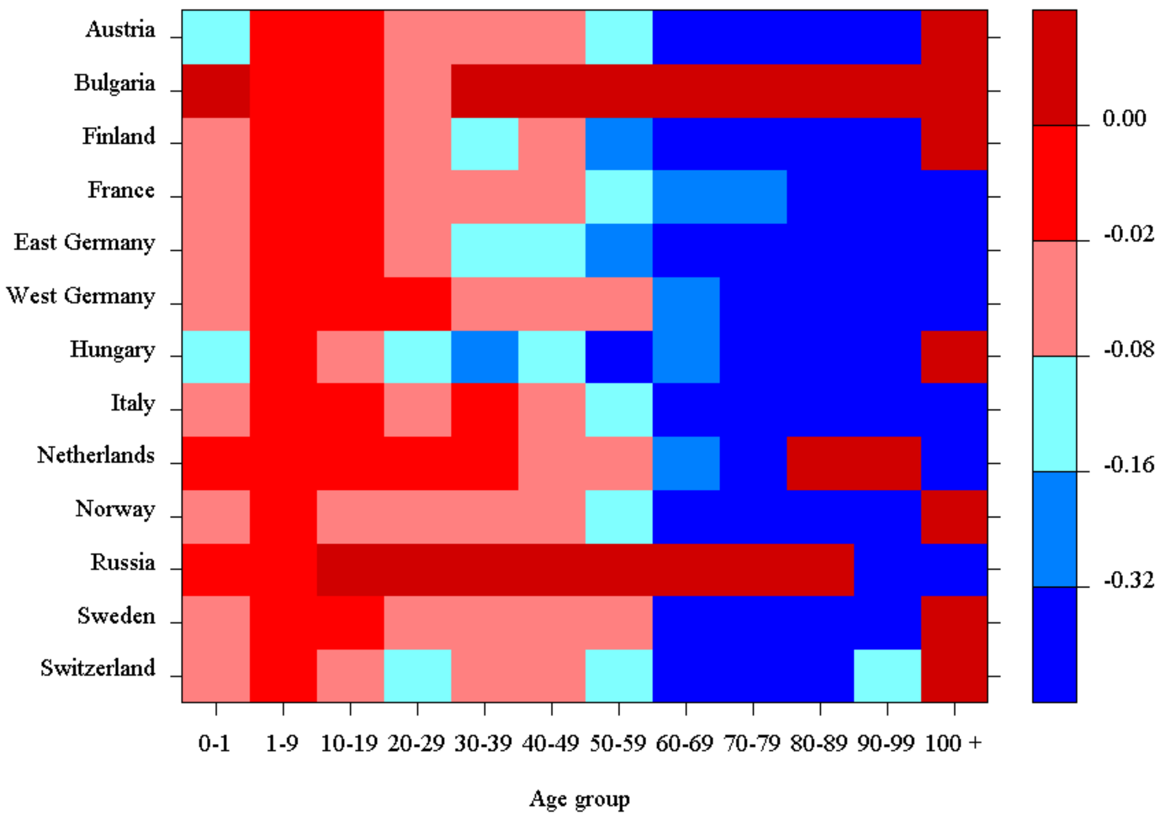
Chapter 8 illustrates how each age group and country contribute to the total change in the crude death rate by using a lexis surface. Depicting the matrices $\dot{U}^*(t+h/2)$ and $\acute{N}^*(t+h/2)$ by using a lexis surface yields some interesting insights. These matrices account for the change over time in the death rates $\dot{U}(t+h/2)$ and in the population size $\acute{N}(t+h/2)$, only in matrix format. The $\acute{N}^*(t + h/2)$ matrix can also be interpreted as a matrix of growth rates, where each cell is equal to $\acute{N}^*_{a,c}(t + h/2) = r_{a,c}(t + h/2)$.

9.2.3 Matrices of Change

Figures 8.1 and 8.2 show the age and country decomposition of the change of CDR into the direct effect and the compositional effect. Each square in these diagrams corresponds to the contribution of one age group in a specific country to the total change in the crude death rate.

Figure 9.1 displays a lexis surface of the change over time in the matrix of death rates $\dot{U}^*(t + h/2)$. The change in the matrix of deaths is represented in the rows by countries and in the columns by age groups. In Figure 8.1 France, East and West Germany, Hungary and

Figure 9.1: Lexis surface of change in the death rates by age groups and selected European countries from 1992 to 1996.



Italy were the main contributors to the average decline in mortality rates. At the same time, it can be seen that other countries, such as Bulgaria, Russia and the Netherlands experienced fewer improvements in their survivorship profile. Although, Figure 9.1 shows similar results, it is difficult to draw conclusions on how a particular country and age group affect the total outcome of the overall crude death rate of these countries. For example, in Figure 8.1 we see that several age groups in Italy contribute to the decline of the CDR, while in Norway few age groups are involved in the increase of this measure. In Figure 9.1 we find a greater number of age groups in Norway than in Italy in which improvements in mortality took place.

In Figure 8.1, the improvements in mortality rates are concentrated in the age groups 40 to 99. At the same time the young and the very old age groups countered the decrease in mortality. Figure 9.1 shows that the improvements in mortality mainly occur between the age groups 60 to 99.

We conclude here that Figure 8.1 is more informative in depicting direct effect in the study of the CDR than $\dot{U}^*(t+h/2)$. Similar conclusions would be reached for other demographic measures decomposed in this book. The direct component is a measure of the change in the variable of interest weighted by its relevance in the population.

9.3 Estimation of Derivatives in Demography

Both direct vs. compositional decomposition and the other formulas for change over time of demographic averages are exact. The implicit assumption is that these functions are differentiable with respect to time. This also implies that the averages are continuous over time. Continuity in itself may not be a problem, but the fact that demographic data are found in chunks poses a problem. It is, therefore, necessary to estimate precise formulas with the available data on the variables under study. However the estimation will contain minor inaccuracies in some of the tables presented above. For example, for the estimation of some formulas we substituted sums for integrals, and the demographic average was calculated as

$$\begin{aligned} \bar{v}(t) &= \frac{\int_0^\omega v(a,t)w(a,t)da}{\int_0^\omega w(a,t)da} \approx \frac{\sum_{a=0}^{\omega} v(a+0.5,t)w(a+0.5,t)}{\sum_{a=0}^{\omega} w(a+0.5,t)} \\ &= \frac{v(0.5,t)w(0.5,t)+...+v(\omega-0.5,t)w(\omega-0.5,t)}{w(0.5,t)+...+w(\omega-0.5,t)}. \end{aligned} \tag{9.12}$$

In this section we show which assumptions were made for calculating the changes over time in the applications presented above. The next subsection explains the different options for estimating derivatives and relative derivatives of functions that are only known in discrete moments. Successively, we illustrate this by case studies of changes in mortality, fertility and growth measures.

9.3.1 Changes over Time

This section presents the estimation for the change over time of demographic variables used in direct vs. compositional decomposition. These estimations depend on the assumptions of the type of change. Therefore, here we present the two types of change that we have used for our applications.

Change in the population is expressed as rate of increase or growth rate. This notion of change can be generalized to other demographic variables. We begin this section by presenting the relationship between a population at two points in time.

Let the population change be denoted as the difference of the population size at two points in time t and $t+h$, $\Delta N(t) = N(t+h) - N(t)$. When this time interval is small enough the crude growth rate is the ratio of the difference $\Delta N(t)$ divided by the person-years lived over the period, approximated by $N(t)h$. The limit of this ratio when h approaches 0 is the instantaneous growth rate at time t

$$r(t) = \lim_{h \to 0} \frac{N(t+h) - N(t)}{N(t)h} = \frac{\dot{N}(t)}{N(t)}, \tag{9.13}$$

where the second equality holds because this limit is equal to the derivative of $N(t)$. Equation (9.13) can be rewritten as the intensity or derivative of the logarithm of $N(t)$,

$$r(t) = \acute{N}(t) = \frac{d \ln [N(t)]}{dt}. \tag{9.14}$$

Based on equation (9.13) a new expression for population change over time can be developed. By integrating both sides of (9.14) from t to $t+h$ we obtain the desired relation

$$\int_t^{t+h} r(x)dx = \ln [N(x)] \Big|_t^{t+h} = \ln \left[\frac{N(t+h)}{N(t)} \right]. \tag{9.15}$$

This expression for the instantaneous population growth rate at time t can also be used to derive changes from other demographic variables.

Exponential Change

When data are available for time t and $t+h$, we generally used the following approximations for the value at the mid-point $t + h/2$. Variables that have a constant positive growth during the interval are considered to be growing exponentially. We can conclude from (9.15) that the growth for those variables is $\int_t^{t+h} r dx = rh$. This accounts for population size, births and others. Analogous to $r(t) = \acute{N}(t)$, for the relative derivative of the function $v(a,t)$ we have

$$\acute{v}(a, t + h/2) \approx \frac{\ln \left[\frac{v(a,t+h)}{v(a,t)} \right]}{h}. \tag{9.16}$$

Dividing by the number of years in the period h implies that the relative change in $\acute{v}(t + h/2)$ is annualized. The value of the function at the mid-point $v\,(a, t + h/2)$ is estimated by

$$v(a, t + h/2) \approx v(a,t) e^{(h/2)\acute{v}(a,t+h/2)}. \tag{9.17}$$

Substituting the right-hand side of (9.16) by $\acute{v}(a, t + h/2)$ in (9.17) yields the equivalent approximation

$$v(a, t + h/2) \approx [v(a,t) v(a, t+h)]^{1/2}. \tag{9.18}$$

This is a standard approximation in demography, as shown in Preston et al. (2001).

The derivative of the function $v(a, t + h/2)$ is estimated from the previous two equations (9.16) and (9.18) by

$$\dot{v}(a, t + h/2) = \acute{v}(a, t + h/2) v(a, t + h/2). \tag{9.19}$$

The equations (9.16), (9.18) and (9.19) are used when the change of the variable under study follows a constant positive growth. For example, the rate of progress in reducing the death rate $\rho(a, t + h/2)$, which was used in Table 4.8 and Figures 6.1 and 6.2, was calculated at time $t + h/2$ as

$$\rho(a, t + h/2) = -\acute{\mu}(a, t + h/2) \approx -\frac{\ln \left[\frac{\mu(a,t+h)}{\mu(a,t)} \right]}{h}. \tag{9.20}$$

Linear Change

The most common way to compute derivatives in numerical analysis is to use finite difference methods (see Judd (1988)). If the variable of interest, for instance $\bar{v}(t)$, has a linear growth, then the approximation of the derivative $\dot{\bar{v}}(t+h/2)$ is the one-sided finite difference formula

$$\dot{\bar{v}}(t+h/2) \approx \frac{\bar{v}(t+h) - \bar{v}(t)}{h}, \tag{9.21}$$

where h is the appropriate step size chosen, which normally is between one and ten years. When we divide by the number of years in the period this implies that the change in the average, $\dot{\bar{v}}(t+h/2)$, is annualized.

For a more general case, if the variable of interest is a function of several variables, such as the age-specific variable $v(a,t)$ at age a and time t, then the one-sided finite difference formula that substitutes the partial derivative with respect to time is

$$\dot{v}(a,t+h/2) = \frac{\partial v(a,t)}{\partial t} \approx \frac{v(a,t+h) - v(a,t)}{h}. \tag{9.22}$$

For example, in Table 7.2 we studied the change over time of the population growth rate of country i, denoted as $r_i(t)$. We assumed that the change is linear

$$\dot{r}_i(t+h/2) = \frac{\partial r_i(t)}{\partial t} \approx \frac{r_i(t+h) - r_i(t)}{h}. \tag{9.23}$$

The value of the function $v(a,t)$ at the mid-point $v\left(a,t+h/2\right)$ is estimated by

$$v(a,t+h/2) \approx \frac{v(a,t+h) + v(a,t)}{2}. \tag{9.24}$$

The following application contrasts the possible errors due to different assumptions of the change over time. Table 9.1 presents the decomposition of the change over time in the crude death rate of the United States. The age-specific death rates are assumed to change linearly and exponentially over time. In both cases the population size is growing exponentially, therefore by contrasting these two columns we observe only the change due to the age-specific death rates. In all the applications presented in the book only two points of the demographic variables over time are known. The straight line between these points is always above the exponential curve between these points. As a result assuming an exponential growth in the death rates implies lower values for the estimated mid-year crude death rate, direct change and compositional component. In Table 9.1 these results are shown in the rows of $d(1990)$, $\bar{\dot{\mu}}$ and $C(\mu,r)$ respectively. However, the estimated change $\dot{d}$ is the same for both assumptions.

Other types of growth rates could be used for estimating the formulas. One example is the s-shaped curve, like the logistic formula. Nevertheless, in the applications presented in this book we only used exponential and linear growth to estimate derivatives.

9.3.2 Mortality Measures

White (2002) has shown that mortality measures have followed linear patterns over the last fifty years. White considered a simple regression model for each mortality measure. Letting

Table 9.1: Crude death rate, $d(t)$, per thousand, and decomposition of the annual change over time from 1985 to 1995 for the United States. The values in the columns are derived from the assumption that the age-specific death rates change exponentially and linearly over time.

United States	linear growth	exponential growth
$d(1990)$	8.954	8.762
$d(1985)$	9.683	9.683
$d(1995)$	7.941	7.941
$\dot{d}(1990)$	-0.174	-0.174
$\bar{\acute{\mu}}$	-0.249	-0.245
$C(\mu, r)$	0.075	0.071
$\dot{d} = \bar{\acute{\mu}} + C(\mu, r)$	-0.174	-0.174

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

k_1 and k_2 be constants the model implemented is

$$mortality\ measure\ in\ year\ t = k_1 + k_2 * year\ t. \tag{9.25}$$

Then, life expectancy and logged age-specific death rates were tested for goodness of fit by models of the type illustrated by (9.25) in 21 developed countries. The averages over countries of straight lines fitting the mortality measures, R^2 were above 0.91 for all the measures. Considering that 1 is the perfect line, these are very good fits. Life expectancy proved to be the most linear of all these measures. It therefore seems appropriate to focus on linear trends when analyzing life expectancy and logged age-specific death rates.

In Tables 6.2 and 6.7 we assumed that life expectancy changed over time following a linear trend as in (9.21)

$$\dot{e}^o(0, t + h/2) = \frac{\partial \dot{e}^o(0,t)}{\partial t} \approx \frac{\dot{e}^o(0, t+h) - \dot{e}^o(0,t)}{h}. \tag{9.26}$$

In Tables 6.1, 6.3, 7.1, 8.2, 8.3 and 8.5 the derivative of the force of mortality (or age-specific death rates) at time $t + h/2$ is estimated using the exponential change in (9.19),

$$\dot{\mu}(a, t + h/2) = \acute{\mu}(a, t + h/2)\mu(a, t + h/2), \tag{9.27}$$

where the force of mortality at time $t + h/2$ is estimated using (9.18)

$$\mu(a, t + h/2) \approx [\mu(a,t)\mu(a, t+h)]^{1/2} \tag{9.28}$$

and the intensity is estimated as (9.16)

$$\acute{\mu}(a, t + h/2) \approx \frac{\ln\left[\frac{\mu(a,t+h)}{\mu(a,t)}\right]}{h}. \tag{9.29}$$

Also the force of mortality is defined as a change. $\mu(a,t)$ is equal to the negative intensity of the survival function $\ell(a,t)$ over age. When data were available for ages a and $a+k$ we used the following approximation for the force of mortality at age $a+k/2$

$$\mu(a+k/2,t) \approx \int_a^{a+k} \mu(x,t)dx = -\frac{\ln\left[\frac{\ell(a+k,t)}{\ell(a,t)}\right]}{k}. \tag{9.30}$$

The force of mortality is normally defined for each age, which means that k would then be 1. equation (9.30) was used in Table 4.8 and Figures 6.1 and 6.2 to estimate the force of mortality.

For Table 4.8 and Figures 6.1 and 6.2 it was necessary to calculate the other functions involved in the decomposition at that age, because the force of mortality in (9.30) is at age $a+k/2$. The survival function $\ell(a,t)$ and the remaining life expectancy $e^o(a,t)$ at age $a+k/2$ were calculated using a formula analogous to (9.18). The lifetable distribution of deaths was calculated as

$$f(a+k/2,t) \approx \mu(a+k/2,t)\ell(a+k/2,t). \tag{9.31}$$

9.3.3 Fertility Measures

Smith et al. (1996) studied the trends in fertility of unmarried women in the United States using a decomposition technique proposed by Das Gupta (1993). They found that between 1960 and 1992 the fertility of Black Americans outside wedlock underwent an exponential growth. Nevertheless, the decomposition technique used does not allow exponential growth.

The formulas presented in Part III allow for different types of change. Fertility changes on a yearly basis can occur at any age, birth order and because of quantum and tempo effects. Here, we took changes over time in births rates by age into account without looking at other types of change. The other possible reasons for change in the fertility rates are beyond the focus of this book. However, decomposition methods are certainly an interesting tool for examining other types of changes in fertility measures.

Derivatives and intensities of fertility measures as shown in Tables 4.6, 6.5, 6.6 and 8.6 are estimated by using exponential growth in (9.16) to (9.19). For example, in Table 6.5 the intensity with respect to time of the birth rates among married women $\acute{b}_m$ was calculated as

$$\acute{b}_m(a,t+h/2) \approx \frac{\ln\left[\frac{b_m(a,t+h)}{b_m(a,t)}\right]}{h}. \tag{9.32}$$

9.3.4 Growth Measures

In Chapter 4 we presented growth rate measures based on several researchers, among them the studies by Preston and Coale (1982) and Arthur and Vaupel (1984). Their work presented generalizations of the stable population model and some relationships that hold among demographic variables. Following these efforts to understand the population growth, Kim (1986) derived the formulas for discrete time and age. We present some of these formulations here.

The population growth rates are calculated as shown in equation (9.16). If the population size was known both at time t and $t+h$ for age a we get

$$r(a, t+h/2) = \acute{N}(a, t+h/2) \approx \frac{\ln\left[\frac{N(a,t+h)}{N(a,t)}\right]}{h}. \tag{9.33}$$

This approximation was applied in all tables that involved a population growth rate in Chapters 4, 6, 7 and 8.

Tables 4.9, 7.2 and 7.3 present the change over time of growth rates. In these examples we assume that the population growth rate changed linearly and we use differences to estimate the change over time

$$\dot{r}_i(t+h/2) = \frac{\partial r_i(t)}{\partial t} \approx \frac{r_i(t+h) - r_i(t)}{h}. \tag{9.34}$$

In Table 4.11 two measures of the population growth rate are shown. The first corresponds to the population growth rate calculated by using equation (9.33) for 1990

$$r(1990) = \acute{N}(1990) = \frac{\ln\left[\frac{N(1995)}{N(1985)}\right]}{10} = 0.506. \tag{9.35}$$

The second measure corresponds to the growth rate calculated as an average of age-specific growth rates, estimated from equation (9.16), weighted by the population size at each age

$$\begin{aligned} r(1990)\ * &= \frac{\int_0^\omega r(a,1990)N(a,1990)da}{\int_0^\omega N(a,1990)da} \\ &= \frac{\int_0^\omega \frac{\ln\left[\frac{N(a,1995)}{N(a,1985)}\right]}{10} N(a,1990)da}{\int_0^\omega N(a,1990)da} = 0.504. \end{aligned} \tag{9.36}$$

In the same Table 4.11 we see that the estimated decomposition corresponds exactly to the value of equation (9.36).

The reason for the difference between (9.35) and (9.36) when estimating population growth rates is due to the use of logarithms in equation (9.16), that is, $\ln[a+b] \neq \ln[a] + \ln[b]$.

For the applications that use exponential growth it is possible to reduce the bias in estimation procedures by redefining the observed changes in the demographic average $\dot{\bar{v}}(t)$. The observed change over time in a demographic variable is calculated as

$$\dot{\bar{v}} = \frac{\partial}{\partial t}\left[\frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}\right] = \int_0^\infty \frac{\partial}{\partial t}\left[\frac{v(x,t)w(x,t)}{\int_0^\infty w(x,t)dx}\right] dx. \tag{9.37}$$

Using the right-hand side of (9.37) reduces considerably the possible bias due to the logarithm function, $\ln[a+b] \neq \ln[a] + \ln[b]$. The right-hand side of (9.37) was implemented in Table 4.11 in order to obtain precise results.

9.4 Conclusion

An interesting debate has arisen in studies of both fertility and mortality concerning cohort and period changes. Bongaarts and Feeney (1998) and (2002) discussed about whether to use adjusted measures of fertility and mortality to understand observed changes. A similar context is found when applying the equations presented in Part III. Our equations are exact, and the techniques are applicable to many circumstances. Nevertheless, it is necessary to look at the particular period under study to understand which kind of assumptions should be made for the changes over time.

In this final chapter we have shown how direct vs. compositional decomposition can be estimated. First, a matrix formulation (9.8) of the main equation was introduced followed by an application of the matrices under study. This formulation also points out which calculations are explicitly required to obtain the decomposition. It involves two vectors that correspond to changes occurring during the period under study. In all the applications of direct vs. compositional decomposition shown in the book we have chosen the mid-point as the moment for representing this change. The other vectors are allocated at that moment, so it is a simple mechanical procedure to apply them in the order specified by equation (9.8). This section also allowed us to re-assess the relevance of the components of change, and of the direct and compositional effects in our decomposition.

This chapter was dedicated to the estimation of derivatives and intensities over time. Two types of change are suggested, but many others could be applied because the formulas are flexible and can be applied to any assumption of the trends over time in the demographic variable under study. We also gave some suggestions on how to reduce the bias in estimation procedures.

Part IV

Evaluation of the Decomposition Methods

10

Decomposition Techniques, Revisited

10.1 Introduction

The last part of this book consists of two chapters. The first chapter, on *Decomposition Techniques, Revisited*, includes four sections. These sections correspond to selected decompositions presented in Part II, which are compared in light of the direct vs. compositional decomposition.

In the first section, we compare Kitagawa's initial formulation with our main equation (6.4). Kitagawa's formulation was further developed by Cho and Retherford, Kim and Strobino, Das Gupta and structural decomposition. In an additional section, we compare the extensions of Kitagawa's formula with the formulations of Chapter 8. Next we include a section on applications of decomposition methods in mortality, fertility and population growth rates, where we compare existing methods with direct vs. compositional decomposition. Finally, alternative decomposition methods, regression decomposition, the purging and delta methods are compared with direct vs. compositional decomposition.

Several of the applications are examined again in these sections. This allows us to distinguish differences or similarities between the methods. We combine some parts of tables from Part II with parts of tables from direct vs. compositional decomposition of the same application.

The second chapter of this part comprises the final conclusions of the book.

10.2 Kitagawa's Decomposition Method, Revisited

The first method introduced in the book is the technique proposed by Kitagawa (1955). Her proposal includes two components presented in equations (3.1) and (3.2). As shown earlier, the first component corresponds to the change in the variable of interest while the second component is the contribution of the change in the structure of the population. In Kitagawa's words, these terms are respectively the rate component and the composition component of change.

Equation (6.4) of direct vs. compositional decomposition also presents two components. Here, the components express a change in the demographic average due to a direct change in the characteristic of interest and a compositional change.

In cases where the variable under study is a demographic rate, as the crude death rate, the direct change of direct vs. compositional decomposition is equal to a rate effect, as in Kitagawa's proposal. In both methods, the compositional term accounts for the total change due to changes in the population composition. Kitagawa's structural change is simply the difference of the normalized weights. In direct vs. compositional decomposition this component is a covariance between the variable of interest and the intensity of the population structure. On one hand, the first method used a more simple term, on the other hand, the term in the second method describes the dynamic relationship between the variables better.

A second option for direct vs. compositional decomposition is introduced in equation (6.9). This equation uses normalized weights and two components

$$\dot{\bar{v}} = \bar{\dot{v}} + \overline{v\acute{c}}.$$

As in Kitagawa's formulation, the compositional component accounts for the effect of change in the normalized weights.

A linear change, as explained in Chapter 9, in both equations (6.4) and (6.9) leads precisely to Kitagawa's expression. As mentioned in Chapter 9, direct vs. compositional decomposition can be estimated assuming many other types of change. Kitagawa's proposal is therefore a particular case of the more general decomposition (6.4).

Table 10.1 presents the results of Tables 3.1 and 6.1 on the Mexican crude death rate (CDR) for both decompositions. Both models yield the same results. Nevertheless, direct vs. compositional decomposition shows that the structure of the population contributes to the change in the CDR through a covariance between age-specific death rates and growth rates.

Table 10.1: Crude death rate, $d(t)$, per thousand, and the annual change over time in 1985-1995 for Mexico. The change is decomposed by the direct vs. compositional decomposition and the method suggested by Kitagawa.

Direct vs. compositional		*Kitagawa*	
$d(1990)$	5.100		
$d(1985)$	5.532	$d(1985)$	5.532
$d(1995)$	4.755	$d(1995)$	4.755
$\dot{d}(1990)$	-0.078	$\Delta d(1985)$	-0.078
$\bar{\dot{\mu}}$	-0.116	$\Delta\ m_a$	-0.116
$C(\mu, r)$	0.039	$\Delta\ \frac{N_a}{N.}$	0.038
$\dot{d} = \bar{\dot{\mu}} + C(\mu, r)$	-0.077	$\Delta d(1985) = \Delta\ m_a + \Delta\ \frac{N_a}{N.}$	-0.078

Source: Tables 3.1 and 6.1.

As shown in Figure 7.3, the mortality increases with age after age 10. The positive covariance in Table 10.1 of 0.038 implies a shift in the population structure towards older ages.

More is learned from direct vs. compositional decomposition by looking at the extensions of the change in the Mexican CDR in Chapter 7 and in Tables 6.4 and 6.3. All these alternative sections used direct vs. compositional decomposition to study age decomposition, relative change and the sex difference of the Mexican CDR, respectively.

10.3 Further Decomposition Research, Revisited

Following Kitagawa's proposal, in Chapter 3 we introduced four general decompositions for the case of numerous compositional components. Analogously, Chapter 8 extended direct vs. compositional decomposition (6.4) to the case of multidimensional decomposition.

Table 10.2 displays the crude death rate, $d_E(t)$, and the decomposition of the change over time for 1992-1996 for selected European countries, which is derived from Tables 3.2, 3.3, 3.4, 3.5 and 8.3. As in Table 10.1, there is hardly any difference in the results of the different

Table 10.2: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1992-1996 for selected European countries. The decompositions are direct vs. compositional decomposition and the methods suggested by Cho and Retherford, Kim and Strobino, Das Gupta and Oosterhaven and Van der Linden.

Direct vs. compositional			*CR*	*KS*	*DG*	*OV*
$\bar{d}_E(1994)$	11.187					
$\bar{d}_E(1992)$	10.916	$\bar{d}_E(1992)$	10.917	10.917	10.917	10.917
$\bar{d}_E(1996)$	11.595	$\bar{d}_E(1996)$	11.596	11.596	11.596	11.596
$\dot{\bar{d}}_E(t)$	0.170	$\Delta\bar{d}_E(1992)$	0.170	0.170	0.170	0.170
$\dot{\bar{m}}$	0.036	Δm_{ac}	0.047	0.047	0.047	0.047
$C(m,r)$	0.133					
$C(m,r_a)$	0.134	$\Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}}$	0.131	0.131	0.129	0.130
$C(m,r_c)$	-0.001	$\Delta\frac{N_{ac}}{N_{a\cdot}}$	-0.008	-0.008	-0.006	-0.007
$\dot{\bar{d}}_E = \dot{\bar{m}} + C(m,r_a) + C(m,r_c)$		$\Delta\bar{d}_E(t) = \Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}} + \Delta\frac{N_{ac}}{N_{a\cdot}} + \Delta m_{ac}$				
	0.169		0.170	0.170	0.170	0.170

Source: Tables 3.2, 3.3, 3.4, 3.5 and 8.3. The headings in the table correspond to CH for Cho and Retherford, KS for Kim and Strobino, DG for Das Gupta, and OV for Oosterhaven and Van der Linden.

decompositions in Table 10.2. Particularly, the first four methods by Cho and Retherford, Kim and Strobino, Das Gupta and Oosterhaven and Van der Linden, respectively, obtain the same direct change and compositional component. The difference in death rates is $\Delta m_{ac} = 0.047$ and the addition of the two compositional components is $\Delta\frac{N_{a\cdot}}{N_{\cdot\cdot}} + \Delta\frac{N_{ac}}{N_{a\cdot}} = 0.123$. The only method that differs is direct vs. compositional decomposition where we get a level-1 effect of

0.036 and a level-2 effect of 0.133. Nevertheless, for direct vs. compositional decomposition the estimated results of Table 8.3 varied slightly from the observed changes.

There is no doubt that the biggest advantage of direct vs. compositional decomposition compared to the other methods is its simplicity. Equation (8.21) is, as (6.4), simple and elegantly expressed as

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}_1) + C(v, \acute{w}_2) + ... + C(v, \acute{w}_n).$$

The first term on the right-hand side captures the change in the characteristic of interest, this is the direct change. The other components are the covariances between the underlying variable of interest and each of the intensities of the weighting function. These are the structural or compositional components of change due to each of the compositional factors. It seems inconvenient to assume that the weighting function can be expressed as a product of weighting functions $w = w_1 w_2 ... w_n$. But this is not the case as is immediately proved.

If instead of equation (6.4) we extend its alternative (6.9), $\dot{\bar{v}} = \bar{\dot{v}} + \overline{v\acute{c}}$, then the assumption of the weighting function as a product of weights is avoided. Equation (6.9) uses the normalized weights as weights. This is shown in its extension in equation (10.3). In the case of two compositional components x and z, the normalized weights are expressed as $c(x, z, t)$, that is the proportion of the total values of the weights that belong to the category x and z at time t, $c(x, z, t) = \frac{w(x,z,t)}{\int_0^\infty \int_0^\infty w(x,z,t)dxdz} = \frac{w(x,z,t)}{w(t)}$. The second component in (6.9) is the average of the variable of interest multiplied by the intensity of the normalized weights, $\overline{v\acute{c}}$. The normalized weights can be further separated into a product of normalized weights as suggested by Kim and Strobino in (3.12) as

$$c(x, z, t) = \frac{w(x, z, t)}{w(t)} = \frac{w(x, z, t)}{w(x, t)} \frac{w(x, t)}{w(t)}, \tag{10.1}$$

where $\frac{w(x,z,t)}{w(x,t)}$ and $\frac{w(x,t)}{w(t)}$ correspond to the weights due to the components z and x, respectively. Let these two terms be $c_z = \frac{w(x,z,t)}{w(x,t)}$ and $c_x = \frac{w(x,z,t)}{w(z,t)}$. Likewise, Das Gupta's equation (3.15) can be used to separate the weights into multiplicative terms.

The property of the relative derivative of a product (8.19) and the average property of separating additions

$$\overline{u_1 + u_2} = \overline{u_1} + \overline{u_2}, \tag{10.2}$$

lead us to achieve the extension of the decomposition when numerous compositional components are included. From equation (6.9) and by using (10.1) and (10.2) we obtain

$$\begin{aligned} \dot{\bar{v}} &= \bar{\dot{v}} + \overline{v\acute{c}} \\ &= \bar{\dot{v}} + \overline{v\left(\acute{c}_z + \acute{c}_x\right)} \\ &= \bar{\dot{v}} + \overline{v\acute{c}_z} + \overline{v\acute{c}_x}, \end{aligned} \tag{10.3}$$

where $\overline{v\acute{c}_z}$ and $\overline{v\acute{c}_x}$ are the compositional components accounting for the change in z and x, respectively. By using the methods proposed by Kim and Strobino, Das Gupta or fellow innovators, such as the log-linear model used in Chapter 8, separating the weighting function is a minor step in the exercise. As shown in Chapter 9, derivatives and relative derivatives with respect to time can be estimated in several ways. These approximations correspond to different types of changes and not only to linear assumptions as assumed in the other methods. Therefore, we conclude that direct vs. compositional decomposition is a general option that allows for diverse types of changes over time.

10.4 Applications of Decomposition Methods, Revisited

In Chapter 4 we presented three sections on decomposition applications in mortality, fertility and population growth. The same three sections are included here as subsections. Each subsection contrasts the decompositions used in these areas of demography with direct vs. compositional decomposition.

10.4.1 Mortality Measures, Revisited

In Chapter 4 we studied mortality measures, such as life expectancy. Here we concentrate on the decompositions of the change over time of this measure.

We start by looking at Keyfitz's equation (4.5) for the relative change in life expectancy. The $\acute{e}^o(0,t)$ is equal to the product of improvements in mortality and the entropy of the survival function,

$$\acute{e}^o(0,t) = \rho(t)\mathcal{H}(t).$$

In the work of Keyfitz (1985), the change over time in life expectancy corresponds to equal proportional changes in mortality at all ages. Keyfitz assumes constant mortality improvement at all ages, $\rho(a,t) = \rho(t)$ for all a. An obvious generalization would then be to allow the improvements to change over age. Vaupel and Canudas Romo (2003) proved that the new proposed decomposition of life expectancy (6.35) is this generalization. We mention here some of the crucial points of that demonstration.

The average number of life-years lost as a result of death, $e^\dagger$, introduced in (6.33) can be written as the product of life expectancy at birth and the entropy of the survival function:

$$e^\dagger(t) = e^o(0,t)\mathcal{H}(t). \tag{10.4}$$

By substituting this result in equation (6.35) and dividing by life expectancy at birth, the general formula for the relative change in life expectancy is obtained

$$\acute{e}^o(0,t) = \frac{\dot{e}^o(0,t)}{e^o(0,t)} = \bar{\rho}(t)\mathcal{H}(t) + \frac{C_f(\rho, e^o)}{e^o(0,t)}. \tag{10.5}$$

No constraints were made on the improvement in mortality in equation (6.35). Consequently, (10.5) is the generalization of Keyfitz's (4.5) since the improvement in mortality here is not necessarily equal for all ages.

The next comparison is between the Vaupel-Canudas decomposition (6.35) and the proposals by Pollard (1982, 1988), Arriaga (1984), Pressat (1985) and Andreev (1982; Andreev et al. 2002). Their work focused on the discrete difference in life expectancy between two periods of time. As already mentioned in Chapter 4, these proposals for the difference in life expectancies were developed independently. However, they all have a similar decomposition for the difference in life expectancies. We present them here as Arriaga's decomposition.

Table 10.3 combines the results of applying Arriaga's method in Table 4.1 and Vaupel and Canudas Romo's (2003) result for Sweden, from 1995 to 2000. A similar application of Vaupel and Canudas Romo's method is found in Table 6.7 for Sweden from 1990 to 1999. When we

Table 10.3: Life expectancy at birth, $e^o(0,t)$, and decomposition suggested by Vaupel and Canudas and Arriaga for the annual change over time from 1995 to 2000 for Sweden.

Vaupel–Canudas		*Arriaga*	
$e^o(0,1998)$	79.262		
$e^o(0,1995)$	78.784	$e^o(0,1995)$	78.784
$e^o(0,2000)$	79.740	$e^o(0,2000)$	79.740
$\dot{e}^o(0,1998)$	0.191	$\Delta e^o(0,1995)$	0.191
$\bar{\rho}$ (%)	1.586		
$e^\dagger$	10.042		
$\bar{\rho}e^\dagger$	0.159	Δ_D	0.007
$C_f(\rho, e^o)$	0.032	Δ_I	0.184
$\dot{e}^o(0,1995) = \bar{\rho}e^\dagger + C_f(\rho, e^o)$	0.191	$\Delta e^o(0,1995) = \Delta_D + \Delta_I$	0.191

Source: Tables 4.1 and Vaupel and Canudas Romo (2003).

compare the two methods, we see that both methods divide the change into two components. Arriaga's decomposition consists of a direct and an indirect component (interaction included here). The second component accounts for all the change; in Table 10.3 this is 0.184 years of the annual change. The direct component is generally not at all significant. Here it was only 0.007 years of the annual change. A component in the Vaupel-Canudas method accounts for the general effect of the reduction in the death rates. The second component, the covariance captures the effect of heterogeneity on mortality improvements over age. The direct component was $\bar{\rho}e^\dagger = 0.159$ while the distribution of the improvements in mortality was $C_f(\rho, e^o) = 0.032$. The two components of the decompositions can be interpreted similarly, but there are big differences in the results of the components. It can, therefore, be concluded that the Vaupel-Canudas decomposition distributes Arriaga's interaction effect into both its components.

Vaupel and Canudas Romo (2003) show that their and Arriaga's formula allocate the same contribution by ages in the total change in life expectancy. But, even the contribution by age of Arriaga's direct and indirect components are far from equal to the results obtained by the direct and compositional components of the Vaupel-Canudas method.

Other tables, which complement each other, are the decomposition of the average age at death in Table 8.4 and the decomposition by age and cause of death proposed by Pollard in Table 4.2.

We conclude here that the new decomposition method complements the previous studies of changes in life expectancy. As in some of the previous decompositions, the Vaupel-Canudas method also allows for an age decomposition, a cause of death decomposition and a decomposition of the difference in life expectancies.

10.4.2 Fertility Measures, Revisited

The second section in Chapter 4 focuses on the study of fertility measures. We will concentrate here on comparing the decompositions of the crude birth rate, as proposed by Zeng et al. (1991) and direct vs. compositional decomposition.

Tables 4.3, 4.4 and 6.5 are combined in Table 10.4. The first two columns display the result of the relative change in the crude birth rate using direct vs. compositional decomposition. The remaining columns show the decomposition suggested by Zeng et al. (1991) for married, unmarried and for all women. All the results have previously been discussed in their respective

Table 10.4: Crude birth rate, $CBR(t)$, in percentage, for the total population and by marital status (married, unmarried and total). The decompositions are direct vs. compositional decomposition for the relative annual change over time in CBR and the method suggested by Zeng for the difference over time, in percentage, from 1992 to 1997, for the Netherlands.

Direct vs. compositional		*Zeng* *Marital status*	*Married*	*Unmarried*	*Total*
$CBR(1995)$	1.258				
$CBR(1992)$	1.295	$CBR_s(1992)$	1.134	0.161	1.295
$CBR(1997)$	1.232	$CBR_s(1997)$	0.997	0.236	1.233
$C\acute{B}R(1995)$	-0.996	$\Delta CBR_s(1992)$	-0.027	0.015	-0.012
$\tilde{\acute{b}}$	-0.285	Δb_{sa}	0.015	0.012	0.027
$\tilde{\acute{c}}_f$	0.060	$\Delta \pi_{fa}$	-0.005	-0.003	-0.008
r_B	-0.212	$\Delta \pi_{sa}$	-0.037	0.007	-0.030
r	0.554				
$r_B - r$	-0.766	$\Delta CBR_s(1992) = \Delta b_{sa} + \Delta \pi_{sa} + \Delta \pi_{fa}$			
$C\acute{B}R = \tilde{\acute{b}} + \tilde{\acute{c}}_f + [r_B - r]$	-0.991		-0.027	0.016	-0.011

Source: Tables 4.3, 4.4 and 6.5. The subindex s corresponds for the marital status: married or unmarried.

chapters. Here we concentrate on some comparisons between these methods. The terms denoting the average of the relative change in birth rates, $\tilde{\acute{b}}$, and the differences in birth rates, Δb_{sa}, have opposite signs. Both depict the change over time of birth rates. If we only look at one of these results, it would lead us to conclude that birth rates are declining or increasing depending on whether we observe the first or second decomposition. The intensity of the birth rates, $\tilde{\acute{b}}$, uses as weights the number of births to women. The weights for the difference in births, Δb_{sa}, are the proportion of women (married or unmarried) to the total population. These different weights account for the discrepancy in the results.

The simplicity of the decomposition proposed by Zeng is generalized in the book by applying it to unmarried women. Further generalization is shown in equations (6.16) and (6.17) where the direct vs. compositional decomposition method is used and the residual term in Zeng's formulation is eliminated. Table 8.6 presents the multidimensional decomposition of the general fertility rate when the compositions due to age, marital status and country are included. We want to stress here that the use of these formulas allows for other types of change in the variables involved in the decomposition. Direct vs. compositional decomposition also permits a study of the relative changes. This is an alternative change that can help us detect variations in rates that are not visible in other decompositions due to their weights.

10.4.3 Growth Measures, Revisited

The growth rate of the population is highly relevant for demographic research. The close relation between growth rates and direct vs. compositional decomposition is already exhibited in Chapter 7. This relation is a simple substitution of any of the systems of the age-specific growth rates, from Chapter 4, in the covariance of direct vs. compositional decomposition. By substituting Preston and Coale's (1982) system in equation (4.30), $r = \nu - \mu$, in the decomposition of the crude death rate

$$\dot{\bar{\mu}} = \bar{\dot{\mu}} + C(\mu, r),$$

we obtain

$$\dot{\bar{\mu}} = \bar{\dot{\mu}} + C(\mu, \nu) - C(\mu, \mu) = \bar{\dot{\mu}} + C(\mu, \nu) - \sigma^2(\mu). \tag{10.6}$$

By substituting Arthur and Vaupel's (1984) system in equation (4.33), $r = r_0 - \varphi$, in the decomposition of the crude death rate we obtain

$$\dot{\bar{\mu}} = \bar{\dot{\mu}} + C(\mu, r_0) - C(\mu, \varphi). \tag{10.7}$$

equations (10.6) and (10.7) explain not only why the structure of the population has changed, but also how this change affects the change in the crude death rate.

Another useful formula is the average growth rate and its change over time. Table 10.5 includes the results of the decompositions proposed by Keyfitz and direct vs. compositional decomposition for the change in the average growth rate of the world, found in Tables 4.9 and 7.2. As a consequence of Keyfitz's (1985) assumption on fixed age-specific growth rates, the observed growth rates for 1980 and 1983 are different from those observed during the period shown in the first two columns. The direct vs. compositional decomposition is another generalization of Keyfitz's results where not only the variance but also a direct change appear. In Table 10.5 the new term is the most relevant of the two components and essential to understand the observed variation in the world's growth rate.

10.5 Alternative Decomposition Methods, Revisited

Chapter 5, on *Alternative Methods of Decomposition*, presented an alternative separation procedure for the case of parametric models. Little connection has been made between these decompositions and direct vs. compositional decomposition. The first two alternative methods were developed for cases that consider samples rather than population totals (from censuses

Table 10.5: Population growth rate of the world, $\bar{r}(t)$, and direct vs. compositional decomposition and the method suggested by Keyfitz for the annual change over time around January 1, 1982.

Direct vs. compositional		*Keyfitz*	
$\bar{r}(1980)$	1.732 %	$\bar{r}(1980)$	1.697 %
$\bar{r}(1983)$	1.711 %	$\bar{r}(1983)$	1.722 %
$\dot{\bar{r}}(1982)$	-0.716 *	$\dot{\bar{r}}(1982)$	0.832 *
$\bar{\dot{r}}$	-1.545 *		
$\sigma^2(r)$	0.829 *	$\sigma^2(r)$	0.832 *
$\dot{\bar{r}} = \bar{\dot{r}} + \sigma^2(r)$	-0.716 *	$\dot{\bar{r}} = \sigma^2(r)$	0.832 *

Source: Tables 4.9 and 7.2.

or vital statistics). Direct vs. compositional decomposition is used for measures of population totals.

The regression decomposition is based on the difference of means, but, if the same population is studied over time, derivatives could be applied. As in equation (5.3), by carrying out the following

$$\begin{aligned}\bar{y}_2 - \bar{y}_1 &= (\alpha_2 - \alpha_1) + \sum_{k=1}^{K} \left(\frac{\bar{x}_{1k} + \bar{x}_{2k}}{2} \right) (\beta_{2k} - \beta_{1k}) \\ &\quad + \sum_{k=1}^{K} \left(\frac{\beta_{1k} + \beta_{2k}}{2} \right) (\bar{x}_{2k} - \bar{x}_{1k}),\end{aligned}$$

and using derivatives, we can write the change over time in groups' means as the sum of changes in intercepts, coefficients and independent variables $\bar{x}_{ik}$,

$$\dot{\bar{y}} = \dot{\alpha} + \sum_{k=1}^{K} \dot{\beta}_k \bar{x}_k + \sum_{k=1}^{K} \beta_k \dot{\bar{x}}_k. \tag{10.8}$$

These components can also be interpreted as the effects of change on the influence of group membership on the dependent variable, $\dot{\alpha} + \sum_{k=1}^{K} \dot{\beta}_k \bar{x}_k$, and as the effect on the dependent variable of changes in the characteristics of the groups over time, $\sum_{k=1}^{K} \beta_k \dot{\bar{x}}_k$. Nevertheless, as with regression decomposition, next to the calculations, a test of statistical hypotheses should also be considered.

The log-linear model formulation used by Clogg (1978) is also suggested as an option for separating the compositional component for direct vs. compositional decomposition in Chapter 8. Again, here it is necessary to include indicators of the precision of the adjusted frequencies.

Both regression decomposition and the purging method are important statistical models that account for sampling variability. Direct vs. compositional decomposition is not immediately applicable in sampling cases. The advantage of the direct vs. compositional decomposition method compared to other decomposition methods mentioned is that there is no need to develop procedures for hypothesis testing or confidence intervals.

The delta method is a general decomposition method, which sometimes consists of terms that are difficult to interpret. Similar to direct vs. compositional decomposition, it uses derivatives. In the direct vs. compositional decomposition method the derivative is always with respect to time. For the delta method, the derivative is with respect to all parameters of the model. If the parameters of the delta model also depend on time, then the delta method could be used as a general formulation of change over time.

10.6 Conclusions

This chapter evaluates the various decomposition methods. The first point to note is that one should talk of complementary methods rather than of competing methods.

Even in the first method with its further extensions we find similarities with direct vs. compositional decomposition. The simplicity of Kitagawa's method is only comparable to the continuous formulation of direct vs. compositional decomposition in equation (6.4). Moreover, it is proven that the earlier method is a particular case of direct vs. compositional decomposition.

We also compared the extensions of Kitagawa's work, by Cho and Retherford, Kim and Strobino, Das Gupta and structural decomposition, in direct vs. compositional decomposition. An alternative formulation of direct vs. compositional decomposition helped us to demonstrate the similarity among all these techniques. The flexibility of direct vs. compositional decomposition is especially useful for adapting techniques of separation used by the other methods. Direct vs. compositional decomposition can also assist in implementing other hypotheses concerning types of changes over time of the variables involved in the decomposition: e.g. linear, exponential, logistic, or others.

Studies of changes in life expectancy, crude birth rate and population growth rate can potentially benefit from the contributions of direct vs. compositional decomposition. However, it will always be in the researcher's interest to find the proper decomposition for the measure under study. Here again, complementary application of the methods is the final advice.

The decomposition methods are applied to population totals, while regression decomposition and the purging method are used in cases of samples of the population. We have suggested some of the possible relations in both directions here by applying direct vs. compositional decomposition method in regression decomposition and by using log-linear models for separating cross-tabulated data into compositional components of direct vs. compositional decomposition. Therefore, more exchange between these methodologies, as those suggested here, can facilitate the study of changes in demographic measures.

11 Conclusions

11.1 Introduction

This final chapter summarizes the research presented in the previous ten chapters. It includes three sections, beginning with this introductory part which reviews both Part II and Part III of the book. Based on the studies of all these methods we have drawn some conclusions about desired *Properties of Decomposition Methods,* which are included in the following section. Finally, the last section presents some concluding remarks on future prospects of decomposition methods.

The word decomposition in demography is not restricted to separations of changes over time. It is used in simple cases for defining an addition as well as in complex cases where derivatives with respect to different variables are found. In this book we have used decompositions for analysis of changes of demographic variables over time.

When decomposing social change over time, it is important to keep in mind that this is a form of data reduction whose objective is to summarize the effects of the components of change.

This has been the concern of many researchers. In demography the studies of decomposition date back to the 1950s when Kitagawa proposed a simple technique that separates the difference in demographic rates. This study inspired further research questions on how to allocate components when numerous variables are involved in the rates of study. Some methods are general decomposition techniques while others are particular applications to demographic variables. In Part II we studied applications of decompositions to measures of mortality, fertility and population growth rates. Finally, we introduced some methods used when analyzing demographic variables with parametric models.

This review of the methods of decomposition set a framework to introduce Part III and *The Direct Versus Compositional Decomposition.* This core part of the book presents Vaupel's (1992) general method of decomposition and this author extensions of Vaupel's method that can also be used in particular areas of demography, such as the study of life expectancy and population growth. We claim and prove in this book that this decomposition technique deals effectively with demographic variables that have different types of changes over time.

The greatest advantage of this direct vs. compositional decomposition is its simplicity. If $\bar{v}$ denotes a demographic average and $\dot{\bar{v}}$ its change over time, then this is simply decomposed as

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}).$$

The change in an average is expressed as the sum of the average change of the variable of interest and the covariance between this variable and the intensity of its weights. In this way, the change over time in the crude death rate is separated into the average change in the death rates and the covariance between the death rates and the age-specific growth rates. Similarly, the change in the average life expectancy of the world is the average of the changes in the country-specific life expectancies and the covariance between these life expectancies and the population growth rates of the countries. The change in the population growth rate of the world is the average change in the countries' population growth and the variance of the growth rates. The average age of the population can be taken as an average of average ages over subpopulations. Its change is due to a change within each of these subpopulations and a change between the subpopulations. This is the average over subpopulations of the covariances between age and age-specific growth rates, and the covariance between the subpopulations' average age and their total growth rate, respectively. The intensity of the crude birth rate is decomposed into terms on the intensity in age-specific fertility rates, intensities of the female ratio and a difference between two average growth rates. Many other applications are possible. By using direct vs. compositional decomposition, the user knows the components that account for the change and the way these components are inter-related.

We have supplied some extensions of direct vs. compositional decomposition here, namely an age, categorical and cause-specific decomposition, and a multidimensional decomposition. These are some of the options that are of interest for demographers. Further extensions are suggested in the book such as the possibility to use this methodology for cases with samples instead of population totals. Another extension could be to examine second derivatives which allow us to study the contribution of the speed of the changes and the contribution of the components to this change.

A final contribution of this method is the possibility of implementing assumptions on the change over time of the demographic variables involved in the decomposition. This flexibility arises when considering the use of continuous change which can be interpreted as linear, exponential, logistic or another type.

From reviewing previous methods and comparing them with direct vs. compositional decomposition we gained more insight into the desired properties of a decomposition method. In the next section we list the most important properties.

11.2 Properties of Decomposition Methods

Several properties are desirable for a decomposition method. The following lists the most relevant properties.

- The decomposition method should allow the study of changes over any parameter: time, populations, sexes, ages, ethnicity, education attainment, etc. For example, when studying the decompositions of the age-specific growth rates we also included changes over age and cohorts.

- The method should also separate the total change into independent terms and avoid interaction effects. For example, when studying demographic averages, changes occurring in the variable of interest and those in the structure of the population should be separated into independent terms. Also residual terms are inconvenient for interpretation and, when present, if they have an important contribution in the change they are also a clear sign of the method's inefficiency.

- The components of the decomposition should represent meaningful demographic terms. This is a key issue in decomposition theory. The aim of this theory is to explain the total change into parts that have a clear interpretation and that explain the dynamics among the variables involved.

- The decomposition should have a simple mathematical expression which is easy to remember. A basic consideration in the use of any method is a formulation that immediately draws attention to its elements. The fewer and simpler these components are, the easier it is to use the method.

- The method should allow further decompositions for age, categorical and for cases of numerous compositional components, among others. For example, when decomposing life expectancy it should allow for a cause of death decomposition. Another example is the case of numerous compositional components. The decomposition should allow the user to choose whether there is a certain hierarchy between the components, or if they are all equally balanced.

- The methodology should be related to other decomposition methods. As in any other theory, decomposition methods are built on previous efforts in this area. The steps in the development of a new general decomposition have to relate to the questions that other, earlier decompositions have aimed at answering. It should study how these previous methodologies have answered the questions and permit implementations of the previous methods in the new technique.

- The decomposition should explicitly show the relations between the variables involved in the dynamic process under study. We are interested in studying the parts that explain the phenomena as well as in examining how these parts relate to each other.

- A decomposition method should have an expression for the relative change, relative to time, to a population, to one of the sexes, to an ethnic group, or to a similarly designated group, depending on the change under study. In the case of changes over time this is the intensity of the demographic average or its growth rate.

- The method should be flexible and allow the change of variables over time to be of different types: linear, exponential, logistic, and others. In many cases this entails suggesting a particular type of change for each of the variables involved in the decomposition, though the change is not necessarily equal change for all.

These are the properties that have become evident during the present study. Additional desired attributes could be listed for every one of the challenges that demographers have solved by using decomposition methods.

The direct vs. compositional decomposition of Part III has proved to fulfill all the properties listed above and, therefore, contributes to a greater understanding of population dynamics.

11.3 Concluding Remarks

Which decomposition method is to be preferred?

This question has been of interest to numerous researchers, including those mentioned in this book. The criterion for choosing the most suitable decomposition technique depends on the demographic phenomena of study and the time of study.

As noted earlier, many methods complement each other by focusing on new aspects of the change that have not been revealed by other methods. A second conclusion, which involves more calculations, is the advice to apply all possible methods. By doing this the analysis will benefit from studying several factors associated with the phenomena being measured. Please note that we have used the word association for the relation between the demographic average and the components, or factors. This is to avoid the misunderstanding of causal relationship. Underlying unobserved heterogeneity of the components can actually be responsible for these associations.

Initially the decomposition methods were used to study the difference in demographic measures between two periods of time. This is an interesting aspect of population dynamics. However, it is important to analyze changes from a wider perspective. The era of the computer and the revolution of information have accelerated the collection of data and thus the amount of demographic information available. Consequently, new information will allow us to modify the assumptions of trends over time in demographic variables so that they become more similar to the real observed paths. The direct vs. compositional decomposition, then, is likely to become the leading methodology for studying change over time in demographic variables, because of its flexibility and ability to effectively capture change over time.

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INDEX

Samenvatting

Inleiding

Een verklaring van demografische processen impliceert vaak een analytische benadering waarbij de processen worden gesplitst in componenten en deelprocessen. Deze benadering is vooral nuttig wanneer beoogd wordt inzicht te verwerven in de bijdrage aan de bevolkingsdynamiek van ieder van de componenten van demografische verandering. Dit boek bespreekt en ontwikkelt methoden om demografische variabelen en hun veranderingen in de tijd te splitsen waardoor het inzicht in de verandering wordt vergroot. De methode staat bekend als decompositietechniek. Deze techniek is onderdeel van de formele demografie.

Demografische processen worden veelal gekarakteriseerd aan de hand van kerncijfers of indicatoren, zoals de levensverwachting, het bruto geboortecijfer, de gemiddelde leeftijd bij geboorte van het eerste kind, en het migratiecijfer. Het voordeel van die indicatoren is dat zij een eenvoudige interpretatie hebben. Een nadeel is echter dat zij bestaan uit één getal dat veelal afhankelijk is van de samenstelling of structuur van de bevolking. Bijvoorbeeld, het bruto geboortecijfer wordt niet alleen beïnvloed door het vruchtbaarheidsniveau, maar ook door de leeftijdsstructuur van de bevolking, verandering in aandeel gehuwden, verandering in opleidingsniveau, etc.. Aangezien de samenstelling van de bevolking verandert, veranderen de indicatoren, ook bij gelijkblijvend gedrag. Dat maakt trendanalyses en comparatief demografisch onderzoek (vergelijking van verschillende populaties) bijzonder moeilijk. Decompositietechnieken bieden de mogelijkheid om de veranderingen in indicatoren uiteen te leggen in bijdragen van verschillende variabelen of factoren.

Ter illustratie volgt een aantal vragen waarbij compositie-effecten een belangrijke rol spelen en die in dit proefschrift aan bod komen. Waarom daalt in Mexico het bruto sterftecijfer terwijl dat in andere landen stijgt? Is de daling een gevolg van de veranderende leeftijdsopbouw van de bevolking? Waarom daalt in Nederland, Denemarken en Zweden het bruto geboortecijfer? Heeft dat te maken met een vruchtbaarheidsdaling onder gehuwde en ongehuwde vrouwen, of is dat een gevolg van een kleiner aandeel gehuwden? Is de verandering in levensverwachting van de bevolking in Europa een gevolg van een algehele sterftedaling in Europa of van een verandering in de leeftijdsopbouw en de verdeling van de bevolking over de landen van Europa? Decompositietechnieken ontrafelen complexe veranderingsprocessen, identificeren de componenten of bouwstenen, en meten de relatieve betekenis van compositie-effecten (structurele verandering) en gedragseffecten (gedragsverandering). In dit boek worden twee componenten onderscheiden: de directe component ('direct component') (resulterend in gedragsefffecten) en compositie-component (compositional component') (resulterend in compositie-effecten).

Decompositiemethoden

Deel II van het boek is een literatuuroverzicht. De voorlopers van decompositiemethoden zijn de standaardisatietechnieken, namelijk de directe en de indirecte standaardisatie. Niettegenstaande deze technieken veel worden toegepast in vergelijkend onderzoek, hebben zij een belangrijk nadeel. De gestandaardiseerde maten, bijvoorbeeld cijfers of 'rates', zijn afhankelijk van de gekozen standaard. Kitagawa (1955) was niet tevreden met de standaardisatietechnieken en volgde een andere benadering in de vergelijking van twee bevolkingen: de decompositiemethode. In die methode worden demografische variabelen of indicatoren gesplitst in een compositie-component en een gedrags-component. De compositie-component kan betrekking hebben op één of meerdere persoonskenmerken. De methode van Kitagawa is beperkt tot één persoonskenmerk. Latere auteurs hebben methoden ontwikkeld om compositie-effecten te meten wanneer de bevolkingsstructuur wordt beschreven aan de hand van meerdere persoonskenmerken. Sommige decompositiemethoden maken daarbij gebruik van regressiemodellen en andere modellen van relaties tussen variabelen. Voorbeelden zijn de 'purging' methode van Clogg (1978) en andere onderzoekers, en de delta methode toegepast door Wilmoth (1988) en anderen. Het literatuuroverzicht van eerdere decompositiemethoden biedt het kader voor de presentatie van de decompositie in directe en compositiecomponenten ('direct vs compositional decomposition').

Decompositie van demografische verandering in directe versus decompositie-componenten

In 1992 introduceerde Vaupel een methode voor de decompositie van veranderingen van demografische variabelen in de tijd. Zijn idee was om de verandering te schrijven als de som van twee componenten: de gemiddelde verandering en de covariantie tussen de bestudeerde variabele en de intensiteit van de wegingsfunctie ('weighting function'). De vergelijking van Vaupel zegt dat de verandering in het gemiddelde van een variabele gelijk is aan de som van twee termen. De eerste term is de gemiddelde verandering in de bestudeerde variabele. Hij meet de directe effecten van de verandering. De tweede component is de covariantie-term die de effecten van verandering in de heterogeniteit van de bevolking meet. De twee componenten vormen de termen van een wiskundig model:

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}),$$

met $\dot{\bar{v}}$ de verandering in het gemiddelde, $\bar{\dot{v}}$ de gemiddelde verandering in de bestudeerde variabele, $C(v, \acute{w})$ de covariantie tussen de bestudeerde variabele en de intensiteit van de wegingsfunctie.

Dit boek beschrijft de methode van Vaupel en een stelt een extensie voor. In de meeste toepassingen in het boek bestaan de compositievariabelen uit leeftijdsamenstelling van de bevolking en omvang van deelbevolkingen. Deelbevolkingen worden afgebakend op basis van persoonskenmerken.

Bijvoorbeeld, Vaupel en Canudas Romo (2003) toonden aan dat de levensverwachting in twee componenten kan worden uiteengelegd. De eerste component meet de verandering in sterftekans en de tweede de leeftijdsverschillen (heterogeniteit) in die veranderingen. Dit boek bevat verscheidene toepassingen van de centrale vergelijking en extensies daarvan.

Extensie van de decompositiemethoden

Decompositiemethoden worden gehanteerd voor onder andere de studie van leeftijdseffecten op demografische veranderingen. In plaats van leeftijd kan een ander persoonskenmerk worden geselecteerd. Aangezien de meeste persoonskenmerken discrete variabelen zijn, wordt gesproken van 'categorical decomposition' van verandering. De studie toont bijvoorbeeld aan op welke wijze de verandering in de groeivoet van de wereldbevolking samenhangt met veranderingen in de verschillende regio's. Een ander voorbeeld dat in deze studie wordt besproken is de verandering in levensverwachting van de bevolking in Japan. De verandering in de totale levensverwachting is opgebouwd uit veranderingen in doodsoorzaken. Bijvoorbeeld, meer dan de helft van de stijging van de levensverwachting wordt verklaard door de vermindering van hart- en vaatziekten.

De decompositie van effecten in directe effecten (bijvoorbeeld gedragsverandering) en compositie-effecten vereist niet zelden de gelijktijdige studie van meerdere persoonskenmerken. In de studie wordt de centrale vergelijking uitgebreid. De methode wordt toegepast om de componenten te achterhalen van de verandering in het bruto sterftecijfer van Europa. De verandering wordt beïnvloed door verandering in de leeftijdsstructuur van de Europese bevolking, verandering in de sterfteniveaus (bruto sterftecijfer) van de verschillende Europese landen, en de verandering in de omvang en leeftijdsstructuur van de bevolking in de landen van Europa.

De wiskundige vergelijking in aanwezigheid van verschillende karakteristieken van de bevolking is

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}_1) + C(v, \acute{w}_2) + ... + C(v, \acute{w}_n),$$

met $\dot{\bar{v}}$ de verandering in het gemiddelde en $\bar{\dot{v}}$ de gemiddelde verandering in de bestudeerde variabele. De verschillende covarianties hebben betrekking op de invloed van de meerdimensionele structuur van de bevolking. $C(v, \acute{w}_1)$ is de compositiecomponent van de verandering veroorzaakt door de factor w_1. $C(v, \acute{w}_2)$ is de compositiecomponent van de verandering veroorzaakt door de factor w_2. Andere covarianties worden op een vergelijkbare wijze gedefiniëerd.

Besluit

In deze studie wordt de decompositiemethode van Vaupel verder ontwikkeld tot een breed toepasbare methode met een aantal interessante eigenschappen. Uit het onderzoek bleek dat deze eigenschappen algemene kenmerken zijn die voor alle decompositiemethoden gewenst zijn. De meest relevante eigenschappen zijn:

- De methode is van toepassing op veranderingen in een groot aantal variabelen: tijd, leeftijd, geslacht, etniciteit, opleidingsniveau, woonplaats, etc.
- De methode omvat, als bijzondere gevallen, een groot aantal decompositiemethoden uit de literatuur.
- De methode biedt de mogelijkheid om hoofdeffecten van compositievariabelen te meten zonder interactie-effecten.
- De methode splitst verandering in componenten met een duidelijke demografische betekenis en vergemakkelijkt daardoor de interpretatie van verandering.
- De methode is gebaseerd op een eenvoudig wiskundig model dat gemakkelijk te onthouden is.
- De methode is bijzonder flexibel en biedt een aanzet voor specificatie van niet-lineare verbanden.

Aanvankelijk werden decompositiemethoden toegepast om verschillen in waarden van demografische variabelen op twee tijdstippen te onderzoeken. Dat is een belangrijk aspect van bevolkingsdynamiek. Belangrijk is echter de verandering vanuit een breder perspectief te benaderen. Twee tijdstippen of twee meetpunten vormen slechts een beperkt beeld van dynamiek. Meerdere meetpunten, d.w.z. een tijdreeks, geven een beter beeld van verandering. De beschikbaarheid van tijdreeksen bieden de mogelijkheid de veronderstellingen betreffende het traject van demografische variabelen te wijzigen waardoor een meer realistisch beeld van verandering ontstaat. Aangezien de methode van direct vs compositional decomposition voldoende flexibel is om verschillende patronen van verandering te vatten, is de kans groot dat deze methode een leidende rol zal gaan spelen bij studies van bevolkingsdynamiek.