Forecasting Births in Post-Transition Populations: Stochastic Renewal with Serially Correlated Fertility

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Demographic forecasting techniques fail with post-transition populations dominated by fluctuating fertility; time series analysis of fertility can improve the forecasts. This article develops the optimal forecast and its variance for births to an age-structured population subject to serially correlated random fertility. The white noise, first-order autoregressive, second-order autoregressive and random walk fertility specifications are analyzed, each leading to different forecasts and very different variances, as shown by illustrative applications to U.S. data, 1917–1972.

1. INTRODUCTION

Demographic predictions have had a poor record in recent decades.[1] Not only have official U.S. predictions been misleading with respect to future levels, they have also failed to anticipate the likely range of error.[2] Nor have recent refinements of technique improved the situation, either with respect to the accuracy of predictions, or the derivation of appropriate confidence intervals (see [32, p. 103], and Sections 9 and 10 of this article). Indeed the continued efforts of demographers to predict population are sometimes justified on the grounds that they facilitate demographic analysis, rather than the reverse [32, 16]. Why have predictions based on demographic analysis been so poor?

Demographic analysis is concerned with precisely defined rates and corresponding subtleties of population structure; it proceeds by successive refinements of measure.[3] This preoccupation with measurement of rates and structures yields two basic capabilities for population prediction: the detection of stable and persistent trends amid apparent variation,[4] and the anticipation of reverberations from a distorted demographic structure.[5]

But in post-transition populations there are no secular trends in fertility, or so one might argue. The adjustment of fertility to declining mortality has been completed, and the positive population growth rates that persist reflect the growth potential of industrial economies, not demographic disequilibria.[6]

While it is possible that advances in contraceptive technology may lead to further fertility declines, this hypothesis is still highly conjectural, given the low levels of fertility achieved in the past. Many demographers concur in this diminished importance of trend—as Ryder [28, p. 116] states: "The future of fertility is likely to be increasingly bound up with questions of fluctuation rather than of trend"; Bogue [2, p. 741]: "In the recent past and in the foreseeable future the annual across-all-ages fluctuations in fertility have and will dominate fertility trends ..."; and Keyfitz [17, p. 361]: "... advanced countries will have an endless series of ups and downs in their births, about an average near replacement level." Indeed one might argue that there has been no secular trend in the total fertility rate for the U.S. since early in this century. Under these circumstances the ability of demographic analysis to detect subtle trends is not useful for population predictions.

What of the second prediction capability of demographic analysis—anticipating the effects of a distorted demographic structure? In fact, these effects are important only under exceptional circumstances. In a closed population subject to a regime of unvarying vital rates, distortions do not occur, and growth is perfectly exponential. On the other hand, if structural distortions do exist, they reflect past variations in vital rates and suggest the likelihood that comparable variations will occur in the future. But in this case—continually varying vital rates—the effect of structural distortions is second-order,

1 Demographers make population projections for several reasons; one is to suggest the likely size and distribution of the population at a future date or a range containing these magnitudes. In this article we are concerned solely with this aspect of projection—which we call "forecasting" or "prediction." JASA has published a number of papers reviewing demographic prediction methods, among them [6, 11, 17]. Other useful reviews are [7, 12].

2 For example, the 1970 prediction anticipated between 7.5 and 7.9 million births during the following two years; the actual number was 7.1 million [43, p. 5].

3 With respect to age, marital status, parity, timing of births, etc.

4 Such as the trend upward in cohort parity progression ratios below the second, and downward in those above the second, noted by Ryder [28].

5 Such as age structure effects, the "making up" of births, etc.

6 Of course a secular change in economic growth could cause a long-run adjustment of fertility.

dwarfed by the immediate effect of variations in the rates themselves.[7] It is only when a population is subjected to a catastrophic shock, which is then followed by relative tranquility, that the effect of structural distortion will dominate—and such situations are very rare.

The central problem of prediction under post-transition conditions is the variability of fertility, and we have seen that conventional demographic analysis contributes little to our understanding of it. Ideally, of course, we would use suitable empirically verified causal models to explain and predict fertility; however, we are still very far from satisfactorily explaining past variation in fertility, which is a necessary prerequisite.[8]

This is the background for the development of stochastic population models which formalize and quantify our ignorance concerning fertility change. At the least such models should help us to avoid confounding short-run fluctuation with long-run change—which demographers have consistently failed to do (see Section 9)[9]—and enable us to develop confidence regions for predictions. They should be able to do so; but, despite a growing body of literature, they have not.[10] Let us briefly review the reasons for this failure.

Stochastic models of the population renewal process have been developed along two lines. One approach interprets the vital rates as invariant probabilities at the individual level, thus focusing on the pure randomness of reproduction and survival in a branching process framework [25, 30, 33]. But for even moderately large populations, these models imply prediction variances which are clearly much too small, so that Schweder comments: "The source of projection deviation must rest mainly on year-to-year variation in death probabilities and birth distributions.... The pure randomness of population dynamics is of minor importance" [30, p. 448].

It is with this "year-to-year variation" in the rates that the second approach is concerned; it treats the vital rates themselves as random variables indexed on time [19, 21, 31, 33]. While this approach appears more promising for developing confidence regions, it has not yet made a practical contribution. This is in part because it concentrates on the demographically interesting but quantitatively unimportant problem of the internal correlations of the population projection matrix, and ignores the autocovariance structure of fertility variation as a process in time.[11]

But as Louis Henry recently pointed out, "Prediction depends on the inertia of demographic phenomena, which has never really been studied" [12, p. 395]. Fertility tends to change smoothly and slowly, a fact recognized implicitly by demographers when they weight heavily its recent levels in forecasts of its future. In the absence of reliable causal models, to ignore the autocovariance structure of fertility is to throw out the most useful available information.

Indeed the principal statistical forecasting method for time series is based entirely on autocovariance structure (see, e.g., [3]). Recently, there have been a number of papers applying these statistical techniques of time series analysis directly to demographic time series, to forecast population growth rates [26, 29] or births [13]. However, the introduction of this new technique has been entirely at the expense of the old, for as Pollard states: "The analysis becomes extremely complex if the model includes an age structure and for this reason, we shall restrict our analysis to a particular model which ignores the problem of age" [26, p. 209]. This research is a useful step in the right direction, but a satisfactory forecasting method would have to incorporate age structure [14].

This article attempts to integrate the forecasting techniques of time series analysis with the model of renewal for an age-structured population subject to stochastic vital rates. The complexities which have prevented such a synthesis in the past are overcome by developing a simple linear stochastic model which approximates the renewal process, and allows straightforward analysis of the effect of serially correlated disturbances on the development of the birth series. Particular attention is given to the case when fertility is a first-order autoregressive (Markov) process, but the second-order autoregressive and random walk processes are also discussed.

2. POPULATION RENEWAL AS A LINEAR STOCHASTIC PROCESS

A closed population is renewed only by births to its members, and each birth may be attributed to some previous birth cohort of the population; thus the identity

$$B_t = \sum_{a=1}^{u} \phi_{a,t} B_{t-a}, \qquad (2.1)$$

where B_t denotes the number of births at time t; $\phi_{a,t}$ denotes the net maternity function for age a at time t; and u denotes the upper limit of the female reproductive ages.[12]

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[7] For example, it can be shown from the model developed in later sections that when the fertility of a population is continually subject to serially uncorrelated variation over a long period, less than three percent of the variance in the difference between births in successive five-year periods is explained by age structure; the remainder is due to current variation of fertility.

[8] There are any number of successful analyses of short-run fluctuation in fertility in terms of economic variables, but these do not begin to explain the bulk of the variation, concentrated in long swings. There have been several attempts by economists to account for this longer-run variation: [8, 9, 10, 21, 23, 27, 44].

[9] Akers argues that "the cohort fertility method differentiates trend from fluctuation by defining trend in terms of completed cohort fertility" [1, p. 414]. But in the U.S., cohort fertility has shown the same long swings as period fertility, although of lesser amplitude. It has not been statistically demonstrated that cohort fertility is more than a moving average of period fertility. It is in any case an arbitrary definition of trend and loses its apparently lower variance as soon as it is translated back into period terms via timing and spacing assumptions.

[10] A successful exception, not taken seriously these days, is the logistic curve. By fitting a stochastic differential equation to more than a century of historical data in most applications, the method brought a strong sense of history to prediction and was able to provide effective confidence regions. Thus the U.S. predictions based on data for 1790 to 1920 contained the next three census counts (i.e., 1930, 1940, and 1950) within the 95 percent confidence region; while retrodiction regions contained the actual population estimates all the way back through 1640! For a discussion, see Davis [5].

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[11] It is perhaps because the internal correlations are high that they appear to be quantitatively unimportant for the birth series.

[12] More precisely, B_t refers to births in the interval $(t, t + 1)$, and $\phi_{a,t}$ is the net fertility rate over the interval $(t, t + 1)$ for those aged $(a - 1, a)$ at time t. Strictly speaking, the equation describes the process of *female* births only; the total number of births may be found by multiplying this by 2.05.

We may regard $\phi_{a,t}$ as an element of a vector of age-specific random variables indexed on time, with expected value ϕ_a.[13] The net reproduction rate at time t, denoted NRR_t, is defined as the sum over all ages of $\phi_{a,t}$; its expected value equals the sum of ϕ_a. Depending on whether this is greater, less than, or equal to unity, the population will grow, decline or remain roughly constant over the long run.[14]

For convenience, we wish to base our analysis on demographically stationary populations, i.e., those for which the expected value of the NRR equals one. This can be done with no loss of generality, because any stochastic net maternity function and associated birth series can be transformed to stationarity in the mean.[15] We will therefore assume

$$\sum_{a=1}^{u} \phi_a = 1. \qquad (2.2)$$

We now wish to approximate the exact relationship expressed in (2.1), which involves variations of the net maternity function over age and time, by a relation involving only NRR_t and the expected values of the net maternity function, ϕ_a. We define the following variables: $x_{a,t} = \phi_{a,t} - \phi_a$; $\epsilon_t = NRR_t - 1$; B is the long-run expected number of births corresponding to some initial sequence of birth cohorts subject to fertility ϕ_a;[16] h_t represents the proportional deviation of B_t from its long-run expected value, i.e., $h_t = (B_t/B) - 1$.

Then (2.1) may be rewritten as follows:

$$B(1 + h_t) = \sum_{a=1}^{u} (\phi_a + x_{a,t})(1 + h_{t-a})B, \qquad (2.3)$$

which on multiplication and simplification yields

$$h_t = \sum \phi_a h_{t-a} + \epsilon_t + \sum x_{a,t} h_{t-a}. \qquad (2.4)$$

The last term represents the second-order effect of the sum of products of deviations. Its magnitude depends on the size of σ_ϵ^2, and its distribution by frequency (spectrum). The contribution of this term in a deterministic model is assessed by Coale [4] and is generally small. We will henceforth ignore it, and work with the linear approximation

$$h_t \doteq \sum \phi_a h_{t-a} + \epsilon_t. \qquad (2.5)$$

If we define $\{w_i\}$ by $w_0 = 1$; $w_i = -\phi_i$ otherwise, then

this may be rewritten as

$$\sum_{i=0}^{u} w_i h_{t-i} \doteq \epsilon_t. \qquad (2.6)$$

We see that proportional variations in the number of births form an autoregressive process in which variations in net fertility represent the random disturbance.

Now any autoregressive process also has a "moving average" representation, so that for appropriate c_j we can write

$$h_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}, \qquad (2.7)$$

where the $\{c_j\}$ can be derived from the $\{w_i\}$ by using (2.7) to eliminate h from (2.6), and noting that the sum of coefficients for each ϵ_{t-j} must be zero [3, p. 46]. This procedure yields:

$$c_0 = 1 \qquad\qquad c_2 = c_0\phi_2 + c_1\phi_1$$
$$c_1 = c_0\phi_1 \qquad\quad c_3 = c_0\phi_3 + c_1\phi_2 + c_2\phi_1 \quad \text{etc.} \quad (2.8)$$

These coefficients have a simple demographic interpretation. They correspond to the progeny of a single birth (c_0) after i periods (for $i \geq 1$), subject to the constant net maternity function ϕ. It is therefore clear that the c_i must converge to a constant value for large i (by the fundamental theorem of stable population theory, and because the NRR = 1); in fact, this constant equals $1/A$ where A is the mean age at childbearing, $\sum a\phi_a$.[17]

We may summarize these results as follows: the series of proportional deviations of births from their long-run trend forms an autoregressive process for which proportional variations in the net reproduction rate are the disturbance. Alternatively, the proportional deviations of births may be regarded as an infinite moving average of past variations in the NRR, weighted by the progeny of a unitary population element.[18]

Under conditions of roughly constant mortality, such as obtain in modern industrial nations (except in time of war), variation in the NRR derives from variation in fertility; we may then interpret ϵ_t in terms of the (transformed) gross reproduction rate: $\epsilon_t \doteq (GRR_t - GRR)/GRR$; or in terms of the total fertility rate: $\epsilon_t \doteq (TFR_t - TFR)/TFR$.[19]

This simple model of the renewal process could be complicated in several ways to reflect more sophisticated demographic effects. Changes in mortality and nuptiality impart a specific form of autocorrelation to the NRR, which can be incorporated in the specification (see [22]). Various kinds of cohort effects related to the "making up" of births, and to target fertility levels may also be incorporated by complicating the specification of ϵ. On balance, however, these demographic effects appear too subtle to have much influence in the present context.

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[13] We thus assume, for the present, that the expected value exists and is independent of time.

[14] This is only approximately true, as noted later.

[15] LeBras [20] has shown that a unique long-run rate of increase may be associated with each random vector ϕ of age-specific net maternity rates, depending only on the probability distribution of ϕ. This long-run rate will not in general be the intrinsic rate of natural increase, r, implied by the vector of expected values, ϕ_a (where $1 = \sum e^{-ra}\phi_a$); but it will be close to r for processes of practical interest, and equal it when there is no serial correlation [33]. The most unfavorable case occurs when net maternity is cyclic with period equal to the mean age of childbearing; then, even if the amplitude of the cycle is .5 when the net reproduction rate is 1.0, the growth rate will deviate by only .0025 from zero (see [20, p. 551] or [4, p. 186]). Thus the growth rate of the stochastic birth series B_t is approximately r. It follows that the transformed series, $B_t^* = e^{-rt}B_t$, generated by the transformed stochastic function $\phi_{a,t}^* = e^{-ra}\phi_{a,t}$, is approximately stationary in the mean over the long run.

[16] See [15, p. 107] or [18].

[17] See, e.g., [15, p. 107] or [18].

[18] For a related but quite different approximation to a birth series as a transformation of moments of past values of the net maternity function, see [20].

[19] This involves one further approximation of negligible importance; we must ignore the interaction of differences in survivorship by age with variations in fertility by age.

3. PREDICTION WHEN VARIATIONS IN FERTILITY ARE SERIALLY UNCORRELATED

The general prediction problem may be stated as follows: we observe a series of births and net reproduction rates up to time t, and seek the best prediction of births for some future period, say time $t + s$. By "best" prediction we understand the one with the smallest error variance, assuming that the model specification is correct and the parameters are known exactly.

To start the analysis, we consider the simple and unrealistic case in which variations in the NRR are serially uncorrelated; the variations (ϵ) are then said to be a "white noise" (WN) process.[20] In this case, once the long-run mean value of the NRR has been determined, its recent past tells us nothing about its likely future. The unconditional expected value of ϵ, by definition 0, is equal to its expectation, conditional on earlier observations. The predicted value of ϵ_{t+s}, conditional on ϵ_{t-j}, $j = 0$, $\cdots$, ∞, is simply zero.

Under these circumstances, it is well known that the best predictor of h_{t+s}, denoted[21] $h_t^*(s)$, is given by

$$h_t^*(s) = \sum_{a=1}^{u} \phi_a h_{t+s-a}, \qquad (3.1)$$

where values of h_{t+s-a} after h_t are to be replaced in the expression by their predicted values, $h_t^*(s - a)$ [3, p. 156]. This corresponds to the conventional demographic projection with constant vital rates.

We now consider the error in this prediction, denoted $e_t(s)$, which by definition equals $h_{t+s} - h_t^*(s)$. It is a well-known result in prediction theory that

$$e_t(s) = \sum_{j=1}^{s} c_{s-j}\epsilon_{t+j}, \qquad (3.2)$$

where the c_i are the coefficients of the moving average formulation given in (2.7). This also makes demographic sense. If the NRR is constant ($\epsilon = 0$), there will be no prediction error; if the NRR does vary, the error each variation generates will be proportional to the progeny of a birth cohort, as indicated by c.

Since the ϵ are serially uncorrelated, we can readily derive the prediction variance, V, from (3.2):

$$V(s) = \sigma_\epsilon^2 \sum_{j=0}^{s-1} c_j^2. \qquad (3.3)$$

Confidence intervals for the birth prediction $h_t^*(s)$ may be derived from (3.3). Table 1 shows illustrative calculations based on U.S. data 1917–1972. We stress that these calculations are purely illustrative, and we will later show that the white noise assumption is not empirically appropriate. Nonetheless, the results are interesting. The last column shows 95 percent confidence intervals for birth predictions over various intervals up to 105 years.[22] Thus the initial prediction would have a confidence in-

terval of plus or minus 34 percent, and the interval would grow very slowly up to 44 percent after 105 years. It is surprising how broad the intervals are initially and by how little they deteriorate over the 100-year period. The amplification of prediction error by the internal structure of the population is not a very important source of error; rather it is the uncertainty about predicting fertility in any given period that predominates.

1. Illustrative Calculation of Variance and Confidence Interval for U.S. Birth Prediction Assuming White Noise Disturbance

Prediction period (years)	Index i	c_i	$S_i = \sum_{j=0}^{i} c_i^2$	Prediction variance[a] $(\sigma_\epsilon^2 S_i)$	Proportional confidence interval[b] $(2\sigma_\epsilon \sqrt{S_i})$
5	0	1.000	1.000	.0289	.340
10	1	.000	1.000	.0289	.340
15	2	.001	1.000	.0289	.340
20	3	.067	1.004	.0290	.341
25	4	.223	1.054	.0305	.349
30	5	.290	1.138	.0329	.363
35	6	.223	1.188	.0343	.371
40	7	.160	1.214	.0351	.375
45	8	.147	1.236	.0357	.378
50	9	.174	1.266	.0366	.383
55	10	.203	1.307	.0378	.389
60	11	.206	1.349	.0390	.395
65	12	.191	1.385	.0400	.400
70	13	.179	1.417	.0410	.405
75	14	.180	1.449	.0419	.409
80	15	.188	1.484	.0429	.414
85	16	.192	1.521	.0440	.419
90	17	.191	1.557	.0450	.424
95	18	.188	1.592	.0460	.429
100	19	.186	1.627	.0470	.434
105	20	.187	1.662	.0480	.438

[a] As proportion of squared predicted value.

[b] Approximate 95% confidence interval as a proportion of the predicted value, plus or minus.

Source: The net maternity function is based on the average fertility for each five-year age group for U.S. women, 1917–1972, combined with mortality rates appropriate for a life expectancy of 70 years, transformed to stationarity. $\sigma_\epsilon^2 = .0289$ is the variance of the total fertility rate, divided by its squared mean, using five-year intervals over the same period. For further details, see text.

The result expressed in (3.3) and used to construct Table 1 can be simplified by a further approximation. Recalling that $c_0 = 1$, and c_i converges to $1/A$ for $i > 0$, we have

$$V(s) \doteq \sigma_\epsilon^2[1 + (s - 1)/A^2]. \qquad (3.4)$$

In developed countries, A is between 25 and 30 years. If, as is usual, we employ a five-year time interval ($\delta = 5$ years), then A will be five or six units. From (3.3) we see that the prediction of births over a five-year period ($s = 1$) will have $V(1) = \sigma_\epsilon^2$; this will increase linearly with the prediction period until it doubles for a period of 130 years ($s = 26$) for which $V(26) = 2\sigma_\epsilon^2$. This illustrates that the increase in prediction error is very, very slow under the white noise specification. Equation (3.4) may also be used to study the effect of age-time units on prediction error.

We have made simplifying assumptions to arrive at the

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[20] This implies that ϵ_t is independent of all h_{t-i} for $i > 0$.

[21] We reserve this notation for the best predictor in the special case when ϵ is white noise, so that $h_t^*(s)$ is a pure age structure effect. More generally, we will denote the best predictor $\hat{h}_t(s)$.

[22] Assuming ϵ is normally distributed.

approximations expressed by (3.3) and (3.4);[23] we now pause to check their accuracy. Sykes [33] derived exact numerical results under the white noise assumption for a particular case, and we can attempt to replicate his results using our approximations. The predicted values must necessarily agree exactly, though the variances will differ. A comparison of calculated variances is shown in Table 2.[24] The agreement is very close, particularly for (3.3) which gives results nearly indistinguishable from the exact ones. Evidently the exact formulation can be greatly simplified with little loss of precision, even over a period as long as a century.

2. Comparison of Approximate and Exact Prediction Variance Based on Sykes' Example

Prediction year (base is 1940)	s	Variance as proportion of squared predicted value		
		Exact-Sykes' Model III	$\sigma_\epsilon^2 \sum_{j=0}^{s-1} c_j^2$ from (3.3)	$\sigma_\epsilon^2[1 + (s-1)/A^2]$ from (3.4)
1955	1	.0513	.0525	.0525
1970	2	.0569	.0575	.0685
1985	3	.0829	.0827	.0846
2000	4	.0954	.0955	.1006
2015	5	.1149	.1142	.1166
2030	6	.1304	.1299	.1326
2045	7	.1482	.1474	.1486

4. FORECASTING BIRTHS WHEN FERTILITY IS AN AUTOREGRESSIVE PROCESS

We now relax the assumption that fertility variations are serially independent and allow them to have an arbitrary autocovariance structure. With considerable generality, let us suppose that proportional deviations of the NRR can be expressed as a linear function of lagged values of a white noise process, η. This assumption covers *all* stationary processes, plus many nonstationary ones.[25] Then,

$$\epsilon_t = \sum_{j=0}^{\infty} d_j \eta_{t-j}, \qquad (4.1)$$

where d_0 is taken equal to unity. Equivalently, in autoregressive form,[26]

$$\epsilon_t = \sum_{j=1}^{n} g_j \epsilon_{t-j} + \eta_t, \qquad (4.2)$$

where the g_j may be derived from the d_j [see (2.8)].

The best predictor of ϵ, denoted $\epsilon_t^*(s)$, and the error variance of this predictor (both conditional on values of ϵ up to time t) may be derived from these equations. Thus from (4.2) we have

$$\epsilon_t^*(s) = \sum_{j=1}^{n} g_j \epsilon_{t+s-j}, \qquad (4.3)$$

inserting predicted values of ϵ when $t + s - j > t$. The actual value of ϵ_{t+s} will be given by

$$\epsilon_{t+s} = \epsilon_t^*(s) + \sum_{j=1}^{s} d_{s-j} \eta_{t+j} \qquad (4.4)$$

from which we see that the prediction variance of $\epsilon_t^*(s)$, denoted $V'(s)$, will be given by

$$V'(s) = \sigma_\eta^2 \sum_{j=0}^{s-1} d_j^2. \qquad (4.5)$$

How is the prediction of fertility related to the prediction of births? From (3.2) and (4.4) we have[27]

$$h_{t+s} = h_t^*(s) + \sum_{j+1}^{s} c_{s-j}[\epsilon_t^*(j) + \sum_{i=1}^{j} d_{j-1}\eta_{t+i}]. \qquad (4.6)$$

The best predictor of h_{t+s}, denoted $\hat{h}_t(s)$, will be given by its expected value, which is the deterministic part of (4.6):

$$\hat{h}_t(s) = h_t^*(s) + \sum_{j=1}^{s} c_{s-j}\epsilon_t^*(j). \qquad (4.7)$$

The predicted deviation of births from long-run trend may thus be approximately decomposed into two additive effects: one due to the initial age structure, the other due to the predicted time path of fertility.

From (4.6) it is also clear that the error of the prediction will be

$$e_t(s) = \sum_{j=1}^{s} c_{s-j} \sum_{i=1}^{j} d_{j-i}\eta_{t+i}. \qquad (4.8)$$

Rearranging terms, we have

$$e_t(s) = \sum_{i=1}^{s} k_{s-i}\eta_{t+i}, \quad \text{where} \quad k_i = \sum_{j=0}^{i} c_j d_{i-j}. \qquad (4.9)$$

Finally from (4.9) it is clear that the variance will be given by

$$V(s) = \sigma_\eta^2 \sum_{i=0}^{s-1} k_i^2. \qquad (4.10)$$

Equations (4.7) and (4.10) provide the best predictor and its error variance for the general case. It is easily confirmed that in the special case when ϵ is itself white noise (corresponding to $d_0 = 1$; for $i > 0$, all g_j, $d_j = 0$), these equations collapse to the results derived in Section 3.

5. FERTILITY AS A FIRST-ORDER AUTOREGRESSIVE, OR MARKOV, PROCESS

Having developed general results for autoregressive fertility, we now consider several special cases in more detail. We have already analyzed the simplest specification—the white noise process. Here we consider the Markov process specification and in subsequent sections we discuss the second-order autoregressive and random

[23] To get (3.3) we assumed $\Sigma x_{a,t}h_{t-a} = 0$; to get (3.4) we also assumed $c_i = 1/A$ for $i > 0$.

[24] Sykes' Model III assumes a population transition matrix of the form $A + \Delta_t$, where A is a traditional deterministic projection matrix, and Δ_t is a sequence of independent matrix random variables with mean zero and covariance matrix Σ [33, p. 122]; this corresponds to the white noise assumption for each vital rate and, hence, for the NRR and ϵ. The derivation of ϕ and σ_ϵ^2 from his data is complicated. In his notation, the matrix A has nonzero elements b_1, b_2, b_3, s_1, and s_2. The implied net maternity function is: $\phi_1 = b_1$; $\phi_2 = s_1 b_2$; $\phi_3 = s_1 s_2 b_3$. The NRR is given by $\phi_1 + \phi_2 + \phi_3$. The mean age of ϕ is easily calculated. Together with NRR this provided an estimate of r, which can be used to transform ϕ and ϵ. Appropriate calculations then give the variance of the transformed NRR, σ_2 and the mean age of the transformed ϕ, A. The original matrix A is given in the middle of page 126; Σ is given in Table 2, page 127.

[25] In particular, it covers all the integrated autoregressive moving average (ARIMA) processes treated by Box and Jenkins [3].

[26] Subject to the condition that none of the roots of the equation $\Sigma d_j z^j = 0$ lies on the unit circle.

[27] Recall that $h_t^*(s)$ denotes the best predictor of h_{t+s} when the disturbance is white noise; it therefore denotes the pure age structure effect, assuming constant vital rates.

walk specifications. None of these is consistent with a deterministic time trend in the mean level of fertility, so none is appropriate for populations which have not yet completed the demographic transition, or for situations in which drastic structural changes are expected permanently to alter future fertility. All specifications except the random walk (which is "nonstationary") also posit the existence of a constant "normal" level of fertility to which the process always tends to return. Particular specifications then concern the pattern of fluctuations about this normal level.

The white noise specification assumes that the near future of fertility is independent of its recent past and thus fails to acknowledge that fertility changes slowly under the impact of slowly changing socioeconomic conditions. The Markov process (with a positive coefficient) improves on this specification by expressing a tendency for deviations from the mean to persist while the series returns gradually to its normal level following a disturbance. Formally,

$$\epsilon_t = \rho\epsilon_{t-1} + \eta_t, \quad 0 \le \rho < 1. \tag{5.1}$$

Here, in the notation of (4.1) and (4.2), $g_1 = \rho$, $g_i = 0$ otherwise; and $d_i = \rho^i$ for all $i \ge 0$, and 0 otherwise. Referring to (4.3) we see that predicted fertility is given by

$$\overset{*}{\epsilon_t}(s) = \rho^s\epsilon_t. \tag{5.2}$$

We begin by considering the effect of autocorrelated fertility on the predicted level of births. From (5.2) and (4.7) we have

$$h_t(s) = \overset{*}{h_t}(s) + \epsilon_t \sum_{i=1}^{s} c_{s-i}\rho^i. \tag{5.3}$$

Recalling that c_0 is always taken equal to 1, and that for $i > 1$, $c_i \doteq 1/A$, this simplifies to

$$\hat{h}_t(s) \doteq \overset{*}{h_t}(s) + \epsilon_t\{\rho^s + (1 - \rho^s)/[A(1 - \rho)]\}. \tag{5.4}$$

The term on the right represents the proportional effect of the fertility prediction on the predicted number of births. This effect converges quite rapidly to its asymptotic value, $\epsilon_t/[A(1 - \rho)]$. For the U.S., with $A \doteq 5$ and $\rho \doteq .6$, the cumulative effect of the fertility prediction is to raise the birth prediction by the proportion $.5\epsilon_t$. For the U.S. over the past 55 years, with five-year time intervals, the standard deviation of ϵ has been about .19, and values greater than .25 have occurred several times. The effect on the prediction of taking account of the autocorrelation of fertility, rather than assuming fertility constant at its mean level, may therefore be considerable, sometimes over ten percent.[28]

Now let us consider the error of the prediction. From (4.10) we get

$$V(s) = \sigma_\eta^2 \sum_{i=0}^{s-1} \left(\sum_{j=0}^{i} c_{i-j}\rho^j\right)^2. \tag{5.5}$$

Using the approximation for c, we find the approxi-

mate recurrence relation for the variance

$$V(s + 1) \doteq V(s) + \sigma_\eta^2\{\rho^s + (1 - \rho^s)/[A(1 - \rho)]\}^2, \tag{5.6}$$

where $V(1) = \sigma_\eta^2$. This allows fairly simple computation. Bearing in mind that $\sigma_\eta^2 = (1 - \rho^2)\sigma_\epsilon^2$, we see from (5.6) that the increase in the variance per increase in the prediction period will fairly rapidly approach its limit:

$$\lim_{s\to\infty} V(s + 1) - V(s) = \sigma_\epsilon^2\{(1 + \rho)/[A^2(1 - \rho)]\} \tag{5.7}$$

This may be compared with (3.4) which shows that when ϵ is white noise the incremental variance is only σ_ϵ^2/A^2, smaller by a factor of $(1 - \rho)/(1 + \rho)$. Recalling that for U.S. data $\rho = .6$, we see that the erroneous assumption that fertility was a white noise process would lead one to underestimate the increase in variance with prediction period by a factor of about 4.

Table 3, based on U.S. data, shows the way in which calculated prediction variance increases with prediction period for four specifications, indexed on the innovation variance, σ_η^2, of each. Because of this indexing, comparison across rows is not appropriate. Reading down the columns for the white noise process and the Markov process, we note that the increase in prediction variance is indeed more rapid for the latter than the former.

3. Growth in Prediction Variance for Births According to Different Fertility Specifications[a]

s	Prediction period (years)	Index of prediction variance [V(s)/σ_η^2 for each specification]			
		White noise	Markov	2nd-order autoregressive	Random walk
1	5	1.000	1.000	1.000	1.000
2	10	1.000	1.360	2.880	2.000
3	15	1.000	1.490	3.611	3.002
4	20	1.004	1.571	3.640	4.143
5	25	1.054	1.726	4.427	5.810
6	30	1.138	2.002	4.991	8.310
7	35	1.188	2.293	5.001	11.564
8	40	1.214	2.526	6.144	15.421
9	45	1.236	2.717	8.415	19.877
10	50	1.266	2.908	9.940	25.098
11	55	1.307	3.100	9.992	31.288
12	60	1.349	3.336	10.422	38.546

[a] These figures are indexes of prediction variance, V(s), taking the first period's variance, σ_η^2, as 1. These first period variances are actually different for each specification, and are given in Table 4. The specifications for fertility variations, ϵ_t, are as follows: $\epsilon_t = \eta_t$ (white noise); $\epsilon_t = .6\epsilon_{t-1} + \eta_t$ (Markov process); $\epsilon_t = 1.37\epsilon_{t-1} - 1.03\epsilon_{t-2} + \eta_t$ (second order autoregressive); $\epsilon_t = \epsilon_{t-1} + \eta_t$ (random walk).

6. FERTILITY AS A SECOND-ORDER AUTOREGRESSIVE PROCESS

While the Markov process tended constantly to return to its mean, the second-order process is more flexible. It can represent cyclic behavior of any periodicity and degree of regularity and can express a tendency for short-run trends to persist. It therefore seems better suited to represent fertility change in a fluctuating economic

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[28] Note, however, that the long-run growth *rate* is unaffected since the *long-run* predicted fertility is unaffected.

environment, characterized for example by business cycles and Kuznets' cycles.

We will not develop general properties of the prediction for this specification; however, the numerical calculations for a particular parameterization are simple, and we give illustrative results for the U.S. The estimated autoregressive parameters for quinquennial averages of U.S. fertility, 1917 to 1972, are as follows, with standard deviations in parentheses:

$$\epsilon_t = 1.37\epsilon_{t-1} - 1.03\epsilon_{t-2} + \eta_t, \quad R^2 = .927. \quad (6.1)$$
$$(.157) \quad (.154)$$

The fit is very good. The estimated coefficients imply a slightly explosive oscillatory pattern, with a period of about 38 years; this evidently conforms closely to the historical long swing in U.S. fertility, with a trough in the 1930s, a peak in the 1950s, and a decline since then.

On narrow statistical grounds, this specification is much superior to the Markov, since the parameter for ϵ_{t-2} has a t-statistic of -6.7. Common sense, however, cautions that the long swing in U.S. fertility, for which we can observe only one cycle, is a feature only of this particular short realization of the process, and not of its underlying structure.

It is interesting to consider the constrained second-order specification proposed by Pollard [26] for annual population growth rate series: $\epsilon_t = k(2\epsilon_{t-1} - \epsilon_{t-2}) + \eta_t$. This specification does very well with the quinquennial U.S. fertility data; we estimated $k = .63$, with a t-statistic of 5.7. When we test this as a constraint on the general second-order process, as just estimated, we reject it at the .05 significance level, but are unable to reject it at the .01 level. Of course, working with only eleven observations, these calculations are illustrative at best.

We now consider the implied prediction variances. In (4.2) we have $g_1 = 1.37$ and $g_2 = -1.03$. The d_j of (4.1) can be calculated recursively as follows:

$$d_0 = 1 \qquad d_2 = g_1 d_1 + g_2 d_0$$
$$\vdots$$
$$d_1 = g_1 d_0 \qquad d_n = g_1 d_{n-1} + g_2 d_{n-2} \quad (6.2)$$

The calculated d_j, together with the c_j given in Table 1, may then be used to derive the k_i as specified in (4.9). Equation (4.10) then permits calculation of the prediction variances.

The resulting calculated variances, for predictions of up to 60 years, are shown in Table 3. Evidently the variance increases much more rapidly with prediction period than it does when fertility is a white noise or Markov process.

7. FERTILITY AS A RANDOM WALK PROCESS

According to Akers, "The most common assumption in making fertility projections is that the present level of period fertility will remain unchanged" [1, p. 417]. While this approach, known as the "period" method, was shown years ago to be potentially misleading, it is not

clear that the more complicated methods have fundamentally altered the basic assumption. We will discuss this point further in Section 9. Here we develop the implicit logic of the period approach.

To believe that fertility is best predicted by its current level, despite large variations in the past, is to reject the notion of a "normal" level to which fertility tends to return. Doubtless no one using the period method would expect fertility to remain constant at the contemporary level; rather they might expect upward or downward variations to be equally likely. The statistical model corresponding to this view is the "random walk":

$$\epsilon_t = \epsilon_{t-1} + \eta_t. \quad (7.1)$$

Fertility in the next period equals fertility in this period plus a random disturbance. The best predictor of fertility for all future periods is simply the last observed value:

$$\overset{*}{\epsilon}_t(s) = \epsilon_t. \quad (7.2)$$

We now consider the nature of the forecast and its variance, if fertility is a random walk process.[29]

According to (4.7), the birth forecast depends partly on age structure and partly on predicted changes in fertility. But under the random walk specification, the base period level of the NRR equals its expected value for all future periods, so we may take $\epsilon_t = 0$ and $\overset{*}{\epsilon}_t(s) = 0$ for all s. Thus the fertility effect drops out of (4.7), and the forecast depends solely on age structure: $\hat{h}_t(s) = \overset{*}{h}_t(s)$.

In the notation of (4.1) and (4.2), we have $d_i = 1$ for all $i \geq 0$; $g_1 = -1$, and $g_i = 0$ otherwise. Thus from (4.10) it follows that

$$V(s) = \sigma_\eta^2 \sum_{i=1}^{s} (1 + \sum_{j=1}^{s-i} c_j)^2. \quad (7.3)$$

Using the approximation for c, this simplifies to

$$V(s) \doteq \sigma_\eta^2 [s + s(s-1)/A$$
$$+ s(s-1)(2s-1)/(6A^2)]. \quad (7.4)$$

From (7.4) we see that the variance grows with the cube of the prediction period.

Table 3 shows the rapidity with which error variance increases. If fertility actually followed this implausible specification, forecasting beyond a few years would be incredibly uncertain.

8. DISCUSSION OF SPECIFICATIONS

We have considered four specifications of the fertility process and their consequences for forecasting births. Table 3 showed the very different rates at which the calculated error variance of the forecast increased with prediction period depending on the specification assumed. Table 4, which gives variances and confidence intervals for forecasts of five and fifty years for each method, also shows the sensitivity of the results to the fertility specifi-

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[29] The unconditional expected value of the NRR does not exist in this model; therefore neither h nor ϵ can be defined, as in Section 2. However the expectation of the NRR at a given time, conditional on a previous value, does exist, and therefore we can apply the analysis of that part in the context of prediction.

cation. Column 1 gives the innovation variances, σ_η^2, which vary by a factor of 14.[30] The last two columns give approximate 95 percent confidence intervals. For the five-year forecast, these range from ± 9 percent for the second-order autoregressive process, to ± 34 percent for the white noise process. For a forecast of 50 years, the intervals range from ± 29 percent for the second-order autoregressive process, to ± 152 percent for the random walk.

4. Variances and Confidence Intervals for U.S. Birth Predictions Under Different Fertility Specifications

Fertility specification	Prediction variance by prediction period[a]		95% confidence interval by prediction period[b]	
	5 years (σ_η^2)	50 years	5 years	50 years
White noise	.0289	.0366	.340	.383
Markov	.0185	.0538	.272	.464
Second-order autoregressive	.0021	.0207	.091	.288
Random walk	.0231	.5803	.304	1.524

[a] As proportion of squared predicted number. The entry for 5 years is simply the innovation variance, σ_η^2.

[b] As proportion of predicted number; given by twice the square root of the variance. Based on Table 3, and the estimated innovation variances, σ_η^2 (plus or minus).

Evidently the sensitivity of the anticipated forecast variance to the specification is very considerable; therefore, it is impossible to form confidence intervals without first determining the autocovariance structure of fertility. In addition, of course, the forecasts themselves will be quite different.

How then may we determine the appropriate specification? Ordinarily we would resolve the question empirically, using statistical techniques to determine an appropriate parametric representation of the process. We have attempted to apply the techniques suggested by Box and Jenkins [3] to the time series of annual observations of the total fertility rate of the U.S. from 1917 to 1972. These techniques involve initially studying the estimated autocorrelation coefficients (the correlogram) and the estimated autoregression coefficients (the g_j in our notation) for the original series, and for its first and second differences. This preliminary examination should suggest an appropriate model. Results of this exercise were disappointing. The first-order Markov process was clearly inadequate. Only after second differencing of the fertility series were the results even roughly amenable to stationary modeling. For theoretical reasons, the severe nonstationarity implied by the necessity of second differencing makes this approach unattractive.

9. OFFICIAL U.S. FORECASTS

Our discussion of the effect of serially correlated fertility on demographic forecasting provides an interesting

context for the review of two further topics: the official U.S. forecasts, which we discuss here, and Sykes' influential pioneering study of stochastic versions of the projection matrix, which we discuss in Section 10.

The U.S. Census Bureau uses complicated techniques for "projection" of births and fertility, based in past years on several different approaches: period analysis, cohort analysis, and the fertility expectations of married couples as measured by surveys. These techniques are complemented by ad hoc smoothing procedures. A number of projections based on different fertility assumptions are made, to give the user a sense of the range, and these different assumptions are chosen on subjective grounds. It appears impossible to interpret these complicated procedures in terms of any simple statistical model, and thereby to derive confidence intervals for the predictions.

However, a cursory look at the outcomes of these different procedures is surprisingly suggestive. Over the past 20 years, there have been nine sets of "projections" published by the Census Bureau. For each set, we have calculated the average level of assumed terminal completed fertility,[31] and compared it to the average total fertility rate for the five years preceding the projection.[32] The result is shown in the figure.

Average Official Projections of Ultimate Completed Fertility Versus Base Period Fertility, U.S., 1953–1972

[a] Average of all distinct completed fertilities at latest projection period. Calculated from [34,35,36,37,38,40,41,42; and 43].

Under the random walk specification and according to the "period" projection method, the projected and base period rates would be the same, so that all points would lie on the diagonal. In fact we see that the points
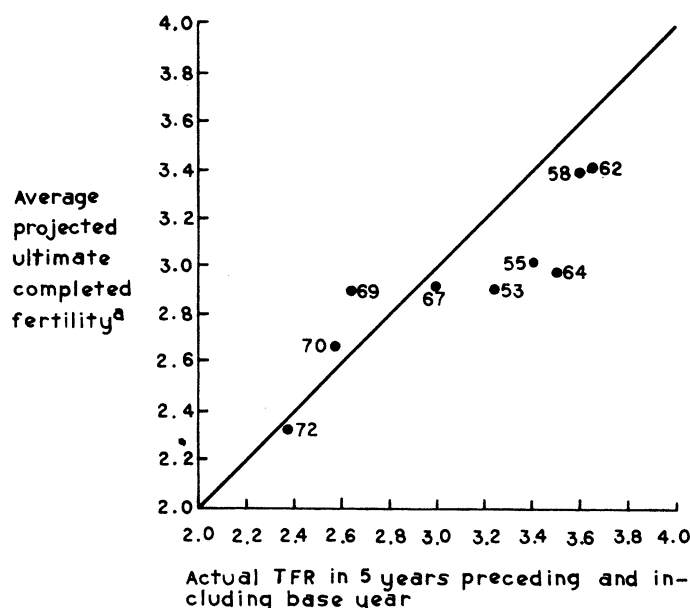
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[30] Using these, the remainder of the table, and the entries for prediction periods other than five or fifty years, can be constructed with the help of Table 3.

[31] For example, the latest set [43] assumed the following terminal completed fertilities: 1.8, 2.1, 2.5, 2.8; the average is 2.3.

[32] For example, for the latest projection (December 1972) we compare the average TFR for 1968 through 1972, which equals 2.33.

for 1958, 1962, 1967, 1970, and 1972 *do* lie near the diagonal, while the other four are not far from it. Thus *in their effects*, the official procedures closely resemble the discredited "period" approach, or random walk model. Despite the impressive technical apparatus employed, the principal input to population projections is the implicit uncritical imputation of a simple and implausible autoregressive structure to the time series of fertility.[33] A small amount of attention to the time structure of fertility variation might be more useful than the enormous effort that has been devoted to the study of the internal structure of population, as affected by past variations, for this latter is only a second-order effect.

Ideally, of course, we would develop a causal theory explaining past variations in fertility and helping us to predict them in the future. In the absence of such an established theory, however, we may still hope to extract useful information about the likely future of fertility from its past, using techniques of time series analysis for an objective analysis of fertility variation as a process in time.

10. SYKES' ILLUSTRATIVE FORECASTS

Sykes' pioneering article [33] developed for the first time the mathematics of population renewal subject to stochastic vital rates. He was the first to point out that the branching process interpretation led to implausibly small variances for moderate sized populations and that, therefore, the time-varying rate interpretation was more appropriate. He also recognized that for practical applications, the serial correlation of disturbances would often be important; however, he did not develop that aspect of the problem in any detail.[34]

Sykes' article also contained illustrative forecasts, with error variances, for the U.S. population, 1940 to 2045. It is with these forecasts that we are concerned here: how well have they performed, in terms of their implied confidence regions, and why? We can easily check them against the actual population data for 1955 and 1970. Table 5 shows the results, for the youngest age group.[35]

The prediction for 1955 was off by twice its supposed standard error, which should happen only one out of 20 times, assuming a normal distribution for the vital rates. The prediction for 1970 was off by three times its standard error, which should happen only very rarely by chance. It is clear therefore that the calculated variance did not provide an adequate guide to the reliability of the pre-

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[33] The use of fertility expectations data, gathered in national surveys, does not really alter this fact (see the points on the figure for 1970 and 1972, which were based on expectations data). It is not clear that the time series of expectations actually leads that of fertility. But even if it did (see [17, p. 355]), the relation between the expectations of current and future couples poses exactly the same prediction problem as that between current and future fertility.

[34] The basic model developed in Sections 2 and 3 of this article is formally a version of Sykes' "Model I" (additive disturbances), although it was derived, via transformation and approximation, from his "Model III" (multiplicative disturbances).

[35] Sykes used a 15-year age-time interval. The age group 0–14 is the closest analog to births in which we are primarily interested. Presumably the predictions for the older age groups, which depended only on mortality, were more accurate.

5. Prediction Errors Based on Sykes for U.S. Women Aged 0–14

Year	(A) Predicted[a]	(B) Actual[b]	(C) = (A)-(B)	(D) Standard[c] error of prediction	(C)/(D) Error/ standard error
1955	16,428	24,031	7,603	3,720	2.04
1970	16,799	28,974	12,175	4,007	3.04

[a] Sykes [33, p. 128].
[b] U.S. Bureau of the Census [39,42].
[c] Square root of variance in Sykes [33, p. 128], Model III.

dictions. This is particularly surprising since the prediction was almost entirely within the sample period.[36]

How did this happen? There are two difficulties with Sykes' analysis. First, he made the empirically inappropriate assumption that vital rates were white noise processes, abstracting from their serial correlation; and second, his numerical illustration was fundamentally inconsistent with his development of the model, as shown here.

Sykes assumed the white noise specification with the comment: "Perhaps, however, it is safe to argue that for projections of human populations with five-year steps the variables Δ_t and Δ_{t+1} would have low correlation compared with the internal correlations within each matrix" [33, p. 123]. But we have shown that we may abstract entirely from the internal correlations of the projection matrix without much affecting the calculated variances (this is perhaps because these correlations are high). Furthermore, we have shown (in Section 5 and Tables 2 and 4) that this is *not* the case for abstraction from the serial correlation of the vital rates; and for the U.S., 1917 to 1972, using "five-year steps" we have calculated a first-order autocorrelation of .6 between successive TFRs. Thus the white noise specification leads to serious errors.

The second difficulty is an internal inconsistency. Sykes attempted to provide confidence intervals for so-called "period" population predictions, in which the base period vital rates are assumed to prevail in the future. We have seen that this procedure is appropriate only when the vital rates are random walk processes, in which case the variances can be derived using (7.3) or (7.4). However Sykes' theoretical derivation of the variance, based on his Modell III, assumed that the vital rates were white noise processes [33, p. 122]. He thus based his illustrative prediction and his prediction variances on two different and contradictory specifications of the fertility processes.

This inconsistency is revealed by the implementation of the model: the prediction is based on the vital rates of 1940, the *base* period, but the prediction variances are based on the variance of the vital rates about their *long-run mean* values over the period 1920 to 1965. To be consistent, the prediction should be based on these long-run values, with the variances as stated; or he should have

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[36] The sample period was 1920 to 1965.

calculated the variance of first differences of the vital rates over time, and derived prediction variances from an appropriate model based on the random walk assumption [such as our (7.3) or (7.4)] leaving the predictions as stated.

These criticisms pertain only to the illustrative calculations and have no implications for the mathematical results in Sykes' article.

11. CONCLUSIONS

Given the fidelity with which demographic forecasts, whatever their provenance, reflect changes in base period fertility, we may conclude that our major source of information about future fertility is not any deep understanding of fertility, but rather its recent past. While the period projection method in its many guises[37] attempts to exploit this source of information, it does so arbitrarily, and is inclined to confound fluctuation with permanent change.

On the other hand, stochastic population models, intended to aid in formulating optimal forecasts and their confidence regions, have been forced by the complexity of the problems either to ignore this information (by assuming the fertility rates are serially independent) or to ignore the age structure of population.

In this article we have developed approximate results for an age-structured population subject to serially correlated random vital rates. We treated the general case in which the fertility process had arbitrary autoregressive structure; we then considered four special cases: white noise, first-order autoregressive (or Markov), second-order autoregressive, and random walk. We concluded that the predictions and their variances were highly sensitive to the autoregressive structure of fertility and, therefore, that if stochastic models are to be used for prediction, they must emphasize this aspect of the problem. Preliminary empirical efforts to model the time series of U.S. fertility, 1917 to 1972, were unsuccessful, but it is clear that *at least* a second-order autoregressive scheme is necessary.

The analysis in this article could be usefully extended in several ways. First, any application of the procedure requires a successful parameterization of the fertility process. Second, fertility variation could be decomposed into the effects of nuptiality and marital fertility, and then births and marriages could be jointly predicted; the basic model is in [22]. Third, the simplifying approximations could be dropped, and each age-specific rate could be analyzed and predicted. Sykes' results could be used to find the covariance structure of the forecasts.

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