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A bivariate model for total fertility rate and mean age of childbearing *

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Annual U.S. total fertility rates and mean ages of childbearing, 1921–1980, are fitted to a bivariate autoregressive model which incorporates regression adjustments for the effects of World War II. The result is a transfer function model with total fertility rate as a predictor of mean age of childbearing. While *contemporaneous* changes in mean age of childbearing tend to be negatively correlated with changes in total fertility rate, there is a tendency for adjustments in mean age to be in the same direction as changes in total fertility in the long run. Point forecasts from the model suggest total fertility will tend to rise but stay below replacement level, and mean age of childbearing will increase very slightly.

Keywords: Bivariate autoregression, Transfer functions, Forecasts

1. Introduction

Two quantities that receive a great deal of attention in practical demographic work are the total fertility rate (TFR) and the mean age of childbearing (MACB). These are especially important quantities in preparing and comparing demographic projections. My colleague John Long, who is Chief of the Populations Projections Branch of the U.S. Census Bureau, pointed out that demographers had noted that contemporaneous changes in TFR and MACB tended to be negatively correlated, but he wondered if time series analysis would shed additional light on the relationship between these two important quantities. This paper reports the results of my investigations. I wish to thank Bill Bell, another colleague at the Census Bureau, for valuable assistance in conducting this research. Most of the computations and all the graphics in this paper were done with the UNIVAC 1100 computer at the Bureau of the Census. Some of the computations were done with the VAX 750 computer at the Statistics Department, University of Wisconsin–Madison.

In Section 2 the notation of the paper is developed and the data introduced. Section 3 contains the model fitting results, Section 4 forecasts from the model, and Section 5 a summary.

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2. Notation and data

Estimates of aggregate age-specific fertility rates are available for white women in the U.S. for ages 14–49 and years 1917–1980. 'Preliminary' data are available for more recent years, but they are subject to substantial revision and are not used in this study. To avoid the effect of World War I only data from 1921 onward are used. Let k denote age of mother, t calendar year, and f_{kt} the number of births per 10,000 women aged k in year t. Figure 1 is a three-dimensional plot of these rates. The distribution of mothers' ages tends to be unimodal with peak in the mid-20s and right skewed. They also exhibit rather dramatic shifts in level over time.

Demographers define the total fertility rate in year t to be the sum of the age-specific rates observed in that year, namely

$$\mathrm{TFR}_t = \sum_{k=14}^{49} f_{k,t}. \tag{1}$$

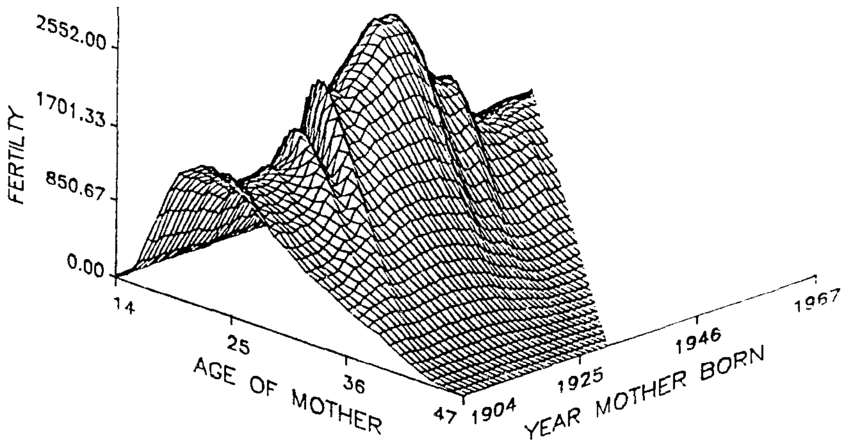

Fig. 1. Fertility rate by birth cohort and age [per 10,000 U.S. (white) women].

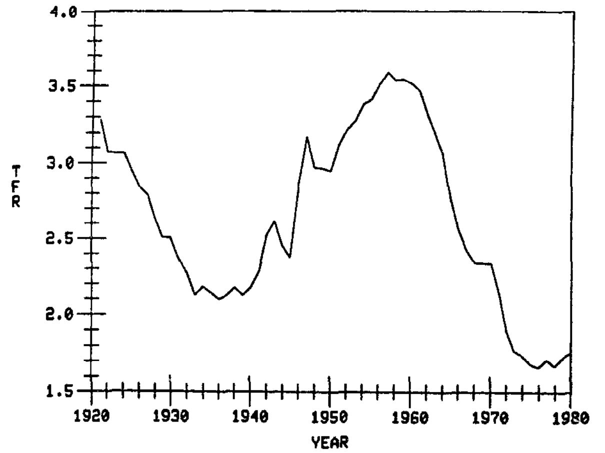

Fig. 2a. Period total fertility rate, white U.S. women, 1921–1980.

If we define normalized fertility rates

$$g_{kt} = f_{kt}/\mathrm{TFR}_t, \tag{2}$$

then the mean age of childbearing is defined by

$$\mathrm{MACB}_t = \sum_{k=14}^{49} (k+\lambda) g_{kt}, \tag{3}$$

where λ is typically chosen to be 0 or 1/2. The value $\lambda = 0$ is used in this paper. The data are listed in Appendix 2.

Figures 2a, b show time series plots of TFR_t and MACB_t for $t = 1921, 1922, \ldots, 1980$. Figure 3 shows time series plots of the annual changes

$$\nabla\mathrm{TFR}_t = \mathrm{TFR}_t - \mathrm{TFR}_{t-1},$$

$$\nabla\mathrm{MACB}_t = \mathrm{MACB}_t - \mathrm{MACB}_{t-1}. \tag{4}$$

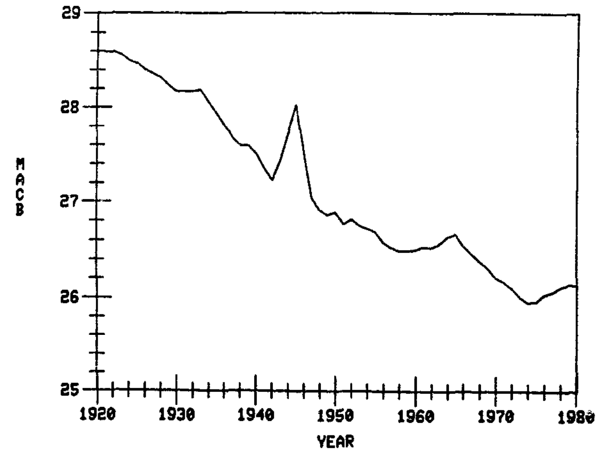

Fig. 2b. Period mean age of child bearing, white U.S. women, 1921–1980.

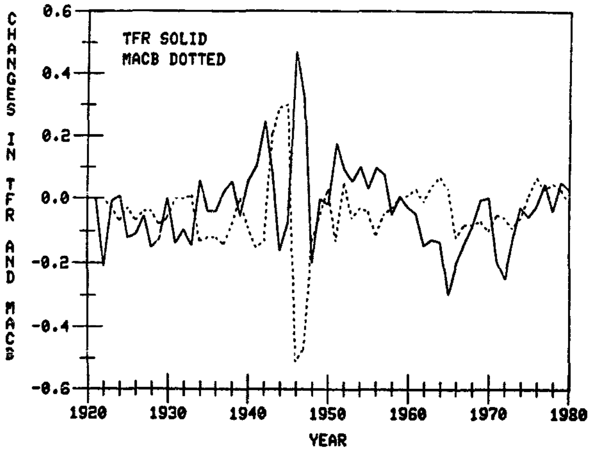

Fig. 3. Annual changes in total fertility rates and mean ages of childbearing 1921–1980, U.S. white women.

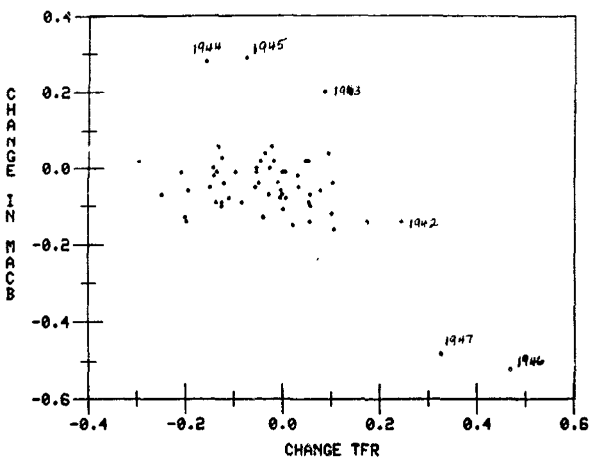

Fig. 4. Scatterplot of change in total fertility rate versus change in mean age of childbearing, 1921–1980, U.S. white women.

Figure 4 is a scatterplot of the changes in MACB versus the changes in TFR. Figures 3 and 4 show that the negative correlation in the changes is very pronounced in the war years 1942–1947. Such extreme behavior in a few points can mask relationships and tendencies in the rest of the data. Thus part of the modeling effort must be devoted to adjusting for the extreme behavior of the war years.

Before we proceed to the modeling it is useful to examine the effects on TFR and MACB of some simple assumptions about the age-specific rates. Let

$$\nabla f_{kt} = f_{kt} - f_{k(t-1)} \quad \text{and} \quad \nabla g_{kt} = g_{kt} - g_{k(t-1)}. \tag{5}$$

Then we see that

$$\nabla \mathrm{TFR}_t = \sum_{k=14}^{49} \nabla f_{kt}, \tag{6}$$

and

$$\begin{aligned} \nabla \mathrm{MACB}_t &= \sum_{k=14}^{49} (k+\lambda) \nabla g_{kt} \\ &= \sum_{k=14}^{49} (k+\lambda)\big[(f_{kt}/TRF_t) \\ &\qquad - (f_{k(t-1)}/TFR_{t-1})\big]. \end{aligned} \tag{7}$$

Example. Suppose $\nabla f_{kt} = \varepsilon/36$, $\varepsilon > 0$, for all k. This is what demographers call a pure period shift. Then $f_{kt} = f_{k(t-1)} + \varepsilon/36$ implies $\mathrm{TFR}_t = \mathrm{TFR}_{t-1} + \varepsilon$, which implies $\nabla \mathrm{TFR}_t = \varepsilon$. Moreover

$$\begin{aligned} \nabla \mathrm{MACB}_t &= (\varepsilon/\mathrm{TFR}_t) \sum_{k=14}^{49} (k+\lambda)\left(\tfrac{1}{36} - g_{k(t-1)}\right) \\ &= (\varepsilon/\mathrm{TFR}_t)\left[(31.5+\lambda) - \mathrm{MACB}_{t-1}\right]. \end{aligned} \tag{8}$$

In practice $\mathrm{MACB}_{t-1} < (31.5+\lambda)$, so $\nabla \mathrm{MACB}_t$ will agree with ε in sign. This means that both $\nabla \mathrm{TFR}_t$ and $\nabla \mathrm{MACB}_t$ shift in the same direction in response to a pure period shift in the age-specific rates.

Example. Assume that there are compensating changes in the age-specific rates, so that $\nabla \mathrm{TFR}_t = 0$. Then $\nabla \mathrm{MACB}_t$ could be positive, negative, or zero, depending on the age distributions in years $t-1$ and t.

3. Modeling

3.1. Model formulation

The extreme movements in the 1940s point to the need to treat World War II as an intervention [see Box and Tiao (1975)]. To do this define

$$\begin{aligned} X_{jt} &= 1 \quad \text{if } t = j, \\ &\, 0 \quad \text{if } t \neq j, \end{aligned} \tag{9}$$

for $j = 1942, \ldots, 1947$, and let

$$.\mathrm{TFR}_t = \mathrm{TFR}_t - \sum_{j=1942}^{1947} \beta_{1j} X_{jt}, \tag{10}$$

$$.\mathrm{MACB}_t = \mathrm{MACB}_t - \sum_{j=1942}^{1947} \beta_{2j} X_{jt}, \tag{11}$$

where the βs are coefficients measuring the effects of the war years. Because total fertility decreased during the war, we expect the βs in the TFR adjustment to be negative. Because mean age of childbearing increased during the war, we expect the βs in the MACB adjustment to be positive.

We hypothesize a bivariate ARIMA model for the vector $\boldsymbol{Z}_t = [.\mathrm{TFR}_t, .\mathrm{MACB}_t]'$. The modeling strategy adopted is to fit a simple model, examine the residuals, and complicate the model only if the residuals suggest model inadequacy. This approach is crucial in multivariate time series modeling because parameters proliferate very rapidly as the order of the model increases. The bivariate ARIMA (1, 1, 0) model is defined by

$$(I-\phi B)\nabla \boldsymbol{Z}_t=\boldsymbol{\alpha}_t, \tag{12}$$

where $I=\mathrm{diag}[1,1]$, $\phi=((\phi_{ij}))$ is a 2×2 matrix of parameters, and $\boldsymbol{\alpha}_t=[\alpha_{1t}, \alpha_{2t}]'$ is a bivariate white noise series with 2×2 covariance matrix $\Sigma_\alpha=((\sigma_{ij}))$. See Tiao and Box (1981) for notation and examples. Fortunately, this simple model proves to be sufficient for our needs.

3.2. Model fitting

The SCA System [Liu and Hudak (1983)] allows the simultaneous estimation of the regression parameters in equations (10) and (11) and the time series parameters in equation (12) in a fully efficient manner. The algorithm used by SCA is based on the work of Hillmer and Tiao (1979). Table 1 gives parameter estimates and selected standard errors. Residual plots, autocorrelations, and cross-correlations suggest no model inadequacy, so there is no reason to entertain a more complicated model. In fact, the time series portion of the model has been simplified, with no deterioration in the fit, by restricting ϕ_{12} to be zero.

Table 1
Parameter estimates with standard errors in parentheses of models specified in equations (10), (11), and (12).

j	TFR	MACB
	$\hat{\beta}_{1j}$(se)	$\hat{\beta}_{2j}$(se)
1942	0.124(0.064)	−0.016(0.045)
1943	0.084(0.106)	0.281(0.074)
1944	−0.200(0.128)	0.636(0.090)
1945	−0.390(0.127)	0.988(0.090)
1946	−0.018(0.104)	0.520(0.075)
1947	0.236(0.063)	0.089(0.046)

$$\hat{\phi}=\begin{bmatrix} 0.701 & 0^{a} \\ (0.095) & (-) \\ 0.154 & 0.681 \\ (0.064) & (0.087) \end{bmatrix}$$

$$\hat{\Sigma}_\alpha=\begin{bmatrix} 0.0056 & -0.0016 \\ -0.0016 & 0.0025 \end{bmatrix}$$

$$\hat{\rho}_{\hat{\alpha}_1,\hat{\alpha}_2}=0.41$$

[a] ϕ_{12} is restricted to zero because its unrestricted estimate is very small.

3.3. Interpretation of regression parameters

The $\hat{\beta}$s in Table 1 may be interpreted to say:

(1) total fertility is above normal levels in 1942 (remember Pearl Harbor was bombed in December of 1941; did young couples anticipate U.S. entry into the war during 1941?);

(2) total fertility drops sharply during 1943–1945, then rises sharply to well above normal levels in 1947;

(3) mean age of childbearing, at normal levels in 1942 and 1947, is well above normal levels during 1943–1946, with a peak in 1945.

The fit of a model with βs for 1948 yielded insignificant estimates for these βs, so the fertility process seems to have returned to normal by 1948. Not for long, however, as further analysis suggests that the 'baby boom' begins showing up in the 1949 statistics!

3.4. Interpretation of time series parameters

The ARIMA model leads to a number of useful conclusions. First, let us write the model as

$$\nabla.\mathrm{TFR}_t=\phi_{11}\ \nabla.\mathrm{TFR}_{t-1}+\alpha_{1t}, \tag{13}$$

$$\nabla.\mathrm{MACB}_t=\phi_{21}\ \nabla.\mathrm{TFR}_{t-1} +\phi_{22}\ \nabla.MACB_{t-1}+\alpha_{2t}. \tag{14}$$

Equation (13) says the first difference of .TFR follows an AR(1) model (i.e., .TFR is univariate ARIMA(1, 1, 0)). Equation (14) relates changes in .MACB to *lags* of changes in .TFR and .MACB. Thus equation (14) yields a handy forecasting equation: knowing $\nabla.\mathrm{TFR}_n$ and $\nabla.\mathrm{MACB}_n$ in year n, we forecast $\nabla.\mathrm{MACB}_{n+1}$ with

$$\hat{\phi}_{21}\ \nabla.\mathrm{TFR}_n+\hat{\phi}_{22}\nabla.\mathrm{MACB}_n =(.154)\nabla.\mathrm{TFR}_n+(.681)\nabla.\mathrm{MACB}_n,$$

and our estimated variance of the forecast error is $\hat{\sigma}_{22}=0.0025$.

It is also handy to be able to estimate the impact of the change $\nabla.\mathrm{TFR}_n=1$ on the sequence of current and future changes $\nabla.\mathrm{MACB}_n$, $\nabla.\mathrm{MACB}_{n+1},\dots$ (compare the first example in Section 2). This is called the impulse response function.

To do this we must write $\nabla.\mathrm{MACB}_t$ as a function of $\nabla.\mathrm{TFR}_t$ and its lags, i.e., as a transfer function. In Appendix 1 it is shown that the model in equations (13) and (14) can be written

$$\nabla.\mathrm{TFR}_t = \phi_{11}\nabla.\mathrm{TFR}_{t-1} + \alpha_{1t}, \tag{15}$$

$$\nabla.\mathrm{MACB}_t = -d\left[1 + \eta B(1-\phi_{22}B)^{-1}\right]\nabla.\mathrm{TFR}_t + (1-\phi_{22}B)^{-1}e_t, \tag{16}$$

where B is the backshift operator defined by $BZ_t = Z_{t-l}$,

$d = -\sigma_{21}/\sigma_{11}$, $\eta = (\phi_{22} - \phi_{11}) - \phi_{21}/d$,

$\mathrm{Var}(e_t) = \sigma_{22}$, and $\mathrm{Cov}(\alpha_{1t}, e_t) = 0$.

Equation (16) may also be written

$$\nabla.\mathrm{MACB}_t = -d\nabla.\mathrm{TFR}_t - d\eta\nabla.\mathrm{TFR}_{t-1} - d\eta\phi_{22}\,\nabla.\mathrm{TFR}_{t-2} - d\eta\phi_{22}^2.\mathrm{TFR}_{t-3} - \cdots + (1-\phi_{22}B)^{-1}e_t. \tag{17}$$

This equation is used to calculate the impulse response function.

The impulse response function is computed by assuming that the noise sequence is zero ($e_t = 0$) and $\nabla.\mathrm{TFR}_t$ is the 'impulse' sequence

$$p_t^{(n)} = \begin{cases} 1 & \text{if } t = n, \\ 0 & \text{if } t \neq n. \end{cases}$$

This implies a permanent shift in level of .TFR. Then we see

$$\begin{aligned} \nabla.\mathrm{MACB}_t &= 0 \text{ for } t < n, \\ \nabla.\mathrm{MACB}_n &= -d, \\ \nabla.\mathrm{MACB}_{n+1} &= -d\eta, \\ \nabla.\mathrm{MACB}_{n+l} &= -d\eta\phi_{22}^{l-1}, \quad l \geqslant 2. \end{aligned}$$

Since $|\phi_{22}| < 1$, $\phi_{22}^{l-1} \to 0$ as $l \to \infty$, so the impact on *changes* in MACB of a level shift in TFR eventually dies way. The cumulative effect upon MACB_{n+l}, $l = 0, 1, 2, \ldots$, is, however, non-zero, as we see by adding the impacts on the changes:

$$\begin{aligned} &-d - d\eta - d\eta\phi_{22} - d\eta\phi_{22}^2 - \ldots - d\eta\phi_{22}^{l-1} - \ldots \\ &= -d - d\eta\left(1 + \phi_{22} + \ldots + \phi_{22}^{l-1}\right) \\ &\to -d - d\eta(1-\phi_2)^{-1} \end{aligned}$$

as $l \to \infty$. This limit is called the 'gain' of the transfer function.

If we use the parameter estimates of our fitted model: $\hat{\phi}_{11} = 0.701$, $\hat{\phi}_{21} = 0.154$, $\hat{\phi}_{22} = 0.681$, $\hat{\sigma}_{11} = 0.0056$, $\hat{\sigma}_{21} = -0.0016$, and $\hat{\sigma}_{22} = 0.0025$ we get $\hat{d} = -(-0.0016/0.0056) = 0.285714$ and $\eta = (0.681 - 0.701) - 0.154/0.285714 = -0.559$. The impulse response function is displayed in Table 2.

Table 2

Estimated impulse response function derived from model in equation (17).

l	Impulse response	Cumulative impulse response
0	−0.2857	−0.2857
1	0.1597	−0.1260
2	0.1088	−0.0172
3	0.0741	0.0569
4	0.0504	0.1073
5	0.0343	0.1416
6	0.0234	0.1650
7	0.0159	0.1809
.	.	.
.	.	.
.	.	.
∞	0	0.2149
Total (gain)	0.2149	

We see that the initial response in ∇.MACB to a positive 'impulse' in ∇.TFR is negative, but the responses thereafter are positive. So .MACB tries to 'recover' from the initial impact of a change in .TFR. In fact, the cumulative impulse response function shows that 'recovery' is so successful that the ultimate effect of a series of positive constant changes in .TFR is a *positive* gain in .MACB of 0.2149.

When we look at Figure 3, our eyes tend to see the initial negative reaction of ∇.MACB to ∇.TFR, but they do not pick up the lagged positive reactions. This is natural because ∇.TFR does not behave like a simple 'impulse' function, and there is noise. The tendencies revealed by the model are clouded by the complicated behavior of ∇.TFR and the noise term $(1-\phi_{22}B)^{-1}e_t$.

The model is consistent with demographers' observation about negative correlation in contemporaneous changes in TFR and MACB, but it also reveals dynamic behavior that is not obvious in the data plots.

4. Forecasts

Table 3 reports forecasts and standard errors of TFR and MACB for the 14 years 1981–1994. The point forecasts of TFR rise from 1.776 to 1.826, and the associated standard errors rise from 0.075 to 0.879. The point forecasts of TFR stay well

Table 3
Forecasts and standard errors of annual U.S. TFR and MACB, 1981–1994, from model in equations (10), (11), and (12).

Year	TFR		MACB	
	Forecast	Std. err.	Forecast	Std. err.
1981	1.776	0.075	26.128	0.050
1982	1.791	0.177	26.130	0.265
1983	1.801	0.261	26.133	0.375
1984	1.809	0.337	26.137	0.463
1985	1.814	0.408	26.141	0.539
1986	1.817	0.475	26.145	0.607
1987	1.820	0.536	26.148	0.669
1988	1.822	0.594	26.150	0.727
1989	1.823	0.648	26.152	0.781
1990	1.824	0.700	26.154	0.832
1991	1.825	0.748	26.155	0.881
1992	1.825	0.794	26.156	0.927
1993	1.825	0.838	26.156	0.971
1994	1.826	0.879	26.157	1.014

below 'replacement level' of about 2.3, but the standard error in 1994 would accommodate a forecasted TFR as low as 1 and as high as 2.7.

The point forecasts of MACB rise from 26.128 to 26.157, and the associated standard errors rise from 0.050 to 1.014. While there is virtually no change in the point forecasts, the standard error in 1994 would accommodate a forecasted MACB as low as 25.143 and as high as 27.171. Judging from the behavior implied by the fitted model we should predict that contemporaneous changes in TFR and MACB will be negatively correlated.

Forecasts for years beyond 1994 were generated from the model. The point forecasts for TFR and MACB remain at 1.826 and 26.128, but the standard errors increase rapidly and are useless as guides to the future behavior of these variables. This phenomenon serves to define a limit to which useful information can be extracted from the demographic data and the modeling approach employed here. Of course, other time series could be added to the model, and the expert judgement of demographers, economists, and sociologists could be blended with the model-based extrapolations, but this would take us beyond the scope of the present paper. See Thompson (1984, 1985) and Thompson and Miller (1984, 1985) for some research on these possibilities.

5. Summary

This paper is a response to a query from a demographer concerning the dynamic behavior of two important demographic quantities: total fertility rate (TFR) and mean age of childbearing (MACB). Annual U.S. estimates of these quantities from the years 1921–1980 are fitted to a model that makes a regression adjustment for the effects of World War II, which would otherwise distort the estimated behavior of the other years, and a bivariate ARIMA time series model. The estimated parameters of the regression adjustments suggest plausible deviations of TFR and MACB from normal levels during the period 1942–1947. The time series model reduces to a transfer function model in which TFR is a function of its own past and noise, whereas MACB is a function of its own past, the past of TFR, and noise. Computation of the impulse response function reveals that a simple level change in TFR produces a contemporaneous change in the opposite direction in MACB but an *eventual* adjustment in MACB in the same direction as the change in TFR. This dynamic tendency in the relationship between TFR and MACB is not discernable in simple graphs of these quantities. Finally, forecasts generated from the fitted model suggest that forecasts beyond 1994 are subject to so much uncertainty as to exceed the limits of any reasonable expectation on the actual behavior of TFR and MACB.

Appendix 1

We wish to show that the model

$$(I - \phi B)\nabla \boldsymbol{Z}_t = \boldsymbol{\alpha}_t, \tag{A.1}$$

with $\phi_{12} = 0$ can be transformed to transfer function form. The trick is to find d such that

$$\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \Sigma_\alpha \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}, \tag{A.2}$$

that is, we wish to diagonalize Σ_α. Now carrying out the matrix multiplication in (A.2), we get

$$\begin{bmatrix} \sigma_{11} & d\sigma_{11}+\sigma_{12} \\ d\sigma_{11}+\sigma_{12} & d^2\sigma_{11}+2d \quad \sigma_{12}+\sigma_{22} \end{bmatrix} = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}. \tag{A.3}$$

This means $v_1=\sigma_{11}$, $d\sigma_{11}+\sigma_{12}=0$, and $v_2=d^2\sigma_{11}+2d\sigma_{12}+\sigma_{22}$. Clearly $d=-\sigma_{12}/\sigma_{11}=-\sigma_{21}/\sigma_{11}$ (because Σ_α is symmetric), and $v_2=-\sigma_{12}^2/\sigma_{11}+\sigma_{22}$.

Now write (A.1) as

$$\nabla \boldsymbol{Z}_t = \boldsymbol{\phi}\nabla \boldsymbol{Z}_{t-1} + \boldsymbol{\alpha}_t \tag{A.4}$$

or

$$\begin{bmatrix} \nabla.\mathrm{TFR}_t \\ \nabla.\mathrm{MACB}_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} \nabla.\mathrm{TFR}_{t-1} \\ \nabla.\mathrm{MACB}_{t-1} \end{bmatrix} + \begin{bmatrix} \alpha_{1t} \\ \alpha_{2t} \end{bmatrix}. \tag{A.5}$$

Multiply through (A.5) by $\left[\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right]$ to get

$$\begin{bmatrix} \nabla.\mathrm{TFR}_t \\ d\nabla.\mathrm{TFR}_t + \nabla.\mathrm{MACB}_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & 0 \\ d\phi_{11}+\phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} \nabla.\mathrm{TFR}_{t-1} \\ \nabla.\mathrm{MACB}_{t-1} \end{bmatrix} + \begin{bmatrix} \alpha_{1t} \\ d\alpha_{1t}+\alpha_{2t} \end{bmatrix}$$
$$= \begin{bmatrix} \phi_{11}\nabla.\mathrm{TFR}_{t-1} \\ (d\phi_{11}+\phi_{21})\nabla.\mathrm{TFR}_{t-1} + \phi_{22}\nabla.\mathrm{MACB}_{t-1} \end{bmatrix} + \begin{bmatrix} \alpha_{1t} \\ e_t \end{bmatrix},$$

where $e_t = d\alpha_{1t}+\alpha_{2t} = -\sigma_{12}\alpha_{1t}/\sigma_{11}+\alpha_{2t}$. This yields the equations

$$\nabla.\mathrm{TFR}_t = \phi_{11}\nabla.\mathrm{TFR}_{t-1} + \alpha_1, \tag{A.6a}$$

$$\nabla.\mathrm{MACB}_t = -d\nabla.\mathrm{TFR}_t + (d\phi_{11}+\phi_{21})\nabla.\mathrm{TFR}_{t-1} + \phi_{22}\nabla.\mathrm{MACB}_{t-1} + e_t. \tag{A.6b}$$

Equation (A.6b) can be written

$$(1-\phi_{22}B)\nabla.\mathrm{MACB}_t = -d\left[1-(\phi_{11}-d^{-1}\phi_{21})B\right]\nabla.\mathrm{TFR}_{t-1} + e_t$$
$$= d\left[1-\phi_{22}B + (\phi_{22}-\phi_{11}-d^{-1}\phi_{21})B\right]\nabla.\mathrm{TFR}_{t-1} + e_t. \tag{A.7}$$

So

$$\nabla.\mathrm{MACB}_t = -d\left[1+\left(\frac{\eta B}{1-\phi_{22}B}\right)\right]\nabla.\mathrm{TFR}_{t-1} + \left(\frac{1}{1-\phi_{22}B}\right)e_t,$$

which is equation (16) in the paper.

Appendix 2

Table A.1
Annual total fertility rate (TFR) and mean age of childbearing (MACB), white U.S. women, 1921–1980. [a]

Row	Year	TFR	MACB
1	1921	3.2816	28.59
2	1922	3.0716	28.59
3	1923	3.0625	28.56
4	1924	3.0693	28.49
5	1925	2.9493	28.46
6	1926	2.8388	28.39
7	1927	2.7830	28.35
8	1928	2.6324	28.31
9	1929	2.5058	28.23
10	1930	2.5055	28.17

Table A.1 (continued)

Row	Year	TFR	MACB
11	1931	2.3687	28.17
12	1932	2.2707	28.17
13	1933	2.1261	28.18
14	1934	2.1814	28.05
15	1935	2.1408	27.93
16	1936	2.1009	27.81
17	1937	2.1211	27.67
18	1938	2.1747	27.59
19	1939	2.1200	27.59
20	1940	2.1770	27.50
21	1941	2.2814	27.35

Table A.1 (continued)

Row	Year	TFR	MACB
22	1942	2.5255	27.22
23	1943	2.6113	27.43
24	1944	2.4517	27.72
25	1945	2.3752	28.02
26	1946	2.8431	27.51
27	1947	3.1674	27.04
28	1948	2.9682	26.91
29	1949	2.9643	26.86
30	1950	2.9454	26.89
31	1951	3.1199	26.76
32	1952	3.2126	26.81
33	1953	3.2690	26.75
34	1954	3.3720	26.72
35	1955	3.4045	26.68
36	1956	3.5043	27.57
37	1957	3.5823	26.52
38	1958	3.5323	26.49
39	1959	3.5367	26.49
40	1960	3.5102	26.50
41	1961	3.4637	26.53
42	1962	3.3201	26.52
43	1963	3.1942	26.56
44	1964	3.0613	26.63
45	1965	2.7644	26.66
46	1966	2.5627	26.54
47	1967	2.4250	26.46
48	1968	2.3411	26.38
49	1969	2.3365	26.31
50	1970	2.3378	26.21
51	1971	2.1436	26.16
52	1972	1.8943	26.10
53	1973	1.7675	26.01
54	1974	1.7396	25.95
55	1975	1.6847	25.96
56	1976	1.6611	26.03
57	1977	1.7080	26.06
58	1978	1.6707	26.11
59	1979	1.7237	26.14
60	1980	1.7543	26.13

[a] *Source*: U.S. Bureau of the Census.

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