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State space models for estimating and forecasting fertility

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Abstract

We introduce multivariate state space models for estimating and forecasting fertility rates that are dynamic alternatives to logistic representations for fixed time points. Strategies are provided for the Kalman filter and for quasi-Newton algorithm initialization, that assure the convergence of the iterative fitting process. The broad impact of the new methodology in practice is shown using data series from Spain, Sweden and Australia, and by comparing the results with a recent approach based on functional data analysis and also with official forecasts. Very satisfactory short- and medium-term forecasts are obtained. Besides this, the new modeling proposal provides practitioners with several suitable interpretative tools, and the application here is an interesting example of the usefulness of the state space representation in modelling real multivariate processes.

Keywords: State space model; Kalman filter; Fertility rates; Demographic forecast; Logistic model; Total fertility rate

1. Introduction

The term 'fertility' refers to the occurrence of births to an individual, a group or an entire population, and is determined by several biological, economic and social factors. The problem of estimating and forecasting fertility parameters is one that has a long tradition in demography. For example, the population projections derived from the fertility, mortality and migration components have always been of critical importance for policy-making because they set the basis for medium- and long-term planning in many fields. In addition, age-fertility rates are used as inputs in many of the most popular population projection models.

Several different approaches have commonly been used for projecting rates in demography. The simplest is to use the average rates from recent years. Another approach is to suppose that the rates in the population to be projected will converge over time with those found in another population, or those chosen by expert judgement.

On the other hand, approaches based on stochastic modeling have also been developed. These approaches have two advantages when compared with the simple extrapolation method: they use more historical information and they provide prediction intervals.

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However, projections from time series models, are often strongly affected by the structure of the models themselves, and by the changes in rates that occur during the base period. Therefore, many demographers support the use of mixed procedures, where external judgments and information on historical errors are included in the models. Interesting recent proposals along this line include those of Alho et al. (2006) and Alders, Keilman, and Cruijsen (2007). Nowadays, there is no unanimity about what the best procedure is, because there are important issues with all of the proposals and more statistical research is necessary. This paper is a contribution to this field.

The simplest way to make stochastic forecasts is to use univariate time series models to analyse separate age-specific rates. However, when taken together, the separate analyses may not yield a plausible age-pattern (the inconsistency problem). Therefore, it seems desirable to use modeling and forecasting methods that capture the smooth shape over age to produce consistent and accurate estimates.

Several different approaches have been developed for analysing fertility and mortality patterns using stochastic models. Here, we comment on the two most widely used: the curve fitting approach, which involves fitting parametric curves to the age-specific rates, and the principal components approach, which involves using a matrix decomposition to obtain a linear transformation of the data with a simplified structure. Among all of the available curve fitting models for fertility rates, the most familiar until recently were the Coale and Trussell (1974) model and the Gamma curve model (Thompson, Bell, Long, & Miller, 1989). Both have been used in many applications since their development, including those by Keilman and Pham (2000) and Scherbov and Vianen (2002), among others. Also, Schmertmann (2003) recently proposed a new model based on constrained quadratic splines.

The second approach uses dimensional reduction techniques to linearly transform the rates. One of the most popular models is the Lee–Carter model for forecasting mortality rates (see Lee & Carter, 1992). Several authors have since extended the Lee–Carter method (Booth, Hyndman, Tickle, & De Jong, 2006, and De Jong & Tickle, 2006, among others). Moreover, Hyndman and Ullah (2007) and Hyndman and Booth (2008) proposed a functional data (FD) approach that can also be considered as a successor of the Lee–Carter model, and which has been applied to the forecasting of both fertility and mortality rates. The main difference between the curve fitting and dimension reduction approaches is that the model for the rates is defined using known parametric functions depending on age for the former, but estimated functions depending on age for the latter. For a review of the different approaches, see Booth (2006).

The aim of this paper is to propose a new approach to forecasting fertility curves that uses state space (SS) modeling. We will use the model to provide short- and medium-term forecasts of age-specific fertility rates and other fertility indices. The SS model is based on the Logistic model (LO) proposed by Rueda and Alvarez (2008). It uses simple functional expressions, and the modeling and forecasting steps are done simultaneously. The parameters of the model can be interpreted as indicators of the level of fertility, and of the shape of the fertility curves. Finally, explanatory variables can also be incorporated into the model easily.

The procedure is validated using data series from different countries and periods, and the results are compared with those obtained using the FD approach and with official forecasts. In all cases studied, very satisfactory results are obtained for both the short- and medium-term forecasts.

The rest of the paper is organised as follows. The LO model is presented in Section 2. The SS model is defined in Section 3, where a strategy is designed for the Kalman filter and for quasi-Newton algorithm initialization in order to ensure convergence of the iterative fitting process. In Section 4, the SS model is applied to data series from Spain, Sweden and Australia; and finally, conclusions are drawn in Section 5.

2. Logistic model for fertility

2.1. Data and initial assumptions

We assume that birth counts and estimates of the population at risk are available from vital registration and population censuses or population registers. To simplify the exposition, single year age data are used, although the approach can also be used for other age groups. Let d be the total number of childbearing ages analysed. In the applications in Section 4, $d = 30$, since the lowest childbearing age is 16 for Spain and Sweden, according to the Eurostat database, and 15 for Australia, with the highest being 45 and 44, respectively. We use the following data: age-specific birth numbers for each calendar year, and age-specific population numbers at 30th June in each year. For each year $t = 1, \ldots, n$, and age $j = 1, \ldots, d$, we use the following definitions:

$b_j(t)$ = Births in calendar year t from females of age j

$w_j(t)$ = Female population of age j exposed to risk in year t (30th June)

$$m_j(t) = \frac{b_j(t)}{w_j(t)}$$

= observed fertility rate for females of age j in calendar year t.

Following the general consensus in actuarial modeling, we assume that births are generated by a Poisson process with intensity $\rho_j(t)$. Under this model, $m_j(t)$ are the MLE of $\rho_j(t)$. The models proposed in this paper give smoothed estimators for $\rho_j(t)$.

In the next subsection, the LO model is defined and some properties of the model are commented on.

2.2. Description and properties of LO models

The r-dimensional logistic model for analysing the fertility curve for a given moment in time, t, is given by:

$$[LO]_r \quad \log(\rho(t)/(1-\rho(t))) = A\beta_t, \tag{1}$$

where $\beta_t = (\beta_0(t), \beta_1(t), \ldots, \beta_{r-1}(t))'$ is the parameter vector and A is a known $d \times r$ design matrix with orthogonal columns defined as a function of the power of age:

$$A = (A_0, A_1, \ldots, A_{r-1}) \qquad A_k = (A_{1k}, \ldots, A_{dk})' \quad 0 \le k \le r-1.$$

The expressions of the first three columns are given below using basic statistics from the age distribution:

$$A_{j0} = 1, \qquad A_{j1} = (j - \overline{a}),$$
$$A_{j2} = (j - \overline{a})^2 - S_a^2,$$
$$A_{j3} = (j - \overline{a})^3 - \frac{K_a}{S_a^2}(j - \overline{a})$$
$$\overline{a} = \frac{1}{d}\sum_{j=1}^{d} j, \qquad S_a^2 = \frac{1}{d}\sum_{j=1}^{d}(j - \overline{a})^2,$$
$$K_a = \frac{1}{d}\sum_{j=1}^{d}(j - \overline{a})^4, \quad j = 1, \ldots, d.$$

The suitability of the fitted model for describing fertility curves is evaluated by Rueda and Alvarez (2008) using data from 226 countries. They compare the fit of the $[LO]_r$ model with that of the quadratic spline (QS) model of Schmertmann (2003) and the CT model form (Coale & Trussell, 1974). The logistic model $[LO]_4$ gives better results than the CT model and comparable results to the QS model in developed countries (the three models are each defined using four parameters). The incorporation of the power of age of higher orders ($[LO]_r, r > 4$) significantly improves the fit in many countries, but a model with fewer parameters suffices for some countries and years. We have decided to work with the model $[LO]_7$ as a standard for the dynamic analysis, for single-year age groups, in the following sections.

The parameters of the model can be interpreted as measures of the level (or period *quantum* $\beta_0(t)$) and shape (or *tempo* $\beta_i(t)$) of fertility curves, as shown by Rueda and Alvarez (2008). For a discussion of the concepts of *tempo* and *quantum* in demography, see Van Imhoff and Keilman (2000) and Sobotka (2003). Then, changes in the total fertility rate (TFR) values in a period can be interpreted as changes in the *quantum* and/or *tempo*, via the changes in the observed beta series. These interpretative properties are used in practice to describe past and future fertility rates using real data in Section 4.

3. Definition of state space models

In this section we present the SS representation for analysing series of rates. SS modelling provides a unified methodology for dealing with a wide range of problems in time series analysis, and allows a considerable amount of flexibility in the specification of the parametric structure for time series processes. In this approach, it is assumed that the development over time of the system under study is determined by an unobserved series of state vectors α_t, which are associated with a series of observations Y_t. The linear SS model can be defined using two equations. The first is known as the observation equation, and expresses the vector observation as a linear function of a state vector plus noise. The second equation, called the state equation, determines α_{t+1} in terms of α_t and a noise term. It is assumed that the initial state vector is uncorrelated with all of the noise terms, which means that the state vector has the Markov property. In a general SS representation, neither the vector observation nor the state vector is assumed to be stationary. A large number of well known time series models have an SS representation. To obtain the estimates of the state vector, the SS methodology uses the well-known Kalman filter. The Kalman filter is a recursive algorithm; that is, it is based on formulae by which we calculate the value at time $t+1$ from earlier values for $t, t-1, \ldots, 1$. These recursions are started up at the beginning of the series using a method called initialization. The Kalman filter provides a unified approach to prediction and estimation for all processes that can be given by an SS representation. When the models depend on unknown parameters, the estimation is provided by maximum likelihood. For the maximization of the log-likelihood, we use a quasi-Newton algorithm that starts with a trial value for the parameter vector. For readers who are not expert in state space modeling, we recommend the book by Commandeur and Koopman (2007) and chapter 8 of the book by Brockwell and Davis (2002). For a more detailed study, see Durbin and Koopman (2001).

To derive the final expression of the Gaussian SS model given below, we have assumed that the logits vector, $Y_t = \left(\log\left(\frac{m_1(t)}{1-m_1(t)}\right), \ldots, \log\left(\frac{m_d(t)}{1-m_d(t)}\right)\right)'$, is conditionally normally distributed with mean $A\beta_t$, and that β_t are independent ARIMA processes. The properties that support these assumptions are the logistic model defined in Eq. (1), the orthogonality of the design matrix A, the standard normal approximation to the Poisson distribution and the time component of the data.

The SS model for fertility is written in the usual form, with two equations. The observation equation has the structure of a linear regression model with coefficients that depend on time, and the state equation represents the development of the system over time, as follows:

$$\begin{array}{lll} \text{Observation equation} & Y_t = B\alpha_t + \varepsilon_t & \varepsilon_t \sim N_d(0, H) \\ \text{State equation} & \alpha_{t+1} = R\alpha_t + \eta_t & \eta_t \sim N_p(0, Q) \\ 1 \le t \le n & \alpha_1 \sim N(a_1, P_1). & \end{array} \tag{2}$$

To obtain the above SS representation we first calculate the logistic estimators fitting the model (1). These estimators are obtained using the standard software for logistic regressions that uses the observed fertility rates and female population figures as inputs, and provides estimators for the mutivariate beta process: $\beta_t = (\beta_0(t), \beta_1(t), \ldots, \beta_6(t))'$ as output. ARIMA processes are then fitted to each component of the output series. The analysis of data sets from different countries points to nonstationary ARIMA (0, 1, 0) or ARIMA (1, 1, 0) models being selected for each component; this is nothing new, as fertility series in the literature have traditionally been fitted using small order ARI processes. The p-dimensional state vector α_t, where $p \geq 7$, and the form of the state equation R, are derived from β_t, and the differences needed to define the ARIMA processes selected. The number of differences considered determines the exact value of p, and the matrix B is obtained from the relationship $A\beta_t = B\alpha_t$, $B = [A, 0]$, where 0 is $d \times (p-7)$ (see Durbin & Koopman, 2001, p. 46).

Moreover, in the SS representation (Eq. (2)), the error terms ε_t and η_t are assumed to be serially independent of each other at all time points, and H and Q are unknown diagonal matrices that do not depend on time and that measure the model errors (disturbance variances). Also, the initial state vector α_1 is assumed to be $N(a_1, P_1)$, independently of $\varepsilon_1, \ldots, \varepsilon_n$, and $\eta_1, \ldots, \eta_n$, where a_1 and P_1 are assumed to be known and must be provided in order to initialize the Kalman filter. a_1 is derived using the logistic estimators and P_1 is initialized as 0.

The vector of model parameters comprises the parameters in R and the $d + p$ disturbance variances in H and Q. Initial estimates of the parameters are needed in order to be able to use a quasi-Newton maximization algorithm to derive MLE. The initial values for the disturbance variances in Q are derived using the logistic estimators. For H, we use the mean value (over the observed times) of the asymptotic estimated Poisson variance matrix for the logistic transformation. The asymptotic Poisson variance of $\log(m(t)/(1 - m(t)))$ is obtained using the Taylor approximation $H_t^p = \text{diag}\,([w_j(t)\rho_j(t)(1-\rho_j(t))^2]^{-1})$, and the estimated variance is given by $\widehat{H}_t^p$: $\widehat{H}_t^p = \text{diag}\,([w_j(t)m_j(t)(1-m_j(t))^2]^{-1})$. Finally, the autoregressive parameters which define the matrix R are initialized using nonnegative values (we use 0.5).

We illustrate the way in which the matrices B and R are derived, and also show how we obtained the initial value a_1, using the model fitted to the Swedish data in Section 4. In this case, the ARIMA processes for the initial series of logistic estimators are: ARIMA (1,1,0) for $\beta_i(t), i \le 3$, and ARIMA (0, 1, 0) (random walk without drift) for $\beta_i(t), i \ge 4$. Then,

$$\alpha_t = (\beta_0(t), \ldots, \beta_6(t),\ \nabla\beta_0(t+1),\ \nabla\beta_1(t+1),\ \nabla\beta_2(t+1),\ \nabla\beta_3(t+1))',$$

where

$$\nabla\beta_i(t+1) = \beta_i(t+1) - \beta_i(t), \quad i = 0, 1, 2, 3.$$

The initialization is given by

$$\alpha_1 = a_1 = \big(\widehat{\beta}_0(1), \ldots, \widehat{\beta}_6(1), \nabla\widehat{\beta}_0(2),\ \nabla\widehat{\beta}_1(2),\ \nabla\widehat{\beta}_2(2),\ \nabla\widehat{\beta}_3(2)\big)',$$

where $\widehat{\beta}_i(1)$ and $\widehat{\beta}_i(2)$ are the logistic estimators given by Eq. (1) for the first two years. Therefore, matrix R is 11×11, and depends on 4 parameters which are the autoregressive coefficients of the beta process models for $\beta_i(t), i = 0, 1, 2, 3$, as follows:

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \phi_3 \end{pmatrix}.$$

B is a 30×11 matrix, where the first 7 columns equal those in matrix A and the last 4 columns have values equal to zero. Q is diagonal 11×11 with 7 parameters:

$$Q = \text{diag}\,(0, 0, 0, 0, \sigma^2_{\eta 5}, \sigma^2_{\eta 6}, \ldots, \sigma^2_{\eta 11}).$$

Alternatively, a model with the same structure as model (2), where $\widehat{H}_t^p = \text{diag}\,([w_j(t)m_j(t)(1-m_j(t))^2]^{-1})$, can be fitted to the data. The results are similar to those obtained using H. To fit the state space model, we use the software package SsfPack in the Ox computing environment (Koopman, Shephard, & Dornik, 1998). The inputs required to start the program are: the observed rates, the exact forms of matrices B and R, the initial values for the parameter vector, and the distribution of the initial state $\alpha_1 \sim N(a_1, P_1)$. The outputs are the parameter estimates and the h-step-ahead forecast, together with their estimated standard errors. We summarize the steps of the fitting process below.

1. Model definition: ARIMA processes for the betas are derived using the logistic estimators from Eq. (1). The selected processes determine the dimensionality of the multivariate alpha process and the exact form of matrices R and B.
2. Kalman filter initialization: an initial estimate of the vector a_1 is derived using the logistic estimators for $t = 1$ and $t = 2$, and P_1 is initialized as 0.
3. Estimation of parameters: MLE are derived using a quasi-Newton maximization algorithm. For the disturbance variances in Q, the values used to initialize the algorithm are derived using the logistic estimators. For H, the mean value of $\widehat{H}_t^p$ (over the observed times) is used. The autoregressive parameters in the matrix R are initialized as 0.5.
4. Forecasting: smoothed estimators for the beta process, forecasted values and prediction intervals are derived using the Kalman filter and smoother.

The iterative estimation process converged in all cases we tried, but this will not necessarily be the case if other initial values are used. In order to measure the prediction capacity, the last three (five) years for each country and period are reserved when the model is fitted, and the corresponding SSE for these three (five) years alone is computed.

Through the selection of specific ARIMA processes fitted to beta series, different future fertility scenarios can be assumed. This means that the SS approach allows demographers to draw the form of the fertility curve for the following years by selecting models that stagnate, accelerate, or decelerate the current trends in fertility levels (controlling $\beta_0(t)$) and other important characteristics (controlling $\beta_i(t), i \ge 1$). This can also be done in a similar way to Lee's (1993) proposal to constrain the ultimate level of the

TFR forecast. Therefore, the SS approach permits changes to be incorporated into the age pattern of fertility which are expected to be different in the future from in the past.

Since in this paper we focus on a comparison between the SS and FD approaches, we next introduce the main features of the FD approach, and the main differences between it and the SS approach. The FD approach uses a model structure similar to that of the SS approach:

$$g(m(j,t)) = \sum_{k=0}^{r-1} \beta_k(t)\phi_k(j) + e_{tj} + \epsilon_{tj},$$

where e_{tj} and ϵ_{tj} are the model and observational error terms respectively, g is a Box–Cox transformation, often the logarithm, and $\phi_k(j)$ is a set of orthonormal basis functions estimated from the data using functional data analysis in a manner similar to that of Ramsay and Silverman (1997, Chapter 6). The prediction intervals in the FD approach are obtained by adding the variances of the beta coefficient forecasts, the observational error and the model error (see Hyndman & Booth, 2008; and Hyndman & Ullah, 2007, for details). The main difference between the SS and FD approaches is that the base functions $\phi_k(j)$ are fixed in the former but data dependent in the latter. In our application at least, this appears to make the SS approach somewhat less dependent on the choice of the data period than the FD approach. Moreover, since in the SS approach the parametric series are interpreted as measures of changes in the levels and shapes of fertility curves, a comparison of the beta series' predictions from different base periods is a good strategy for selecting a reasonable base period for medium-term forecasts: long enough to provide good estimators but also short enough to reject any data that are not relevant for the near future. We illustrate these ideas in the next section by the analysis of real data series.

4. Examples

The cases of Spain, Australia and Sweden will be analyzed in this section. These countries have been selected for several reasons. The analysis of Spanish fertility rates is interesting because the drop in the birth rate in Spain has occurred in part because of the adoption of general European patterns, but with three essential differences: a much greater relative and absolute drop, a noticeably smaller final TFR, and a considerably lower fertility rate below the age of thirty (Cabré, 2003, and Fernández de la Mora y Varela, 2000). For the case of Sweden, demographic researchers have paid a great deal of attention to the study of fertility in the country, both because good quality data are available, and because Sweden was one of the first countries where fertility levels under the replacement level (2.1 in developed countries) were observed (see Andersson, 2004; Hoem, 2005; and Kohler & Ortega, 2002, and references therein. Finally, the data from Australia have been analysed extensively by an important group of demographers and statisticians from the country who have produced several of the most interesting recent papers in the field (Booth, 2006; Hyndman & Booth, 2008; Hyndman & Ullah, 2007). This research checked the proposed FD approach using Australian series. Thus, a fair comparison with results from the SS approach is both feasible and of special interest in this case.

The European data have been obtained from the Eurostat database[1] for 1971–2005 for Spain and 1955–2005 for Sweden. The Australian data for 1921–2003 come from the R package addb.[2] The demography R package[3] is used to implement the FD approach (Hyndman, 2006) and obtain summary statistics that are useful for comparing different aspects of the two approaches. Moreover, official TFR forecasts from Eurostat, the Spanish National Statistical Institute, and the Australian Bureau of Statistics are also used as a comparison with the predicted TFR obtained from the SS approach.

For Spain, as only short series are available, one set of forecasts up to 2020 is provided using the complete period 1971–2005. For Sweden, two sets of predictions up to 2020 were constructed: one based on the annual data series 1955–2005 and the other based on annual figures observed during the period 1975–2005. For Australia, the country with the longest series, we construct three series of predictions based on data from the periods 1921–2003, 1955–2003 and 1975–2003.

1 http://epp.eurostat.ec.europa.eu.

2 http://robjhyndman/software/addb.

3 http://robjhyndman/software/demography.

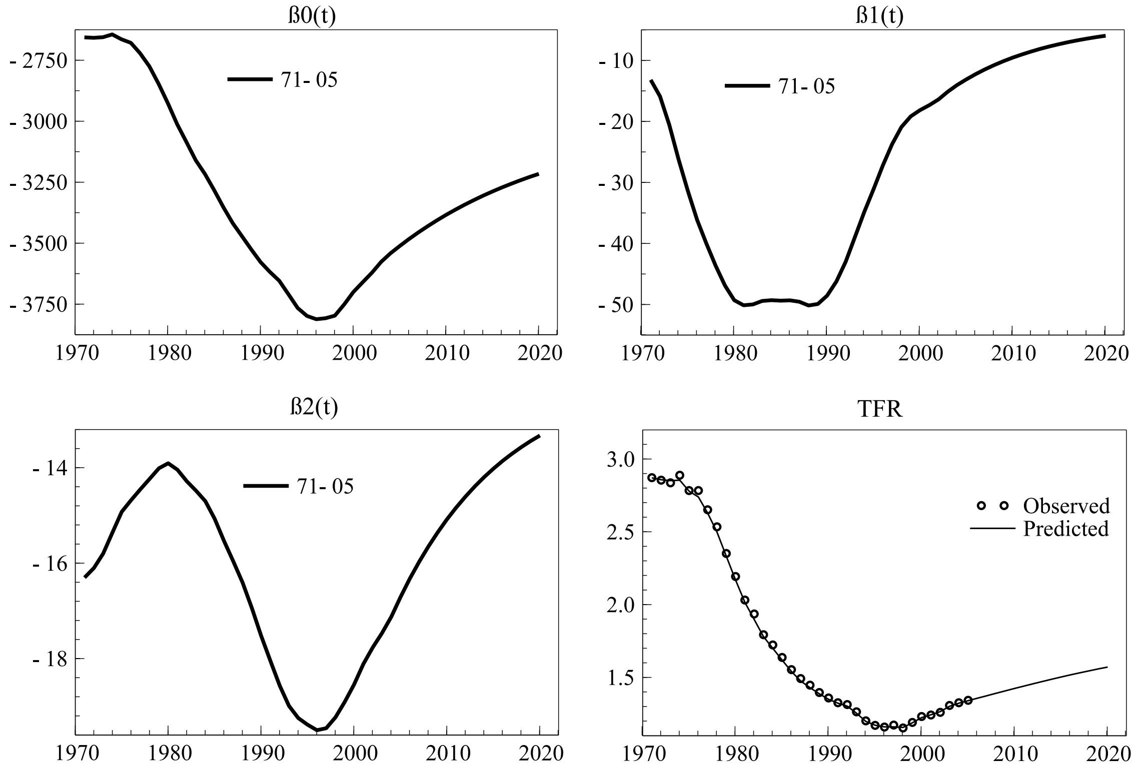

Fig. 1. Spain: SS beta series. Observed and predicted TFR values.

The general features of the past and future fertility rates for Spain, Sweden and Australia are analyzed in Section 4.1 based on the beta series. Short- and medium-term forecasts are also included, together with 80% prediction intervals. In Section 4.2, the official forecasts and forecasts from the FD approach are compared with the SS forecasts from different base-periods, using the TFR series. The prediction capacity is also calculated, reserving the last three (five) years in each country.

4.1. Past and future fertility using the state space approach

In the following presentation we will discuss the interpretative properties of the beta series estimates obtained by fitting model (2) to real data. The R matrix is derived using the longest period for each country. This matrix depends on several autoregressive parameters which are then estimated for each period. Alternatively, different ARIMA models could have been selected for each period. However, we have found that the predictions are quite similar with different ARIMA models, which all fit the data reasonably well. We reproduce here only the most significant estimated series $\beta_i(t)$, $i = 0, 1, 2$. We also use the TFR series to illustrate the comments.

In Spain, the TFR has dropped from values around 2.9 in 1975 to values around 1.34 in 2005, which is one of the lowest values in the world. Fig. 1 illustrates this. The beta series, which explain that the fertility change in Spain is due mainly to a *quantum* effect ($\beta_0(t)$), are also drawn in Fig. 1. Also, changes in the shape of the fertility curve have been observed over the last 15 years, as is illustrated by the $\beta_1(t)$ trend over this period. As a consequence, lower TFR values have been observed than would be expected without shape changes. For instance, Fig. 1 shows that while β_1 (1991) and β_1 (2002) are quite similar, TFR (1991) is larger than TFR (2002).

The opinion of Bijak (2004), among other experts, is that a slow recovery of the fertility level is expected.

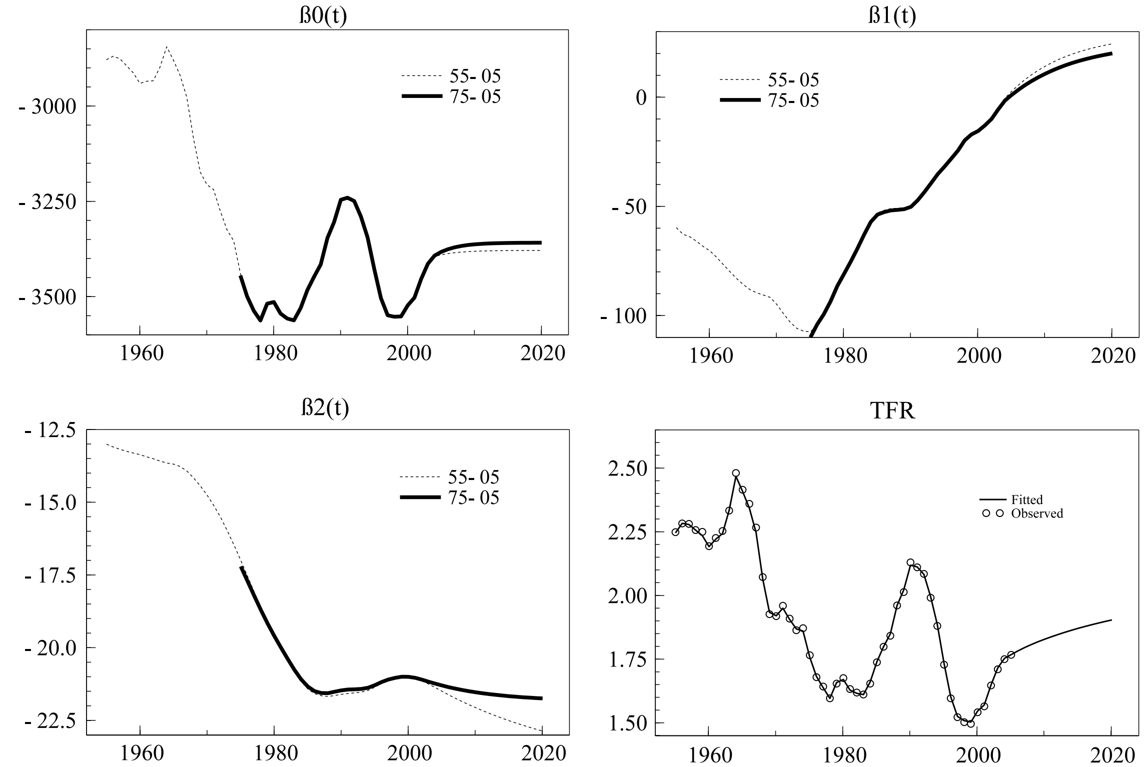

Fig. 2. Sweden: SS beta series. Observed and predicted TFR values.

This is the scenario that our forecast gives: the $\beta_i(t)$ trends show that the recovery predicted by the SS approach will be due mainly to the increasing values of $\beta_0(t)$. The TFR for 2020 is 1.57. A slight increase in other $\beta_i(t)$ values is also predicted in the future. The result is that higher rates are predicted in the early- to mid-twenties, relative to the rates in 2005. This is an interesting feature of the Spanish fertility curves that started in 2000, fits the model and is predicted to continue in the future (see Fig. 4).

Swedish fertility (Fig. 2) has behaved quite differently: after decreasing in the early seventies, the trend stopped in around 1977, and for several years, the TFR remained more or less constant. It then increased in the late 1980s, before decreasing again in the 1990s, reaching 1.51 in 1996. Since then, the TFR has increased to 1.75 in 2005. In Sweden, changes in fertility have been connected with important changes in the level ($\beta_0(t)$), but the mean age of childbearing has also been increasing since 1975, with a stable period in the late 1980s (the same pattern as $\beta_1(t)$). Important changes have also been observed in $\beta_2(t)$ between 1970 and the late 1980s. Again, as in Spain, over the period 1995–2002 lower TFR values have been observed than would be expected without shape changes.

We have used two base periods to obtain the forecast for Sweden: 1955–2005 and 1975–2005. The results are very similar, regardless of the base period used (see Fig. 2). The *quantum* component is predicted to be stable in the future. The longer base period is more informative in forecasting future trends, which we take into account when obtaining the fertility parameters for 2020. The TFR value is expected to increase very slowly as a consequence of changes in the *tempo* component. Bijak (2004) says that for Sweden one can expect the recent high TFR (1.85 in 2006) values to be quite good predictors of comparatively high fertility rates in the future, and other experts share this opinion.

Australian fertility rates have experienced important changes in both the level and shape components over the long period starting from 1921. The trends in

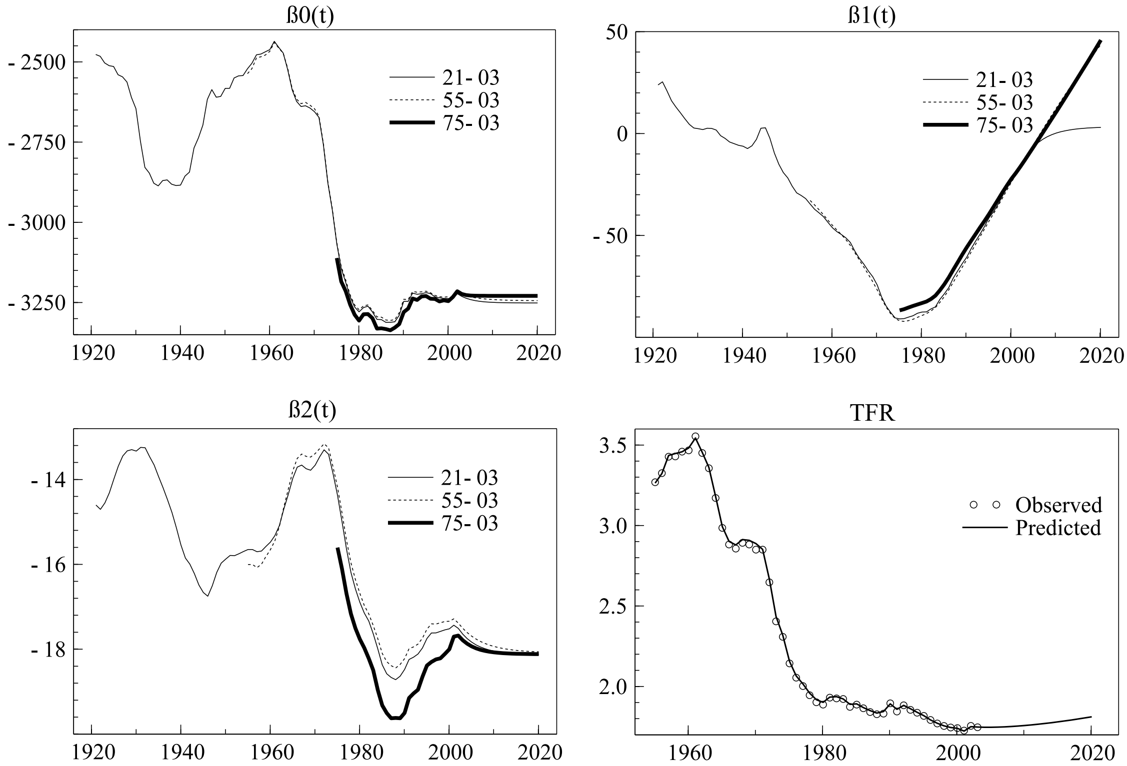

Fig. 3. Australia: SS beta series. Observed and predicted TFR values.

the beta series illustrate this fact (see Fig. 3). Having reached a TFR of 3.0 during the early 1920s, Australian fertility was relatively low during the 1930s, falling to 2.1 children per woman in 1934. In 1961, it peaked at 3.5 children per woman. Since then, fertility has declined to 1.73 in 2001 (followed by 1.76 in 2002 and 1.75 in 2003). However, the *quantum* component $\beta_0(t)$ decreased until the late eighties, and has since increased. The low TFR values observed since 1990 are again a consequence of changes in *tempo*. The reverse effect is observed in the period 1940–1960.

Australian fertility data are available for analysis for the period since 1921, and the SS model is fitted using three different base periods: 1921–2005, 1955–2005 and 1975–2005. As in Sweden, small differences are observed in the predictions, depending on the base period selected (Fig. 3). As the current childbearing behavior is very different from that of women in the 1930s, and the data from 1955 give us a sufficiently long series and good estimators, we have selected this period as the base period to get and interpret future fertility parameters. A behavior similar to that of Sweden is predicted for future Australian fertility. The TFR value will increase as a consequence of increases in the *tempo* component to 1.81 in 2020 (see Fig. 3). Besides this, it is known that the TFR has

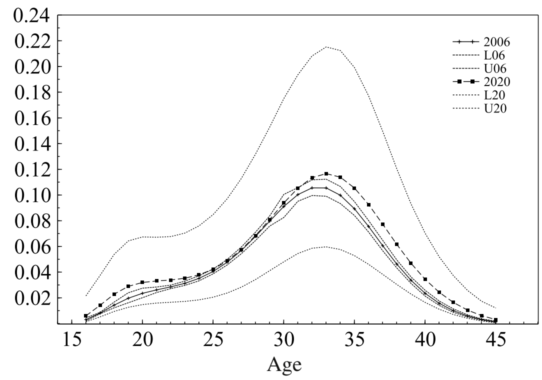

Fig. 4. Spain: Forecast fertility rates for 2006 and 2020, along with 80% prediction intervals.

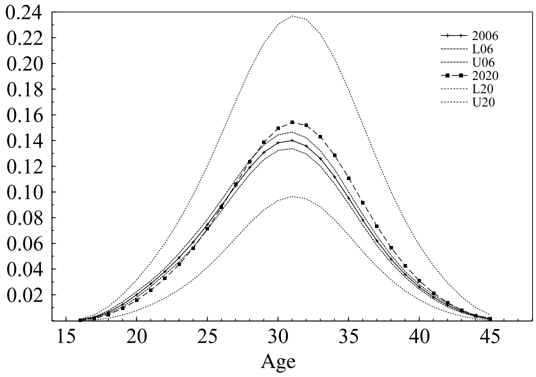

Fig. 5. Sweden: Forecast fertility rates for 2006 and 2020, along with 80% prediction intervals.

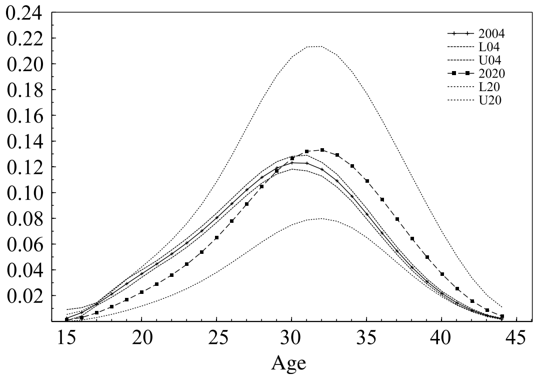

Fig. 6. Australia: Forecast fertility rates for 2004 and 2020, along with 80% prediction intervals.

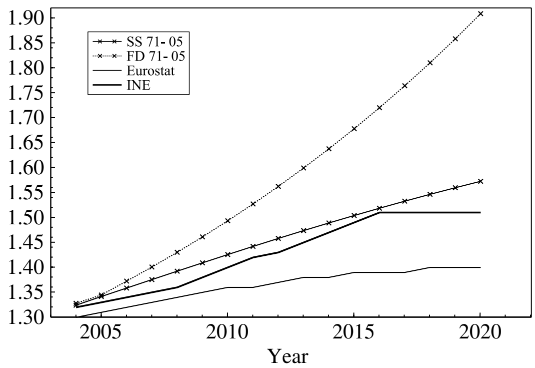

Fig. 7. Spain: SS, FD and official forecasts from Eurostat and INE (Spanish Statistical National Institute) of TFR for 2006–2020.

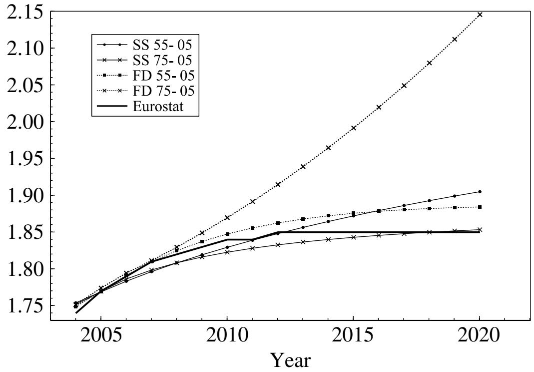

Fig. 8. Sweden: SS, FD and official forecasts from Eurostat of TFR for 2006–2020, using different base periods.

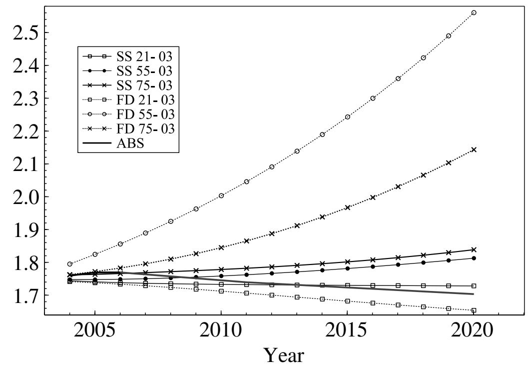

Fig. 9. Australia: SS, FD and official forecasts from the Australian Bureau of Statistics (ABS) of TFR for 2006–2020, using different base periods.

had an upward trend, reaching 1.81 babies per woman in 2006. Hugo (2007), together with other experts, says that the most reasonable interpretation of recent trends is that there is a degree of stability around 1.8 births per woman. The forecasted values from the SS approach are also interpreted in the same way.

Figs. 4–6 show the forecast fertility curves, along with 80% prediction intervals, for one-step-ahead forecasts and forecasts for 2020 for the three countries under study. The Spanish pattern is significantly different from the other two countries for ages 15 to 30. This agrees with the hypothesis of several experts that pronounced regional differences in European fertility are likely to prevail.

4.2. A comparative study

Figs. 7–9 show the forecasted TFR values obtained using the SS and FD approaches, as well as the official forecasts. The FD approach has been implemented

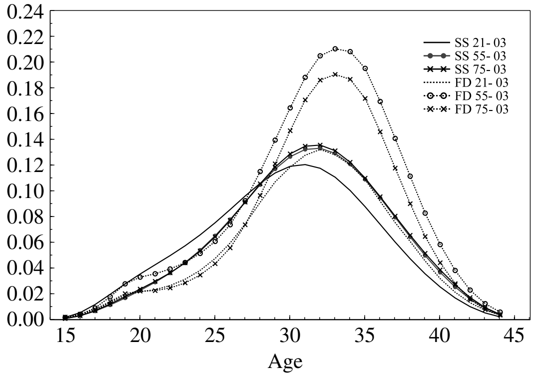

Fig. 10. Australia: SS and FD forecasts for 2020 using different base periods.

Table 1
Prediction capacity of the SS and FD approaches (SSE for the last three years).

Country and period	State space	Functional data
75–00 Australia	**0.000396**	0.001264
55–00 Australia	**0.000398**	0.003290
21–00 Australia	**0.000368**	0.000528
71–02 Spain	0.000220	**0.000166**
75–02 Sweden	0.000801	**0.000702**
55–02 Sweden	**0.000726**	0.002067

Table 2
Prediction capacity of the SS and FD approaches (SSE for the last five years).

Country and period	State space	Functional data
75–98 Australia	**0.00084921**	0.00186771
55–98 Australia	**0.00107156**	0.0179468
21–98 Australia	**0.00108486**	0.00127282
71–00 Spain	0.00625957	**0.00296848**
75–00 Sweden	0.00621306	**0.00611426**
55–00 Sweden	**0.00653331**	0.00921913

using $k = 6$ basis functions and state space exponential smoothing time series models. Different base periods up until 2005 for Spain and Sweden and 2003 for Australia have been considered. The official forecasts have been obtained using data up to 2004. To simplify the graphical representation, we have only drawn the series for the medium fertility assumptions. It is interesting to note that the recent TFR values in the three countries are higher than those observed in 2004, being 1.33 (2004), 1.34 (2005) and 1.37 (2006) in Spain, 1.75 (2004), 1.77 (2005) and 1.85 (2006) in Sweden, and 1.76 (2004), 1.77 (2005) and 1.81 (2006) in Australia. This suggests that the next official forecasts are likely to be higher.

For the three countries and selected base periods from Section 4.1, the SS approach provided higher TFR forecasts for 2020 than the official forecasts, but was still close to them. In addition, the forecasted TFR values for alternative base periods are fairly close to each other in all cases. On the other hand, the FD approach also gives values which are close to the official forecasts in the cases of Sweden and Australia for selected periods (1921–2003 in Australia and 1955–2005 in Sweden). However, the TFR forecasts for Spain are far from the results obtained by either the official forecasts or the SS approach, and the influence of the base period is also stronger, as the forecasted TFR values from different periods are all quite different. The choice of the base period also has consequences for the predicted age patterns. Let us consider the example of Australia. Fig. 10 shows the forecast fertility schedules for 2020 using data from the periods 21–03, 55–03 and 75–03. The differences in the FD forecast patterns are stronger than those for the SS approach. Moreover, in the last case, the differences in the predicted slope for $\beta_1(t)$ explain the differences in the forecast patterns. Those from the 75–03 and 55–03 data, where the predicted slope of $\beta_1(t)$ is increasing, result in a pattern that corresponds to an increasing trend in the mean age of fertility. However, using the 21–03 data, where the predicted slope of $\beta_1(t)$ is more or less constant, results in a pattern that corresponds to no trend in the mean age of fertility.

Finally, in Table 1 we have included the SSE values for 2003–2005 for Spain and Sweden and 2000–2003 for Australia, after the model is fitted, reserving the last three years; while Table 2 shows the results of reserving five years for each country and period. The prediction capacity for the short term across periods and countries is very high with both the SS and FD approaches, with the SS approach being the best predictor when longer periods are used.

5. Conclusions

From a statistical point of view, the SS approach has the advantage that the modeling and forecasting steps are done simultaneously and simple functional expressions are used; the model permits the analysis of parity-age-specific data or grouped data and the inclusion of covariables. From a demographic point of view, a useful feature is that the most important parameters have natural interpretations. In this paper, we have only begun to exploit these possibilities by explaining past and future fertility rates and selecting the base period.

We have focused in this paper on a comparison between the SS and FD approaches from a statistical point of view. For some periods the FD approach has led to implausible forecasts. In practice, such forecasts would not be used, and the forecasting approach itself would be modified in one of the many available ways. However, a potential advantage of the SS approach is that it seems to be less sensitive to the choice of the data period.

In carrying out the analysis, we have used the SsfPack software of Koopman et al. (1998). The software can be obtained freely from http://www.ssfpack.com. SsfPack is a suite of C routines for performing computations involving the statistical analysis of univariate and multivariate models in state space form. SsfPack is a module for Ox, which is an object-oriented statistical system. We have prepared programs for analysing and forecasting fertility using the logistic-SS models based on this software framework, and both the sample programs and our advisor are available from the authors by email request.

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