Stochastic Population Forecasts for the United States: Beyond High, Medium, and Low

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Conventional population projections use "high," "medium," and "low" scenarios to indicate uncertainty, but probability interpretations are rarely given, and in any event the resulting ranges for vital rates, births, deaths, age groups sizes, age ratios, and population size cannot possibly be probabilistically consistent with one another. This article presents and implements a new method for making stochastic population forecasts that provide consistent probability intervals. We blend mathematical demography and statistical time series methods to estimate stochastic models of fertility and mortality based on U.S. data back to 1900 and then use the theory of random-matrix products to forecast various demographic measures and their associated probability intervals to the year 2065. Our expected total population sizes agree quite closely with the Census medium projections, and our 95 percent probability intervals are close to the Census high and low scenarios. But Census intervals in 2065 for ages 65+ are nearly three times as broad as ours, and for 85+ are nearly twice as broad. In contrast, our intervals for the total dependency and youth dependency ratios are more than twice as broad as theirs, and our ratio for the elderly dependency ratio is 12 times as great as theirs. These items have major implications for policy, and these contrasting indications of uncertainty clearly show the limitations of the conventional scenario-based methods.

KEY WORDS: Demographic forecasting; Population projection; Stochastic demography.

Population forecasts are needed for many purposes, ranging from assessment of future environmental pressures to planning for manpower, education, or pension systems. Population forecasts often miss the mark, but that hardly distinguishes them from other kinds. Missing the mark is inevitable in an uncertain and changing world. More disturbing, perhaps, is that the high and low brackets provided with demographic forecasts often fail to contain the actual values only a few years into the forecast, and, as with the U.S. Bureau of the Census forecast of 1967, the actual births may fall outside the brackets for virtually the entire forecast period. Population forecasts by any name have an obligation to provide sound indications of their associated uncertainties. Progress has recently been made toward this goal in the special case of forecasts of the total population, as will be reviewed later. However, less has been accomplished along these lines for the age detail of forecasts. We believe that the methods described in this article make substantial progress in this important area.

There are numerous sources of error in demographic forecasts, but by far the most important is uncertainty about future rates. In the first part of this article we present a method for carrying out fully stochastic population projections when vital rates have been separately modeled as stochastic processes. In the second part, we model indices of fertility and mortality for the United States as time series leading to forecasts with confidence regions and use these models to generate forecasts of births, deaths, population age group sizes, and the ratios of age group sizes, such as the old age dependency ratio. We implement a stochastic population projection, with the exception of net migration, which is treated deterministically. Our forecasts are not intended to be merely illustrative of the new methods we have developed, but aspire to be taken seriously as detailed, realistic, and plausible.

Our work builds on two research traditions: (1) the use of statistical time series methods to forecast vital rates, and (2) the use of random-matrix products to describe population renewal with stochastic rates. The first draws on work by Lee (1974, 1981, 1992, 1993), Carter and Lee (1986, 1992), Lee and Carter (1992), Alho and Spencer (1985, 1990), Alho (1985, 1991, 1992), Saboia (1977), MacDonald (1979), and others. As for the second, early work was done by Sykes (1969) and Pollard (1968, 1973), but the analyses were highly stylized and did not adequately deal with time-varying rates. The demographic use of random-matrix products rests on Cohen's (1977) random ergodic theorem, and there is now an extensive literature on the subject (e.g., Cohen, Kesten, and Newman 1986; Tuljapurkar 1990). The approach that we take is based on the perturbation theory of random-matrix products as developed by Tuljapurkar (1982, 1990). We obtain a straightforward numerical algorithm to compute moments of interesting population variables over time. The algorithm works recursively, but we cannot describe the evolution of population moments by a closed system of recursion equations. An important feature of our new method is that it generates probabilistically consistent projection intervals for all items forecast.

We begin by considering sources of uncertainty in demographic forecasts and methods to deal with them. Next we develop our analytical procedure for computing stochastic forecasts, a procedure that may be used with many stochastic models of vital rates in addition to the ones we use later. Then we turn to specific time series models of vital rates and to the details of our forecast for the United States.

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1. APPROACHES TO UNCERTAINTY IN DEMOGRAPHIC FORECASTS

1.1 Sources of Error in Demographic Forecasts

1.1.1 Individual Level Randomness. Demographic rates are functions of probabilities of events for individuals. Even if the true vital rates were perfectly known in advance, forecasts for finite populations would be uncertain. Sykes (1969) and Pollard (1973), among others, have shown that population variation due to this kind of uncertainty is negligible, except in very small populations. We will therefore ignore it.

1.1.2 Data Errors. Errors in the raw data underlying forecasts lead to several kinds of forecast error. First, errors in estimation of the initial (jump-off) population lead to permanent errors in the forecasts. Though not negligible, such errors are not a serious problem in countries with well-developed statistical systems. Second, errors in measurement of base period vital rates lead to more serious errors in the forecast, for they determine the rate at which the population is believed to be growing, and thus their effects grow over time as a proportion of the levels forecast. Third, errors in measurement of the trends in vital rates may lead to the greatest forecast errors, because they affect the rate of change of the population growth rate. Errors in measurement of the level or trend in vital rates are still important for a number of populations but not for the industrial world, and probably not for a majority of the nations in the Third World (see Inoue and Yu 1979). We can assume that this will remain the case in the future. Therefore, our method does not deal with data errors.

1.1.3 Changing Vital Rates. Doubtless the most important source of error in population forecasts is uncertainty about future rates, because these rates are changing over time in ways that so far have been difficult to predict or even to explain after the fact. In the United States, the gyrations of fertility were, until recently, the main problem for forecasting. In recent decades, however, variations in the pace and age pattern of mortality decline have become increasingly important, as has migration.

1.2 Methods to Quantify Uncertainty

Forecasters have attempted to capture uncertainty from some or all of these sources in various ways.

1.2.1 The High-Medium-Low Approach. Virtually all population forecasts (except for a few in scholarly journals) treat uncertainty in the following conventional way. They construct high, medium, and low sets of assumptions about fertility, mortality, and migration and generate projections based on these, assuming that the rates hew exactly to the assumed trajectories in each year. McNown, Rogers, and Knudsen (1993) have made an advance by selecting projection scenarios based on stochastic models, but the essential logical structure of the forecasts is not thereby altered. In particular, the degrees of uncertainty associated with vital rates, numbers of events, age group sizes, age group ratios, and total population sizes are all exactly the same. But on reflection we can see that all these uncertainties should really be different (Lee 1981, 1992). For example, the uncertainty of fertility rates is compounded in births, because after about 15 years into a forecast, the population of reproductive-age women to which the rates are to be applied is also uncertain, because it depends on earlier fertility forecasts. Further, the uncertainty of births is not the uncertainty of age group sizes, because the size of an age group depends on a (weighted) sum of previous births, and we would expect some of the random ups and downs in the numbers of births to cancel.

Shrewd forecasters, of course, recognize the problem, but not much can be done about it within the conventional methodology. It is unfortunate when the actual number of births falls outside the high–low forecast range before the forecast has even been published, as it has at times in the past, or when the actual birth trajectory turns out to lie outside the forecast brackets over most of the time horizon. Births in 1990 exceeded the 1989 "high" forecast of the U.S. Bureau of the Census (1989) by 5%, and births from 1971 to 1990 fell below the lowest series of the census forecast (U.S. Bureau of the Census 1967) by 15%–20% in the last decade. Yet such discrepancies should not be surprising, because the high–low range on fertility and mortality must indicate not how high these might be in individual years, but rather how high and how low they might be on average over many years. If we take the high and low fertility forecasts to provide, say, a 90% confidence interval for fertility in any particular year, then we should take the corresponding high and low population totals to provide a much higher level of confidence—perhaps a 98% confidence interval, for example.

1.2.2 Stochastic Analyses.

1.2.2.1 Ex Post Analysis

Keyfitz (1981) and Stoto (1983) pioneered the analysis of ex post errors in United Nations forecasts for individual countries, as a basis for attaching confidence intervals to current forecasts. Lee (1991) supplemented their analyses by examining forecasts with horizons of 50–350 years. Keyfitz and Stoto concluded that the standard deviation of growth rate forecast errors was in the range of .3%–.5% per year; Lee found an average error of about 1% per year. This approach is very simple to apply and captures most sources of error. These advantages are balanced against a disadvantage—it provides intervals only for the population total, not for age groups or ratios, and not for the vital rates themselves (although extensions to these variables might be possible). It also does not reflect any changes either in forecasting methods or in the demographic environment.

1.2.2.2 Stochastic Simulation

If age detail is desired, one possibility is to do many stochastic simulations of population trajectories, drawing vital rates according to an assumed probability distribution at each step. Pflaumer (1988) has done this for the United States, concluding that after 100 years, the standard error of the forecast is 8% of the expected value. This has the advantages of being conceptually simple and of enabling the calculation of any desired confidence interval from the output. But to do this correctly, one should start by modeling the vital rates as stochastic processes and then estimate the parameters of the model on actual historical data, so that one gets the right distributions for the simulation. Pflaumer's distribution for U.S. fertility, uniform over a total fertility rate (TFR) of 1.6–2.3, appears excessively narrow. Furthermore, he assumed no autocorrelation in fertility, which leads to severe understatement of the uncertainty in the population forecast. In a later section we use stochastic simulation as a check on the results of our analytically derived confidence intervals.

1.2.2.3 Stochastic Models of the Growth Rate

Raymond Pearl, in the early twentieth century, generated population forecasts using fitted logistic curves and subsequently provided confidence intervals. More recently, Cohen (1986) in a wise and insightful paper, developed and applied a simple maximum likelihood estimator for the long-run growth rate of a population governed by a stochastic sequence of Leslie matrices and confidence bounds for it. These have a solid statistical basis and are relatively simple to use. But they have the disadvantage of applying only to the total population, and neither to its age components nor to the vital rates. Consequently, one does not know whether growth (or the absence of it) occurs because mortality is declining or fertility is rising or whether the population is gaining children or elderly members. Additionally, it cannot be efficient to ignore the initial age distribution of the population, even if one is only forecasting total population size. The future expected trajectory must be proportional to the initial "stable equivalent population," and the percentage bias from ignoring this will never wear off. An alternative procedure, which suffers from all the same drawbacks, is to use standard time series methods to forecast the logarithm of total population size, with associated confidence intervals. This gives point forecasts very similar to the Cohen method, but far wider confidence bands (Pflaumer 1992; in the latter paper, the alternate forecast based on population size rather than its logarithm also has wider bands).

1.2.2.4 Stochastic Leslie Matrix With Fitted Models of Vital Rates

The last approach, the one that we use in this article, is to model age-specific fertility, mortality, and (in principle) migration rates as stochastic processes and fit the models to historical data. Then, drawing on analysis of the stochastic renewal process (Tuljapurkar 1990), one can generate a fully stochastic forecast with associated confidence intervals. The confidence intervals must be calculated separately for each item of interest: population size, the vital rates, age group sizes, and age group ratios, for example. Although some progress has been made toward forecasts of this kind (Alho 1992; Alho and Spencer 1990; Sykes 1969), and Alho and Spencer have developed an approximate algorithm to carry out the calculations, we believe that this article represents the first full implementation of the approach, and makes a substantial advance. At the same time, some previous analyses consider sources of error that we ignore, such as data errors and specification error.

This approach has a number of advantages. It gives separate forecasts for each vital rate and for any desired population element or ratio. The forecasts and their confidence intervals are probabilistically consistent. Because the vital rates are explicitly forecast and have their own confidence intervals, their paths and intervals can be independently evaluated drawing on extraneous information. The disadvantage of this approach is its complexity. It is difficult to model the vital rates and to calculate the confidence intervals. Furthermore, though the approach can deal with both stochastic disturbances to the vital rates and error arising in estimation of coefficients, it does not deal with specification error and error arising from historical discontinuities.

2. COMPARISONS OF FORECASTS AND CONFIDENCE BOUNDS

Before we present our methods in detail, it should be helpful to compare point forecasts and confidence intervals derived from the sources and methods just discussed, including those for the method developed in this article. Table 1 displays these alternative forecasts and forecast intervals. The forecast marked LE is based on Cohen's (1986) logarithmic estimator and is presented over a 75-year horizon for comparative purposes, although we stress that Cohen recommends against using the method to forecast over time horizons longer than one generation, or about 25 years, so this application violates his prescriptions.

There are a number of points to note about the forecasts summarized in the table. First, they disagree strongly about the expected population size in 2065, ranging from 296 to

Table 1. Summary of Alternative Population Forecasts and 95 Percent Confidence Intervals for the United States, From c 1990 to c 2065

Forecaster	Population c 2065	% Half-width	% High	% Low
Soc Sec (1989)[a]	324	18	+20	−16
Soc Sec (1991)[a,d]	351	20	+24	−17
USCB (1989)[a]	296	42	+54	−30
USCB (1992)[c]	413	40	+45	−35
Pflaumer (1988)	301	16	+16	−16
Logarithmic Estimate[f]	620	12	+13	−11
Pflaumer (1992)[e]				
ARIMA log(Pop)	680	21	+23	−19
ARIMA Pop	443	39	+38	−39
Stoto (1983)[b]	n.a.	52	+57	−46
Lee–Tuljapurkar	398	44	+53	−35

[a] The Social Security forecasts attach no probabilistic interpretation to their high-low bands. The earlier USCB (Spencer, 1989) suggests tentatively that their band may cover roughly a 95% range for horizons of 50 years or more. The later USCB (Day 1992) suggests no probabilistic interpretation for horizons over 20 years. The other five forecasts explicitly claim 95% coverage.

[b] Stoto's method generates intervals, but no point forecasts. The figures shown reflect his "optimistic" result of .003 for the standard deviation of the rate forecast error for developed country populations. For the United States alone, he found .005, but believes this to be too pessimistic. The Lee–Tuljapurkar interval is expressed relative to the expected value of population. The Cohen model was fit on 1915 to 1990.

[c] The 1992 USCB forecasts extend only to 2050. The numbers given above were obtained by extrapolating the 2050 figures based on the ratio of population size in 2050 to that in 2035.

[d] The 1991 Social Security forecast is for 2060.

[e] The log(Population) model is (1, 1, 0), and the Population model is (2, 2, 0). Both are fit to population from 1900 to 1988.

[f] The Logarithm Estimate forecast uses Cohen's (1986) method over a 75 year horizon, although we emphasize that Cohen himself recommends against its use over horizons longer than 25 years. Following a procedure suggested by Cohen, we based the analysis on growth rates for the years 1915 to 1990, excluding earlier years for which the growth rate was higher. We used Method (2) of Cohen (1986); we subsequently learned of the "Correction" (Cohen 1988), but as this was said to make only a very slight difference, we did not change our original forecasts.

633 million. The two methods based on direct forecasts of the growth rate—LE and Pflaumer log(Population) model—lead to far higher numbers for 2065 than do the component forecasts. Second, they disagree strongly about the width of the confidence band. The band's *half-width* defined as half the difference between the high and low forecasts, divided by the middle forecast, provides a convenient measure. This ranges from a low of 12% for LE to a high of 52% for Stoto. Third, every confidence band excludes at least two point forecasts. The LE band covers no point forecast except its own and the Pflaumer log(Population) forecast. The 1992 Census and the Lee–Tuljapurkar confidence bands cover all point forecasts except LE and Pflaumer log(Population). It is surprising that the best forecasts of informed professionals should so often be viewed by colleagues as falling entirely outside the range of plausibility (as defined by 95% confidence intervals). Fourth, note that eight of the ten bands are asymmetric, with the upper bracket farther from the point forecast than the lower bracket.

3. A METHOD FOR STOCHASTIC PROJECTION

The method we derive here is designed to use a wide range of stochastic models for fertility and mortality (not just the ones we develop for the United States later on), and produce forecasts and forecast errors for population age segments.

3.1 The Role Of Random Matrix Products

For clarity, we first consider a single-sex (female only) population projection. Results for both sexes are presented later. In each period t, age-specific fertility and mortality rates are combined in the standard way (e.g., Keyfitz 1968) to produce a population projection (Leslie) matrix $\mathbf{X}_t$. This matrix acts on a vector $\mathbf{N}_{t-1}$, containing population numbers at time $t-1$ (of the single sex) arrayed by age, to generate the population vector at time t. Net immigration is included by adding a vector $\mathbf{I}_t$ of net numbers of immigrants by age. The basic recursion is

$$\mathbf{N}_t = \mathbf{X}_t\mathbf{N}_{t-1} + \mathbf{I}_t. \tag{1}$$

Let time $t = 0$ be the launch time for our forecast, so that the population vector $\mathbf{n}_0$ is known, along with age-specific fertility and mortality at $t = 0$. (To the extent possible, we use upper-case letters for random variables and lower-case letters for deterministic quantities. Bold type denotes a vector or matrix.) We assume that the forecaster has obtained stochastic models for the age specific fertility and mortality rates, and that these models, starting from initial conditions at the launch time, will generate future stochastic sequences of vital rates. (Examples of such models include the time series models given later in this article and the risk factor based diffusion model of mortality of Manton, Stallard, and Singer 1992.)

For every stochastic sequence of rates, iterating equation (1) yields

$$\mathbf{N}_t = \mathbf{Y}(t, 1)\mathbf{n}_0 + \mathbf{Y}(t, 2)\mathbf{I}_1 + \mathbf{Y}(t, 3)\mathbf{I}_2 + \cdots + \mathbf{I}_t. \tag{2}$$

Here we use time-ordered matrix products, defined as

$$\mathbf{Y}(t, s) = \mathbf{X}_t\mathbf{X}_{t-1}\ldots\mathbf{X}_s, \quad \text{for} \quad t > s. \tag{3}$$

Because equation (2) is an exact solution of the population recursion (1), a complete stochastic forecast is obtained if we can compute averages and other moments of the matrix products in the equation. This computation is difficult for three reasons. First, the population projection matrices $\mathbf{X}_t$ for $t > 0$ are random, in that their elements are constructed from the random rates. They are also serially correlated, because the age-specific fertility and mortality rates usually display serial autocorrelation over time. Second, the matrices do not commute, so the time ordering in equations (2) and (3) is critical. Third, we are often interested in forecasting functions of the vector in (1) (e.g., the old-age dependency ratio), and so we have to deal with nonlinear functions of an already complicated random vector.

3.2 An Expansion Method

The method we use to deal with equation (2) is a perturbation expansion (originally developed for stationary random matrices; see Tuljapurkar 1982, 1990). Related expansions have been used in studying physical systems (van Kampen 1981) and products of random evolutions (Pinsky 1991); all of these are extensions of the delta method in statistics (e.g., Cramer 1946). We expect that the models that generate forecasts of vital rates will contain a secular (deterministic) component and a random component. Accordingly, we decompose the random population projection matrix $\mathbf{X}_t$ at time t into an average and a random deviation:

$$\mathbf{X}_t = \mathbf{b}_t + \mathbf{Z}_t, \tag{4}$$

where $E\mathbf{Z}_t = 0$; $E(\cdot)$ indicates an expectation. Both the deterministic matrices $\mathbf{b}_t$ and the stochastic matrices $\mathbf{Z}_t$ may be explicitly time-dependent.

Using equation (4), we now expand the matrix products in equation (3). The expansion separates out terms that contain zero, one, two, and so on, of the deviations $\mathbf{Z}_t$. We call these terms zeroth order, linear, quadratic, and so on, in the stochastic deviations. For integers t, s, we can expand the random-matrix products as follows:

$$\begin{aligned}\mathbf{Y}(t, s) &= \prod_{i=s}^{t} \mathbf{X}_i \\ &= \prod_{i=s}^{t} (\mathbf{b}_i + \mathbf{Z}_i) \\ &= a(t, s) + \mathbf{S}_1(t, s) + \mathbf{S}_2(t, s) + \cdots.\end{aligned} \tag{5}$$

The first term has no $\mathbf{Z}_t$ factors and is a deterministic product defined by

$$\begin{aligned}\mathbf{a}(t, s) &= \mathbf{b}_t\mathbf{b}_{t-1}\ldots\mathbf{b}_s, && \text{for} \quad t > s \\ &= \mathbf{b}_t, && \text{for} \quad t = s \\ &= \mathbf{I}, && \text{for} \quad t < s.\end{aligned} \tag{6}$$

Here $\mathbf{I}$ is the identity matrix. The linear and quadratic terms are

$$\mathbf{S}_1(t, s) = \sum_{i=s}^{t} \mathbf{a}(t, i+1)\mathbf{Z}_i\mathbf{a}(i-1, s)$$

and

$$\mathbf{S}_2(t, s) = \sum_{i=s}^{t-1} \sum_{j=1}^{t-i} \mathbf{a}(t, i+j+1)\mathbf{Z}_{i+j}\mathbf{a}(i+j-1, i+1) \times \mathbf{Z}_i\mathbf{a}(i-1, s). \quad (7)$$

Higher-order terms can be written out similarly.

Armed with this expansion, we return to equation (2) and collect all terms of order zero, one, two, and so on, from the right of the equation, by using (5) every time a matrix product appears. This yields the basic expansion

$$\mathbf{N}_t = \mathbf{n}_t + \Delta_{1t} + \Delta_{2t} + \Delta_{3t} + \Delta_{4t} + \cdots, \quad (8)$$

where the subscripted terms Δ_{it} are of order $i = 1, 2, \ldots$. It is useful to write out explicitly the first few terms:

$$\mathbf{n}_t = \mathbf{a}(t, 1)\mathbf{n}_0 + \sum_{j=1}^{t} \mathbf{a}(t, j+1)\mathbf{I}_j,$$

$$\Delta_{1t} = \mathbf{S}_1(t, 1)\mathbf{n}_0 + \sum_{j=1}^{t-1} \mathbf{S}_1(t, j+1)\mathbf{I}_j,$$

and

$$\Delta_{2t} = \mathbf{S}_2(t, 1)\mathbf{n}_0 + \sum_{j=1}^{t-2} \mathbf{S}_2(t, j+1)\mathbf{I}_j. \quad (9)$$

The crucial thing to note is that the terms in the expansion (8) provide a decomposition of the moments of the random vector $\mathbf{N}_t$ in terms of moments of successively higher order of the random component $\mathbf{Z}_t$ of the projection matrix. For example, if we consider the average $E\mathbf{N}_t$, then $E\Delta_{1t}$ contains the contribution from the average of the random terms and is zero by definition (4); the term $E\Delta_{2t}$ contains all contributions from pairwise moments (variances, covariances, and autocovariances) of the random terms, and so forth.

3.3 Systematic Approximation

We now assume that the vital rates are modeled so as to capture the principal temporal trend in the deterministic component of the projection matrix, so that we may reasonably view the stochastic component as a small perturbation in each period t. This assumption is consistent with the modeling strategy usually adopted by demographers (and others). The expansion in equation (8) provides a systematic way of defining approximations to the full stochastic vector $\mathbf{N}_t$.

Let us define a quadratic approximation to the forecast vector,

$$\mathbf{N}_t^Q = \mathbf{n}_t + \Delta_{1t} + \Delta_{2t}. \quad (10)$$

This approximation by definition includes all two-time moments (covariances and autocovariances) of all vital rates over the forecast period. To include all fourth-order multitime moments, we would use a quartic approximation $\mathbf{N}_t^F$, which consists of the five terms written out explicitly in equation (8).

Any approximation that stops with moments of some fixed order will be inaccurate by an amount that depends on the relative size of the higher moments that are neglected, and also will become increasingly inaccurate as the time of the forecast increases (see Tuljapurkar 1992 for a fuller discussion of this point). Because the expansion (8) allows us to work consistently at any order of approximation, the decision to stop at a particular order will be based on a trade-off between the accuracy of the approximation and computational efficiency. For forecast horizons of 50–75 years, experience with U.S. forecasts (described later) leads us to recommend the quadratic approximation $\mathbf{N}_t^Q$.

The expansion method provides a useful perspective on some earlier approximations that have been proposed for stochastic population forecasts. Lee (1974), considering a nearly stationary population, used a linearization method with just the first two terms in the quadratic approximation equation (10). Davis (1988) proposed a closed-form recursive method involving the first two moments of the population vector; the limitations of such recursive methods were discussed by Tuljapurkar (1992).

Alho and Spencer (1991) and Alho (1992) have proposed a method for computing approximations to $\mathbf{N}_t$. Their method appears to treat survival in a manner consistent with our quadratic approximation, but only for the first 15 years or so of a forecast. They explicitly write components of the forecast population in terms of starting values and period survival rates and compute averages using only second-order moments of survival rates, which happens automatically in (10). But their method becomes problematic for forecasts longer than 15 years, because at that point they must include births to individuals born after the launch time. Alho and Spencer deal with this complexity by assuming that after time $t = 16$ they can omit "various covariance terms" (Alho 1992, p. 309). Specifically, they ignore serial covariances between mortality values separated by more than half a generation or so and ignore the serial correlation between fertility at time t and all previous population vectors $\mathbf{N}_{t-s}$, $1 \le s \le (t-1)$. This amounts to discarding a substantial number of terms that contribute to our quadratic approximation (10).

In our expansion, the matrix products automatically generate all multitime moments of the desired order. For example, consider the quadratic forecast approximation. Recalling equation (7), the linear Δ_{1t} is a sum of terms, each containing exactly one deviation matrix $\mathbf{Z}_i$ with $1 \le i \le t$. We can generate Δ_{1t} recursively, because the basic projection equation (1) implies that

$$\Delta_{1t} = \mathbf{Z}_t\mathbf{n}_{t-1} + \mathbf{b}_t\Delta_{1t-1}.$$

Recall that we begin with models that forecast the vital rates, so the deterministic matrices b_t can be computed for all t, as can moments of the stochastic matrix $\mathbf{Z}_t$. To automate the calculation of terms in Δ_{it}, for $i > 1$, we use the recursive structure

$$\Delta_{it} = \mathbf{Z}_t\Delta_{i-1t-1} + \mathbf{b}_t\Delta_{it-1}.$$

The nonlinear term in $\mathbf{N}_t^Q$ is Δ_{2t}, which involves all ordered pairwise products that contain two deviation matrices $\mathbf{Z}_i$ and $\mathbf{Z}_j$ with $i > j$. To compute averages, use the vital rate models to compute covariances involving one or more elements of different $\mathbf{Z}_i$, with $1 \le i \le t$. This is straightforward with time series models generating the vital rates and should be feasible for many other models.

Thus, to quadratic order, the vector $\mathbf{N}_t^Q$ can be completely decomposed into stochastic vectors whose entries contain known coefficients and zero, one, or two stochastic factors. A similar decomposition can be obtained for the quartic $\mathbf{N}_t^F$, though it involves vastly many more terms.

3.4 Moments Of The Quadratic Forecast

Making a forecast requires that we compute moments for components of the population vector and for nonlinear functions of it. To show how we do this, consider our recommended quadratic forecast $\mathbf{N}_t^Q$. Dropping the superscript, we compute the average population vector in quadratic order as

$$E\mathbf{N}_t = \mathbf{n}_t + E\Delta_{2t}.$$

Covariances between population age segments are computed using the Kronecker (tensor) product, indicated by the symbol $\otimes$. To quadratic order,

$$E\mathbf{N}_t \otimes \mathbf{N}_t = E\Delta_{1t} \otimes \Delta_{1t}.$$

To present formulas for other interesting quantities, we need additional notation. For a vector $\mathbf{v}$ with components $\{v_i\}$, let

$$|\mathbf{v}| = \sum_i v_i.$$

The scalar product of $\mathbf{v}$ with a vector $\mathbf{w}$ is written as $(\mathbf{v}, \mathbf{w})$.

Consider now the total population number $P_t = |\mathbf{N}_t|$. We demonstrate later that the distribution of P_t becomes increasingly skewed, and approximately lognormal, as the forecast period t increases. It thus is useful to forecast $\log P_t$, and to quadratic order (via Taylor expansion around $\mathbf{n}_t$), we have

$$E(\log P_t) = \log|\mathbf{n}_t| + \frac{|E\Delta_{2t}|}{|\mathbf{n}_t|} - \frac{|E\Delta_{1t} \otimes \Delta_{1t}|}{2|\mathbf{n}_t|^2},$$

with variance

$$\mathrm{var}(\log P_t) = \frac{|E\Delta_{1t} \otimes \Delta_{1t}|}{|\mathbf{n}_t|^2}.$$

Another class of objects to forecast includes proportions of the population in certain age segments, or ratios such as the old-age dependency ratio. These take the form

$$q(\mathbf{N}_t) = \frac{(\mathbf{u}, \mathbf{N}_t)}{(\mathbf{w}, \mathbf{N}_t)},$$

with weight vectors $\mathbf{u}$, $\mathbf{w}$ (each having at least one nonzero element). Define the vector

$$\mathbf{z}_t = \mathbf{u} - q(\mathbf{n}_t)\mathbf{w}.$$

To quadratic order, we have

$$Eq(\mathbf{N}_t) = q(\mathbf{n}_t) + \frac{E(\mathbf{z}, \Delta_{2t})}{(\mathbf{w}, \mathbf{n}_t)} - \frac{E(\mathbf{z} \otimes \mathbf{w}, \Delta_{1t} \otimes \Delta_{1t})}{(\mathbf{w}, \mathbf{n}_t)^2}$$

and the variance

$$\mathrm{var}\, q(\mathbf{N}_t) = \frac{E(\mathbf{z} \otimes \mathbf{z}, \Delta_{1t} \otimes \Delta_{1t})}{(\mathbf{w}, \mathbf{n}_t)^2}.$$

Similar expansions have been obtained (but are not reported) for the quartic approximation $\mathbf{N}_t^Q$.

3.5 Two Sex Projection

In practice we are interested in projecting both sexes. Demographers have various ways of dealing with reproduction when both sexes are considered, and there is as yet no accepted method for dealing with the problems of pair formation (see Pollak 1990). In keeping with virtually all forecasters, we use a linear model in which births are calculated by applying age-specific fertility rates to the female population and there is a fixed sex ratio at birth. This model does make the unrealistic assumption that the relative numbers of males to females does not affect realized fertility. We write the age vector at time t of females as $\mathbf{N}_t^F$ and of males as $\mathbf{N}_t^M$. Let the random projection matrix for females, containing both fertility and survival rates by age, be $\mathbf{X}_t^F$, and let the random projection matrix for males, containing only survival rates by age, be $\mathbf{X}_t^M$. Age-specific net immigration vectors for each sex are identified similarly. Then the two-sex population recursion is

$$\mathbf{N}_t^F = \mathbf{X}_t^F \mathbf{N}_{t-1}^F + \mathbf{I}_t^F$$

and

$$\mathbf{N}_t^M = \mathbf{X}_t^M \mathbf{N}_{t-1}^M + \mathbf{I}_t^M + \mathbf{B}_t.$$

Here $\mathbf{B}_t$ is a vector with only one entry, representing males born in that period. It is computed from the number of female births and the sex ratio at birth (taken as 1.05).

Applying the foregoing expansion method to equation (11) (the earlier one-sex expansion carries over to the female population; once that is written, an expansion of products of $\mathbf{X}_t^M$ yields the male population), we obtain a quadratic forecast approximation analogous to (10) written as

$$\mathbf{N}_t^F = \mathbf{n}_t^F + \Delta_{1t}^F + \Delta_{2t}^F$$

and

$$\mathbf{N}_t^M = \mathbf{n}_t^M + \Delta_{1t}^M + \Delta_{2t}^M. \quad (12)$$

Forecasts of functions of the two-sex population are computed as in the preceding subsection.

4. U.S. FORECASTS, STEP 1: MODELING THE AGE-TIME STRUCTURE OF FERTILITY AND MORTALITY

We seek a parsimonious model expressing the variation in fertility and mortality over age and time. Our strategy for both is the same, and was described by Lee and Carter (1992). We represent fertility at age x and time t, $F_{x,t}$, as approximately equal to a pure age-specific term, a_x, plus a term expressing the interaction of a period-specific effect, F_t, and an age schedule of fertility changes, b_x:

$$F_{x,t} = a_x + F_t b_x.$$

Schedules a_x and b_x are to be fixed over time, whereas F_t will be assumed to vary randomly over time. The level of the age schedule b_x is arbitrary, so it can be normalized to sum to unity; in this case the total fertility rate in year t is just $a + F_t$, where a is the sum of the a_x. For mortality, period variations are described by an index K_t in an identical model. We use the singular value decomposition (SVD) to find least squares estimates of a_x and b_x (using just the first elements from the decomposition and discarding the other components).

The model can be fit to the matrix of original rates or to transformations of the rates. The most common transforms in situations of this sort are the log of the rates or the logit of the rates. For fertility, an SVD analysis of the original series is most satisfactory (see Lee 1993). For mortality, we fit the logs of the rates, because the exponential trends in the rates have been very regular and because this transform guarantees that forecasts will not generate negative rates. Gomez (1990) independently arrived at the same model for mortality through exploratory analysis of the matrix of Norwegian mortality rates.

In effect, these fitted models describe single parameter families of model life tables (see Coale and Demeny 1983, for example) and model fertility schedules, tailor made to the U.S. experience. Our strategy is then to analyze and forecast the time series of the single time-varying parameters, K_t, F_t, for mortality and fertility. Once these have been forecast, the implied age schedules of fertility and mortality, along with their confidence intervals, can be readily derived. Empirical work confirms that these models capture the overwhelming portion of the temporal variation in the rates. For example, for mortality from 1933–1987, this model accounts for 97.5% of the total variance of the rates about their age-specific means (Lee and Carter 1992).

4.1 The Fitted Age–Time Structure Models

Using the procedures we have described, models were estimated for U.S. fertility and mortality, based on data from 1917–1987 for fertility and 1933–1987 for mortality. Then in a second stage, the time-varying parameters (F_t and K_t) were reestimated (using the age coefficients a_x and b_x estimated in the first stage) over the expanded ranges 1909–1990 and 1900–1989, such that the fitted model predicted exactly the observed number of births and deaths.

4.1.1 Mortality. Box–Jenkins methods were used to forecast the mortality index, K_t, for both sexes together, which is well modeled as a random walk with drift (see Lee and Carter 1992). Figure 1 (based on that paper) plots the second-stage estimates of the mortality index K_t from 1900–1989. Note that the mortality index has been trending downward at a fairly constant rate, with fluctuations about the trend. The near linearity of K_t corresponds to a near constancy of the exponential rate of decline of each age-specific death rate.

Forecasts of K_t are then used, together with the SVD-fitted coefficients a_x and b_x, to generate age-specific death rates for each five-year age group in each year. The implied point forecast for life expectancy (sexes combined) in 2065 is 86.1, with a confidence interval of 81–90 years. The fitted model for K_t is given in the Appendix.

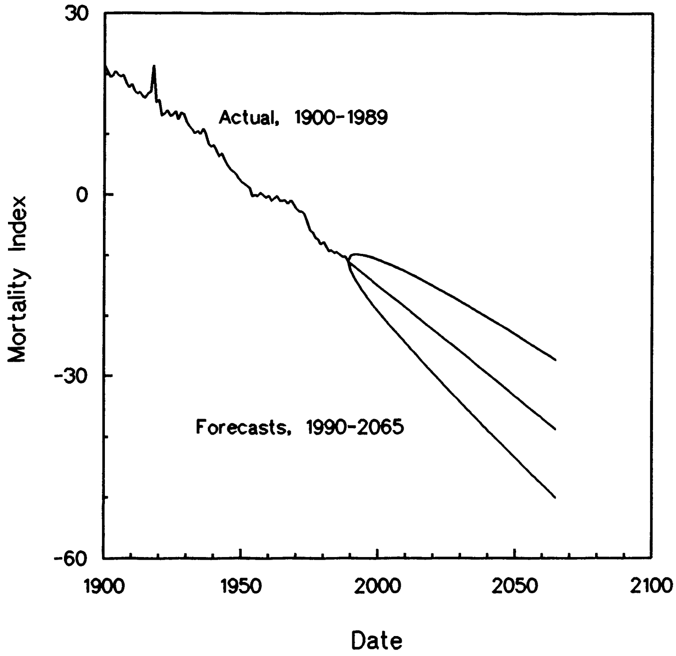

Figure 1. Base Period Values (1900–1989) and Forecast (1990–2065) for the Mortality Index K_t With 95% Probability Intervals. K_t is modeled as a random walk with drift, with a dummy for the influenza epidemic of 1918. The probability intervals do not reflect uncertainty about the estimated drift. This is based on the work of Lee and Carter (1992).

4.1.2 Fertility. Conventional time series models for fertility can lead to unrealistic forecasts (including negative fertilities), so we examined two alternative models that incorporate prior information (as discussed by Lee [1993]). In both alternatives we constrain the models so that they yield a prescribed ultimate average value of the total fertility rate. We used a mean of 2.1, chosen in part because it is close to the ultimate level of the 1992 Census projection and in part because many demographers view such an assumption as appropriate. Both alternative models use age-specific coefficients from an SVD decomposition of the historical age-specific fertility rates.

Our first alternative was a Box–Jenkins model of the time series F_t, but with a constrained mean. Our second alternative includes an additional constraint, that the total fertility rates lie between prespecified upper and lower bounds, $U = 4.0$ and $L = 0$. This constraint is achieved by making a logistic transform of F_t, denoted by G_t. The series G_t is then modeled and forecast with a constrained mean.

Residuals from the fitted linear mean-constrained models are highly nonnormal. Fertility spikes after each World War are one important source of nonnormality. Though residuals from the modeled G_t process are also significantly nonnormal, they are far less so than residuals from the linear model. Therefore, the models with constrained bounds appear superior.

But when we forecast fertility using these alternative models, it turns out that the standard error of the forecast based on the bound- and mean-constrained models is about 50% wider than for the mean-only–constrained models. This happens because the nonlinear transform converts the normal distribution on forecasted G_t into a more rectangular distribution on forecasts of F_t, so that the variance of F_t is higher. This substantially widens the probability bands on forecasts of population size and other quantities of interest.

More research on these matters is called for. It is possible that stochastic trend models, proposed by Lee (1993), may help resolve these difficulties. Here we will focus on the mean-only–constrained forecast with an ultimate average of 2.1 as our preferred forecast. (Note that small differences in the average level of fertility will have large effects over the long term. For example, an average completed fertility of 2.1 children per woman implies an intrinsic population growth rate .0017 per year higher than would 2.0 children per woman; in a stable population, after 75 years, the population would be 14% higher.)

Figure 2 plots the second-stage estimates of $F_t + A$, corresponding to within-sample fitted values of the total fertility rate. Though the fertility index fluctuates a great deal over the period since 1909, it is not visually clear that it has a trend. But the partial data available before 1909 do show a downward trend. Note also the striking hook at the end of the series, where the total fertility rate began a rapid rise in the last three years, peaking in 1990 at about 2.1.

The figure also shows the forecast and its confidence brackets, based on the linear model with a long-run mean of 2.1. Though the point forecasts are determined largely by the constraints, the fitted model provides valuable and essential information about the variance of the stochastic disturbance term and the autocovariation in forecast errors. It is important to note that these confidence bounds of necessity are far broader than bounds for an unconstrained conventional model that would bracket the long-run average possibilities (see Lee 1992). We give the actual fitted model in the Appendix.

4.2 Are the Fertility and Mortality Processes Interrelated?

It is important to consider the possibility that the disturbances in the fertility and mortality processes are associated with one another. Such an association is very strong in historical data for all countries (Galloway 1988; Lee 1981). It comes about because unobserved shocks to health affect both mortality and fecundity (or coital frequency), as may variations in weather and economic shocks. We examined the residuals of the preferred fertility model in relation to the residuals from the mortality model and found no relationship, probably because the short-run variation in mortality has become so small as epidemic mortality has declined. This simplifies the subsequent analysis considerably.

5. U.S. FORECASTS, STEP 2: THE QUADRATIC FORECAST APPROXIMATION

5.1 From Time Series to Rates

The time series models generate random sequences of the fertility index F_t and mortality index K_t starting from initial data at the launch point $t = 0$. These are linear ARMA models that can be solved (see the Appendix) and written in the form

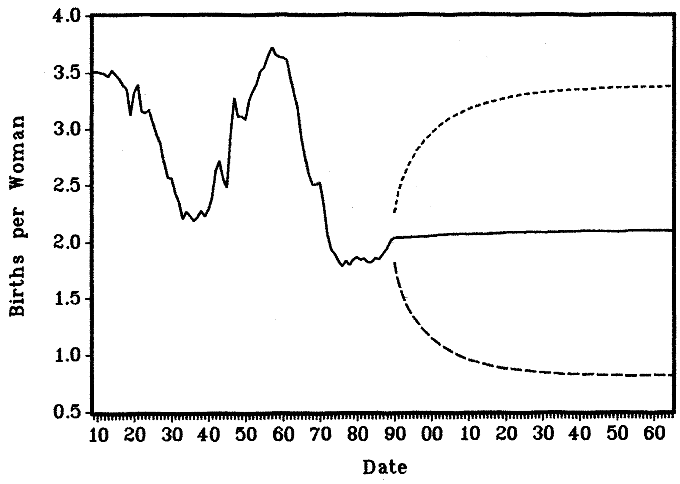

Figure 2. Base Period Values (1909–1989) and Forecast (1990–2065) for the Fertility Index, F_t, Plus a, With 95% Probability Intervals. F_t was modeled as a (1, 0, 1) process with the mean of F_t plus a constrained to equal 2.1. Upper and lower bounds were not imposed.

$$F_t = \bar{F}_t + \xi_t$$

and

$$K_t = \bar{K}_t + \eta_t.$$

Here we have separated the averages, $\bar{F}_t = E(F_t)$ and $\bar{K}_t = E(K_t)$, from the deviations ξ_t and η_t. The deviations are serially autocorrelated, but at every t and s the variables ξ_s and η_t are independent.

Next we use standard transformations (see the Appendix) along with the SVD equation (1), to generate age-specific survivorship l_{xt} (the fraction of a cohort that survives to age x after birth) and fertility F_{xt}, then combine these to obtain the elements of the population projection matrix. In this step we use linearization; for example, the survivorship l_{xt} is a function of K_t, say $l_{xt} = a_x(K_t)$. We use a Taylor expansion in every such case, to write

$$l_{xt} \simeq a_x(\bar{K}_t) + \frac{\partial a_x}{\partial K_t}(\bar{K}_t)\eta_t.$$

The fertility F_{xt} is already a linear function of F_t in our model. Linearization has little effect on the forecast rates over an 80-year forecast horizon; we compared rates forecast with and without linearization and found that the differences were very small.

The mortality model was fitted using data for both sexes combined. We modeled the sex differential in mortality by estimating a constant positive male differential (k_{M}) added to K_t to yield a forecast male index and a constant positive female differential (k_{F}) subtracted from K_t to get the forecast female mortality index. Carter and Lee (1992) studied alternative models.

5.2 From Rates to Projection Matrices

For females, the projection matrix $\mathbf{X}_t^{\mathrm{F}}$ has nonzero elements in the first row (representing fertility and its random deviations) and the subdiagonal (representing survival and its random deviations). For males, the projection matrix $\mathbf{X}_t^{\mathrm{M}}$ has nonzero elements only along the subdiagonal. Using the aforementioned linearization for survival rates, these matrices take the form

$$\mathbf{X}_t^{\mathrm{F}} = \mathbf{b}_t^{\mathrm{F}} + \eta_t \mathbf{c}_t^{\mathrm{F}} + \xi_t \mathbf{d}_t^{\mathrm{F}}$$

and

$$\mathbf{X}_t^{\mathrm{M}} = \mathbf{b}_t^{\mathrm{M}} + \eta_t \mathbf{c}_t^{\mathrm{M}}. \qquad (13)$$

The structure of these matrices is described in the Appendix. It is important to recognize that the top row of $\mathbf{X}_t^{\mathrm{F}}$ has both fertility and survival elements in it, so the forecast dynamics must account for an interaction between variations in fertility and mortality.

5.3 Immigration and Initial Vectors

Our general procedure accommodates arbitrary net immigration by age, sex, and time. For the U.S. forecasts, we chose the Census Bureau's 1989 Ultimate Middle Migration levels (U.S. Bureau of the Census 1989, p. 159). Initial age compositions for males and females were taken from the 1989 tabulation in the Census Bureau's 1989 projections (U.S. Bureau of the Census 1989, p. 40).

5.4 Recursive Forecast Calculations

The vital rate models that we use lead to a relatively simple structure for the random matrix expansions of Section 3. In particular, equation (12) for the quadratic forecast approximation implies the following recursive equations. Recalling that Δ_{it}^{F} are the random components of the quadratic approximation to $\mathbf{N}_t^{\mathrm{F}}$, we have

$$\Delta_{1t}^{\mathrm{F}} = \mathbf{b}_t^{\mathrm{F}} \Delta_{1t-1}^{\mathrm{F}} + \eta_t \mathbf{c}_t^{\mathrm{F}} \mathbf{n}_t^{\mathrm{F}} + \xi_t \mathbf{d}_t^{\mathrm{F}} \mathbf{n}_t^{\mathrm{F}}$$

and

$$\Delta_{2t}^{\mathrm{F}} = \mathbf{b}_t^{\mathrm{F}} \Delta_{2t-1}^{\mathrm{F}} + \eta_t \mathbf{c}_t^{\mathrm{F}} \Delta_{1t-1}^{\mathrm{F}} + \xi_t \mathbf{d}_t^{\mathrm{F}} \Delta_{1t-1}^{\mathrm{F}}.$$

The random components of the male vector obey the equations

$$\Delta_{1t}^{\mathrm{M}} = \mathbf{b}_t^{\mathrm{M}} \Delta_{1t-1}^{\mathrm{M}} + \eta_t \mathbf{c}_t^{\mathrm{M}} \mathbf{n}_t^{\mathrm{M}} + \rho \mathbf{e}_1 \Delta_{1t}^{\mathrm{F}}(1)$$

and

$$\Delta_{2t}^{\mathrm{M}} = \mathbf{b}_t^{\mathrm{M}} \Delta_{2t-1}^{\mathrm{M}} + \eta_t \mathbf{c}_t^{\mathrm{M}} \Delta_{1t-1}^{\mathrm{M}} + \rho \mathbf{e}_1 \Delta_{2t}^{\mathrm{F}}(1).$$

Here ρ is the ratio of males to females at birth (assumed fixed), the vector $\mathbf{e}_1$ has its first component equal to unity and all other components equal to zero, and $\Delta_{1t}^{\mathrm{F}}(1)$ indicates the first (i.e., birth) component of the vector Δ_{1t}^{F}. It should be apparent that Δ_{1t}^{F} can be treated as an array of vector coefficients of the random variables ξ_j and η_j, for $1 \le j < t$. Similarly, Δ_{2t}^{F} can be treated as an array of vector coefficients of all ordered pairs of the ξ_i and η_i.

To compute the quadratic approximation, we recursively generate and store all the vector coefficients. The time series models are used to compute moments such as, for example, $E[\xi_i \xi_j]$. At each forecast period t, we use the methods of Section 3 to calculate expectations and variances for total population, population age segments, and ratios of age segments.

5.5 Simulations

Along with our forecasts we present, for comparison and validation, results of full stochastic simulations of single-sex and two-sex projections. There were done by implementing the actual population recursions (11) using random number routines from the NAG (Numerical Algorithms Group) program library. The simulation results in each case are drawn from 750 independent projections over the full forecast period. The large number of runs was used to reduce sampling error (see Sec. 6.1.2 for details).

5.6 The Quartic Approximation to the Forecast

The foregoing discussion reveals the difficulty of making a quartic forecast—the number of terms in an object such as Δ_{4t}^{F} grows explosively as the forecast period t gets longer. In the spirit of enquiry we implemented a one-sex quartic forecast and compared it to the quadratic forecast and simulations. The quartic forecast does improve accuracy; 60 years into the forecast, the quartic approximation for the population variance is within 5% of a simulation and the quadratic approximation is in error by nearly 15%. But the improvement is mainly in the projection of confidence bounds and is small in relative terms. It is our judgment that the improvement does not justify the extraordinary increase in the computational effort required. To show what we mean, a two-sex quadratic forecast over 80 years runs on a Sun Sparc 2 in a few minutes (even with the rudimentary programming skill of the writers), much faster than a simulation. On the other hand, a quartic forecast for one sex took approximately 10 days of processor time. We believe that the quartic program can be made faster but only with considerable effort.

6. DISCUSSION OF U.S. FORECASTS

We now focus on our projections of the course of the U.S. population over the next 80 years. Results for the two-sex projection using the quadratic analysis and a 5-year projection interval are presented in Figures 3–12. The discussion interweaves two themes: (1) the accuracy of the analytically based projection method, as evaluated by comparison with simulations of the full model, and (2) a substantive discussion of the features of the forecast.

6.1 Total Population

6.1.1 Estimators. Figure 3 plots four quantities over the forecast period: the average population EP_t and another estimator of population size defined as $\exp(E \log P_t)$, each computed by the quadratic analysis and by simulation. It is obvious that compared to the simulations, the quadratic forecast does a very good job of predicting the mean population over the entire period.

Notwithstanding this result, we must recall an important feature of the population recursions—they are multiplicative. In consequence, the total population P_t will tend to a lognormal distribution over time. This tendency is rigorously established in the theory of stationary stochastic rates (e.g., Tuljapurkar and Orzack 1980) and we should see it here.

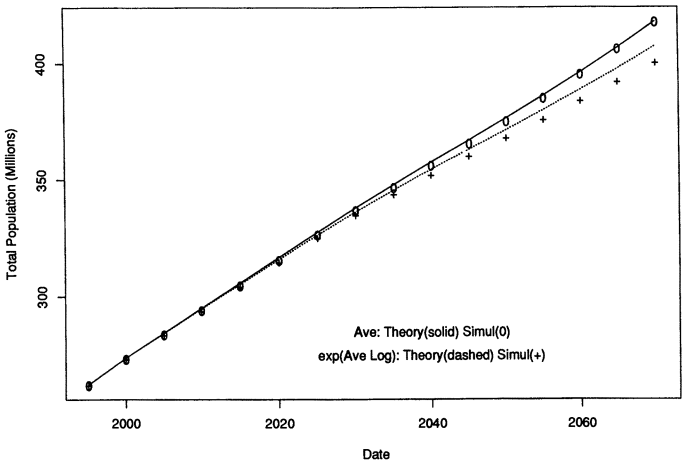

Figure 3. Total Population Forecasts for the United States by the Quadratic Method Compared With Simulation Results. Values are shown for average population EP_t and $\exp(E[\log P_t])$.

In Figure 4 we display histograms of P_t from the simulations at forecast times 20 years apart; the early development of strong skewness is obvious. A quantile plot against a standard normal (not shown) confirms that P_t is not normally distributed. On the other hand, histograms of $\log P_t$ at the same intervals (not shown) are much closer to normal, and this is confirmed by a quantile plot. Drawing on the theory of stochastic renewal (e.g., Tuljapurkar 1990), we conclude that the skewness of the distribution of P_t must be taken into account in determining confidence intervals for projections of total population.

It is relevant here to consider the impact of Jensen's inequality (Billingsley 1979), which guarantees that $\exp(E \log P_t) \leq E(P_t)$. Accordingly, Figure 3 includes plots of $\exp(E \log P_t)$. The difference between the simulation results for EP_t and $\exp(E \log P_t)$ is about 5% at the end of the interval. The quadratic analysis does not do as well at forecasting $\exp(E \log P_t)$ as it does with the mean, but in fact the error is about 1% at 2040 and rises to a little under 2% by 2065. Thus one may forecast EP_t directly by the quadratic method, or first forecast $E \log P_t$, and then exponentiate it to predict total population. We prefer the second method and use it to forecast population totals, although the difference between the two methods is small.

6.1.2 Accuracy of Quadratic Forecast. The performance of the quadratic forecasts of location appears rather good (an error of less than 2% at the end of an 80-year forecast). (Note that the sampling errors in the simulation, which is being used as the standard here, are small: thus the sample standard deviation of the estimates of $E(P_t)$ at forecast periods of 20, 50, and 80 years are 67.93, 1,966.86, and 4,457.57 [in thousands]. The sampling error at 80 years is slightly larger than 1% of the estimated mean.) The quadratic forecast does not do as well with the variance. We have compared the quadratic forecasts of variance (P_t) and variance $(\log P_t)$ against the simulations. The standard deviation predicted by the theory underestimates that obtained from the simulation in both cases. For the log, which is the worst of the two and also the one of interest to us, the theory predicts a standard deviation below the simulation value by nearly 2% in 2040 and by almost 15% in 2070. The error grows by about .45% per year starting about 50 years into the forecast. For the first 50 years, the error is small and grows at only .04% per year. We judge these to be acceptable errors.

A somewhat different issue is how well one can do with a deterministic forecast—that is, ignore the stochastic terms in the recursion (11) and just use the time-varying deterministic matrices. Recall the first equation in Section 3.4; the left side is the quadratic forecast, and the first term on the right side is the deterministic forecast. For U.S. rates, we find that the deterministic forecast turns out to accurately predict $\exp(E \log P_t)$. As a predictor of this quantity, the deterministic forecast has an error nearly 25% higher than the quadratic forecast over the first 70 years, but the latter has an error of less than 1.5% over this period. Note, however, that this amazing performance of the deterministic forecast reflects the particular circumstances of the United States; with different regimes of vital rates, we have found cases where the deterministic forecast has an error many times that of the quadratic forecast.

Overall, the quadratic point forecast is extremely accurate over 80 years; the quadratic uncertainty forecasts are very accurate for 50-year horizons and are moderately accurate between 50 and 80 years into the forecast.

6.1.3 Features Of The Forecast. In the following discussion, the reader may find it helpful to refer again to Table 1, for total population forecasts from various methods, and to Table 2, which lists quadratic forecasts and confidence ranges for various objects. A recurring feature in our forecasts is the effect of the initial age distribution, which is relatively "young" with the baby boom cohorts. As these people age, they contribute substantially, first and briefly to births and then to older age classes. In terms of total population, their aging is seen in the slowing down of the rate of change of total population that occurs around 2030 in Figure 3.

Figure 5 displays our estimator of total population, $\exp(E \log P_t)$, along with 95% prediction intervals. These intervals are based on a lognormal distribution of P_t; we first use the quadratic theory to compute the standard deviation σ_{Lt} of $\log P_t$, then define the prediction interval

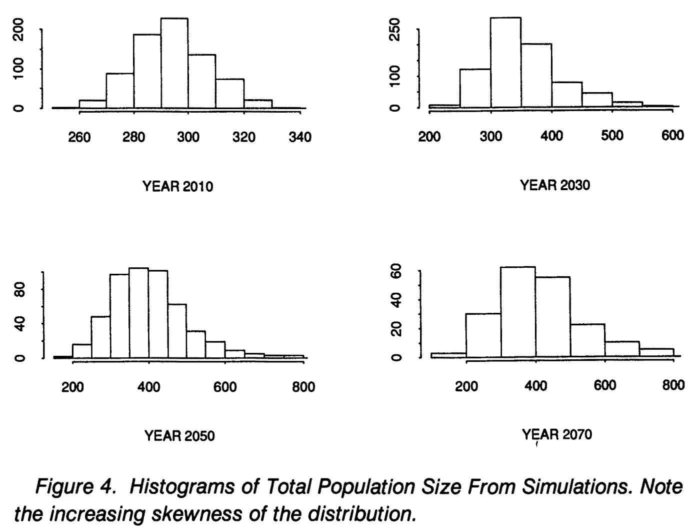

Figure 4. Histograms of Total Population Size From Simulations. Note the increasing skewness of the distribution.

as $\exp(E \log P_t \pm 2\sigma_{Lt})$. Note that the confidence bands are strikingly asymmetric; however, the distribution of P_t is rather skewed, so there is not much weight in the upper reaches of the fan. A characteristic feature of the confidence bands is that they grow fairly slowly till about 2030 and then start to widen much more rapidly. This accumulation of uncertainty is driven by variance in fertility, as we add random births to random births.

Also displayed in Figure 5 are the high, medium, and low estimates from the 1992 Census and 1992 Social Security forecasts. The trend in the forecasted "best" location is similar for all three projections through 2050. At that time, our forecast and prediction interval are remarkably similar to those obtained by the Census, but Social Security has much narrower prediction intervals (because they choose their high and low scenarios to indicate uncertainty in old-age dependency ratios). At 2070 we have only the Social Security forecast to compare; it is substantially different, with a medium predicted population that is only 87% of our point forecast. Our forecast is for an overall gain of about 55% in total population by the end of the period. The slowing down of growth that we predict at about 2030 is much steeper in the Social Security forecast, due to the different assumptions they make about future fertility. The Social Security projection bands are also much narrower at the end of the period, with a half-width of 20%, compared to a half-width of 44% for our forecast.

In 2040, our lower confidence limit is at 76.7% of our predicted population, but the upper confidence limit is at 130%. By the end of the period, the lower confidence limit has fallen to 63.5% of the predicted size, but the upper confidence limit has risen to 157%. The Census Bureau's projections show a similar asymmetry, whereas the Social Security projections are only slightly asymmetric by comparison.

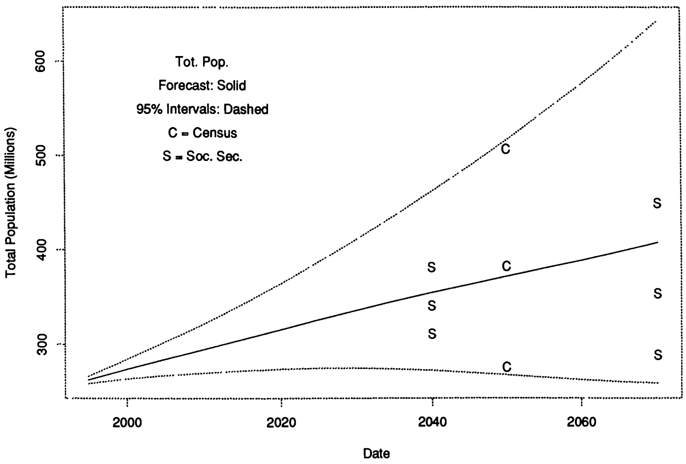

Figure 5. Total Population Forecasts for the United States by the Quadratic Method. The points marked C are Census Bureau forecasts; those marked S are Social Security forecasts.

Recall that the uncertainty in our fertility forecast ranged from .81 to 3.38 TFR. The range for the Census Bureau is far narrower, from 1.833 to 2.522 TFR. Because of the far greater range in our fertility forecast, one might expect the degrees of uncertainty in the forecasts of population size to differ far more than they do. Recall, however, that under the Census "high" assumption, fertility is at the upper bound every single year. In contrast, our upper bound on fertility merely says that in any individual year there is a 2.5% chance that fertility might be this high. To generate the population corresponding to our upper 95% confidence limit in 2065, the TFR would need to average 2.45 or so, rather than 3.38.

Table 2. Forecasts by Quadratic Method for the United States

	1990	*2000*	*2010*	*2020*	*2030*	*2040*	*2050*	*2060*	*2070*
POP pr	248254.000	273786.867	294763.542	315981.489	336343.189	354575.189	371540.843	388680.470	407476.824
POP lo	248254.000	263596.643	269839.215	273883.995	274815.809	272312.886	267599.843	262490.187	258711.879
POP hi	248254.000	284371.028	321990.063	364549.601	411645.682	461687.899	515854.556	575535.830	641784.841
$P > 64$ pr	30993.000	35215.579	39404.330	51779.452	68268.913	76245.290	80766.817	87793.245	94530.815
$P > 64$ lo	30993.000	34526.311	37743.366	48950.497	63868.828	70125.506	73156.684	78765.411	80898.677
$P > 64$ hi	30993.000	35918.607	41138.388	54771.899	72972.130	82899.143	89168.596	97855.819	110460.088
$P < 20$ pr	71357.000	77813.459	78652.873	78145.783	80624.854	82183.444	84133.732	86682.596	89106.348
$P < 20$ lo	71357.000	70901.830	62728.260	56475.499	54853.640	53183.594	52279.997	52081.656	51951.276
$P < 20$ hi	71357.000	85398.846	98620.215	108131.198	118503.843	126996.277	135395.662	144270.997	152834.385
$P > 84$ pr	3135.000	4755.872	6695.037	7753.425	9648.998	15233.935	21411.162	23001.684	25128.268
$P > 84$ lo	3135.000	4516.147	6005.419	6624.311	8022.026	12506.600	17237.394	18006.599	19439.688
$P > 84$ hi	3135.000	5008.321	7463.846	9074.996	11605.940	18556.023	26595.543	29382.421	32481.480
ODR pr	.212	.220	.226	.285	.377	.409	.419	.447	.467
ODR lo	.212	.214	.213	.259	.316	.305	.271	.255	.257
ODR hi	.212	.225	.238	.310	.439	.514	.566	.640	.677
YDR pr	.367	.487	.461	.451	.475	.475	.472	.479	.480
YDR lo	.367	.423	.314	.258	.258	.255	.254	.259	.260
YDR hi	.367	.551	.609	.643	.692	.695	.691	.700	.700
TDR pr	.579	.707	.687	.735	.852	.884	.891	.927	.947
TDR lo	.579	.642	.539	.550	.656	.687	.675	.674	.681
TDR hi	.579	.771	.835	.921	1.048	1.082	1.108	1.180	1.213
OODR pr	.021	.030	.039	.043	.054	.083	.113	.121	.129
OODR lo	.021	.028	.033	.034	.039	.054	.064	.057	.053
OODR hi	.021	.032	.044	.053	.070	.112	.163	.184	.204

NOTE: The notations pr, lo, and hi stand for predicted, lower 95% prediction interval, and upper 95% prediction interval. POP is total population in thousands; $P > 64$ means the number of individuals over 64. DR is a dependency ratio, with O for old, Y for young, OO for oldest-old, and T for total.

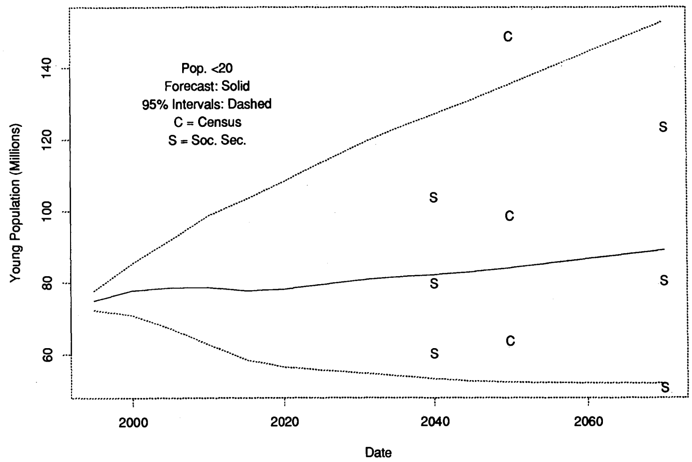

Figure 6. Forecasts by Quadratic Method of the Young (Under 20) Population. The points marked C are Census Bureau forecasts; those marked S are Social Security forecasts.

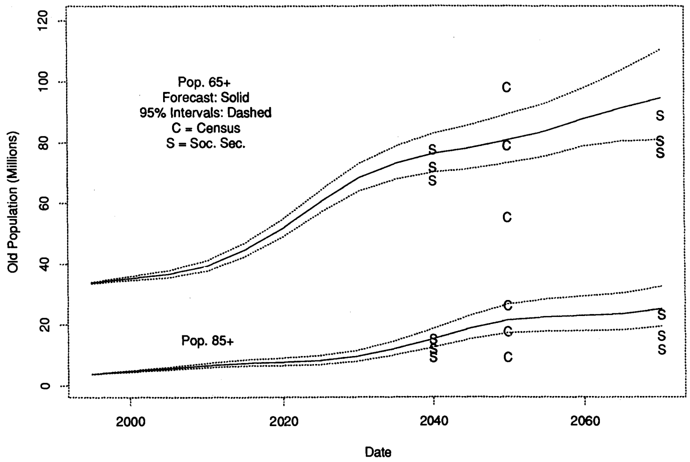

Figure 7. Forecasts by Quadratic Method of the Old (Over 65) and Oldest-Old (Over 85) Population. The points marked C are Census Bureau forecasts; those marked S are Social Security forecasts.

The difference is due to the cancellation of errors, our model anticipates that in some years fertility will be high and in other years it will be low (see Lee 1993).

6.2 Forecasts of Population Age Segments

In the subsequent discussion we focus on the quadratic forecasts. We have found that the accuracy of the quadratic method relative to simulations, for forecasts of all age segments and age ratios, is as good or better than that for total population.

Figures 6 and 7 display forecasts and confidence intervals of the size of the segment of the population that is in the age groups 0 to 19 (young), 65 and over (old), and 85 and over (oldest-old). Forecasts and high–low intervals are also displayed for the 1992 Census and 1991 Social Security forecasts. As mentioned earlier, the effect of the starting age structure is seen in our projections for all three segments. In the young population there is a slight rise followed by a dip around 2015; in the old population we see a rapid rise starting at about that time and lasting through about 2035; finally, the oldest-old projection is a delayed, attenuated version of the curve for the old.

The uncertainty is greatest for the young; this is the uncertainty of fertility at work. It is smallest for the old, whose numbers depend mainly on mortality variation, which is least uncertain for ages through about 75. Uncertainty increases again for the oldest-old, because there is increasing uncertainty about the rate of mortality advance after age 75. On this general pattern, all the forecasts agree.

The Census intervals for the under-20 population are as large for our stochastic forecast as they were for total population. But their medium forecast number of young is substantially higher (by about 20%) than our point forecast in 2050. The difference is likely due to the higher levels of immigration assumed by the Census in the 1992 forecast compared to the 1989 assumptions. They also have much wider prediction intervals for both the old and the oldest-old. Social Security has the smallest projection intervals, except for the oldest-old. Our stochastic projections attach very low probabilities to the Social Security low numbers for the elderly.

6.3 Dependency Ratios

Figures 8, 9, and 10 plot forecasts of four dependency ratios. The denominator in every case is the working age population between 20 and 64. The numerators are the populations of young (under 20), the old (over 64), all nonworking individuals (defined as those under 20 and over 65), and the oldest-old (over 85).

6.3.1 Old-Age Dependency Ratio. The upper triple of lines in Figure 8 plots the old-age dependency ratio. This ratio is of particular interest, because the payroll tax rate for Social Security and Medicare is roughly proportional to it. Our point forecast is for the ratio to rise in two steps to a level in 2069 about 2.5 times as high as it is today. The shape and level of the trajectory is very similar to that in the Social Security forecasts. The Census Bureau forecasts show an implausibly low level of uncertainty, because their "high" and "low" scenarios are chosen to indicate uncertainty in population size and hence combine low mortality with high fer-

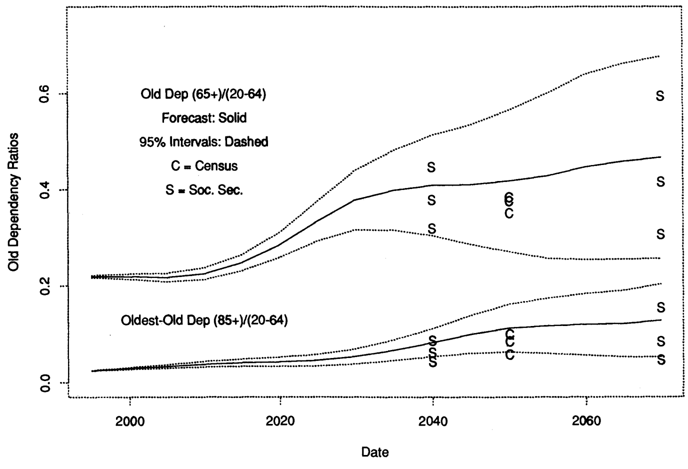

Figure 8. Forecasts by Quadratic Method of the Old and Oldest-Old Dependency Ratios. The points marked C are Census Bureau forecasts; those marked S are Social Security forecasts.

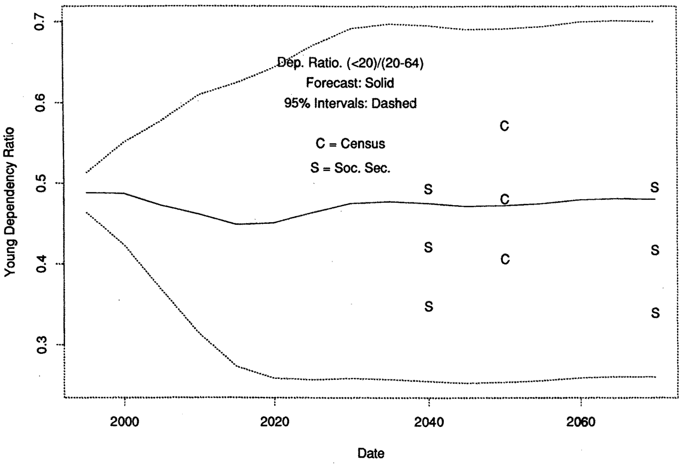

Figure 9. Forecasts by Quadratic Method of the Young Dependency Ratio. The points marked C are Census Bureau forecasts; those marked S are Social Security forecasts.

tility, unlike the Social Security forecasts. Until 2010, all the uncertainty is due to mortality, because no births have yet reached age 20 and the working ages. After 2010, and particularly after 2030, the uncertainty in the forecast increases very rapidly indeed, due to the great uncertainty about fertility, which drives the denominator of the ratio. Note, however, that the lower bound of the confidence bracket is about at the level of the dependency ratio of today, no lower; even with a return to high fertility and with slower mortality decline, we are unlikely to avoid population aging. The upper bound would represent roughly a *tripling* of the dependency ratio. The forecast calls attention to the possibility of an enormous increase in the cost of public sector pensions.

The Social Security forecasts are consistent with ours but again have smaller prediction intervals. The interesting feature of the Census forecasts is that they have the smallest prediction intervals for the old (and oldest-old) dependency ratios, in contrast to their very wide intervals for the numbers in those population segments. This contrast results from the particular fixed pattern of covariation built into their fixed trajectories of vital rates. On the other hand, the Social Security forecasts have the opposite feature (i.e., higher uncertainty in ratios than in numbers), because they choose combinations of rates that maximize the uncertainty of ratios.

6.3.2 Proportion of Oldest-Old. The rapid and sustained decline in mortality rates at the oldest ages has led to a historical high in the proportion of people over age 85. The lower triple of lines in Figure 8 displays our forecasts of the dependency ratio, expressing the oldest-old as a proportion of the working-age population. This ratio is significant as a measure of the potential burden of morbidity and long-term care expenses. The predicted ratio rises rather slowly, reaching a level of 12% by the end of the period. There is substantial uncertainty in the later years, and the upper confidence bound is as high as 20%. That would indeed be a new phase in the "graying" of the U.S. population.

6.3.3 Youth Dependency Ratio. The youth dependency ratio (Fig. 9) fluctuates about a constant level. Although fertility is forecast to rise to a TFR of 2.1, our current ratio is heavily influenced by the passing of the baby boom through the prime reproductive ages, a passage that leads to declining ratios in the early 21st century. The great uncertainty about the ratio closely mirrors the uncertainty in the fertility forecast.

6.3.4 Total Dependency Ratio. We have seen that there is great uncertainty associated with the forecast of both the elderly and the youth dependency ratios. Both the elderly and youth ratios might rise to .7. This raises the nightmare possibility that some generations might be caught in a squeeze with both many children and many elderly to support, and about one and one-half dependents for each working age adult to support. But could this happen, or is there sufficient negative covariation in the two ratios to rule out this possibility? Figure 10 is fairly reassuring on this count. We see, first, that the expected value of the ratio rises only moderately, and second, that the confidence bounds are tighter than would have been found by combining the separate intervals for the youth ratio and the old age ratio. Though the future is highly likely to bring a deterioration in the ratio from its current level, and though the ratio could rise by as much as 75% toward a level of 1.2, this is less than one might have feared. Both the Census Bureau and Social Security appear to underestimate substantially the uncertainty in this ratio, by 50% or more.

7. CONCLUSION

We have presented methods and results for population forecasts based on a stochastic Leslie matrix in which the vital rates are modeled as stochastic time series. Comparison of our population size forecasts and intervals with those prepared by others indicates a high degree of consistency with the latest Census forecasts and bounds and considerable agreement with the bounds derived from ex post analysis of Census forecasts by Stoto (1983) and by the Census Bureau itself (1989). Intervals differ greatly from those of Social Security, because they choose their scenarios to generate vari-

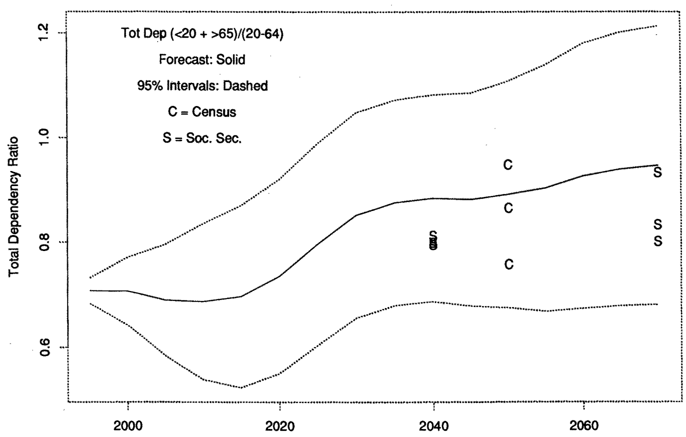

Figure 10. Forecast by Quadratic Method of the Total Dependency Ratio. The points marked C are Census Bureau forecasts; those marked S are Social Security forecasts.

ance in the old-age proportion, not the number. Both intervals and bounds differ strongly from extrapolations based on rates of growth (LE; see Sec. 2 and also Pflaumer 1992), perhaps because that procedure takes no account of the necessarily diminishing influence of mortality decline on growth rates. An additional drawback of these latter methods, of course, is that they cannot provide forecasts of population components.

An important pay-off to our elaborate forecasting strategy comes with the calculation of probabilistically consistent intervals for population age segments and ratios, which no other method can provide. Relative to our intervals, we find serious inconsistencies in those given for population ratios and components by the Census Bureau and Social Security. For example, though we agree quite closely with the Census intervals for total population and likewise for numbers under age 20, the Census interval for the age 65 and older population is 2.7 times as broad as ours and for the age 85 and older population is nearly twice as broad as ours. At the same time, our interval for the total dependency rate is 2.3 times as broad as theirs, for the youth dependency ratio 2.6 times as broad, and for the elderly dependency ratio fully 12 times as broad! These items have major implications for policy, and forecasting them is one of the most important goals of the entire forecasting enterprise. Another distinctive feature of our approach is the effect of stochastic variations over time; for example, our fertility model forecasts a wide range for the TFR, but our forecast population interval would be achieved with a much smaller (by a factor of four) range of fixed fertilities. These contrasting indications of uncertainty show clearly the limitations of the conventional scenario-based methods and the advantages of the approach that we have taken.

APPENDIX: DETAILS OF DEMOGRAPHIC MODELS

The mortality factor is modeled by

$$K_t = K_{t-1} - z + \varepsilon_t,$$

with $z = .365$ and $\varepsilon_t \sim N(0, \sigma)$, $\sigma = .651$. We rewrite this as

$$K_t = \bar{K}_t + \eta_t,$$

with $\bar{K}_t = \bar{K}_{t-1} - z$ and $\eta_t = \sum_{i=0}^{t} \varepsilon_i$.

The fertility factor is modeled by the mean-constrained autoregressive moving average (ARMA) process,

$$F_t = cF_{t-1} + F(1 - c) + u_t + du_{t-1},$$

with $c = .9676$ and $d = .47978$ and with the ultimate mean level set so that $(F + a) = 2.1$ and $u_t \sim N(0, \sigma_u)$, $\sigma_u = .110663$. We rewrite this as

$$F_t = \bar{F}_t + \xi_t$$

and

$$\xi_t = u_t + (c + d) \sum_{i=0}^{t} c^{t-i-1} u_i .$$

Central death rates, m_{xt} at time t for age x, are constructed from the model

$$\log m_{xt} = \alpha_x + \beta_x K_t .$$

These are mapped into survival rates s_{xt} by standard functions (e.g., Shryock, Siegel, and Stockwell 1976; Smith 1992). For ages 0–1 year,

$$s_1 = \frac{2 - m_1}{2 + m_1};$$

for 1–4 years,

$$s_2 = \exp[-4m_1 - (4)^3(.008)m_1^2];$$

and for all later age groups,

$$s_x = \exp[-5m_x - (5)^3(.008)m_x^2].$$

Survivorship (l_{xt}) and lifetable (L_{xt}) functions are constructed in the standard way. The projection matrix survival rates are given by ratios of the L_{xt}. At each step of these transformations, linearization is carried out (by Taylor expanding, each rate about its average value, as in the third equation of Sec. 5.1, and retaining terms linear in the deviations about the mean) to obtain the coefficients of the stochastic terms.

Fertility rates F_{xt} are combined with survival rates to produce entries in the first row of the female projection matrix (as described in Keyfitz 1968).

The projection matrices in equation (13) have the following structure. The matrix $\mathbf{b}_t^F$ has a standard Leslie matrix structure, with average fertility projection elements in the first row, average survival projection elements in the subdiagonal, and zeros elsewhere. The matrix $\mathbf{c}_t^F$ also has nonzero elements only in the first row and subdiagonal; the first row contains combinations of fertility and mortality rates, and the subdiagonal contains mortality terms, all emerging from the assembly of the projection matrix as coefficients of the stochastic mortality component η_t. The matrix $\mathbf{d}_t^F$ has nonzero elements only in the first row, and these emerge from the assembly process as coefficients of the stochastic fertility component ξ_t.

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