The threshold age of the lifetable entropy

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Abstract

BACKGROUND
Indicators of relative variation of lifespans are important because they capture the dimensionless shape of aging. They are markers of inequality at the population level and express the uncertainty at the time of death at the individual level. In particular, the lifetable entropy $\bar{H}$ represents the elasticity of life expectancy to a change in mortality and it has been used as an indicator of lifespan variation. However, it is unknown how this measure changes over time and whether a threshold age exists, as it does for other lifespan variation indicators.

RESULTS
The time derivative of $\bar{H}$ can be decomposed into changes in life disparity $e^{\dagger}$ and life expectancy at birth e_o. Likewise, changes over time in $\bar{H}$ are a weighted average of age-specific rates of mortality improvements. These weights reflect the sensitivity of $\bar{H}$ and show how mortality improvements can increase (or decrease) the relative inequality of lifespans. Further, we prove that in the assumption that mortality is reduced at all ages, $\bar{H}$, as well as $e^{\dagger}$, has a threshold age below which saving lives reduces entropy, whereas improvements above that age increase entropy.

CONTRIBUTION
We give a formal expression for changes over time of $\bar{H}$, and provide a formal proof of the existence of a unique threshold age that separates reductions and increases in lifespan variation as a result age-specific mortality improvements.

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1 Relationship

The lifetable entropy is a dimensionless indicator of the relative variation in the length of life compared to life expectancy at birth (Keyfitz 1968, 1977; Demetrius 1974, 1978). It is usually defined as

$$\bar{H}(t) = -\frac{\int_0^\infty \ell(a,t)\,\ln \ell(a,t)\,da}{\int_0^\infty \ell(a,t)\,da} = \int_0^\infty c(a,t)\,H(a,t)\,da = \frac{e^\dagger(t)}{e_o(t)}\ ,$$

where $e^\dagger(t) = -\int_0^\infty \ell(a,t)\,\ln \ell(a,t)\,da$ is the life disparity or number of life-years lost as a result of death (Vaupel and Canudas-Romo 2003), $e_o(t) = \int_0^\infty \ell(a,t)\,da$ is the life expectancy at birth at time t, $\ell(a,t)$ is the lifetable survival function, $c(a,t) = \ell(a,t)\,/\,\int_0^\infty \ell(x,t)\,dx$ is the population structure, and $H(a,t) = \int_0^a \mu(x,t)\,dx$ is the cumulative hazard to age a, where $\mu(x,t)$ is the force of mortality (hazard rate or risk of death) at age x at time t. Note that $\bar{H}(t)$ can be interpreted as an *average value* of $H(a,t)$ in the population at time t.

Goldman and Lord (1986) and Vaupel (1986) proved that

$$e^\dagger(t) = \int_0^\infty d(a,t)\,e(a,t)\,da\ ,$$

where $d(a,t)$ represents the distribution of deaths and $e(a,t) = \int_a^\infty \ell(x,t)\,dx\,/\,\ell(a,t)$ the remaining life expectancy at age a at time t. This provides an alternative expression for the lifetable entropy as

$$\bar{H}(t) = \frac{\int_0^\infty d(a,t)\,e(a,t)\,da}{\int_0^\infty \ell(a,t)\,da}\ .$$

Let $\dot{\bar{H}}$ denote the partial derivative of $\bar{H}$ with respect to time.[1] We define $\rho(x) =$

[1] In the following, a dot over a function will denote its partial derivative with respect to time t, but

$-\dot{\mu}(x) \,/\, \mu(x)$ as the age-specific rates of mortality improvements. Then, the relative derivative of $\bar{H}$ can be expressed as a weighted average of $\rho(x)$,

$$(1) \qquad \dot{\bar{H}} \,/\, \bar{H} = \int_0^\infty \rho(x)\, w(x)\, W(x)\, dx \;,$$

with weights

$$w(x) = \mu(x)\, \ell(x)\, e(x) \qquad \text{and} \qquad W(x) = \frac{1}{e^\dagger} \left(H(x) + \bar{H}(x) - 1 \right) - \frac{1}{e_o} \,.$$

Function $\bar{H}(x)$ is the lifetable entropy conditioned on surviving to age x, defined as

$$\bar{H}(x) = \frac{e^\dagger(x)}{e(x)} = \frac{\int_x^\infty d(a)\, e(a)\, da}{\int_x^\infty \ell(a)\, da} \,.$$

where $e^\dagger(x) = \int_x^\infty d(a)\, e(a)\, da \,/\, \ell(x)$ refers to life disparity above age x, and $e(x)$ is the remaining life expectancy at age x.

Note that the lifetable entropy $\bar{H}$ is a measure of relative lifespan variation. Thus, higher values represent more variation, whereas lower values denote less variation of lifespans. If mortality improvements over time occur at all ages, there exists a unique threshold age a^H that separates *positive* from *negative* contributions to the lifetable entropy $\bar{H}$ resulting from those mortality improvements. This threshold age a^H is reached when

$$(2) \qquad H\left(a^H\right) + \bar{H}\left(a^H\right) = 1 + \bar{H} \,.$$

variable t will be omitted for simplicity.

2 Proof

Fernández and Beltrán-Sánchez (2015) showed that the relative derivative of $\bar{H}$ can be expressed as

$$\dot{\bar{H}} \,/\, \bar{H} = \frac{\dot{e}^{\dagger}}{e^{\dagger}} - \frac{\dot{e}_o}{e_o} \,, \tag{3}$$

This formula indicates that relative changes in $\bar{H}$ over time are given by the difference between relative changes in $e^{\dagger}$ (dispersion component) and relative changes in e_o (translation component). We will first provide expressions for $\dot{e}_o$ and $\dot{e}^{\dagger}$ to prove that (1) and (3) are equivalent. Next, we will prove the existence of threshold age for $\bar{H}$ and its uniqueness.

2.1 Relative changes over time in $\bar{H}$

Vaupel and Canudas-Romo (2003) showed that changes over time in life expectancy at birth are a weighted average of the total rates of mortality improvements, given by

$$\dot{e}_o = \int_0^{\infty} \rho(x)\, w(x)\, dx \,, \tag{4}$$

where $\rho(x) = -\dot{\mu}(x) \,/\, \mu(x)$ are the age-specific rates of mortality improvement, and $w(x) = \mu(x)\, \ell(x)\, e(x) = d(x)\, e(x)$ is a measure of the importance of death at age x.

Since $d(x) = \mu(x)\, \ell(x)$ and $\ell(x)\, e(x) = \int_x^{\infty} \ell(a)\, da$, the partial derivative with respect to time of $e^\dagger = \int_0^\infty d(x)\, e(x)\, dx$ can be expressed as

$$\begin{aligned}
\dot{e}^\dagger &= \int_0^\infty \dot{\mu}(x)\, \ell(x)\, e(x)\, dx + \int_0^\infty \mu(x) \int_x^\infty \dot{\ell}(a)\, da\, dx \\
&= -\int_0^\infty \rho(x)\, w(x)\, dx + \int_0^\infty \dot{\ell}(a) \int_0^a \mu(x)\, dx\, da \\
&= -\int_0^\infty \rho(x)\, w(x)\, dx - \int_0^\infty \int_0^a \dot{\mu}(x)\, dx\, \ell(a)\, H(a)\, da\ ,
\end{aligned}$$

where $H(a)$ is the cumulative hazard to age a. By reversing the order of integration and doing some additional manipulations, we get

$$\begin{aligned}
\dot{e}^\dagger &= -\int_0^\infty \rho(x)\, w(x)\, dx - \int_0^\infty \dot{\mu}(x) \int_x^\infty \ell(a)\, H(a)\, da\, dx \\
&= -\int_0^\infty \rho(x)\, w(x)\, dx + \int_0^\infty \rho(x)\, w(x)\, \frac{\int_x^\infty \ell(a)\, H(a)\, da}{\ell(x)\, e(x)}\, dx \\
&= \int_0^\infty \rho(x)\, w(x) \left(\frac{\int_x^\infty \ell(a)\big(H(a) - H(x) + H(x)\big)\, da}{\ell(x)\, e(x)} - 1 \right) dx \\
&= \int_0^\infty \rho(x)\, w(x) \left(H(x)\, \frac{\int_x^\infty \ell(a)\, da}{\ell(x)\, e(x)} + \frac{\int_x^\infty \ell(a)\big(H(a) - H(x)\big)\, da}{\ell(x)\, e(x)} - 1 \right) dx \\
&= \int_0^\infty \rho(x)\, w(x) \left(H(x) + \frac{\int_x^\infty \ell(a)\big(H(a) - H(x)\big)\, da}{\ell(x)\, e(x)} - 1 \right) dx\ .
\end{aligned} \tag{5}$$

In Proposition 1 in the Appendix, we prove that

$$e^\dagger(x) = \frac{1}{\ell(x)} \int_x^\infty d(a)\, e(a)\, da = \frac{1}{\ell(x)} \int_x^\infty \ell(a)\big(H(a) - H(x)\big)\, da\ . \tag{6}$$

Replacing (6) in (5) yields

$$\begin{aligned}
\dot{e}^\dagger &= \int_0^\infty \rho(x)\, w(x) \left(H(x) + \frac{e^\dagger(x)}{e(x)} - 1 \right) dx \\
&= \int_0^\infty \rho(x)\, w(x) \left(H(x) + \bar{H}(x) - 1 \right) dx \; .
\end{aligned} \tag{7}$$

Finally, replacing the expressions of $\dot{e}_o$ and $\dot{e}^\dagger$ from (4) and (7) in (3), we get

$$\begin{aligned}
\dot{\bar{H}} / \bar{H} &= \frac{1}{e^\dagger} \int_0^\infty \rho(x)\, w(x) \left(H(x) + \bar{H}(x) - 1 \right) dx - \frac{1}{e_o} \int_0^\infty \rho(x)\, w(x)\, dx \\
&= \int_0^\infty \rho(x)\, w(x) \left(\frac{1}{e^\dagger} \left(H(x) + \bar{H}(x) - 1 \right) - \frac{1}{e_o} \right) dx \\
&= \int_0^\infty \rho(x)\, w(x)\, W(x)\, dx \; ,
\end{aligned}$$

which proves (1) and shows that relative changes over time in the lifetable entropy $\bar{H}$ are the average of the rates of mortality improvement weighted by the product $w(x)\, W(x)$.$\square$

2.2 The threshold age for $\bar{H}$

Using (1), changes over time in the lifetable entropy $\bar{H}$ are given by

$$\dot{\bar{H}} = \bar{H} \int_0^\infty \rho(x)\, w(x)\, W(x)\, dx \; . \tag{8}$$

Whenever $\dot{\bar{H}} > 0$, lifespan inequality increases over time, whereas $\dot{\bar{H}} < 0$ implies that variation of lifespans decreases over time. Because $\ell(x)$ is a positive function bounded between 0 and 1, we have that $\bar{H} > 0$. Moreover, assuming age-specific death rates $\mu(x)$ improve over time at all ages, then $\dot{\mu}(x) < 0$ and $\rho(x) > 0$ at any age x. Therefore, (8) implies that

1. Those ages x in which $w(x)\,W(x) > 0$ will contribute *positively* to the lifetable entropy $\overline{H}$ and increase lifespan variation;

2. Those ages x in which $w(x)\,W(x) < 0$ will contribute *negatively* to the lifetable entropy $\overline{H}$ and favor lifespan equality;

3. Those ages x in which $w(x)\,W(x) = 0$ will have no effect on the variation over time of $\overline{H}$.

Our goal is to prove that if mortality improvements occur for all ages and $\rho(x) > 0$, there exists a unique threshold age a^H such that $w\left(a^H\right)\,W\left(a^H\right) = 0$. That threshold age will separate *positive* from *negative* contributions to $\overline{H}$ resulting from mortality improvements.

The product $w(x)\,W(x)$ can be re-expressed as

$$\begin{aligned} w(x)\,W(x) &= \mu(x)\,\ell(x)\,e(x)\,\left(\frac{1}{e^{\dagger}}\left(H(x) + \overline{H}(x) - 1\right) - \frac{1}{e_o}\right) \\ &= \frac{\mu(x)\,\ell(x)\,e(x)}{e^{\dagger}}\left(H(x) + \overline{H}(x) - \overline{H} - 1\right)\ . \end{aligned}$$

Since $\mu(x)$, $\ell(x)$, $e(x)$ and $e^{\dagger}$ are all positive functions, the threshold age of $\overline{H}$ occurs whenever

$$g(x) := H(x) + \overline{H}(x) - \overline{H} - 1 = 0\ . \tag{9}$$

When x is close to 0, $g(x)$ takes negative values since

$$g(0) = H(0) + \overline{H}(0) - \overline{H} - 1 = 0 + \overline{H} - \overline{H} - 1 = -1 < 0\ .$$

Likewise, $g(x)$ takes positive values when x becomes arbitrary large. Note that $\overline{H}$ does not depend on age, and therefore

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(H(x) + \overline{H}(x)\right) - \overline{H} - 1 = \infty$$

because $\lim_{x\to\infty} H(x) = \infty$. By definition, $\overline{H}(x) \geq 0$ for all x, so regardless of the behavior of $\overline{H}(x)$ when x is arbitrarily large, the limit of $g(x)$ tends to infinity. Hence, given that $g(0) = -1$ and $\lim_{x\to\infty} g(x) = \infty$, in a continuous framework the intermediate value theorem guarantees the existence of at least one age a^H at which $g(a^H) = 0$.

Moreover, as shown in Proposition 2 in the Appendix, $g(x)$ is a strictly increasing function, and therefore a one-to-one function assuming continuity. As a result, there is a unique threshold age a^H that separates *positive* from *negative* contributions to the lifetable entropy $\overline{H}$, and that threshold age is reached when

$$w(x)\,W(x) = 0 \Longleftrightarrow g(x) = 0 \Longleftrightarrow H(x) + \overline{H}(x) = 1 + \overline{H}\ ,$$

which proves (2). □

3 Related results

Demographers have developed a battery of indicators to measure how lifespans vary in populations (van Raalte and Caswell 2013; Colchero et al. 2016). The most common indexes are the variance (Edwards and Tuljapurkar 2005; Tuljapurkar and Edwards 2011), standard deviation (van Raalte, Sasson, and Martikainen 2018) and coefficient of variation (Aburto et al. 2018) of the age at death distribution, the Gini coefficient (Shkolnikov, Andreev, and Begun 2003; Gigliarano, Basellini, and Bonetti 2017; Archer et al. 2018), the Theil index (Smits and Monden 2009), and the years of life lost (Vaupel, Zhang, and van Raalte 2011; Aburto and van Raalte 2018), among others. However, only few studies have analytically derived formulas for the *threshold age* below and above which mortality improvements respectively decrease and increase lifespan variation. Zhang and Vaupel (2009) showed that the threshold age ($a^\dagger$) for life disparity ($e^\dagger$) occurs when $H(x) + \bar{H}(x) = 1$. Similarly, Gillespie, Trotter, and Tuljapurkar (2014) determined a threshold age for the variance of the age at death distribution. Van Raalte and Caswell (2013) also showed that it is possible to determine the threshold age by performing an empirical sensitivity analysis of lifespan variation indicators.

In this article, we contribute to the lifespan variation literature by deriving the threshold age a^H for the lifetable entropy $\bar{H}$. This age separates negative from positive contributions of age-specific mortality improvements. We analytically proved its existence and—in the assumption that mortality improves over time for all ages—also its uniqueness. In Section 4 we empirically show that it differs from the threshold age of $e^\dagger$.

4 Applications

The code and data to reproduce the results and graphs presented in this section are publicly available through the repository in the link `https://bit.ly/2wqzOFp`.

4.1 Numerical findings

Figure 1 depicts the threshold ages of the two related measures, the life disparity $e^\dagger$ and lifetable entropy $\bar{H}$. Calculations were performed using data from the Human Mortality Database (2018) for females in the United States and Italy in 2005. The blue line represents $g(x)$ from Equation (9). The threshold age a^H occurs when $g(x)$ crosses zero. The red and grey lines display the same functions that Zhang and Vaupel (2009) used to find the threshold age for $e^\dagger$ rescaled to fit in the graph. The intersection of these two lines denotes the threshold age $a^\dagger$. Finally, the dashed black line depicts the life expectancy at birth. Vaupel, Zhang, and van Raalte (2011) noted that $a^\dagger$ tends to fall just below e_o. The threshold age for the lifetable entropy a^H is greater than $a^\dagger$ and is very close above life expectancy for these countries. Note the similarity between the formulas for $a^\dagger$, given by $H(a^\dagger) + \overline{H}(a^\dagger) = 1$, and a^H, given by $H(a^H) + \overline{H}(a^H) = 1 + \overline{H}$.

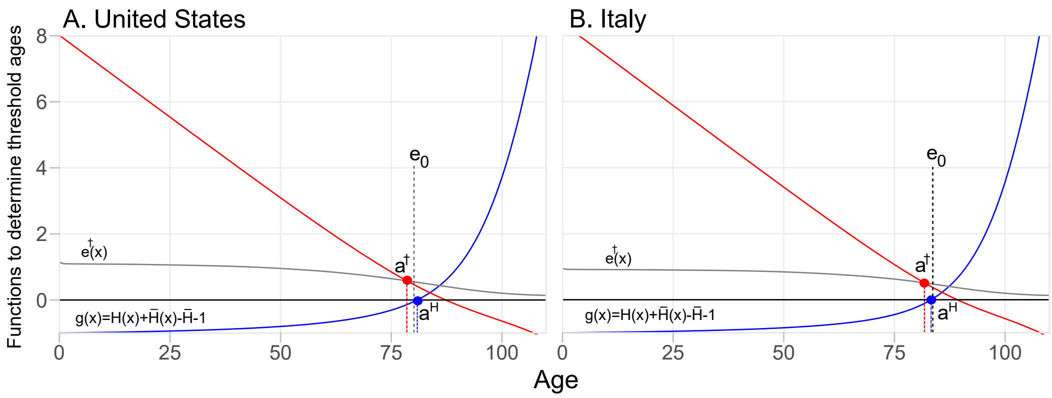

Figure 1: Threshold ages for life disparity ($a^\dagger$) and for the lifetable entropy (a^H). United States and Italy in 2005. Values in Panel A: e_o = 80.13, $a^\dagger$ = 78.51, and a^H = 80.86. Values in Panel B: e_o = 83.67, $a^\dagger$ = 81.76, and a^H = 83.28. Note: functions to determine the threshold age for $e^\dagger$ where rescaled by a factor of 1/10 for comparability. Source: Human Mortality Database (2018)

Panels A and B in Figure 2 illustrate the evolution over time of the threshold ages for $e^\dagger$ and $\overline{H}$ in French and Swedish females, respectively. We chose these countries because they portray large series of reliable data available at the Human Mortality Database (2018).

Values for $a^\dagger$ are close to life expectancy throughout the period. However, around 1950 there is a crossover between $a^\dagger$ and e_o such that $a^\dagger$ remained close to life expectancy, but below it. This result shows that the threshold age $a^\dagger$ being below life expectancy is

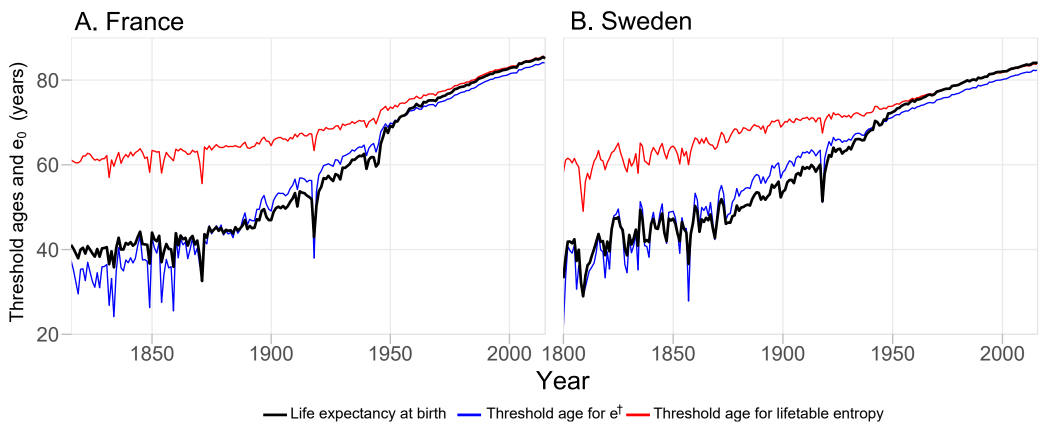

Figure 2: Threshold ages for life disparity ($a^\dagger$) and for the lifetable entropy (a^H) compared to life expectancy at birth. French and Swedish females, 1800–2016. Source: Human Mortality Database (2018)

a modern feature of ageing populations with high life expectancy. From the beginning of the period of observation to the 1950s, the threshold age for the lifetable entropy was above life expectancy for both countries. During some periods $a^\dagger$ was roughly constant whereas life expectancy trended upwards. After the 1950s, a^H converged towards life expectancy.

4.2 The threshold age of the lifetable entropy within the Gompertz mortality model

We further analyze the relationship between a^H and e_o assuming the force of mortality follows a Gompertz distribution with hazard $\mu(x) = \alpha\, e^{\beta x}$, where $x \geq 0$ denotes the age and $\alpha, \beta > 0$ are parameters. In Proposition 3 in the Appendix, we prove that in the Gompertz model the threshold age a^H of the lifetable entropy $\overline{H}$ is proportional to life expectancy at birth e_o by a factor δ, which only depends on parameters α and β, and the Euler-Mascheroni constant $\gamma \approx 0.57722$. A value of δ close to 1 indicates that mortality is roughly following a Gompertz model.

Figure 3 shows the evolution of factor δ for French and Swedish females. The observation that this value converges towards 1 could be explanatory for the convergence of the threshold age and life expectancy at birth in modern mortality profiles. It also indicates that modern mortality schedules are roughly Gompertzian. Therefore, it can be speculated that differences between these two measures in earlier years are a consequence of a big proportion of deaths occurring in ages where the force of mortality does not follow a Gompertz, such as in infancy. This is consistent with historical patterns which suggest that, since hunter-gathers to modern populations, death rates have decreased at all ages, but mostly at younger ones (Burger, Baudisch, and Vaupel 2012).

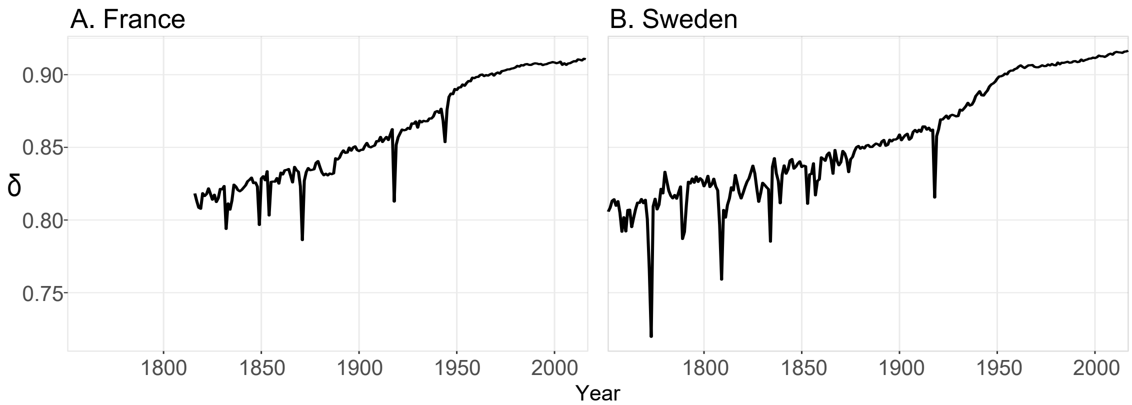

Figure 3: Factor value δ for threshold age under Gompertz distribution for French and Swedish women

4.3 Decomposition of the relative derivative of $\bar{H}$

The relative derivative of $\bar{H}$ defined in Equation (1) can be decomposed between components before and after the threshold age a^H as follows:

$$
\begin{aligned}
\dot{\bar{H}} \,/\, \bar{H} &= \int_0^\infty \rho(x)\, w(x)\, W(x)\, dx \\
&= \int_0^{a^H} \rho(x)\, w(x)\, W(x)\, dx + \int_{a^H}^\infty \rho(x)\, w(x)\, W(x)\, dx \\
&= \underbrace{\left\{ \frac{\dot{e}^\dagger[x|x<a^H]}{e^\dagger} - \frac{\dot{e}_o[x|x<a^H]}{e_o} \right\}}_{\textit{Early life component}} + \underbrace{\left\{ \frac{\dot{e}^\dagger[x|x>a^H]}{e^\dagger} - \frac{\dot{e}_o[x|x>a^H]}{e_o} \right\}}_{\textit{Late life component}}
\end{aligned} \tag{10}
$$

If mortality reductions occur at every age, the *early life component* in Equation (10) is always negative (contributing to reduce entropy) while the *late life component* is positive (contributing to increasing entropy). Thus, it is clear that a negative relationship between life expectancy and entropy over time occurs if the early life component outpaces the late life component. This decomposition is based on the additive properties of the derivatives of life expectancy and $e^\dagger$ as previously shown by Vaupel and Canudas-Romo (2003) and Fernández and Beltrán-Sánchez (2015).

5 Conclusion

Several authors have been interested in decomposing changes over time in life expectancy (Arriaga 1984; Vaupel 1986; Pollard 1988; Vaupel and Canudas-Romo 2003; Beltrán-Sánchez, Preston, and Canudas-Romo 2008; Beltrán-Sánchez and Soneji 2011). Most recently, scholars have also investigated how life disparity fluctuations over time can be decomposed (Zhang and Vaupel 2009; Wagner 2010; Shkolnikov et al. 2011; Aburto and van Raalte 2018; Aburto and Beltrán-Sánchez 2019). Here, we bring both perspectives together and shed light on the dynamics behind changes in the lifetable entropy.

Keyfitz (1977) proposed $\bar{H}$ as a lifetable function "that measures the change in life expectancy at birth consequent on a proportional change in age-specific rates" (p. 413). Since then, several authors have been interested in this measure and its use (Demetrius 1978, 1979; Mitra 1978; Goldman and Lord 1986; Vaupel 1986; Hakkert 1987; Hill 1993; Fernández and Beltrán-Sánchez 2015). Even though the lifetable entropy and $e^{\dagger}$ are both measures of lifespan variation, their demographic interpretation differs. The former is defined as the elasticity of life expectancy due to changes in death rates (Keyfitz 1968) whereas the later one refers to the average years lost due to death (Vaupel, Zhang, and van Raalte 2011). The life table entropy measures relative variability while $e^{\dagger}$ measures absolute lifespan variation. Therefore the lifetable entropy is appropriate to compare different shapes of age-at-death distributions across different species and over time (Baudisch 2011; Wrycza, Missov, and Baudisch 2015), while $e^{\dagger}$ has been used to obtain insights about lifespan variation in different countries and in sub-population groups, for instance by occupational class or income (van Raalte, Martikainen, and Myrskyl 2014; Brønnum-Hansen 2017). Both measures are meaningful and complementary, but the calculation of their threshold ages should be performed accordingly to interpret correctly changes of age patterns of mortality.

In this article, we uncovered the mathematical regularities behind the changes over time in the lifetable entropy. In particular, this study contributes to the existing literature by showing that (1) the lifetable entropy can be decomposed as a weighted average of rates of mortality improvements, and (2) there exists a unique threshold age that separates positive from negative contributions to lifetable entropy as a result of reductions in mortality over time.

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Appendix

Proposition 1. *Let* $e^\dagger(x) = \int_x^\infty d(a)\, e(a)\, da \,/\, \ell(x)$ *be a measure of lifespan disparity above age* x*, where* $d(a)$ *accounts for the distribution of deaths,* $e(a)$ *the remaining life expectancy at age* a*, and* $\ell(x)$ *is the probability of surviving from birth to age* x*. Then,*

$$e^\dagger(x) = \frac{1}{\ell(x)} \int_x^\infty \ell(a)\, \big(H(a) - H(x)\big)\, da \,, \tag{A1}$$

where $H(x)$ *is the cumulative hazard to age* x*.*

Proof. Note that

$$\frac{1}{\ell(x)} \int_x^\infty \ell(a)\, \big(H(a) - H(x)\big)\, da = \frac{1}{\ell(x)} \int_x^\infty \ell(a) \int_x^a \mu(y)\, dy\, da \,,$$

where function $\mu(y)$ is the force of mortality or hazard rate. By reversing the order of integration, and using that $e(y) = \int_y^\infty \ell(a)\, da \,/\, \ell(y)$ and $d(y) = \mu(y)\, \ell(y)$, we get

$$\begin{aligned}
\frac{1}{\ell(x)} \int_x^\infty \ell(a) \int_x^a \mu(y)\, dy\, da &= \frac{1}{\ell(x)} \int_x^\infty \mu(y) \int_y^\infty \ell(a)\, da\, dy \\
&= \frac{1}{\ell(x)} \int_x^\infty \mu(y)\, \ell(y)\, e(y)\, dy \\
&= \frac{1}{\ell(x)} \int_x^\infty d(y)\, e(y)\, dy \\
&= e^\dagger(x) \,,
\end{aligned}$$

which proves (A1). □

Proposition 2. *Let* $\ell(x)$ *be the probability of surviving from birth to age* x*. Let* $\bar{H}$ *be the lifetable entropy and* $\bar{H}(x) = e^\dagger(x) \,/\, e(x)$ *the lifetable entropy conditioned on reaching age* x*. Let* $H(x)$ *be the cumulative hazard to age* x*. Then,* $g(x) = H(x) + \bar{H}(x) - 1 - \bar{H}$ *is a strictly increasing function.*

Proof. In order to demonstrate that $g(x)$ is a strictly increasing function it is sufficient to show that its first derivative is always positive. We must prove that

$$\frac{\partial}{\partial x}\, g(x) = \frac{\partial}{\partial x} \left(H(x) + \bar{H}(x) - 1 - \bar{H}\right) = \frac{\partial}{\partial x}\, H(x) + \frac{\partial}{\partial x} \bar{H}(x) > 0 \tag{A2}$$

for all ages x.

By the fundamental theorem of calculus,

$$\frac{\partial}{\partial x}\, H(x) = \frac{\partial}{\partial x} \int_0^x \mu(a)\, da = \mu(x) \,, \tag{A3}$$

whereas

$$\frac{\partial}{\partial x}\overline{H}(x) = \frac{\partial}{\partial x}\left(\frac{e^{\dagger}(x)}{e(x)}\right) = \frac{1}{e(x)^2}\left(e(x)\,\frac{\partial}{\partial x}\,e^{\dagger}(x) - e^{\dagger}(x)\,\frac{\partial}{\partial x}\,e(x)\right) \, .$$

First, note that

$$\begin{aligned}
\frac{\partial}{\partial x}\,e^{\dagger}(x) &= \frac{\partial}{\partial x}\left(\frac{1}{\ell(x)}\int_x^{\infty} d(a)\,e(a)\,da\right) \\
&= \frac{1}{\ell(x)^2}\left(\ell(x)\,\frac{\partial}{\partial x}\left(\int_x^{\infty} d(a)\,e(a)\,da\right) - \int_x^{\infty} d(a)\,e(a)\,da\,\frac{\partial}{\partial x}\,\ell(x)\right) \\
&= \frac{1}{\ell(x)^2}\left(\ell(x)\,\big(-d(x)\,e(x)\big) - \int_x^{\infty} d(a)\,e(a)\,da\,\big(-\mu(x)\,\ell(x)\big)\right) \\
&= -\frac{\mu(x)\,\ell(x)\,e(x)}{\ell(x)} + \mu(x)\,\frac{\int_x^{\infty} d(a)\,e(a)\,da}{\ell(x)} \\
&= \mu(x)\,\big(e^{\dagger}(x) - e(x)\big) \, .
\end{aligned} \tag{A4}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial x}\,e(x) &= \frac{\partial}{\partial x}\left(\frac{1}{\ell(x)}\int_x^{\infty} \ell(a)\,da\right) \\
&= \frac{1}{\ell(x)^2}\left(\ell(x)\,\frac{\partial}{\partial x}\left(\int_x^{\infty} \ell(a)\,da\right) - \int_x^{\infty} \ell(a)\,da\,\frac{\partial}{\partial x}\,\ell(x)\right) \\
&= \frac{1}{\ell(x)^2}\left(\ell(x)\,\big(-\ell(x)\big) - \int_x^{\infty} \ell(a)\,da\,\big(-\mu(x)\,\ell(x)\big)\right) \\
&= e(x)\,\mu(x) - 1 \, .
\end{aligned} \tag{A5}$$

Therefore, using (A4) and (A5), we get

$$\begin{aligned}
\frac{\partial}{\partial x}\overline{H}(x) &= \frac{1}{e(x)^2}\left(e(x)\,\mu(x)\,\big(e^{\dagger}(x) - e(x)\big) - e^{\dagger}(x)\big(e(x)\,\mu(x) - 1\big)\right) \\
&= \frac{1}{e(x)^2}\,\big(e^{\dagger}(x)\,e(x)\,\mu(x) - e(x)^2\,\mu(x) - e^{\dagger}(x)\,e(x)\,\mu(x) + e^{\dagger}(x)\big) \\
&= \frac{e^{\dagger}(x)}{e(x)^2} - \mu(x) \, .
\end{aligned} \tag{A6}$$

Finally, replacing (A3) and (A6) in (A2) yields

$$\frac{\partial}{\partial x}\,g(x) = \mu(x) + \frac{e^{\dagger}(x)}{e(x)^2} - \mu(x) = \frac{e^{\dagger}(x)}{e(x)^2} > 0 \, ,$$

which holds true for all ages since by definition $e^{\dagger}(x) > 0$ for all $x \geq 0$. Hence, $g(x)$ is a strictly increasing function. □

Proposition 3. *Assume the force of mortality follows a Gompertz distribution with hazard* $\mu(x) = \alpha\, \mathrm{e}^{\beta x}$, *where* $x \geq 0$ *denotes the age and* $\alpha, \beta > 0$ *are parameters. Suppose mortality improvements over time occur at all ages and therefore there is a unique threshold age* a^H

that separates positive from negative contributions to the lifetable entropy $\bar{H}$. Then, a^H is approximately proportional to the life expectancy at birth e_o.

Proof. The cumulative hazard of the Gompertz model is given by

$$H(x) = \frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x} - 1\right) \ ,$$

where $x \geq 0$ denotes the age and $\alpha, \beta > 0$ are parameters. Following Wrycza (2014), the lifetable entropy can be expressed in terms of the Gompertz parameters as

$$\bar{H} = \frac{1}{\beta}\left(\frac{1}{e_o} - \alpha\right) \ ,$$

where e_o is the life expectancy at birth. Plugging these two expressions into function $g(x)$ from Equation (9) yields

$$g(x) = \frac{1}{\beta}\left(\alpha\,\mathrm{e}^{\beta x} - \frac{1}{e_o}\right) + \bar{H}(x) - 1\ . \tag{A7}$$

From Equation (A1) in Proposition 1, the lifetable entropy conditioned on surviving to age x can be expressed as

$$\bar{H}(x) = \frac{e^{\dagger}(x)}{e(x)} = \frac{\int_x^{\infty} \ell(a)\big(H(a) - H(x)\big)\,da}{\int_x^{\infty} \ell(a)\,da}\ .$$

Using the above expressions in terms of the Gompertz parameters, it holds that the lifetable entropy conditioned on surviving to age x is

$$\begin{aligned}\bar{H}(x) &= \frac{\int_x^{\infty} \ell(a)\,\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta a} - \mathrm{e}^{\beta x}\right)da}{\int_x^{\infty} \ell(a)\,da} = \frac{\int_x^{\infty} \ell(a)\,\alpha\,\mathrm{e}^{\beta a}\,da}{\beta\int_x^{\infty} \ell(a)\,da} - \frac{\alpha}{\beta}\,\mathrm{e}^{\beta x}\\ &= \frac{\int_x^{\infty} \ell(a)\,\mu(a)\,da}{\beta\,e(x)\,\ell(x)} - \frac{\alpha}{\beta}\,\mathrm{e}^{\beta x} = \frac{1}{\beta}\left(\frac{1}{e(x)} - \alpha\,\mathrm{e}^{\beta x}\right)\ .\end{aligned} \tag{A8}$$

The last step in (A8) uses the product $\ell(a)\,\mu(a)$ as the age-at-death distribution, which then implies that $\int_x^{\infty} \ell(a)\,\mu(a)\,da = \ell(x)$. Thus, $g(x)$ in (A7) reduces to

$$g(x) = \frac{1}{\beta}\left(\frac{1}{e(x)} - \frac{1}{e_o}\right) - 1\,, \tag{A9}$$

where $e(x)$ is the remaining life expectancy at age x. Equation (A9) implies that the threshold age a^H of the lifetable entropy $\bar{H}$ under the Gompertz model occurs whenever

$$e(x) = \frac{e_o}{\beta\,e_o + 1}\ .$$

Following Missov and Lenart (2013), the remaining life expectancy at age x in the

Gompertz case can be approximated by

$$e(x) \approx \frac{1}{\beta}\, \mathrm{e}^{\alpha/\beta} \left(-\gamma - \ln(\alpha/\beta) - \beta\, x \right) \,, \tag{A10}$$

where $\gamma \approx 0.57722$ is the Euler-Mascheroni constant. Hence, the threshold age occurs whenever

$$\begin{aligned} e(x) \approx \frac{1}{\beta}\, \mathrm{e}^{\alpha/\beta} \left(-\gamma - \ln(\alpha/\beta) - \beta\, x \right) &= \frac{e_o}{\beta\, e_o + 1} \\ \Longleftrightarrow x &= -\frac{\mathrm{e}^{-\alpha/\beta}\, e_o}{\beta\, e_o + 1} - \frac{1}{\beta} \left(\gamma + \ln(\alpha/\beta) \right) \,. \end{aligned}$$

Note that (A10) implies that $e_o \approx \mathrm{e}^{\alpha/\beta} \left(-\gamma - \ln(\alpha/\beta) \right) / \beta$. Using this approximation,

$$\begin{aligned} a^H &\approx -\frac{\mathrm{e}^{-\alpha/\beta}\, e_o}{\mathrm{e}^{\alpha/\beta}\, (-\gamma - \ln(\alpha\,/\,\beta)) + 1} + \frac{e_o}{\mathrm{e}^{\alpha/\beta}} \\ &= \frac{e_o}{\mathrm{e}^{\alpha/\beta}} \left(\frac{1}{\mathrm{e}^{\alpha/\beta}\, (\gamma + \ln(\alpha\,/\,\beta)) - 1} + 1 \right) \\ &= e_o \left(\frac{\gamma + \ln(\alpha/\beta)}{\mathrm{e}^{\alpha/\beta}\, (\gamma + \ln(\alpha\,/\,\beta)) - 1} \right) \\ &= e_o \cdot \delta \,, \end{aligned} \tag{A11}$$

which proves that the threshold age a^H of the lifetable entropy $\overline{H}$ for the Gompertz model is (approximately) proportional to e_o by a factor δ that only depends on parameters α, β and γ. □