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Gompertz–Makeham life expectancies: Expressions and applications

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ABSTRACT

In a population of individuals, whose mortality is governed by a Gompertz–Makeham hazard, we derive closed-form solutions to the life-expectancy integral, corresponding to the cases of homogeneous and gamma-heterogeneous populations, as well as in the presence/absence of the Makeham term. Derived expressions contain special functions that aid constructing high-accuracy approximations, which can be used to study the elasticity of life expectancy with respect to model parameters. Knowledge of Gompertz–Makeham life expectancies aids constructing life-table exposures.

1. Introduction

Parametric models of human mortality date back to Gompertz (1825) and his insight that death rates at adult ages increase exponentially with age. Makeham (1860) added an age-independent constant that, on the one hand, accounts for mortality that is not related to aging and moreover, statistically speaking, introduces a third parameter that improves the model fit.

In human populations, the overestimation of observed death rates at ages 80+ by the Gompertz–Makeham (GM) curve inspired the study of models that account for unobserved heterogeneity (Beard, 1959), i.e. models in which the study population is assumed to be stratified according to an unobserved measure of individual susceptibility to death. Vaupel et al. (1979) introduced a positive random variable Z, called frailty, that modulates individual hazards. The resulting marginal distribution, a continuous mixture for the baseline mortality distribution with respect to the mixing frailty distribution, describes the process at the level of the population. The simplest (in terms of frailty distribution choice) model (Vaupel et al., 1979) that accurately captures observed mortality dynamics at adult, old, and oldest-old ages (see, for example, Missov and Finkelstein (2011) and Missov and Vaupel (2013)) is the gamma-Gompertz–Makeham (ΓGM) model or its special case (when $c = 0$) the gamma-Gompertz (ΓG) model. Within its framework individual frailty Z is described by a p.d.f.

$$\pi(z) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}, \quad k, \lambda > 0.$$

Frailty is considered to be fixed, i.e. one's frailty is initialized at the starting age of study by a value that remains the same throughout one's life. The force of mortality and the survival function of an individual with frailty $Z = z$ at age x is given, respectively, by

$$\mu(x \mid z) = z\, a e^{bx} + c \tag{1}$$

and

$$s(x \mid z) = \exp\left\{-z\,\frac{a}{b}\left(e^{bx} - 1\right) - cx\right\}, \tag{2}$$

where $a, b > 0$ are the Gompertz parameters and $c \geq 0$ stands for the level of age-independent extrinsic mortality (Kirkwood, 1985). When $c = 0$, $\mu(x \mid z)$ follows a Gompertz curve. Otherwise $\mu(x \mid z)$ has a GM shape.

The distribution of lifetimes in a ΓG mixture model is described by a survival function

$$s(x) = \int_0^\infty s(x \mid z)\,\pi(z)dz = e^{-cx}\left(1 + \frac{a}{b\lambda}(e^{bx} - 1)\right)^{-k}. \tag{3}$$

As a result remaining life expectancy at age x is expressed by the integral

$$e(x) = \int_x^\infty e^{-ct}\left(1 + \frac{a}{b\lambda}(e^{bt} - 1)\right)^{-k} dt. \tag{4}$$

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In this article we focus on remaining life expectancy $e(x)$ at age x in four nested models: a Gompertz or GM baseline with (gamma-distributed) or without unobserved heterogeneity. In each of these four settings we first derive analytical expressions $e(x)$ and construct high-accuracy approximations that do not include special functions. Second, we study the elasticities of $e(x)$ with respect to all model parameters a, b, c, k, λ. Finally, we present problems, in which knowledge of Gompertz–Makeham life-expectancy expressions might be useful: (i) estimating age-specific exposures in life tables; (ii) assessing the onset of senescent mortality and the start of mortality deceleration.

2. Life expectancy: exact expressions and approximations

In this section we consider three special cases of (1) that cover the four nested models of interest: (1) when Z has a degenerate distribution concentrated at 1 and $c = 0$ (Gompertz), (2) when Z has a degenerate distribution concentrated at 1 and $c > 0$ (GM), and (3) when Z is gamma-distributed (ΓG when $c = 0$ and ΓGM when $c > 0$). In each case we first derive analytical expressions for (remaining) life expectancy and then construct high-accuracy approximations based on the properties of the resulting special functions.

2.1. Gompertz life expectancy and its approximation

In the Gompertz case, when the force of mortality is given as

$$\mu_G(x) = ae^{bx},$$

Missov and Lenart (2011) showed that the corresponding remaining life expectancy at age x can be expressed by

$$e_G(x) = \frac{1}{b}\, e^{\frac{a}{b}} E_1\left(\frac{a}{b}\, e^{bx}\right), \tag{5}$$

where $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ denotes the exponential integral. As shown by Abramowitz and Stegun (1965, 5.1.11)

$$E_1(t) = -\gamma - \ln t - \sum_{n=1}^{\infty} \frac{(-1)^n\, t^n}{n \cdot n!},$$

so if $t = ae^{bx}/b$ is close to 0, then $e_G(x)$ can be approximated by

$$e_G(x) \approx \frac{1}{b}\, e^{\frac{a}{b}} \left(-\gamma - \ln \frac{a}{b}\right). \tag{6}$$

2.2. GM life expectancy and its approximation

In the Gompertz–Makeham case, when the force of mortality is given as

$$\mu_{GM}(x) = ae^{bx} + c,$$

remaining life expectancy at age x equals

$$e_{GM}(x) = \frac{1}{b}\, e^{\frac{a}{b}} \left(\frac{a}{b}\right)^{\frac{c}{b}} \Gamma\left(-\frac{c}{b}, \frac{a}{b}\, e^{bx}\right), \tag{7}$$

where $\Gamma(s, z) = \int_z^\infty t^{s-1} e^{-t} dt$ denotes the upper incomplete gamma function (see B.1). Note that

$$E_1(z) = \lim_{s\to 0} \int_z^\infty t^{s-1}\, e^{-t} dt = \lim_{s\to 0} \Gamma(s, z),$$

i.e. $e_G(x)$ is a degenerate form of $e_{GM}(x)$ when the Makeham term equals zero.

If a is close to 0, life expectancy at birth $e_{GM}(0)$ can be approximated by

$$e_{GM}(0) = \frac{1}{c} - \frac{\left(\frac{a}{b}\, e^{\gamma-1}\right)^{\frac{c}{b}}}{c\left(1 - \frac{c}{b}\right)}, \tag{8}$$

where $\gamma \approx 0.57722$ is the Euler–Mascheroni constant.

For parameter values corresponding to mortality patterns in modern societies ($0 < \frac{a}{b}\, e^{bx} \leq 1$ and $0 < \frac{c}{b} \leq 0.1$), the incomplete gamma function $\Gamma\left(-\frac{c}{b}, \frac{a}{b} e^{bx}\right)$ can be approximated by (see B.2)

$$\Gamma(s, z) = \frac{1}{s + s^2} \exp\left\{(1-\gamma)s + 0.3225 s^2\right\} - \sum_{k=0}^{\infty} (-1)^k \frac{z^{s+k}}{k!(s+k)}, \tag{9}$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta function and $0.3225 \approx \frac{\zeta(2)-1}{2}$. The closer the z-argument of the upper incomplete gamma function to 0, i.e. at younger ages, the fewer terms of $\sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{s+k}}{k!(s+k)}$ we need to use. To achieve a desired accuracy ε, the number of terms m in the latter series to be taken into account can be determined by

$$\frac{z^{s+m+1}}{(m+1)!(s+m+1)} \leq \varepsilon.$$

2.2.1. Example

Fitting a GM model by maximizing a Poisson likelihood (see (C.1) in Appendix C) for the 2007 United States death counts (ages 30 and above), we get the following parameter estimates: $\hat{a}_{30} = 0.00046$, $\hat{b} = 0.094$ and $\hat{c} = 0.0007$. If we want to measure remaining life expectancy at age 30, by calculating $\hat{s} = -0.0074$ and $\hat{z} = 0.0049$ from the fitted parameters and setting the error of approximation to the sum by $\varepsilon = 0.001$,

$$\frac{0.0049^{-0.0074+m+1}}{(m+1)!(-0.0074+m+1)} \leq 0.001$$

solving this inequality for m gives $m = 0.24$. Rounding up to the first integer, as only integers are allowed for k in (9), yields $m = 1$. Similarly, when the Makeham term is close to zero, in the exponential part of (9) the $\zeta(n)$-term can be left out of the approximation. In this case, the approximation error is 0.0014. Adding the two errors together, by using

$$\Gamma(s, z) = \frac{e^{(1-\gamma)s}}{s + s^2} - \frac{z^s}{s} + \frac{z^{s+1}}{s+1} \tag{10}$$

instead of (9) approximates $\Gamma(s, z)$ with an error of less than 0.0024.

To illustrate approximation quality, we calculate the exact remaining life expectancy at age 30 for the United States (2007), Japan and Germany (2009), as well as Sweden (2010) by (7), and compare it to approximation (7) taking into account (10) (see Table 1). Please note that the approximation error of (10) is inflated by the multiplicative terms preceding the upper incomplete gamma function in (7), leading to the error of not more than 0.02.

2.3. ΓGM life expectancy and its approximations

Suppose in a population the force of mortality and the survival function for an individual with frailty $Z = z$ are given by (1) and (2), respectively. Then the remaining period life expectancy at age

Table 1
Exact and approximate values for remaining Gompertz–Makeham life expectancy at age 30.

	Sweden 2010	Germany 2009	Japan 2009	USA 2007
exact	51.69	49.97	52.91	48.42
approx	51.70	49.99	52.92	48.44

Table 2
Gamma-Gompertz–Makeham parameter estimates.

	Sweden 2010	Germany 2009	Japan 2009	USA 2007
$\hat{a}_{30}$	8.68E−05	1.65E−04	1.45E−04	3.49E−04
$\hat{b}$	0.127	0.117	0.112	0.101
$\hat{\gamma}$	0.100	0.097	0.093	0.110
$\hat{c}$	0.0005	0.0007	0.0006	0.0010

x is calculated by (4). Following the idea of Missov (2013), it can be derived (see A.1) that

$$e(x) = \frac{e^{-(bk+c)x}}{bk+c}\left(\frac{b\lambda}{a}\right)^k \times {}_2F_1\left(k+\frac{c}{b}, k; k+\frac{c}{b}+1; \left(1-\frac{b\lambda}{a}\right)e^{-bx}\right) \tag{11}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gaussian hypergeometric function, i.e.

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j\,(\beta)_j}{(\gamma_j)}\,\frac{z^j}{j!}, \tag{12}$$

which is defined for $\gamma > \beta > 0$ (see, for example, Bailey (1935)). For $n \in \mathbb{N}(m)_n = m(m+1)\dots(m+n-1)$ denotes the Pochhammer symbol, $(m)_0 = 1$.

Setting up $x = 0$ and applying linear transformation formula 15.3.5 from Abramowitz and Stegun (1965), we reduce (11) to the ΓGM life expectancy at birth

$$e(0) = \frac{1}{bk+c}\,{}_2F_1\left(k; 1; k+1+\frac{c}{b}; 1-\frac{a}{b\lambda}\right), \tag{13}$$

which for $c = 0$ is equal to the expression for ΓG life expectancy in Missov (2013).

Using properties of the hypergeometric function and assuming parameter values corresponding to contemporary human mortality data (see A.2), we get the following approximation to ΓGM life expectancy

$$e(x) \approx \frac{\left[\left(1-\frac{a}{b\lambda}\right)e^{\frac{c}{k}x}\right]^{-k}}{bk+c}\left[\left(\frac{bk}{c}+1\right)+(-z)^{-\frac{c}{b}} \times \frac{\Gamma\left(1+k+\frac{c}{b}\right)\Gamma\left(-\frac{c}{b}\right)}{\Gamma(k)}\right]. \tag{14}$$

2.3.1. Example

In order to illustrate the accuracy of approximation, we fitted a ΓGM hazard to the data for the United States (2007), Japan (2009), Germany (2009), and Sweden (2010), ages 30–110, by maximizing a Poisson likelihood for the death counts. The resulting parameter estimates are presented in Table 2.

$\hat{\gamma} = \hat{k}/\hat{\lambda}^2$ denotes the estimated variance of the gamma distribution. Using (11) for the exact result, (A.6) for the approximate value, and (14) for a cruder approximation, we get the following life-expectancy values (see Table 3).

Table 3
Exact and approximate values for remaining gamma-Gompertz–Makeham life expectancy at age 30.

	Sweden 2010	Germany 2009	Japan 2009	USA 2007
exact	52.380	50.824	53.825	49.697
approx_1	52.379	50.823	53.824	49.691
approx_2	52.375	50.812	53.814	49.663

Table 4
Elasticities of Gompertz–Makeham parameters.

Elasticity of	Modern society		Hunter–gatherers
	Homogeneous	Heterogeneous	
a	−0.12	−0.12	−0.07
b	−0.87	−0.85	−0.575
c	−0.038	−0.028	−0.355
k		−0.13	
λ		0	
k^*		−0.01	

3. Elasticity of life expectancy with respect to model parameters

Approximations (A.6) and (14) to the ΓGM life expectancy, though accurate, give little insight about the impact of model parameters on $e(x)$. A simple approximation to the ΓG life expectancy at birth for $k \in \mathbb{N}$

$$e(0) \approx \frac{1}{b}\left[\ln\frac{b\lambda}{a} - \sum_{i=1}^{k-1}\frac{1}{i}\right] \quad a \to 0,$$

suggested by Missov (2013), illustrates that $e(0)$ is much more sensitive to changes in the rate of aging b in comparison to changes in the initial level of mortality a.

We can illustrate this formally by calculating the elasticities of life expectancy with respect to the GM parameters, i.e. the ratio of the relative change in life expectancy to the relative change in a, b, and c. We define elasticity $E_{y,x}$ of y with respect to x by

$$E_{y,x} = \frac{\partial y}{\partial x}\cdot\frac{x}{y}.$$

Calculating the elasticities of (8) gives

$$E_{e_{GM},a} = \frac{\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}}{1+\frac{b}{c}\left[\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}-1\right]}$$

with respect to a,

$$E_{e_{GM},b} = \frac{\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}\left[c-2b+(c-b)\ln\left(\frac{a}{b}e^{\gamma-1}\right)\right]}{\left(\frac{b}{c}-1\right)\left[c+b\left(\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}-1\right)\right]}$$

with respect to b, and

$$E_{e_{GM},c} = \frac{c^2+b^2-2bc+\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}\left[-b^2+2bc+c(b-c)\ln\left(\frac{a}{b}e^{\gamma-1}\right)\right]}{(b-c)\left[c+b\left(\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}-1\right)\right]}$$

with respect to c.

Substituting $a = 0.00002$, $b = 0.1$ and $c = 0.0007$ in the equations above for modern societies, and $a = 0.0000067$, $b = 0.125$, $c = 0.011$ for hunter–gatherers (Gurven and Kaplan, 2007) shows that life expectancy is highly inelastic to small changes in a or c and more sensitive to changes in b (see Table 4).

That is, in the case of heterogeneous modern societies, a 10% increase in a would decrease life expectancy by 1.2%, a 10% increase

Table 5
Elasticities of Gompertz–Makeham parameters as Chebyshev polynomials.

Elasticity w.r.t.	In terms of c	
	Homogeneous	Heterogeneous
a	$-0.12 + 5c - 5.62c^2$	$-0.13 + 5c - 5.6c^2$
b	$-0.88 + 45.65c - 48.07c^2$	$-0.87 + 58.22c - 66.07c^2$
c	$-47.65c + 53.69c^2$	$-63.22c + 71.67c^2$
k		$-0.14+5.22c-5.84c^2$
λ		$0.87c - c^2$
k^*		$-0.01+0.22c-0.24c^2$
	In terms of a	
a	$-0.1 + 0.27a + 0.32a^2$	
b	$-0.87 - 0.29a - 0.34a^2$	
c	$-0.03 + 0.02a + 0.03a^2$	

in b would decrease life expectancy by 8.5%, and a 10% increase in c would decrease life expectancy by 0.03%. The elasticities of these parameters do not differ considerably in the homogeneous case. An increase in the shape parameter k of the heterogeneity distribution decreases the proportion of "low-frailty" individuals. In accordance with this, elasticities show that if k increases by 10%, life expectancy would decrease by 1.3%. Note that in this case the average frailty of the population would increase. An increase in the scale parameter λ of the frailty distribution would have a very small positive effect on life expectancy as it lowers mean frailty. We denote by k^* the parameter of the gamma distribution when $k = \lambda$. In this case, an increase in k^* decreases the variance of frailty in the population while its mean stays the same. As a result life expectancy mildly decreases—a 10% increase in k^* lowers life expectancy by 0.1%. Note, however, that the elasticities are not constant: for a higher value of a or c, life expectancy can be more elastic, as it is in the case of hunter–gatherers.

Elasticities could be approximated by Chebyshev polynomials of the first kind. The latter provide an approximation that is close to the polynomial of best approximation to a continuous function under the maximum norm. A second degree polynomial approximation shows the sensitivity of the elasticities to changes in c (Table 5). Substituting a value for c or a in these formulae gives the elasticity of life expectancy with respect to any of its parameters. As Table 5 shows, a change in a leaves elasticities undisturbed. Looking at the row for b, we can see that if c increases, a change in b loses its importance in determining life expectancy. Nevertheless, for $c \approx 0$, a change in the rate of aging b would have the largest impact on life expectancy. Fig. 1 presents the elasticity of period life expectancy for Swedish males from 1800 to 2010 with respect to ΓGM model parameters (a, b, c, and $k = \lambda$). While the share of a (around 10%) and k (very small—around 1%) stay almost constant over time, the relative importance of b from the beginning of the 20th century to the 1950s increased (at the expense of c) from around 50% to more than 85%. After the 1950s the shares of a, b, c, and k stay almost the same at levels of 12%, 84%, 3%, and 1%, respectively. Note that b reflects the improvements in age-specific mortality rates. Since the latter have been decreasing at an almost linear pace since the 1950s (Tuljapurkar et al., 2000; Vaupel, 1986), it is not surprising that increases of life expectancy are mostly due to improvements in the age-specific mortality rates reflected in b.

4. Using GM life expectancies to estimate exposures

Derived expressions for GM life expectancies can be used to estimate life-table exposures. If age x is continuous, exposures can be derived analytically via the respective special functions, introduced in the previous sections (see Lenart, 2012, p. 13). However, demographic data are aggregated age- or interval-wise

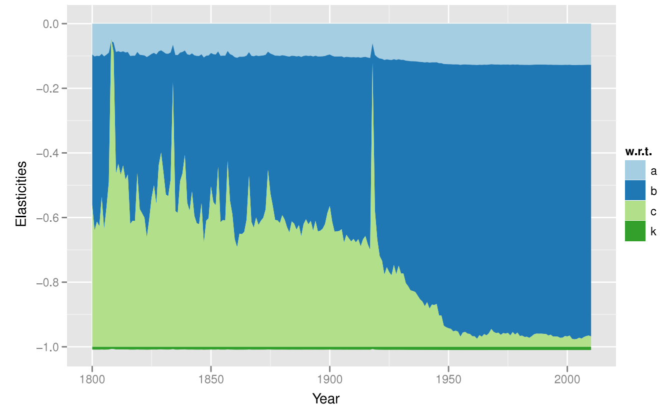

Fig. 1. Elasticity of remaining life expectancy at age 25, Swedish males, 1800–2010, with respect to ΓGM model parameters in the case when $k = \lambda$.

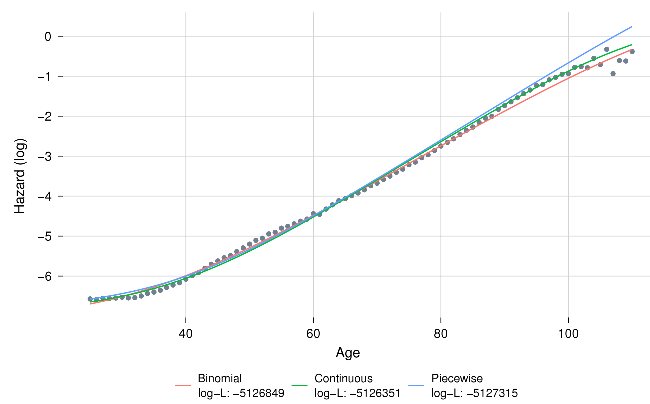

Fig. 2. US male mortality, 2007, fitted when exposures were calculated by assuming constant hazard within a year (blue), ΓGM parametric continuous hazard (green), and ΓGM parametric discrete hazard (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

and exposures need to be estimated discretely. Suppose $N(x)$ is the population size at age x for a given cohort. Then the corresponding exposure, i.e. the number of person–years lived by these $N(x)$ individuals between ages x and $x + 1$, can be expressed as

$$E(x) = N(x) \int_x^{x+1} s(t)\, dt = N(x)\, [e(x) - e(x + 1)]. \tag{15}$$

This holds if we assume that within each age-interval $[x, x + 1)$, death counts $D(x)$ are (known and) Poisson distributed and we maximize a Poisson likelihood (see Appendix C) to estimate the ΓGM parameters

$$\ell(\mu|D, E) \propto \sum_x [D(x) \ln \mu(x; \theta) + E(x)\mu(x; \theta)]\,,$$

or, alternatively, if we maximize a binomial likelihood

$$\ell_{\text{Bin}}(q \mid D, N) = \sum_x \{D(x)\ \ln q(x) + [N(x) - D(x)]\ \ln(1 - q(x))\},$$

where $q(x) = 1 - \exp\left\{-\int_x^{x+1} \mu(x)dx\right\}$ is the probability of death in $[x, x + 1)$.

Thus, knowing population counts $N(x)$ and remaining life expectancy $e(x)$, we can use (15) to calculate exposures $E(x)$. Fig. 2 compares the fits to 2007 US male log-mortality rates if we use three different expressions for exposures $E(x)$ in a ΓGM setting: the one in (15) (red), the expression involving a hypergeometric function when time is continuous (green), and an expression obtained under the assumption that in $[x, x+1)$ the hazard is piecewise-constant $\mu(x) = D(x)/N(x)$ (blue) resulting in

$$E(x) = \frac{1}{\mu(x)} N(x) \left(1 - e^{-\mu(x)}\right).$$

The latter is used to calculate exposures in the Human Mortality Database (HMD, 2013). Fig. 2 suggests that calculating exposures by taking advantage of the expressions for GM life expectancies might be a reasonable alternative, especially at ages 90+.

5. Discussion

Life expectancy at birth $e(0)$ in a probabilistic mortality model is simply the expected value of the underlying (mixture) distribution. In this sense our results characterize the first moment of the Gompertz, GM, ΓG and ΓGM distributions. Remaining life expectancy $e(x)$, on the other hand, provides the expected lifespan of survivors to age x, i.e. the first moment of the conditional underlying distribution. The analytical expressions for life expectancy (5), (7), (11) and (13) contain special functions (the exponential integral, the upper incomplete gamma function, and the hypergeometric function) with series representations that allow approximations of first and second order when the initial level of mortality a is small. Note that the exponential integral in (5) reduces analytically from the upper incomplete gamma function in (7) when the Makeham term c approaches zero. The link between the hypergeometric life-expectancy representation of the ΓG/ΓGM to $e(x)$ of the Gompertz/GM model is obtained by taking the limit of (11) when $k = \lambda \to \infty$.

The sensitivity analysis we perform aims at identifying those model parameters, the changes in which affect life expectancy the most. Both theoretically (by looking at the elasticities) and empirically (by looking at Swedish males) it turned out that life expectancy changes (improvements) are predominantly due to changes in b (the individual rate of aging, slowed down by age-specific mortality improvements). Changes in a affect at a constant and moderate pace life expectancy values, while changes in c have influenced life expectancy substantially by the beginning of the 20th century after which they gradually lost their importance. Changes in $k = \lambda$, which is the reciprocal of squared coefficient of variation of $\Gamma(k, \lambda = k)$, play a minor role in life expectancy dynamics over time.

ΓGM life expectancies can be also useful in estimating age-specific exposures in a life table. The conceptual difference with the standard procedure applied in the HMD is that we do not assume a piecewise-constant hazard, but rather a sensible parametric structure, which might be preferable at the oldest ages.

6. Conclusion

This paper studies the average life span under the Gompertz–Makeham model, which describes the pattern of adult human deaths. We consider both the case of homogeneous and heterogeneous populations and derive analytical solutions for $e(x)$. The ΓGM life expectancy is expressed in terms of a hypergeometric function that reduces to an upper incomplete gamma function in the homogeneous case when frailty Z is assumed to be common for all individuals: $k = \lambda \to \infty$. When, in addition, the Makeham term c does not play an important role, i.e. $c \to 0$, life expectancy is proportional to an exponential integral. The three $e(x)$-expressions in terms of special functions can be calculated by simpler high-accuracy approximations. They aid interpretation of the role of model parameters on the values of $e(x)$, especially in the homogeneous case when the elasticities of the Gompertz–Makeham parameters are approximated by Chebyshev polynomials of the first kind. In both the homogeneous and the heterogeneous case the rate of aging b has the largest impact on life expectancy. Finally, model-based life expectancies can serve as a tool for estimating age-specific exposures in a life table.

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Appendix A

A.1. Gamma-Gompertz–Makeham life expectancy

Theorem 1. *Life expectancy of the gamma-Gompertz–Makeham mortality model at age x equals*

$$e(x) = \frac{\left(\frac{b\lambda}{a} e^{-bx}\right)^k e^{-cx}}{bk + c} \left(1 - \left(1 - \frac{b\lambda}{a}\right) e^{-bx}\right)^{-k}$$
$$\times {}_2F_1\left(k, 1; k + 1 + \frac{c}{b}; \frac{\left(1 - \frac{b\lambda}{a}\right) e^{-bx}}{\left(1 - \frac{b\lambda}{a}\right) e^{-bx} - 1}\right)$$
$$a, b, k, \lambda > 0, c \geq 0.$$

Proof. We will first show that remaining life expectancy in the gamma-Gompertz–Makeham model can be expressed by the upper incomplete Beta function. Then we will represent the solution in terms of the more general Gaussian hypergeometric function.

As in (4), remaining life expectancy at age x is defined by[1]

$$e(x) = \int_x^\infty e^{-ct} \left(1 + \frac{a}{b\lambda}(e^{bt} - 1)\right)^{-k} dt.$$

By substituting $h = \frac{a}{b\lambda}$ and $q = e^{bt}$, we get

$$e(x) = \frac{1}{b} \int_{e^{bx}}^\infty q^{\frac{-c}{b} - 1} (1 + h(q - 1))^{-k} dq. \tag{A.1}$$

Reorganizing (A.1) yields

$$e(x) = \frac{1}{b}(1 - h)^{-k} \int_{e^{bx}}^\infty q^{\frac{-c}{b} - 1} \left[1 + \left(\frac{h}{1 - h}\right) q\right]^{-k} dq.$$

Substituting $u = -\frac{gq}{gq+1}$, where $g = \frac{h}{1-h}$ gives

$$e(x) = \frac{1}{b}(1 - h)^{-k} \int_{\frac{ge^{bx}}{ge^{bx}+1}}^1 \left(-\frac{u}{g(u-1)}\right)^{\frac{-c}{b} - 1}$$
$$\times \left[1 + g\left(-\frac{u}{g(u-1)}\right)\right]^{-k} \frac{1}{g(u-1)^2} du,$$

and by substituting $s = -\frac{c}{b}$, we get:

$$e(x) = \frac{1}{b}(1 - h)^{s-k} h^{-s} \int_{\frac{ge^{bx}}{ge^{bx}+1}}^1 u^{s-1}(1 - u)^{k-s-1} du,$$

[1] Note that the remaining life expectancy (expected value) of a (gamma-) Gompertz–Makeham distribution is the Laplace transform of the (gamma-) Gompertz survival function.

or

$$e(x) = \frac{1}{b}(1-h)^{s-k}h^{-s}\bar{B}_{\frac{ge^{bx}}{ge^{bx}+1}}(s, k-s), \tag{A.2}$$

where $\bar{B}_x(\alpha, \beta)$ denotes the upper incomplete Beta function.[2]

Please note that the Beta function is defined only for positive arguments, and because $s = -\frac{c}{b}$ is generally negative, analytic continuation of the Beta function to the negative plane is necessary. Also, 0 is out of the domain of the Beta function but as $c = 0$ connects the Gompertz–Makeham force of mortality to the Gompertz case, it is more advantageous to represent the Beta function in a more general function.

The upper incomplete Beta function is defined as

$$\bar{B}_x(\alpha, \beta) = \int_x^1 t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Substituting $t = \frac{1}{1-u}$ yields

$$\bar{B}_x(\alpha, \beta) = \int_{\frac{x-1}{x}}^0 -\frac{1}{u}\left(\frac{1}{1-u}\right)^{\alpha}\left(\frac{u}{u-1}\right)^{\beta-1} du. \tag{A.3}$$

Expanding (A.3) around $u = 0$ gives

$$\bar{B}_x(\alpha, \beta) = \int_{\frac{x-1}{x}}^0 (-u)^{\beta-1}\sum_{n=0}^{\infty}(\alpha+\beta)_n \frac{u^n}{n!} du,$$

where $(\alpha+\beta)_n$ is the Pochhammer symbol: $(\alpha+\beta)_n = (\alpha+\beta)(\alpha+\beta+1)\dots(\alpha+\beta+n-1)$.

Performing integration, finally gives

$$\bar{B}_x(\alpha, \beta) = \frac{1}{\beta}\left(\frac{1}{x}-1\right)^{\beta} {}_2F_1\left(\beta; \alpha+\beta; 1+\beta; 1-\frac{1}{x}\right). \tag{A.4}$$

Substituting (A.4) back to (A.2) completes the proof. □

A.2. Approximations for gamma-Gompertz–Makeham $e(x)$

First, we will use a linear transformation of the hypergeometric function and some of its properties (Abramowitz and Stegun, 1965, 15.3.4, 15.3.8, 15.1.8). For $z = e^{-bx}\left(1-\frac{b\lambda}{a}\right)$ we have

$$\begin{aligned}{}_2F_1\left(k; k+\frac{c}{b}; 1+k+\frac{c}{b}; z\right) &= (1-z)^{-k}\left(\frac{bk}{c}+1\right)\\ &\times {}_2F_1\left(k, 1, 1-\frac{c}{b}, \frac{1}{1-z}\right)\\ &+(-z)^{-k-\frac{c}{b}}\frac{\Gamma\left(1+k+\frac{c}{b}\right)\Gamma\left(-\frac{c}{b}\right)}{\Gamma(k)}.\end{aligned} \tag{A.5}$$

The z-argument of the hypergeometric function in the first additive term on the right-hand side of (A.5) approaches 0 (for reasonable parameter values for humans). This implies that the general term of the series (12) will tend to 0. Thus we can use the first two terms of the hypergeometric series on the right-hand side of (A.5) as an approximation:

$$\begin{aligned}{}_2F_1\left(k; k+\frac{c}{b}; k+1+\frac{c}{b}; z\right) &\approx (1-z)^{-k}\\ &\times\left(\frac{bk}{c}+1\right)\left(1+\frac{k}{1-\frac{c}{b}}\frac{1}{1-z}\right)\\ &+(-z)^{-k-\frac{c}{b}}\frac{\Gamma\left(k+1+\frac{c}{b}\right)\Gamma\left(-\frac{c}{b}\right)}{\Gamma(k)}.\end{aligned}$$

As a result, for $z = e^{-bx}\left(1-\frac{b\lambda}{a}\right)$ the corresponding approximation of $e(x)$, which contains also the multiplicative terms in the right-hand side of (13), is given by:

$$\begin{aligned}e(x) \approx \frac{\left(\frac{b\lambda}{a}e^{-bx}\right)^k e^{-cx}}{bk+c}&\Bigg[(1-z)^{-k}\left(\frac{bk}{c}+1\right)\\ &\times\left(1+\frac{k}{1-\frac{c}{b}}\frac{1}{1-z}\right)\\ &+(-z)^{-k-\frac{c}{b}}\frac{\Gamma\left(k+1+\frac{c}{b}\right)\Gamma\left(-\frac{c}{b}\right)}{\Gamma(k)}\Bigg].\end{aligned} \tag{A.6}$$

When a is close to 0, $(1-z) \approx (-z)$, and $1/(1-z) \approx 0$, we have

$$\begin{aligned}e(x, y) \approx \frac{\left[\left(1-\frac{a}{bk}\right)e^{\frac{c(y)}{k}x}\right]^{-k}}{bk+c(y)}&\Bigg[\left(\frac{bk}{c(y)}+1\right)\\ &+(-z)^{-\frac{c(y)}{b}}\frac{\Gamma\left(1+k+\frac{c(y)}{b}\right)\Gamma\left(-\frac{c(y)}{b}\right)}{\Gamma(k)}\Bigg].\end{aligned}$$

Appendix B

B.1. Gompertz–Makeham life expectancy

Theorem 2. *Life expectancy of the Gompertz–Makeham mortality distribution at age x is*

$$e_{GM}(x) = \frac{1}{b}e^{\frac{a}{b}}\left(\frac{a}{b}\right)^{\frac{c}{b}}\Gamma\left(-\frac{c}{b}, \frac{a}{b}e^{bx}\right) \quad a, b > 0, c \geq 0,$$

where $\Gamma(s, z) = \int_z^{\infty} t^{s-1}e^{-t}dt$ *denotes the upper incomplete gamma function.*

Proof. The individual force of mortality and the survival function for a Gompertz–Makeham homogeneous population is given by (1) and (2) for $z = 1$, i.e.

$$\mu_{GM}(x) = ae^{bx}+c \qquad s_{GM}(x) = \exp\left\{-\frac{a}{b}\left(e^{bx}-1\right)-cx\right\}.$$

Remaining life expectancy at age x can be calculated by

$$e_{GM}(x) = \int_x^{\infty} s_{GM}(t, y)\, dt. \tag{B.1}$$

Substituting $s = e^{bt}$ and denoting $\alpha = \frac{a}{b}$ reduces (B.1) to

$$e_{GM}(x) = \frac{e^{\alpha}}{b}\int_{e^{bx}}^{\infty} e^{-\alpha s}q^{-\frac{c}{b}-1}\, dq.$$

Finally we substitute $u = \alpha s$ and get

$$\begin{aligned}e_{GM}(x) &= \frac{e^{\alpha}}{b}[\alpha]^{\frac{c}{b}}\int_{\alpha e^{bx}}^{\infty} e^{-u}u^{-\frac{c}{b}-1}\, du\\ &= \frac{e^{\alpha}}{b}[\alpha]^{\frac{c}{b}}\Gamma\left(-\frac{c}{b}, \alpha e^{bx}\right),\end{aligned}$$

which is equivalent to

$$e_{GM}(x) = \frac{1}{b}e^{\frac{a}{b}}\frac{a}{b}^{\frac{c}{b}}\Gamma\left(-\frac{c}{b}, \frac{a}{b}e^{bx}\right). \quad \square$$

B.2. Approximation to the upper incomplete gamma function

The upper incomplete gamma function, $\Gamma(s, z)$ (Abramowitz and Stegun, 1965, 6.5.3) is

$$\Gamma(s, z) = \Gamma(s) - \gamma(s, z), \tag{B.2}$$

[2] As there is no standard notation for the upper incomplete Beta function, we used $\bar{B}_x(\alpha, \beta)$ denoting in this manner the complementary tail of the lower incomplete Beta function, $B_x(\alpha, \beta)$.

where $\Gamma(s)$ stands for the (complete) gamma function and $\gamma(s,z) = \int_0^z t^{s-1} e^{-t} dt$ denotes the lower incomplete gamma function.

The power series expansion of the gamma function (Abramowitz and Stegun, 1965, 6.1.33) is

$$\ln \Gamma(1+s) = -\ln(1+s) + s(1-\gamma) + \sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\frac{s^n}{n} \quad |s| < 2, \tag{B.3}$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta function, or by exponentiating and using the recurrence relation of the gamma function, $\Gamma(1+s) = s\Gamma(s)$ (Abramowitz and Stegun, 1965, 6.1.15):

$$\begin{aligned}\Gamma(s) &= \frac{1}{s}\exp\left\{-\ln(1+s) + s(1-\gamma) + \sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\frac{s^n}{n}\right\} \\ &= \frac{1}{s+s^2}\exp\left\{s(1-\gamma) + \sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\frac{s^n}{n}\right\}.\end{aligned} \tag{B.4}$$

The closer s to 0, the faster the speed of convergence of the series in (B.4). For $\frac{c}{b} = 0.1$, which is an extremely high value of the human force of mortality, the error made by using only the first term of the series (for $n = 2$) is lower than 0.001

$$\Gamma_{\text{exact}}(-0.1) = -10.68629$$
$$\Gamma_{\text{approx}}(-0.1) = -10.68555.$$

If $\frac{c}{b} \leq 0.0031$,[3] then the approximation error made by (B.3) without any summation terms is lower than 0.001.

$$\Gamma_{\text{exact}}(-0.003) = -333.9135$$
$$\Gamma_{\text{approx}}(-0.003) = -333.9126.$$

For the lower incomplete gamma function, we first take advantage of the series expansion of e^{-t} and then integrate:

$$\begin{aligned}\gamma(s,z) &= \int_0^z t^{s-1}e^{-t}\,dt = \int_0^z \sum_{k=0}^{\infty}(-1)^k\frac{t^k}{k!}t^{s-1}\,dt \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{z^{s+k}}{k!(s+k)}.\end{aligned} \tag{B.5}$$

Substituting (B.4) and (B.5) into (B.2) gives

$$\Gamma(s,z) = \frac{1}{s+s^2}\exp\left\{s(1-\gamma) + \sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\frac{s^n}{n}\right\} - \sum_{k=0}^{\infty}(-1)^k\frac{z^{s+k}}{k!(s+k)}.$$

[3] It is the typical case for all modern societies.

Appendix C. Poisson log-likelihood for count data

Life tables contain aggregated data, in which death counts $D(x)$ and exposures $E(x)$ at each age x (in the absence of explanatory variables) are the only sources of information about the mortality trajectories of (real or synthetic) cohorts. Demographers use a standard assumption (Brillinger, 1986) that death counts $D(x)$ are generated by a Poisson distribution with a rate parameter $\mu(x;\theta)\,E(x)$, i.e. $D(x) \sim \text{Poisson}(\mu(x;\theta)\,E(x))$, where $\mu(x;\theta)$ is the force of mortality at age x. As the density of Poisson(λ) equals

$$f(x) = \frac{\lambda^x}{x!}e^{-\lambda},$$

the log-likelihood of $D = (D_0, D_1, \ldots, D_\omega)^T$ to be maximized given exposures $E = (E(0), \ldots, E(\omega))^T$, where ω is the last attainable age, will be given by

$$\ell(\theta \mid D, E) = \sum_{x=0}^{\omega}[D(x)\ln\mu(x;\theta) + E(x)\mu(x;\theta)]\,. \tag{C.1}$$

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