The age separating early deaths from late deaths

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Abstract

There is one unique age separating early deaths from late deaths such that averting an early death decreases life disparity, but averting a late death increases inequality in lifespans.

1. The relationship

We measure life disparity by life expectancy lost due to death, $e^{\dagger} = \int_0^{\omega} e(x,t)f(x,t)dx$,

where $e(a,t) = \dfrac{\int_a^{\omega} l(x,t)dx}{l(a,t)}$ is remaining life expectancy at age a and time t,

$l(a,t) = \exp(-\int_0^a \mu(x,t)dx)$ gives the probability of survival to age a and $\mu(a,t)$

denotes the age-specific hazard of death. The life table distribution of deaths is given by $f(a,t) = l(a,t)\mu(a,t)$. Maximum lifespan is denoted by ω.

Consider the increase in $e^{\dagger}$ due to reductions in mortality,

$$(1) \qquad g(a,t) = \frac{de^{\dagger}(t)}{-d\mu(a,t)/\mu(a,t)} = f(a,t)k(a,t),$$

where

$$(2) \qquad k(a,t) = e^{\dagger}(a,t) - e(a,t)(1 - H(a,t)),$$

where $H(a,t) = \int_0^a \mu(x,t)dx$ is the cumulative hazard function and

$e^{\dagger}(a,t) = \dfrac{\int_a^{\omega} e(x,t)f(x,t)dx}{l(a,t)}$ is life expectancy lost due to death among people

surviving to age a. The function $g(a,t)$ measures how much $e^{\dagger}$ will be increased by a proportional reduction in mortality at age a and time t. Because f is always positive, if k is negative, then the change decreases life disparity; if k is positive, then the change increases life disparity. If k is negative at younger ages and positive at older ages, then there is some age $a^{\dagger}$ at which k equals zero. This is the age that separates early deaths from late deaths. We prove below that $a^{\dagger}$ exists under conditions that generally characterize modern human populations. Furthermore we prove that if $a^{\dagger}$ exists then there is one and only one age, $a^{\dagger}$, at which k equals zero.

2. Proof

For notational simplicity, the time subscript t can be dropped without confusion. Let

(3) $\qquad k(a) = e^{\dagger}(a) - e(a)(1 - H(a))$,

so that

(4) $\qquad k(0) = e^{\dagger}(0) - e(0)$.

We consider three cases.

Case 1: $k(0) < 0$.

At advanced ages, as $a \to \omega$, both $e^{\dagger}(a)$ and $e(a)$ approach 0, but $H(a)$ approaches $+\infty$, and thus $k(\omega) > 0$. The function $k(a)$ is continuous on $[0, \omega]$. According to the intermediate value theorem, there exists at least one point, say $a^{\dagger}$, in $[0, \omega]$ such that $k(a^{\dagger}) = 0$.

It is readily shown that there is only one $a^{\dagger}$ in $[0, \omega]$ such that $k(a^{\dagger}) = 0$. If there were more than one point at which $k(a)$ equals zero, then the derivative of $k(a)$ at some of these points would be positive and at others negative, because the continuous function $k(a)$ must go up and down to cross zero more than once. If the derivative of $k(a)$ is always positive when $k(a) = 0$, then there is only one point at which $k(a)$ crosses zero.

The derivative of (Eq. 3) is given by

$$\frac{dk(a)}{da} = \frac{de^{\dagger}(a)}{da} - (1 - H(a))\frac{de(a)}{da} - e(a)\frac{d(1 - H(a))}{da}$$

$$= -\mu(a)e(a) + \mu(a)e^{\dagger}(a) - (1 - H(a))(-1 + \mu(a)e(a)) + e(a)\mu(a)$$

(5) $\qquad = \mu(a)e^{\dagger}(a) + (1 - H(a))(1 - \mu(a)e(a)).$

When $k(a) = 0$ it follows from (3) that $1 - H(a) = \dfrac{e^{\dagger}(a)}{e(a)}$. Substituting this into (5) yields

(6) $\qquad \left.\dfrac{dk(a)}{da}\right|_{a=a^{\dagger}} = \mu(a)e^{\dagger}(a) + \dfrac{e^{\dagger}(a)}{e(a)}(1 - \mu(a)e(a)) = \dfrac{e^{\dagger}(a)}{e(a)} > 0$.

Case 2: $k(0) = 0$.

In this case, $a^{\dagger} = 0$. We need to show that this is the only value of $a^{\dagger}$, i.e., the only age when $k=0$. It follows from (5) that if $k(0) = 0$ then

$$\left.\frac{dk(a)}{da}\right|_{a=0} = 1 .$$

Hence $k(a)$ becomes positive as age increases from zero. If there were an age above zero when $k(a) = 0$, then the derivative of k at this age would have to be zero or negative. But as shown in (6), the derivative has to be positive at any age when $k(a) = 0$. This contradiction implies that the value of $a^{\dagger} = 0$ is unique in the case when $k(0) = 0$.

Case 3: $k(0) > 0$.

As noted above in Case 2, if there were an age when $k(a) = 0$ then the derivative of k at this age would have to be zero or negative. But as shown in (6), the derivative has to be positive at any age when $k(a) = 0$. This contradiction implies that there is no age that separates early from late deaths when $k(0) > 0$: averting a death at

any age would increase life disparity. Hence in this case it is convenient to set $a^\dagger$ equal to zero by definition. Q.E.D.

We have computed the value of $k(0)$ for all 5830 life tables since 1840 in the Human Mortality Database (2008), the life tables used in this article. We have also computed the value of $k(0)$ for the 3404 life tables in the Human Life-Table Database (2008). In every case $k(0) < 0$. The closest approach to zero was found for females in 1911-1921 in India: for this population $e(0) = 23.33$, $e^\dagger = 23.08$, so that $k(0) = -0.22$. Goldman and Lord (1986), however, provide two examples of life tables for which $k(0)$ is positive. Both pertain to selected populations in rural areas of China in the period 1929-31. One is for females (Barclay et al., 1976) and the other is for males (Coale and Demeny, 1983). For the Chinese women $e(0) = 21.00$ and $e^\dagger = 21.73$. For the Chinese males, $e(0) = 17.43$ and $e^\dagger = 22.17$.

3. History and Related Results

Following the notion of life table entropy H (Keyfitz, 1977), Vaupel (1986) derived the mathematical expression for $e^\dagger$, and showed that $H = e^\dagger / e^0$, which facilitates understanding of why H measures the percentage increase in life expectancy generated by a decrease in the mortality rate of one percent. Furthermore, the increase in life expectancy is given by the product of life disparity and the rate of progress in reducing age-specific death rates (Vaupel and Canudas-Romo, 2003).

Vaupel (1986) found that, if the force of mortality follows a Gompertz curve, say $\mu(a,t) = \mu(0,t)e^{ba}$, then $e^\dagger \approx 1/b$, where b was traditionally interpreted as the rate of aging. This, offering an alternative interpretation for b, suggests that, the less lifespan disparity is, the faster the population age.

In addition to $e^\dagger$, several other measures of the life disparity in a lifetable have been proposed (Cheung et al., 2005). These include the variance in the age at death, the standard deviation, the standard deviation above age 10 (Edwards and Tuljapurkar, 2005), the inter-quartile range (Wilmoth and Horiuchi, 1999), the Gini coefficient (Shkolnikov et al., 2003), and the entropy of the lifetable (Keyfitz, 1977). These measures are highly correlated with each other. In particular, the correlation of $e^\dagger$ with the other measures never falls below 0.964, according to our calculations based on 2915 period life tables from 1840-2007 available from the Human Mortality Database (2008). Hence $e^\dagger$ can be viewed as a surrogate for the other measures. We prefer $e^\dagger$ because of its desirable mathematical properties, used above, and because it can be readily explained and interpreted.

4. Applications

At the threshold age $a^\dagger$, the change in $e^\dagger$ resulted from mortality decline can be decomposed into two components

$$(7) \qquad \dot{e}^\dagger(t) = \frac{de^\dagger(t)}{dt} = \int_0^{a^\dagger(t)} g(a,t)\rho(a,t)da + \int_{a^\dagger(t)}^{\omega} g(a,t)\rho(a,t)da,$$

where $\rho(a,t)$ is the rate of progress in reducing death rates. The first term in the right side of (7) represents the compression of mortality at younger ages, and the second term the expansion of mortality at older ages. The balance of the two components

determine whether the whole population experiences mortality compression or expansion.

Analogously, the increase in life expectancy at birth $\dot{e}(0,t)$ can be broken into two parts,

$$(8) \qquad \dot{e}(0,t) = \int_0^{a^\dagger} e(x,t)f(x,t)\rho(x,t)dx + \int_{a^\dagger}^{\omega} e(x,t)f(x,t)\rho(x,t)dx,$$

where the first term captures the contribution of averting early deaths to increases in $e(0)$, while the second that of decreasing mortality among the elderly or very elderly. In a recent study, Vaupel et al. (2008) showed that the countries benefiting from the longest life expectancies are those that have succeeded in reducing disparities in how long individuals live by averting early deaths.

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