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A NEW LOOK AT ENTROPY AND THE LIFE TABLE

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Demographers borrow ideas and tools from statisticians and mathematicians in attempts to understand relationships among demographic parameters. One such idea is that of *entropy*, or the related idea of information, which is used primarily in the field of statistical mechanics. During the past decade, the concept of entropy has been adapted by several demographers as a way to measure the dispersion of the net maternity function and the convexity of the life table. The advantages of this type of measure is that it can help us to assess the impact of changes in demographic behavior on vital rates, specifically, the effect of variations in fertility on the intrinsic rate of growth (Demetrius 1979; Demetrius and Ziehe 1984), of changes in mortality on life expectancy (Keyfitz 1977, Demetrius 1979) and of changes in marriage dissolution rates on the average duration of marriage (Goldman 1984).

In this paper we focus on applications of entropy to life tables or survivorship curves. After presenting a brief derivation of entropy, to be denoted by H, we indicate an error in the characterization of H found in the earlier papers noted above. Specifically, whereas previous research suggests that H ranges between zero and one inclusive, we show that, in theory, H can assume any nonnegative value; we even find empirical life tables for which H exceeds unity. In addition, we obtain a new mathematical expression for H, as applied to life tables, one that leads directly to an intuitive interpretation of entropy and makes apparent the conditions under which H exceeds unity.

DERIVATION OF H

Keyfitz (1977:62–72) has shown how the logarithm of the survivorship curve [ln $l(x)$] contains useful information about the effects of small changes in death rates on life expectancies. Consider the overall death rate at age x, $\mu(x)$, and suppose that it changes at all ages by 100δ percent. Then, the new death rate, $\mu^*(x) = \mu(x)(1 + \delta)$, gives rise to a new survivorship curve ($l_0 = 1$),

$$l_x^* = e^{-\int_0^x \mu(a)(1+\delta)da} = l_x^{1+\delta} , \tag{1}$$

and a new life expectancy,

$$e_0^* = \int_0^{\omega} l_a^{1+\delta} da , \tag{2}$$

where ω is the oldest age of life. To determine the effect of δ on e_0^*, we consider the derivative of e_0^* with respect to δ:

$$\frac{de_0^*}{d\delta} = \int_0^{\omega} [\ln l_a]\, l_a^{1+\delta} da . \tag{3}$$

For values of δ near zero, we have

$$\frac{de_0^*}{d\delta} \approx \int_0^\omega [\ln l_a] l_a da \tag{4}$$

or

$$\frac{\Delta e_0}{e_0} \approx \left[\frac{\int_0^\omega (\ln l_a)\, l_a da}{\int_0^\omega l_a da}\right] \delta \, . \tag{5}$$

Defining H as minus the expression in brackets (so as to make H positive),

$$H = \frac{-\int_0^\omega [\ln l_a] l_a da}{\int_0^\omega l_a da}, \tag{6}$$

we have

$$\frac{\Delta e_0}{e_0} \approx -H\delta \, . \tag{7}$$

The definition of entropy in (6) differs from that associated with net maternity functions (Demetrius 1979) and from that applied to general probability density functions (see, for example, Theil 1972).

Equation (7) tells us that for a small proportional reduction δ in the death rate at all ages, the proportional increase in life expectancy can be simply approximated as H times δ. The value of H depends directly on the concavity of the survivorship curve: if all people die in a narrow range of ages before ω ($\omega - \varepsilon$ to ω), then l_x equals unity and $\ln l_x$ equals zero for all ages below $\omega - \varepsilon$, and H is close to zero. Hence, a change in $\mu(x)$ has almost no effect on life expectancy. If the risk of dying is constant at all ages, the survivorship curve declines exponentially, H equals unity (as integration in equation (6) will show) and a change δ in the death rates is reflected by an equal proportionate change in e_0. For a life table that declines linearly with age, the value of H lies halfway between those described above, i.e., $H = 1/2$.

Theoretically, values of H need not bear any particular relationship to life expectancy. This can be seen by noting that H equals unity for all exponential survivorship curves, $l_x = e^{-\lambda x}$, regardless of the value of λ. Nevertheless, empirically derived correlations between e_0 and H are quite large (Keyfitz 1977). For example, as life expectancy improved in the United States from a value of 55 years in 1940 to 74 years in 1981, the value of H declined from 0.37 to 0.16. For recent mortality experiences in developed countries, H values appear to be below 0.2; in the future, uniform proportionate reductions in mortality will result in very modest increases in life expectancy.

FINDING BOUNDS FOR *H*

In previous research, the exponential survivorship curve is presented as one with maximum entropy. That is, it has been suggested that a constant proportionate mortality change has the largest impact on life expectancy when the risk of mortality, and, hence, the life expectancy, is the same at all ages. This, in fact, is not the case.

An examination of the definition of H in equation (6) shows that H can be viewed as a weighted average of $\ln l_a$, the weight being equal to the normalized survivorship curve. We note that as l_a declines from one to zero over the age span, $-\ln l_a$ increases from zero to infinitely large values. (We assume that the radix of the life table equals unity, i.e., $l_0 = 1$.) Therefore, there is no apparent mathematical explanation of why H should be bounded from above.

Figure 1 presents a life table in which the proportion $(1 - \kappa)$ of births die at age κ and the remaining proportion κ of survivors live until ω when everyone dies. The quantity H for this life table can be evaluated by dividing $l(x)$ into its two components as follows:

$$H = -\left[\frac{\int_0^\kappa (\ln 1)dx + \int_\kappa^\omega \kappa(\ln \kappa)dx}{\int_0^\kappa 1\, dx + \int_\kappa^\omega \kappa\, dx} \right], \tag{8}$$

which reduces to

$$H = \frac{-(\omega - \kappa)\ln \kappa}{1 + \omega - \kappa}. \tag{9}$$

As κ approaches zero, the denominator in (9) approaches $1 + \omega$; the numerator, on the other hand, becomes infinitely large. Hence, by choosing a sufficiently small value of κ, we can obtain a value of H as large as desired. Later we indicate that we can achieve the same result with a family of continuous and differentiable survivorship curves.[1]

RE-EXPRESSING H

A re-expression of the numerator of H gives us a new interpretation of entropy that is much more satisfying than the notion of a weighted average of $\ln l_a$. This re-expression, which is derived below, also enables us to understand the conditions under which entropy exceeds unity—i.e., life tables for which reductions in mortality of a fixed percent can result in even larger improvements in overall life expectancy.

We denote the numerator of H by N,

$$N = -\int_0^\omega l_a \ln l_a\, da, \quad l_0 = 1. \tag{10}$$

Integration by parts (with $u = \ln l_a$, $dv = -l_x dx$, and $v = \int_x^\omega l_y dy$) yields

$$N = \left[\lim_{x \to \omega} (\ln l_x) \int_x^\omega l_y dy \right] - \ln l_0 \int_0^\omega l_y dy - \int_0^\omega \frac{l'_x}{l_x} \left[\int_x^\omega l_y dy \right] dx, \tag{11}$$

where l'_x is the derivative of l_x with respect to x. The second term is zero since $\ln l_0 = 0$; evaluation of the first term is more difficult because it is the product of a term that becomes infinitely large and one that approaches zero. Since l_x is a nonincreasing positive function, the first term is always less than or equal to

$$\lim_{x \to \omega} (\ln l_x) \cdot l_x \cdot (\omega - x). \tag{12}$$

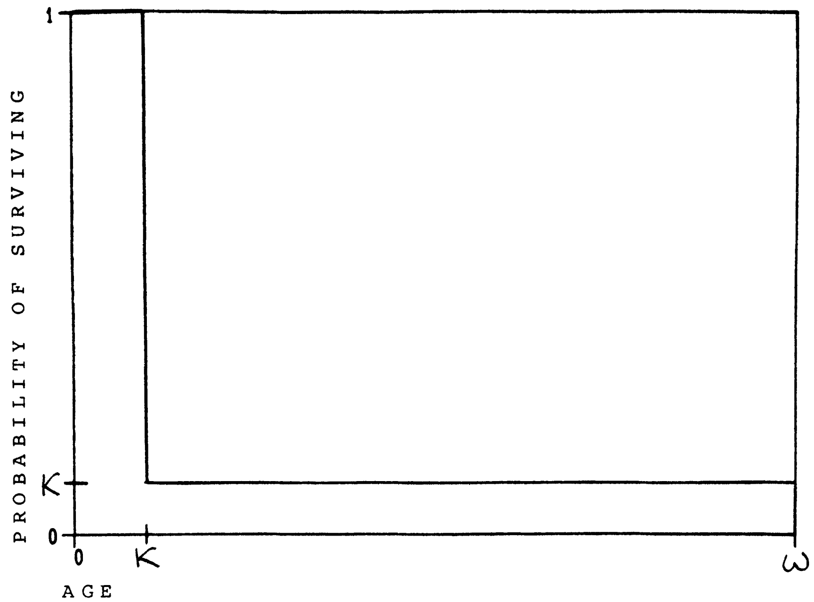

Figure 1.—Hypothetical survivorship curve with (1-κ) percent of the population dying at exact age κ

Applying L'Hôpital's rule to the product ($l_x \cdot \ln l_x$), we find that ($l_x \cdot \ln l_x$) approaches zero as l_x approaches zero (see, for example, Buck 1965:93). Hence the entire first term approaches zero and

$$N = -\int_0^\omega \frac{l'_x}{l_x} \left[\int_x^\omega l_y dy \right] dx. \tag{13}$$

We convert this expression into common demographic language by noting that $-l'_x$ is d_x where $d_x\, dx$ is the number of deaths between ages x and $x + dx$ (or when $l_0 =$ 1, the probability of dying between these ages) and that $\int_x^\omega l_y dy\ / l_x$ is e_x, the expectation of life at age x. Consequently, the quantity H equals

$$H = \frac{\int_0^\omega d_x e_x dx}{\int_0^\omega l_x dx} = \frac{\int_0^\omega d_x e_x dx}{e_0}. \tag{14}$$

Since $\int_0^\omega d_x = 1$ (i.e., $l_0 = 1$), the numerator of H can be viewed as a weighted average of life expectancies at age x. In fact, it denotes the average years of future life that are lost by the observed deaths.[2] (One can also view the numerator as the average number of years a person could expect to live, given a second chance on life.) H will be greater than unity whenever the average years lost by deaths exceed life expectancy at birth.

LIFE TABLES FOR WHICH H EXCEEDS UNITY

It is apparent from (14) that if life expectancy at age x is a monotonically decreasing function, H is less than unity. Similarly, if e_x is constrained to be nonincreasing, the maximum entropy value will be unity. As shown earlier, one type of life table for which H equals unity is an exponential survivorship function. This is obvious from (14), since if $l_x = e^{-\lambda x}$, $e_x = 1/\lambda$ for all x, and the numerator and denominator of H are equal. On the other hand, if e_x is a monotonically increasing function, H will always exceed unity. An example of such a function is $l_x = 1/(1+x)^2$, $x \geq 0$. Integration of l_x from x to ∞ yields an e_x value of $(1+x)$—each additional year that a person survives, he or she can expect to live yet one more year. Evaluation of either (14) or (6) yields a value for H of two. According to this life table, a given improvement in mortality (proportionately equal at all ages) translates into twice as large an increase in life expectancy. This is not a surprising result given the nature of the e_x function. Figure 2 shows this survivorship curve along with several life tables for which H ranges between zero and one.

Consider the family of survivorship curves, $l_x = 1/(1+x)^\alpha$ $\alpha > 1$, $x \geq 0$, for which the special case $\alpha = 2$ is described above. For $\alpha > 1$, e_x is an increasing function—$e_x = (1 + x)/(\alpha-1)$—and H is, therefore, larger than unity. It can be shown that H is equal to $\alpha/(\alpha-1)$. Hence, we can obtain a life table with as large an H value as desired by choosing a value of α very close to unity. Note that when $\alpha = 1$, both life expectancy and H are undefined.

In actual populations, we never obtain continuously increasing e_x curves. Nor, for that matter, do we find continuously decreasing ones. Most frequently, e_x rises during the early years of life when individuals are subject to the high mortality risks of infancy and childhood. In modern developed countries where life expectancies at birth exceed 70 years, e_x rises for only the first year or for an even shorter period. When life expectancy at birth is extremely low, i.e., about 20 to 25 years, e_x rises or remains relatively constant for as long as ten years (see Coale and Demeny 1983). For the latter type of life table in which infant mortality removes about one-third to one-half of births in the first few years of life, the quantities denoting future years lost by deaths at age x ($d_x e_x$) are greater than the number of survivors at age x (l_x) for small values of x (young ages). Although the inequality reverses at older ages, the contribution of the first few values is sufficient for H to exceed unity.

Two such life tables are those estimated for selected rural areas of China during the period 1929–1931 (South model life tables, females; Barclay et al. 1976) and the first level of the East model life tables for males presented in Coale and Demeny (1983). These two survivorship curves and the corresponding H values, estimated numerically to be 1.07 and 1.32, respectively, are shown in figure 3. These life tables are contrasted with one for the U.S. population in 1980. The life expectancy (e_x) curve for the model life table is drawn in figure 4. We note the extreme conditions under which H values exceed unity: infant mortality rates of about 350 and 500 deaths per 1000 births in China and in the model schedule, respectively, and large increases in life expectancy during the first few years of life—e.g., from about 17 to 40 during the first five years in the model life table. We observe that although the exponential survivorship curve (figure 2) and the life table for rural China (figure 3) have almost the same estimate of H, this value is attained in very different ways. In the former case, values of $d_x e_x$ are equal to those of l_x at all ages; in the latter case, values of $d_x e_x$ are much higher than l_x in the first few years of life and lower than l_x in all remaining ages.

Although there are few empirical life tables of national populations for which H is greater than one, if we use the l_x function to describe the survivorship experience of select cohorts or of nonhuman populations, there are probably many such life tables. For example, survivorship after difficult surgical operations such as heart transplants would be described by life tables similar in shape to those in figure 3. Life tables for some animal populations such as those of the oyster, British robin, and song sparrow

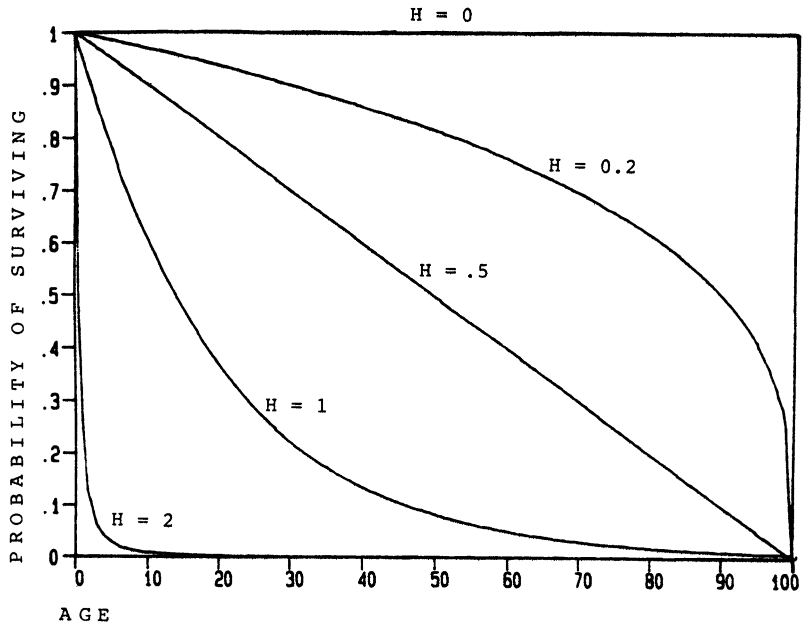

Figure 2.—Hypothetical survivorship curves (l_x) and corresponding H values

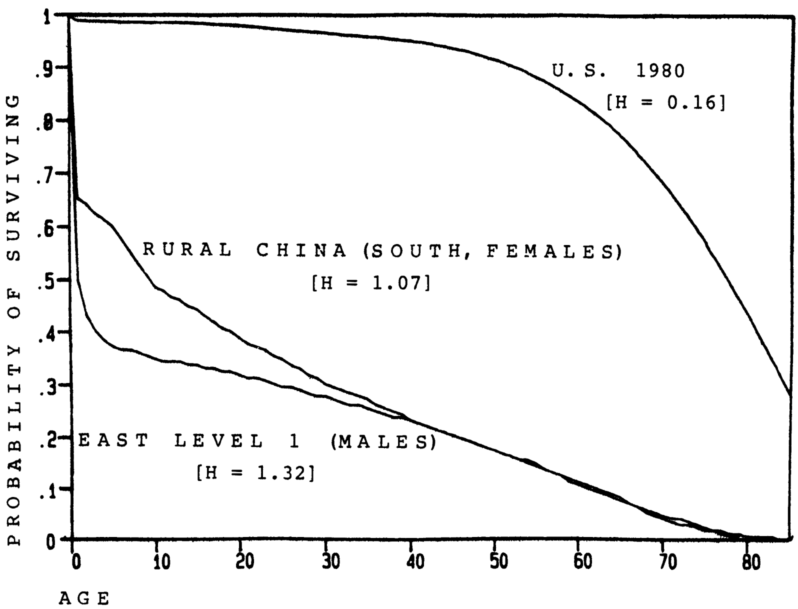

Figure 3.—Probabilities of surviving l_x and corresponding H values

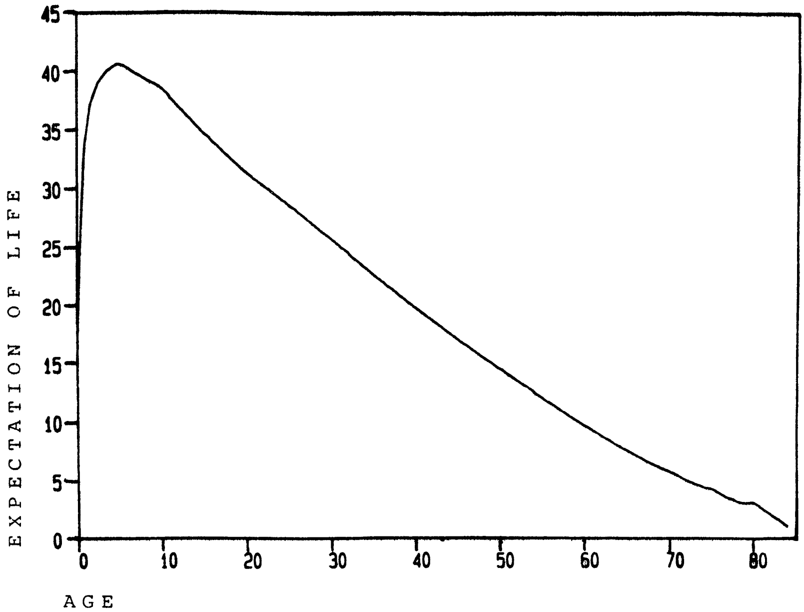

Figure 4.—Expectation of life at successive ages (e_x), East model life table

(Deevey 1947; 1950) most likely have H values greater than unity. The application of H to describe life tables of marriage dissolution indicates that the high risks of divorce at early marriage durations results in H values as large as 0.7 for U.S. data on marriage dissolution in the late 1970s (Goldman 1984). Since divorces are rare in the first year of marriage because of various legal restrictions, life tables for separation of spouses or partners might be characterized by H values near or above unity in countries with high marital instability.

CONCLUSIONS

The mathematical derivations presented above have afforded us a new look at the entropy of the life table. Rather than being expressed as a weighted average of logarithms of the survivorship curve, H can be expressed as a weighted average of life expectancy at different ages relative to life expectancy at birth. For most observed mortality experiences, life expectancy rises only modestly for the first year or several years of life and declines thereafter. The expected number of years of life lost by deaths in such populations falls far below life expectancy at birth, and H values are considerably below unity. Improvements in mortality of a certain percentage δ at all ages result in much smaller than a δ percentage increase in life expectancy.

On the other hand, it is clearly theoretically possible, and has been observed empirically, that life tables may have entropy values above one. Such life tables are characterized by extremely high death rates in the young ages. Individuals who survive the first year or first few years of life can expect to live many more years than can newborns. Hence, it is not surprising that, for these populations, reductions in mortality by a fixed factor at all ages can result in even larger gains in life expectancy.

NOTES

[1] In response to an earlier version of this manuscript, McClean (1985) demonstrated that H never exceeds $\ln(\omega/e_0)$, where ω is the oldest age of life. More recently, Demetrius (1985) demonstrated that H never exceeds the quantity $(1+\ln \mu/e_0)$, where μ is the mean age of the stationary population (i.e., $\mu = \int_0^\infty x l_x dx/e_0$). Hence, if the mean age of the stationary population is less than the value of life expectancy, H will be bounded above by unity.

[2] Keyfitz (1977: 73) derives a quantity *Dep* which denotes the average deprivation of life for persons aged x and is described by an expression similar to that for N in (13).

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REFERENCES

Barclay, G. W., A. J. Coale, M. A. Stoto, and T. J. Trussell. 1976. A reassessment of the demography of traditional rural China. *Population Index* 42:606–635.

Buck, R. C. 1965. *Advanced Calculus*. Second Edition. New York: McGraw-Hill Book Company.

Coale, A. J., and P. Demeny. 1983. *Regional Model Life Tables and Stable Populations*. Second Edition. New York.

Deevey, E. S. 1947. Life tables for natural populations of animals. *Quarterly Review of Biology* 22:283–314.

———. 1950. The probability of death. *Scientific American* 182:58–60.

Demetrius, L. 1979. Relations between demographic parameters. *Demography* 16:329–338.

Demetrius, L., and M. Ziehe. 1984. The measurement of Darwinian fitness in human populations. *Proceedings of the Royal Society of London B 222:* 33–50.

———. 1985. Personal communication.

Goldman, N. 1984. Changes in widowhood and divorce and expected durations of marriage. *Demography* 21:297–307.

Keyfitz, N. 1977. *Applied Mathematical Demography*. New York: John Wiley.

McClean, S. 1985. Personal communication.

Theil, H. 1972. *Statistical Decomposition Analysis*. Amsterdam: North-Holland Publishing Company.