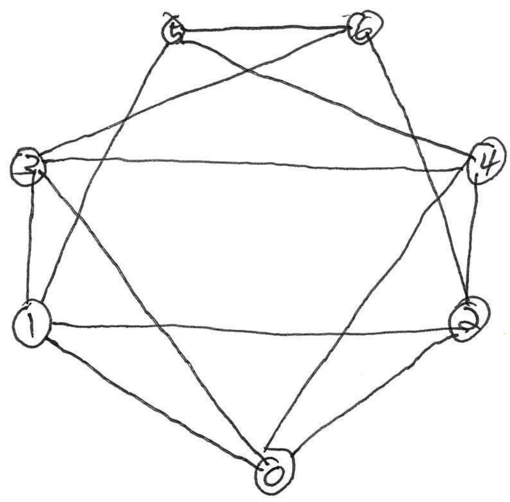
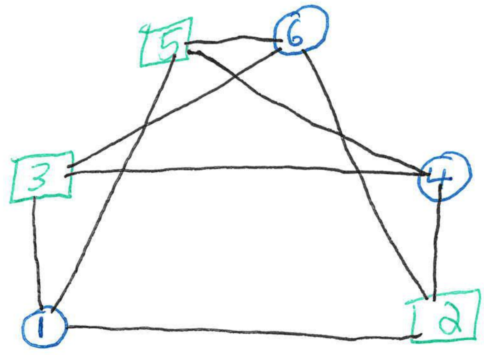
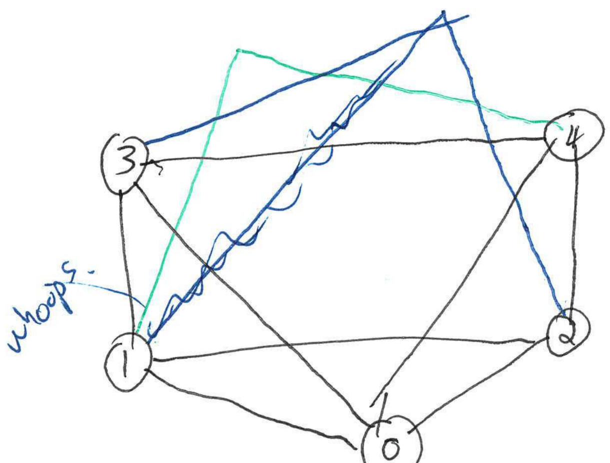
Question 1 Using Kuratowski's theorem, prove the following graph is non-planar





Deleting vertex 0 shows that $K_{3,3}$ is an honest subgraph of G with 1, 4, 6 coloured blue, and 2, 3, 5 coloured green.

Another method is to find a subgraph homeomorphic to K_5 using the vertices 0-4. The only edges of K_5 missing in G are 1-4, which we can get passing through 5 (in green), and 3-2 which we can get passing through 6 (in blue)

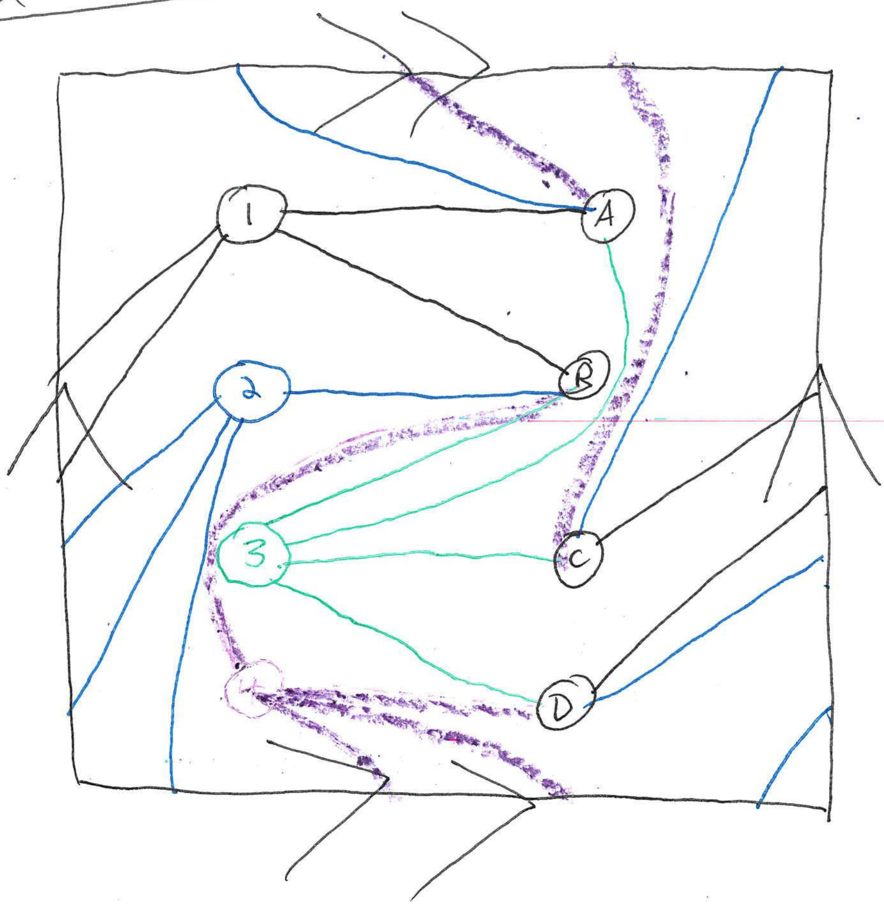


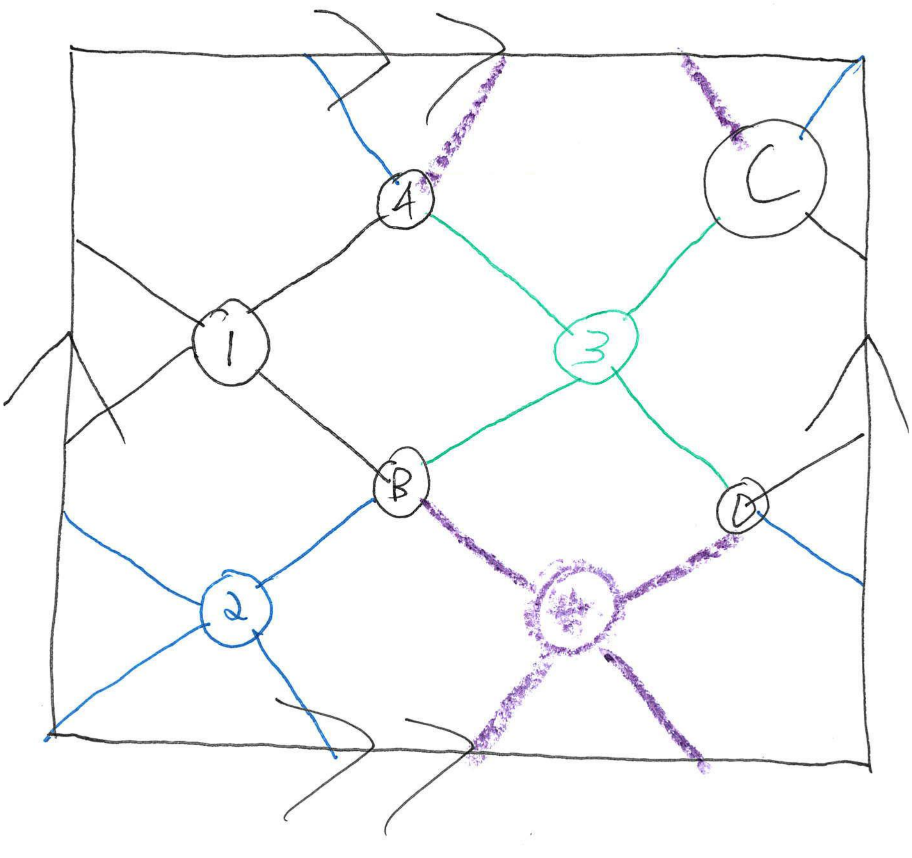
Question 2

Two drawings, one ugly, one pretty.

"Red" vertices are 1-4, "Blue" vertices are A, B, C, D.

Edges coloured according to which of 1-4 they map to.





Question 3

~~Theo~~ The genus $g(\Gamma)$ is the minimum number m so that Γ can be drawn on a surface of genus m without any edges crossing, but not on a surface of genus k for any $k<m$.

Prove that $g(K_n) \geq \frac{(n-3)(n-4)}{12}$

Proof We need to prove that if we can draw K_n on a surface of genus g, then $g \geq \frac{(n-3)(n-4)}{12}$

We have two tools: Euler characteristic and handshaking.

Suppose that we have K_n drawn on an orientable surface S. We'd like to us K_n to calculate the Euler characteristic $\chi(S)$, but to do so requires that each component of $S \setminus K_n$ is a disk.

I was ok with you assuming this. In any case, if some of the "faces" of $S \setminus K_n$ aren't disks, we can just replace them with disks and only lower the genus.

~~In any ca~~

So, assume K_n is drawn on S, a surface of genus g, with f faces, each a disk.

On the one hand, we have:

$$2-2g = \chi(S) = v-e+f.$$

K_n has n vertices, so $v=n$

K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, so $e = \frac{n(n-1)}{2}$, and so we have

$$(*).\quad 2-2g = n - \frac{n(n-1)}{2} + f.$$

On the other hand, we can relate f and n via handshaking between edges and faces.

$$\sum_{f \text{ a face}} d(f) = 2e = 2\frac{n(n-1)}{2} = n(n-1)$$

But K_n is a simple graph, so each face has at least 3 sides: $d(f) \geq 3$. So

$$n(n-1) = \sum_f d(f) \geq \sum_f 3 = 3f$$

So $f \leq \frac{n(n-1)}{3}$

Substituting this with $(*)$ gives

$$2-2g = n - \frac{n(n-1)}{2} + f \leq n - \frac{n(n-1)}{2} + \frac{n(n-1)}{3}$$

Solving for g gives

$$g \geq 1 - \frac{n}{2} + \frac{n(n-1)}{4} - \frac{n(n-1)}{6} = \frac{12 - 6n + 3n(n-1) - 2n(n-1)}{12}$$

$$= \frac{12 - 6n + 3n^2 - 3n - 2n^2 + 2n}{12} = \frac{n^2 - 7n + 12}{12}.$$

$$= \frac{(n-3)(n-4)}{12}$$ as desired.