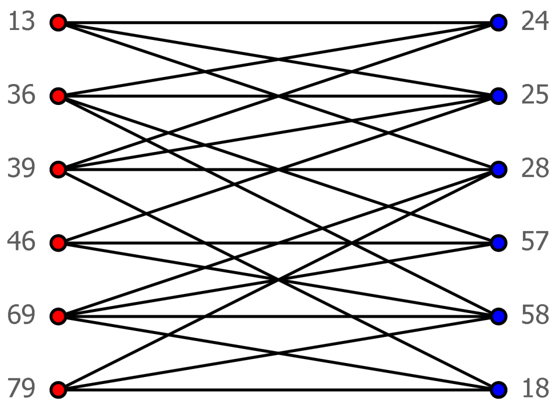
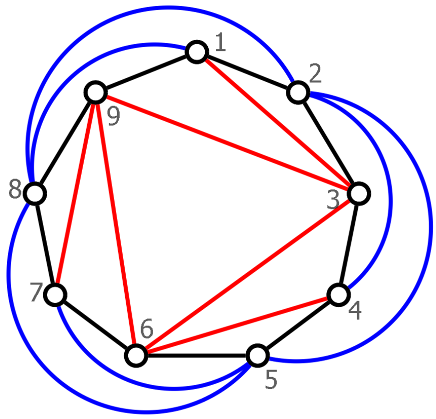
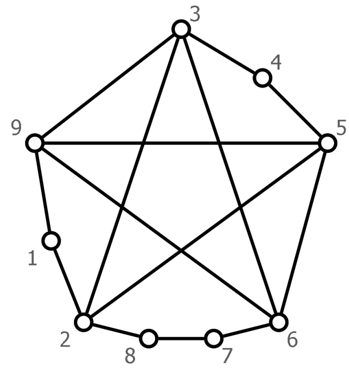
QUESTION 28 Draw the incompatibility graph for the Hamilton cycle 1234567891. This is bipartite as shown, so the graph is planar. Draw with the red edges inside and the blue edges outside.





To make the graph non-planar, we need to add an edge incompatible with at least one blue edge and at least one red edge. For example, 59 is incompatible with 36 (red) and 28 (blue). When 59 is added, the graph contains a subdivision of K_5 as shown. (In fact adding any new edge would work; the original graph has $21 = 3 \times 9 - 6$ edges, which is the most a simple planar graph can have. The subdivision you get will depend on which edge you add; often it will be a subdivision of $K_{3,3}$ instead.)



QUESTION 29 Since each vertex has degree at least 3, the degree sum is at least $3v$, so $2e \geqslant 3v$ and consequently

$$v \leqslant \tfrac{2e}{3}\,. \tag{1}$$

Suppose there are t faces of degree 3 and d of degree 10. By face-handshaking

$$2e = 3t + 10d\,. \tag{2}$$

Euler's formula gives

$$\begin{aligned} e &= f + v - 2 \\ &= v + t - d + 2 \\ &\leqslant \tfrac{2e}{3} + t + d - 2 \qquad \text{using (1)} \\ \therefore\quad \tfrac{e}{3} &\leqslant t + d - 2 \\ \therefore\quad e &\leqslant 6t + 6d - 12\,. \end{aligned}$$

Combining this with (2), we get

$$\begin{aligned} 3t + 10d &\leqslant 6t + 6d - 12 \\ \therefore\quad 3t &\geqslant 4d + 12 \\ \therefore\quad t &\geqslant \tfrac{4d}{3} + 4 > d\,. \end{aligned}$$

If every vertex has degree 3 then we have equality in (1) and hence at every stage. So $3t = 4d+12$. Since $4d + 12$ is a multiple of 4, so is $3t$ and so t must be a multiple of 4. Likewise $4d = 3t - 12$ and $3t - 12$ is a multiple of 3, so $4d$ and d must be multiples of 3.

A dodecahedron has 12 faces of degree 5 and 20 vertices. If we start from a solid dodecahedron and slice off a small piece at each vertex, we create 20 new faces of degree 3 and the original faces now have degree 10. So $t = 20$ and $d = 12$.

QUESTION 30 By handshaking, $2e = d_1 v$ so $v = \frac{2e}{d_1}$. Similarly, using face-handshaking, $f = \frac{2e}{d_2}$. By Euler's formula,

$$\begin{aligned} e + 2 &= v + f \\ &= 2e\left(\frac{1}{d_1} + \frac{1}{d_2}\right) \end{aligned}$$

so

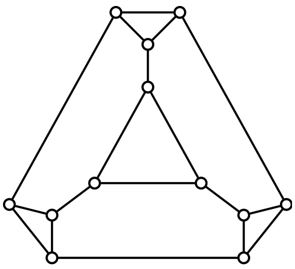
$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{e+2}{2e} > \frac{1}{2}\,.$$

If $d_1, d_2 \geqslant 4$ then we would have $\frac{1}{d_1} + \frac{1}{d_2} \leqslant \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, a contradiction. So one of them must be 3, and then the other must be less than 6, since $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. The possibilities are:

d_1	3	3	3	4	5
d_2	3	4	5	3	3
example	tetrahedron	cube	dodecahedron	octahedron	icosahedron

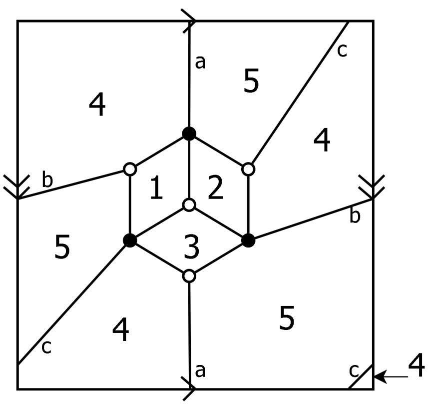
QUESTION 31 Suppose there are t faces of degree 3 and h of degree 6. Since it is 3-regular, handshaking gives $2e = 3v$ so $v = 2e/3$.
Euler's formula gives

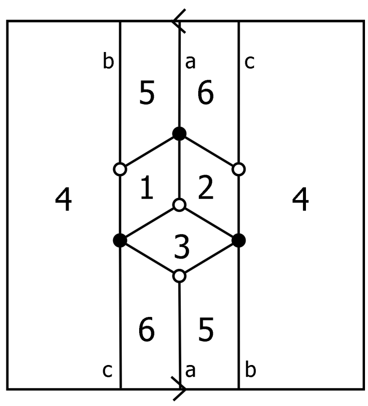
$$\begin{aligned} e &= v + f - 2 \\ &= \tfrac{2e}{3} + t + h \\ \therefore \quad \tfrac{e}{3} &= t + h - 2 \\ \therefore \quad 2e &= 6t + 6h - 12\,. \end{aligned}$$



However, face-handshaking gives $2e = 3t + 6h$, so we must have $3t + 6h = 6t + 6h - 12$, ie $3t = 12$, so there are 4 faces of degree 3. An example is the truncated tetrahedron shown above: $t = h = 4$.

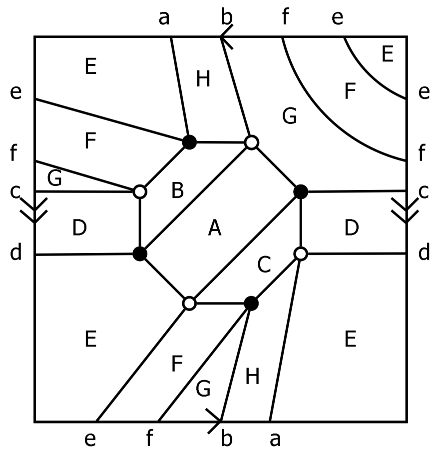
QUESTION 32 We may draw $K_{3,4}$ on the torus and Möbius strip as shown. The faces are numbered.



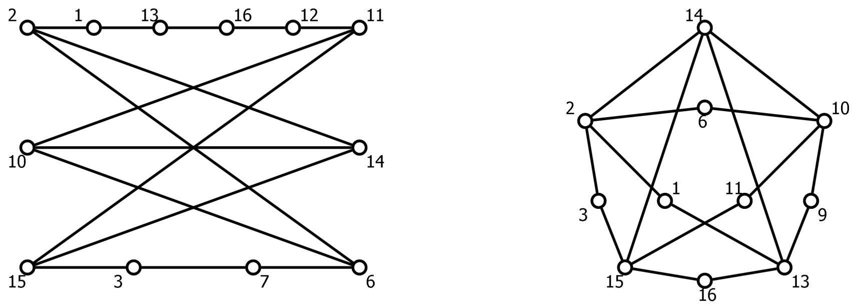


In each case $v = 7$ and $e = 12$. For the torus, $f = 5$ and so $v + f - e = 0$; for the Möbius strip $f = 6$ and so $v + f - e = 1$.

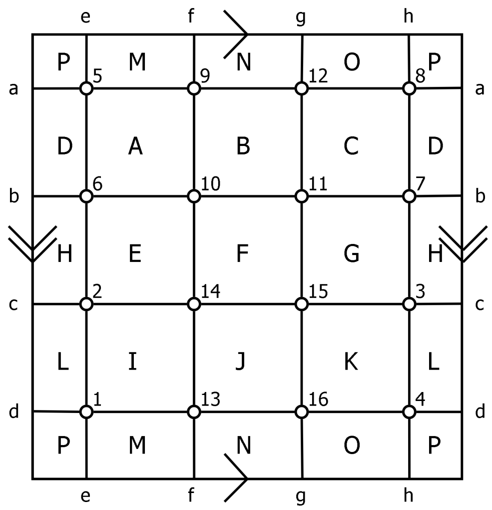
QUESTION 33 We may draw $K_{4,4}$ on the Klein bottle as shown; edges are marked with lower-case letters and faces with capital letters. There are 8 faces, 8 vertices and 16 edges, so $v + f - e = 0$.



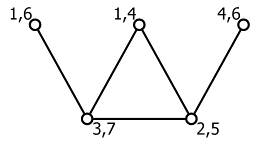
QUESTION 34 Subgraphs which are subdivisions of $K_{3,3}$ (left) and K_5 (right) are given below.



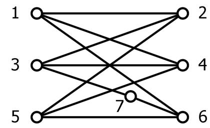
One way to draw this graph on a torus is as shown. There are 16 faces, A–P, 16 vertices, and 32 edges (since the graph is 4-regular, $2e = 4v$). So $v + f - e = 16 + 16 - 32 = 0$.



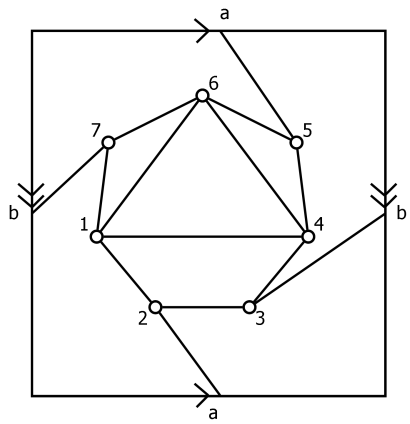
QUESTION 35 (a) Carrying out the planarity algorithm with Hamilton cycle 1234567, we get the incompatibility graph H shown. H is not bipartite, since it contains a 3-cycle, so G is not planar.

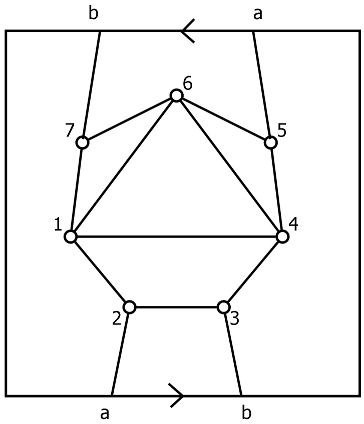


There is a subgraph which is a subdivision of $K_{3,3}$, as shown to the right.



The graph may be drawn on the torus and Möbius strip as shown below.





QUESTION 36 Draw the graph on the surface of genus g, and let f be the number of faces. By the generalisation of Euler's formula, $v + f - e = 2 - 2g$, so

$$g = \frac{1}{2}e - \frac{1}{2}v - \frac{1}{2}f + 1\,.$$

Since every cycle has length at least 4 (and $v \geq 3$), every face has degree at least 4. By face-handshaking, $2e \geq 4f$, i.e. $f \leq e/2$. Consequently,

$$\begin{aligned} g &\geq \frac{1}{2}e - \frac{1}{2}v - \frac{1}{4}e + 1 \\ &= \frac{1}{4}e - \frac{1}{2}v + 1\,, \end{aligned}$$

as required.