Practice exam Solutions.

Question 1, part 1: A graph is regular if every vertex has the same degree. Draw two non-isomophic simple, regular graphs, each with 8 vertices and 12 edges, and justify your answer.

Pf: It may help when trying to draw graphs ~~that~~ handshaking implies $24 = 2 \cdot e(V) = \sum_{V} d(v) = 8 \cdot d$, where d is the degree of the vertices.

It turns out there are 5 isomorphism classes of such graph:

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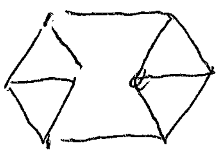

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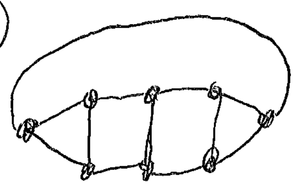

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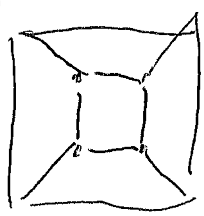

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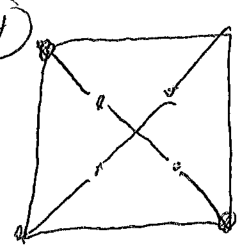

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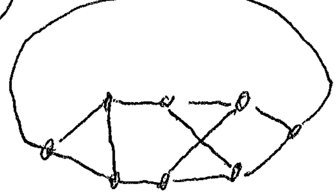


They can be shown to be non isomorphic by consider cycles:

1 Has 4 3-cycles
2 Has 2 3-cycles
5 Has 1 3-cycles

3+4 have no 3-cycles.

So the only possible isomorphisms are 3 and 4, but 3 Has 6 4cycles and is bipartite, while 4 has only one 4-cycle and has 5-cycles.

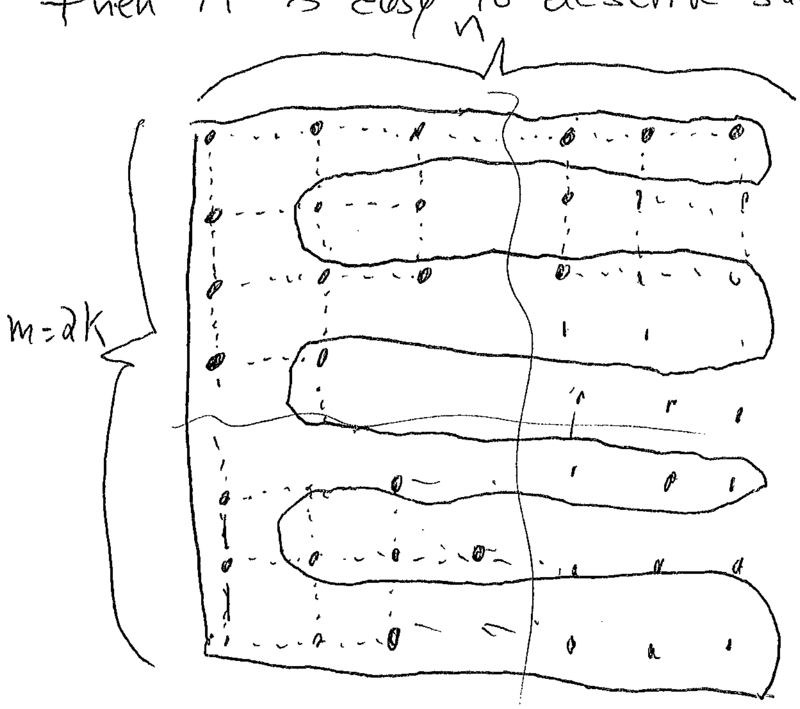
Question 1, Part 2: A town's roads form a rectangular grid, m east-west roads and n north-south roads, with $m, n > 1$. At each road sits a cafe, so there are mn cafes. For which values of m, n is it possible to set off from one of the cafes and visit each other cafe just once before returning to its starting point?

Solutions: The town is a graph, with the cafes as vertices, we are looking for a Hamiltonian cycle.

The graph is bipartite, and so if it has an odd number of vertices it cannot be Hamiltonian - a cycle visiting all the vertices would have odd length, but bipartite graphs only have even cycles. Since the graph has $m \cdot n$ vertices, we see that if both m and n are odd, it is not possible.

If, however, one of m, n is even (say $m = 2k$), then it is easy to describe such a hamiltonian cycle:

~~In words, let the vertices be ordered pairs (a,b), $1 \le a \le n$, $1 \le b \le m = 2k$.~~



Question 1, part 3:

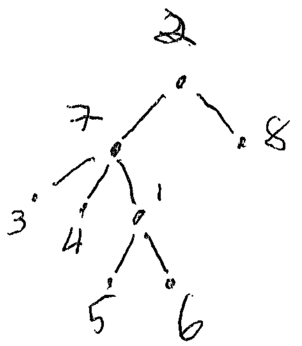
Draw the tree with Prüfer code 7, 7, 1, 1, 7, 2

Solution: The Prüfer code ~~has~~ of a tree with n vertices has length $n-2$; our code has length 6, so our tree will have 8 vertices.

The code records the "parent vertices", we fill in a table giving the "leaf" vertices by taking the smallest # in $\{1, \dots, 8\}$ not yet used and that doesn't appear in remaining code:

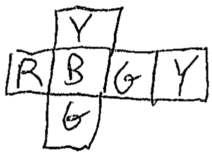
Parent	7	7	1	1	7	2
Leaf	3	4	5	6	1	7

The resulting 6 columns are 6 of the edges. The two #'s not used as leaves - 2 and 8, are the remaining edge. This gives

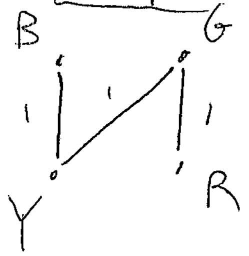


Question 2, Part i: Solve the following 4-cubes problem:

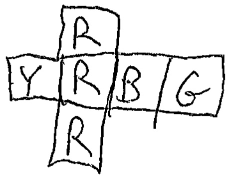
Cube 1:



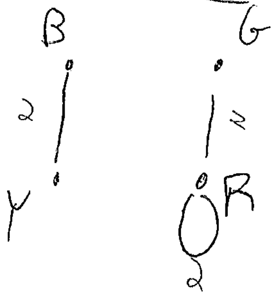
Graph



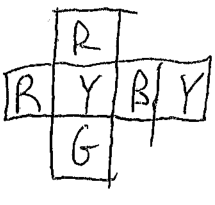
Cube 2



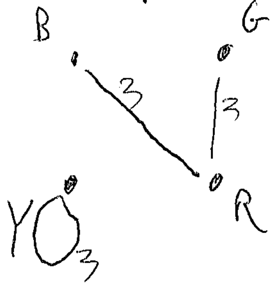
Graph



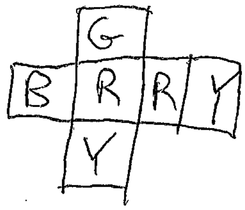
Cube 3

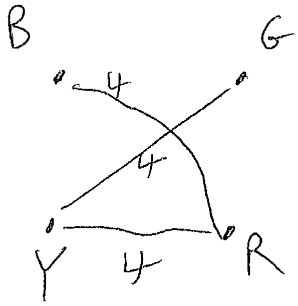


Graph

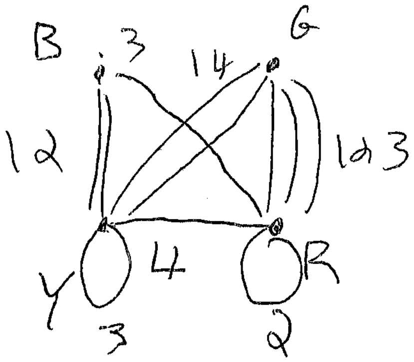


Cube 4



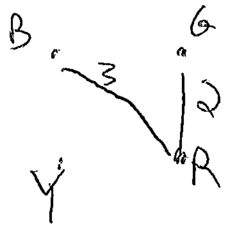


To each cube, we make a graph, with edges between colours that are on opposite faces. ~~This~~ Putting these together into 1 graph with labelled edges, we need to find two subgraphs that each use every vertex and one edge of each label, and has deg 2 at each vertex.

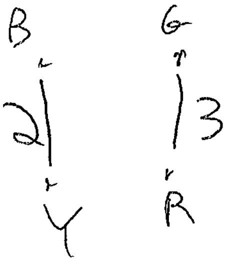


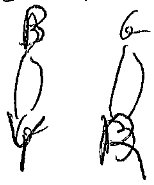
Note that we can't use either loop ~~in~~ in either of the subgraphs - the other edges would then either be a triangle (but no B-G edge), or consist of all loops and multiple edges, which can be ruled out. So we know each subgraph contains the edge G-R, and would have to start

Subgraph 1 start



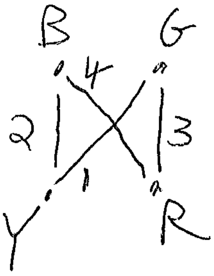
Subgraph 2 start



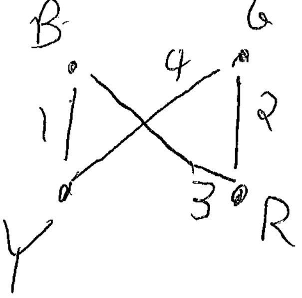
Subgraph 2 must thus either be a 4 cycle or  - the later can be ruled out since cube 4 contains no B-~~Y~~ or G-R edges.

Question 2, part i, continued:

Since B-G is not in Graph 1 or 4, we see that subgraph 2 must be



Now, to complete subgraph 1: The edge YG ~~is also~~ from cube 1 is already used in subgraph 2, and we can't use the edge G-R from cube 1 because R already has degree 2. Thus we must use B-Y from cube 1, and hence YG from cube 4, giving



for subgraph 1.

Making subgraph 1 Top/Bottom, and subgraph 2 Front/Back, gives the following solution:

Side	Cube 1	2	3	4
Top	B	G	R	Y
Bottom	Y	R	B	G
Front	Y	B	G	R
Back	G	Y	R	B

Note: There were other ways to find the solution, and this solution has more work/words than needed in your solution.

Question 2, part ii): The solution is in Lecture 13

Question 2, part iii)

Let S be a closed, compact surface, and Γ a graph drawn on S so that every component of $S \setminus \Gamma$ is a disk.

Let v be the # of vertices of Γ

e be the # of edges of Γ

and f be the # of components of $S \setminus \Gamma$.

Then the euler characteristic $\chi(S)$ is defined to be
$$\chi(S) := v - e + f.$$

$$\chi(\text{sphere}) = 2, \quad \chi(\text{Torus}) = 0, \quad \chi(\mathbb{RP}^2) = 1$$

Part iv) Use Euler characteristic to show that $K_{3,3}$ isn't planar,

Pf: Suppose $K_{3,3}$ is drawn on S^2; each component of ~~$S^2 \setminus K_{3,3}$~~ $S^2 \setminus \Gamma$ ~~minus any corner~~ is a disk for any connected Γ. Let f be the # of faces of our drawing of $K_{3,3}$ on $S^2 \setminus \Gamma$.

$K_{3,3}$ has 6 vertices and 9 edges, so we have
$$2 = v - e + f = 6 - 9 + f = f - 3,$$ so we must have 5 faces.

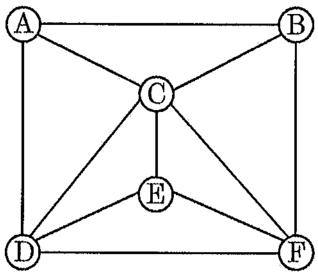
But $K_{3,3}$ is simple and bipartite, so every face of $K_{3,3}$ must have at least 4 sides. Using handshaking between faces and edges:
$$18 = 2e = \sum_{f \text{ face}} d(f) \geq \sum_{f \text{ face}} 4 = 4f,$$ so $f \leq \frac{18}{4} = 4.5 < 5$, a contradiction.

~~MAS341 Graph Theory~~
~~2012 exam solutions~~

Question 3

(i)(a) It is not Eulerian because A, B, C and E have odd degrees.

(i)(b) It is planar; a plane drawing is given below.

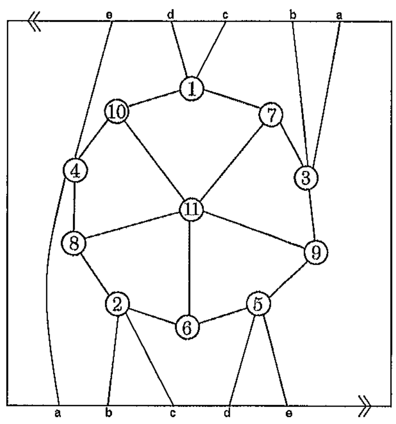


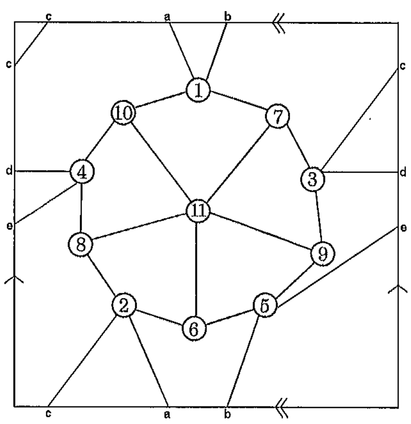
(i)(c) It is Hamiltonian: ABCFEDA is a Hamilton cycle.

(i)(d) It is not bipartite since it contains a triangle ABC.

(ii)(a) It is not planar: the subgraph obtained by deleting 11 is a subdivision of K_5, and no graph which contains a subdivision of K_5 can be planar.

(ii)(b) and **(ii)(c)**

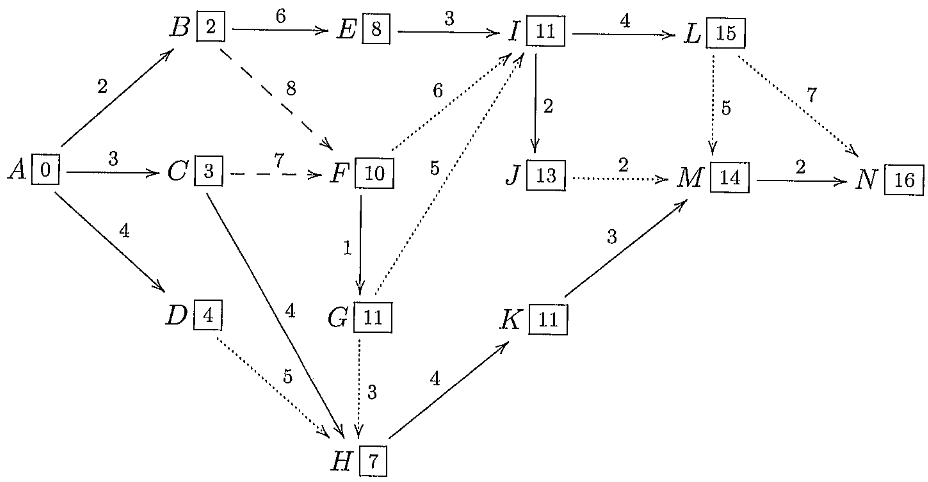




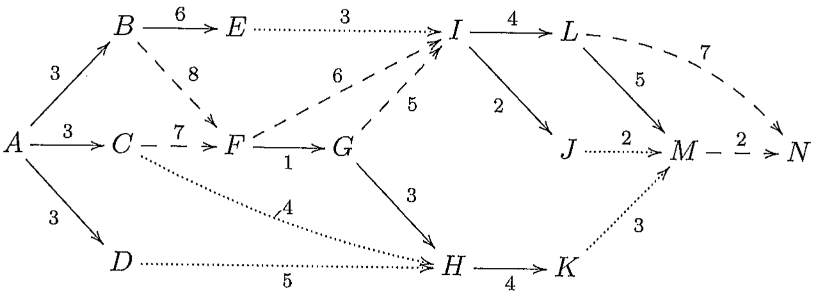
(iii) The possible orders of the components are $(5,1)$, $(4,2)$ and $(3,3)$. There are three different trees on 5 vertices, two on 4 vertices, and one each on 3, 2 and 1 vertex, so there are 6 different forests.

Question 4

(i)(a) In each case the arc used to reach each vertex in the optimal route is indicated by a solid line. Where there is a choice of optimal route this is indicated by dashed lines. Dotted lines are not in any optimal route.
The shortest path algorithm gives the following:



The unique shortest path is ACHKMN, length 16.
For the longest path algorithm, first sort into columns. The algorithm gives the following:



with lengths

B	C	D	E	F	G	H	I	J	K	L	M	N
2	3	4	8	10	11	14	16	18	18	20	25	27

There are eight longest paths: ABFILN, ABFILMN, ABFGILN, ABFGILMN, ACFILN, ACFILMN, ACFGILN and ACFGILMN, each of length 27.

(i)(b) Any arc not on a longest path may be increased (by 1) or decreased without changing l. Increasing any arc on a longest path will increase l, but decreasing an arc will only decrease l if that arc is on every longest path. The only arc which is on every longest path is IL.

(ii)(a) The nearest vertex to U is V, so add V to get UVU. The shortest edge between a vertex already in and a new vertex is VW so add V after W to get UVWU. Next the shortest such edge is VX so add X after V to get UVXWU. Next is XY, so add Y after X to get UVXYWU; finally add Z after Y to get UVXYZWU, total length 61, as an upper bound.

(ii)(b) Omitting U, the remaining lengths in order are VW, XY, VX, YZ, WX, ... We may use the first four of these without creating a cycle, so the minimal spanning tree has total length $6 + 10 + 11 + 11 = 38$. We then add the two shortest lengths from U to get our lower bound of $38 + 4 + 9 = 51$.

Question 5

(i) Expanding the polynomial gives

$$k(k-1)(k-2)^2(k^2-4k+5) = k^6 - 9k^5 + 33k^4 - 61k^3 + 56k^2 - 20k\,.$$

The degree is 6 and the second coefficient -9 so it has six vertices and nine edges. The chromatic number of a graph is the smallest number of colours in any vertex colouring. The chromatic number of G is therefore the least positive integer k such that $P_G(k) > 0$; in this case $k = 3$.

(ii)(a) If G is a simple graph and v a vertex of degree 2 whose neighbours, x and y, are adjacent then $P_G(k) = (k-2)P_{G-v}(k)$, since any k-colouring of $G-v$ has x and y different colours, and so may be extended to a k-colouring of G in $k-2$ ways (giving v any colour other than those used at x and y). Successively removing A, C, E and G in this way gives $P_H(k) = (k-2)^4 P_{C_4}$. The contraction–deletion relation says that $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$. Deleting any edge of C_4 gives the path P_4, and contracting any edge gives K_3. $P_{P_4}(k) = k(k-1)^3$, since P_4 is a tree, and $P_{K_3}(k) = k(k-1)(k-2)$. Consequently

$$\begin{aligned} P_{C_4}(k) &= k(k-1)^3 - k(k-1)(k-2) \\ &= k(k-1)(k^2-3k+3)\,, \end{aligned}$$

and so $P_H(k) = k(k-1)(k-2)^4(k^2-3k+3)$.

(ii)(b) In $P_H(4) = 1344$ ways.

(ii)(c) The chromatic index of a graph is the least number of colours required to colour the edges such that edges with a vertex in common get different colours. H needs at least 4 colours, since the 4 edges from B need different colours, and this is sufficient (colour the edges round the cycle ABCDEFGHA alternately red and blue and the edges round the cycle BDFHB alternately green and yellow).

(ii)(d) Two. Colour the four triangular faces red and the square face and the external face blue.