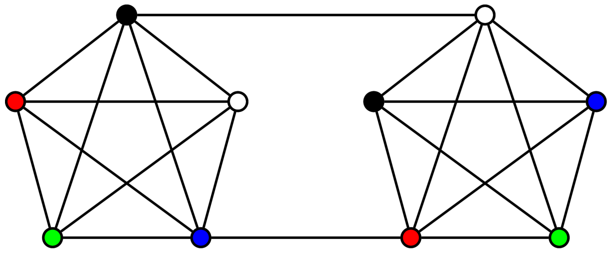
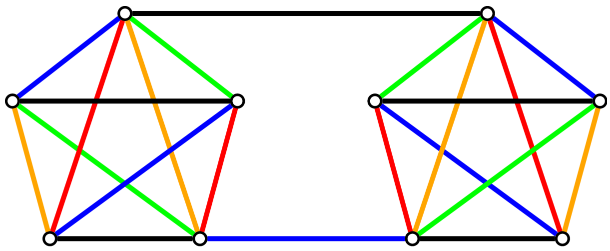
Problem book solutions 42–51

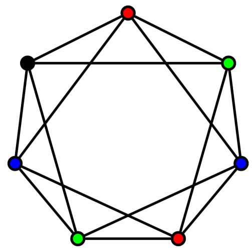
QUESTION 42 The graph contains K_5 as a subgraph, so its chromatic number is at least 5. A colouring with 5 colours exists, so $\chi(G) = 5$.



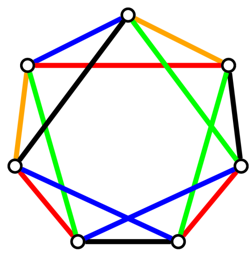
G has maximum degree 5, so $\chi'(G) \geqslant 5$. An edge colouring with 5 colours exists, so $\chi'(G) = 5$.



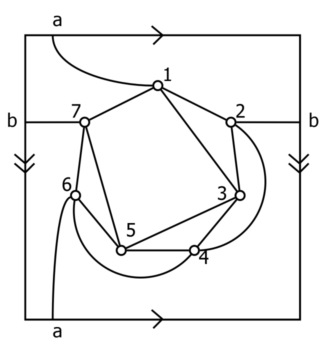
QUESTION 43 (a) Suppose there is a colouring with three colours. 1, 2 and 3 must all be different colours, say red, green and blue respectively. Now 4 is adjacent to 2 and 3, so must be red; 5 is adjacent to 3 and 4, so must be green; 6 is adjacent to 4 and 5 must be blue; and 7 is adjacent to 5 and 6 so must be red. But this is not a valid colouring, since 7 and 1 are adjacent. So no colouring with three colours is possible. There is a 4-colouring, so $\chi(G) = 4$.

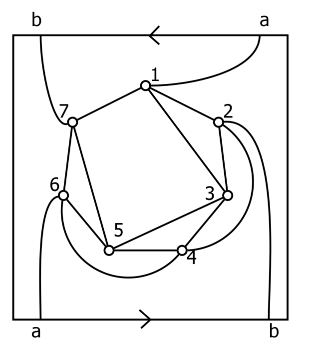


(b) Since there are only seven vertices, among any four edges some two will have a vertex in common. So no four edges can be the same colour in any edge-colouring. Suppose we have an edge colouring with 4 colours. Each colour is used on at most 3 edges, so there are at most 12 edges coloured. But G has 14 edges, so another colour is required. It can be 5-edge-coloured, so $\chi'(G) = 5$.



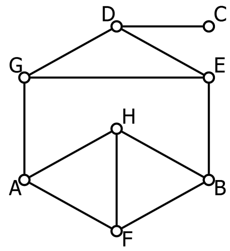
The graph can be drawn on the torus and Möbius strip, as shown below.





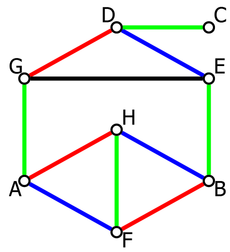
QUESTION 44

Draw a graph M whose vertices are the teams and whose edges are the matches. An allocation of weeks to matches avoids clashes if and only if it is an edge colouring of the graph, so the minimum number of weeks is $\chi'(M)$.

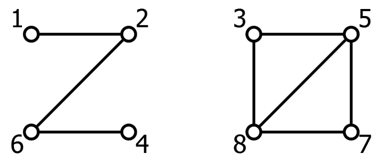


We will show that $\chi'(M) > 3$. Suppose we can colour the edges with red, green and blue. AH, HF and FA need three different colours, as each meets both of the others, so without loss of generality assume they are coloured red, green and blue respectively. Now AG must be green, HB must be blue, and FB must be red. Considering colours used at B, BE must be green.

Now GD, DE and GE must be three different colours, but none of them can be green because each meets either AG or BE. So there is no edge colouring with three colours. An edge-colouring with four colours may be obtained by colouring GD red, DE blue, GE black and DC green, as shown, so $\chi'(M) = 4$.



QUESTION 45 Draw a graph, with the vertices being exams and two vertices are connected by an edge if there is a student taking both exams.



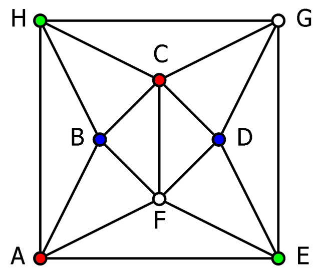
Write G_1 for the component containing 1,2,4 and 6, and G_2 for the other, with G being the whole graph. $P_G(k) = P_{G_1}(k)P_{G_2}(k)$. G_1 is a tree so $P_{G_1}(k) = k(k-1)^3$; G_2 is two copies of K_3 glued along an edge so $P_{G_2}(k) = \frac{1}{k(k-1)}P_{K_3}(k)^2 = k(k-1)(k-2)^2$, and so $P_G(k) = k^2(k-1)^4(k-2)^2$.

$P_G(2) = 0$ but $P_G(3) = 144$, so three days are required and there are 144 ways. One example is to hold 1,5 and 6 on Monday; 2,4 and 8 on Tuesday; and 3 and 7 on Wednesday.

QUESTION 46

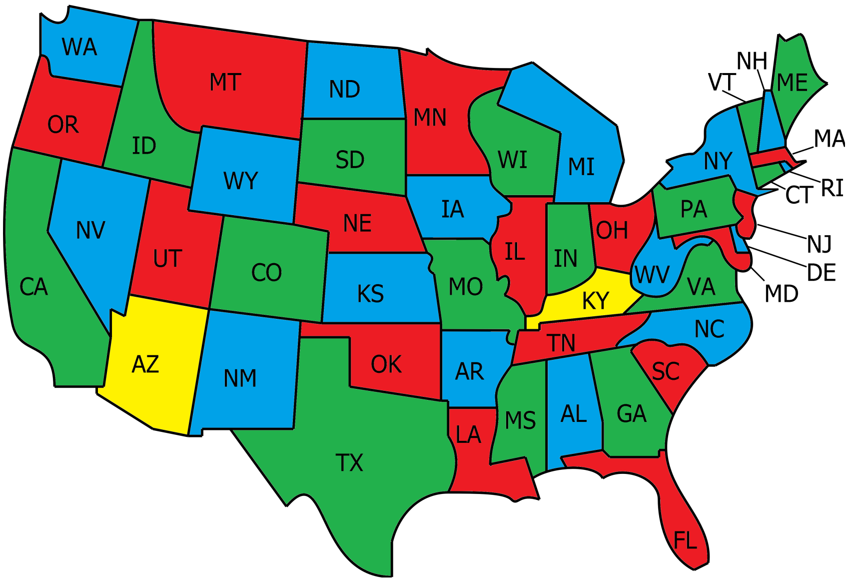
Draw a graph M whose vertices are the foods, two vertices being adjacent if the corresponding foods must be in different compartments. An acceptable assignment of foods to compartments is a colouring of the graph M, so the minimum number of compartments required is $\chi(M)$.

ABCDEA is an odd cycle, so at least 3 colours must be used for these vertices. F is adjacent to all of them, so requires a fourth colour. A colouring with 4 colours exists as shown, so we have $\chi(M) = 4$. A possible placing is to put A and C in compartment 1, B and D in compartment 2, E and H in compartment 3, and F and G in compartment 4.

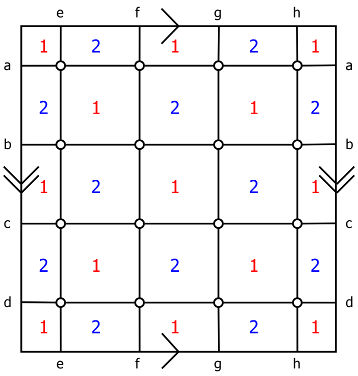


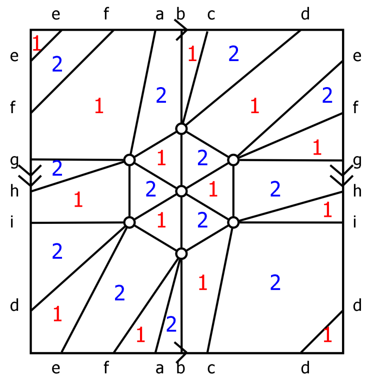
QUESTION 47 If we try to colour California, Oregon, Idaho, Utah and Arizona with two colours, the colours would have to alternate as each state has a common boundary with the previous one. But California and Arizona also have a common boundary. So these five states need at least three colours. Each of them borders Nevada, so all four colours must be used for these six states. So at least one of them is yellow.

Similarly, Tennessee, Missouri, Illinois, Indiana, Ohio, West Virginia and Virginia need at least three colours, and they all border Kentucky, which needs a fourth colour. So one of these eight states must be yellow. Since there is no overlap between the two groups of states, any colouring must have at least two different yellow states. The colouring shown has exactly two.

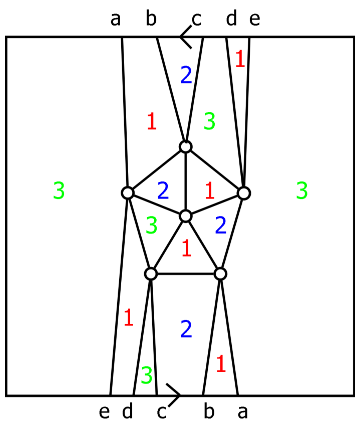


QUESTION 48 In (a) and (b), two colours are sufficient.

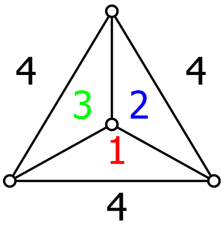


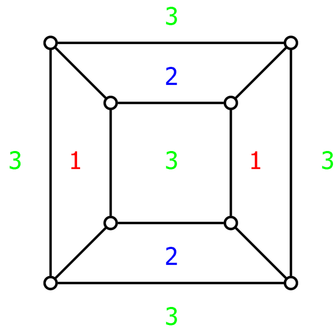


For (c), the five faces around any vertex form an odd cycle, so three colours are required. Three colours are sufficient as shown.

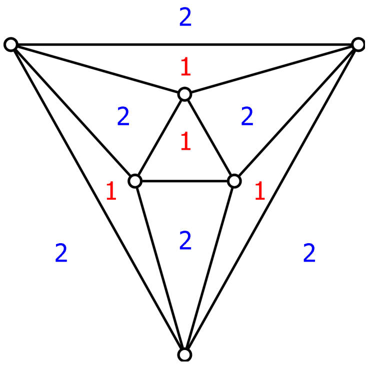


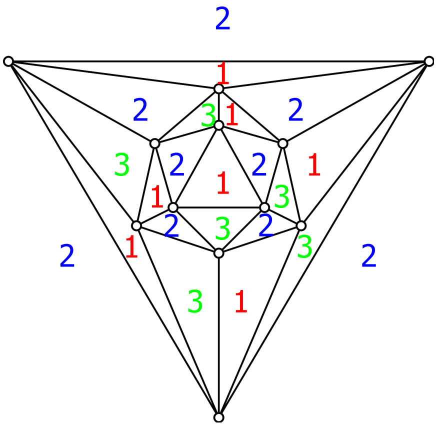
QUESTION 49 For the tetrahedron, four colours are required as every pair of faces meets. For the cube, the three faces around any vertex form an odd cycle and so at least three colours are required.



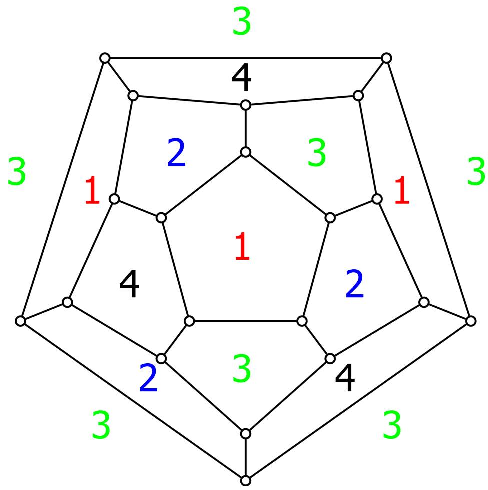


Two colours are sufficient for the octahedron. For the icosahedron, the five faces surrounding any vertex form an odd cycle and so three colours are required.

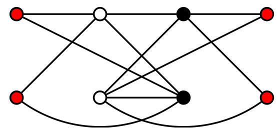




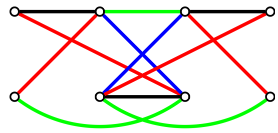
For the dodecahedron, the five faces surrounding the central face in the picture form an odd cycle, so three colours are required for these faces. The central face meets all of them so must be given a different colour, so four colours are required.



QUESTION 50

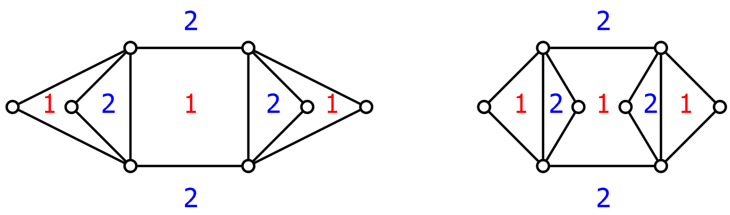


a) G contains a triangle, so $\chi(G) \geqslant 3$. A colouring with 3 colours is shown, so $\chi(G) = 3$.



G contains a vertex of degree 4, so $\chi'(G) \geqslant 4$. An edge colouring with 3 colours is shown, so $\chi'(G) = 4$.

b) G' has all degrees even, so we know it may be face coloured with two colours. There are various possibilities for G', two of which are given below.



QUESTION 51 A 4-colouring of the map is given below.