

Graph theory Practice Exam

Choose 4 Questions

Question 1

1 5 Marks

Recall that a graph is regular if every vertex has the same degree. Draw two non-isomorphic simple, regular graphs, each with exactly 8 vertices and 12 edges, and justify that your graphs are not isomorphic.

2 8 Marks

A town's roads form a rectangle grid, m east-west roads and n north-south roads, where $m, n > 1$. At each road junction is a cafe, so there are mn cafes. For which values of m, n is it possible to set off from one of the cafes and visit every other cafe just once before returning to the starting point? Every time you pass through a road junction, you must visit its cafe.

3 5 Marks

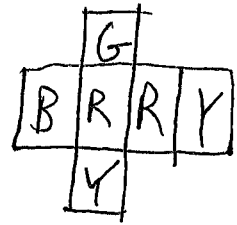
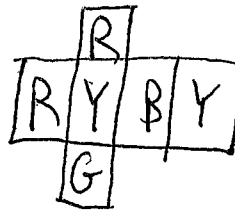
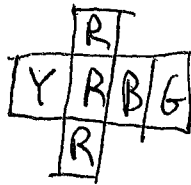
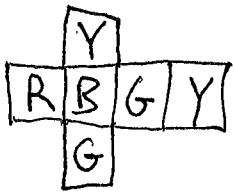
Draw the tree with Prufer code 7,7,1,1,7,2

- (ii) Seven ice hockey teams, A–G, are required to play thirteen matches as given in the table below (a cross in the table indicates that those teams must play each other).

A	×	×	–	–	–	×
B	×	×	–	×	–	
C		×	×	–	–	
D			×	×	–	
				×	×	
					×	
						×

The matches are to be scheduled so that no team plays more than one match in any week. Relate this problem to a graph, and state the parameter of the graph which gives the minimum number of weeks needed. Determine (with justification) the minimum number of weeks, and give an example of a schedule which achieves this. *(7 marks)*

i) Solve the following 4-cubes problem (9)



(5)

ii) By considering a Hamiltonian cycle, prove that $K_{3,3}$ is not planar

(6)

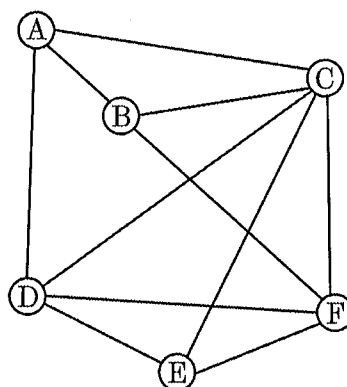
iii) Define the Euler characteristic of a closed, compact surface. What is the Euler characteristic of the sphere S^2 ? Of the torus? of the real projective plane \mathbb{RP}^2 ?

(5)

iv) Using Euler characteristic of S^2 , give another proof that $K_{3,3}$ isn't planar.

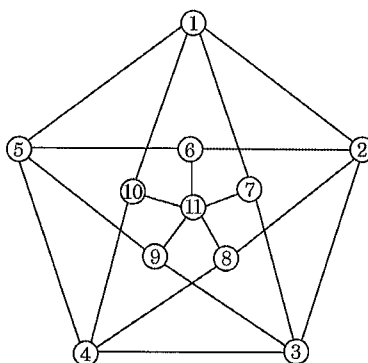
Question 2

- (iii) By drawing forests, determine the number of non-isomorphic forests with exactly 6 vertices and two trees. (5 marks)



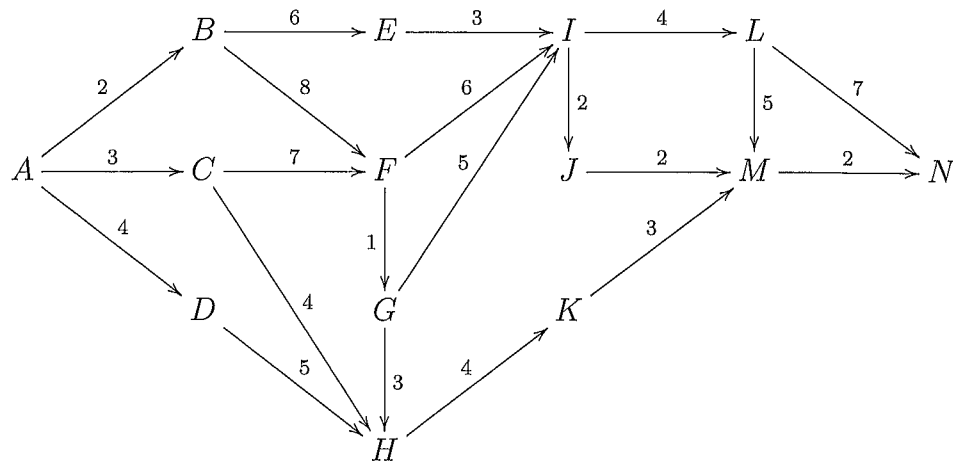
- Is this graph Eulerian? Explain your answer. *(2 marks)*
- Is this graph planar? Explain your answer. *(2 marks)*
- Is this graph Hamiltonian? Explain your answer. *(2 marks)*
- Is this graph bipartite? Explain your answer. *(2 marks)*

- (iii) By drawing forests, determine the number of non-isomorphic forests with exactly 6 vertices and two trees. (5 marks)



- (a) Is this graph planar? Explain your answer. (4 marks)
- (b) Show how this graph can be drawn on the Möbius band without any of its edges crossing. You should begin by drawing the cycle 1-7-3-9-5-6-2-8-4-10-1 as a regular polygon. Your drawing *must* use the given numbering of the vertices. (4 marks)
- (c) Show how this graph can be drawn on the torus without any of its edges crossing. You should begin by drawing the cycle 1-7-3-9-5-6-2-8-4-10-1 as a regular polygon. Your drawing *must* use the given numbering of the vertices. (4 marks)

- 3 (i) Consider the following diagram.



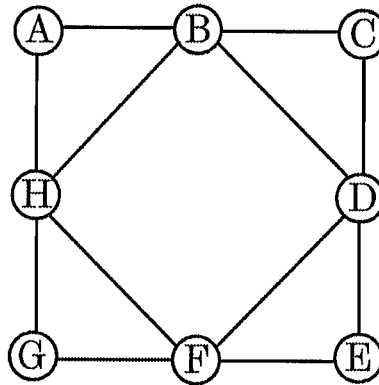
- (a) Use the shortest and longest path algorithms to determine all shortest and longest paths from A to N . State the length s of the shortest paths and the length l of the longest paths. **(14 marks)**
- (b) Which length can neither be increased nor decreased, without altering l ? **(3 marks)**
- (ii) The distances between six towns, U , V , W , X , Y , and Z are given in the table below.

U					
4	V				
9	6	W			
15	11	12	X		
18	16	15	10	Y	
23	17	16	15	11	Z

- (a) Starting at U , use “the heuristic algorithm for finding a good upper bound for the travelling salesman problem” on these towns. State the order in which you add the towns to the circuit. **(4 marks)**
- (b) By initially omitting U , give a good lower bound for the travelling salesman problem for these towns. **(4 marks)**

- 5 (i) Let G be a graph with chromatic polynomial $k(k-1)(k-2)^2(k-4k+5)$. How many vertices and edges does this graph have? Define the chromatic number of a graph. What is the chromatic number of G ? (5 marks)

- (ii) Consider the graph H shown below.



- (a) What is the chromatic polynomial of H ? (12 marks)
- (b) In how many ways can this graph H be coloured with four colours? (2 marks)
- (c) Define the chromatic index of a graph. What is the chromatic index of H ? (4 marks)
- (d) What is the minimum number of colours needed to give a face colouring of the graph H ? (2 marks)

End of Question Paper