QUESTION 23 a) Remove C and use Kruskal's algorithm to find a minimum-weight spanning tree in the remaining vertices. The shortest edges are

MY	BM	BL	BY	BE	EL	$\cdots$
64	80	105	130	157	172	$\cdots$

so we add MY, BM and BL. We skip BY, because it would create a cycle, and add BE to complete our minimal spanning tree

$$\begin{tikzcd} E \arrow[r, dash, "157"] & B \arrow[r, dash, "105"] \arrow[d, dash, "80"] & L \\ & M \arrow[r, dash, "64"] & Y \end{tikzcd}$$

of total length 406.

We then add the two shortest edges from C, 54 and 100, to get a lower bound of 560.

b) Removing M, the shortest remaining edges are

CL	BC	BL	BY	CY	BE	$\cdots$
54	100	105	130	150	157	$\cdots$

so we add CL and BC, skip BL as it would create a cycle, add BY, skip CY and add BE to get

$$\begin{tikzcd} Y \arrow[r, dash, "130"] & B \arrow[r, dash, "54"] \arrow[d, dash, "157"] & C \arrow[r, dash, "100"] & L \\ & E & & \end{tikzcd}$$

of total length 441.

We then add the two shortest edges from M, 64 and 80, to get a lower bound of 585. This is a better bound.

c) Starting from B, the shortest edge is BM, so our first partial tour is just B⇄M. The shortest edge from either of these to a new vertex is MY, so add Y after M to get BMYB. The shortest edge from one of these to a new vertex is BC, so add C after B to get BCMYB. Next the shortest edge to a new vertex is CL, giving BCLMYB, then BE, so our final tour is

$$\begin{tikzcd} B \arrow[r, "157"] & E \arrow[r, "220"] & C \arrow[d, "54"] \\ Y \arrow[u, "130"] & M \arrow[l, "64"'] & L \arrow[l, "185"'] \end{tikzcd}$$

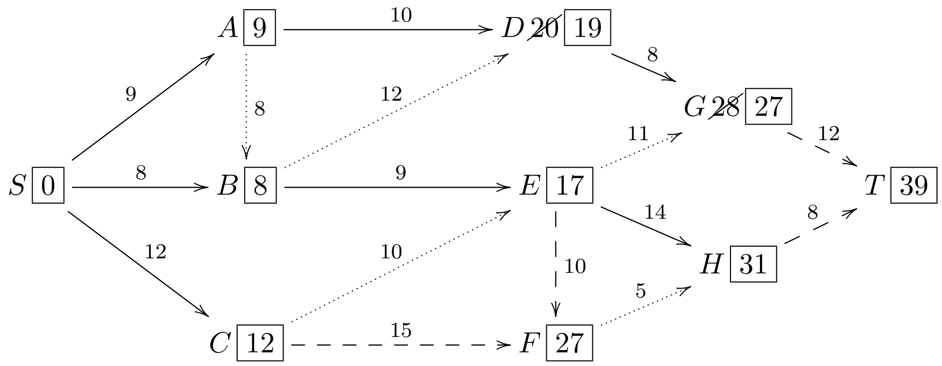
of total length 810.

Drawing approximate geographical positions and going around the outside gives the route shown

$$\begin{tikzcd} & M \arrow[r, "64"] & Y \arrow[d, "150"] \\ & B \arrow[u, "80"] & C \arrow[d, "54"] \\ E \arrow[ur, "157"] & & L \arrow[ll, "172"'] \end{tikzcd}$$

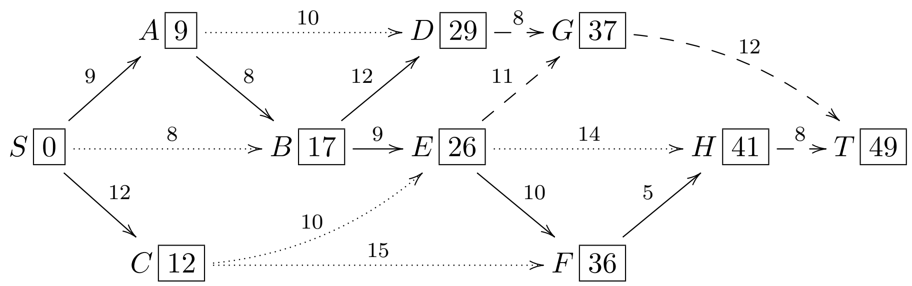
of total length 677.

QUESTION 24 a) Running the shortest path algorithm we mark vertices in the order B, A, C, E, D, F, G, H, T, (or ...G, F,...) obtaining the following diagram. Arcs we use to get to each vertex are solid; where there is a choice of which to use they are dashed; dotted arcs are not used.



There are two shortest paths: SADGT and SBEHT.

b) First we rearrange the vertices into columns so that each column contains the vertices only reachable from earlier columns. So S goes in the first column, A and C in the second, B in the third and so on. Then run the longest path algorithm column by column to get the following.



There are three longest paths: SABDGT, SABEGT and SABEFHT.

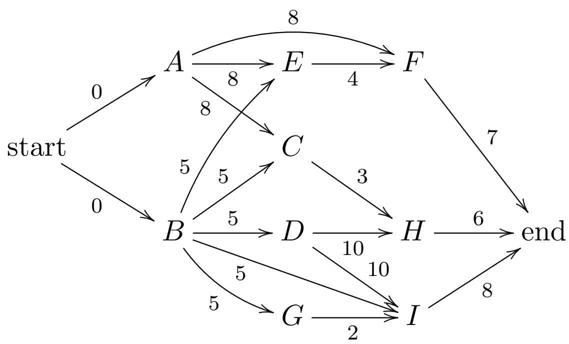
c) Any arc on any shortest path will do, since shortening the arc shortens that path. So SA, AD, DG, GT, SB, BE, EH and HT.

d) This will only happen if the arc is on every longest path, for otherwise there is a longest path which remains unchanged when we shorten the arc. So only SA and AB.

e) This will only happen if the arc is on every shortest path, for otherwise there is a shortest path which remains unchanged. But there are no such arcs.

f) Any arc on any longest path will do, since lengthening the arc lengthens that path. So SA, AB, BD, BE, DG, EG, EF, FH, GT and HT.

QUESTION 25 a) First sort the activities into columns as before, adding an arc for each precedence with weight according to the activity the arc goes from.



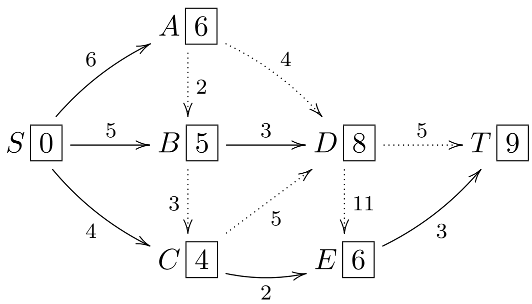
b) Going forwards from "start" we get the earliest start times.

start	A	B	C	D	E	F	G	H	I	end
0	0	0	8	5	8	12	5	15	15	23

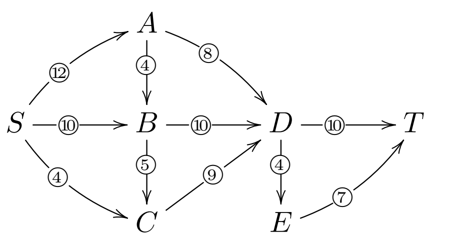
The shortest time to completion is 23. Going backwards from "end" gives the latest start times if we are to finish in 23 days.

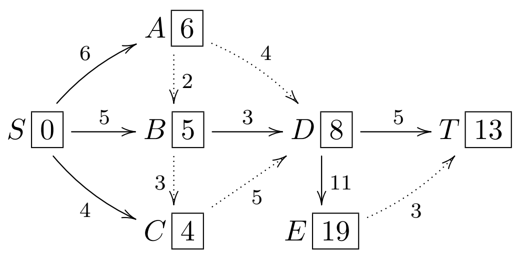
start	A	B	C	D	E	F	G	H	I	end
0	4	0	14	5	12	16	13	17	15	23

QUESTION 26 First we use the shortest path algorithm on the network with costs on the edges (the uncircled numbers).

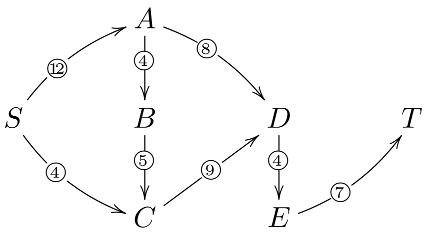


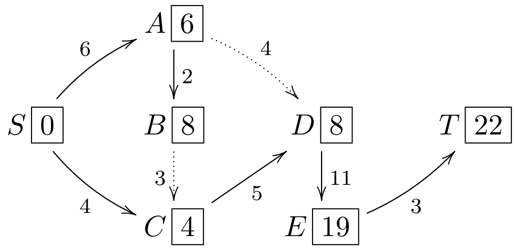
The shortest path is SCET. Water can flow along this path at rate 8 (the minimum capacity of any of the arcs on it). Once a flow of 8 is running along this path, the revised capacities are as shown. CE now has capacity 0, so is removed, and we look for the shortest path (for the costs) in what remains.



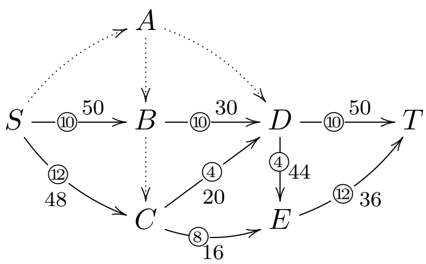


The shortest path is SBDT, and we can add a flow of 10 along this path. We do this, reducing the capacities along the path, and carry on.

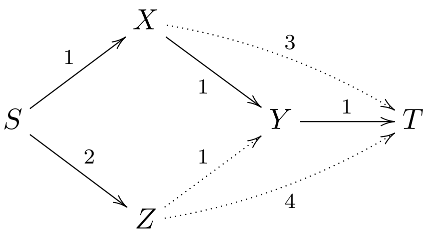


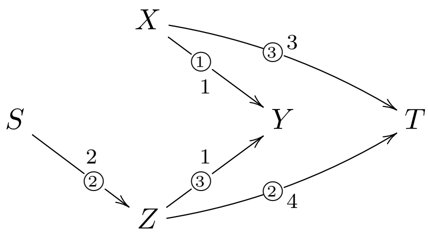


The shortest path is SCDET, and we can add a flow of 4 along this path. Once we do this we will be finished, since there is no longer a path to T. The final flow we achieve is shown below (a circled number indicates the amount flowing along that arc, with the uncircled number being the total cost for that arc). The flow is 22 and the total cost 294.



QUESTION 27 The shortest path is SXYT, length 3. When 2 units of flow are added along this path, the revised capacities are as shown.





There is only one path remaining, SZT, of length 6. We can put 2 units of flow along this, to get the 4-unit flow shown below left, of total cost 18. However, the flow shown below right also has 4 units flowing, at a total cost of 16 units, so is better.