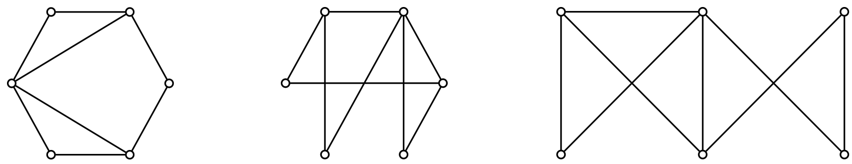
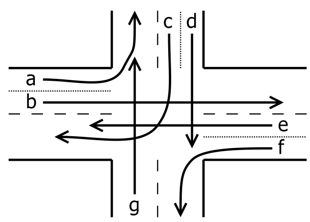
Graph theory problems book

1. For each of the following, either give an example of such a graph or prove that no such graphs exist:
   (a) a graph with six vertices whose degrees are 5, 5, 4, 3, 2, 2;
   (b) a graph with six vertices whose degrees are 5, 5, 4, 3, 3, 2;
   (c) a graph with six vertices whose degrees are 5, 5, 5, 5, 3, 3;
   (d) a simple graph with six vertices whose degrees are 5, 5, 5, 5, 3, 3.
2. (2003) Determine which of the following graphs are isomorphic to one another.



3. Determine the number of non-isomorphic simple graphs with seven vertices such that each vertex has degree at least five.
4. A simple graph G has n vertices and is isomorphic to its complement. Prove that $n \equiv 0$ or 1 (mod 4), and determine the number of edges G must have. Is this condition on the number of edges sufficient for G to be isomorphic to its complement? Provide an example of a graph with more than one vertex which is isomorphic to its complement.
5. (2000) Explain why every graph has an even number of vertices of odd degree.
6. Prove that a simple graph with more than one vertex must have two vertices of the same degree. (*Hint:* use the pigeonhole principle.) Is this necessarily true for a non-simple graph?
7. Consider the traffic flow shown. Draw the *compatibility graph* G whose vertices are the flows a–g, two vertices being joined by an edge if and only if both can flow at once without crashing. By picking out a set of complete subgraphs of G which together contain every vertex of G, obtain a traffic-light sequence in which each flow operates for at least 20 seconds every minute.



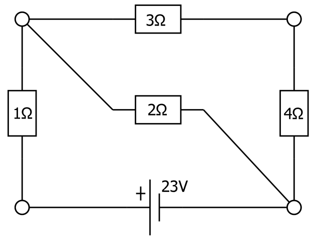
8. Define the graph G_n whose vertices are the positive divisors of n, in which two divisors are joined by an edge if their ratio is a prime number. What does G_{60} look like? Explain why G_n is always bipartite. When is G_n a tree?
9. Show how to deduce the number of components of a forest which has v vertices and e edges. Give an example of a forest with 12 vertices and 9 edges.
10. (2008) Determine the number of spanning trees of the complete bipartite graph $K_{2,4}$. How many non-isomorphic spanning trees does it have?
11. What is the maximum number of edges a bipartite simple graph with n vertices can have (a) when n is even and (b) when n is odd?
12. Draw a spanning tree of the graph of the cube, and the fundamental cycles of this spanning tree.

13. (2005) Does hexanol, with formula $C_6H_{13}OH$, have a molecular model which is a tree?

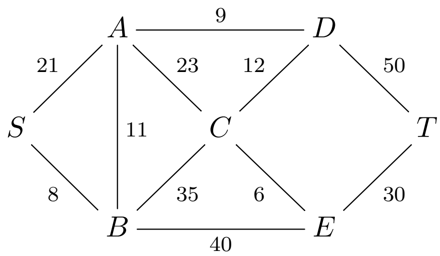
14. Determine the number of isomers of heptane, C_7H_{16}.

15. (1993) Determine whether each of the braced frameworks shown is rigid. In the case of a rigid framework, is the bracing a minimum bracing? In the case of a non-rigid bracing, show how the last two rows can be distorted.

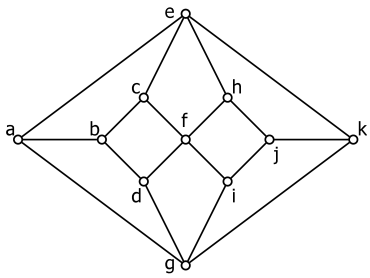
16. (1993) Show a spanning tree and a system of fundamental cycles for the graph of the electrical circuit shown, and find the currents in the various parts of the circuit.



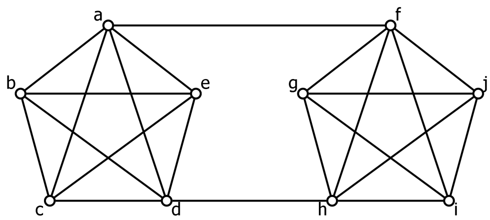
17. Solve the Chinese postman problem for the weighted graph shown, with a sorting office at S.



18. (2011) Is the graph shown (a) Eulerian, (b) semi-Eulerian, (c) Hamiltonian, (d) semi-Hamiltonian? Justify your answers.



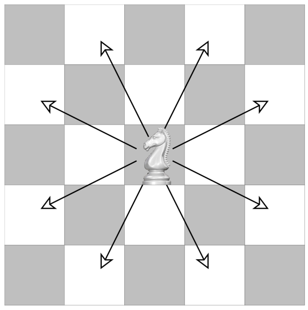
19. Is the graph shown Eulerian? Determine the number of Hamilton cycles. (Two cycles count as the same if they use the same edges – starting at a different vertex or going backwards doesn't make it a different cycle.)



20. A set of dominoes consists of 28 rectangular tiles. Each tile is divided into two squares, and each square has from zero to six spots. Every possible combination occurs on one tile. Is it possible to arrange the tiles end-to-end in a ring so that wherever two squares on different tiles touch they have the same number of spots?

21. A mouse eats a $3 \times 3 \times 3$ cube of cheese divided into 27 $1 \times 1 \times 1$ cubes, eating one $1 \times 1 \times 1$ cube at a time. Any two consecutive $1 \times 1 \times 1$ cubes he eats have a 1×1 square face in common. By considering a bipartite graph, decide whether he can eat the central small cube last.

22. Consider an $m \times m$ chessboard, where $m \geq 4$. The graph G has the squares as its vertices, with two squares joined by an edge if and only if a knight can move from one to the other in a single move (see diagram for how a knight may move). Is G bipartite? Show that G is not Hamiltonian if m is odd. Determine the number of edges of G.



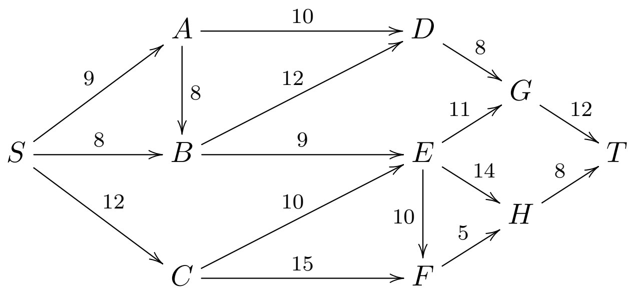
Challenge: show that G is Hamiltonian when $m = 8$.

23. (1993) The distances between six cities are given in miles in the following table:

Birmingham					
100	Cambridge				
157	220	Exeter			
105	54	172	London		
80	151	236	185	Manchester	
130	150	287	193	64	York

Find lower bounds for the solution of the travelling salesman problem for these cities by removing (a) Cambridge, (b) Manchester. Which of these two lower bounds is the better? Use the nearest insertion heuristic algorithm beginning with Birmingham to find an upper bound. Use your geographical intuition to find a better upper bound.

24. (1992) Consider the following network:



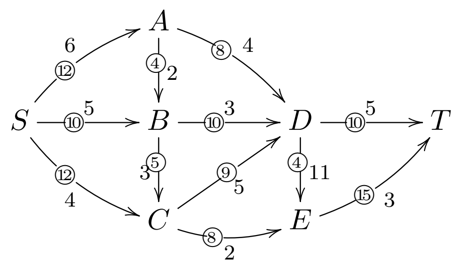
(a) Find all shortest paths from S to T and determine the length s of these.
(b) Find all longest paths from S to T and determine the length l of these.
(c) For which arcs (if any) is it true that shortening that arc by one unit, leaving all others unaltered, will decrease s?
(d) For which arcs (if any) is it true that shortening that arc by one unit, leaving all others unaltered, will decrease l?
(e) For which arcs (if any) is it true that lengthening that arc by one unit, leaving all others unaltered, will increase s?
(f) For which arcs (if any) is it true that lengthening that arc by one unit, leaving all others unaltered, will increase l?

25. (2006) A project consists of nine activities, A, . . . , I. The duration in days of the activities and the precedence relations are given in the following table:

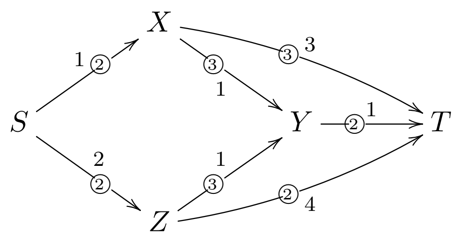
activity	A	B	C	D	E	F	G	H	I
duration	8	5	3	10	4	7	2	6	8
preceding activities	—	—	A,B	B	A,B	A,E	B	C,D	B,D,G

Use Fulkerson's algorithm to construct an activity network for this project. Determine the smallest number of days needed to complete the project, and the earliest and latest starting times of each activity in order to complete the project in the shortest possible time.

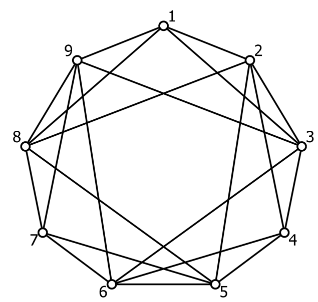
26. Use the algorithm described in lectures to find the minimum-cost maximum flow from S to T in the following network. The circled numbers are the capacities and each uncircled number is the cost per unit flow along that arc.



27. Consider the following network (again, the circled numbers are capacities and the uncircled numbers are costs). Explain why at most 4 units can flow from S to T. Use the algorithm on this network to obtain a 4-unit flow, and calculate its total cost. Now find another 4-unit flow with a smaller total cost, thus showing that the algorithm does not find the absolute minimum-cost flow in this case.



28. (1992) Use the planarity algorithm to show that the given graph is planar, and draw a plane graph isomorphic to it. How might you obtain a non-planar graph by the insertion of one extra edge? For your non-planar graph, find a subgraph which is a subdivision of $K_{3,3}$ or K_5.



29. (1993) A connected plane graph has faces of degrees 3 and 10 only, and every vertex has degree at least 3. Prove that it must have fewer faces of degree 10 than of degree 3. If

every vertex has degree 3, prove that the number of faces of degree 10 must be a multiple of 3 and the number of faces of degree 3 must be a multiple of 4.

How many faces of degree 3 and how many faces of degree 10 does a truncated dodecahedron possess? (A *truncated dodecahedron* is obtained by slicing off each vertex of a dodecahedron to give a triangle.)

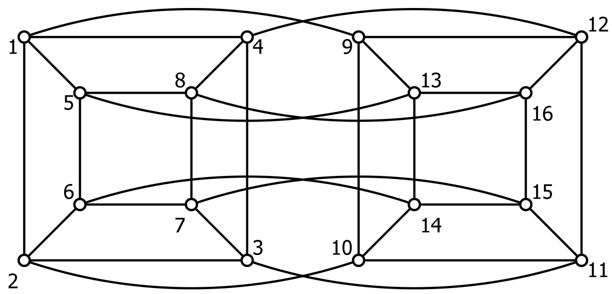
30. A connected plane graph has every vertex of the same degree $d_1 > 2$ and every face of the same degree $d_2 > 2$. Show that $\frac{1}{d_1} + \frac{1}{d_2} > \frac{1}{2}$, and find all pairs (d_1, d_2) satisfying this inequality. For each such pair, give an example of such a graph.

31. A 3-regular connected plane graph has faces of degrees 3 and 6 only. Show that there are precisely four faces of degree 3 and give an example of such a graph which has some faces of degree 6.

32. Show how $K_{3,4}$ may be drawn on a torus and embedded in a Möbius strip without its edges crossing. What is $v - e + f$ in each case?

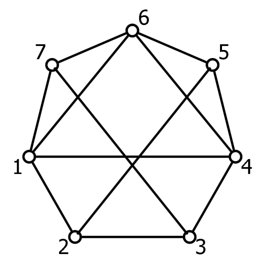
33. Embed $K_{4,4}$ in a Klein bottle without its edges crossing, and work out $v-e+f$. (Suggestion: start by arranging the red and blue vertices alternating around a regular octagon.)

34. The graph of the 4-dimensional cube is shown below. Find a subgraph which is a subdivision of $K_{3,3}$, and another which is a subdivision of K_5. Draw the graph on a torus without its edges crossing, using the labelling of the vertices shown, and verify the formula $v - e + f = 0$.



35. (2009) Let G be the graph shown. Show that G is not planar in two ways: (a) by using the planarity algorithm; (b) by finding a subgraph of G which is a subdivision of $K_{3,3}$ or K_5.

Show how G may be drawn on a torus and embedded on a Möbius strip without its edges crossing. (Your drawings should include the given labelling of the vertices.)

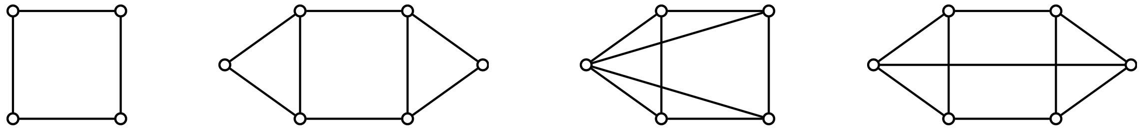


36. (2011) Let g be the genus of a connected graph with $v \geq 3$ vertices and e edges, and with every cycle of length at least 4. Prove that

$$g \geq \frac{1}{4}e - \frac{1}{2}v + 1\,.$$

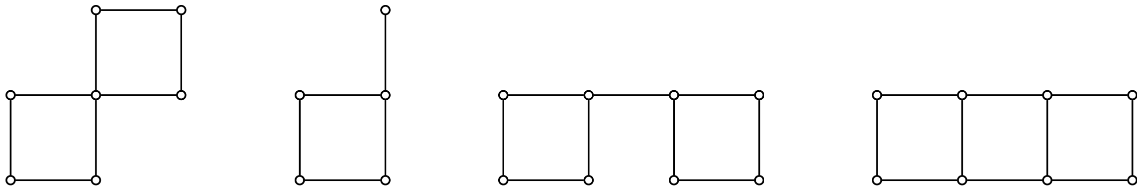
37. (2005) The simple, connected planar graph G has v vertices (with $v \geq 3$) and e edges, and no cycles of length 3. Prove that $e \leq 2v - 4$, and deduce that G must have a vertex of degree 3 or less. Hence prove by induction on v that G can be vertex-coloured using four colours.

38. (2005) Determine the chromatic polynomials of each of the graphs shown, factorising your polynomials where possible.



(a) (b) (c) (d)

39. Find the chromatic polynomials of the following graphs.



(a) (b) (c) (d)

40. For each of the following polynomials, either find a simple graph with that chromatic polynomial or give a reason why no such graph exists.

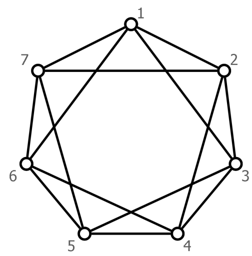
(a) $k(k-1)(k^2+2k-1)$	(d) $k(k-1)^2(k^2-3k+1)$
(b) $k(k-1)(k-2)^2$	(e) $k(k-1)^2(k^2-3k+2)$
(c) $k(k-1)^2(k-3)$	(f) $k(k-1)^2(k^2-3k+3)$

41. (2001) Is it true that if two simple graphs have the same chromatic polynomial then (a) they have the same number of edges, (b) they have the same number of vertices, (c) they are isomorphic? Give reasons for your answers.

42. Determine the chromatic number and chromatic index of the graph in Q19.

43. Let G be the graph shown. What is (a) $\chi(G)$, (b) $\chi'(G)$?

Show how G may be drawn on a torus and embedded on a Möbius strip without its edges crossing. (Your drawings should include the given labelling of the vertices.)



44. (2001) Eleven football games are to be arranged among eight teams A to H as follows.

A plays F,G,H	D plays C,E,G	F plays H
B plays E,F,H	E plays G	

If no team can play more than once a week, what is the minimum number of weeks needed to schedule all the games? Justify your answer.

45. Eight students A–H each have to choose two courses from a list of eight options 1–8. They choose as follows.

A : 1,2	B : 2,6	C : 3,5	D : 4,6
E : 5,7	F : 7,8	G : 5,8	H : 3,8

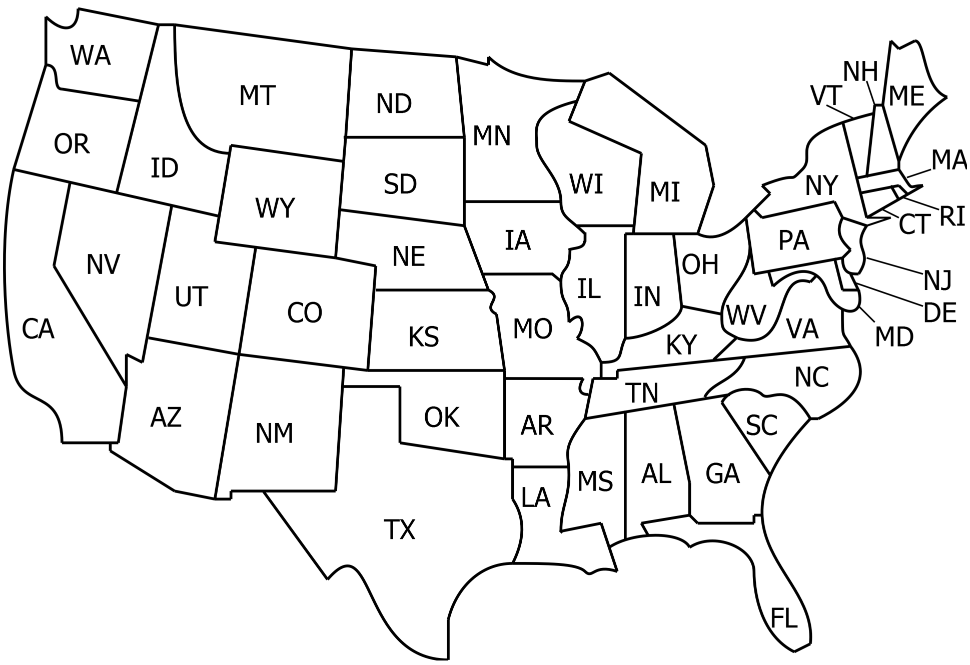
You have to timetable the examinations in such a way that no student has to take two exams on the same day. What is the smallest number of days you need, and in how many ways can you fit the exams into these days? Describe one way.

46. (2001) Eight foods A to H are to be put in refrigerated compartments in a supermarket. Because of various regulations, some cannot share the same compartment with others, as indicated by crosses in the following table.

A	×	−	−	×	×	−	×
	B	×	−	−	×	−	×
		C	×	−	×	×	×
			D	×	×	×	−
				E	×	×	−
					F	−	−
						G	×
							H

Determine the smallest number of compartments needed to display the foods and find a possible placing of the foods in the compartments.

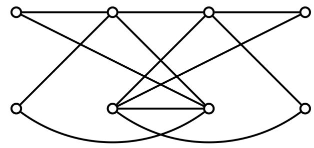
47. Suppose the 48 contiguous states of the USA are to be coloured with the four colours blue, green, red and yellow so that states with a common boundary are given different colours. What is the minimum number of times yellow must be used? Give a colouring with this many yellow states, and a reason why it is the minimum. (A simplified map is provided; territorial waters in the great lakes are included in some states. Meeting at a corner – UT & NM or CO & AZ – doesn't count.)



48. What is the minimum number of colours required to colour the faces of (a) the 4-dimensional cube graph (see Q34) on a torus, (b) K_7 on a torus, (c) K_6 embedded in a Möbius strip, so that faces with a common boundary are coloured differently?

49. What is the minimum number of colours required to colour the faces of the graphs of the five Platonic solids so that faces with a common boundary are coloured differently?

50. Let G be the graph shown and G' be a plane drawing of G. Determine the chromatic number and chromatic index of G. What is the minimum number of colours required to colour the faces of G' such that faces with a common boundary have different colours?



51. A challenge: colour the following map using four colours. Martin Gardner presented this in his *Scientific American* column on April Fools' day 1975 (the year before the four-colour theorem was finally proved) claiming it was a counterexample.