MAS-439: SOLUTIONS TO PROBLEM SET 1

QUESTION 1

Each part gives a subset of a commutative ring. Determine, with proof, whether or not each subset is a subring.

Part 1. The subset of $\mathbb{Q}$ consisting of numbers with odd denominators, when written in lowest terms.

Proof. Recall that a subset S is a subring of R if and only if it contains 1_R, and is closed under addition, taking negatives, and multiplication.

Let $\mathbf{Odd} \subset \mathbb{Q}$ be the subset of $\mathbb{Q}$ consisting of those numbers which have an odd denominator when written in lowest terms; we show that **Odd** satisfies these properties, and hence is a subring.

- *Identity* When written in lowest terms, the identity in $\mathbb{Q}$ is $1/1$ and hence in **Odd**.
- *Negatives* If $a/b \in \mathbf{Odd}$, its negative is $(-a)/b$, which has the same denominator and hence in **Odd**.
- *Addition* This is the most delicate; suppose a/b and c/d are both in **Odd**. Their sum is $a/b + c/d = (ad + bc)/bd$, which has an odd denominator, but this expression may not be in lowest terms. However, when we simplify to lowest terms, the resulting denominator will be a factor of the starting denominator, and hence will still be odd.
- *Multiplication* If a/b and c/d are in **Odd**, their produc is ac/bd, which has an odd denominator. Again, this may not be in lowest terms, but when we simplify we will only divide by things.

$\square$

Part 2. Let $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$, and let

$$E = \{a + b\omega | a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

Proof. Again, we have to check four things:

- *Identity* The multiplicative identity $1 = 1 + 0 \cdot \omega \in E$.
- *Closed under negatives* If $z = a + b\omega \in E$, then $a, b \in \mathbb{Z}$, and so $-z = -a + (-b)\omega \in E$.
- *Closed under addition* If $z = a + b\omega$ and $w = c + d\omega$ are both in E, then $a, b, c, d \in \mathbb{Z}$, and $z + w = (a + c) + (b + d)\omega \in E$.

- *Closed under multiplication* If $z = a + b\omega, w = c + d\omega$ are in E, then

  $$z \cdot w = ab + (ad + bc)\omega + bd\omega^2$$

  does not immediately appear to be in E.

  However, since $\omega^3 = 1$ and $\omega \neq 1$, we see that $0 = (\omega^3 - 1)/(\omega - 1) = \omega^2 + \omega + 1$. Thus, we may substitute $\omega^2 = -1 - \omega$ into our expression for $z \cdot w$ to obtain

  $$z \cdot w = (ab - bd) + (ad + bc - bd)\omega$$

  and since $a, b, c, d \in \mathbb{Z}$, we have $z \cdot w \in E$.

□

Part 3. Polynomials $f \in \mathbb{R}[x]$ satisfying $f(0) = 0$.

Proof. This is not a subring of $\mathbb{R}[x]$, as it does not contain the identity function 1; 1 evaluated at $x = 0$ is just 1. □

Part 4. Polynomials $f \in R[x]$ satisfying $f'(0) = 0$.

Proof. Let

$$\mathbf{D}_0 \subset \mathbb{R}[x] = \{f \in \mathbb{R}[x] | f(0) = 0\}$$

we show that $\mathbf{D}_0$ is a subring by checking the four properties:

- *idenity* The identity in the ring $\mathbb{R}[x]$ is the constant polynomial 1. The derivative of any constant function is 0, and so in particular the derivative of 1 gives 0 when evaluated at 0, and hence is in $\mathbf{D}_0$.
- *closed under negatives* If $f(0) = 0$, then $(-f)(0) = -f(0) = 0$.
- *closed under addition* The derivative is linear; if f, g are both in $\mathbf{D}_0$, then

  $$(f + g)\prime(0) = f'(0) + g'(0) = 0 + 0 = 0$$

  and so $f + g \in \mathbf{D}_0$.
- *closed under multiplication* This uses the product rule for the derivative: $(fg)' = f'g + fg^prime$. In particular, using this, if $f, g \in \mathbf{D}_0$, we calculate:

  $$(fg)'(0) = f'(0)g(0) + f(0)g'(0) = 0 \cdot g(0) + f(0) \cdot g(0) = 0$$

  since anything times 0 is 0.

□

QUESTION 2

Let R be a nontrivial commutative ring. An element $r \in R$ is a *zero divisor* if there exists $s \in R, s \neq 0$, with $r \cdot s = 0$. Show that for a nontrivial commutative ring R, the element $x \in R[x]$ is not a zero divisor.

Proof. We need to show that for any nonzero element $r = a_0 + a_1 x + a_2 x + \cdots + a_n x^n \in R[x], a_i \in R$, then $x \cdot r$ is not a zero divisor.

If R is a nontrivial ring, then elements of $R[x]$ are zero if and only if each of the $a_i = 0$. Since $r \neq 0$, there is some i with $a_i \neq 0$. Then we have

$$x \cdot r = x \cdot a_0 + a_1 x + \cdots + a_n x^n = a_0 x + a_1 x^2 + \cdots + a_n x^{n+1}$$

and, since $a_i \neq 0$, we have $x \cdot r \neq 0$, and r is not a zero divisor.

This question probably should have included a part b; a monic polynomials is one whose leading coefficient is 1, i.e., $f = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Show that if R is a nontrivial ring, then any monic polynomial is not a zero divisor in $R[x]$.

□

PROBLEM 3