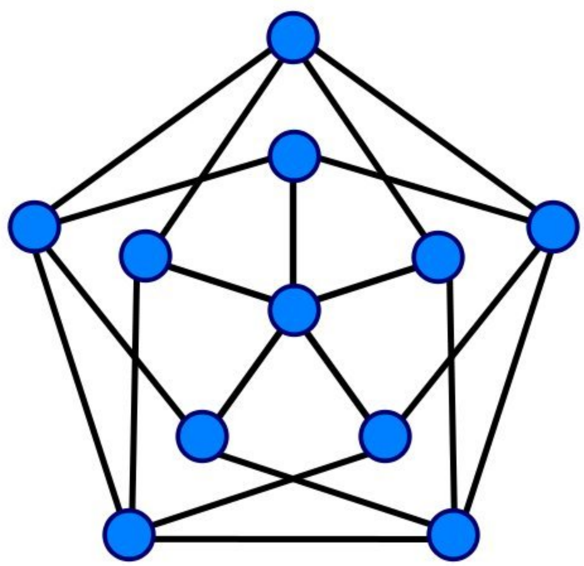
PROBLEM SET 1 SOLUTIONS
MAS341: GRAPH THEORY

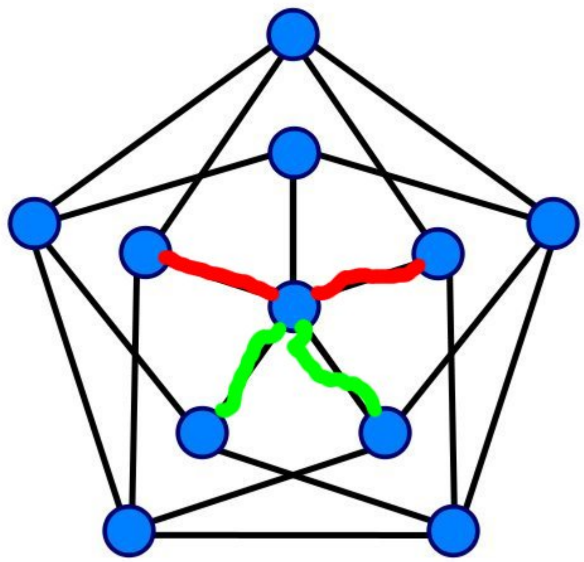
1. QUESTION 1

Find a Hamiltonian cycle in the following graph:

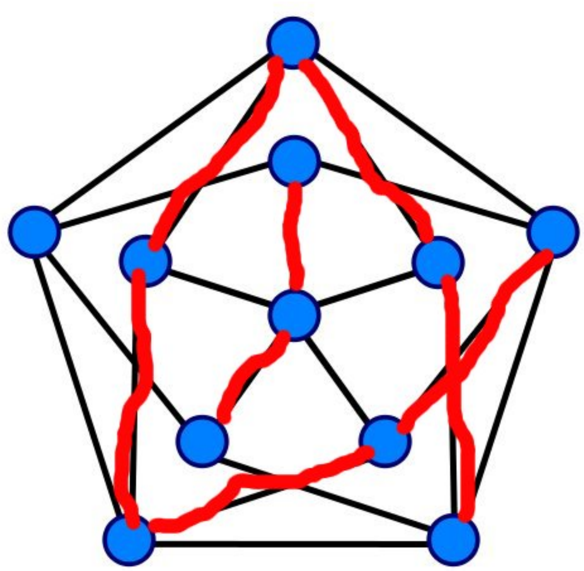


Proof. Can be done by trial an error. Here we find the path using some helpful observations.

Up to symmetry, there are two possibilities for the path as it goes through the central vertex: either it comes and leaves by two vertices next to each other on the circle (for example, the green edges in the next graph) , or it by vertices not next to each other on the circle (for example, the red edges in the next graph).

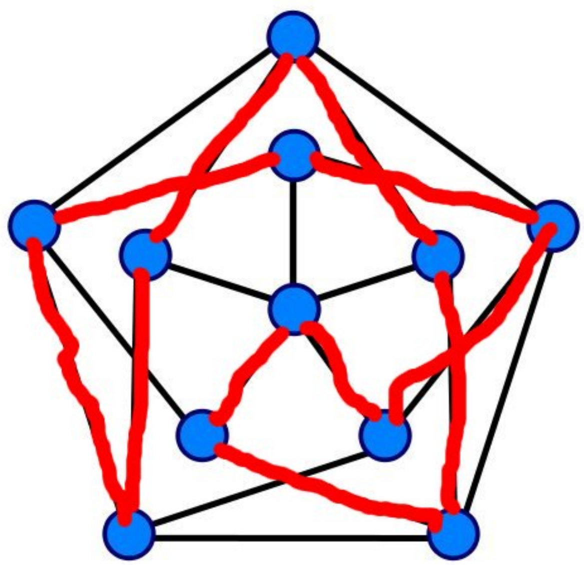


In any case, the path will visit two of the 5 vertices adjacent to the central vertex, and miss 3 of these vertices. Since these vertices all have degree 3 – for the three vertices adjacent to the central vertex in the graph, but not in the Hamiltonian cycle, we then know that any Hamiltonian path must use the other two edges in this ring. If we consider the case where the path at the central vertex does not use edges "next" to each other, we get see that the path must use all the red edges in the following graph:



Now consider how the Hamiltonian path behaves at the leftmost vertex. It cannot use the edges in the outer pentagon, as those both go to edges that the Hamiltonian path already enters and leave. Hence it must use teh two edges going into the middle layer, but this makes a cycle of length 4, not using all the vertices.

Hence, the Hamiltonian path must leave and enter the central vertex from consecutive edges. As in the previous case, we then now the behavior of the Hamiltonian cycle at the other 3 vertices in the middle ring, and from there onyl a few possibilities are left to investigate, leading to a solution similar to teh following:



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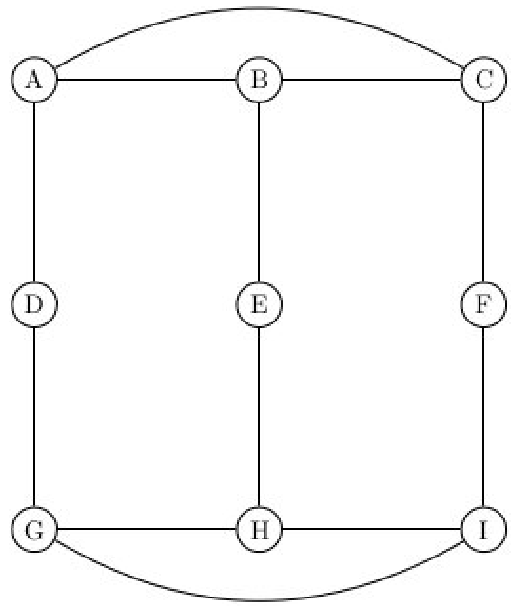
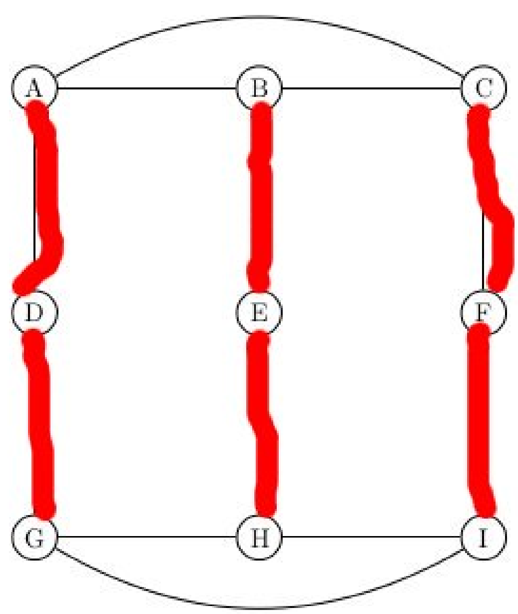
2. Question 2

Prove that the following graph is not Hamiltonian:



2.1. **Proof 1:** Since the vertices D, E and F all have degree two, we know the Hamiltonian path must pass straight through each of them, yielding the following:



Now consider how the path continues at vertex A – either it can go to vertex B or vertex C. If it goes to vertex B, then there is nowhere else for the path to go to from vertex C, and similarly, if A is connected to C, there is nowhere else for the path to connect to at vertex B.

2.2. **Proof 2:** Since the middle vertices D, E, F all have degree two, the path throw these must connect the "top" (A, B, C) to the "bottom" (G, HI).

Consider the path starting from A – it must pass throw each of D, E, F exactly once, and hence must switch between the top and bottom of the graph exactly three times. This means it would end on the bottom, and be unable to connect back to vertex A.

2.3. **Proof 3:** This proof starts by mimicing the proof that the Petersen graph is not Hamiltonian.

The graph has 9 vertices and 12 edges, so if there were a Hamiltonian cycle, it would use all but three edges. The Hamiltonian cycle would be a 9-cycle; each of these additional edges would "split" the nine cycle into two cycles, with total length 11 (since the two vertices on the "new" edges are each used twice). Since the graph is simple, there are no loops or multiple edges, and hence the possibilities are a 3 cycle and an 8 cycle, a 4 cycle or a 7 cycle, or a 5-cycle and a 6-cycle. See the pictures in Lecture

The graph has no 4 or 5 cycles, and hence each edge must split it into a 3 cycle and an 8 cycle. However, we need to add 3 edges, each making a 3 cycle, but the graph itself only has 2 three cycles, and hence the graph cannot have a Hamiltonian cycle.

Other arguments at the end can also work.

3. Question 3

Prove that a bipartite graph with an odd number of vertices is not Hamiltonian.

Proof. Since the graph has an odd number of vertices, and a Hamiltonian cycles goes through all of them, the graph would have a cycle with an odd number of vertices. But we proved in class that a graph is bipartite if and only if it does not have any odd cycles.

That is a complete proof, but it was fine to give other proofs, that essential reproduced some or all of the proof we gave in class. In fact, by doing this, we can prove something stronger. Suppose we have a bipartite graph, with vertices colored red and blue, and suppsoe the graph is Hamiltonian, and consider the Hamiltonian cycle. The vertices in the cycle must alternate red and blue, and so there are an equal number of red vertices and blue vertices. Not that this is stronger than just asking that the graph have an even number of vertices – e.g., the complete bipartite graph $K_{3,5}$ has an even number of vertices, but the above argument proves it does not have a Hamiltonian cycle.

$\square$