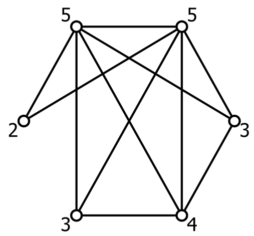
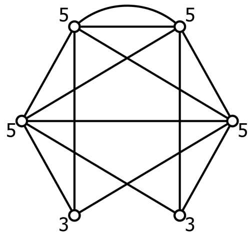
Problem book solutions 1–22

QUESTION 1 (a) Impossible: by handshaking the degree sum must be even but $5+5+4+3+2+2 = 21$.

(b)

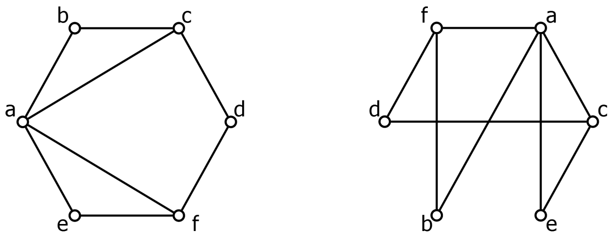


(c)

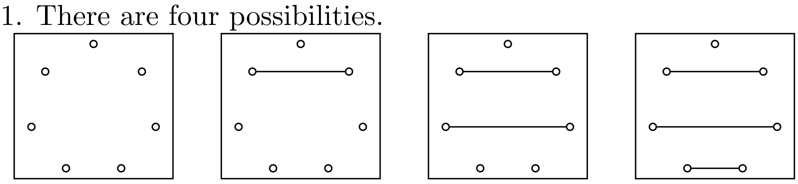


(d) Impossible. Each vertex of degree 5 must be adjacent to every other vertex, so if there are 4 vertices of degree 5 then every other vertex must have degree at least 4.

QUESTION 2 The third graph has two adjacent vertices of degree 2, but neither of the other graphs does. So the third graph cannot be isomorphic to either of the first two.
The other two graphs are isomorphic: the diagram shows how the vertices correspond.



QUESTION 3 Simple graphs G and H are isomorphic if and only if the complements G^{C} and H^{C} are isomorphic. So the number of non-isomorphic possibilities for the graph in question is the same as the number of non-isomorphic possibilities for its complement.
The complement of a 7-vertex graph with every degree at least 5 is a 7-vertex graph with every degree at most 1. There are four possibilities.



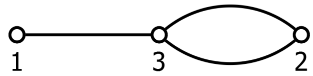
QUESTION 4 For every edge xy in K_n, either xy is an edge of G or xy is an edge of G^{C}, but not both. So $e(G) + e(G^{\mathsf{C}}) = e(K_n) = n(n-1)/2$. Since $G \cong G^{\mathsf{C}}$, we must have $e(G) = e(G^{\mathsf{C}})$, so $e(G) = n(n-1)/4$. If $n \equiv 2$ or $n \equiv 3 \pmod 4$ this is impossible, since $n(n-1)/4$ is not an integer. So $n \equiv 0$ or $n \equiv 1 \pmod 4$.
The cycle C_5 is isomorphic to its complement. In fact such a graph exists for any n which is 0 or 1 mod 4. Try to prove this by induction: if $G \cong G^{\mathsf{C}}$ and G has n vertices, can you find a way to construct a graph H with $n+4$ vertices such that $H \cong H^{\mathsf{C}}$?

QUESTION 5 Write n_i for the number of vertices of degree i. By handshaking, $2e(G) = n_1 + 2n_2 + 3n_3 + \cdots$. So $n_1 + 2n_2 + 3n_3 + \cdots$ is even. Since $n_1 + 2n_2 + 3n_3 + \cdots \equiv n_1 + n_3 + n_5 \cdots \pmod 2$, also $n_1 + n_3 + n_5 + \cdots$ is even. But that is the number of vertices of odd degree.

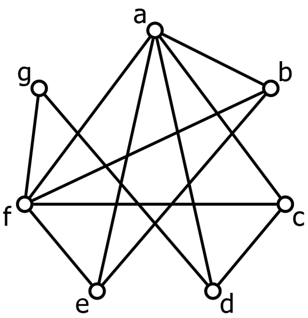
QUESTION 6 Let G be a simple graph with n vertices, where $n \geqslant 2$. Each vertex has degree at least 0, and at most $n-1$ (since G is simple), so there are n possibilities. If every degree is different there must be one vertex of each possible degree. But then there is a vertex x with

$d(x) = 0$ and a vertex y with $d(y) = n - 1$. Since $n \geqslant 2$, these are different. Now x is not adjacent to any other vertex, so x and y are not adjacent, but y is adjacent to every other vertex, so they are, which is a contradiction. So there must be two vertices of the same degree.

If G is not simple this need not be true.



QUESTION 7 Two flows are compatible if they can both go simultaneously without crashing.

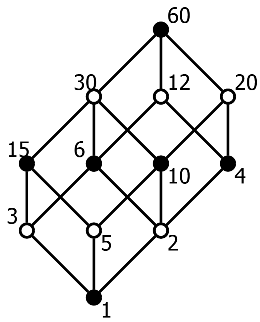


a, b, e and f form a complete subgraph, so all can go simultaneously. c and d also form a complete subgraph, and g is a single-vertex complete subgraph, so one possible solution is as follows.

For the first 20s allow a, b, e and f to go;
for the next 20s allow c and d;
for the next 20s allow g only; repeat.

QUESTION 8 In this arrangement, parallel lines correspond to multiplication or division by the same prime.

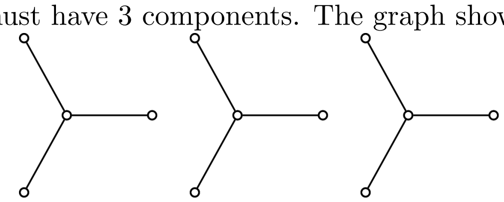
G_n is always bipartite, because we can colour an integer red if it is a product of an odd number of (not necessarily distinct) primes, and blue if it is 1 or a product of an even number of primes (shown in the image with white and black respectively). Then any edge in G_n must go between a red vertex and a blue vertex.



If p and q are distinct primes dividing n then 1, p, pq and q form a 4-cycle in G_n, so it is not a tree. Consequently, for G_n to be a tree we must have $n = 1$ or $n = p^a$ for some prime p and $a > 0$. G_1 is clearly a tree, and G_{p^a} is a path on $a + 1$ vertices, so a tree.

QUESTION 9 Each component of a forest is connected and has no cycles, so is a tree. So if the forest has k components, it is a disjoint union of k trees, $T_1, \ldots, T_k$. Write v_i for the number of vertices of T_i. Then $v_1 + v_2 + \cdots + v_k = v$. Since T_i is a tree, it has $v_i - 1$ edges. So $e = v_1 - 1 + v_2 - 1 + \cdots + v_k - 1 = v - k$, and consequently $k = v - e$.

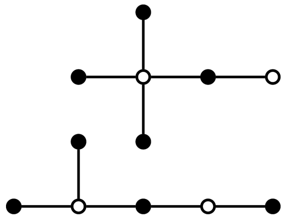
For $v = 12$ and $e = 9$ we must have 3 components. The graph shown is an example.



QUESTION 10 A spanning tree must have 5 edges, since $K_{2,4}$ has 6 vertices. There are 8

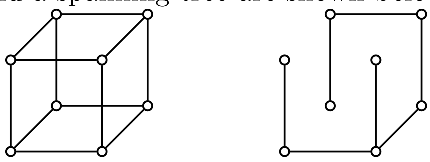
edges in $K_{2,4}$, so we must remove 3. We cannot remove both edges from any blue vertex, since this would disconnect the graph, so we must remove one edge each from three blue vertices. Any way of doing this gives a spanning tree, and there are 4 choices for the three blue vertices, and 2 choices of edge from each one, so there are $4 \times 2^3 = 32$ spanning trees.

There are only two non-isomorphic possibilities, as shown (we must have one blue vertex joined to both reds, and the others joined to one each, so either all three are joined to the same red, or two are joined to one and one to the other).

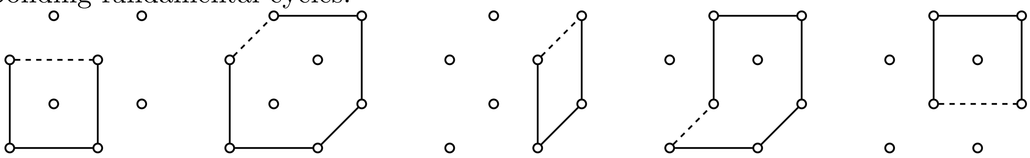


QUESTION 11 Suppose we have r red vertices and b blue vertices, where $r + b = n$. There is at most one edge from each red vertex to each blue vertex, so $e(G) \leq rb$. Now $4rb = (r+b)^2 - (r-b)^2 = n^2 - (r-b)^2$. If n is even then, since $(r-b)^2 \geq 0$, we have $e(G) \leq n^2/4$; this bound is attainable since $K_{\frac{n}{2},\frac{n}{2}}$ is a bipartite graph with $n^2/4$ edges. If n is odd then r and b must be different integers, so $(r-b)^2 \geq 1$ and $e(G) \leq (n^2-1)/4$. In this case $K_{\frac{n-1}{2},\frac{n+1}{2}}$ is a bipartite graph with this many edges.

QUESTION 12 The cube and a spanning tree are shown below.



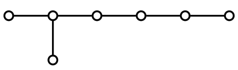
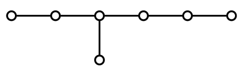
The cube has 12 edges and the spanning tree has 7 branches. So there are 5 chords, and 5 corresponding fundamental cycles.



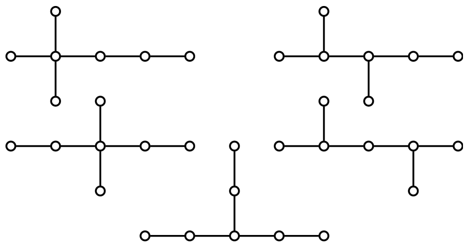
QUESTION 13 A connected graph on n vertices is a tree if and only if it has $n-1$ edges. Carbon has valency 4, oxygen 2 and hydrogen 1. So the graph of this molecule has 21 vertices and the number of edges is given by half the sum of the degrees, i.e. $(6 \times 4 + 2 + 14)/2 = 20$. It is connected, so it must be a tree.

QUESTION 14 First note that the molecule has 23 vertices and $(16 + 4 \times 7)/2 = 22$ edges, so is a tree. Removing the hydrogens leaves a carbon backbone. This must be a 7-vertex tree, with no vertex of degree higher than 4. We first find the 7-vertex trees systematically by classifying trees according to the length of the longest path.

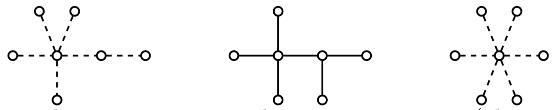
Longest path has seven vertices:

Longest path has six vertices:  

Longest path has five vertices:

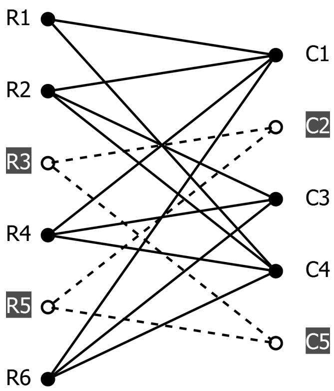


Longest path has four or three vertices:

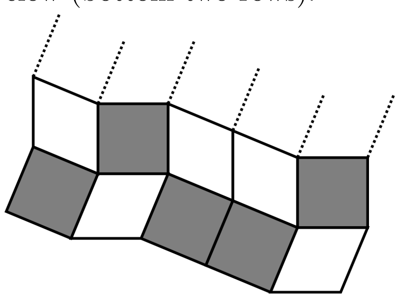


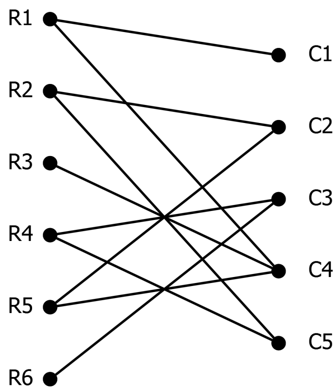
So there are 11 trees, up to isomorphism. Nine of them have maximum degree at most 4 (those with solid lines). For each of these there is a unique way to attach hydrogens to ensure every carbon has degree 4. Consequently there are 9 isomers of heptane.

QUESTION 15 Draw a row and column graph for each.



This is not connected – R3, R5, C2 and C5 form a separate component. It can be deformed, as shown below (bottom two rows).





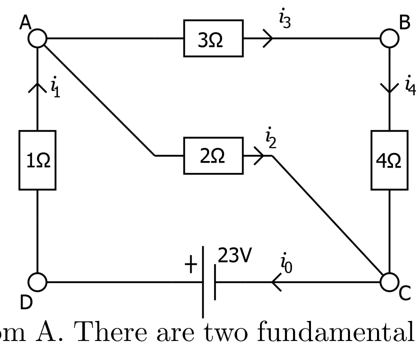
This is connected. It has 10 edges and 11 vertices, so must be a tree. Consequently the framework is a minimal bracing.

QUESTION 16 Label the currents as shown. Using Kirchoff's first law at D, B and A gives

(1) $$i_0 = i_1$$

(2) $$i_3 = i_4$$

(3) $$i_1 = i_2 + i_3\,.$$



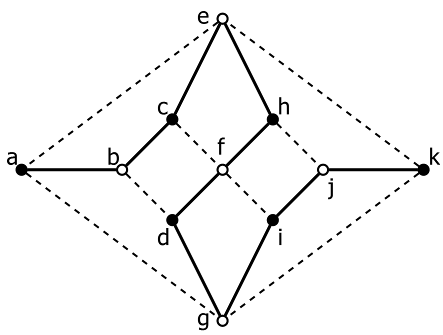
Next choose a spanning tree; we will choose the three edges from A. There are two fundamental cycles for this spanning tree: ABC and ACD. From these we get two equations using Kirchoff's second law.

$$3i_3 + 4i_4 - 2i_2 = 0 \tag{4}$$
$$2i_2 - 23 + i_1 = 0\,. \tag{5}$$

Substituting (2) into (4) and rearranging gives $i_2 = 7i_3/2$. Substituting this into (3) gives $i_1 = 9i_3/2$. Substituting these into (5) gives $23i_3/2 = 23$, so $i_3 = i_4 = 2$A, $i_2 = 7$A, $i_0 = i_1 = 9$A.

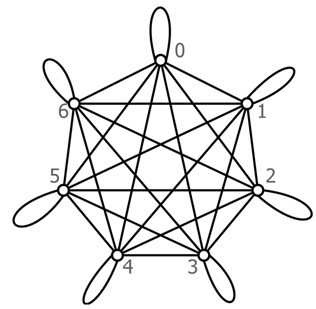
QUESTION 17 This graph is semi-Eulerian, with only D and E having odd degree. Writing l for the total length of all the edges, and p for the length of the shortest path from D to E, the optimal length of a postman route is $l + p$. Here the shortest path from D to E is DCE, so $p = 18$. The total length $l = 245$, so the shortest route has length 263. Such a route is an Euler trail in the graph obtained by adding extra copies of edges DC and CE. SABCADCDTECEBS is such a route.

QUESTION 18 (a),(b) No, for it to be Eulerian or semi-Eulerian it would need at most two vertices of odd degree, but a–d and h–k all have degree 3.
(c) No, it is bipartite as shown, so cannot contain an odd cycle. But the number of vertices is odd, so there cannot be a Hamilton cycle.
(d) Yes, a Hamilton path from a to k is shown (solid edges).



QUESTION 19 It's not Eulerian or semi-Eulerian, as a, d, f and h all have odd degree. Any Hamilton cycle must pass from left to right and back again, so must use af and hd. So it goes af(gij)hd(bce)a, where the vertices inside a pair of brackets may occur in any order. So there are 3! possibilities for (gij) and 3! for (bce), giving $3!^2 = 36$ possibilities.

QUESTION 20 Each domino is an unordered pair of numbers from $\{0, 1, 2, 3, 4, 5, 6\}$, including the possibility of both numbers being the same. So we may represent them as the edges of the graph shown (i.e. K_7 with a loop at each vertex).
Every degree is 8, so the graph is Eulerian. Thus there is a closed trail using every edge exactly once, which corresponds to a circuit using every domino where the ends of the dominoes match up.



QUESTION 21 Represent the cheese by a graph where the vertices are small cubes, and two vertices are adjacent if the corresponding small cubes share a face. This graph is bipartite, with the corner cubes and the centres of the faces red and the rest blue. The mouse eats cubes which form a Hamilton path. Any Hamilton path must alternate colours, and since there are 14 red vertices and 13 blue vertices it must start and end with a red vertex. But the centre cube corresponds to a blue vertex, so the mouse cannot eat that cube last.

QUESTION 22 G has m^2 vertices and is bipartite, as the normal chessboard colouring shows: a knight's move must be from a black square to a white square or vice versa. If there is a Hamilton cycle, it must alternate black and white squares, so the total number of squares must be even. If m is odd, so is m^2, so G is not Hamiltonian.

For $m \geqslant 4$, mark off the first and last two rows and columns.

2	3	$\cdots$ 4 $\cdots$	3	2
3	4	$\cdots$ 6 $\cdots$	4	3
$\vdots$	$\vdots$	$\ddots$ $\quad$ $\cdot^{\cdot^{\cdot}}$	$\vdots$	$\vdots$
4	6	8	6	4
$\vdots$	$\vdots$	$\cdot^{\cdot^{\cdot}}$ $\quad$ $\ddots$	$\vdots$	$\vdots$
3	4	$\cdots$ 6 $\cdots$	4	3
2	3	$\cdots$ 4 $\cdots$	3	2

$\leftarrow \underset{\text{columns}}{m-4} \rightarrow$

The number in each region gives the number of possible knight's moves from each square in that region, i.e the degree of the corresponding vertex in G_m. So we have 4 vertices of degree 2, 8 of degree 3, $4 + 4(m-4)$ of degree 4, $4(m-4)$ of degree 6, and $(m-4)^2$ of degree 8.

By handshaking, we get

$$\begin{aligned} 2e &= 2 \times 4 + 3 \times 8 + 4 \times (4 + 4(m-4)) + 6 \times 4(m-4) + 8 \times (m-4)^2 \\ &= 32 + 16m - 48 + 24m - 96 + 8m^2 - 64m + 128 \\ &= 8m^2 - 24m + 16 = 8(m-1)(m-2)\,, \end{aligned}$$

so there are $4(m-1)(m-2)$ edges.