Problem book solutions 37–41

QUESTION 37 Take a plane drawing of G with f faces. Since G is simple with no cycles of length 3, every face has degree at least 4. So the sum of the face degrees is at least $4f$, so $2e \geq 4f$, i.e. $f \leq e/2$. By Euler's formula,

$$e = f + v - 2 \qquad\qquad \leq v - 2 - e/2\,,$$

so $e/2 \leq v - 2$, i.e. $e \leq 2v - 4$. If every vertex has degree 4 or more, the degree sum would be at least $4v$, so we would have $e \geq 2v$. This is not the case, so we must have a vertex of degree 3 or less.

We wish to show that every simple connected planar graph with n vertices and no cycle of length 3 is 4-colourable. The statement is trivial for $n \leq 3$. Suppose it is true for all $m \leq n$, and take a simple planar graph G with $n + 1$ vertices and no cycle of length 3. G has a vertex x of degree at most 3; remove this vertex to get G'. G' is still planar and has no cycle of length 3. Every component of G' can be coloured with four colours by the induction hypothesis; take such a colouring. Now at most three colours are used on the neighbours of x, so one of the four colours is available to be used at x. This gives a colouring for G. So if true for all $m \leq n$ the result is also true for $n + 1$, hence the result is true for all n by induction.

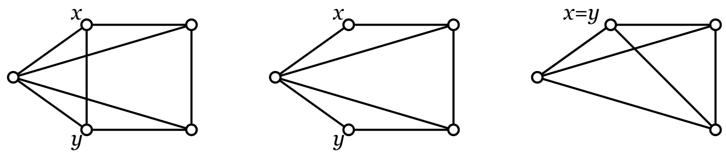
QUESTION 38 Write G_a, G_b, G_c and G_d for the four graphs. If e is any edge of G_a, $G_a - e \cong K_3$ and $G_a/e \cong P_4$ which is a tree. The chromatic polynomial of K_3 is $k(k-1)(k-2)$, and that of any tree with n vertices is $k(k-1)^{n-1}$, so

$$\begin{aligned} P_{G_a}(k) &= P_{P_4}(k) - P_{K_3}(k) \\ &= k(k-1)^3 - k(k-1)(k-2) \\ &= k(k-1)(k^2 - 3k + 3)\,. \end{aligned}$$

G_b consists of K_3 glued along an edge to another graph which in turn consists of K_3 and G_a glued along an edge. So

$$\begin{aligned} P_{G_b}(k) &= \frac{1}{k(k-1)} P_{K_3}(k) \left(\frac{1}{k(k-1)} P_{K_3}(k) P_{G_a}(k) \right) \\ &= \frac{(k(k-1)(k-2))^2 k(k-1)(k^2-3k+3)}{(k(k-1))^2} \\ &= k(k-1)(k-2)^2(k^2 - 3k + 3)\,. \end{aligned}$$

Let x and y be the marked vertices of G_c.



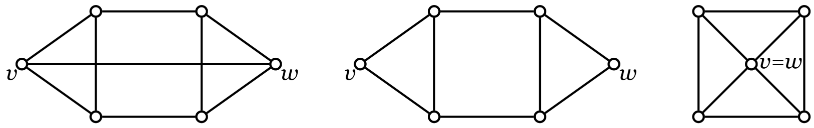
$G_c - xy$ consists of K_3 glued along an edge to another graph which consists of two copies of K_3 glued along an edge, so

$$\begin{aligned} P_{G_c - xy}(k) &= \frac{1}{k(k-1)} P_{K_3}(k) \left(\frac{1}{k(k-1)} P_{K_3}(k)^2 \right) \\ &= k(k-1)(k-2)^3\,. \end{aligned}$$

$G_c/xy \cong K_4$, so its chromatic polynomial is $k(k-1)(k-2)(k-3)$. So

$$\begin{aligned}
P_{G_c}(k) &= P_{G_c - xy}(k) - P_{G_c/xy}(k) \\
&= k(k-1)(k-2)^3 - k(k-1)(k-2)(k-3) \\
&= k(k-1)(k-2)(k^2-5k+7)\,.
\end{aligned}$$

Let v and w be the marked vertices of G_d.



$G_d - vw \cong G_b$ and $G_d/vw \cong G_c$, so

$$\begin{aligned}
P_{G_d}(k) &= P_{G_b}(k) - P_{G_c}(k) \\
&= k(k-1)(k-2)^2(k^2-3k+3) - k(k-1)(k-2)(k^2-5k+7) \\
&= k(k-1)(k-2)(k^3-3k^2+3k-2k^2+6k-6-k^2+5k-7) \\
&= k(k-1)(k-2)(k^3-6k^2+14k-13)\,.
\end{aligned}$$

QUESTION 39 As in the previous question, $P_{C_4}(k) = k(k-1)(k^2-3k+3)$.

Now (a) is two copies of C_4 glued together at a vertex, so has chromatic polynomial

$$\frac{1}{k}P_{C_4}(k)^2 = k(k-1)^2(k^2-3k+3)^2\,.$$

(b) is C_4 and K_2 glued together at a vertex, so has chromatic polynomial

$$\frac{1}{k}P_{C_4}(k)P_{K_2}(k) = k(k-1)^2(k^2-3k+3)\,.$$

(c) is C_4 and the graph in (b) glued together at a vertex, so has chromatic polynomial

$$\frac{1}{k}P_{C_4}(k)k(k-1)^2(k^2-3k+3) = k(k-1)^3(k^2-3k+3)^2\,.$$

(d) is C_4 glued along an edge to another graph, which in turn consists of two copies of C_4 glued along an edge. So its chromatic polynomial is

$$\frac{1}{k(k-1)}P_{C_4}(k)\left(\frac{1}{k(k-1)}P_{C_4}(k)^2\right) = k(k-1)(k^2-3k+3)^3\,.$$

QUESTION 40 Multiplying out (a), we get $k^4 + k^3 - 3k^2 + k$. This cannot be the chromatic polynomial of any graph, since the second coefficient of the chromatic polynomial is minus the number of edges, so cannot be positive.

(b) is the chromatic polynomial of the graph consisting of two copies of K_3 joined along an edge.

Neither (c) nor (d) can be the chromatic polynomial of any graph, since substituting $k = 2$ gives -2 ways to colour the graph with 2 colours.

(e) factorises to $k(k-1)^3(k-2)$, which is the chromatic polynomial of P_3 and K_3 glued together at a vertex: $\frac{1}{k}k(k-1)^2k(k-1)(k-2) = k(k-1)^3(k-2)$.

(f) is the chromatic polynomial of the graph consisting of K_2 and C_4 glued together at a vertex, since $P_{C_4}(k) = k(k-1)(k^2 - 3k + 3)$ from Q38.

QUESTION 41 (a) Yes, the number of edges of a graph is minus the coefficient of the second-highest power of k in the chromatic polynomial; since the two graphs have the same polynomial, they have the same coefficients.

(b) Yes, the number of vertices is the degree of the polynomial.

(c) No, since any two trees with n vertices have the same polynomial $k(k-1)^{n-1}$. So (for example) P_n and $K_{1,n-1}$ are not isomorphic if $n > 3$, but have the same chromatic polynomial.