# Randomly Activated Block-Iterative Saddle Projective Splitting for Monotone Inclusions

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# Problem statement

- $\mathcal{H}$ is an Euclidean space.
- $M\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a monotone operator with $\operatorname{zer} M \neq \varnothing$.
- **Problem:** Find $\bar{x} \in \mathcal{H}$ such that $0 \in M\bar{x}$.
- **Applications:**
  - Dynamical systems
  - Games
  - Variational inequalities
  - Machine learning
  - Data science
  - Image processing
  - Optimization
  - etc.

# Splitting methods

Several methods have been proposed to solve it by exploiting specific attributes of the operators.

$$\text{Let } \begin{cases} W\colon H \to 2^{H} \text{ be maximally monotone,} \\ C\colon H \to H \text{ be } \alpha\text{-cocoercive,} \\ Q\colon H \to H \text{ be } \beta\text{-Lipschitzian.} \end{cases} \tag{1}$$

Then we can use

- Forward-backward algo. if $M = W + C$ .
- Forward-backward-forward algo. if $M = W + Q$.
- Forward-backward-half-forward algo. if $M = W + C + Q$.
- Among many others.

# Splitting methods

- As shown in (Combettes, Acta Numer., 2024), all the examples come from the same geometric framework: Let $x_0 \in \mathsf{H}$ and iterate

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
H_n \text{ is a closed half-space such that } \operatorname{zer} M \subset H_n \\
p_n = \operatorname{proj}_{H_n} x_n \\
\lambda_n \in \left]0, 2\right[ \\
x_{n+1} = x_n + \lambda_n (p_n - x_n).
\end{array}
\right.
\end{array}
\tag{2}
$$

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\end{array}
\tag{2}
$$

- Stochasticity in splitting can be introduced in:

  (i) random block-iterative implementations,

  (ii) random relaxation parameters $(\lambda_n)_{n \in \mathbb{N}}$.

**Our objective is to implement (i) and (ii) to the following highly structured composite multivariate inclusion problem.**

# Multivariate monotone inclusion

$I$ and $K$ are finite. For every $\mathsf{i} \in I$ and every $\mathsf{k} \in K$,

$$\begin{cases} \mathsf{H}_\mathsf{i} \text{ is an Euclidean space,} \\ \mathsf{A}_\mathsf{i} \colon \mathsf{H}_\mathsf{i} \to 2^{\mathsf{H}_\mathsf{i}} \text{ is maximally monotone,} \\ \mathsf{C}_\mathsf{i} \colon \mathsf{H}_\mathsf{i} \to \mathsf{H}_\mathsf{i} \text{ is cocoercive,} \\ \mathsf{Q}_\mathsf{i} \colon \mathsf{H}_\mathsf{i} \to \mathsf{H}_\mathsf{i} \text{ is monotone and Lipschitzian,} \\ \\ \mathsf{G}_\mathsf{k} \text{ is an Euclidean space,} \\ \mathsf{B}_\mathsf{k}^m \colon \mathsf{G}_\mathsf{k} \to 2^{\mathsf{G}_\mathsf{k}} \text{ and } \mathsf{D}_\mathsf{k}^m \colon \mathsf{G}_\mathsf{k} \to 2^{\mathsf{G}_\mathsf{k}} \text{ are maximally monotone,} \\ \mathsf{B}_\mathsf{k}^c \colon \mathsf{G}_\mathsf{k} \to \mathsf{G}_\mathsf{k} \text{ and } \mathsf{D}_\mathsf{k}^c \colon \mathsf{G}_\mathsf{k} \to \mathsf{G}_\mathsf{k} \text{ are cocoercive,} \\ \mathsf{B}_\mathsf{k}^l \colon \mathsf{G}_\mathsf{k} \to \mathsf{G}_\mathsf{k} \text{ and } \mathsf{D}_\mathsf{k}^l \colon \mathsf{G}_\mathsf{k} \to \mathsf{G}_\mathsf{k} \text{ are monotone and Lipschitzian,} \\ \\ \mathsf{L}_{\mathsf{ki}} \colon \mathsf{H}_\mathsf{i} \to \mathsf{G}_\mathsf{k} \text{ is linear.} \end{cases}$$

# Multivariate monotone inclusion

We set $\mathbf{H} = \bigoplus_{i \in I} H_i$ and $\mathbf{G} = \bigoplus_{k \in K} G_k$.

$$(\forall i \in I) \quad R_i \colon \mathbf{H} \to H_i, \tag{3}$$

and we assume that $\mathbf{R} \colon \mathbf{H} \to \mathbf{H} \colon \mathbf{x} \mapsto (R_i \mathbf{x})_{i \in I}$ is monotone and Lipschitzian.

**Problem 1**

The objective is to

$$\text{find } \overline{\mathbf{x}} \in \mathbf{H} \text{ such that } (\forall i \in I) \ \ 0 \in A_i \overline{x}_i + C_i \overline{x}_i + Q_i \overline{x}_i + R_i \overline{\mathbf{x}}$$
$$+ \sum_{k \in K} L_{ki}^* \Bigg( \Big( (B_k^m + B_k^c + B_k^l) \,\square\, (D_k^m + D_k^c + D_k^l) \Big) \Bigg( \sum_{j \in I} L_{kj} \overline{x}_j \Bigg) \Bigg),$$

where $M_1 \,\square\, M_2 = (M_1^{-1} + M_2^{-1})^{-1}$ denotes the parallel sum of $M_1$ and $M_2$.

# Multivariate monotone inclusion

- Modern large-scale problems require complex models.
- It unifies and extends the models in the literature.

# Multivariate monotone inclusion

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- It unifies and extends the models in the literature.

**To solve Problem 1, we consider the following saddle form.**

# The saddle form

We set $\underline{\mathbf{X}} = \mathbf{H} \oplus \mathbf{G} \oplus \mathbf{G} \oplus \mathbf{G}$. The *saddle operator* is

$$\underline{\mathcal{S}} \colon \underline{\mathbf{X}} \to 2^{\underline{\mathbf{X}}} \colon (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}^*) \mapsto$$

$$\Bigg( \bigtimes_{i \in I} \Big( A_i x_i + C_i x_i + Q_i x_i + R_i \mathbf{x} + \sum_{k \in K} L_{ki}^* v_k^* \Big),$$

$$\bigtimes_{k \in K} \big( B_k^m y_k + B_k^c y_k + B_k^l y_k - v_k^* \big),$$

$$\bigtimes_{k \in K} \big( D_k^m z_k + D_k^c z_k + D_k^l z_k - v_k^* \big),$$

$$\bigtimes_{k \in K} \Big\{ y_k + z_k - \sum_{i \in I} L_{ki} x_i \Big\} \Bigg),$$

and the *saddle form* is to

$$\text{find } \overline{\underline{\mathbf{x}}} \in \underline{\mathbf{X}} \text{ such that } \underline{\mathbf{0}} \in \underline{\mathcal{S}} \overline{\underline{\mathbf{x}}}. \tag{4}$$

# The saddle form

- It follows from (Bùi-Combettes, Math. Oper. Res., 2022) that

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}^*) \in \operatorname{zer} \underline{\mathcal{S}} \Rightarrow \mathbf{x} \text{ solves Problem 1.}$$

# The saddle form

- It follows from (Bùi-Combettes, Math. Oper. Res., 2022) that

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}^*) \in \operatorname{zer} \underline{\mathcal{S}} \Rightarrow \mathbf{x} \text{ solves Problem 1.}$$

- **Strategy:** Solve the saddle form.
  - $\underline{\mathcal{S}}$ is the product of a big number of operators.
  - We construct random half-spaces by using $\underline{\mathcal{S}}$.
    - We use them to develop a random block-iterative algorithm.

# Algorithm

Set,

- for every $\mathrm{i} \in I$, $x_{\mathrm{i},0} \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}_\mathrm{i})$.
- for every $\mathrm{k} \in K$, $\{y_{\mathrm{k},0}, z_{\mathrm{k},0}, v^*_{\mathrm{k},0}\} \subset L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{G}_\mathrm{k})$.

Iterate as follows

# Algorithm

for $\mathrm{n} = 0, 1, \ldots$

$$
\begin{array}{l}
\text{for every } \mathrm{i} \in I_{\mathrm{n}} \\
\quad l^*_{\mathrm{i,n}} = \mathrm{Q}_{\mathrm{i}} x_{\mathrm{i,n}} + \mathrm{R}_{\mathrm{i}} \boldsymbol{x}_{\mathrm{n}} + \sum_{\mathrm{k}\in K} \mathrm{L}^*_{\mathrm{ki}} v^*_{\mathrm{k,n}};\ a_{\mathrm{i,n}} = \mathrm{J}_{\gamma_{\mathrm{i,n}} \mathrm{A}_{\mathrm{i}}} \big(x_{\mathrm{i,n}} + \gamma_{\mathrm{i,n}} (\mathrm{s}^*_{\mathrm{i}} - l^*_{\mathrm{i,n}} - \mathrm{C}_{\mathrm{i}} x_{\mathrm{i,n}})\big); \\
\quad a^*_{\mathrm{i,n}} = \gamma^{-1}_{\mathrm{i,n}} (x_{\mathrm{i,n}} - a_{\mathrm{i,n}}) - l^*_{\mathrm{i,n}} + \mathrm{Q}_{\mathrm{i}} a_{\mathrm{i,n}};\ \xi_{\mathrm{i,n}} = \|a_{\mathrm{i,n}} - x_{\mathrm{i,n}}\|^2; \\
\text{for every } \mathrm{i} \in I \smallsetminus I_{\mathrm{n}} \\
\quad a_{\mathrm{i,n}} = a_{\mathrm{i,n-1}};\ a^*_{\mathrm{i,n}} = a^*_{\mathrm{i,n-1}};\ \xi_{\mathrm{i,n}} = \xi_{\mathrm{i,n-1}}; \\
\text{for every } \mathrm{k} \in K_{\mathrm{n}} \\
\quad u^*_{\mathrm{k,n}} = v^*_{\mathrm{k,n}} - \mathrm{B}^l_{\mathrm{k}} y_{\mathrm{k,n}};\ w^*_{\mathrm{k,n}} = v^*_{\mathrm{k,n}} - \mathrm{D}^l_{\mathrm{k}} z_{\mathrm{k,n}};\ b_{\mathrm{k,n}} = \mathrm{J}_{\mu_{\mathrm{k,n}} \mathrm{B}^m_{\mathrm{k}}} \big(y_{\mathrm{k,n}} + \mu_{\mathrm{k,n}} (u^*_{\mathrm{k,n}} - \mathrm{B}^c_{\mathrm{k}} y_{\mathrm{k,n}})\big); \\
\quad d_{\mathrm{k,n}} = \mathrm{J}_{\nu_{\mathrm{k,n}} \mathrm{D}^m_{\mathrm{k}}} \big(z_{\mathrm{k,n}} + \nu_{\mathrm{k,n}} (w^*_{\mathrm{k,n}} - \mathrm{D}^c_{\mathrm{k}} z_{\mathrm{k,n}})\big);\ e^*_{\mathrm{k,n}} = \sigma_{\mathrm{k,n}} \big(\sum_{\mathrm{i}\in I} \mathrm{L}_{\mathrm{ki}} x_{\mathrm{i,n}} - y_{\mathrm{k,n}} - z_{\mathrm{k,n}} - \mathrm{r}_{\mathrm{k}}\big) + v^*_{\mathrm{k,n}}; \\
\quad q^*_{\mathrm{k,n}} = \mu^{-1}_{\mathrm{k,n}} (y_{\mathrm{k,n}} - b_{\mathrm{k,n}}) + u^*_{\mathrm{k,n}} + \mathrm{B}^l_{\mathrm{k}} b_{\mathrm{k,n}} - e^*_{\mathrm{k,n}};\ t^*_{\mathrm{k,n}} = \nu^{-1}_{\mathrm{k,n}} (z_{\mathrm{k,n}} - d_{\mathrm{k,n}}) + w^*_{\mathrm{k,n}} + \mathrm{D}^l_{\mathrm{k}} d_{\mathrm{k,n}} - e^*_{\mathrm{k,n}}; \\
\quad \eta_{\mathrm{k,n}} = \|b_{\mathrm{k,n}} - y_{\mathrm{k,n}}\|^2 + \|d_{\mathrm{k,n}} - z_{\mathrm{k,n}}\|^2;\ e_{\mathrm{k,n}} = \mathrm{r}_{\mathrm{k}} + b_{\mathrm{k,n}} + d_{\mathrm{k,n}} - \sum_{\mathrm{i}\in I} \mathrm{L}_{\mathrm{ki}} a_{\mathrm{i,n}}; \\
\text{for every } \mathrm{k} \in K \smallsetminus K_{\mathrm{n}} \\
\quad b_{\mathrm{k,n}} = b_{\mathrm{k,n-1}};\ d_{\mathrm{k,n}} = d_{\mathrm{k,n-1}};\ e^*_{\mathrm{k,n}} = e^*_{\mathrm{k,n-1}};\ q^*_{\mathrm{k,n}} = q^*_{\mathrm{k,n-1}};\ t^*_{\mathrm{k,n}} = t^*_{\mathrm{k,n-1}}; \\
\quad \eta_{\mathrm{k,n}} = \eta_{\mathrm{k,n-1}};\ e_{\mathrm{k,n}} = \mathrm{r}_{\mathrm{k}} + b_{\mathrm{k,n}} + d_{\mathrm{k,n}} - \sum_{\mathrm{i}\in I} \mathrm{L}_{\mathrm{ki}} a_{\mathrm{i,n}}; \\
\text{for every } \mathrm{i} \in I \\
\quad p^*_{\mathrm{i,n}} = a^*_{\mathrm{i,n}} + \mathrm{R}_{\mathrm{i}} \boldsymbol{a}_{\mathrm{n}} + \sum_{\mathrm{k}\in K} \mathrm{L}^*_{\mathrm{ki}} e^*_{\mathrm{k,n}}; \\
\Delta_{\mathrm{n}} = -(4\alpha)^{-1} \big(\sum_{\mathrm{i}\in I} \xi_{\mathrm{i,n}} + \sum_{\mathrm{k}\in K} \eta_{\mathrm{k,n}}\big) + \sum_{\mathrm{i}\in I} \langle x_{\mathrm{i,n}} - a_{\mathrm{i,n}} \mid p^*_{\mathrm{i,n}} \rangle \\
\qquad + \sum_{\mathrm{k}\in K} \big(\langle y_{\mathrm{k,n}} - b_{\mathrm{k,n}} \mid q^*_{\mathrm{k,n}} \rangle + \langle z_{\mathrm{k,n}} - d_{\mathrm{k,n}} \mid t^*_{\mathrm{k,n}} \rangle + \langle e_{\mathrm{k,n}} \mid v^*_{\mathrm{k,n}} - e^*_{\mathrm{k,n}} \rangle\big); \\
\theta_{\mathrm{n}} = 1_{[\Delta_{\mathrm{n}} > 0]} \Delta_{\mathrm{n}} \Big/ \Big(\sum_{\mathrm{i}\in I} \|p^*_{\mathrm{i,n}}\|^2 + \sum_{\mathrm{k}\in K} \big(\|q^*_{\mathrm{k,n}}\|^2 + \|t^*_{\mathrm{k,n}}\|^2 + \|e_{\mathrm{k,n}}\|^2\big) + 1_{[\Delta_{\mathrm{n}} \leqslant 0]}\Big); \\
\lambda_{\mathrm{n}} \in L^\infty(\Omega, \mathcal{F}, \mathrm{P}; \,]0, +\infty[\,) \\
\text{for every } \mathrm{i} \in I \\
\quad x_{\mathrm{i,n+1}} = x_{\mathrm{i,n}} - \lambda_{\mathrm{n}} \theta_{\mathrm{n}} p^*_{\mathrm{i,n}}; \\
\text{for every } \mathrm{k} \in K \\
\quad y_{\mathrm{k,n+1}} = y_{\mathrm{k,n}} - \lambda_{\mathrm{n}} \theta_{\mathrm{n}} q^*_{\mathrm{k,n}};\ z_{\mathrm{k,n+1}} = z_{\mathrm{k,n}} - \lambda_{\mathrm{n}} \theta_{\mathrm{n}} t^*_{\mathrm{k,n}};\ v^*_{\mathrm{k,n+1}} = v^*_{\mathrm{k,n}} - \lambda_{\mathrm{n}} \theta_{\mathrm{n}} e_{\mathrm{k,n}};
\end{array}
$$

# Algorithm

for $\mathrm{n} = 0, 1, \ldots$

for every $\mathrm{i} \in I_\mathrm{n}$

$l^*_{\mathrm{i,n}} = \mathrm{Q}_\mathrm{i} x_{\mathrm{i,n}} + \mathrm{R}_\mathrm{i} \boldsymbol{x}_\mathrm{n} + \sum_{\mathrm{k} \in K} \mathrm{L}^*_{\mathrm{ki}} v^*_{\mathrm{k,n}}$; $a_{\mathrm{i,n}} = \mathrm{J}_{\gamma_{\mathrm{i,n}} \mathrm{A}_\mathrm{i}} \big(x_{\mathrm{i,n}} + \gamma_{\mathrm{i,n}} (\mathrm{s}^*_\mathrm{i} - l^*_{\mathrm{i,n}} - \mathrm{C}_\mathrm{i} x_{\mathrm{i,n}})\big)$;

$a^*_{\mathrm{i,n}} = \gamma^{-1}_{\mathrm{i,n}} (x_{\mathrm{i,n}} - a_{\mathrm{i,n}}) - l^*_{\mathrm{i,n}} + \mathrm{Q}_\mathrm{i} a_{\mathrm{i,n}}$; $\xi_{\mathrm{i,n}} = \|a_{\mathrm{i,n}} - x_{\mathrm{i,n}}\|^2$;

for every $\mathrm{i} \in I \smallsetminus I_\mathrm{n}$

$a_{\mathrm{i,n}} = a_{\mathrm{i,n-1}}$; $a^*_{\mathrm{i,n}} = a^*_{\mathrm{i,n-1}}$; $\xi_{\mathrm{i,n}} = \xi_{\mathrm{i,n-1}}$;

for every $\mathrm{k} \in K_\mathrm{n}$

$u^*_{\mathrm{k,n}} = v^*_{\mathrm{k,n}} - \mathrm{B}^l_\mathrm{k} y_{\mathrm{k,n}}$; $w^*_{\mathrm{k,n}} = v^*_{\mathrm{k,n}} - \mathrm{D}^l_\mathrm{k} z_{\mathrm{k,n}}$; $b_{\mathrm{k,n}} = \mathrm{J}_{\mu_{\mathrm{k,n}} \mathrm{B}^m_\mathrm{k}} \big(y_{\mathrm{k,n}} + \mu_{\mathrm{k,n}} (u^*_{\mathrm{k,n}} - \mathrm{B}^c_\mathrm{k} y_{\mathrm{k,n}})\big)$;

$d_{\mathrm{k,n}} = \mathrm{J}_{\nu_{\mathrm{k,n}} \mathrm{D}^m_\mathrm{k}} \big(z_{\mathrm{k,n}} + \nu_{\mathrm{k,n}} (w^*_{\mathrm{k,n}} - \mathrm{D}^c_\mathrm{k} z_{\mathrm{k,n}})\big)$; $e^*_{\mathrm{k,n}} = \sigma_{\mathrm{k,n}} \big( \sum_{\mathrm{i} \in I} \mathrm{L}_{\mathrm{ki}} x_{\mathrm{i,n}} - y_{\mathrm{k,n}} - z_{\mathrm{k,n}} - \mathrm{r}_\mathrm{k} \big) + v^*_{\mathrm{k,n}}$;

$q^*_{\mathrm{k,n}} = \mu^{-1}_{\mathrm{k,n}} (y_{\mathrm{k,n}} - b_{\mathrm{k,n}}) + u^*_{\mathrm{k,n}} + \mathrm{B}^l_\mathrm{k} b_{\mathrm{k,n}} - e^*_{\mathrm{k,n}}$; $t^*_{\mathrm{k,n}} = \nu^{-1}_{\mathrm{k,n}} (z_{\mathrm{k,n}} - d_{\mathrm{k,n}}) + w^*_{\mathrm{k,n}} + \mathrm{D}^l_\mathrm{k} d_{\mathrm{k,n}} - e^*_{\mathrm{k,n}}$;

$\eta_{\mathrm{k,n}} = \|b_{\mathrm{k,n}} - y_{\mathrm{k,n}}\|^2 + \|d_{\mathrm{k,n}} - z_{\mathrm{k,n}}\|^2$; $e_{\mathrm{k,n}} = \mathrm{r}_\mathrm{k} + b_{\mathrm{k,n}} + d_{\mathrm{k,n}} - \sum_{\mathrm{i} \in I} \mathrm{L}_{\mathrm{ki}} a_{\mathrm{i,n}}$;

for every $\mathrm{k} \in K \smallsetminus K_\mathrm{n}$

$b_{\mathrm{k,n}} = b_{\mathrm{k,n-1}}$; $d_{\mathrm{k,n}} = d_{\mathrm{k,n-1}}$; $e^*_{\mathrm{k,n}} = e^*_{\mathrm{k,n-1}}$; $q^*_{\mathrm{k,n}} = q^*_{\mathrm{k,n-1}}$; $t^*_{\mathrm{k,n}} = t^*_{\mathrm{k,n-1}}$;

$\eta_{\mathrm{k,n}} = \eta_{\mathrm{k,n-1}}$; $e_{\mathrm{k,n}} = \mathrm{r}_\mathrm{k} + b_{\mathrm{k,n}} + d_{\mathrm{k,n}} - \sum_{\mathrm{i} \in I} \mathrm{L}_{\mathrm{ki}} a_{\mathrm{i,n}}$;

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$p^*_{\mathrm{i,n}} = a^*_{\mathrm{i,n}} + \mathrm{R}_\mathrm{i} \boldsymbol{a}_\mathrm{n} + \sum_{\mathrm{k} \in K} \mathrm{L}^*_{\mathrm{ki}} e^*_{\mathrm{k,n}}$;

$$\Delta_\mathrm{n} = -(4\alpha)^{-1} \big( \textstyle\sum_{\mathrm{i} \in I} \xi_{\mathrm{i,n}} + \sum_{\mathrm{k} \in K} \eta_{\mathrm{k,n}} \big) + \sum_{\mathrm{i} \in I} \langle x_{\mathrm{i,n}} - a_{\mathrm{i,n}} \mid p^*_{\mathrm{i,n}} \rangle + \sum_{\mathrm{k} \in K} \big( \langle y_{\mathrm{k,n}} - b_{\mathrm{k,n}} \mid q^*_{\mathrm{k,n}} \rangle + \langle z_{\mathrm{k,n}} - d_{\mathrm{k,n}} \mid t^*_{\mathrm{k,n}} \rangle + \langle e_{\mathrm{k,n}} \mid v^*_{\mathrm{k,n}} - e^*_{\mathrm{k,n}} \rangle \big);$$

$$\theta_\mathrm{n} = 1_{[\Delta_\mathrm{n} > 0]} \Delta_\mathrm{n} \Big/ \Big( \textstyle\sum_{\mathrm{i} \in I} \|p^*_{\mathrm{i,n}}\|^2 + \sum_{\mathrm{k} \in K} \big( \|q^*_{\mathrm{k,n}}\|^2 + \|t^*_{\mathrm{k,n}}\|^2 + \|e_{\mathrm{k,n}}\|^2 \big) + 1_{[\Delta_\mathrm{n} \leqslant 0]} \Big);$$

$\lambda_\mathrm{n} \in L^\infty(\Omega, \mathcal{F}, \mathrm{P}; \,]0, +\infty[)$

for every $\mathrm{i} \in I$

$x_{\mathrm{i,n+1}} = x_{\mathrm{i,n}} - \lambda_\mathrm{n} \theta_\mathrm{n} p^*_{\mathrm{i,n}}$;

for every $\mathrm{k} \in K$

$y_{\mathrm{k,n+1}} = y_{\mathrm{k,n}} - \lambda_\mathrm{n} \theta_\mathrm{n} q^*_{\mathrm{k,n}}$; $z_{\mathrm{k,n+1}} = z_{\mathrm{k,n}} - \lambda_\mathrm{n} \theta_\mathrm{n} t^*_{\mathrm{k,n}}$; $v^*_{\mathrm{k,n+1}} = v^*_{\mathrm{k,n}} - \lambda_\mathrm{n} \theta_\mathrm{n} e_{\mathrm{k,n}}$;

# Control rule

- **Assumption:** There exists $\mathsf{N} \in \mathbb{N} \smallsetminus \{0\}$ such that $I_0 = I, K_0 = K$, and

$$(\forall \mathsf{n} \in \mathbb{N}) \quad \begin{cases} (\forall \mathsf{i} \in I) \;\; \mathsf{P}\left(\left[\mathsf{i} \in \bigcup_{\mathsf{j}=\mathsf{n}}^{\mathsf{n}+\mathsf{N}-1} I_{\mathsf{j}}\right]\right) \geqslant \pi_{\mathsf{i}} > 0 \\ (\forall \mathsf{k} \in K) \;\; \mathsf{P}\left(\left[\mathsf{k} \in \bigcup_{\mathsf{j}=\mathsf{n}}^{\mathsf{n}+\mathsf{N}-1} K_{\mathsf{j}}\right]\right) \geqslant \zeta_{\mathsf{k}} > 0. \end{cases} \tag{5}$$

- Particularly true if $I_{\mathsf{n}} = \{i_{\mathsf{n}}\}$ and $K_{\mathsf{n}} = \{k_{\mathsf{n}}\}$ with $i_{\mathsf{n}} = \mathrm{uniform}(I)$ and $k_{\mathsf{n}} = \mathrm{uniform}(K)$.

# Super relaxation

- **Assumption:** $\inf_{n \in \mathbb{N}} \mathsf{E}(\lambda_n(2-\lambda_n)) > 0$.
- When the relaxation parameters are deterministic, this condition reduces to

$$(\exists\, \varepsilon \in \left]0,1\right[)(\forall n \in \mathbb{N}) \quad \lambda_n \in [\varepsilon, 2-\varepsilon]\,. \tag{6}$$

- In the stochastic case, we can construct relaxation in which

$$(\forall n \in \mathbb{N}) \quad \mathsf{P}([\lambda_n > 2]) > 0. \tag{7}$$

# Convergence

Denote as $\mathscr{P}$ the set of solutions to Problem 1. Then there exists a $\mathscr{P}$-valued random variable $\overline{\boldsymbol{x}}$ such that

$$(\forall i \in I) \quad \begin{cases} x_{i,n} \to \overline{x}_i \ \text{ P-a.s.} \\ x_{i,n} \to \overline{x}_i \ \text{ in } \ L^1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^N). \end{cases} \tag{8}$$

# Example on minimization

# Product space stochastic gradient algorithm

Let $\alpha \in \,]0,+\infty[$ and, for every $k \in \{1,\ldots,p\}$, let $g_k\colon \mathbb{R}^N \to \mathbb{R}$ be convex, differentiable, and such that $\nabla g_k$ is $1/\alpha$-Lipschitzian. The task is to

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \;\; \frac{1}{p}\sum_{k=1}^{p} g_k(x). \tag{9}$$

- If p is huge, we may want to use a stochastic method to split the sum.
- **First idea:** Stochastic gradient method with uniform sampling from $\{1,\ldots,p\}$.
  - It requires control over the variance.
  - Only convergence of the function values.

# Product space stochastic gradient algorithm

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- If $p$ is huge, we may want to use a stochastic method to split the sum.
- **First idea:** Stochastic gradient method with uniform sampling from $\{1,\ldots,p\}$.
  - It requires control over the variance.
  - Only convergence of the function values.
- **Second idea:** The saddle form.
  - Convergence to a solution without extra assumptions.

# Product space stochastic gradient algorithm

Set $x_0$, $(y_{k,0})_{k\in K}$, $(z_{k,0})_{k\in K}$, and $(v^*_{k,0})_{k\in K}$ be in $L^2(\Omega,\mathcal{F},\mathsf{P};\mathbb{R}^N)$. Iterate

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
l^*_n = \sum_{k\in K} v^*_{k,n};\\
\xi_n = \|\gamma_n l^*_n\|^2;\\
\text{if } k = k_n\\
\left\lfloor
\begin{array}{l}
b_{k,n} = y_{k,n} + \mu_{k,n}\big(v^*_{k,n} - \nabla g_k(y_{k,n})\big);\\
q^*_{k,n} = \mu_{k,n}^{-1}(y_{k,n} - b_{k,n}) - \sigma_{k,n}(x_n - y_{k,n} - z_{k,n});\ t^*_{k,n} = -v^*_{k,n} - \sigma_{k,n}(x_n - y_{k,n} - z_{k,n});\\
\eta_{k,n} = \|b_{k,n} - y_{k,n}\|^2 + \|\nu_{k,n}v^*_{k,n}\|^2;\ e_{k,n} = b_{k,n} + z_{k,n} + \nu_{k,n}v^*_{k,n} + \gamma_n l^*_n - x_n;
\end{array}
\right.\\
\text{else}\\
\left\lfloor
\begin{array}{l}
b_{k,n} = b_{k,n-1};\ q^*_{k,n} = q^*_{k,n-1};\ t^*_{k,n} = t^*_{k,n-1};\ \eta_{k,n} = \eta_{k,n-1};\ e_{k,n} = b_{k,n} + z_{k,n} + \nu_{k,n}v^*_{k,n} + \gamma_n l^*_n - x_n;
\end{array}
\right.\\
p^*_n = \sum_{k\in K}\big(\sigma_{k,n}(x_n - y_{k,n} - z_{k,n}) + v^*_{k,n}\big);\\
\Delta_n = -(4\alpha)^{-1}\big(\xi_n + \sum_{k\in K}\eta_{k,n}\big) + \langle \gamma_n l^*_n \mid p^*_n\rangle\\
\qquad + \sum_{k\in K}\Big(\langle y_{k,n} - b_{k,n} \mid q^*_{k,n}\rangle - \langle \nu_{k,n}v^*_{k,n} \mid t^*_{k,n}\rangle - \big\langle e_{k,n} \,\big|\, \sigma_{k,n}(x_n - y_{k,n} - z_{k,n})\big\rangle\Big);\\
\theta_n = 1_{[\Delta_n>0]}\Delta_n \Big/ \Big(\sum_{i\in I}\|p^*_{i,n}\|^2 + \sum_{k\in K}\big(\|q^*_{k,n}\|^2 + \|t^*_{k,n}\|^2 + \|e_{k,n}\|^2\big) + 1_{[\Delta_n\leqslant 0]}\Big);\\
\lambda_n \in L^\infty(\Omega,\mathcal{F},\mathsf{P};\,]0,+\infty[)\\
x_{n+1} = x_n - \lambda_n\theta_n p^*_n;\\
\text{for every } k \in K\\
\left\lfloor
\begin{array}{l}
y_{k,n+1} = y_{k,n} - \lambda_n\theta_n q^*_{k,n};\ z_{k,n+1} = z_{k,n} - \lambda_n\theta_n t^*_{k,n};\ v^*_{k,n+1} = v^*_{k,n} - \lambda_n\theta_n e_{k,n};
\end{array}
\right.
\end{array}
\right.
\end{array}
$$

In addition suppose that $\inf \mathsf{E}(\lambda_n(2-\lambda_n)) > 0$. Then $(x_n)_{n\in\mathbb{N}}$ converges P-a.s. and converges in $L^1(\Omega,\mathcal{F},\mathsf{P};\mathsf{H})$ to a $\mathscr{P}$-valued random variable.

# References & Acknowledgment

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