Geometrical scheme for stochastic Fejér monotonicity

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Background

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- The sets of strong and weak sequential cluster points of a sequence $(x_n)_{n\in\mathbb{N}}$ in H are denoted by $\mathfrak{S}(x_n)_{n\in\mathbb{N}}$ and $\mathfrak{W}(x_n)_{n\in\mathbb{N}}$, respectively.

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Problem 1 covers many problems in analysis and optimization.

Example I

For every $n \in \mathbb{N}$, let $T_n : \mathsf{H} \to \mathsf{H}$ be a *firmly nonexpansive operator*, i.e.,

$$(\forall (x, y) \in \mathsf{H} \times \mathsf{H}) \quad \|T_n x - T_n y\|^2 \leqslant \langle T_n x - T_n y \mid x - y \rangle. \tag{1}$$

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The common fixed point problem, i.e.,

$$\text{Find } \bar{x} \in \mathcal{H} \text{ such that } (\forall n \in \mathbb{N})\ T_n \bar{x} = \bar{x},$$

is an example of Problem 1.

Example II

Let $f\colon \mathcal{H} \to]-\infty, +\infty]$ be a proper, lower semicontinuous convex function.

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$$\underset{x \in \mathcal{H}}{\text{minimize}} \ \ f(x).$$

is an example of Problem 1.

Fejér monotonicity

A sequence $(x_n)_{n\in\mathbb{N}}$ of vectors in $\mathcal{H}$ is said to be Fejér monotone with respect to Z if

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3. Suppose that $\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset Z$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to some vector in Z.

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\begin{array}{l}
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\left\lfloor
\begin{array}{l}
t_n^* \in \mathcal{H} \text{ and } \eta_n \in \mathbb{R} \text{ satisfy} \\
\qquad \begin{cases}
\alpha_n = \begin{cases} \dfrac{\langle x_n \mid t_n^* \rangle - \eta_n}{\|t_n^*\|^2}, & \text{if } t_n^* \neq 0 \text{ and } \langle x_n \mid t_n^* \rangle > \eta_n; \\ 0, & \text{otherwise.} \end{cases} \\
(\forall z \in Z) \quad \left\langle z \,\middle|\, \alpha_n t_n^* \right\rangle \leqslant \alpha_n \eta_n
\end{cases} \\
x_{n+1} = x_n - \alpha_n t_n^*.
\end{array}
\right.
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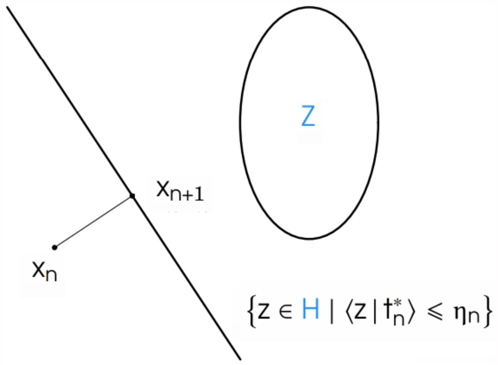
$$

Fejérian scheme



Fejérian scheme

Main issue: How to construct/select t_n^* and η_n such that

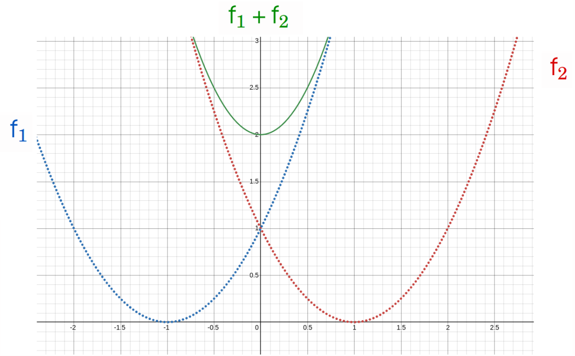
$$\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset Z \quad ?$$

Fejérian scheme

Stochastic algorithms

When transforming those algorithms into stochastic algorithms, **Fejér monotonicity is no longer true**.

Let $f_1 \colon x \mapsto (x+1)^2$ and $f_2 \colon x \mapsto (x-1)^2$. The task is to minimize $f_1 + f_2$ using the stochastic gradient method.



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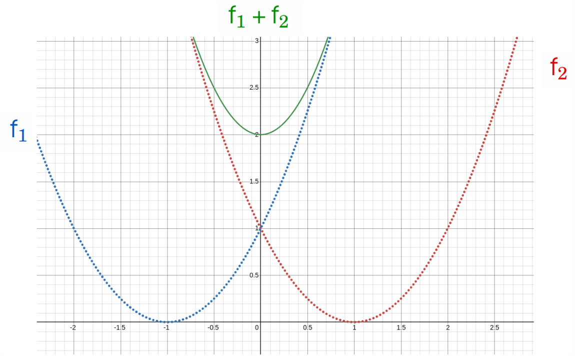

No Fejér monotonicity (in general)!

Background

- The underlying probability space is $(\Omega, \mathcal{F}, \mathsf{P})$. Denote as $\mathcal{B}_{\mathsf{H}}$ the Borel σ-algebra of H. An H-valued random variable is a measurable mapping $x\colon (\Omega, \mathcal{F}) \to (\mathsf{H}, \mathcal{B}_{\mathsf{H}})$.

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- Given $\mathsf{p} \in [1, +\infty[$, $L^{\mathsf{p}}(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$ denotes the space of H-valued random variables $x\colon (\Omega, \mathcal{F}) \to (\mathsf{H}, \mathcal{B}_{\mathsf{H}})$ such that

$$\mathsf{E}\|x\|^{\mathsf{p}} < +\infty.$$

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Further,

$$(\forall \mathsf{A} \in \mathcal{B}_{\mathsf{H}}) \quad L^{\mathsf{p}}(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{A}) = \{x \in L^{\mathsf{p}}(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}) \mid x \in \mathsf{A}\ \mathsf{P}\text{-a.s.}\}$$

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A sequence $(x_n)_{n\in\mathbb{N}}$ of H-valued random variables is said to be stochastic quasi-Fejér monotone with respect to Z if there exists a sequence of $[0,+\infty[$-valued random variables $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$(\forall z \in Z)(\forall n \in \mathbb{N}) \quad \begin{cases} \mathcal{X}_n = \sigma(x_0,\ldots,x_n) \\ \mathsf{E}\big(\|x_{n+1}-z\| \,\big|\, \mathcal{X}_n\big) \leqslant \|x_n - z\| + 2\varepsilon_n. \end{cases}$$

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We propose a geometrical framework...

Let $x_0 \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$. Iterate

$$
\begin{array}{l}
\text{for } \mathsf{n} = 0, 1, \ldots \\
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\begin{array}{l}
\mathfrak{X}_\mathsf{n} = \sigma(x_0, \ldots, x_\mathsf{n}) \\
t_\mathsf{n}^* \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}),\ \eta_\mathsf{n} \in L^1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}),\ \text{and } \varepsilon_\mathsf{n} \in L^1(\Omega, \mathcal{F}, \mathsf{P}; [0, +\infty[) \\
\quad \text{satisfy} \\
\qquad \begin{cases}
1_{[t_\mathsf{n}^* \neq 0]} \eta_\mathsf{n} / (\|t_\mathsf{n}^*\| + 1_{[t_\mathsf{n}^* = 0]}) \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}) \\
\alpha_\mathsf{n} = \dfrac{1_{[t_\mathsf{n}^* \neq 0 \text{ and } \langle x_\mathsf{n} | t_\mathsf{n}^* \rangle > \eta_\mathsf{n}]} \big(\langle x_\mathsf{n} \mid t_\mathsf{n}^* \rangle - \eta_\mathsf{n}\big)}{\|t_\mathsf{n}^*\|^2 + 1_{[t_\mathsf{n}^* = 0]}} \\
(\forall \mathsf{z} \in \mathsf{Z}) \ \ \langle \mathsf{z} \mid \mathsf{E}(\alpha_\mathsf{n} t_\mathsf{n}^* \mid \mathfrak{X}_\mathsf{n}) \rangle \leqslant \mathsf{E}(\alpha_\mathsf{n} \eta_\mathsf{n} \mid \mathfrak{X}_\mathsf{n}) + \varepsilon_\mathsf{n} \ \text{P-a.s.}
\end{cases} \\
x_{\mathsf{n}+1} = x_\mathsf{n} - \alpha_\mathsf{n} t_\mathsf{n}^*.
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$$

Some results

1. Let $n \in \mathbb{N}$ and $z \in Z$. Then
   $E(\|x_{n+1} - z\|^2 \mid \mathcal{X}_n) \leqslant \|x_n - z\|^2 + 2\varepsilon_n$ P-a.s.

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2. Let $n \in \mathbb{N}$ and $z \in L^2(\Omega, \mathcal{X}_n, P; Z)$. Then
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3. Let $n \in \mathbb{N}$ and $z \in L^2(\Omega, \mathcal{X}_n, P; Z)$. Then
$\|x_{n+1} - z\|^2_{L^2(\Omega,\mathcal{F},P;H)} \leqslant \|x_n - z\|^2_{L^2(\Omega,\mathcal{F},P;H)} + 2E\varepsilon_n$.

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Suppose that $\sum_{n\in\mathbb{N}} \varepsilon_n < +\infty$ P-a.s. Then the following hold:

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3. Suppose that $\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset \mathsf{Z}$ P-a.s. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly P-a.s. to a Z-valued random variable.

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4. Let x be an H-valued random variable. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly P-a.s. to x if and only if $(x_n)_{n\in\mathbb{N}}$ converges strongly in $L^1(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$ to x.

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