412 Individual HW4

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Problem 1

We have that $irr(\sqrt{2}, \mathbb{Q}) = x^2 - 2$, it is clear that this polynomial is irreducible by Eisenstein, choose $p = 2$. Also, $irr(\sqrt[3]{5}, \mathbb{Q}) = x^3 - 5$, which is also irreducible by Eisenstein and choosing $p = 5$. Thus, by divisibility of finite field towers theorem, neither of these fields can be contained in eachother. Thus, since it must be that $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{5}) : \mathbb{Q}(\sqrt[3]{5})]$ and $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{5}) : \mathbb{Q}(\sqrt{2})]$, the degree of the extension will be 6.

Problem 2

Problem 3

We see that $3x^5 - 4x + 2$ is irreducible in $\mathbb{Q}[x]$, due to Eisenstein's criterion, choosing $p = 2$. Therefore, $irr(\alpha, \mathbb{Q}) = x^5 - \frac{4}{3}x + \frac{2}{3}$, and therefore the degree of $\mathbb{Q}(\alpha)$ is 5. Thus, if K was some field between $\mathbb{Q}$ and $\mathbb{Q}(\alpha)$, $[K : \mathbb{Q}] = 1, 5$ due to divisibility. If $[K : \mathbb{Q}] = 1$, then $K = \mathbb{Q}$. Otherwise, suppose $[K : \mathbb{Q}]$ has degree 5. Then $[\mathbb{Q}(\alpha) : K]$ has degree 1, and $K = \mathbb{Q}(\alpha)$.

Problem 4

Consider the field $F(\alpha)$. Every element in this field can be written as $a_{n-1}\alpha^{n-1} + \cdots a_0$ for $a_i \in F$ and some n. Since $\alpha \in D$ and $a_i \in D$, any element of $F(\alpha) \in D$ and therefore $F(\alpha) \subset D$. Since $F(\alpha)$ is a field, $\frac{1}{\alpha} \in F(\alpha) \in D$. Therefore, for any $\alpha \in D$, $\frac{1}{\alpha} \in D$. Since D is an integral domain, and contains all multiplicative inverses, it is a field.

Problem 5

Let $[F(\alpha) : F] = 2k + 1$. Exclude the trivial case where $k = 0$ as then $\alpha \in F$ and $F(\alpha) = F = F(\alpha^2)$. Otherwise, the elements of $F(\alpha) = a_{2k}\alpha^{2k} + \cdots + a_0$ with $a_i \in F$. Since $k > 0$, clearly $\alpha^2 \in F(\alpha)$ and therefore we get the tower of fields $F \subset F(\alpha^2) \subseteq F(\alpha)$. Suppose $F(\alpha) \neq F(\alpha^2)$. Then $\alpha \notin F(\alpha^2)$. In $F(\alpha^2)[x]$ we have the polynomial $x^2 - \alpha^2$ which has α as a root. Since the degree of α over $F(\alpha^2)$ cannot be 1 as we supposed $F(\alpha) \neq F(\alpha^2)$, it has degree 2. Therefore, we have that $[F(\alpha) : F] = [F(\alpha) : F(\alpha^2)][F(\alpha^2) : F]$, $2k + 1 = 2 \cdot [F(\alpha^2) : F]$. But this cannot be the case, so $F(\alpha) = F(\alpha^2)$.

Problem 6

Since α_1 is algebraic over K by definition, we know that $K \subset K(\alpha_1)$ is a finite extension with $[K(\alpha_1) : K] = deg(irr(\alpha_1, K))$. Then, we have $K \subset K(\alpha_1) \subset K(\alpha_1)(\alpha_2) = K(\alpha_1, \alpha_2)$. Since K is a subset of $K(\alpha_1)$, we have that $deg(irr(\alpha_2, K(\alpha_1))) \leq deg(irr(\alpha_2, K))$, and we know that $deg(irr(\alpha_2, K))$ is finite again by the definition. Proceeding with this process up to k, we conclude that

$$[K(\alpha_1, \cdots, \alpha_k) : K] = \prod_{i=1}^{k} [K(\alpha_1, \cdots, \alpha_i) : K(\alpha_1, \cdots, \alpha_{i-1})] \leq \prod_{i=1}^{k} deg(irr(\alpha_i, K))$$

and thus the extension is finite.

Problem 7

Problem 8

We know that it is possible to construct the regular pentagon in a circle from class. After we have done this, we can create the other 5 points of the 10-gon by bisecting each side of the pentagon. We can then connect all 10 points with the straight edge giving us the 10-gon. So the 10 gon is constructible.

Problem 9

The internal angles of the 9-gon are 20 degrees, so constructing the 9-gon would be equivalent to trisecting the 60 degree angle, which is not possible. Thus, the 9-gon is not constructible.

Problem 10

Let $x^2 = y$. Then, we have $y^2 - 6x + 2 = 0$. By the quadratic formula, this holds for $y = 3 \pm \sqrt{7}$, and thus $x^4 - 6x^2 + 2$ will hold for $x = \sqrt{3 \pm \sqrt{7}}$. Computing $x^4 - 6x^2 + 2$ for $x = \sqrt{3 - \sqrt{7}}$ yields:

$$(3 - \sqrt{7})^2 - 6(3 - \sqrt{7}) + 2$$

$$9 - 6\sqrt{7} + 7 - 18 - 6\sqrt{7} + 2 = 0$$

and $\sqrt{3 - \sqrt{7}}$ is clearly constructible by the "IPhone calculator" fact. Thus, this polynomial has a constructible root.

Problem 11

Problem 12

After guessing and checking the first 3-4 terms we guess that $cos\left(\frac{2\pi}{2^{n+2}}\right) = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}$ n times. We prove this via induction. For the base case, $n = 1$, we have $cos\left(\frac{2pi}{8}\right) = cos\left(\frac{\pi}{4}\right) = \sqrt{2}$ which holds. Then, suppose that this formula holds for n. We will use the half angle formula to show that it holds for $n + 1$.

$$cos\left(\frac{2\pi}{2^{(n+1)+2}}\right) = \sqrt{\frac{1 - cos\left(\frac{2\pi}{2^{n+2}}\right)}{2}}$$

using our inductive hypothesis we have:

$$cos\left(\frac{2\pi}{2^{(n+1)+2}}\right) = \sqrt{\frac{1 - \frac{\sqrt{2+\sqrt{2+\cdots}} \text{ } n \text{ times}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{2 + \sqrt{2 + \cdots}} \text{ } n+1 \text{ times}}{4}} = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}} \text{ } n+1 \text{ times}}$$

as desired.