MA411 HW1

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Exercise 1.8.4

a) Define $f^{-1}(x) = y$ such that $f(y) = x$. We see that this function is well defined for all $x \in B$, because since f is onto, every x has $y \in A$ such that $f(y) = x$, and the fact that f is 1-1 ensures that y is unique.

b) If f has an inverse f^{-1}, then it is 1-1, because for f^{-1} to be well defined it means each input must correspond to one output. Likewise, since f^{-1} is defined for all $x \in B$, it means that every element $x \in B$ has a $y \in A$ such that $f(y) = x$. So f is onto and 1-1.

c)

Exercise 1.8.5

a) Neither 1-1 or onto.

b) 1-1, but not onto

c) 1-1 and onto

Exercise 1.8.9

We first prove the right side of the iff: We perform a proof by contraposition. Given that a function from A to B, $|A| = |B|$, is not onto, it is not 1-1. There are n elements in n. Given that it is not onto, it maps to a set of size at most $n - 1$. Map the first $n - 1$ elements of A in a 1-1 fashion. Then n^{th} element must be mapped to an element which is mapped to by some other element in A making the function not 1-1. For the second part of the proof, we again perform contraposition. Given that a function from A to B, $|A| = |B|$, is not 1-1, it is not onto. Given that it is not 1-1, there exist $x, y \in A$ such that $f(x) = f(y)$. Assume the other $n - 2$ values map to $n - 2$ values in the B. Additionally, x and y map to 1 additional value in B. So f reaches $n - 1$ elements in B whereas B has size n. So f cannot be onto.

Exercise 1.8.17

Prove that $|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$

Exercise 1.8.18

a) Not a binary operation, choose $a = 1$, $b = 2$, then $a, b \in \mathbb{Z}$ but $a \circ b = \frac{1}{2} \notin \mathbb{Z}$

b) Is a binary operation, we can recursively use the fact that an integer times an integer is an integer.

c) Is not a binary operation, choose $a = 1, b = 2$. It is clear that $\sqrt{2} \notin \mathbb{Z}$

Problem 7

Choose $f : S \mapsto T$, $g : T \mapsto U$, where S, T, U are sets and f and g are bijections. Compose f and g to create a function $g \circ f : S \mapsto U$. We will prove that this is a bijection. First, we will show that $g \circ f$ is surjective. Choose an arbitrary element $x \in U$. Since g is a bijection, and therefore a surjection, we know that there is an element $y \in T$ such that $g(y) = x$. Furthermore, since f is a bijection and also therefore a surjection, we know that there exists $z \in S$ such that $f(z) = y$. Clearly then, for every $x \in U$ there exists $z \in S$ such that $g(f(z)) = x$ and so $g \circ f$ is surjective. Now, we will show that it is injective. Choose $x \in U$, and suppose that there exist $z_1 \neq z_2 \in S$ such that $g(f(z_1)) = g(f(z_2)) = x$. Since g is injective, we know that $f(z_1) = f(z_2)$, and since f is also injective, we know that z_1 must be equal to z_2. Here we have a contradiction, showing us that there is a unique $z \in S$ for every $x \in U$ therefore making $g \circ f$ injective. Since $g \circ f$ is injective and surjective, it is a bijection.

Exercise 2.1.8

$$(ab)^{-1} = b^{-1}a^{-1}$$
$$(ab)^{-1}(ab) = b^{-1}a^{-1}(ab)$$
$$e = b^{-1}(a^{-1}a)b$$
$$e = b^{-1}eb$$
$$e = b^{-1}b$$
$$e = e$$

Exercise 2.1.9

Let $a^{-1} = b$:

$$(b)^{-1} = a$$
$$b(b)^{-1} = ba$$
$$b(b)^{-1} = a^{-1}a$$
$$e = e$$

Problem 10

We suppose $x \circ a = x \circ b$. Since G is a group, and $x \in G$, x has an inverse x^{-1}. We multiply the equation on both sides by x^{-1}. $x^{-1} \circ x \circ a = x^{-1} \circ x \circ b$. Since groups are associative we may perform the multiplications in any order. This leaves us with $e \circ a = e \circ b$, where e is the identity element of G. Since for any $y \in G$, $e \circ y = y$, $a = b$.

Problem 11

Let $f = x^2, g = -x, h = x$

Exercise 2.1.10

$$(ab)^2 = a^2b^2$$
$$(ab)(ab) = a^2b^2$$
$$a^{-1}(ab)(ab) = a^{-1}a^2b^2$$
$$(a^{-1}a)b(ab) = (a^{-1}a)a(b^2)$$
$$eb(ab) = e(a)b^2$$
$$ba(bb^{-1}) = (ab)(bb^{-1})$$
$$bae = abe$$
$$ba = ab$$

Problem 13

First, we see that the identity element is the null set. This is because for any $A \in 2^S$, $(A \cup \varnothing) - (A \cap \varnothing) = (A) - \varnothing = A$. Second, we see that for any $A \in 2^S$, the inverse of A is itself, as $(A \cup A) - (A \cap A) = (A) - (A) = \varnothing$. Third, we must prove associativity. Take $A, B, C \in 2^S$:

Then: We write some proof that shows that these two sets are equal.

End of proof.