411 Individual HW6

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Problem 1

a) Since $\mathbb{Z}_{21}$ is Abelian by listing the left cosets we are implicitly listing the right cosets and so we will only list the left cosets. H is also normal for the same reason. The cosets are: $\langle 7\rangle, 1+\langle 7\rangle, 2+\langle 7\rangle, 3+\langle 7\rangle, 4+\langle 7\rangle, 5+\langle 7\rangle, 6+\langle 7\rangle$

b) $\mathbb{Z}^*{}_{13}$ is also abelian and therefore any subgroup will be normal. Same as the last we will only list the right cosets. The subgroup generated by 3 is $\{1, 3, 9\}$. So the cosets will be $\langle 3\rangle, 2\langle 3\rangle, 4\langle 3\rangle, 7\langle 3\rangle$.

c) We can see that $\langle (1\ 2), (2\ 3)\rangle$ generates a subgroup isomorphic to S_3 in S_4. Then we have that the left cosets are $\langle (1\ 2), (2\ 3)\rangle, (1\ 4)\langle (1\ 2), (2\ 3)\rangle, (2\ 4)\langle (1\ 2), (2\ 3)\rangle, (3\ 4)\langle (1\ 2), (2\ 3)\rangle$ after some computation. This subgroup is not normal, take the coset $(2\ 4)\langle (1\ 2), (2\ 3)$. We know that $(2\ 4)$ is its own inverse. Conjugate $(1\ 2)$, giving $(2\ 4)(1\ 2)(2\ 4) = (1\ 4)$, which is not in the subgroup. So it is not normal.

d) Let $D_4 = \{Id, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2), (1\ 3), (2\ 4), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}$. Choose representatives of right cosets $e, (1\ 2), (1\ 4)$. Then we have:

$$D_4 e = D_4$$

$$D_4(1\ 2) = \{(1\ 2), (1\ 3\ 4), (1\ 4\ 2\ 3), (2\ 4\ 3), (1\ 2\ 3), (1\ 4\ 2), (3\ 4), (1\ 3\ 2\ 4)\}$$

$$D_4(1\ 4) = \{(1\ 4), (2\ 3\ 4), (1\ 2\ 4\ 3), (1\ 3\ 2), (1\ 4\ 3), (1\ 2\ 4), (2\ 3), (1\ 3\ 4\ 2)\}$$

We then calculate the left cosets:

$$eD_4 = D_4$$

$$(1\ 2)D_4 = \{(1\ 2), (2\ 3\ 4), (1\ 3\ 2\ 4), (1\ 4\ 3), (1\ 3\ 2), (1\ 2\ 4), (3\ 4), (1\ 4\ 2\ 3)\}$$

$$(1\ 4)D_4 = \{(1\ 4), (1\ 2\ 3), (1\ 3\ 4\ 2), (2\ 4\ 3), (1\ 3\ 4), (1\ 4\ 2), (1\ 2\ 4\ 3), (2\ 3)\}$$

We observe that the left and right cosets give diferrent partitions of S_4 and thus D_4 is not a normal subgroup of S_4.

Problem 2

a) We examine the normal subgroups of S_4. We note that for a subgroup to be normal, it must contain all or no members of a conjugacy class, as otherwise it is not closed under conjugation which is the requirement for a normal subgroup. We also note that proper subgroups of S_4 have orders of 2, 3, 4, 6, 8, 12. All elements of order 2 constitute subgroups of order 2, but none of these subgroups are normal as they are generated by elements of the form $(a\ b)$ or $(a\ b)(c\ d)$ which are part of conjugacy classes larger than 1 in S_4. Thus these groups are not closed under conjugation and not normal. We observe that we have a subgroup of S_4 consisting of the union of elements of the form $(a\ b)(c\ d)$ of which there are 3, and identity. We prove that this is a subgroup as follows. First, it contains the identity, e. Also, each element is its own inverse. Lastly, we check closure:

$$(1\ 2)(3\ 4)(1\ 2)(3\ 4) = e$$

$$(1\ 2)(3\ 4)(1\ 3)(2\ 4) = (1\ 4)(2\ 3)$$
$$(1\ 2)(3\ 4)(1\ 4)(2\ 3) = (1\ 3)(2\ 4)$$
$$(1\ 3)(2\ 4)(1\ 2)(3\ 4) = (1\ 4)(2\ 3)$$
$$(1\ 3)(2\ 4)(1\ 3)(2\ 4) = e$$
$$(1\ 3)(2\ 4)(1\ 4)(3\ 4) = (1\ 2)(3\ 4)$$
$$(1\ 4)(2\ 3)(1\ 2)(3\ 4) = (1\ 3)(2\ 4)$$
$$(1\ 4)(2\ 3)(1\ 3)(2\ 4) = (1\ 2)(3\ 4)$$
$$(1\ 4)(2\ 3)(1\ 4)(2\ 3) = e$$

So this is a subgroup, and it is also normal as the conjugate of any element in it will also be in it as it contains all members of each conjugacy class included. We notice that this group is isomorphic to V_4.

b) We examine the normal subgroups of A_4, or those groups that are closed under conjugation. By including all elements which have the same cycle structure in A_4 in a subgroup of A_4, we guarantee that it will be closed under conjugation, as conjugation does not modify cycle structure. Since A_4 has order 12, its proper subgroups will have orders 2, 3, 4, 6. We know that there are 8 3-cycles, which may be written as products of 2 transpositions, and 3 products of disjoint transpositions, along with the identity. We observe that under the map f given by $e \to Id, a \to (1\ 2)(3\ 4), b \to (1\ 4)(2\ 3), c \to (1\ 3)(2\ 4)$, the union of the products of two disjoint transpositions and the identity is isomorphic to the Klein group. This group is also normal as since all products of two disjoint transpositions are included in the subgroup, any conjugation of these elements will yield another element of this type and thus will be closed under conjugation.

We note that each of the elements of this group form cyclic subgroups of order 2, however, these groups are not normal as:

$$(1\ 2\ 3)(1\ 2)(3\ 4)(1\ 3\ 2) = (1\ 4)(2\ 3) \notin \langle (1\ 2)(3\ 4) \rangle$$
$$(1\ 2\ 3)(1\ 3)(2\ 4)(1\ 3\ 2) = (1\ 2)(3\ 4) \notin \langle (1\ 3)(2\ 4) \rangle$$
$$(1\ 2\ 3)(1\ 4)(2\ 3)(1\ 3\ 2) = (1\ 3)(2\ 4) \notin \langle (1\ 4)(2\ 3)$$

We also have 4 cyclic subgroups: $\langle (1\ 2\ 3) \rangle, \langle (1\ 2\ 4) \rangle, \langle (1\ 3\ 4) \rangle, \langle (2\ 3\ 4) \rangle$. We see however, that these are not closed under conjugation, take for instance $(2\ 3\ 4)(1\ 2\ 3)(2\ 4\ 3) = (1\ 3\ 4) \notin \langle (1\ 2\ 3) \rangle, (1\ 2\ 3)(1\ 2\ 4)(1\ 3\ 2) = (2\ 3\ 4) \notin \langle (1\ 2\ 4) \rangle, (2\ 3\ 4)(1\ 3\ 4)(2\ 4\ 3) = (1\ 4\ 2) \notin \langle (1\ 3\ 4) \rangle, (1\ 2\ 3)(2\ 3\ 4)(1\ 3\ 2) = (1\ 4\ 3) \notin \langle (2\ 3\ 4) \rangle$.

So the only non trivial normal subgroup in A_4 is the subgroup isomorphic to the klein group.

c) We write D_4 as $\{Id, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2), (1\ 3), (2\ 4), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}$. By Lagrange's theorem, we know that subgroups must have order $1, 2, 4, 8$. The only group with order 1 is the trivial group, and the only group with order 8 is D_4 itself. So we search for groups with orders 2 and 4. For groups with order 2 we have the reflections $\{e, (1\ 3)\}, \{e, (2\ 4)\}, \{e, (1\ 2)(3\ 4)\}, \{e, (1\ 4)(2\ 3)\}$. None of these groups are normal, simply try conjugating the first two by $(1\ 2)(3\ 4)$ and the last two by $(1\ 3)$- they are not closed under conjugation and not normal. The last group of order 2 is the group $\{e, (1\ 3)(2\ 4)\}$. Since we proved in the last homework that this is the center of D_4, it is automatically a normal subgroup.

For the subgroups of order 4, we first have the group of rotations. As it is a subgroup and therefore closed, conjugations of any element by another rotation will yield a rotation. As we showed in the last homework as well, conjugation of a rotation by a reflection is also a rotation. Thus, this subgroup is normal. From class, we also observe that there are two subgroups which are isomorphic to the Klein group, $\{Id, (1\ 3)(2\ 4), (1\ 3), (2\ 4)\}, \{Id, (1\ 3)(2\ 4), (1\ 2)(3\ 4), (1\ 4)(2\ 3)\}$. As $(1\ 3)(2\ 4)$ is in the center, it is conjugate to itself. The other elements which are isomorphic to a, b in the Klein group, are conjugate to eachother as they have the same cycle decompositions, 1 and 2 disjoint transpositions respectively.

d) We examine the normal subgroups of Z_{23}^*. We note that since Z_{23}^* is Abelian, any subgroup will be normal. We also notice that by Lagrange's theorem, the subgroups must have orders 1, 2, 11, or 22, as the order of the group is 22. We notice that $\langle 5\rangle$ generates the group and thus all subgroups must also be cyclic. Thus, we find the normal subgroups:

$$\langle 5\rangle, \langle 2\rangle, \langle 22\rangle, e$$

.

Problem 3

a) Take an element in the center of S_n, τ. Then $\tau\sigma = \sigma\tau$ for all $\sigma \in S_n$. This can also be stated as $\tau = \sigma\tau\sigma^{-1}$. However, as we have shown, $\sigma\tau\sigma^{-1}$ maps τ We notice that for any of the types of cycle structures, as we have shown in the homework, there exists a permutation in S_n that maps τ to any other permutation with the same cycle structure under conjugation. Thus, if τ is not the only permutation with a given cycle structure in S_n, this equation will not always hold and it cannot be in the center.

   We will show that in S_n, the only cycle structure that corresponds to only 1 element is simply zero cycles, or the identity. Take a permutation which is not the identity. Then this permutation contains at least one disjoint cycle. If we consider elements mapped to themselves as "1-cycles", we see that for any such permutation swapping two elements in different cycles (while not swapping 1-cycles with eachother) produces a different permutation with the same cycle structure, and is therefore conjugate to the first permutation and therefore not in the center.

   We must also say that this proof does not hold for S_2, in this case both elements $e, (1,2)$ are conjugate to only themselves and thus S_2 is its own center.

b) y Let's imagine a matrix $X \in Z(GL_3(\mathbb{R}))$. Then $XY = YX$ for any Y in $GL_3(\mathbb{R})$. Let X have row vectors x_1, x_2, x_3 and column vectors x'_1, x'_2, x'_3. We define an arbitrary Y similarly. We then have the condition that $XY = YX$ if and only if $x_i \cdot y'_j = x'_j \cdot y_i$ for all $i, j \in \{1, 2, 3\}$. Let's examine these dot products.

$$x_i \cdot y'_j = x_{i1} \cdot y_{1j} + x_{i2} \cdot y_{2j} + x_{i3} \cdot y_{3j}$$

$$x'_j \cdot y_i = y_{i1} \cdot x_{1j} + y_{i2} \cdot x_{2j} + y_{i3} \cdot x_{3j}$$

Problem 4

a) x

b) y

Problem 5

a) Yes, it's a homomorphism. Choose $a, b \in \mathbb{Z}$. Then $f(a+b) = 5(a+b)$ and $f(a)+f(b) = 5a+5b = 5(a+b)$. Since the identity in $\mathbb{Z}$ is 0, the kernel is just zero.

b) Yes, it's a homomorphism. Choose $a, b \in \mathbb{R}^*$. Then $f(ab) = |ab|$, and $f(a) \cdot f(b) = |a||b| = |ab|$. The kernel of this homomorphism is $\{-1, 1\}$.

c) Yes, it's a homomorphism. Choose $x, y \in \mathbb{R}$. Then $f(x+y) = e^{x+y}$, and $f(x) \cdot f(y) = e^x e^y = e^{x+y}$. Since the multiplicative identity is 1, f has the kernel 0 since $0 = ln(1)$.

d) No, it's not a homomorphism. Choose $x = 1, y = 2$. Then $f(1+2) = 3^{43} \neq 1 + 2^{43} = f(1) + f(2)$.

e) Yes, it's a homomorphism. Let $x, y \in \mathbb{Z}_{43}$. Then $(x+y)^4 3 = \sum_0^{43} \binom{43}{k} x^{43-k} y^k$. Then:

$$(x+y)^{43} = \sum_0^{43} \binom{43}{k} x^{43-k} y^k$$

$$(x+y)^{43} = x^{43} + \sum_1^{42} \binom{43}{k} x^{43-k} y^k + y^{43}$$

We observe that within the range $1 \leq k \leq k-1$,

$$\binom{43}{k} = \frac{43 \cdot 42 \cdot 41 \cdots 2 \cdot 1}{(k \cdot k-1 \cdots 2 \cdot 1)(43-k \cdot 43-k-1 \cdots 2 \cdot 1)}$$

$$\binom{43}{k} = \frac{43 \cdot 42 \cdots 43-k+1}{k \cdot k-1 \cdots 1}$$

Loosely, since $k < 43$ and 43 is prime, it cannot be adding any "divisibility" to the numerator and so we pull it out. Thus, we have:

$$(x+y)^{43} \ mod \ 43 = x^{43} + 43 \sum_1^{42} \frac{42 \cdot 41 \cdots 43-k+1}{k \cdot k-1 \cdots 1} x^{43-k} y^k + y^{43} \ mod \ 43 \equiv x^{43} + y^{43} \ mod \ 43$$

So we have showed the property of homomorphism. Now we must determine the kernel of f. $f(x) = x^{43} mod 43$ Since the identity is 0, we find all x such that

$$x^{43} mod 43 = 0$$

. We see that $x^{43} = xx^{42}$ and so x^{43} is a multiple of x and therefore equal to $xmod43$. Thus the kernel is only $x = 0$.

f) Yes, it's a homomorphism. This is because of the fact that for any two matrices $X, Y \in GL_n(\mathbb{R})$, $det(XY) = det(X)det(Y)$. The function may also never map to zero, which is not in $\mathbb{R}^*$ because matrices in $GL_n(\mathbb{R})$ have non-zero determinants. Its kernel is the special linear group, which are matrices with determinant 1.

Problem 6

a) The cosets of $\langle(1\ 2)\rangle$ are $\langle(1\ 2)\rangle = \{Id, (1\ 2)\}$, $(1\ 3)\langle(1\ 2)\rangle = \{(1\ 3), (1\ 2\ 3)\}$, $(2\ 3)\langle(1\ 2)\rangle = \{(2\ 3), (2\ 1\ 3)\}$. We find that:

$$\{Id, (1\ 2)\} \star \{Id, (1\ 2)\} = \{Id, (1\ 2)\}$$
$$\{Id, (1\ 2)\} \star \{(1\ 3), (1\ 2\ 3)\} = \{(1\ 3), (1\ 2\ 3), (1\ 3\ 2), (2\ 3)\}$$
$$\{Id, (1\ 2)\} \star \{(2\ 3), (2\ 1\ 3)\} = \{(2\ 3), (2\ 1\ 3), (1\ 2\ 3), (1\ 3)\}$$

$$\{(1\ 3), (1\ 2\ 3)\} \star \{(1\ 3), (1\ 2\ 3)\} = \{Id, (1\ 2), (2\ 3), (2\ 1\ 3)\}$$
$$\{(1\ 3), (1\ 2\ 3)\} \star \{Id, (1\ 2)\} = \{(1\ 3), (1\ 2\ 3)\}$$
$$\{(1\ 3), (1\ 2\ 3)\} \star \{(2\ 3), (2\ 1\ 3)\} = \{(2\ 3), (2\ 1\ 3), Id, (1\ 2)\}$$

$$\{(2\ 3), (2\ 1\ 3)\} \star \{(2\ 3), (2\ 1\ 3)\} = \{Id, (1\ 2), (1\ 3), (1\ 2\ 3)\}$$
$$\{(2\ 3), (2\ 1\ 3)\} \star \{Id, (1\ 2)\} = \{(2\ 3), (2\ 1\ 3)\}$$
$$\{(2\ 3), (2\ 1\ 3)\} \star \{(1\ 3), (1\ 2\ 3)\} = \{(1\ 3), (1\ 2\ 3), Id, (1\ 2)\}$$

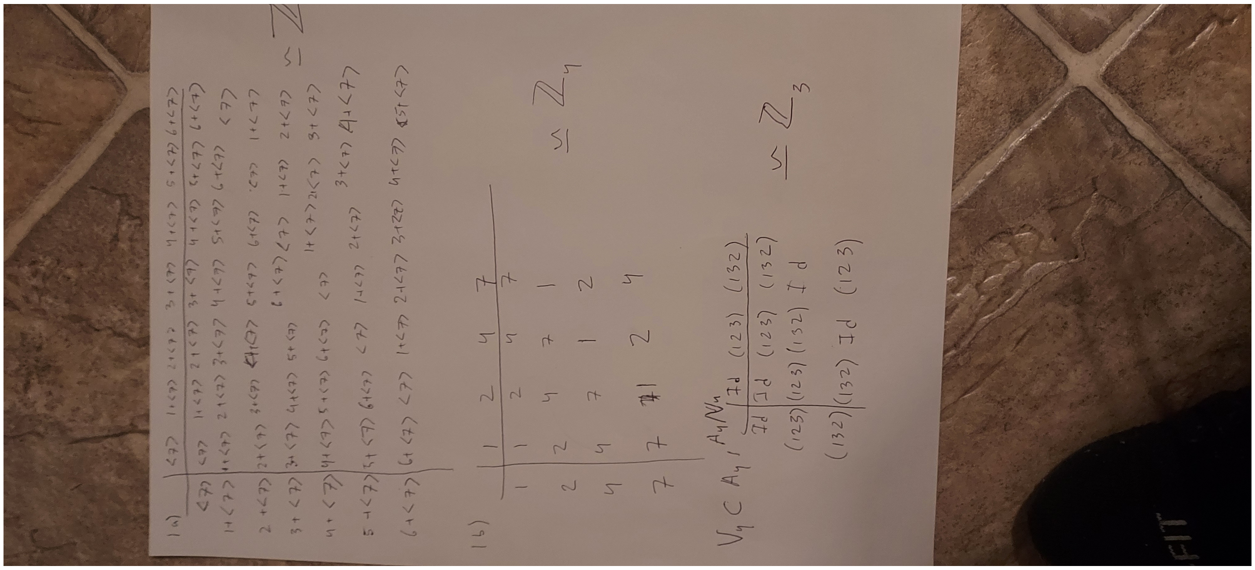

Figure 1: Enter Caption

b) Let there be two left cosets C_1, C_2 and choose representatives c_1, c_2. Then every element in C_1 can be written as c_1h_1 for some $h_1 \in H$, and every element in C_2 can be written as c_2h_2 for some $h_2 \in H$. We then have $C_1 \star C_2 = \{c_1h_1c_2h_2 | h_1, h_2 \in H\}$. If we fix $h_1 = h$ we see that $c_1hc_2 \in G$ and thus is the representative of some coset $(c_1hc_2)H$. Thus, $C_1 \star C_2$ is the union of cosets with representatives c_1hc_2, $h \in H$.

c) To do this we will prove that if H is normal then $c_1h_1c_2H = c_1h_2c_2H$, for $h_1, h_2 \in H$ and c_1, c_2 as representatives of C_1 and C_2. To do this we must just prove that $c_1h_1c_2 \in c_1h_2c_2H$, or that there exist $h, \bar{h} \in H$ such that $c_1h_1c_2h = c_1h_2c_2\bar{h}$. Let $\bar{h} = e$. We then solve for h and get $h = c_2^{-1}h_1^{-1}h_2c_2$. Since $h_1^{-1}h_2 \in H$ by closure, and then $c_2^{-1}h_1^{-1}h_2c_2 \in H$ as H is normal. Thus we have shown that if H is normal, then the product $C_1 \star C_2$ results in a single coset.

We proceed to proving the opposite direction. We have that $C_1 \star C_2 = c_1hc_2H$ for some $h \in H$. Since modulating h does not modify the coset, choose h to be e, and then $C_1 \star C_2 = c_1c_2H$. Thus, there exists some h such that we have $c_1h_1c_2h_2 = c_1c_2h$ for any $h_1, h_2 \in H$. This simplifies to $h_1c_2h_2 = c_2h$, and then $c_2^{-1}h_1c_2 = hh_2$. Since any $g \in G$ is the representative of some coset, and $hh_2 \in H$ by closure, and since we have made h_1 arbitrary, we have shown that conjugation of any $h \in H$ by any $g \in G$ is also in H, the condition for a normal subgroup.

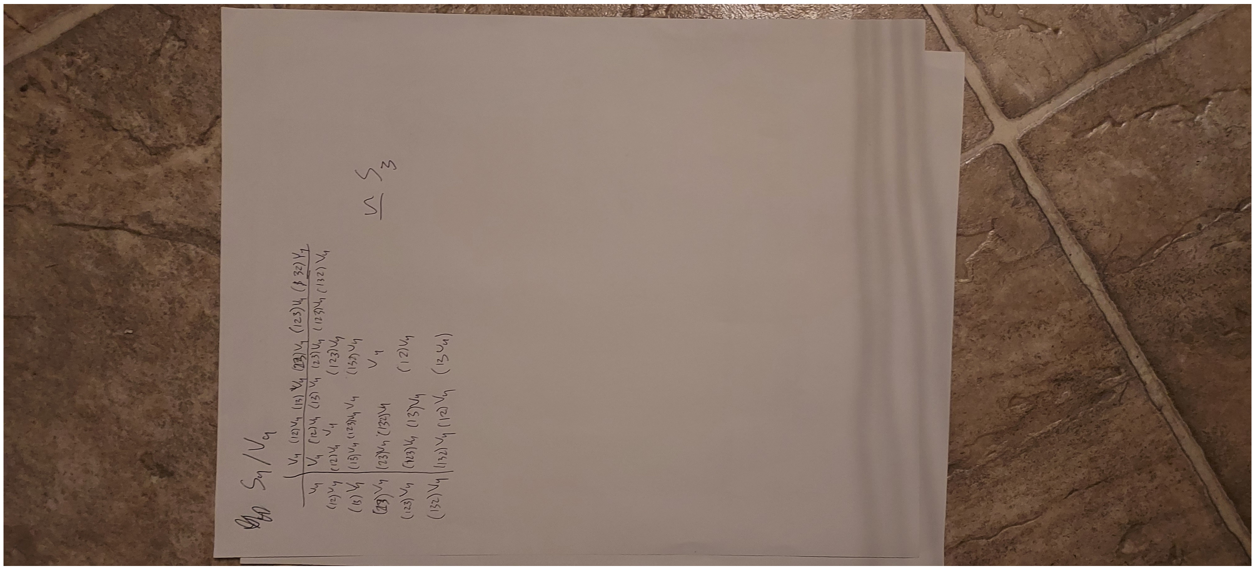

Figure 2: Enter Caption

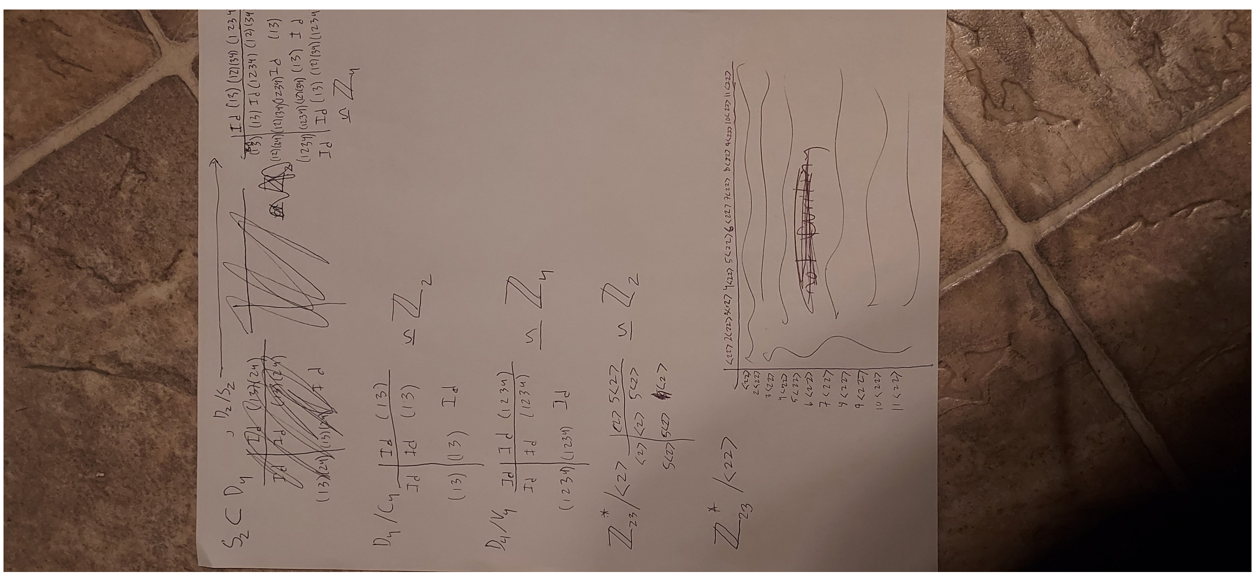

Figure 3: Enter Caption