Fourier Analysis of Continued Fraction Representations of Non-Trivial Riemann Zeta Zeros

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ABSTRACT

The Riemann zeta function is the analytic continuation of the infinite series of the reciprocals of the natural numbers raised to a given power s, the complex variable of the function. The function trivially has zeros at negative even integers, but its other, non-trivial roots are of the most interest to modern mathematics. These non-trivial zeros are directly related to the distribution of prime numbers and as such gaining a better understanding of their behavior would be critical to securing many current Internet encryption methods. Additionally, Riemann's hypothesis that all non-trivial zeros have real part ½ is revered as the "Greatest Unsolved Problem in Mathematics" [1], and finding a pattern among the roots we've already calculated may provide a step in unlocking it and thus have a profound impact on many fields, particularly modern number theory. Through analyzing simple infinite continued fraction and generalized continued fraction representations of the imaginary components of known zeros, this investigation would hopefully yield equations to accurately approximate the n^{th} root of the function, a tool that would drastically aid in current methods for root-finders of the zeta function.

INTRODUCTION

Continued fractions, in the simplest sense, are another kind of number. Any rational number can be represented as a finite simple continued fraction, while any irrational number can be represented by an infinite simple continued fraction. Some irrationals exhibit patterns in the terms of their simple CFs, while others, such as π, have had no such pattern discovered. However, continued fractions may also use functions to generate the terms of their numerators and partial denominators. Called generalized continued fractions, several of them have been found that elegantly converge to π [2].

While a simple algorithm can be applied to generate the simple CF terms for any real number, there is no known algorithm to find a generalized CF that converges to a given real number, though infinitely many seem to exist. In fact, the convergence problem of generalized CFs is still a relatively open problem in mathematics, and as such our investigation into generalized CF representations of zeta zero approximations will be fairly arbitrary in its basis and numerical in approach.

$$x = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}} \qquad \begin{array}{l} b_n = A_i n^2 + B_i n + C_i \\ a_n = D_i n + E_i \end{array}$$

$$n = 1 \cdots 15$$

$$a_0, b_0 \in \{1, 2, 3, 5, 7, 11, 13, 17, 19\}$$

$$A_i, B_i, C_i, D_i, E_i \in \left\{ \begin{array}{c} -19, -17, -13, -11, -7, \\ -5, -3, -2, -1, 0, \\ 1, 2, 3, 5, 7, 11, 13, 17, 19 \end{array} \right\}$$

Figure 1: Generalized continued fraction search definitions

Since we are limited by computation time, we necessarily have to restrict our search space in looking for best-fit generalized CFs for a given imaginary component of a root. Unfortunately, most of these decisions are fairly arbitrary due to the little understanding we have in this area of mathematics. Despite this, several factors still guided our decisions in refining the search space. For one, we chose to fit the numerators to a quadratic and the partial denominators to a linear function due to the fact that an elegant generalized continued fraction for π is of the form $\pi = [3; 1,\{(2n+1)^2|6\}]$ [2]. Note that the coefficients of any quadratic or linear term may be 0, so we aren't strictly limiting our search to polynomials of these orders. The set of possible initial coefficients consisted of the primes in [2, 19] and {0, 1}, with the set of possible function coefficients also including this set's negation. This is arguably the most arbitrary decision in establishing our search parameters, but was ultimately made due to the exponential increase in computation time from adding more coefficients and was guided by the zeta function's connection to the distribution of the primes.

METHODS AND MATERIALS

Before examining patterns in best-fit generalized continued fraction representations for the zeta zeros, we first investigated simple continued fraction representations for the same numbers. As previously mentioned, obtaining the infinite simple CF up to n terms is a fairly simple process. Through a simple recurrence relationship, the nth convergent of a continued fraction can be easily calculated, and this algorithm was implemented in Python. All computation was performed using the imaginary components of the zeta zeroes previously calculated to 1024 digits of precision and Python's built-in decimal library, providing arbitrarily high precision. The first 1024 terms of the simple continued fraction representation for each of the first 100 imaginary components were calculated, and further data, such as the longest recurring subsequence length and maximum term value, were also extracted.

Upon first glance, most of the data we gathered in this investigation appeared chaotic and random. In the interest of gaining insight to potential order behind this wave-like chaos, the discrete Fourier transform was applied to these data sets, again implemented in Python. All analysis from this simple CF portion of the project was also checked in Wolfram's Mathematica due to its built-in functions that make most of it trivial.

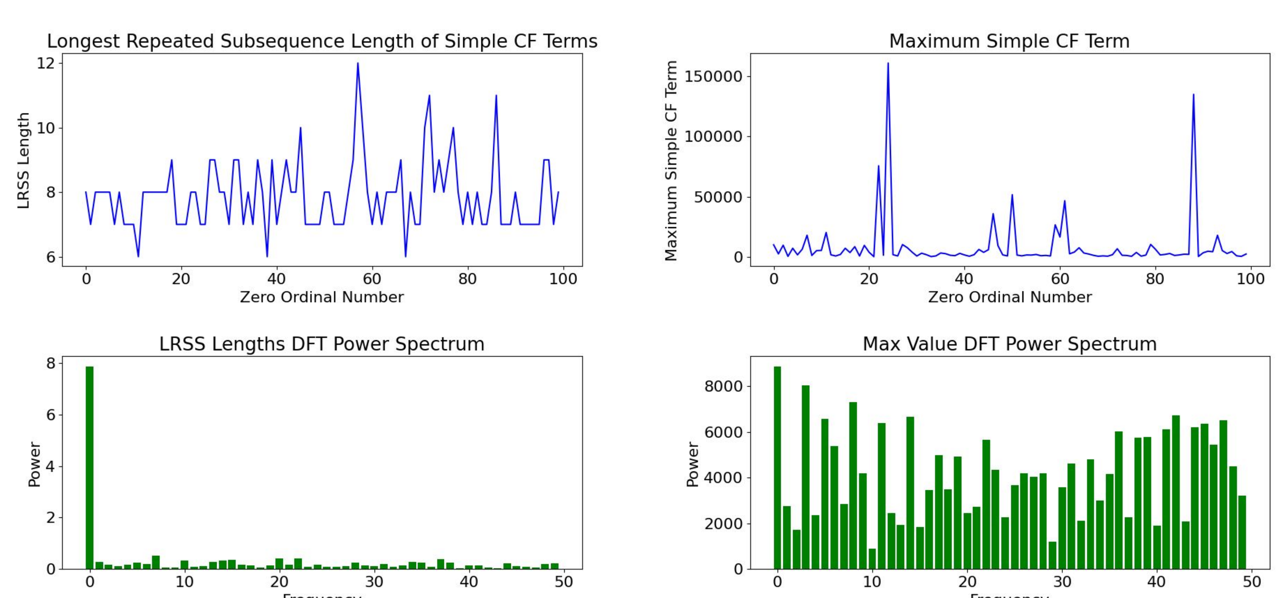

Figure 2: Simple continued fraction term Fourier analysis

Upon completing analysis of this simple CF data, the Python code was modified to allow for "decoding" generalized continued fractions to n iterations as well, using similar recurrence formulas. The core of the generalized CF investigation was the actual brute force search through all possible permutations of the aforementioned coefficient sets. For a given permutation of coefficients, the generalized continued fraction of the given form was iterated to $n = 15$, and stored in memory should this convergent be the closest to the current zero being searched for. The search was parallelized to take advantage of the multi-core CPU it was run on.

When the brute force search completed, the set of coefficients that resulted in the closest convergent to the desired zero was recorded to a text file. This was repeated for the first 51 (indexed 0 to 50) zeta zeros' imaginary components. The improvement of storing all possible convergents to file and searching this for the best-fit was considered, but decided against due to the small scope of this project. The discrete Fourier transform was again applied to each set of best-converging coefficients to hopefully discover a simple pattern among the chaotic sequence.

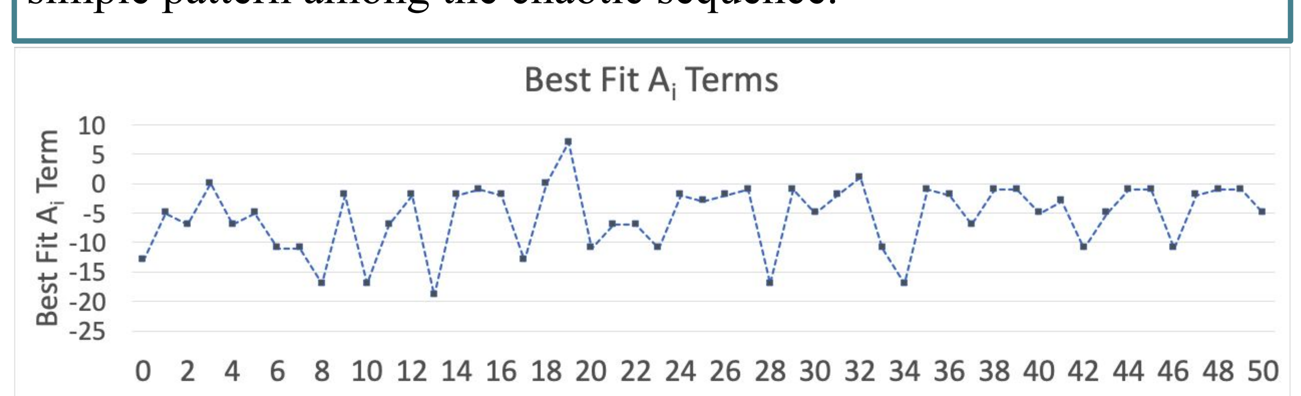

Figure 3: Best-fit generalized continued fraction A_i coefficient data

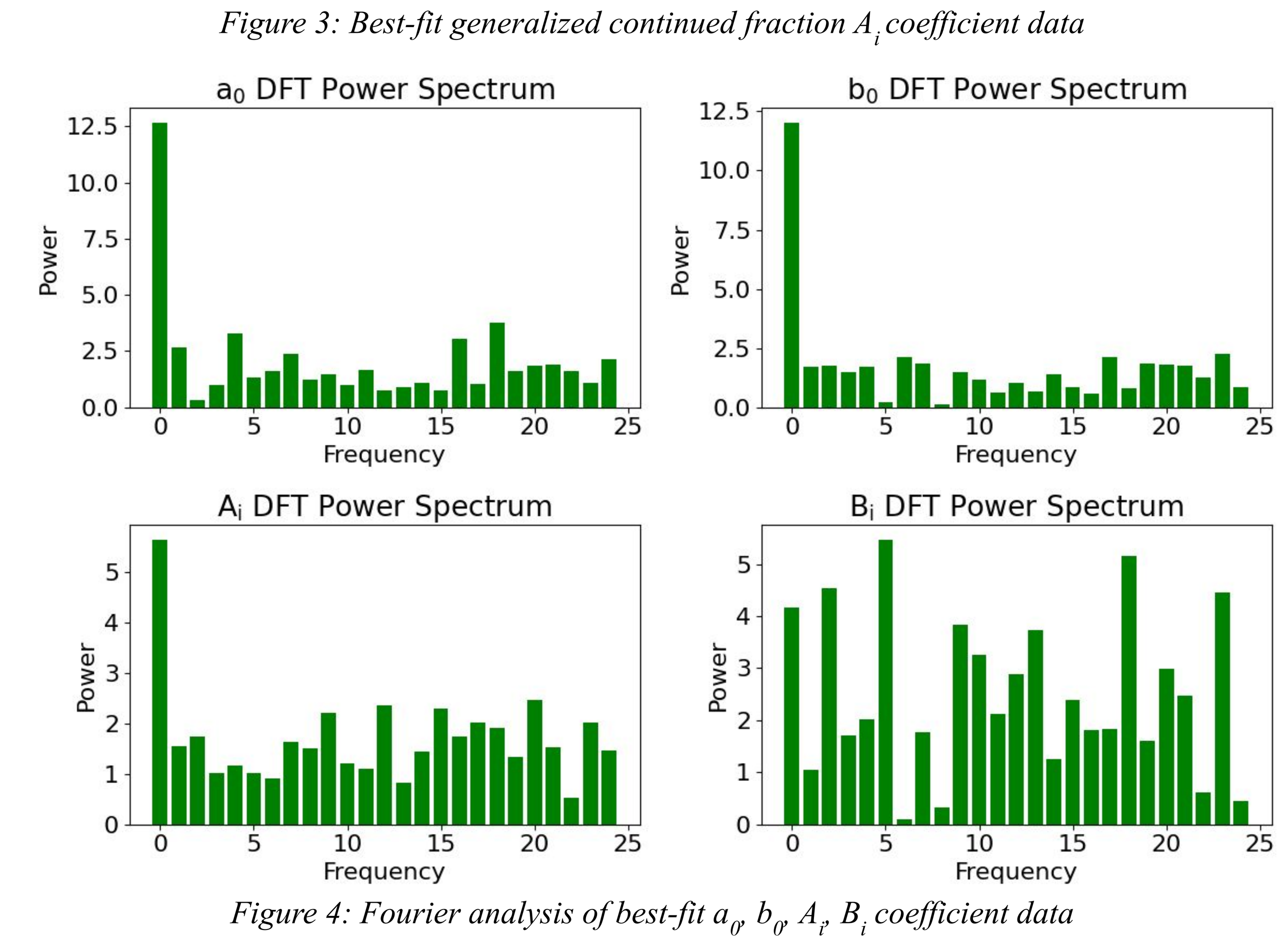

Figure 4: Fourier analysis of best-fit a_0, b_0, A_i, B_i coefficient data

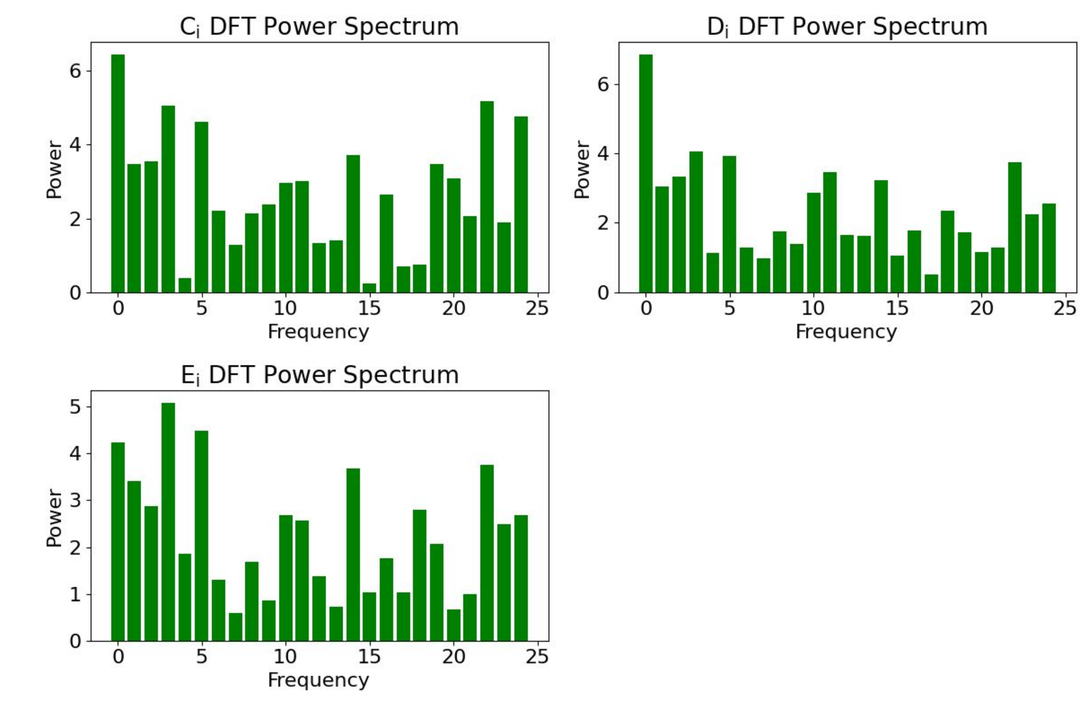

Figure 4: Fourier analysis of best-fit C_i, D_i, E_i coefficient data

a0	b0	A	B	C	D	E	delta
13	11	5	5	0	-17	19	1.14143411e-5
17	7	7	-13	-17	7	-3	-5.68749862e-6
17	11	-5	3	-7	5	-1	-3.22279516e-8

```
Time elapsed: 8819.464145898819 seconds
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Figure 5: Abbreviated console output from early version search script for root 1

CONCLUSIONS

As expected, no pattern emerged in the Fourier analysis of the truncated simple continued fraction terms of the first 100 zeta zeros' imaginary components. Unfortunately, the same appeared to be so for the Fourier analysis of the coefficients of the best-fit generalized continued fraction for the first 51 roots. The terms of the real DFTs indicate signals that don't consist of solely integer frequencies, and the DFT power spectrums show no dominant frequency.

However, several conclusions can be reached from this outcome that can be used in further research on this topic. As mentioned, our constraints to the set of coefficients were arbitrary, and so methods might be employed to more intelligently limit or even calculate a best-fit coefficient, especially a_0 as it will typically be very close to the integer component of the desired convergent. Additionally, the set of integer coefficients could be drastically expanded in range if insight could be gained into determining the necessary conditions for a generalized CF of this form to converge at all, or if the search was able to be parallelized to a greater degree. All source code and data for this project is available at www.github.com/jmaff/bnl-hsrp-2020

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[2] Lange, L. J. "An Elegant Continued Fraction for π." *The American Mathematical Monthly*, vol. 106, no. 5, 1999, pp. 456–458. *JSTOR*, www.jstor.org/stable/2589152.

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