📖

Linear Algebra unit 3

Date: @today

Lane: *Linear Algebra*

Topic: Gaussian elimination, LU decomposition, the four subspaces, Inverses

What it covers:

- inverses
- row operations to solve linear equations - homogeneous and inhomogeneous solutions
- LU decomposition, matrix inverses
- The four subspaces as defined by LU decomposition

Requires:

Definition of linear combinations, matrix multiplication, linear equations.

Required by, used by:

Constructive methods for finding subspaces, matrix rank, determinants & subspace dimensions. Used in other systems of equations, eg. ODEs. Also generalizes to propagation order in graphs.

Summary: Useful properties of Determinants:

To review, here are several equivalent methods for computing matrix determinants (without resorting to permutation-based formulas) that are also useful for computing the rank of a matrix: (From A.J. Laub, *Matrix Analysis for Scientists and Engineers,* (2005), SIAM, p. 5).

the more useful properties of determinants. Note that this is not a minimal set, i.e., several properties are consequences of one or more of the others.

1. If A has a zero row or if any two rows of A are equal, then $\det A = 0$.
2. If A has a zero column or if any two columns of A are equal, then $\det A = 0$.
3. Interchanging two rows of A changes only the sign of the determinant.
4. Interchanging two columns of A changes only the sign of the determinant.
5. Multiplying a row of A by a scalar α results in a new matrix whose determinant is $\alpha \det A$.
6. Multiplying a column of A by a scalar α results in a new matrix whose determinant is $\alpha \det A$.
7. Multiplying a row of A by a scalar and then adding it to another row does not change the determinant.
8. Multiplying a column of A by a scalar and then adding it to another column does not change the determinant.
9. $\det A^T = \det A$ ($\det A^H = \overline{\det A}$ if $A \in \mathbb{C}^{n\times n}$).
10. If A is diagonal, then $\det A = a_{11}a_{22}\cdots a_{nn}$, i.e., $\det A$ is the product of its diagonal elements.
11. If A is upper triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$.
12. If A is lower triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$.
13. If A is block diagonal (or block upper triangular or block lower triangular), with square diagonal blocks $A_{11}, A_{22}, \ldots, A_{nn}$ (of possibly different sizes), then $\det A = \det A_{11} \det A_{22} \cdots \det A_{nn}$.
14. If $A,\ B \in \mathbb{R}^{n\times n}$, then $\det(AB) = \det A \det B$.
15. If $A \in \mathbb{R}_n^{n\times n}$, then $\det(A^{-1}) = \frac{1}{\det A}$.
16. If $A \in \mathbb{R}_n^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$, then $\det \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] = \det A \det(D - CA^{-1}B)$.
Proof: This follows easily from the block LU factorization

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

17. If $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}_m^{m\times m}$, then $\det \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] = \det D \det(A - BD^{-1}C)$.
Proof: This follows easily from the block UL factorization

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ C & D \end{bmatrix}.$$

Inverses

Linear transformations go from one finite n- dimensional vector space to another. (is a strong restriction that is relaxed when we get to non-invertible and non-square matrices.)

💡 General principle — Once we find a basis for a vector space, we know everything about it

- Once we pick a basis (any spanning, linear independent set will do) and know how the basis is transformed, we know how any vector is transformed. Think of it as a transform from the basis in one vector space to another. Examples of linear transformations — scaling, rotation, permutation. Once we know how a
- A “non-singular” square matrix has a unique inverse, that commutes with the matrix.

If the matrix is invertible, then it’s left inverse and right inverse are equal. Proof:.

B = BI = B(AC) = (BA)C = IC = C

Matrix form of systems of linear equations.

As we’ve seen the solution of a set of linear equations is equivalent to finding if a given set of vectors is a linear combination of the column vectors of the matrix. *Row operations* on a matrix are equivalent to transforming the equations so that the solution (itself a row vector) is invariant to the operations.

Row operations. Elimination

Row operations are those allowed transformations that preserve the row space - where both the matrix rows and the solution vector “live.” Note the similarity (but not equivalence!) to the matrix transformations that preserve determinant values.

💡 There only one general transformation rule, called a “row operation” —- Add a multiple of one row to another. So scaling a row is just a version of this, as is swapping rows.

By reducing the A matrix to ***U*** - in *upper triangular form* via row operations, (and recording the transformations to perform it in a lower triangular matrix *L*) we can reduce ***A*** to an equivalent matrix, in the sense of having an equivalent solution to ***A****x=b* that is easier (few operations) to solve.

The result will be a matrix ***U*** with *pivots* along it’s diagonal.

If any pivots are unavoidably zero, they indicate basis vectors that are not in the row space, and form its complement - the *null space.* Vectors in the null space are mapped into the zero vector (in the column space).

Elimination steps leave the row space (and hence, it’s complement, the nullspace) unchanged. The nullspace and rowspace partition $\mathbb{R}^n$. Hence elimination finds a basis for the row space, formed by the non-zero rows of ***U***. A basis for the null space is the set orthogonal to the row space basis. (The basis for the nullspace comes from the zero-pivot elements of x, in Ux = 0. To find these set each element of x = 1, leaving the others equal to 0, and solve for the vector x. The resulting basis vectors are the nullspace.

LU decomposition

Elimination process, as described is also a method to factor any matrix as the product of two matrices, one lower triangular ***L,*** and one upper triangular, ***U.***

Take the example from Strang, Chapter 2.2:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The three necessary row operations: (They can be read off the entries in the L matrix:)

1. Subtract 2 X row1 from row 2
2. Add row 1 to row 3
3. Subtract 2 times row 2 from row 3.

Solutions when A is not invertable.

💡 matrix rank = number of linearly independent rows of the matrix ***U*** (also = number of linearly independent columns)

Note that in this case a zero pivot appears in columns 2 and 4. This is not evident before taking the elimination steps. From their appearance we can see immediately that

- The rank of the matrix = 2
- the nullspace has dimension 1 and
- the left nullspace (complement of the column space) has dimension 2.

Using LU decomposition to solve Ax = b for x.

So once we have ***L*** and ***U*** from ***A***:

$$\begin{gathered} \mathbf{A}x = b \\ \text{solve for c: } \mathbf{L}c = b \\ \mathbf{LU} = \mathbf{L}c \rightarrow \mathbf{U}x = c \end{gathered}$$

Solve for c via forward substitution, then for *x* via back substitution.

4 spaces

With a square matrix, where all pivots are non-zero (so all other full rank conditions apply), the solution to Ax=b is unique, and the matrix A is “invertible.” Then the transform given by the matrix is “1-to-1”, meaning each vector in the domain (the row space) maps to a single non-zero vector in the column space.

When the row space is not "full rank" - that is when the null space has non-zero dimension, either because the row space is deficient, or because there are more columns than rows (the system of equations is "underdetermined") it will be discovered during row elimination by zeros in the "echelon" form of matrix ***U.*** This gives a method to come up with solutions when

- There are more than one solutions to Ax=0
- There are no solutions to Ax=b

Mappings of the 4 spaces.

The row dimension (on the right) decomposes into the rowspace (red) and null space (light green), and the column dimensions into the column space (tan) and the left nullspace (emerald).

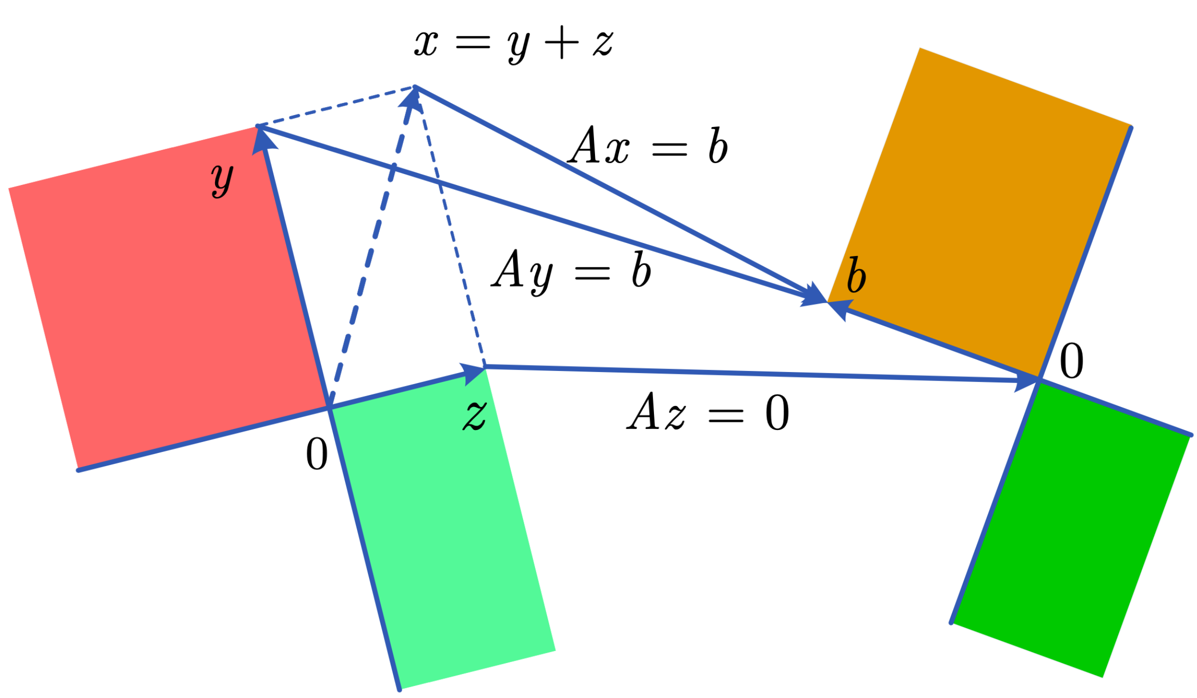


Image from Strang, ch. 2.4

Rectangular matrix LU decomposition

The homogeneous system when m<n, must have a non-zero (e.g. positive dimension) null space as a subspace.

If a particular solution exists if and only if b is in the column space of A. And the full solution is this vector plus the nullspace.

In summary the four spaces are:

1. column space of A: $\mathbb{R}(A)$ — *range* of A
2. null space of A $\mathbb{N}(A)$
3. row space of A: $\mathbb{R}(A^T)$
4. left nullspace of A $\mathbb{N}(A^T)$

Note $\mathbb{R}^n$, where the rows live, is the domain of ***A***, $\mathbb{R}^m$, where the columns live, is the range.

💡 Theorem dim($\mathbb{R}(A)$) == dim($\mathbb{R}(A^T)$) == "rank" of the matrix.

$\mathbb{N}(A)$ & $\mathbb{R}(A^T)$ are subspaces of $\mathbb{R}^n$, (They are orthogonal spaces)

$\mathbb{N}(A^T)$ and $\mathbb{R}(A)$ are subspaces of $\mathbb{R}^m$,(They are orthogonal spaces)

- row space of ***A*** and ***U*** are the same subspace (The non-zero rows of U span the space)

If the rank = m (number of rows) then the null space is = 0, and the column space is spans m (perhaps linearly dependently??) If rank < m then there are zero rows in U (and some particular solutions won't exist).

Inverses, revisited

Gauss-Jordan method to find A inverse.

Start with a block matrix of *A, I* and use row operations to reduce *A* to *I*.

$$\left[\mathbf{A}, \mathbf{I}\right] \rightarrow \left[\mathbf{U}, \mathbf{L}^{-1}\right] \rightarrow \left[\mathbf{I}, \mathbf{A}^{-1}\right]$$

But typically one uses ***LU*** decomposition instead of ***A*** inverse, for computational convenience.

What about when inverses don't exist

Subsequent topic: Is there is within a non-invertable matrix, a linear transform from the row space to the column space, ignoring the nullspaces that is 1-to-1?

References

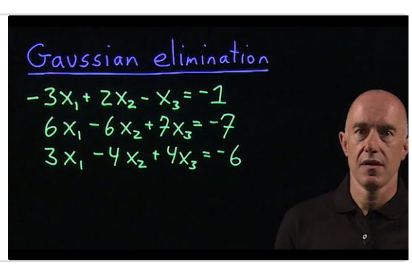
Read Strang, ch 1.4- 1.6 & 2.2 - 2.4, 2.6

Here are some videos on Gaussian Elimination

Gaussian elimination | Lecture 10 | Matrix Algebra for Engineers

We solve a system of three equations with three unknowns using Gaussian elimination (also known as Gauss elimination or row reduction).

https://www.youtube.com/watch?v=RgnWMBpQPXk



Here's G. Strang's lecture on it:

2. Elimination with Matrices.

MIT 18.06 Linear Algebra, Spring 2005

Instructor: Gilbert Strang

View the complete course: http://ocw.mit.edu/18-06S05

https://www.youtube.com/watch?v=QVKj3LADCnA&t=56s