ISE 201 8 Feb '24

Linear Algebra HW 2

- 1. How does the term "dimension" apply to (i.e. what is the dimension of) the vector space of a singular matrix? To a non-singular matrix?

- 2. What is $\det|-\bold{A}|$? If one negates all the elements in a matrix how does that change the value of the determinant?

1 - Singular → linear dependent columns
Hence the columns' vector space dimension < number of columns

Nonsingular → column vector space dim. equals number of columns
"matrix is full rank"

2 - Since multiply a column by -1 changes the sign of the determinant, multiplying all columns by -1 changes the sign by $(-1)^n$
n = number of columns.

3. For the types of matrices - diagonal, triangular, symmetric, permutation - is multiplication closed? Is it commutative?

Note, a permutation matrix consists of just 1s and 0s, with no more than a single 1 in any row or column. The identity matrix -- 1s along the diagonal and 0s elsewhere is the identity for permutations. Pre-multiplying by a permutation matrix swaps rows, and post-multiplying swaps columns. There are just two 2X2 permutation matrices:

Question: In general, for the class of n X n matrices how many different permutation matrices are there?

diagonal: closed & commutative
triangular: closed
symmetrix: closed & commutative
permutation: closed

Since permutation matrices are 1-to-1 with list permutations, the number of permutations is the same as the number of list orderings: n!

4. For these three vectors, $(3,5,0), (-2,-2,0), (0,0,1)$of length 3, not scalar multiples of each other:

- Show they span $\mathbb{R}^3$, e.g. conventional 3-space.
- Show they are linearly independent
- Show that any non-zero linear combination of them, can replace one of the two vectors and preserve the "span" and "linearly independent" properties.

Using row elimination: (swapping col 1 & 2 :)

$$\begin{bmatrix} -2 & 3 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformed matrix spans the same row & column spaces

- Linear independence follows since

$$\begin{bmatrix} -2 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ implies $x_1 = x_2 = x_3 = 0$

- span follows since

$$\begin{bmatrix} -2 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$ has a solution for any b_1, b_2, b_3.

- since row elimination operations create an equivalent basis the span & linear independence properties are preserved.

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

- 5. Show that the common formula for a 2 X 2 determinant $ \begin{bmatrix} a & b \\c & d\end{bmatrix} = ad- bc$ is consistent with the 8 properties of determinants in the class notes.
- 6. Prove from the 8 properties that $0= \det|\bold{M}|$ $0 = \det|M|$
  - For a matrix ***M*** with a column of zeros.
  - For a matrix ***M*** with two columns that are equal.

Restate these results as properties of a set of vectors in a vector space.

Identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1(1) - 0(0) = 1$

skew $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = 1(1) - 0(x) = 1$

rotate $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0(0) - (-1)1 = 1$

swap $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0(0) - (1)1 = -1$

scale $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = xy - 0(0) = xy$

row op $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ - a skew

transpose $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = ad - cb$

matrix multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{bmatrix}$

$\stackrel{?}{=} (ad - cb)(AD - CB)$

6. For a matrix w/ a col of zeros
Note that multiplying any col by $0 \rightarrow \det|M| = 0$
Also a zero vector creates linear <u>dependence</u> among the columns $\rightarrow \det|M| = 0$

Two equal columns can be reduced to create one zero column by subtracting one from the other. And a zero column $\rightarrow \det|M| = 0$
Note: this is another example of linearly <u>dependent</u> columns

8. Why does a permutation matrix P have a non-zero determinant?

We use 4: row swaps change the sign of det|A|.
swap the rows of P to create a diagonal matrix, with ones on the diagonal:

$$\begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & & \vdots \\ \vdots & & 1 & \\ & \cdots & & 1 \end{bmatrix}$$

How does one determine it's determinant?

By keeping track of the number n of swaps to turn it into the identity matrix, so its determinant:

$$\det|P| = (-1)^n$$

One way to count swaps is to convert the matrix to a list, with the column assigned the index of the position of the 1. eg.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow (2\ 1)$

Take this list & split it into sub-lists that end w/ an out of order entry ("cycles")

$(2\ 3\ 4\ 1)(6\ 5)(8\ 9\ 7)$ Count the number of swaps to create each cycle.

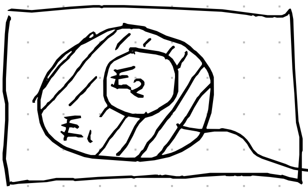
$3 + 1 + 2$

The sum of the swaps in each cycle equals the total number of swaps (the "parity") of the permutation.

Probability HW 2

Probability algebra

1. Show that if $E_2 \subset E_1$ then $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$. (The backslash is the set difference operator, not the probability conditioning symbol.)



$E_1 \setminus E_2$

$$P(E_1 \setminus E_2) = P(E_1 \cap E_2^c)$$
$$= P(E_1) + P(E_2^c) - P(E_1 \cup E_2^c)$$

These are the same

- 2. The law of *inclusion - exclusion* for 2 events is

$$ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) $$

- The expression for the union of 3 events has seven terms. Derive it from the basic laws of probability.
- Do you see the pattern? Explain how one derives the number of terms in the case of the union of 10 events.

Inclusion-Exclusion for 3 events.

$$P(A \cup D) = P(A) + P(D) - P(A \cap D)$$

Let $D \to B \cup C$

$$P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$$

Note that

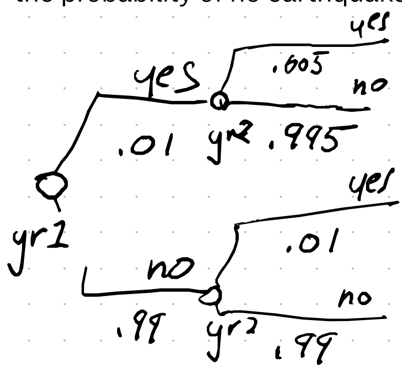
$$P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)$$

So

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

The pattern is that for n events the formula contains the set of all subsets as terms of which there are 2^n — one on the left hand side of the expression, the other $2^n - 1$ on the right.

3. The probability of an earthquake in the Bay Area in the next year assuming we don't have one this year is (say, for sake of argument) 1%. The probability of an earthquake in the year following a year with an earthquake is lower, say, 0.5%. What is the probability of no earthquake in 2 years?



$(0.99)(0.99)$

4. Bayes' Rule is sometimes expressed as a ratio between two events (i.e. two hypotheses). For two hypotheses, H_1, H_2 whose likelihoods, in theory, generate the same observable data, D, write the expression for the ratio of the posteriors of the hypotheses given the data. Why might this be more convenient to apply than the conventional form of Bayes Rule? What other probabilities need to be assumed?

$$P(H_1|D) = \frac{P(D|H_1)\,P(H_1)}{P(D)}$$

$$P(H_2|D) = \frac{P(D|H_2)\,P(H_2)}{P(D)}$$

The ratio

$$\frac{P(H_1|D)}{P(H_2|D)} = \frac{P(D|H_1)}{P(D|H_2)} \frac{P(H_1)}{P(H_2)}$$

This "prior ratio" does not come from D so it needs to be assumed.