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Linear Algebra unit 2

Date: @today

Lane: *Linear Algebra*

Topic: Matrices, vector sub spaces, determinants, Gaussian elimination

What it covers:

- Vector sub-spaces
- Linear independence - the dimension of a vector (sub) space.
- Deriving matrix multiplication
- Properties of determinants.

Requires:

Definition of a vector space, of an inner space, of linear combinations.

Required by, used by:

Understanding the math for everything that has more than one dimension. Understanding tabular data.

Specific uses

- What is the dimension of a set of vectors, of a matrix & how is this measured.

Vector sub-spaces

For a set of vectors in a vector space, all their linear combinations - -that is the *closure* of all linear combinations, is also a vector space. We say the combinations *span* the subspace. The simple case of a set of just one vector with scaling creates a subspace that is an infinite line in both positive and negative directions through the origin. Clearly a subspace is a subset of the vector space where the vectors live, but the opposite is not true — a subset of the vector space (e.g. the positive quadrant) is not a subspace.

To prove some subset is a subspace, it is only necessary to show that it is closed under vector addition and scalar multiplication and it contains the origin vector. Of course the full vector space meets the requirement to be a (improper) sub-space.

Linear Independence and sub-space Dimension

To repeat, a *spanning* set of vectors is able to generate via linear combinations any vector in the sub-space. And one can create sets vectors within the subspace that are linearly independent by removing vectors from the set.

For a linearly *dependent* set, any vector in the set can be expressed as a linear combination of the rest. For a linearly *independent* set, any subset is also linearly independent.

💡 Definition: A ***basis*** is a set of vectors in a subspace that both *spans* the space and is *linearly independent.*

Theorem: Any vector is a unique linear combination of the basis vectors for that sub-space.

Theorem: All bases for a sub-space (there will be many) have the same number of vectors. This is proved by swapping a vector from one basis to the other, so that the basis preserves linear independence and span properties.

So, if the length of a basis is a constant for any basis for the vector (sub-)space!

💡 Definition: The size of a space's basis set is its ***dimension*.**

Matrix multiplication - 3 ways to look at it.

1. **As a linear combination of columns**

Let's re-label a linear combination with scalars x_j and column vectors A_j.

$$\mathbf{Ax} = \sum_j x_j A_J = x_1 A_1 + x_2 A_2 + \ldots x_p A_p$$

Equivalently, in matrix notation this is written concisely as $\mathbf{Ax}$ where $\mathbf{A}$ consists of "stacking" the columns A_j to create an *n X p* matrix and x is a length *p column vector.* We say the result is "in the column space (a subspace) of the matrix ***A***.

Take this another step and create a second *p X m* matrix by stacking the x-s to create a matrix $\mathbf{B}$. Then $\mathbf{C} = \mathbf{AB}$ is their *matrix product.*

2. **As all pair-wise inner products between rows of A and columns of B**

Considering each term in the product matrix of ***A*** which is *m* by *n*, with ***B*** which is *n* by *p*. Then their product, ***C*** is the inner product of all rows in A by all columns in B:

$$c_{m,p} = \sum_{1 \le j \le n} a_{m,j} b_{j,p} =< A_{m,.}, B_{.,p} >$$

where $A_{m,.}$ is the *m-th* row, and $B_{.,p}$ is the *p-th* column. Note that for this to work the number of columns in ***A*** must equal the number of rows in ***B***.

3. **Each row of C is a combination of the rows of B as specified by the row of A.**

Equivalently, each row in C is the product of a row of ***A*** times the matrix ***B.***

Linear combinations explain why matrix multiplication works the way it does. Note that definition 3. is equivalent to just switching "column" for "row" in definition 1. Since a matrix transpose $\mathbf{B} = \mathbf{A}^T$ is simply reflecting the matrix elements around the diagonal , $b_{j,i} = a_{i,j}$ by keeping track of how the transpose of a matrix switches rows and columns, it follow that.

$$(AB)^T = B^T A^T$$

Some examples of matrices, and how they multiply

Consider these square matrici:

1. Diagonal matrix
2. Triangular matrix
3. Symmetric matrix

Restricting matrices to one class creates an "algebra" specific to that class.

Determinants: The "volume" of a matrix

For any set of n vectors of length n, there is an equivalent number analogous to the "volume" of a parallelepiped formed from the vectors. In the xy plane this means the deteminant is the area of a polygon formed by a pair of vectors. & subspa Typically one writes the vectors in an n-by-n matrix. This value is called the *determinant* of the matrix:

$$v = \det|\mathbf{M}|$$

1. In the simplest case, for the unit vectors, of length 1, they form an n-dimensional cube with volume = 1.
2. "offsetting" or "skewing" the volume - e.g. think of the cube with the same height, but with parallelogram faces, does not change the volume.
3. Neither does rotating all the n-vectors by the same amount.
4. Swapping the the order of two vectors changes the sign of the value. For other reorderings, the sign change equals $(-1)^n$ where *n* is the number of pair-wise swaps. (the "parity of the permutation").
5. Scaling any one vector scales the value by that amount. So scaling all n-vectors equally by *x* changes the value by x^n. It follows that the determinant of a diagonal matrix is just the product of the diagonal terms.
6. Adding a multiple of one row to a different vector leaves the value of the determinant unchanged.
7. $\det|\mathbf{M}| = \det|\mathbf{M}^T|$. So all row-wise properties also apply column-wise.

8. The definition of matrix multiplication reduces to “scalar” multiplication of determinants: $\det|\mathbf{AB}| = \det|\mathbf{A}|\det|\mathbf{B}|$. Note how matrix multiplication is consistent with the above-mentioned properties.

In the case of a diagonal matrix, the determinant equals the product of the diagonal entries. This is a consequence of 1. - a matrix of unit vectors’ determinant = 0, and 5 - scaling any vector similarly scales the determinant’s value.

$0 = \det|\mathbf{M}|$ is equivalent to the n- vectors being linearly dependent. *A matrix with linearly dependent columns is called* ***singular.*** *The opposite (a non-negative determinant) is called* ***non-singular****, or of* ***full rank.***

References

A playlist of James Maynard’s lectures. Just good background on various topics.

A James Maynard Playlist

Four films from Fields Medal winning Oxford Mathematician James Maynard - an interview about his medal, his Public lecture on Prime Numbers and two student l...

https://www.youtube.com/playlist?list=PL4d5ZtfQonW12i4MCd8xYwrUGyRF9dmdi