Final Exam Problems

You have one hour to complete this exam. You need to complete just 4 of the following 6 problems. Each problem is worth 6 points (with one point given for free, to make 25 points total).

Please type your answers to each question. You can leave answers as fractions of integers: you do not need to reduce them to decimal values. To enter a matrix, format each line on a separate row. You are allowed to refer to any of the distributed class notes, but not any other sources.

Any calculations you need to do should be doable by hand, but if you want to run python in your browser, you can go to

`https://pyodide.org/en/stable/console.html`

1. Statistics 1: Testing

Derive an *estimation interval* from the measurements in this dataset, where these are the statistics of the measurements, $x_i, i = 1 \ldots 10$ (that have conveniently been computed for you.)

statistic	value

- $\sum_{i=1}^{10} x_i = 80$
- $\sum_{i=1}^{10} x_i^2 = 1540$
- $\min_i x_i = -2.5$
- $\max_i x_i = 19.1$
- $n = 10$

Your interval for the sample mean should cover plus or minus 2 standard deviations.

1. What is

- The sample mean, $\overline{x}$?
- The variance of a sample point $\sigma^2_{x_i}$?
- The standard deviation of the sample mean $\sigma_{\overline{x}}$?

2. Based on the sample statistics what are the endpoints of the interval?
3. Making the assumption that the sample mean is well approximated by a normal probability distribution, what is the probability (approximately) that the "true" value of the sample mean falls within the the interval?
4. Consider that this data was measured for an experiment to determine if the difference of the sample mean from an actual value from zero is statistically significant. (In statistical terms the "null hypothesis" is that the actual value is zero.) What can you conclude about statistical significance? Be careful to state this into precise statistical terms.

2. Statistics 2: Entropy

Here are 4 distributions over 2 discrete states.

1. Order these distributions from smallest to largest entropy. (You should be able to do this by visual inspection.)

	State	
Variable	E	Not E
A	0.4	0.6
B	0.1	0.9
C	0.5	0.5
D	0.0	1.0

Here is a joint probability distribution over 2 of the binary variables A and B.

Variable	A	
B	4/10	6/10
1/10	4/100	6/100
9/10	36/100	54/100

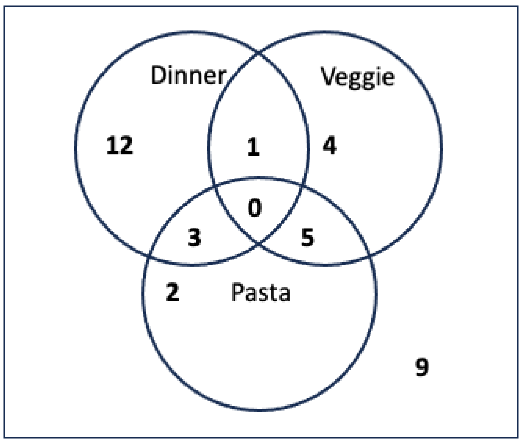
2. What is the rank of the joint probability matrix?

3. Express the joint probability P(A,B) in terms of P(A) and P(B)
4. What is the conditional entropy H (A | B) in terms of H(A) and H(B)?
5. What is the joint entropy H(A,B) in terms of H(A) and H(B)?
6. What is the mutual information of MI(A;B)?

3. Probability 1: events

Penelope has kept track of her last 36 meals. This diagram shows how the meals are distributed among these three events: 16 counted as "Dinner", 10 meals included "Pasta, and 10 were "Veggie". One can compute a fractional probability for composite events using "not", "and", "or" from the counts in

this diagram:



For example

- “Pasta” give “Dinner” is P(P | D) = 3/15
- “Dinner and Veggie” is P(D ∩ V) = 1/36.

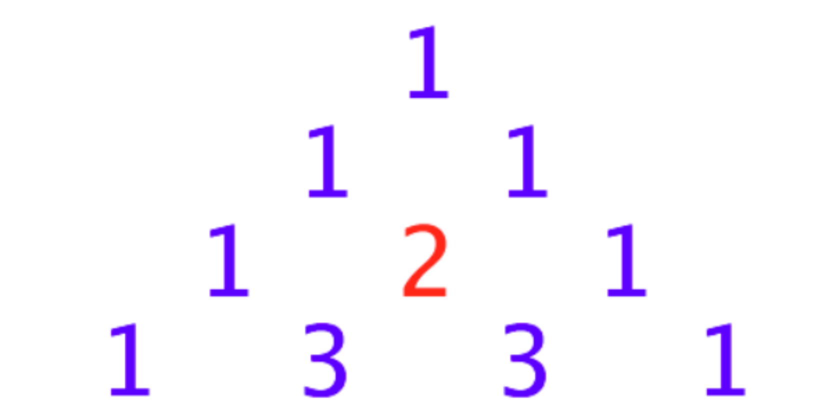
Write the expression for the probability of meals for each of these composite events

1. “Pasta or Veggie”
2. “Not Dinner”
3. “Dinner” and “Pasta” and “Veggie”
4. “Veggie” given “Not Dinner”
5. “Not Dinner” given “Not Pasta” or “Not Veggie”
6. If I were to categorize 10 events out of the events that are “Not Dinner” as “Lunch” and further that all P(V | L) = 4/10 and P(V | not L) = 5/26 what is P(L | V)?

4. Probability 2: binomial coefficients

1. Write out the the entries $\binom{n}{k}$ for the first 6 rows of Pascal’s Triangle, starting with n = 0, for row n, and index k:

Eg.



and so on.

2. What is $\binom{6}{5}$?
3. In general, what is $\binom{n}{n-1}$?

2. What is the relationship between the k-th and k-th +1 entry in row n and the kth+1 entry in row n+1?
3. Persephone is planning her new restaurant menu of 3 course meals. She wants only enough items on her menu so there are no more than 10 combinations (in any order) of 3 course meals. How many different items does she need to make?
4. If Persephone decides only to serve 2 course meals, how does the answer change?

5. Linear Algebra 1: Projection operator

1. Project this vector in 3 space $(1, 1, 1)$ on the subspace defined by this pair:

$$(2^{-1/2}, -2^{-1/2}, 0), (-2^{-1/2}, -2^{-1/2}, 0)$$

(Please show how one can calculate the result, even if you can solve it by inspection).

2. Show how this is computed if instead the basis for the subspace is

$$(2^{-1/2}, -2^{-1/2}, 0), (2^{1/2}, 0, 0)$$

In case you need it, the formula to invert a 2 × 2 matrix is

$$\textbf{When Matrix A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3. What is it about case 1.) that makes it so much easier?

6. Linear Algebra 2

For this matrix

$$\begin{bmatrix} 1, 2 \\ 2, 1 \end{bmatrix}$$

1. Is it singular? Is it invertible? How do you know?

2. & 3. Factor it into lower and upper triangular form

4. & 5 Find it’s eigenvalues, and eigenvectors (they will be orthonormal).

6. If the matrix is extended with these columns, what is the dimension of it’s nullspace?

$$\begin{bmatrix} 1, 2, 1, 1 \\ 2, 1, 1, 0 \end{bmatrix}$$