Linear Algebra unit 1

Date: @today

Lane: *Linear Algebra*

Topic: Vectors, vector space, linear forms, linear independence

What it covers:

- What does "algebra" mean?
- The basic definition of vectors and vector spaces, to distinguish it from matrix algebra overall.
- Linear forms - combinations of vectors that are linear dependent and independent. Dimension
- How this leads natually to defining matrix multiplication

Requires:

Facility with conventional algebra.

Required by, used by:

Understanding the math for everything that has more than one dimension. Understanding tabular data.

Specific uses

  - building a "design matrix" for regression.
  - semantic embedding vectors.

Vector Spaces

- The common example of a vector is an n-tuple of numbers, eg. $\mathbf{v} = (x_1, x_2, \ldots x_n)$ where the elements are real (scalar) values. The elements form a *field* (e.g. real numbers are a "field", but there are others, less obvious, like elements that are polynomials — another field). We speak of a "vector space over the field of real numbers."
- Can we define a vector space over the field whose elements are just *0,1?*
- Vectors are objects with both length and direction.
- By thinking of vector spaces abstractly (not just as tuples of numbers), we can determine their properties in more general cases.
- Why is such a simple concept interesting? **Linear combinations of vectors define *dimension.*** Note that the number of elements, *n,* in a vector is *not* the same thing as "dimension" in this sense.
- It can be helpful to get insights by thinking of vectors in 2 or 3 - dimension space. But they become useful when there are many (say thousands of) dimensions. Some of their properties even transfer over to infinite dimensional ("Hilbert") spaces.
- Thinking in coordinate space, vectors are "points". All vectors start at the origin. (This is different than a vector field, e.g. in fluid dynamics, or electrodynamics.)

Vectors: addition & scalar multiplication

See Boyd, "Applied Linear Algebra", Sections 1.1 - 1.3

Vectors are added element-by-element. The first element in one vector is added to the first in the other, to give the first element in the sum. Vectors need to be of the same length.

Multiplication between a number, a "*scalar*", and a vector scales each element in the vector by the number it is multiplied by. This changes the vector's length. Conceivably there are several other ways that vectors might be multiplied, for example the way complex numbers multiply, or the way an *inner product* is formed. When inner products are included in the definition of a vector space we have something new - an "*inner product space.*"

Vector algebra is illustrated by the "parallelegram" diagram.

- *To qualify as a vector space, for any sum of two vectors in the space, the result is also in the space. Similarly for any product, the product is also in the space. This property of closure is necessary to constitute a vector space.*

Definition of a vector space

In addition to *closure,* the vectors must be constructed so that — while preserving closure:

$$\text{For scalar values } a, b \text{ and vectors } \mathbf{u},\ \mathbf{v},\ \mathbf{w} :$$

1. Addition and scalar multiplication are distributive:

$$a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$
$$(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$$

2. Addition is commutative and associative:

$$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$$
$$(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$$

3. Vector identity (the "origin"), and inverse:

$$\text{There are a vectors } -\mathbf{v}, \mathbf{0} :$$
$$\mathbf{v}+\mathbf{0} = \mathbf{v}$$
$$\mathbf{v}+-\mathbf{v} = \mathbf{0}$$

4. Multiplicative identity:

$$1\mathbf{v} = \mathbf{v}.$$

Other properties are inherited from the vector's *field,* e.g. associativity of multiplication. Note we don't need to define a multiplicative inverse!

Some additional properties

Based on these defining axioms we can determine other properties. The vector $\mathbf{0}$ is an n-tuple of zero length, known as the "origin" to be distinguished from the number (scalar value) *0.* They are related.

$$\mathbf{0} = 0\mathbf{v}$$

The additive identity is unique. (There is only one point called the origin.)

$$\text{Assuming two additive identities } \mathbf{0}, \mathbf{0}'$$
$$\mathbf{0} = \mathbf{0}+\mathbf{0}' = \mathbf{0}'+\mathbf{0} = \mathbf{0}'$$

Linear combinations

Given *p* vectors and *p* scalars the form a linear combination

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + \ldots a_p\mathbf{v_p}$$

Definition: **linear independence.** For some set p of vectors, $0 = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \ldots a_p\mathbf{v_p}$ implies all a_i are zero. If there do exist a_i not all zero then the set is **linearly dependent.** Examples - the set of "unit" vectors, $e_i = (0 \ldots 1_i \ldots 0)$.

This example shows that the *n unit* vectors are independent. Consider just the *i*-th row in this combination, $\mathbf{0} = \sum_j a_j\mathbf{e_j}$.

$$0 = a_j 1$$

For each element $a_j = 0$.

Now consider adding one more term to the linear combination that duplicates one of the unit vectors. $\mathbf{0} = \sum_j a_j\mathbf{e_j} + b_k\mathbf{e_k}$. A non-zero solution is $a_j = b_k$. Hence the enlarged set of unit vectors is linearly *dependent.*

Since any vector can be expressed as a linear combination of the unit vectors we can construct a similar test for any set.

Inner product spaces

If we add an inner product "multiplication" operation to a linear vector space, the result is called a *inner product space.* In addition to concepts of length and direction found in a vector space, an inner product space introduces the concept of *angle.* Sometimes called a "dot-product", common notation is to use angle brackets for inner products:

$$< u, v >= \mathbf{u}^T\mathbf{v} = \sum u_i v_i$$

Matrix multiplication

Let's re-label a linear combination with scalars x_j and column vectors A_j.

$$\sum_j x_j A_J = x_1 A_1 + x_2 A_2 + \ldots x_p A_p$$

Equivalently, in matrix notation this is written concisely as $\mathbf{Ax}$ where $\mathbf{A}$ consists of "stacking" the columns A_j to create an *n X p* matrix and x is a length *p column vector.*

Take this another step and create a second *p X m* matrix by stacking the x-s to create a matrix $\mathbf{B}$. Then $\mathbf{AB}$ is their *matrix product.*

Linear combinations explain why matrix multiplication works the way it does. By keeping track of how the transpose of a matrix switches rows and columns, it follow that.

$$(AB)^T = B^T A^T$$

Assignment

Read Boyd, "Applied Linear Algebra" chapter 1.

Read G. Strang "Linear Algebra 3rd Ed. chapter 1.4 & 2.1

1. Linear Algebra HW1

References

Serge Lang "Linear Algebra".

serge-lang-linear-algebra : Free Download, Borrow, and Streaming : Internet Archive

serge lang linear algebra

https://archive.org/details/serge-lang-linear-algebra/page/n3/mode/2up

Intro lecture to Linear Algebra at Oxford

Linear Algebra II: Oxford Mathematics 1st Year Student Lecture - James Maynard

Our latest student lecture features the first lecture in the second term (1st Year) introductory course on Linear Algebra from leading Oxford Mathematician James Maynard.

https://www.youtube.com/watch?v=pQhVDRojC1U

Vector spaces, intro

Linear Algebra notes

Linear Algebra HW 1