Expressing linear equations in Matrix form

$$Ax = b \longrightarrow x = A^{-1}b$$

$$\underbrace{A^{-1}A}_{I}\,x = A^{-1}b$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Cancel out "c" – reduce matrix to upper triangular form – is equivalent to these steps:

$$[ax_1 + bx_2 = e]\; c/a$$

$$\cancel{cx_1} + \frac{bc}{a}x_2 = ec/a$$

$$\cancel{cx_1} + dx_2 = f$$

$$dx_2 - \frac{bc}{a}x_2 = -ec/a + f$$

Solution by Gaussian Elimination

$$\underbrace{\begin{bmatrix}1&&\\-2&1&\\&&1\end{bmatrix}}_{L_1^{-1}}\begin{bmatrix}1&3&3&2\\2&6&9&5\\-1&-3&3&0\end{bmatrix} \rightarrow \begin{bmatrix}1&3&3&2\\0&0&3&1\\-1&-3&3&0\end{bmatrix} \qquad L_1^{-1}A = U_1$$

$$\underbrace{\begin{bmatrix}1&&\\&1&\\+1&&1\end{bmatrix}}_{L_2^{-1}}\begin{bmatrix}1&3&3&2\\0&0&3&1\\-1&-3&3&0\end{bmatrix} \rightarrow \begin{bmatrix}1&3&3&2\\0&0&3&1\\0&0&6&2\end{bmatrix} \qquad L_2^{-1}U_1 = U_2$$

$$\underbrace{\begin{bmatrix}1&&\\&1&\\&-2&1\end{bmatrix}}_{L_3^{-1}}\begin{bmatrix}1&3&3&2\\0&0&3&1\\0&0&6&2\end{bmatrix} \rightarrow \underbrace{\begin{bmatrix}1&3&3&2\\0&0&3&1\\0&0&0&0\end{bmatrix}}_{U} = U \qquad L_3^{-1}U_2 = U$$

$$L_1 L_2 L_3 = L$$

$$\begin{bmatrix}1&&\\2&1&\\&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\-1&&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\&2&1\end{bmatrix} = \begin{bmatrix}1&0&0\\2&1&0\\-1&2&1\end{bmatrix}$$

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homogeneous solution

$$Ux = 0$$

$$\begin{bmatrix}1&3&3&2\\0&0&3&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow x_h = x_2\begin{bmatrix}-3\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\0\\-1/3\\1\end{bmatrix}$$

both vectors are $\perp$ to rows of U!

For $x_2=1$, $x_4=0$ $\rightarrow \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}$ $\qquad x_2=0$, $x_4=1$ $\rightarrow \begin{bmatrix}-1\\0\\-1/3\\1\end{bmatrix}$

inhomogeneous solution

$$Lc = b \rightarrow c = \begin{bmatrix}b_1\\b_2-2b_1\\b_3-2b_2+5b_1\end{bmatrix}$$

Solve $Ux = c$ with $x_2 = x_4 = 0$

to get $x = U^{-1}c + \underline{x}$

Pivots in the "echelon" upper triangular Matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

$Ax = b$

$Lc = b$

$LUx = b$

$\cancel{L}Ux = \cancel{L}c$

$Ux = c$

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & +1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$