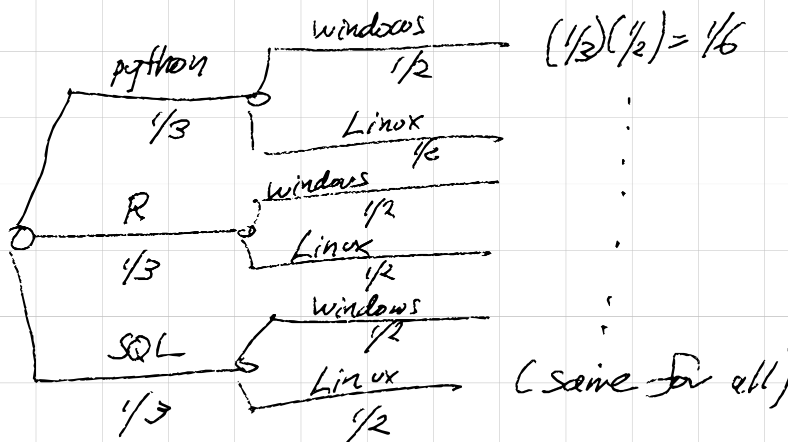
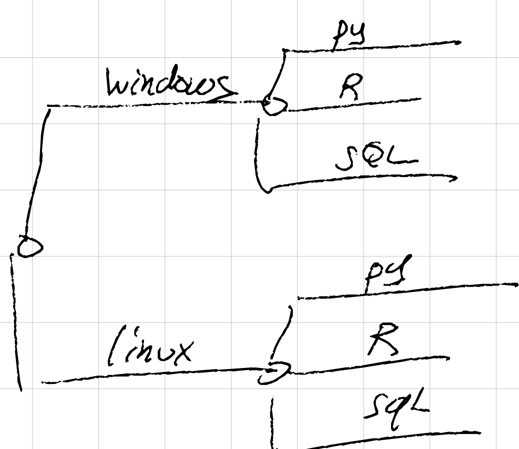
Probability hw1

1. Draw a probability tree for the sets {python, R, SQL} and {Windows, Linux}. (Say you are working on what IT support resources will be required for each combination.) Assume “indifference” among items, and assign probabilities to each branch and leaf.



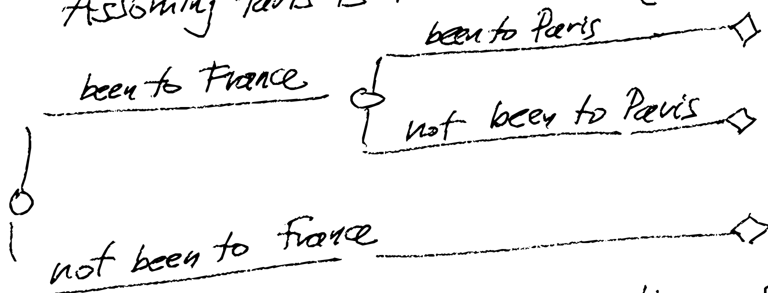
Note this tree also works:



- 2. Consider these two events:
  - Visited Paris
  - Have not been to France.

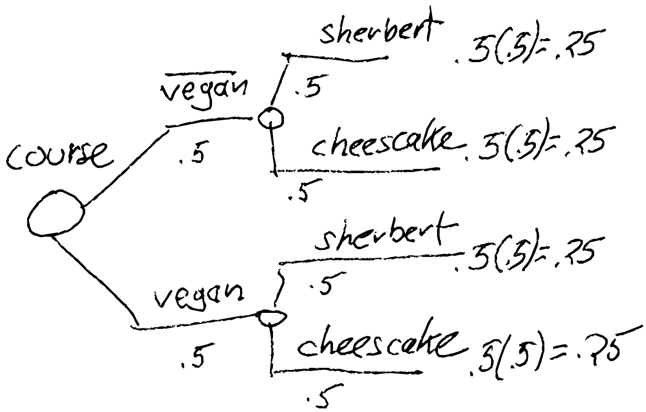
Are they E*xhaustive? Mutually exclusive?* Draw the tree to show your sample space.

Assuming Paris is in France (not Texas ☺)



- The two events are not exhaustive since they do not include "been to France & not Paris"
- The two events are mutually exclusive since it is not possible to both 'have been in France & been to Paris" and "not been in France"

- Assume you are catering a large event and have to decide on course combinations. Your main course combinations are either vegan or non-vegan, and your deserts are either cheese cake or (presumably vegan) lemon sherbert. Draw the tree of possible menu combinations, and assign probabilities, assuming "indifference."
  - Why is indifference a questionable assumption for planning the numbers of each combination to order.
  - What might be an alternative?
  - What would the tree look like if vegans don't eat cheesecake?



— "Indifference" is questionable since the probability of choosing desert may be dependent on the choice of course. Also we may have knowledge about the proportion of vegans from which we can estimate probabilities.

— The alternative is that dessert probabilities differ depending on course choice (but not necessarily on course choice probabilities.

— If vegans don't eat cheesecake then the probability on the lowest branch $\rightarrow$ 0.

Permutations.

- How many ways are there to select 4 cards from a 52 card deck? ① What if the deck has only 13 cards? ② How do the counts change if we don't care what the order of the cards selected is? ③ ④

① $_{52}P_4 = \frac{52!}{(52-4)!} = 6{,}497{,}400$ ③ $\binom{52}{4} = 270{,}725$

② $_{13}P_4 = \frac{13!}{(13-4)!} = 17{,}160$ ④ $\binom{13}{4} = 715$

- Find *n* that makes this expression true: $_{n+1}P_3 = {}_nP_4$

Expand the permutations as polynomials in n:
Note that k is the order of the polynomial — the number of terms:

$$_nP_k = n(n-1)\dots(n-k+1)$$

three terms

$_{n+1}P_3 \quad (n+1)(n)(n-1)$

four terms

$_nP_4 = n(n-1)(n-2)(n-3)$

e.g. $n=4$
$n-k+1 = 4-3+1 = 2$
$4(3)(2)$

$n=3$, $n-k+1$
$3-3+1=1$
$(3)(2)(1)$

$$(n+1)(\cancel{n})(\cancel{n-1}) = \cancel{n}(\cancel{n-1})(n-2)(n-3)$$

$n+1 = n^2-5n+6$
$0 = n^2-6n+5$

$-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$

$3 \mp \sqrt{9-5} = 5, 1$
2

What if $n=1$? $_1P_3$ $\quad$ $_2P_4$

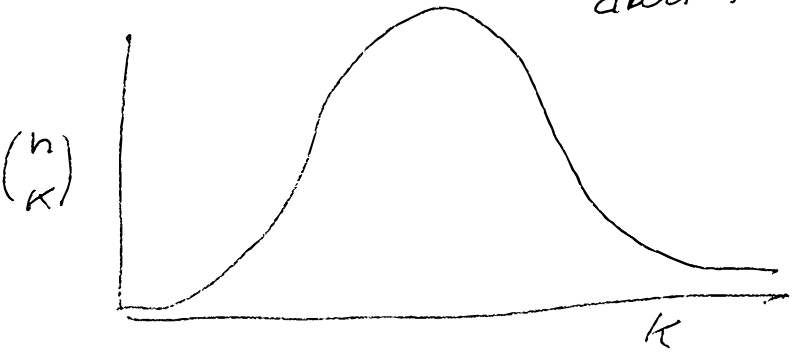
i.e. choose 3 items in order from a set of 1 (ü).

$1(0)(0) \qquad 2(0)(0)$

Binomial coefficients

- Compute the $\binom{n}{k}$ for $n = 0..10$, $0 \le k \le n$
- values
- to create “Pascal’s Triangle.” Plot the values of $\binom{n}{k}$ for each value of *n*. (The look something like distributions.) See the colab notebook ISE-201/notebooks/binomial-coefs.ipynb

The take-away is that, as $n \to 10$ the coefs take this shape: (You'll see a lot more about this later).



Sasha has to compute the number of ways to pick subsets of all-but-one day of the year from a year of 365 days, but discovers that his computer fails to compute *365!* Find him a workable solution.

The number of ways to pick "all-but-one" is the same as the number of ways to pick just one—just complement the membership in the set. In general, this is just the relation

$$\binom{n}{1} = \binom{n}{n-1}$$

So Sasha's solution = 365

Complete the proof that we started in class. (Hint - use n(n-1)! = n!)

To Prove:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Example of an algebraic proof

$$= \frac{n-1}{k}\binom{n-2}{k-1} + \binom{n-1}{k-1}$$

Apply 'move out of ( ) rule, to convert brackets to common denominator.

$$= \frac{n-1}{k} \frac{(n-2)!}{(n-k-1)!(k-1)!} + \frac{(n-1)!}{(n-k)!(k-1)!}$$

Apply def - convert to factorials.

$$= \frac{(n-1)!}{k(n-k-1)!(k-1)!} + \cdots$$

Since $m(m-1)! = m!$

$$= \frac{(n-1)!}{(k-1)!}\left[\frac{1}{k(n-k-1)!} + \frac{1}{(n-k)!}\right]$$

Remove common factor.

$$= \frac{(n-1)!}{(k-1)!}\left[\frac{n-k}{k(n-k)!} + \frac{1}{(n-k)!}\right]$$

Similarly: $(n-k)(n-k-1)! = (n-k)!$

$$= \frac{(n-1)!}{(k-1)!}\left[\frac{n-k+k}{k(n-k)!}\right] = \frac{(n-1)!}{(k-1)!}\left[\frac{n}{k(n-k)!}\right]$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Done!

Linear Algebra HW 1

Vector spaces

- To be a Vector Space, all operations on its vectors need to be *closed*, meaning that the result is also a member of the vector space.
    - If the numbers used for scalars and elements of tuples in the vector space (we speak of the *field over which* the vector space is defined) are restricted to the integers, is the vector space closed?
    - If we define a vector space over the non-negative rational numbers is it closed?

If the field is $\mathbb{Q}$ - integers, then addition and multiplication on integer is closed — so its vectors are closed

I think if restricted to positive rationals, the additive inverse to any vector v does not exist, so the vector space is not closed.

For the vector v in a vector space, suppose both the vectors $u \ \text{and}\ u\prime$ are additive inverses to $v, \ \text{e.g.} v + u = \bold{0}$. Prove that the additive inverse to v is unique.

- To prove uniqueness you want to show that $u = u\prime$. Use axioms from the definition of a vector space to show this algebraically. g

We need to show $u = u'$.

Assume as additive inverses: $0 = v + u'$

$0 = v + u$

Then

$$u = u + 0 = u + (v + u') = (u + v) + u' = (v + u) + u' = 0 + u' = u'$$

So $u = u'$

Linear combinations

1 • For some given values of the b_is, and the vectors $A_1 = (1,0,0), A_2 = (1,1,0), A_3=(1,1,1)$ is there a linear combination (e.g. a tuple of the *xs),* such that $x_1A_1 + x_2A_2 + x_3A_3 = (b_1, b_2, b_3)$? What happens if all b_is, ****are zero?

2 • Is there a linear combination of these vectors **with non-zero coefficients** that equals the zero vector? What if we add a fourth vector $A_4 = (0,0,1)$ to the linear combination?

3 • For the 2-tuples $B_1 = (a, 7), B_2 = (7, b)$, such that $x_1 B_1 + x_2B_2 = \bold{0}$, for what values of a and b do $x_1\ \text{and}\ x_2$ have a non-zero solution?

• What do the 3 problems above look like in matrix notation?

(1) This matrix equation is solved by back substitution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$x_3 = b_3$

$x_2 = b_2 - b_3$

$x_1 = b_1 - b_2 - b_3$

(2) If all b_i are zero, then starting with the last row - solve:

if x_3 & $x_2 = 0$ then $x_1 = 0$

if $x_3 = 0$ then $x_2 = 0$

implies $x_3 = 0$

This is true because the columns are linearly independant.

If instead we add a fourth column, the columns become linearly dependent, so

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

that $0 = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$

and $A_1 + A_2 + A_3 = A_4$

implies $x_1 = x_2 = x_3 = -x_4$

(3)
$$\begin{bmatrix} a & 7 \\ 7 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} ax_1 + 7x_2 = 0 \\ 7x_1 + bx_2 = 0 \end{matrix} \rightarrow \begin{matrix} \frac{a}{7} = -\frac{x_2}{x_1} \\ \frac{b}{7} = -\frac{x_1}{x_2} \end{matrix}$$

$$\rightarrow \frac{a}{7} = \frac{7}{b} \rightarrow ab = 49$$

So any a, b with that product makes the two columns linearly dependent, e.g.

$$\begin{bmatrix} 1 & 7 \\ 7 & 49 \end{bmatrix} \text{ with } 7x_1 = -x_2 \quad \text{or} \quad \begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix} \text{ with } x_1 = -x_2.$$

(4) See above.