HW 4 solutions due 22 Sept '24

- For any row *i*, of a matrix \$M\$, and column *j* of its inverse \$M^{-1}\$ show why they must be orthogonal if i ≠ j.
- If the matrices A and B's product is AB = 0, show that the column space of B is contained in the nullspace of A.

Matrices

1. We need to show that, ~~for~~ M and its inverse M^{-1} the inner product of their columns $= 1$ if the column indexes are the same, and $= 0$ if they are different. This follows from the relation

$$MM^{-1} = I.$$

where I is the diagonal matrix with 1s on the diagonal and zeros elsewhere. By def. of matrix multiplication, an on-diagonal entry is the inner product of the same row index of M & column index of M^{-1}. Conversely off-diagonals imply the indexes are different.

2. This follows from the definition of A's nullspace, which are all vectors x s.t. $Ax = 0$. $AB = 0$ means for each column B_j of B:

$$AB_j = 0$$

2, cont.

The columns of B span B's column space, so any linear combination of the B_j's would be mapped into 0.

Inner Products

- Show that when two vectors align their inner product is the product of their lengths.
- For a vector of all ones of length *n*, e.g. $(1_1, 1_2, \ldots 1_n)$ compute its angle with the x axis.

1. Two vectors align when the angle they form $= 0$. Using the def. of inner product for u, v

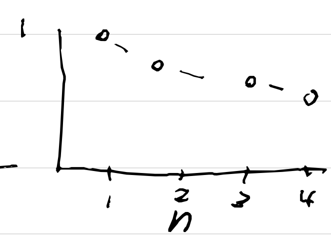
$$\langle U, V\rangle = \|A\| \|B\| \cos\theta$$
$$= \|A\| \|B\| \quad \text{Since } \cos 0 = 1.$$

2. The 'x-axis' $e_x = (1, 0, \ldots 0)$, for $v = (1, 1, \ldots 1)$

$$\cos\theta = \frac{\langle e_x, v\rangle}{\|e_x\| \|v\|}$$

$$= \frac{1 \cdot 1 + 1 \cdot 0 + \ldots 1 \cdot 0}{(1)\sqrt{1^2 + 1^2 + \ldots 1^2}}$$

$$= \frac{1}{\sqrt{n}}$$



- Is a projection matrix invertible? Why or why not?
- For $P = A(A^TA)^{-1}A^T$. Using this definition prove the two properties of projection operators: $P = P^2\ \ \text{and} \ P= P^T$.
- Show how the Projection operator simplifies when the columns of *A* are orthogonal and of unit length.

1. Intuitively P "throws away" the orthogonal component of its argument:
$I-P$ gives the orthogonal component since its orthogonal to P:

$$P(I-P) = P - P^2 = P - P = 0$$

If the orthogonal component is zero (e.g. x lies in P's column space) then

$$Px + (I-P)x = x$$

$$Px + 0 = x \Rightarrow P = I$$

2. Show P is nilpotent - once v is projected in P's space, repeating the projection doesn't change it.

$$A(A^TA)^{-1}A^T \cdot A(A^TA)^{-1}A^T =$$

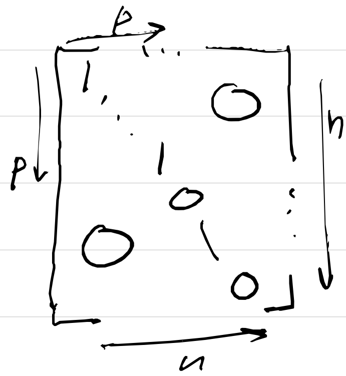
$$A\cancel{(A^TA)^{-1}A^TA}(A^TA)^{-1}A^T =$$

$$A\,I\,(A^TA)A^T = A(A^TA)^{-1}A^T \checkmark$$

Show $P = P^T$

$$\left(A(A^TA)^{-1}A^T\right)^T = \left(\left(A^TA\right)^{-1}A^T\right)^T A^T$$
$$= A^{TT}\left(A^TA\right)^{-1^T} A^T$$
$$= A(A^TA)^{-1}A^T \quad \text{since } (A^TA)^{-1} \text{ is symmetric}$$

3. When A's columns are ortho-normal, and $p < n$ then A^TA is $p \times p$ identity matrix so $A(A^TA)^{-1}A^T \rightarrow AA^T$ which is $n \times n$ with 1s along p diagonal elements & zeros elsewhere: For instance:

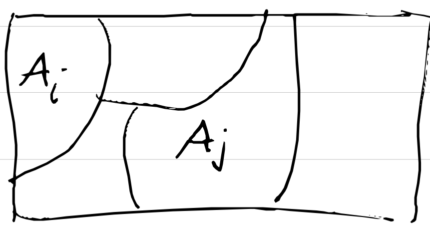
$P =$ 

So Pb just selects the first p elements of b

Show why two events that are mutually exclusive cannot be probabilistically independent.

Intuitively knowing one event occurs implies any other would not, which is informative.

Partition the state space into events E_i



Such that

$$A_i \cap A_j = \emptyset$$

The test for independence is

$$P(A_i)\,P(A_j) = P(A_i / A_j)\,P(A_j)$$

$$= \frac{P(\overset{0}{A_i \cap A_j})}{\cancel{P(A_j)}}\,\cancel{P(A_j)} = 0$$

So this implies that one or both events cannot occur.

For an example of visualizing data, see the notebook "visualizing_dependent_data.ipynb