Midterm Solutions

1).

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 - R_1$$

$$\begin{bmatrix} 1 & & \\ +1 & 1 & \\ & & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = A$$

Since U has 3 pivots, its full rank, so A is non singular and its nullspace is just the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, of

dim = 0.

2

2). Pivots:

$$\begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Left null space [last row] :

2. null space [columns 2 & 4]

3 - # of pivots
  - dim of column space
  - dim of row space
  - rank of matrix

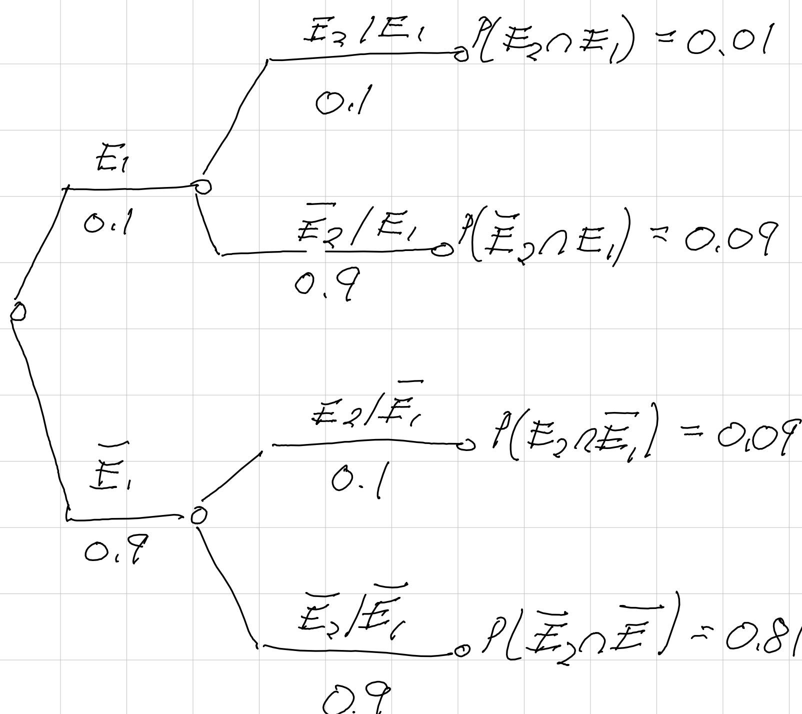
4 - number of rows
5 - number of columns

3) Since $F_1 = \bar{E}_1$, $F_2 = \bar{E}_2$

① $P(E_1 \cap F_2) = P(E_1 \cap \bar{E}_2) = 0.09$

② $P(E_1 \cup E_2) =$
$P(E_1 \cap E_2) + P(E_1 \cap \bar{E}_2) + P(\bar{E}_1 \cap E_2) =$
$0.01 + 0.09 + 0.09 = 0.19$

③ $P(E_1 \cup F_1) = P(E_1 \cup \bar{E}_1) = P(\Omega) = 1$



2, cont: With dependent events:

If $P(\bar{E}_2 | E_1) = 1$

then $P(\bar{E}_2 | E_1) = 1$, $P(E_2 | E_1) = 0$

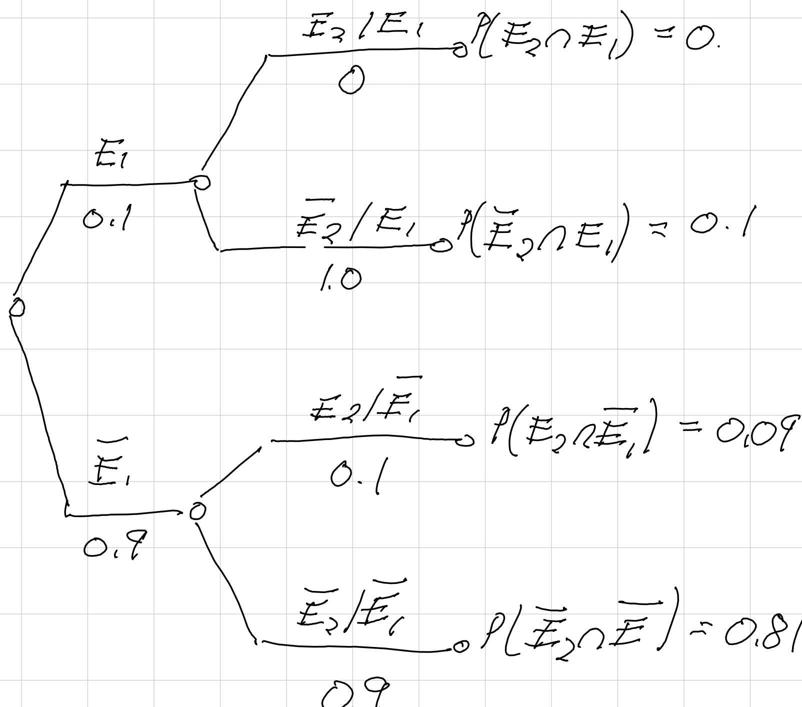
① $P(E_1 \cap E_2) = P(\bar{E}_1 \cap \bar{E}_2) = 0$

② $P(E_1 \cup E_2) =$

$P(E_1 \cap E_2) + P(E_1 \cap \bar{E}_2) + P(\bar{E}_1 \cap E_2) =$

$0 + 0.1 + 0.09 = 0.19$

③ $P(E_1 \cup \bar{E}_1) = P(E_1 \cup \bar{E}_1) = P(\Omega) = 1$



4. Combinatorics

a.

$P(\text{Alice} = \text{President}) = 1/10$

$P(\text{Bob} = \text{VP}) = 1/9$

$P(\text{Carol} = \text{Secretary}) = 1/8$

$$P(A) \cdot P(B) \cdot P(C) = 1/(10 \cdot 9 \cdot 8) = \frac{7!}{10!}$$

$$= \frac{1}{720}$$

b. All possibilities of filling the slate are 3!

$$\frac{\text{\# of ways of "winning"}}{\text{\# of possible slates}} = \frac{3!\,7!}{10!} = \frac{1}{\binom{10}{3}}$$

$$= \frac{3 \cdot 2}{10 \cdot 9 \cdot 8} = \frac{1}{120}$$

5. All possible subsets of 7 items minus the null set

$$2^7 - 1 = 127$$

All possible combinations of 6 items (we include just the bun as one):

$$2^6 = 64$$

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6. We need to find the two vectors orthogonal to the rows of

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Set the non-pivot cols = 1, one at a time:

Solution 1. Set $x_3 = 1$, $x_4 = 0$ $\Rightarrow$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Check

$$\underline{1\;1\;0\;0}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \qquad \underline{0\;1\;0\;1}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

6. continued: Set $x_3 = 0$ $x_4 = 1$ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Check

$$\underbrace{1\,1\,0\,1}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underbrace{0\,1\,0\,1}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So the nullspace basis is all vectors

$$a_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

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7. Any vector made of $\sqrt{2}, -\sqrt{2}, 0$ in any order is $\perp$ to $(\sqrt{3}, \sqrt{3}, \sqrt{3})$. So take $(\sqrt{2}, -\sqrt{2}, 0)$.

Since these two vectors are orthogonal the $(A^TA)^{-1}$ matrix is diagonal.

$$A^T x = \begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}$$

So to complete the projection: $A^T A x =$

$$\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{3} & -\sqrt{2} \\ \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}$$

8. Duplicate 2 columns in A to make the Gram matrix singular

Let A be orthonormal, but with the first two columns identical $= \begin{bmatrix} a_1 & a_1 & a_2 & a_3 & \dots \end{bmatrix}$

The Gram matrix will be

$$\begin{bmatrix} 1 & 1 & 0 & \cdot\cdot & 0 \\ 1 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

with the first two columns & rows identical.

$\Rightarrow$ the columns are linearly dependent.

9. For matrices of the form

$$\begin{bmatrix} 0 & \cdots & & 1 \\ 0 & & 1 & 0 \\ \vdots & & \iddots & \\ \vdots & \iddots & & \\ 1 & \cdots & & 0 \end{bmatrix}$$

To compute the determinant we count the row swaps to convert it to an identity matrix. Each swap flips the det. sign

a. For 3x3:

$$\begin{array}{cccc} & \text{col } 1 & \text{col } 2 & \text{col } 3 \\ & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 1 & 2 \\ 3 & 3 & 2 & 1 \end{array}$$

3 swaps

So $\det(I) \cdot (-1)^3 = -1$

b. Replace a "1" by a scales the determinant by a.

c. Setting an off diagonal element in the identity matrix creates a triangular matrix whose off-diagonal elements do not change the determinant.

d. Setting a diagonal element to zero in an identity matrix $\Rightarrow$ det $\rightarrow 0$

e. For an $n \times n$ matrix, work recursively starting with the $n-1 \times n-1$ matrix and count the number of swaps to move a new column from the first to last row:

$$\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix}$$

If there are n columns, then $n-1$ swaps are needed. If $n-1$ is even, the sign doesn't change and vice versa, so as n increases the sign flips every $n \rightarrow n+2$

This is the pattern:

n	det
1	1
2	−1
3	−1
4	1
5	1
6	−1
7	−1

One solution is

$$\det = (-1)^{1+(n-1) \bmod 2}$$