Midterm practice problems

Probability 1

1.48. An experiment consists of drawing 3 cards in succession from a well-shuffled ordinary deck of cards. Let A_1 be the event "king on first draw," A_2 the event "king on second draw," and A_3 the event "king on third draw." State in words the meaning of each of the following:

(a) $P(A_1 \cap A'_2)$, (b) $P(A_1 \cup A_2)$, (c) $P(A'_1 \cup A'_2)$, (d) $P(A'_1 \cap A'_2 \cap A'_3)$, (e) $P[(A_1 \cap A_2) \cup (A'_2 \cap A_3)]$.

Answer

1.48. (a) Probability of king on first draw and no king on second draw.

(b) Probability of either a king on first draw or a king on second draw or both.

(c) No king on first draw or no king on second draw or both (no king on first and second draws).

(d) No king on first, second, and third draws.

(e) Probability of either king on first draw and king on second draw or no king on second draw and king on third draw.

Probability 2

1.38. A shelf has 6 mathematics books and 4 physics books. Find the probability that 3 particular mathematics books will be together.

All the books can be arranged among themselves in ${}_{10}P_{10} = 10!$ ways. Let us assume that the 3 particular mathematics books actually are replaced by 1 book. Then we have a total of 8 books that can be arranged among themselves in ${}_8P_8 = 8!$ ways. But the 3 mathematics books themselves can be arranged in ${}_3P_3 = 3!$ ways. The required probability is thus given by

$$\frac{8!\,3!}{10!} = \frac{1}{15}$$

Probability 3

1.13. One bag contains 4 white balls and 2 black balls; another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that (a) both are white, (b) both are black, (c) one is white and one is black.

Let W_1 = event "white ball from first bag," W_2 = event "white ball from second bag."

(a) $P(W_1 \cap W_2) = P(W_1)P(W_2 \mid W_1) = P(W_1)P(W_2) = \left(\frac{4}{4+2}\right)\left(\frac{3}{3+5}\right) = \frac{1}{4}$

(b) $P(W'_1 \cap W'_2) = P(W'_1)P(W'_2 \mid W'_1) = P(W'_1)P(W'_2) = \left(\frac{2}{4+2}\right)\left(\frac{5}{3+5}\right) = \frac{5}{24}$

(c) The required probability is

$$1 - P(W_1 \cap W_2) - P(W'_1 \cap W'_2) = 1 - \frac{1}{4} - \frac{5}{24} = \frac{13}{24}$$

Probability 4

1.20 Events E, F, and G form a list of mutually exclusive and collectively exhaustive events with $P(E) \neq 0$, $P(F) \neq 0$, and $P(G) \neq 0$. Determine, for each of the following statements, whether it must be true, it might be true, or it cannot be true:

a E', F', and G' are mutually exclusive.
b E', F', and G' are collectively exhaustive.
c E' and F' are independent.
d $P(E') + P(F') > 1.0$.
e $P(E' + EF' + EFG') = 1.0$.

Linear Algebra 1

LU decomposition

1.5.4 Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

Answers:

1.5.4 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$.

Linear Algebra 2

Show that (1,1,0,(1,0,1),(0,1,1) is a basis in $\mathbb{R}^3$:

1. By showing that the vectors are linearly independent
2. By showing that these vectors span the space of the 3 unit vectors e_1, e_2, e_3 .

Linear Algebra 3

Find a condition on the lengths $||u|| \text{and} ||v||$ such that $u + v$ is orthogonal to $u - v$.