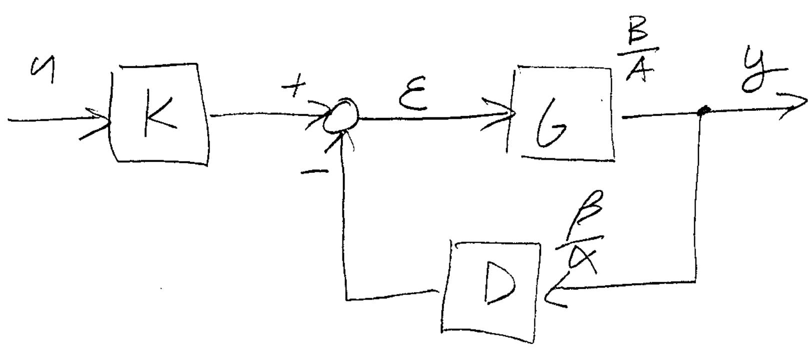
Direct Design

(e.g.) $$G(s) = \frac{1}{s^2}$$

roots: $z = 0.6 \pm 0.4i$ (designer specified)

$$G(z) = (1 - z^{-1})\, \mathcal{Z}\left[\frac{G(s)}{s}\right] \quad \text{(ZOH)}$$

$$G(z) = \frac{T^2}{2} \cdot \frac{z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2}}$$ (refer to ZOH section)



$$\frac{y}{u} = H = K \cdot \frac{G}{1 + DG}$$

Let $G = \frac{B}{A}$

$D = \frac{\beta}{\alpha}$

$$H = K \cdot \frac{B \cdot \alpha}{A\alpha + B\beta}$$

(cont)

(cont)

For the given roots

$$z = 0.6 \pm 0.4\hat{i}$$

we want to perform pole placement

Notice that if the poles are found the zeros are also found,

$$H = K \cdot \frac{B\alpha}{A\alpha + B\beta} \quad \begin{array}{l} \text{zeros} \\ \text{poles} \end{array}$$

$$(z - 0.6 + 0.4i)(z - 0.6 - 0.4i)$$

$$z^2 - 1.2z + 0.520$$

Recall the Diophantine Equation

$$\underset{(n-1)}{\alpha}\,\underset{(n)}{A} + \underset{(n-1)}{\beta}\,\underset{(n)}{B} = \underset{(2n-1)}{D}$$

where D is our characteristic polynomial

The orders must be checked, (include zero)

$$G(z) = \frac{B}{A} \cdot \frac{z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2}} \quad \begin{array}{l} (2) \\ (2) \end{array}$$

$$D = z^2 - 1.2z + 0.52 \quad (2)$$

The orders do not satisfy Diophantine for $n = 2$

$$2 \cdot n - 1 = 3$$

D needs another term.

(cont)

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Adding another root at zero fixes D

$$D = z^3 - 1.2z^2 + 0.52 \cdot z \qquad (2n-1) \checkmark$$

$$G(z) = \frac{B}{A} = \frac{0 \cdot z^2 + z + 1}{z^2 - 2z + 1} \cdot \left(\frac{T^2}{2}\right)$$

$$= \frac{b_0 z^2 + b_1 z + b_2}{a_0 z^2 + a_1 z + a_2}$$

$$D(z) = \frac{\beta}{\alpha} = \frac{\beta_0 z + \beta_1}{\alpha_0 z + \alpha_1} \qquad (\text{1st order}) \checkmark$$

2nd order Sylvester Matrix

Let $T = 0.2$

$$\frac{T^2}{2} = 0.020$$

$$E = \begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0.02 & 0 \\ -2 & 1 & 0.02 & 0.02 \\ 1 & -2 & 0 & 0.02 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(cont)

$$M = E^{-1} \cdot D$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = E^{-1} \cdot D \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$ ← notice order

(Perform calculations in Matlab)

$$M = E^{-1} \cdot D \qquad D = \begin{bmatrix} 0 \\ 0.52 \\ -1.2 \\ 1.0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.32 \\ 1.0 \\ -16.0 \\ 24.0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

$$D(z) = \frac{\beta}{\alpha} = \frac{\beta_0 \cdot z + \beta_1}{\alpha_0 \cdot z + \alpha_1}$$

$$\boxed{D(z) = \frac{24 \cdot z - 16}{z + 0.32}}$$

(cont)

K still needs to be chosen such that the error is zero for a step input,

$$H = K \cdot \frac{B\,\alpha}{A \cdot \alpha + B\beta}$$

$$= K \frac{\overset{B}{(z+1)}\;\overset{\alpha}{(z+0,32)} \cdot \overset{T^2/2}{0.02}}{z^3 - 1,2z^2 + 0,52z}$$

desired poles w/ added root

$$\lim_{z \to 1} K \cdot \frac{(z+1)(z+0,32)0,02}{z^3 - 1,2z^2 + 0,52 \cdot z} = 1$$

$$K = \frac{1 - 1,2 + 0,52}{(1+1)(1+0.32)0,02}$$

$$= \frac{0,320}{0,0528}$$

$$\boxed{K = 6,06}$$

Now K and D(z) can be used to build the controller