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Laplace of unit step

(e.g.)

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} \cdot u(t)\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$\int e^u\, du = e^u$$

$$u = -st$$

$$du = (-s)\,dt$$

$$= -\frac{1}{s} \int_0^\infty e^{-st} \cdot (-s)\, dt$$

$$= -\frac{1}{s} \cdot e^{-s \cdot t} \Big|_0^\infty$$

$$= -\frac{1}{s} \cdot \left[e^{-\infty} - e^{0} \right]$$

$$= -\frac{1}{s} \left[0 - 1 \right]$$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}} \quad \checkmark$$

State Space → Transfer function

(e.g.) $\dot{x} = Ax + Bu$

$y = Cx + Du$

$$\underset{2\times 1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} + \underset{2\times 1}{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \cdot \underset{1\times 1}{u}$$

$$y = \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \cancel{+ 0 \cdot u}$$

1×1

Recall Laplace

$$\mathcal{L}\{\dot{x}\} = s \cdot \mathcal{L}\{x\} - x(0)$$

$$\mathcal{L}\{x(t)\} = X(s)$$

$$s \cdot X(s) = A \cdot X(s) + B\,u(s)$$

$$s \cdot X - AX = Bu$$

$$s \cdot I \cdot X - Ax = Bu$$

$$(sI - A)X = Bu$$

$$\boxed{X = (sI - A)^{-1} \cdot Bu}$$

(cont)

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(cont)

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

$$y = C \cdot X$$

$$y = C \cdot \left[(sI - A)^{-1} \cdot Bu \right]$$

$$\frac{y}{u} = C \cdot (sI - A)^{-1} \cdot B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s+10 & -1 \\ 0.02 & s+2 \end{bmatrix}^{-1} \right) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\left. \begin{array}{l} \det = (s+10)(s+2) + 0.02 \\ \dfrac{1}{\det} \begin{bmatrix} s+2 & 1 \\ -0.02 & s+10 \end{bmatrix} \end{array} \right\} \text{inverse}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ -0.02 & s+10 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}}{s^2 + 12s + 20.02}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2s+20 \end{bmatrix}}{s^2 + 12s + 20.02}$$

$$\boxed{G(s) = \frac{2}{s^2 + 12s + 20.02}}$$

Matlab

ss2tf(A,B,C,D);

= [num, den]

* solution is unique

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Transfer Function to State Space

(e.g.)

$$G(s) = \frac{10}{s^3 + 6s^2 + 11s + 6}$$

$$= \frac{10}{(s+3)(s+2)(s+1)}$$

$$\frac{10}{(s+3)(s+2)(s+1)} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$A = \frac{10}{(-1)(-2)} \qquad B = \frac{10}{-1}$$

$$A = 5 \qquad B = -10$$

$$C = \frac{10}{2}$$

$$C = 5$$

$$G(s) = \frac{5}{s+3} - \frac{10}{s+2} + \frac{5}{s+1}$$

$$\frac{y}{u} = \frac{5}{s+3} - \frac{10}{s+2} + \frac{5}{s+1}$$

$$y = \frac{5}{s+3} \cdot u - \frac{10}{s+2} \cdot u + \frac{5}{s+1} \cdot u$$

(cont)

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(cont)

substitute with x

setup as passthru

$$y = \frac{5}{s+3}\cdot u - \frac{10}{s+2}\cdot u + \frac{5}{s+1}\cdot u$$

$x_1 \qquad x_2 \qquad x_3$

$$x_1 = \frac{5}{s+3}\cdot u \qquad x_2 = -\frac{10}{s+2}\cdot u \qquad x_3 = \frac{5}{s+1}\cdot u$$

$$x_1(s+3) = 5\cdot u \qquad x_2(s+2) = -10u \qquad x_3(s+1) = 5\cdot u$$

$$s\cdot x_1 = 5\cdot u - 3x_1 \qquad s\cdot x_2 = -10u - 2\cdot x_2 \qquad s\cdot x_3 = 5\cdot u - x_3$$

$$\left(\mathcal{L}\{\dot{x}\} = s\cdot X(s) - x(0)\right)$$

assume $x(0) = 0$

$$\dot{x}_1 = -3\cdot x_1 + 5u \qquad \dot{x}_2 = -2x_2 - 10u \qquad \dot{x}_3 = -x_3 + 5u$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + \underbrace{\cancel{D}u}_{0}$$

$$\dot{x} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 5 \\ -10 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

pass thru

Solution is not unique, assumptions were made

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Z - Transform

$$X(z) = Z\{x[k]\} = \sum_{n=-\infty}^{\infty} x[k]\, z^{-k}$$

e.g.) <u>delta</u>

$$Z\{\delta[k]\} = \sum_{k=-\infty}^{\infty} \delta[k] \cdot z^{-k}$$

$$= \sum_{k=0}^{0} \delta[0] \cdot z^{-0}$$

$$= \boxed{1}$$

e.g.) <u>unit step</u>

$$Z\{u[k]\} = \sum_{k=-\infty}^{\infty} u[k] \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots$$

(geometric series)

$$= \frac{1}{1 - z^{-1}} = \boxed{\frac{z}{z-1}}$$

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Geometric Series

$$S_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + r^3 + \ldots r^n$$

$$r \cdot S_n = r + r^2 + r^3 + \ldots + r^{n+1}$$

$$r \cdot S_n - S_n = -1 + r^{n+1}$$

$$S_n (r-1) = -1 + r^{n+1}$$

$$S_n = \frac{-1 + r^{n+1}}{r-1}$$

$$\boxed{S_n = \frac{1 - r^{n+1}}{1-r} = \sum_{k=0}^{n} r^k}$$

$$S_\infty = \sum_{k=0}^{\infty} r^k = \frac{1 - r^{\infty+1}}{1-r}$$

$$\boxed{\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}}$$

(infinite geometric series)

$$\boxed{\sum_{k=0}^{\infty} r^{-k} = \frac{1}{1-r^{-1}} = \frac{r}{r-1}}$$

s-domain $\rightarrow$ z-domain
Zero Order Hold

(e.g.)

$$G(s) = \frac{1}{s^2}$$

$$G(z) = (1 - z^{-1})\, \mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$$= (1 - z^{-1})\, \mathcal{Z}\left[\frac{1}{s^3}\right]$$

$X(s)$	$X(z)$
$\frac{2}{s^3}$	$\frac{T^2 z^{-1} (1 + z^{-1})}{(1 - z^{-1})^3}$

$$= (1 - z^{-1}) \cdot \frac{1}{2} \cdot \mathcal{Z}\left[\frac{2}{s^3}\right]$$

$$= \cancel{(1 - z^{-1})}\, \frac{1}{2} \cdot \frac{T^2 \cdot z^{-1} (1 + z^{-1})}{(1 - z^{-1})^{\cancel{3}\,2}}$$

$$G(z) = \frac{T^2 \cdot z^{-1} (1 + z^{-1})}{2 (1 - z^{-1})^2}$$

$$= \frac{T^2}{2} \cdot \frac{(z^{-1} + z^{-2})}{(1 - z^{-1})^2}$$

$$\boxed{G(z) = \frac{T^2}{2} \cdot \frac{z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2}}}$$

Mapping (s-domain → z-domain)
Forward

e.g.) $$D(s) = \frac{10(s+0.1)}{s+1} \qquad T=1$$

$$s = \frac{z-1}{T}$$

$$D(z) = \frac{10\left(\frac{z-1}{T}+0.1\right)}{\frac{z-1}{T}+1}$$

$$= \frac{10(z-1+0.1\cdot T)}{z-1+T}$$

$$= \frac{10\cdot z-10+T}{z-1+T}$$

$(T=1)$

$$= \frac{10\cdot z-9}{z}$$

$$\boxed{D(z) = 10-9z^{-1}}$$

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Mapping (S-domain to Z-domain)
Backward

(e.g.) $$D(s) = \frac{10(s+0.1)}{s+1} \qquad s = \frac{z-1}{z \cdot T}$$

$$D(z) = \frac{10\left(\frac{z-1}{z \cdot T} + 0.1\right)}{\frac{z-1}{z \cdot T} + 1}$$

$$= \frac{10(z-1+0.1 \cdot zT)}{z-1+zT}$$

$$= \frac{10z - 10 + zT}{z(1+T)-1}$$

$$= \frac{z(10+T)-10}{z(1+T)-1}$$

Let $T = 1$

$$D(z) = \frac{11 \cdot z - 10}{2z - 1}$$

$$\boxed{D(z) = \frac{11 - 10z^{-1}}{2 - z^{-1}}}$$

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Algorithm (z-domain) to Difference Equation

(e.g.)

$$\frac{z^2 - 0.95z + 0.7}{z^2 - 0.85z + 0.98}$$

$$\frac{u(z)}{\mathcal{E}(z)} = \frac{1 - 0.95 \cdot z^{-1} + 0.7 z^{-2}}{1 - 0.85 z^{-1} + 0.98 z^{-2}}$$

$$u(z) - u(z)0.85 z^{-1} + u(z) 0.98 z^{-2}$$
$$= \mathcal{E}(z) - \mathcal{E}(z) 0.95 z^{-1} + \mathcal{E}(z) 0.7 z^{-2}$$

$$u(z) = u(z) 0.85 z^{-1} - u(z) 0.98 z^{-2} + \mathcal{E}(z) - \mathcal{E}(z) 0.95 z^{-1}$$
$$+ \mathcal{E}(z) 0.7 z^{-2}$$

$$\boxed{u(k) = u(k-1) 0.85 - u(k-2) 0.98 + \mathcal{E}(k) - \mathcal{E}(k-1) 0.95 + \mathcal{E}(k-2) 0.7}$$

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Algorithm (z-domain) to Difference Equation

(e.g.) $$D(z) = 10 - 9z^{-1}$$

$$\frac{u(z)}{\mathcal{E}(z)} = 10 - 9 \cdot z^{-1}$$

$$u(z) = 10 \cdot \mathcal{E}(z) - 9 \cdot \mathcal{E}(z)\, z^{-1}$$

$$\boxed{u(k) = 10 \cdot \mathcal{E}(k) - 9 \cdot \mathcal{E}(k-1)}$$

(inverse z-transform)

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(e.g.) $$D(z) = \frac{11 - 10z^{-1}}{2 - z^{-1}}$$

$$\frac{u(z)}{\mathcal{E}(z)} = \frac{11 - 10z^{-1}}{2 - z^{-1}}$$

$$u(z)\left[2 - z^{-1}\right] = \mathcal{E}(z)\left[11 - 10z^{-1}\right]$$

$$2 \cdot u(z) - u(z)z^{-1} = 11 \cdot \mathcal{E}(z) - 10 \cdot \mathcal{E}(z)z^{-1}$$

$$2 \cdot u(z) = u(z)z^{-1} + 11 \cdot \mathcal{E}(z) - 10 \cdot \mathcal{E}(z)z^{-1}$$

$$u(z) = \frac{1}{2}\left[u(z)z^{-1} + 11\,\mathcal{E}(z) - 10\,\mathcal{E}(z)z^{-1}\right]$$

$$\boxed{u(k) = \frac{1}{2}\left[u(k-1) + 11 \cdot \mathcal{E}(k) - 10 \cdot \mathcal{E}(k-1)\right]}$$