11/22/13

$G(s) \to G(z)$ (ZOH)

(e.g.) $G(s) = \dfrac{1}{s(s+1)}$ $T = 1$

$$G(z) = (1 - z^{-1})\, \mathcal{Z}\left[\frac{G(s)}{s}\right]$$

$$= (1 - z^{-1})\, \mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$C = 1$

Let $s = 1$

$$\frac{1}{2} = A + B + \frac{C}{2}$$

$$A + B = 0$$

Let $s = 2$

$$\frac{1}{s+1} = A \cdot s + B + \frac{C \cdot s^2}{s+1}$$

$$\frac{1}{3} = 2A + B + \frac{4}{3}$$

$$2A + B = -1$$

$$A + B = 0$$

$A = -1$

$B = 1$

$$G(z) = (1 - z^{-1})\, \mathcal{Z}\left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right]$$

(cont)

(cont)

$$G(z) = (1 - z^{-1})\left[\frac{-1}{1-z^{-1}} + \frac{T \cdot z^{-1}}{(1-z^{-1})^2} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

$(T=1)$

$$= (1-z^{-1})\left[\frac{-1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2} + \frac{1}{1-e^{-1}z^{-1}}\right]$$

$$= -1 + \frac{z^{-1}}{(1-z^{-1})} + \frac{1-z^{-1}}{1-e^{-1}z^{-1}}$$

$$= \frac{-(1-z^{-1})(1-e^{-1}z^{-1}) + z^{-1}(1-e^{-1}z^{-1}) + (1-z^{-1})^2}{(1-z^{-1})(1-e^{-1}z^{-1})}$$

$$= \frac{-\cancel{1} + z^{-1}(1-e^{-1}) - e^{-1}z^{-2} + z^{-1} - e^{-1}z^{-2} + \cancel{1} - 2z^{-1} + z^{-2}}{1 - z^{-1} - e^{-1}z^{-1} + e^{-1}z^{-2}}$$

$$= \frac{z^{-1}e^{-1} + z^{-2}(1-2e^{-1})}{1 - z^{-1}(1+e^{-1}) + z^{-2} \cdot e^{-1}}$$

$$\boxed{G(z) = \frac{0.368\,z^{-1} + 0.264\,z^{-2}}{1 - 1.368\,z^{-1} + 0.368\,z^{-2}}}$$

Final Value Theorem

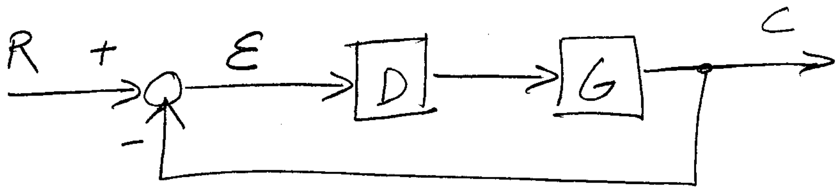
10/2/12

[Ogata 199]
[Franklin 225]

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot F(s)$$

$$= \lim_{z \to 1} (1 - z^{-1}) \cdot F(z)$$

R	s	z
step	$\frac{1}{s}$	$\frac{1}{1 - z^{-1}}$
ramp	$\frac{1}{s^2}$	$\frac{T \cdot z^{-1}}{(1 - z^{-1})^2}$
accel	$\frac{1}{s^3}$	



$$\mathcal{E} = R \cdot \frac{1}{1 + DG}$$

s-domain

$$\mathcal{E}_{ss} = \lim_{s \to 0} s \cdot R \cdot \frac{1}{1 + DG}$$

z-domain

$$\mathcal{E}_{ss} = \lim_{z \to 1} (1 - z^{-1}) R \cdot \frac{1}{1 + DG}$$

$$\mathcal{E}_{ss_{step}} = \lim_{z \to 1} \cancel{(1 - z^{-1})} \frac{1}{\cancel{(1 - z^{-1})}} \cdot \frac{1}{1 + DG}$$

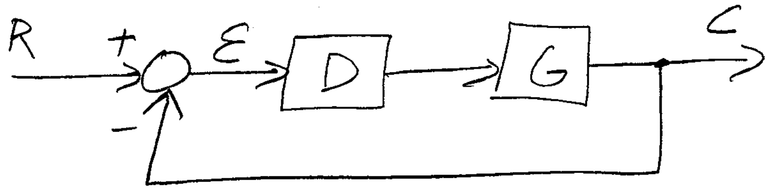
$$= \frac{1}{1 + K_p}$$

$$\mathcal{E}_{ss_{ramp}} = \lim_{z \to 1} (1 - z^{-1}) \frac{T \cdot z^{-1}}{(1 - z^{-1})^2} \cdot \frac{1}{1 + DG}$$

$$= \lim_{z \to 1} \frac{T \cdot z^{-1}}{(1 - z^{-1})(1 + DG)} = \frac{1}{K_v}$$

Steady State Error, ramp

$$G(s) = \frac{1}{s(s+1)} \qquad D(s) = \frac{149s + 880}{s+47}$$



$$H = \frac{C}{R} = \frac{DG}{1+DG} \qquad E = \frac{1}{1+DG} = 1 - H$$

$$E_{ss} = \lim_{s \to 0} s \cdot R \cdot \frac{1}{1+DG} \qquad R = \frac{1}{s^2} \text{ (ramp)}$$

$$= \lim_{s \to 0} s \cdot \frac{1}{s^2} \cdot \frac{1}{1+DG}$$

$$= \lim_{s \to 0} \frac{1}{s} \cdot \frac{1}{1+DG}$$

$$= \lim_{s \to 0} \frac{1}{s} \cdot \frac{1}{1 + \frac{149s+880}{(s+47)s(s+1)}}$$

$$= \lim_{s \to 0} \frac{1}{\cancel{s}} \cdot \frac{(s+47)\cancel{s}(s+1)}{(s+47)s(s+1) + 149s + 880}$$

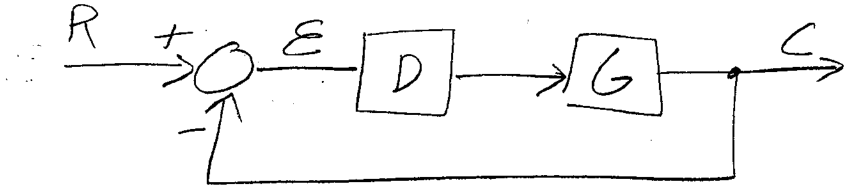
$$= \lim_{s \to 0} \frac{(s+47)(s+1)}{s^3 + 48s^2 + 47s + 149s + 880}$$

$$= \frac{47}{880}$$

$$\boxed{E_{ss\,\text{ramp}} = 0.053}$$

Steady State Error, step

$$G(s) = \frac{1}{s(s+1)} \qquad D(s) = \frac{149s+880}{s+47}$$



$$H = \frac{C}{R} = \frac{DG}{1+DG} \qquad \varepsilon = \frac{1}{1+DG} = 1 - H$$

$$\varepsilon_{ss_{step}} = \lim_{s\to 0} \cdot s \cdot R \cdot \frac{1}{1+DG} \qquad R = \frac{1}{s} \quad \text{(step)}$$

$$= \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{1}{1+DG}$$

$$= \lim_{s\to 0} \frac{1}{1+DG}$$

$$= \lim_{s\to 0} \frac{1}{1 + \frac{149s+880}{(s+47)\,s(s+1)}}$$

$$= \lim_{s\to 0} \frac{(s+47)\cdot s\,(s+1)}{(s+47)s(s+1) + 149s + 880}$$

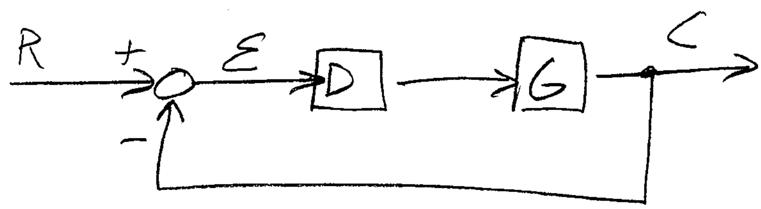
numerator $\to 0$

$\to 0$

$$\boxed{\varepsilon_{ss_{step}} = 0}$$

steady state error, ramp

$$G(s) = \frac{1}{s(s+1)} \qquad D(s) = \frac{20s^2+20s}{s^2+4s}$$



$$\frac{C}{R} = H = \frac{DG}{1+DG} \qquad \mathcal{E} = \frac{1}{1+DG} = 1-H$$

$$\mathcal{E}_{ss\,ramp} = \lim_{s\to 0} s \cdot R \cdot \frac{1}{1+DG} \qquad \frac{1}{s^2} \quad \text{(ramp)}$$

$$= \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{1}{1+DG}$$

$$= \lim_{s\to 0} \frac{1}{s} \frac{1}{1+\frac{20s^2+20s}{(s^2+4s)\,s\,(s+1)}}$$

$$= \lim_{s\to 0} \frac{1}{\cancel{s}} \frac{(s^2+4s)\cancel{s}(s+1)}{(s^2+4s)s(s+1) + 20s^2+20s}$$

$$= \lim_{s\to 0} \frac{(s^2+4s)(s+1)}{(s^2+4s)\cdot s(s+1) + 20s^2+20s}$$

$$= \lim_{s\to 0} \frac{\cancel{s}(s+4)(s+1)}{\cancel{s}\left[(s^2+4s)(s+1)+20s+20\right]}$$

$$= \frac{4}{20}$$

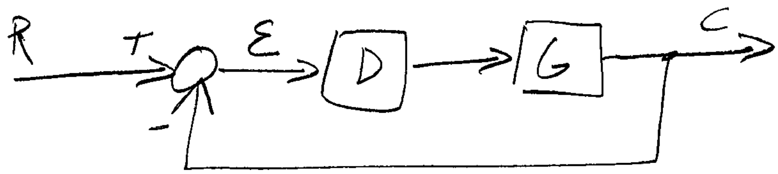
$$\boxed{\mathcal{E}_{ss\,ramp} = 0.200}$$

11/24/13

steady state response, step

PID

$$G(s) = \frac{1}{s^2+0.4s} \qquad D(s) = \frac{48s^2+197s+875}{s}$$



$$H = \frac{C}{R} = \frac{DG}{1+DG} \qquad E = \frac{1}{1+DG} = 1-H$$

$$E_{ss\,step} = \lim_{s\to 0} s \cdot R \cdot \frac{1}{1+DG} \qquad \frac{1}{s} \quad \text{(step)}$$

$$= \lim_{s\to 0} \cancel{s} \cdot \frac{1}{\cancel{s}} \cdot \frac{1}{1+\frac{48s^2+197s+875}{s(s^2+0.4s)}}$$

$$= \lim_{s\to 0} \frac{s(s^2+0.4s)}{s(s^2+0.4s)+48s^2+197s+875}$$

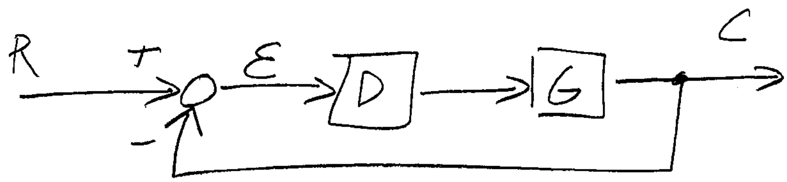
$$\boxed{E_{ss\,step} = 0 \qquad \text{agrees with plot}}$$

11/24/13

steady state response, ramp

PID

$$G(s) = \frac{1}{s^2 + 0.4 s} \qquad D(s) = \frac{48 s^2 + 197 s + 875}{s}$$



$$\frac{C}{R} = H = \frac{DG}{1 + DG} \qquad \mathcal{E} = \frac{1}{1 + DG}$$

$$\frac{1}{s^2} \quad \text{(ramp)}$$

$$\underset{\text{ramp}}{\mathcal{E}_{ss}} = \lim_{s \to 0} s \cdot R \cdot \frac{1}{1 + DG}$$

$$= \lim_{s \to 0} s \cdot \frac{1}{s^2} \cdot \frac{1}{1 + \dfrac{48 s^2 + 197 s + 875}{s(s^2 + 0.4 s)}}$$

$$= \lim_{s \to 0} \frac{1}{\cancel{s}} \cdot \frac{\cancel{s}(s^2 + 0.4 s)}{s(s^2 + 0.4 s) + 48 s^2 + 197 s + 875}$$

$$= \lim_{s \to 0} \frac{s^2 + 0.4 s}{s^3 + 48.4 s^2 + 197 s + 875}$$

$$= \boxed{0}$$

But from the plot of the ramp response it is clearly not zero!

A shortcoming of the final value theorem?