Problem 1: A second-order multi-point method for solving the differential equation

$$\frac{df(x)}{dx} = A(f(x), x)$$

can be derived using center differencing to approximate the derivative instead of using forward differencing, as discussed in class. We will be interested in writing down a formula for $f(x+\Delta x)$ given knowledge of $f(x)$ and $f(x-\Delta x)$, and then analyzing the the stability of this scheme.

a) Write down an approximation for $f(x+\Delta x)$ using the center differencing formula. You can proceed by writing down the formula you derived on the midterm and solving for $f(x+\Delta x)$ in terms of $f(x-\Delta x)$ and $A(f(x), x)$. For this method, we will therefore need information about the function at two points, $f(x-\Delta x)$, and $f(x)$ in order to compute $A(f(x), x)$.

b) Derive an equation for the error $\epsilon(x+\Delta x)$ for the multi-point method. To do so, use the fact that the numerical solution is only an approximation of the true solution, and so includes some error, i.e. $f(x) = f_{\text{true}}(x) + \epsilon(x)$. Write an equation for how the error at one timestep is related to the error at past timesteps. You should taylor expand the function A to linear order in ϵ. You can then assume that $f_{\text{true}}(x)$ obeys the equation you found in part (a) to eliminate several terms.

c) Determine whether the method is stable or unstable when applied to decaying, growing, and oscillating differential equations. Do so by considering the case where $A(f(x), x) = \lambda f$, with $\lambda < 0$, $\lambda > 0$, and $\lambda = \pm i\omega$. For stability of this method, we would like the conditions $|\epsilon(x+\Delta x)| \leq |\epsilon(x)|$ *and* $|\epsilon(x)| \leq |\epsilon(x-\Delta x)|$ to hold. Is there any (non-zero) Δx which can guarantee this is always true for a given λ?
Hint: To demonstrate a method is unstable for a given λ, it is sufficient to find a counter-example.

Problem 2: Complete the Jupyter notebook assignment.