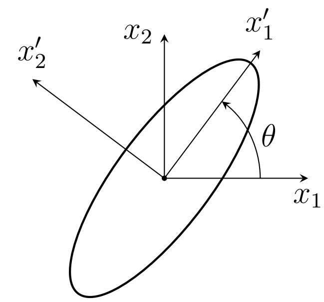
Problem 1: Consider the ellipse given by

$$14x_1^2 - 4x_1x_2 + 11x_2^2 = 25.$$



This ellipse is shown at the right. We want to find the principle axes of this ellipse represented by x_1' and x_2' in the figure. In this problem **all calculations should be done by hand** unless otherwise noted. If your linear algebra is rusty, feel free to use any resources you like as a reference.

a) We can write the equation for the ellipse as a matrix equation of the form

$$\vec{x}^{\mathsf{T}}\mathsf{A}\vec{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 25.$$

Notice that I have written A as a *symmetric* matrix. Write down the matrix A.

b) Calculate the eigenvalues, λ_n, of A. Clearly show how you calculate the eigenvalues. [Note: Despite how it may appear, the coefficients of the ellipse have been chosen to lead to simple eigenvalues.]

c) Calculate the eigenvectors, $\vec{v}_n$, of A. Remember that eigenvectors are normalized so that $|\vec{v}_n| = 1$. [Note: Again, the coefficients have been chosen so that the eigenvectors will have "simple" values.]

d) Consider another matrix B, which contains the eigenvectors of A as its columns. Show that B diagonalizes the matrix A. To do this, calculate the product $\mathsf{D} = \mathsf{B}^{\mathsf{T}}\mathsf{A}\mathsf{B}$. [Note: What diagonal matrix should this product produce? Make sure your answer agrees with this.]

e) Since A is symmetric, this matrix B should be orthogonal. This means $\mathsf{B}^{-1} = \mathsf{B}^{\mathsf{T}}$. To show this calculate the product $\mathsf{B}^{\mathsf{T}}\mathsf{B}$. [Note: What should the answer be? Make sure your answer agrees with this.]

f) The matrix B is a rotation matrix that transforms between the two coordinate systems shown in the figure. Explicitly, $\vec{x}' = \mathsf{B}^{\mathsf{T}}\vec{x}$. Determine the angle, θ, between the original x_1-axis and the new x_1'-axis.

g) Finally, write the ellipse in standard form in terms of the principle axes

$$\left(\frac{x_1'}{\alpha_1}\right)^2 + \left(\frac{x_2'}{\alpha_2}\right)^2 = 1.$$

In other words, determine α_1 and α_2.

Problem 2: Even when a matrix is not singular, it can still be difficult to work with numerically. The condition number of a matrix, A, is denoted by $\kappa(\mathsf{A})$. If κ is large we call the matrix *ill conditioned*. Roughly, if the condition number is of the form $\kappa(\mathsf{A}) \sim 10^k$ then we expect to lose up to k digits of accuracy in addition to the normal loss of accuracy in an algorithm involving A. We will calculate the condition number as[1]

$$\kappa(\mathsf{A}) = \frac{\max\left(|\lambda(\mathsf{A})|\right)}{\min\left(|\lambda(\mathsf{A})|\right)},$$

where $\lambda(\mathsf{A})$ are the eigenvalues of A. The Hilbert matrix is an example of a non-singular but ill conditioned matrix. It has elements $A_{ij} = 1/(i+j+1)$ so that $A_{00} = 1$, $A_{01} = 1/2$, *etc.* In python we may easily construct this matrix by hand but we do not need to! We can instead use `scipy.linalg.hilbert()`. [Note: A number of other, standard matrices are defined too.]

a) Calculate the condition number $\kappa(A)$ for the 10×10 Hilbert matrix. Since A is symmetric you should use `scipy.linalg.eigh` for this purpose.

b) Calculate the inverse of A using `scipt.linalg.inv` and determine the maximum absolute error in $\mathsf{A}^{-1}\mathsf{A} - \mathsf{I}$.

c) An alternative way to calculate the inverse is to use the matrix of eigenvectors, B. Since A is symmetric we know that $\mathsf{B}^\mathsf{T}\mathsf{A}\mathsf{B} = \mathsf{D}$ is the diagonal matrix of the eigvenvalues. With this we may calculate the inverse as

$$\mathsf{A}^{-1} = \mathsf{B}\mathsf{D}^{-1}\mathsf{B}^\mathsf{T}.$$

Using this expression repeat the previous part. We should find in these two parts that the maximum error is huge compared to the expected numerical error. *Hint: Calculating the inverse of a diagonal matrix is very easy to do by hand.*

Problem 3: Complete the Jupyter notebook assignment.

[1] This is not the most general definition but is sufficient for our purposes.