In [1]:

```julia
using LinearAlgebra
using FFTW
using Plots
using BenchmarkTools
using OrdinaryDiffEq

include("../../code/TaylorFourier.jl");


```

The main problem of artificial satellite theory (in Stiefel-Scheifele VOP formulation)

Written as a system of ODEs for a fictitious time τ (with the physical time t as an additional state variable):

$$\begin{aligned}
\frac{d}{d\tau} q &= r\,v, \qquad q(0) = q_0, \\
\frac{d}{d\tau} v &= -\frac{\mu}{r^2} q - r\,\nabla V(q), \qquad v(0) = v_0 \\
\frac{d}{d\tau} t &= r, \qquad t(0) = t_0
\end{aligned} \tag{1}$$

where

$$q_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{pmatrix}, \qquad r = \|q\|,$$

and

$$V(q) = \frac{C}{2\,r^3}\left(3\,\sin^2\theta - 1\right), \qquad \sin\theta = \frac{z}{r}, \qquad C = J_2\,\mu\,R_e^2.$$

Constant parameters of the problem: $R_e = 6378.135 km$, $\mu = 398600 km^3/s^2$, $J_2 = 0.0010826157$.

The energy $\mathcal{E}(q,v) := \frac{1}{2}\langle v, v\rangle - \frac{\mu}{r} + V(q)$ is an invariant of the equations of motion. In addition, the z-component of the angular momentum

$$x\,\dot{y} - \dot{x}\,y$$

is also a first integral.

In [2]:

```
function Energy(q,v,parms)
    μ = parms[1]
    C = parms[2]
    return 0.5*dot(v,v) - μ/norm(q)  +  V(q,parms)
end

function V(q,parms)
    C = parms[2]
    z = q[3]
    r = norm(q)
    sinth = z/r
    return C*(3*sinth^2-1)/(2*r^3)
end

function ZAngularMomentum(q,v)
    return q[1]*v[2] - q[2]*v[1]
end
```

Out[2]:

```
ZAngularMomentum (generic function with 1 method)
```

Following~\cite{Stiefel-Scheifele1971}, $q(\tau)$ and $t(\tau)$ in (1) can be obtained as follows:

$$
\begin{aligned}
&q(\tau) = L(u(\tau))u(\tau), \qquad v(\tau) = \frac{2}{\|u(\tau)\|^2} L(u(\tau))u'(\tau),\\
&u(\tau) = \cos(\omega\tau)\alpha(\tau) + \omega^{-1}\sin(\omega\tau)\beta(\tau), \qquad u'(\tau) = -\omega\, \sin(\omega\tau)\alpha(\tau) + \cos(\omega\tau)\beta(\tau),
\end{aligned}
$$

where

$$
\omega = \sqrt{-\frac{1}{2}\,\mathcal{E}(q_0, v_0)}, \qquad L(\mathbf{u}) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \end{pmatrix},
$$

and $(\alpha(\tau), \beta(\tau), t(\tau))$ is the solution of the 8-dimensional ODE system

$$
\begin{aligned}
\frac{d}{d\tau}\alpha &= \omega^{-1}\sin(\omega\tau)\nabla R(\cos(\omega\tau)\alpha + \omega^{-1}\sin(\omega\tau)\beta), \qquad \alpha(0) = u_0,\\
\frac{d}{d\tau}\beta &= -\cos(\omega\tau)\nabla R(\cos(\omega\tau)\alpha + \omega^{-1}\sin(\omega\tau)\beta), \qquad \beta(0) = w_0,\\
\frac{d}{d\tau}t &= \|\cos(\omega\tau)\alpha + \omega^{-1}\sin(\omega\tau)\beta\|^2, \qquad t(0) = t_0.
\end{aligned} \tag{2}
$$

Here, $R(u) = \frac{1}{4}\,\|u\|^2\, V(L(u)u)$, that is,

$$
R(u) = \frac{1}{8\,r^2}\left(3\,(\sin(\theta))^2 - 1\right),
$$

where $r = u_1^2 + u_2^2 + u_3^2 + u_4^2$ and $\sin(\theta) = 2\,(u_1u_3 + u_2u_4)/r$. As for the vectors $u_0, w_0 \in \mathbb{R}^2$,

- $u_0 \in \mathbb{R}^4$ is chosen in such a way that
$$
q_0 = L(u_0)u_0,
$$
- $w_0 \in \mathbb{R}^4$ is determined as
$$
w_0 = \frac{1}{2}\,L(u_0)^T\,\dot{q}_0.
$$

Obviously, there are infinitely many vectors u_0 that satisfy $q_0 = L(u_0)u_0$. The function $\chi(q)$ implemented below computes one of them.

In [3]:

```
function χ(q)
    x = q[1]
    y = q[2]
    z = q[3]
    r = sqrt(x^2+y^2+z^2)
    if x >= 0
        aux = r + x
        u1 = 0.5*sqrt(aux)
        u4 = u1
        u2 = (y*u1 + z*u4)/aux
        u3 = (z*u1 - y*u4)/aux
    else
        aux = r - x
        u2 = 0.5*sqrt(aux)
        u3 = u2
        u1 = (y*u2 + z*u3)/aux
        u4 = (z*u2 - y*u3)/aux
    end
    return [u1, u2, u3, u4]
end

L(u) = [u[1] -u[2] -u[3]  u[4]
     u[2]  u[1] -u[4] -u[3]
     u[3]  u[4]  u[1]  u[2] ];
```

Let us check it for a randomly chosen q_0:

In [4]:

```
q0 = rand(3)
u0 = χ(q0)
norm(q0 - L(u0)*u0)
```

Out[4]:

```
6.206335383118183e-17
```

In [5]:

```
v0 = rand(3)
w0 = 0.5*(L(u0)')*v0
norm(2/dot(u0,u0)*L(u0)*w0 - v0)
```

Out[5]:

```
6.938893903907228e-17
```

Numerical solution with 9th order explicit RK method (Vern9)

We next implement in an efficient way the system of ODEs defined in (2).

In [6]:

```
function fODE!(dU, U, parms, τ)
    # Efficient implementation of  the differential equations for (α, β, t)
    # α = U[1:4],   dα/dτ = dU[1:4]
    # β = U[5:8],   dβ/dτ = dU[5:8]
    # t = U[9],     dt/dτ = dU[9]
    C = parms[2]
    ω = parms[3]
    θ = ω*τ
    c = cos(θ)
    s = sin(θ)/ω
    #u = c * α + s * β
    u1 = c * U[1] + s * U[5]
    u2 = c * U[2] + s * U[6]
    u3 = c * U[3] + s * U[7]
    u4 = c * U[4] + s * U[8]
    z = 2*(u1*u3 + u2*u4)
    r = u1^2 + u2^2 + u3^2 + u4^2
    w = 1/r^3
    sth = z/r
    A = 0.5*C*w*(1 - 6*sth^2)
    B = 1.5*C*w*sth
    gradR1  = A*u1+B*u3
    gradR2  = A*u2+B*u4
    gradR3  = A*u3+B*u1
    gradR4  = A*u4+B*u2
    dU[1]  = s*gradR1
    dU[2]  = s*gradR2
    dU[3]  = s*gradR3
    dU[4]  = s*gradR4
    dU[5]  = -c*gradR1
    dU[6]  = -c*gradR2
    dU[7]  = -c*gradR3
    dU[8]  = -c*gradR4
    dU[9]  = r
    return nothing
end
```

Out[6]:

```
fODE! (generic function with 1 method)
```

We now consider the initial state of a Geostationary satellite (Montenbruck 2000, pg. 116) and solve (2) numerically with an explicit RK method of order 9 due to Verner.

In [7]:

```
μ = 398600.8
R_e = 6378.135
ϵ = 0.0010826157
C = μ * R_e^2 * ϵ
#q0 = [0., 37947.73745727695, 0.]
#v0 = [3.297676220718193,0., 0.8244190551795483]
# Geostationary satellite (Montenbruck pg. 116)
q0 = [4.21491336e4,0.0,0.0]                                       # km
v0 = [0.0, 3.075823259987749,0.0010736649055318406 ]    # km/s


ω = sqrt(-Energy(q0,v0,[μ,C])/2)

u0 = χ(q0)
w0 = 0.5*(L(u0)') * v0

α0  = u0
β0  = w0
orbit_period = 2*π/ω

t0 = 0.
n_orbits = 400
tspan = (t0,t0+n_orbits*orbit_period)
U0 = [α0; β0; t0]

parms = [μ,C,ω]

prob = ODEProblem(fODE!, U0, tspan, parms);


M = 16
times = range(tspan[1],tspan[2],length=Int64(n_orbits*2M)+1)

tol = 1e-13
@time sol = solve(prob, Vern9(), abstol=tol, reltol=tol,saveat=times);
```

```
  4.712204 seconds (10.70 M allocations: 569.754 MiB, 3.92% gc time,
99.29% compilation time)
```

Let us check the errors in the conservarion of energy and the z component of the angular momemtum.

In [8]:

```
function qfromU(U, τ, parms)
    ω = parms[3]
    θ = ω*τ
    c = cos(θ)
    s = sin(θ)
    α = U[1:4]
    β = U[5:8]
    u = c * α + (s/ω) * β
    q = L(u)*u
    return q
end


function vfromU(U, τ, parms)
    ω = parms[3]
    θ = ω*τ
    c = cos(θ)
    s = sin(θ)
    α = U[1:4]
    β = U[5:8]
    u = c * α + (s/ω) * β
    r = dot(u,u)
    w = -ω * s * α + c * β
    v = 2/r*L(u)*w
    return v
end
```

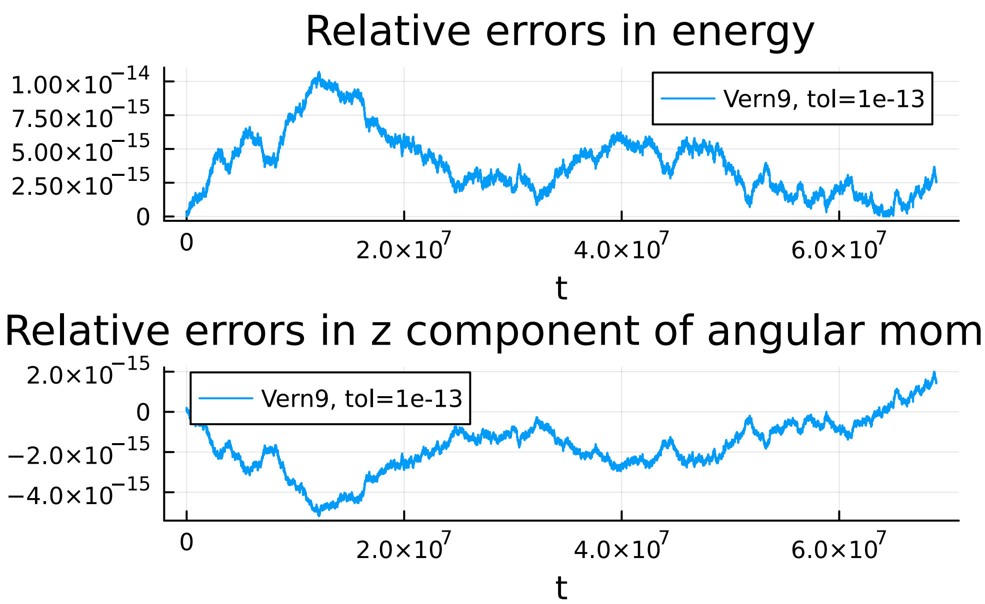
Out[8]:

```
vfromU (generic function with 1 method)
```

In [22]:

```
E0 = Energy(q0,v0,parms)
qq = map((U,τ) -> qfromU(U,BigFloat(τ),parms), sol.u, sol.t)
vv = map((U,τ) -> vfromU(U,BigFloat(τ),parms), sol.u, sol.t)
tt = [U[9] for U in sol.u]
Eerrs = map((q,v) -> abs(Energy(q,v,parms) / E0 - 1), qq, vv)



pl1 = plot(title="Relative errors in energy", xlabel="t")
plot!(pl1, tt, Eerrs,label="Vern9, tol=1e-13")

A0 = ZAngularMomentum(q0,v0)
Aerrs = map((q,v) -> ZAngularMomentum(q,v) / A0 - 1, qq, vv)

pl2 = plot(title="Relative errors in z component of angular momentum", xlabel=
plot!(pl2, tt, Aerrs,label="Vern9, tol=1e-13")

plot(pl1,pl2, layout=(2,1), size=(500,300))
```

Out[22]:



Global errors in positions (q) and physical time (t)

In order to estimate the global errors in q adn t of the numerical solution obtained above, we will compute a new numerical approximation with high precision: tol=1e-20, and higher precision arithmetic (BigFloat).

In [10]:

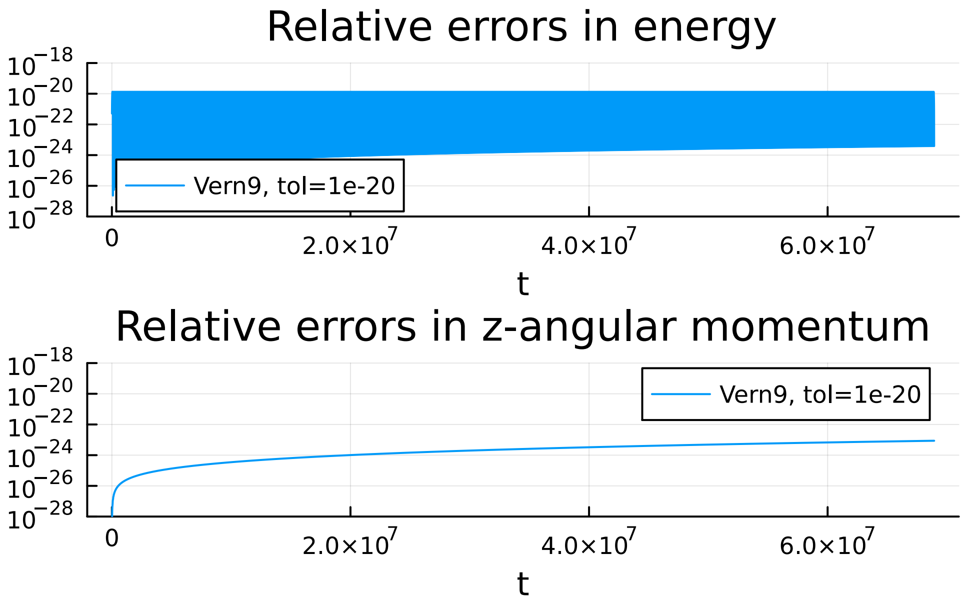
```
prob_BF = ODEProblem(fODE!, BigFloat.(U0), BigFloat.(tspan), BigFloat.(parms))

@time sol_ex = solve(prob_BF, Vern9(), abstol=1e-20, reltol=1e-20,saveat=times
```

```
 44.347835 seconds (472.26 M allocations: 21.264 GiB, 8.81% gc time,
15.47% compilation time)
```

In [28]:

```
qq_ex = map((U,τ) -> qfromU(U,τ,parms), sol_ex.u, sol_ex.t)
vv_ex = map((U,τ) -> vfromU(U,τ,parms), sol_ex.u, sol_ex.t)
tt_ex = [U[9] for U in sol_ex.u]
EE_ex = map((q,v) -> abs(Energy(q,v,parms)), qq_ex, vv_ex)
Eerrs_ex = abs.(EE_ex ./ EE_ex[1] .- 1)
AA_ex = map((q,v) -> abs(ZAngularMomentum(q,v)), qq_ex, vv_ex)
Aerrs_ex = abs.(AA_ex ./ AA_ex[1] .- 1)


pl1 = plot(title="Relative errors in energy", xlabel="t", yscale=:log10, ylims
plot!(pl1, tt_ex,Eerrs_ex,label="Vern9, tol=1e-20", legend=:bottomleft)


pl2 = plot(title="Relative errors in z-angular momentum", xlabel="t", yscale=:
plot!(pl2, tt_ex,Aerrs_ex,label="Vern9, tol=1e-20")

plot(pl1,pl2, layout=(2,1), size=(500,300))
```

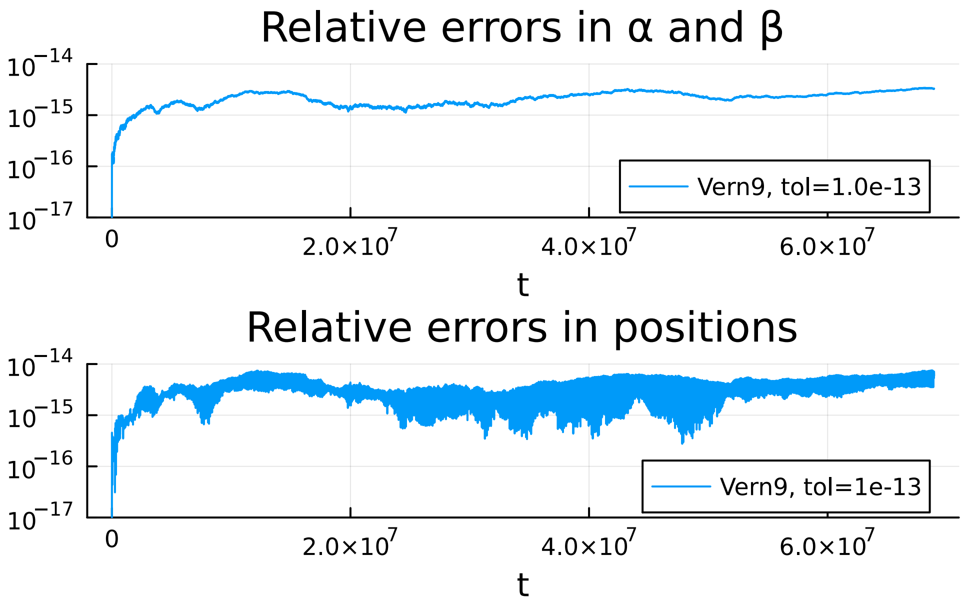
Out[28]:



In [23]:

```
errors = map((u,u_ex) -> norm(u[1:8]-u_ex[1:8])/norm(u_ex[1:8]), sol.u, sol_ex

pl1 = plot(tt_ex, errors, title="Relative errors in α and β", label="Vern9, to
    yscale=:log10, ylims=(1e-17,1e-14),xlabel="t", legend=:bottomright);

qerrs = norm.(qq-qq_ex) ./ norm.(qq_ex) .+ eps(0.01)
pl2 = plot(title="Relative errors in positions", xlabel="t", yscale=:log10, yl
plot!(pl2, tt_ex, qerrs,  label="Vern9, tol=1e-13", legend=:bottomright)

plot(pl1, pl2, layout = (2,1), size=(500,300))
```

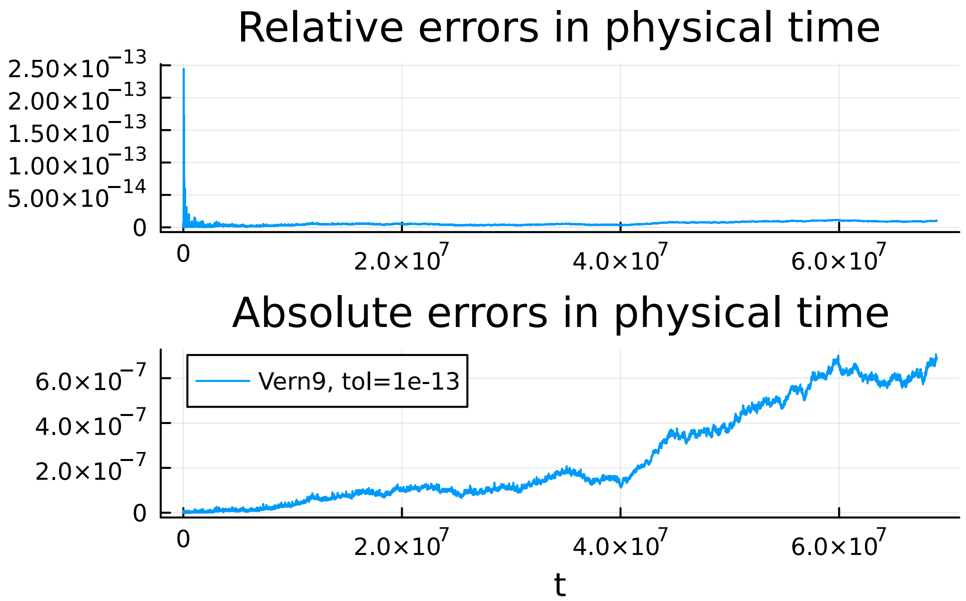
Out[23]:



In [27]:

```
t_rel_errs =  abs.(tt-tt_ex) ./ abs.(tt_ex)
pl1 = plot(tt_ex, t_rel_errs, legend=false, title="Relative errors in physical

terrs = norm.(tt-tt_ex)
pl2 = plot(title="Absolute errors in physical time", xlabel="t")
plot!(pl2, tt_ex, terrs,  label="Vern9, tol=1e-13")

plot(pl1,pl2, layout=(2,1), size=(500,300))
```

Out[27]:



Taylor-Fourier integration

We next integrate the problem above with our Taylor-Fourier integrator

In [14]:

```
include("J2_VOP_ODE_TF.jl")  # include the functions that define the ODE as re
                             # our TaylorFourier integrator
```

Out[14]:

```
J2_VOP_cache_init (generic function with 1 method)
```

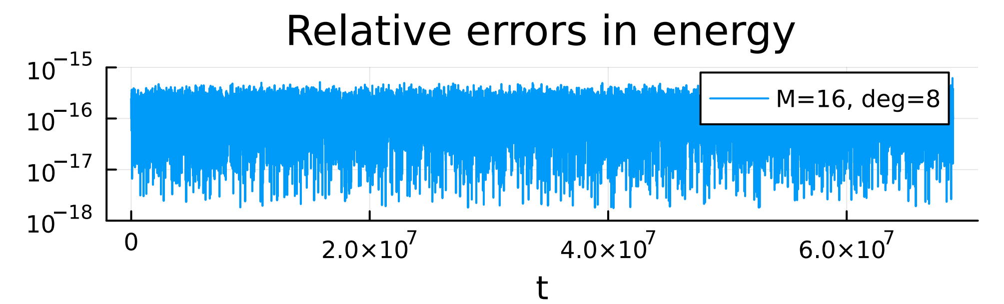
In [16]:

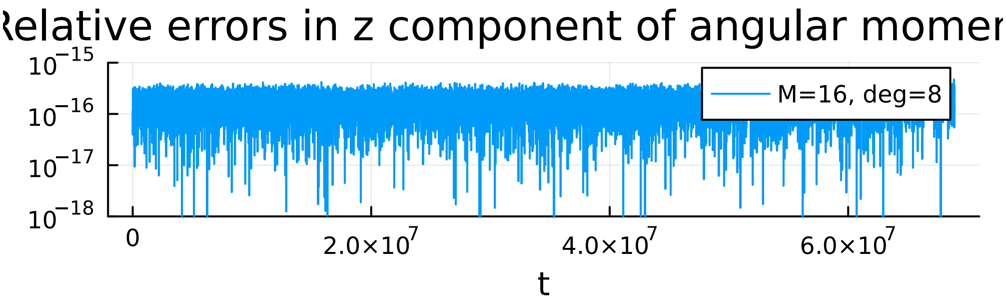
```
deg = 8
tf_cache = J2_VOP_cache_init(U0,parms,ω,deg,M)
prob_TF = PeriodicODEProblem(TF_ODE!, tf_cache, U0, ω)
sol_TF = TaylorFourierSolve(prob_TF,deg,M,n_orbits);
```

In [25]:

```
E0 = Energy(q0,v0,parms)
qq_TF = map((U,τ) -> qfromU(U,BigFloat(τ),parms), sol_TF.u, sol_TF.t)
vv_TF = map((U,τ) -> vfromU(U,BigFloat(τ),parms), sol_TF.u, sol_TF.t)
tt_TF = [U[9] for U in sol_TF.u]
Eerrs = map((q,v) -> abs(Energy(q,v,parms) / E0 - 1), qq_TF, vv_TF) .+ eps(0.0



pl1 = plot(title="Relative errors in energy", xlabel="t", yscale=:log10, ylims
plot!(pl1, tt_TF, Eerrs,label="M=$M, deg=$deg")

A0 = ZAngularMomentum(q0,v0)
Aerrs = map((q,v) -> ZAngularMomentum(q,v) / A0 - 1, qq_TF, vv_TF) .+ eps(0.01

pl2 = plot(title="Relative errors in z component of angular momentum", xlabel=
                  yscale=:log10, ylims=(1e-18,1e-15))
plot!(pl2, tt_TF, Aerrs,label="M=$M, deg=$deg")

plot(pl1,pl2, layout=(2,1), size=(500,300))
```

Out[25]:

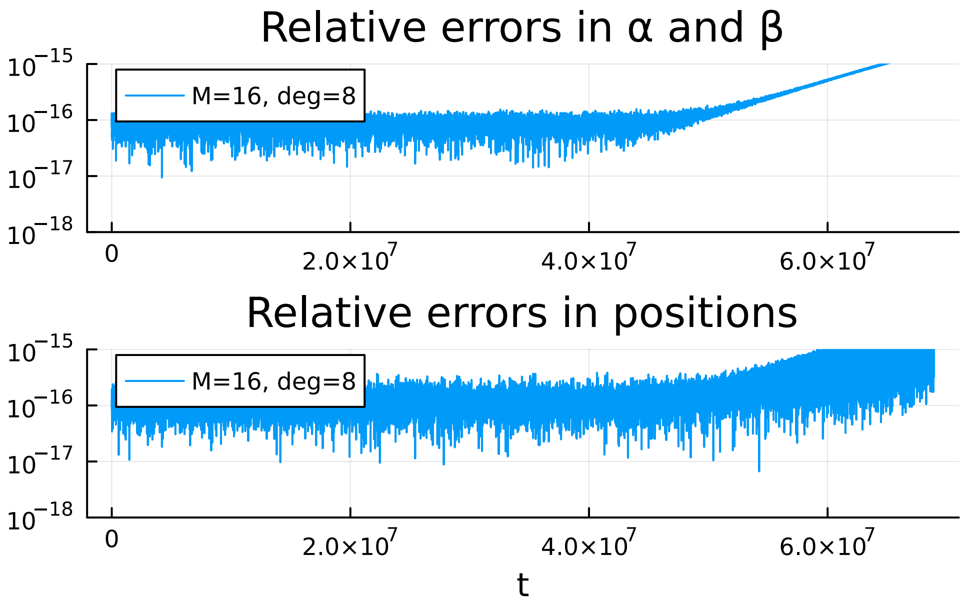




In [24]:

```
aberrors = map((u,u_ex) -> norm(u[1:8]-u_ex[1:8])/norm(u_ex[1:8]), sol_TF.u, s

pl1 = plot(tt_ex, aberrors, title="Relative errors in α and β", label="M=$M, d
             yscale=:log10, ylims=(1e-18,1e-15));

qerrs = norm.(qq_TF-qq_ex) ./ norm.(qq_ex)
pl2 = plot(title="Relative errors in positions", xlabel="t", yscale=:log10, yl
plot!(pl2, tt_ex, qerrs,  label="M=$M, deg=$deg")

plot(pl1, pl2, layout = (2,1), size=(500,300))
```

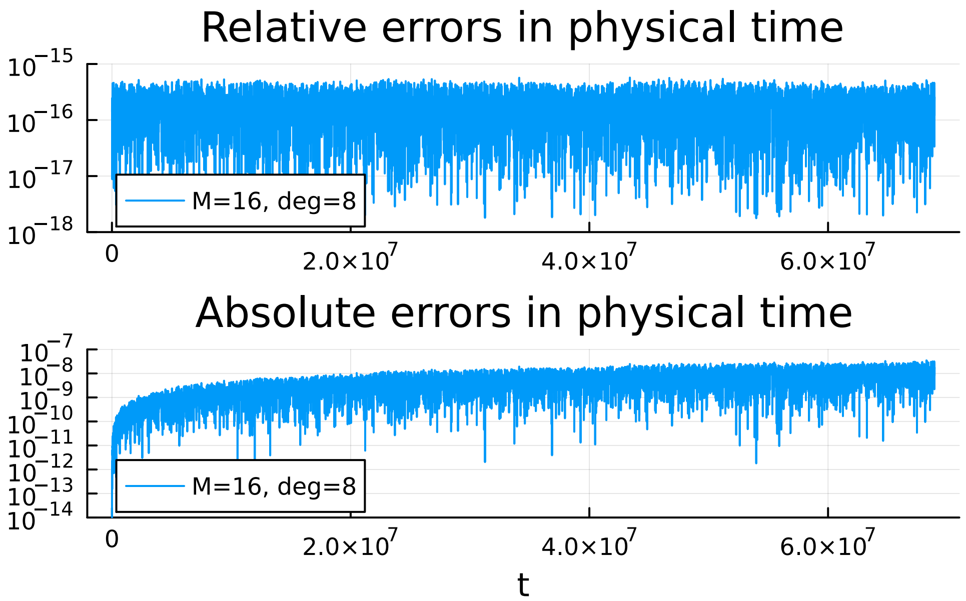
Out[24]:



In [26]:

```
t_rel_errs =  abs.(tt_TF-tt_ex) ./ abs.(tt_ex) .+ eps(0.01)
pl1 = plot(tt_ex, t_rel_errs, title="Relative errors in physical time",  label
              yscale=:log10, ylims=(1e-18,1e-15), legend=:bottomleft);

terrs = norm.(tt_TF-tt_ex) .+ eps(0.01)
pl2 = plot(title="Absolute errors in physical time", xlabel="t",
               yscale=:log10, ylims=(1e-14,1e-7), legend=:bottomleft)
plot!(pl2, tt_ex, terrs,  label="M=$M, deg=$deg")

plot(pl1,pl2, layout=(2,1), size=(500,300))
```

Out[26]:



In [20]:

```
@btime solve(prob, Vern9(), abstol=1e-13, reltol=1e-13,saveat=times)
@btime TaylorFourierSolve(prob_TF,deg,M,n_orbits);
```

```
  30.123 ms (85970 allocations: 11.10 MiB)
  2.413 ms (13260 allocations: 2.06 MiB)
```

In this example, our TaylorFourier integrator gives more precision with 15 times less CPU time.

In [ ]:

```

```