PHYS S123: College Physics I

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Contents

1 COURSE OVERVIEW

1.1 How do different disciplines view the world?

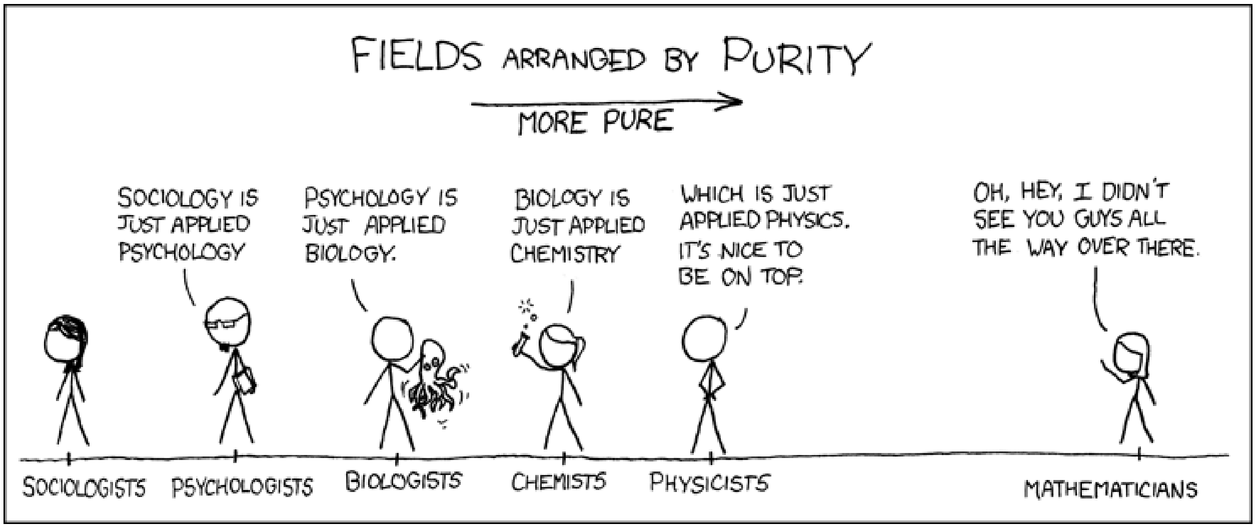

Figure 1: From xkcd.com

1.1.1 Biology

(The focus here is on biology, but very similar comments could be made for geology and environmental science.)

Biology: The study of living organisms. The approach taken by biology is guided by and constrained by the fact that the subject is about living organisms.

1. Much of biology is **complex**
   - first steps are about identification, classification, and description of phenomena
   - describe phenomena before looking for explanations of how it works
   - huge vocabulary and many concepts to learn
2. Biology depends on **history**
   - organisms are connected through a common, unbroken history that affects how things are today — not the case for chemistry and physics
   - evolution "explains" why a particular organism solves biological problems in a given way
3. Biology looks for **mechanism**
   - Not just "What is life?", but also and "How does it work?"
   - function of organs, genes, proteins and how do they affect organisms
4. Biology is **multi-scaled**
   - organisms can be considered at many scales
   - atomic and molecular scale (biochemistry)
   - internal structure and functioning of organs (physiology)

- as part of large system in space (ecology) and time (evolution)
- relation between scales can be treated by reductionism or emergence — going to smaller scales to explain something (reductionism), or seeing new phenomena arise as one goes to a larger scale (emergence)

5. Biology is **integrative**
   - biological phenomena emerge from and must be consistent with the principles of chemistry, physics, and math
   - chemistry and physics constrain how an organism can behave or evolve

1.1.2 Chemistry

Chemistry: The study of the composition and properties of substances and various elementary forms of matter.

Chemistry starts with the idea that all matter is made up of certain fundamental pieces (i.e., atoms) and is about the ways those elements combine to form more complex structures - molecules. But chemistry is not just about building molecules. It's about what you can do with that knowledge in our macroscopic world.

1. Chemistry is about **how atoms interact to form molecules** — Understanding the basic principles of how atoms interact and combine is a fundamental starting point for chemistry.
2. Chemistry is about **developing higher-level principles and heuristics** — Because there are so many different kinds of molecules possible, chemistry develops higher-level ideas that help you think about how complex reactions take place.
3. Chemistry frequently **crosses scales**, connecting the microscopic with the macroscopic, trying to learn about molecular reactions from macroscopic observations and figuring our what is possible macroscopically from the way atoms behave. The connections are indirect, can be subtle, and may involve emergence.
4. Chemistry often assumes a **macroscopic** environment — Much of what chemistry is about is not just idealized atoms interacting in a vacuum, but is about lots of atoms interacting in an environment, such as a liquid, gas, or crystal. In a water-based environment, the availability of H^+ and OH^- ions from the dissociation of water molecules in the environment plays an important role, while in a gas-based environment, the balance of partial pressures is critical.
5. Chemistry often **simplifies** — In chemistry, you often select the dominant reactions to consider, idealize situations and processes in order to allow an understanding of the most important features.

For a chemist, most of what happens in biology (or geology or environmental science) is "macroscopic" — there are lots and lots of atoms involved — even though you might need a microscope to study it. In introductory chemistry you often assume that reactions are taking place at standard temperature and pressure (300 K and 1 atm).

1.1.3 Physics

Physics: The study of matter and its motion through space and time.

The goal of physics is to find the fundamental laws and principles that govern all matter — including biological organisms. Those laws and principles can lead to many types of complex and apparently different phenomena.

Physics emphasizes four scientific skills that may seem different to what you see in introductory biology and chemistry classes but that are very useful:

1. Search for simplest possible example (**“toy model”**)
    - clearly illustrate key principles
    - build-up from simple models to analyze more complex situations
2. Physicists **quantify** their view of the real world
    - not satisfied until something can be quantified
    - purely qualitative reasoning can be misleading
3. Physicists **think with equations**
    - use equations to organize qualitative knowledge and to determine how things happen, what matters, and how much
    - go back and forth between concepts and math
4. Physicists deal with realistic situations by **modeling and approximating**
    - identify what matters most in a complex situation and create a “simple” model
    - the art of physics: figuring out what can be ignored without losing what you want to look at
    - “Physics should be as simple as possible, but not simpler.”

This way of doing science is a bit different from the way biology is often done — but elements of this approach and the constraints imposed on biology by the laws of physics are becoming increasingly important.

1.1.4 Math

Math is a bit different from the sciences. What is math anyway??? How do mathematicians think about the world?

- math is about relationships, patterns, logic
- abstract and rigorous
- not “about” anything in the physical world
- would math exist without humans?
- but — a lot of relationships in science can be modeled by mathematical relationships

Math as taught in math classes often is primarily about the abstract relationships — learning how to use the tools of math. Introductory physics is an excellent place to become comfortable with applying math to real world problems.

1.2 Why physics?

Why are you required to take physics?

Math majors: (1) apply your knowledge to “real world” problems and (2) gain an understanding of where the equations come from that you are learning how to solve and analyze

Biology / environmental science: (1) the core principles that control and organize our understanding of biology and the environment include physics and (2) the skills and competencies developed in physics are of great use in the life and environmental sciences, especially at the more advanced levels.

Useful skills that we will focus on:

1. **Problem solving — modeling and mechanistic reasoning:** People are needed to figure out things that are not trivially obvious. Looking at a situation that is different from one you might have seen before, finding the similarities and differences, figuring out the knowledge and tools you need to resolve it, that's problem solving. A lot of what you will learn in this class will be through problem solving. Not just learning something and giving it back, but learning something and figuring out how to use it in new non-obvious situations.

   I will not expect you to memorize equations for exams, and I may ask exam questions that differ somewhat from homework assignments.

2. **Discourse — learning to talk the talk:** A critical idea in any science is that we use the community of scientists in a discipline to get a broader, more complete, and more accurate view of the world than any one individual can get. We're each limited in our experience and knowledge. Every scientific discipline relies heavily on the interaction of scientists with each other.

   We will do a lot of in-class group work, both in lab and lecture. In addition, your homework will include problems that may be too hard and time-consuming for a single individual to do easily by themselves. You are encouraged to work in groups. The point of this is for you to learn to "talk the talk" — to learn to ask questions of each other until you all understand what's going on better than any of you would individually.

3. **Reasoning from principle:** Physics has been really good at finding universal (or nearly universal) principles that hold in a very wide variety of circumstances; things like energy conservation, conservation of charge, and Newton's laws — our framework for describing and building models of motion.

   By starting from basic principles and building upward, we will begin to make sense of complex knowledge in a coherent way.

4. **Quantification of experience — mathematization, estimation, and scales:** All sciences are becoming increasingly quantitative. As a scientist you need to understand math and be able to interpret the implications of the math. Learning to use math in science (rather than just as pure math) can be quite tricky. Adding a physical interpretation to our symbols can actually change the way we think about the math.

   We'll use a lot of math of this course, so much that some of you may complain that physics is basically just another math course. In some sense that's true. We will use math to solve scientific questions, and will also learn to estimate as a way to (1) determine "what matters most" and (2) check that our solutions make sense.

5. **Multiple-representation translation:** All sciences represent the complex information that they convey in a variety of forms – words, equations, pictures, graphs, and animations. The tricky thing is to learn to create a single enriched physical picture in your mind that blends all these different representations, linking them together.

By focusing on "simple" situations, we'll gain an ability to think about science from multiple perspectives. We'll make sketches of physical processes, describe the processes using equations, and interpret the equations by drawing graphs.

6. **Understanding measurement:** All sciences rely heavily on measurement and data, none of which is perfectly reliable. Every experiment or measurement includes some model, both of the system being measured and of the process yielding the measurements. Understanding how measurement works — and therefore what the measurement tells you, when it can be trusted, and when it can go astray — is very important to a practicing scientist.

   We'll gain a better understanding of measurements and data through lab experiments, which will illustrate that even simple systems can involve complex issues of measurement.

You'll also work on developing these skills in your upper-division classes. Physics is a great place to practice since physics digs down to underlying simple principles and universal constraints. The physics we are learning here is, in the end, rather simple.

2 INTRODUCTION TO KINEMATICS

Objectives:

1. Course introduction

2. Key terminology

3. Graphing motion

2.1 Overview

We'll start the semester by learning about kinematics. Kinematics is the description of motion, and has the same origins as the word cinema. In a few weeks we'll discuss dynamics, or causes of motion. When we combine kinematics with dynamics, we are studying mechanics. In introductory physics, we'll cover classical mechanics — one of the oldest branches of science.

Classical mechanics: large, slow moving objects
Quantum mechanics: small objects
General relativity: fast moving objects

[Insert diagram of physics disciplines.]

2.2 Types of motion

motion: change in object's position with time

trajectory: path along which an object travels

For now, we'll only worry about rigid body motion (i.e., no deformation).

Four special cases of motion:

1. straight line

2. projectile motion (influenced by gravity)

3. circular motion (e.g., planets, satellites)

4. rotational motion

2.3 Representing motion

Motion diagrams are used to represent where an object is at different points in time.

[Insert diagram of person running.]

For motion that does not involve deformation or rotation, we can use a *particle model*. Models are simplifications of reality that allow us to focus on the important physics. In a particle model, we assume that all of the mass of an object is focused at a single point and that the all parts of the object move in the same direction with the same speed. In class I'll often use particles or boxes to represent much more complicated objects.

[Insert example particle model.]

More commonly, we will use graphs to describe the motion of an object.

[Insert example graph.]

You will need to become comfortable going back and forth between motion diagrams and position-time, velocity-time, and acceleration-time graphs.

2.4 Terms used to describe motion

- position = location of an object

  [Insert 1-D diagram; where is the object on this line?]

  position can be positive *or* negative – this is important
  units = [L]

  We represent motion with a position-time graph. [Insert position-time graph.]

- displacement = change in position, [L]

$$\Delta x = x_f - x_i$$

  Can be positive *or* negative (give example).

- velocity = rate of change of position, [L]/[T]

$$v = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

  (Technically, this is the average velocity during some time interval Δt. We'll come back to this.) Velocity can ALSO be positive *or* negative. What does the sign tell you?

  Velocity can be represented with a velocity-time graph.

[Insert velocity-time graph, based on position-time graph.]

Note that we can go from velocity to displacement (i.e., *change* in position) by rearranging the equation for velocity:

$$\Delta x = v\Delta t$$

which is the same thing as calculating the area under the velocity-time graph.

[Insert diagram.]

Average velocity during the first interval:

$$v_1 = \frac{x_1 - x_0}{t_1 - t_0} \Rightarrow \text{ slope of a straight line}$$

$$v_1 > 0$$

Average velocity during the second interval:

$$v_2 = \frac{x_2 - x_1}{t_2 - t_1} < 0$$

- speed = magnitude of velocity, [L]/[T], always positive
- instantaneous velocity (i.e., the slope of the line that is tangent to the velocity curve). For those of you that have taken calculus, this is a derivative.

  In practice, data is not continuous; velocity calculated from data is an average over some time interval.
- acceleration = rate of change of velocity, [L]/[T^2]

  $$a_{avg} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

  Technically, this is the average acceleration over the time period Δt. As with velocity, we are often interested in the instantaneous acceleration, which is the slope of the line that is tangent to the velocity curve.

Acceleration is represented with an acceleration-time graph.

[Insert acceleration-time graph.]

Not surprisingly, given what we just saw with displacement, it is easy to go from acceleration to a change in velocity:

$$\Delta v = a\Delta t$$

Acceleration can also be positive *or* negative. What does the sign mean?

Simple rule for determining whether an object is accelerating or decelerating:
If $a \cdot v > 0$ the object is accelerating, if $a \cdot v < 0$ the object is decelerating

2.4.1 Example #1: Going from position to acceleration

The position of a particle is given in the position-time graph below. Draw the corresponding velocity-time and acceleration-time graphs.

3 KINEMATIC EQUATIONS

Objectives:

1. Describing motion with graphs
2. Transforming between displacement, velocity, and acceleration

Demos:

1. Falling washers to demonstrate that gravitational acceleration is constant

3.1 Constant velocity

We saw that $\Delta x = x_f - x_i$ and $v = \Delta x/\Delta t$. v as defined here are the average velocity over some time period Δt. To compute the instantaneous velocity, let $\Delta t \to 0$. In other words, velocity is the slope of the position-time graph.

[Insert position-time graph]

[Insert velocity-time graph derived from previous graph]

That is pretty straightforward, and is just based on the definitions of Δx and v. Note that the velocity is just the slope of the position-time graph. What if we want to go the other direction?

[Insert velocity-time graph with constant velocity]

For the case of constant velocity (and therefore no acceleration), we can write

$$\Delta x = v\Delta t$$

Previously we defined $\Delta x = x_f - x_i$ and $\Delta t = t_f - t_i$. However, we can apply this equation for all increments of time by replacing x_f with x and t_f with t. This means

$$x - x_i = v(t - t_i) \Rightarrow x = v(t - t_i) + x_i$$

which is the equation for a straight line. To create the position-time graph, we need to also know the object's position at t_i.

[Insert position-time graph derived from previous graph]

Notice anything else about the relationship between velocity and displacement? From the graphs, we can see that displacement is the area under the velocity-time graph. This is always the case, regardless if the velocity is constant or not.

This linear relationship between velocity and position assumes that velocity is constant, which is rarely true. If the velocity is varying, you can calculate the position by breaking the velocity-time graph into many small time intervals, computing the displacement during each of those time intervals, and cumulatively adding the displacements (i.e., computing an integral).

To recap: velocity is the slope of the position-time graph, and displacement equals the area under the velocity-time graph.

3.2 Constant acceleration

We defined acceleration as $a = \Delta v / \Delta t$; as defined here a is the average acceleration over some time interval Δt. To computer the instantaneous velocity, let $\Delta t \to 0$. If the position-time graph is not straight, then the velocity will vary with time and therefore the acceleration is non-zero. Based on everything that I've already said, you should be able to figure out how to generate an acceleration-time graph based on a position-time graph.

[Insert smooth position-time graph, velocity-time graph, and acceleration-time graph]

The relationship between acceleration and velocity is identical to the relationship between velocity and position. Recall that

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

As before, replace v_f with v and t_f with t and rearrange, to find that

$$v = a(t - t_i) + v_i$$

If acceleration is constant, then velocity follows a straight line. Although acceleration doesn't have to be constant, we will find many examples where it is (essentially) constant. So okay, how do we relate displacement to acceleration?

[Insert example with constant acceleration]

Since the acceleration is constant, we can calculate the velocity-time graph using the equation above, assuming that we know v_i.

[Insert linear velocity based on acceleration-time graph]

As before, the displacement is the area under the velocity-time graph. The area consists of two parts: a rectangle and a triangle. Therefore,

$$\Delta x = v_i \Delta t + \frac{1}{2}(v_f - v_i)\Delta t.$$

But we've already seen that $\boxed{\Delta v = a\Delta t}$, so this becomes

$$\boxed{\Delta x = v_i \Delta t + \frac{1}{2}a\Delta t^2}$$

We can derive one more relationship that comes in handy when we don't know Δt.

$$a = \frac{v_f - v_i}{\Delta t} \Rightarrow \Delta t = \frac{v_f - v_i}{a}.$$

Inserting this into the previous equation gives

$$\Delta x = v_i \frac{v_f - v_i}{a} + \frac{1}{2}a\left(\frac{v_f - v_i}{a}\right)^2.$$

Rearranging and cancelling terms, we find that

$$\boxed{2a\Delta x = v_f^2 - v_i^2}$$

To reiterate, for constant acceleration we have the following kinematic equations:

$$\Delta v = a\Delta t$$
$$\Delta x = v_i \Delta t + \frac{1}{2}a\Delta t^2$$
$$2a\Delta x = v_f^2 - v_i^2$$

What happens if $a = 0$? Is this consistent with what we found previously?

3.3 Example #1: Airplane crossing a runway

A 747 has a length of 59.7 m. The plane lands on a runway that intersects another runway. The width of the intersection is 25.0 m. The plane decelerates through the intersection at 5.70 m/s^2 and clears the intersection with a final speed of 45.0 m/s. How long does it take the plane to clear the intersection.

Approach to solving problems:

1. Draw a diagram if applicable.
2. Write down what is known.
3. Write down what you want to find out.
4. Try to figure out what equations to use. This is often the most difficult part. Keep in mind what equations we have available to us. Often we will be making a choice from just a few equations.
5. Solve problem algebraically as much as possible. This is easier than carrying numbers around, makes it easier for others (especially me) to figure out what you did, and sometimes results in a simple and elegant algebraic solution.
6. After arriving at a solution, check that it makes sense.

Given:
$x_i = 0.0$ m
$x_f = 25.0 \text{ m} \ + \ 59.7 \text{ m} \ = \ 84.7$ m
$v_f = 45.0$ m/s
$a = -5.70$ m/s^2

Want to know Δt; note that v_i is not given.

We can calculate v_i from $2a\Delta x = v_f^2 - v_i^2$.

$$v_i = \sqrt{v_f^2 - 2a\Delta x}$$

Then, using $a = \Delta v/\Delta t$,

$$\Delta t = \frac{\Delta v}{a} = \frac{v_f - v_i}{a} = \frac{v_f - \sqrt{v_f^2 - 2a\Delta x}}{a} = 1.7 \text{ s}$$

Does this make sense? How might you check? Use approximation and check units.

3.4 Gravitational acceleration
Free fall is an important example of motion under constant acceleration. Near the surface of the Earth, objects accelerate downward at about $g = 9.81$ m/s^2.

1. $g > 0$ (always positive); it is a magnitude
2. If the coordinate system points upward, then $a = -g$
3. $g = 9.81$ m/s^2 only on Earth, and only near the surface of the Earth
4. Use kinematic equations for constant acceleration.

Example: A ball is shot vertically from the ground at a speed of 50 m/s. (1) What elevation will the ball reach? (2) How long will it take the ball to hit the ground? (3) What will its speed be when it hits the ground?

Given:
$v_i = 50$ m/s

$x_i = 0$ m
$a = -g = -9.81$ m/s^2

(1) The ball will have a speed of 0 m/s when it reaches its peak. Therefore,

$$v_{\text{peak}}{}^2 - v_i^2 = 2a\Delta y$$

$$\Delta y = \frac{-v_i{}^2}{2a} = 127 \text{ m}$$

(2) To calculate the time to peak, we'll again make use of the fact that $v_{\text{peak}} = 0$.

$$a = \frac{\Delta v}{\Delta t}$$

$$\Delta t = \frac{\Delta v}{a} = \frac{v_{\text{peak}} - v_i}{a} = \frac{-v_i}{a} = 5.1 \text{ s}$$

(3)To calculate the speed when it hits the ground, we can use the kinematic equation that doesn't include Δt. That way we don't have to calculate how long it takes the ball to travel up and down.

$$v_f{}^2 - v_i{}^2 = 2a\Delta y$$

Since $\Delta y = 0$,

$$v_f{}^2 = v_i{}^2$$

and so

$$v_f = \pm v_i.$$

The direction that the ball is travelling has changed, so

$$v_f = -v_i = 50 \text{ m/s}.$$

4 MOTION IN TWO DIMENSIONS

Objectives:

- Vectors
- Projectile motion

Demonstrations:

- Free-fall apparatus: one marble falls straight down while the other is shot horizontally
- Ball shot vertically from cart moving on a horizontal track

4.1 Kinematic equations in 1D

In the first two lectures I introduced the kinematic equations and variables used to describe motion in 1-dimension for the case of constant acceleration. Recall:

$$\Delta v = a\Delta t$$

$$\Delta x = v_i \Delta t + \frac{1}{2} a \Delta t^2$$

$${v_f}^2 - {v_i}^2 = 2a\Delta x$$

We saw that one special case of constant acceleration is that of gravity near the Earth's surface, for which $g = 9.81$ m/s^2. g is always positive and always points downward. Acceleration may be $\pm g$, depending on the orientation of the coordinate system.

We also saw that if $a = 0$, this reduces to the kinematic equation for constant velocity, and so

$$\Delta x = v\Delta t$$

We have been using x to define our coordinate system. The coordinate system can point in any convenient direction, and sometimes we will use y or z to indicate a distance along an axis.

4.2 Vectors

We will often want to describe motion in 2-dimensions, and sometimes in 3-dimensions. In these instances we will need to use *vectors*, and we will need to make use of y and/or z. It is pretty straightforward to generalize what we have already learned to describe motion in 2- and 3-dimensions. Before continuing, we also need to remember what is meant by a vector.

Vector: a geometric quantity having both magnitude *and* direction.

[Insert diagram of a vector.]

Kinematic variable	1D (scalar quantity)	2D (vector quantity)
position	x	$\vec{x} = \langle x, y \rangle$
displacement	Δx	$\Delta \vec{x} = \langle \Delta x, \Delta y \rangle$
velocity	$v = \frac{\Delta x}{\Delta t}$	$\vec{v} = \langle v_x, v_y \rangle = \langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \rangle$
acceleration	$a = \frac{\Delta v}{\Delta t}$	$\vec{a} = \langle a_x, a_y \rangle = \langle \frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \rangle$

We can do this as long as the x- and y-axes are orthogonal/perpendicular to each other. For example, if an object moves in the x-direction, its y-position doesn't necessarily change at the same time. This doesn't mean that motion in the x-direction is independent of motion in the y-direction.

[Insert diagram showing displacement vector. Should have $\vec{x_1}$ and $\vec{x_2}$]

Initial position: $\vec{x}_i = \langle x_i, y_i \rangle$
Magnitude: $\vec{x}_i = |\vec{x}_i| = \sqrt{{x_i}^2 + {y_i}^2}$
Angle: $\tan \theta_i = y_i / x_i$
If angle is known: $x_i = \cos \theta_i$ and $y_i = \sin \theta_i$

After some time, the object moves to position $\vec{x}_f$. The displacement is

$$\Delta \vec{x} = \vec{x}_f - \vec{x}_i = \langle x_f, y_f \rangle - \langle x_i, y_i \rangle = \langle x_f - x_i, y_f - y_i \rangle.$$

You have to add or subtract vector components! If this is confusing, ask questions and read your textbook. We will use vectors throughout the semester.

You can also add vectors graphically. Slide the tail of one vector to the tip of the other vector. For subtraction, the easiest way is to recall that subtraction is the same as adding a negative number. This applies also for vectors.

[Insert diagrams showing addition and subtraction of vectors.]

4.3 Representing motion in 2-dimensions

In the majority of the problems that we will consider we will be able to treat motion in the x- and y- directions as being *independent*. In this case, we can directly use our kinematic equations to describe 2-dimensional motion. Projectile motion of hard, metal spheres is a situation where we can treat the horizontal and vertical components of motion separately.

[Show video of ball being shot vertically from moving cart.]

Motion in x-direction	**Motion in y-direction**
$\Delta v_x = a_x \Delta t$	$\Delta v_y = a_y \Delta t$
$\Delta x = v_{x,i}\Delta t + \frac{1}{2}a_x \Delta t^2$	$\Delta y = v_{y,i}\Delta t + \frac{1}{2}a_y \Delta t^2$
${v_{x,f}}^2 - {v_{x,i}}^2 = 2a_x \Delta x$	${v_{y,f}}^2 - {v_{y,i}}^2 = 2a_y \Delta y$

We really haven't added all that much complexity here. All we're saying is that the equations that we developed for 1-dimensional motion can be generalized to describe motion in 2-dimensions.

4.4 Demo: Free-fall apparatus

Last class we started talking about projectile motion, which involves motion of objects in two-dimensions. I'd like to continue that discussion with a demo, after which we'll discuss other types of two-dimensional motion.

Which marble will hit the ground first? The one that falls straight down, or the one that is kicked by the spring?

Turns out that they hit the ground at the same time! They both start with no vertical velocity, and gravity acts downward on both marbles equally.

If we measure the height that the marbles are dropped from, and the distance that the projectile marble travels, can we calculate the marble's initial speed?

First, let's consider motion in the $y-$direction to determine the time it takes the marble to hit the ground.

$$\Delta y = v_{y,i}\Delta t + \frac{1}{2}a\Delta t^2$$

But $v_{y,i} = 0$, $\Delta y = -H$, and, if the y points upward, $a = -g$.

$$-H = -\frac{1}{2}g\Delta t^2$$

$$\Delta t^2 = \frac{2H}{g}$$

$$\Delta t = \sqrt{\frac{2H}{g}}$$

Now let's use the horizontal distance that the marble travelled to calculate its initial velocity. There is no acceleration in the $x-$direction.

$$v = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{\sqrt{\frac{2H}{g}}} = \Delta x\sqrt{\frac{g}{2H}}$$

4.5 Example: What angle to throw a ball?

You throw a ball on a level field. At what angle from horizontal should you throw the ball to get the maximum distance out of the throw?

Let's call V the initial *speed*. It is a positive value and doesn't indicate the direction of motion. We'll define x to be the horizontal direction and y to be the vertical direction. So this gives

$$\vec{v}_i = \langle v_x, v_y\rangle = \langle V\cos\theta_i, V\sin\theta_i\rangle$$

First we need to figure out how long it takes the ball to hit the ground. We only need to worry about the $y-$direction, because the motion in the $x-$direction doesn't affect how long it takes the ball to hit the ground (as long as we don't have to worry about lift).

Let's call Δt the time to hit the ground, in which case $\Delta y = 0$.

$${v_{y,f}}^2 - {v_{y,i}}^2 = 2a_y\Delta y \Rightarrow v_{y,f} = -v_{y,i}$$

Interesting!

$$\Delta v_y = a_y\Delta t$$

$$v_{y,f} - v_{y,i} = -2v_{y,i} = -2V\sin\theta_i = -g\Delta t$$

$$\Delta t = \frac{2V\sin\theta_i}{g}$$

Now, how far does the ball travel in the $x-$direction during this amount of time? The velocity in the $x-$direction is constant, so

$$\Delta x = v_x\Delta t = \frac{2V^2\cos\theta_i\sin\theta_i}{g}$$

For what θ_i is Δx a maximum? Make a plot.

[Insert plot of Δx.]

Maximum occurs when $\theta_i = 45°$.

(This has assumed that air resistance is negligible, and so v_x is constant. If we had accounted for air resistance, which is a *really* difficult problem, we would have found that $\max \theta_i < 45°$.)

5 MOTION IN TWO DIMENSIONS II

Objectives:

- Selection of coordinate system
- Object moving up/down a ramp

First review gravitational acceleration and projectile motion with a couple of examples.

5.1 Example: Projectile motion

Part 1

You are throwing a ball from the top of a 10-m high cliff. You throw the ball horizontally at a speed of $V_i = 30$ m/s. How fast is the ball travelling when it hits the ground?

What are we looking for?

$$v = \sqrt{v_{x,f}^2 + v_{y,f}^2}$$

For the $x-$direction, we have

$$v_{x,f} = v_x = 30 \text{ m/s}$$

For the $y-$direction, we know that $\Delta y = -10$ m, $a_y = -9.81$ m/s^2 and $v_{y,i} = 0$. From kinematics, we have

$$\Delta v_y = a_y \Delta t$$

$$\Delta y = v_{y,i}\Delta t + \frac{1}{2}a_y \Delta t^2$$

Solve for Δt and plug the answer into the second equation, giving

$$v_{y,f}^2 - v_{y,i}^2 = 2a_y \Delta y$$

which we can reduce to

$$v_{y,f} = \sqrt{2a_y \Delta y} = 14.0 \text{ m/s}$$

Plugging back into the equation for speed, we find

$$v = 33.1 \text{ m/s}$$

Part 2

At what angle relative to horizontal should you throw the ball to maximize the speed at which it hits the ground? Upward, downward, or horizontally?

For the $x-$direction, we have a constant velocity of

$$v_x = V_i \cos\theta$$

For the $y-$direction, we have

$$2a\Delta y = {v_{y,f}}^2 - {v_{y,i}}^2$$

The velocity in the $y-$direction changes with time. $a = -g$, and $\Delta y = -10$ m.

$$2gH = {v_{y,f}}^2 - (V_i \sin\theta)^2 \Rightarrow {v_{y,f}}^2 = 2gH + (V_i \sin\theta)^2$$

We don't need to take the square root to find $v_{y,f}$, because we would want to square it in the next step.

The speed that the rock hits the ground at will be

$$V_f = \sqrt{{v_x}^2 + {v_{y,f}}^2}$$

So, inserting the above results

$$V_f = \sqrt{{V_i}^2 \cos^2\theta + 2gH + {V_i}^2 \sin^2\theta} = \sqrt{{V_i}^2(\cos^2\theta + \sin^2\theta) + 2gH} = \sqrt{{V_i}^2 + 2gH}$$

It doesn't matter what angle you throw the ball at, just throw it fast!

This is a great example of why it pays off to do the algebra before plugging in any numbers. We ended up with an elegant, insightful, and surprising solution.

5.2 Objects moving along ramps

Another type of horizontal motion that we'll encounter frequently is that of objects moving along ramps. So let's consider a box sliding down a ramp.

[Insert diagram of box sliding on a ramp.]

Gravity acts downward on the ramp, but the object will move down the ramp (not just vertically downward). To simplify the problem, we can make use of vector components and the fact that we are free to orient our coordinate system whatever way we choose. We will be turning a 2-dimensional problem into a one-dimensional problem.

Steps:
(1) Let x arbitrarily point down the ramp, and y point perpendicular up from the ramp.
(2) Split g into vector components. One component points in the $+x-$direction, the other points in the $-y-$direction. Show that $a_x = g\sin\theta$.
(3) The ramp doesn't allow for motion in the $y-$direction, so we only have to deal with motion in the $x-$direction. This has become a one-dimensional problem.

Let's say that the box is released from a height of H, and that the ramp has an angle of θ. What is the speed of the box when it reaches the end of the ramp?

Given:
$a_x = g\sin\theta$
$x_i = 0$

$x_f = \frac{H}{\sin\theta}$
$v_i = 0$

Want to know v_f. This is another problem where it pays to go through the algebra.

$$v_f{}^2 - v_i{}^2 = 2a\Delta x = 2g\sin\theta\frac{H}{\sin\theta} = 2gH$$

$$v_f = \sqrt{2gH}$$

The final speed depends *only* on the height from which the box is released!

5.3 Example: Box sliding along track
A slightly more difficult problem:

A block slides along a frictionless track with speed $V = 2$ m/s. Assume that it turns all corners smoothly with no loss of speed.

[Insert diagram of block on track.]

(a) What is the maximum height of the ramp that the block can slide up?

To solve this, we will rotate the coordinate system so that x points up hill. This will become a one-dimensional problem.

Given:
$v_i = V$
$a = -g\sin\theta$
$\Delta x = \frac{H}{\sin\theta}$

If it just reaches the top of the hill, then $v_f = 0$

We only need one equation to solve this.

$$2a\Delta x = v_f{}^2 - v_i{}^2$$

$$-2g\sin\theta\frac{H}{\sin\theta} = -v_i{}^2$$

$$2gH = v_i{}^2$$

$$H = \frac{{v_i}^2}{2g} = \frac{(2\ \mathrm{m/s})^2}{2 \times 9.81\ \mathrm{m/s^2}} \approx 0.2\ \mathrm{m}$$

6 RELATIVE MOTION

Objectives:

1. Relative position, velocity, and acceleration in 1-D and 2-D

6.1 Relative motion in 1-D

In physics we often encounter problems where we need to know the motion of an object relative to another object. In order to talk about relative motion, we need to be comfortable with vectors.

Visit ophysics.com to do exercises on vector components and vector addition and subtraction.

1. Motion of an object is always relative to some reference frame.
2. Reference frames *can* move. (If a reference frame is accelerating, then we have to use Einstein's theory of general relativity.)

You've all experienced relative motion in one way or another. For example, if you're sitting in a parked car, and the car next to you starts to move, you sometimes get the sensation that you've started moving.

We are going to start with relative motion in one-dimension, because this is easier to understand than motion in two-dimensions.

For example, Alex and Barbara each have their own reference frame; they are located at the origin of their reference frames. We will assume that Alex and Barbara are stationary and facing the same direction. There is an object located at point P; it also has a reference frame that is oriented in the same direction as Alex and Barbara's reference frames.

[Insert diagram of Alex and Barbara and object at point P.]

For Alex, P is at a positive position, whereas P is at a negative position for Barbara. We can relate the relative position of P to Alex with the relative position of P to Barbara.

$$x_{pa} = x_{pb} + x_{ba}$$

The order of indices matters. Note that $x_{pa} = -x_{ap}$. Why does this expression make sense?

What if point P is moving? Can we write down a similar expression using velocities? What are v_{pa} and v_{pb}?

$$v_{pa} = \frac{\Delta x_{pa}}{\Delta t}$$

$$v_{pb} = \frac{\Delta x_{pb}}{\Delta t}$$

$$v_{ba} = \frac{\Delta x_{ba}}{\Delta t}$$

We already saw that $x_{pa} = x_{pb} + x_{ba}$, so

$$v_{pa} = \frac{\Delta(x_{pb} + x_{ba})}{\Delta t} = \frac{\Delta x_{pb} + \Delta x_{ba}}{\Delta t} = \frac{\Delta x_{pb}}{\Delta t} + \frac{\Delta x_{ba}}{\Delta t}$$

$$\boxed{v_{pa} = v_{pb} + v_{ba}}$$

What if point P is accelerating? What is the relative acceleration of point P?

$$a_{pa} = \frac{\Delta v_{pa}}{\Delta t}$$

$$a_{pb} = \frac{\Delta v_{pb}}{\Delta t}$$

$$a_{ba} = \frac{\Delta x_{ba}}{\Delta t}$$

And now combining these, like we did when going from displacement to velocity,

$$\frac{\Delta v_{pa}}{\Delta t} = \frac{\Delta(v_{pb} + v_{ba})}{\Delta t} = \frac{\Delta v_{pb} + \Delta v_{ba}}{\Delta t} = \frac{\Delta v_{pb}}{\Delta t} + \frac{\Delta v_{ba}}{\Delta t}$$

$$a_{pa} = a_{pb} + a_{ba}$$

We will only deal with situations in which the reference frames are NOT accelerating. This means that $a_{ba} = 0$, and therefore that

$$\boxed{a_{pb} = a_{pa}}$$

Furthermore, if A and B are moving at the same velocity, i.e. $v_{ba} = 0$, then

$$\boxed{v_{pa} = v_{pb}}$$

6.2 Example #1: Boat traveling upriver

A boat travels upriver at 14 km/h relative to the water. The water flows 9 km/h relative to the ground.

(a) What are the magnitude and direction of the boat's velocity relative to the ground?
(b) A child walks to from the bow to the stern at 6 km/h. What are the magnitude and direction of the child's velocity relative to the ground?

Have students work on these.

(a) Given: $v_{bw} = -14$ km/h; $v_{wg} = 9$ km/h. Want to find v_{bg}.

$$v_{bg} = v_{bw} + v_{wg} = -14 + 9 = -5 \text{ km/h}$$

So the boat travels at 5 km/h upstream.

(b) Given: $v_{bg} = -5$ km/h; $v_{cb} = 6$ km/h. Want to find v_{cg}.

$$v_{cg} = v_{cb} + v_{bg} = 6 - 5 = 1 \text{ km/h}$$

6.3 Relative motion in 2-D

Relative motion is exactly identical in two dimensions, but now we have to use vectors. So, for example,

$$\vec{x}_{pa} = \vec{x}_{pb} + \vec{x}_{ba}.$$

You are trying to cross a river with a boat and would like to be exactly on the opposite side of the river. Your boat can travel 20 m/s (relative to the water). The river is flowing 2 m/s and is 1000 m wide. What angle should you leave shore at, and how long will it take you to reach the other side? (Note that ophysics has a simulation that shows this as well.)

[Insert diagram showing what happens if you go straight across the river.]

If you head straight across the river, it will take you 50 s to cross the river ($\Delta t = \Delta x/v_x$), you will end up 100 m downstream from your objective.

[Insert diagram showing angle θ.]

$$\vec{v}_{bw} + \vec{v}_{wo} = \vec{v}_{bo}$$

$$\langle V\cos\theta, V\sin\theta\rangle + \langle 0, -V_w\rangle = \langle V_o, 0\rangle$$

We want to find θ and V_o; the latter will tell us how long it takes to cross the river. This vector equation is the same thing as writing down two equations (with two unknowns).

$$V\cos\theta = V_o$$

$$V\sin\theta - V_w = 0$$

From the second equation,

$$\theta = \sin^{-1}\left(\frac{V_w}{V}\right) \approx 5.7^\circ$$

Inserting this into the first equation gives

$$V_o = 19.9 \text{ m/s}$$

So,

$$\Delta t = \frac{\Delta x}{V_o} = \frac{1000 \text{ m}}{19.9 \text{ m/s}} = 50.3 \text{ s}$$

6.4 Example #2: hockey player and puck

A hockey player is skating due south at 7.0 m/s. A puck is passed to him with a speed of 11.0 m/s and direction 22° west of south. What are the magnitude and direction (relative to due south) of the puck's velocity, relative to the hockey player?

Given:
$V_h = 7.0$ m/s
$V_p = 11.0$ m/s at angle of 22° west of south

We want to know $\vec{v}_{ph}$.

$$\vec{v}_{ph} = \vec{v}_{pi} + \vec{v}_{ih} = \vec{v}_{pi} - \vec{v}_{hi}$$

$$\vec{v}_{ph} = \langle -V_p \sin\theta, -V_p \cos\theta \rangle + \langle 0, -V_h \rangle = \langle -V_p \sin\theta, -V_p \cos\theta \rangle - \langle 0, -V_h \rangle$$

$$\vec{v}_{ph} = \langle -V_p \sin\theta, -V_p \cos\theta + V_h \rangle = \langle -4.12, -3.2 \rangle$$

speed = 5.2 m/s
angle = 52.2°

7 CIRCULAR AND ROTATIONAL MOTION I

Objectives:

1. Definitions
2. Kinematic equations
3. Rolling motion

We have covered straight line motion and projectile motion. This week we'll describe circular motion and rotational motion. Then we'll ask what causes motion—or rather, changes in motion.

7.1 Basic definitions

[Demo: ball on a string]

To describe motion around a circle, it is convenient to define position using the angle from the positive $x-$axis.

[Insert diagram defining θ.]

Angular position is defined as $\theta = s/r$, in radians. Radians are *not* units! However, it is often convenient to report a measurement as being in radians so that it is clear that the number represents an angle. $\theta > 0$ if counterclockwise from the positive $x-$axis. Sometimes we will want to know the distance that an object has travelled along a circle, in which case we might use $s = r\theta$. (Next class we'll relate angular quantities to linear quantities.)

How do we convert between radians and degrees? 1 rad $= \pi/180$

From here, we can probably guess how to define angular displacement, angular velocity, and angular acceleration.

Angular displacement: $\Delta\theta = \theta_f - \theta_i$, [rad]
Angular velocity: $\omega = \frac{\Delta\theta}{\Delta t}$, [rad/s]
Angular speed or angular frequency: $|\omega|$, [rad/s]
Angular acceleration: $\alpha = \frac{\Delta\omega}{\Delta t}$, [rad/s^2]

Like before, angular velocity is the slope of the angular position-versus-time graph, and angular acceleration is the slope of the angular velocity-versus-time graph. Conversely, angular displacement is the area under the angular velocity-versus-time graph, and the change in angular velocity is the area under the angular acceleration-versus-time graph.

What is meant by positive or negative angular velocity? What is meant by positive or negative angular acceleration?

7.2 Kinematic equations for circular motion

I won't go through the derivations for the kinematic equations for circular motion because they are exactly analogous to what we did for one-dimensional motion. For constant angular acceleration:

$$\Delta\omega = \alpha\Delta t$$

$$\Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha\Delta t^2$$

$${\omega_f}^2 - {\omega_i}^2 = 2\alpha\Delta\theta$$

When $\alpha = 0$, this reduces to

$$\Delta\theta = \omega\Delta t.$$

Note that if $\Delta t = T$, where T is the period (i.e., the time to complete one revolution), then $\Delta\theta = 2\pi$ and therefore

$$2\pi = \omega T \Rightarrow \omega = \frac{2\pi}{T}$$

Frequency is defined as the number of revolutions per unit time and is given by $f = 1/T$, which means that we can also write that

$$\omega = 2\pi f$$

7.3 Example #1: shaft of an elevator motor

The shaft of an elevator motor turns clockwise at 180 rpm for 10 s, is at rest for 15 s, then turns counterclockwise at 240 rpm for 12.5 s. What is the angular displacement of the shaft during this motion. Draw angular position and angular velocity graphs for the shaft's motion.

Angular velocity during the first interval: $\omega_1 = -6\pi$ rad/s
Angular displacement during the first interval: $\Delta\theta_1 = \omega_1\Delta t_1 = -60\pi$ rad

Angular velocity during the third interval: $\omega_3 = 8\pi$ rad/s
Angular displacement during the third interval: $\Delta\theta_3 = \omega_3\Delta t_3 = 100\pi$ rad

Net displacement: 40π rad

[Insert position-time and velocity-time graphs.]

7.4 Rolling motion (w/out slipping)

Rolling of wheels and other objects involves rotational motion. Depending on our frame of reference, we observe it either as rotation around the point where the wheel touches the ground or the center of the wheel. Let's first consider this from a static reference frame to see how the wheel's velocity is related to the angular velocity.

[Derive these equations using a demo.]

When the wheel has undergone one revolution, the center of the wheel will have traveled a distance equal to the circumference of the wheel, meaning that

$$\Delta x = 2\pi R$$

For the case of uniform motion (i.e., constant velocity),

$$v = \frac{\Delta x}{\Delta t}$$

If the wheel has undergone one revolution, then $\Delta t = T$. Putting this together,

$$v = \frac{2\pi}{T} R = \omega R,$$

which is referred to as the rolling constraint. Note that ω is the angular velocity of the wheel around it's axis, but in our reference frame the wheel appears to be rotating around the point where it touches the ground.

[Diagram showing rolling motion in our reference frame.]

In our reference frame, rolling motion can be viewed as a sum of rotational and translation motion.

[Diagram depicting rotational + translational motion.]

Note that the velocity of the center of the wheel in our reference frame is the same as the tangential velocity for the rotational component of motion. If instead we are in a reference frame that moves with the wheel, we will only see rotational motion and the wheel will appear to rotate around its axis.

7.5 Example #2: chain speed on a bicycle

The wheels on a road bike are around 0.70 m in diameter. In order to travel 9 m/s (20 mph), how fast must the chain move? Consider two different sprockets, one with a diameter of 7 cm and another with a diameter of 14 cm. It is easiest to view this problem from the moving reference frame.

[Diagram of wheel and sprocket.]

The entire wheel must rotate with the same angular velocity. During one rotation, a point on the edge of the wheel travels a distance of $2\pi R$, whereas a point on the sprocket travels $2\pi r$. The velocity of the edge of the wheel is

$$v_w = \frac{2\pi R}{T} = \omega R$$

as we've already seen. The velocity of the point on the sprocket is

$$v_a = \frac{2\pi r}{T} = \omega r$$

Solving both for ω and setting them equal to each other,

$$\omega = \frac{v_a}{r} = \frac{v_w}{R}$$

$$v_a = v_w \frac{r}{R}$$

Plugging in values, we get 0.9 m/s for the small sprocket and 1.8 m/s for the large sprocket. The large sprocket is better for accelerating, but it is difficult to move fast with it because the chain must travel much faster than when it is on the small sprocket.

8 CIRCULAR AND ROTATIONAL MOTION II

Recall terms used to describe circular motion (e.g., a satellite orbiting the Earth).

- Angular position, θ [rad], measured counterclockwise from positive $x-$axis; recall that $\theta = s/r$
- Angular displacement, $\Delta\theta$ [rad]
- Angular velocity, $\omega = \frac{\Delta\theta}{\Delta t}$ [rad/s]
- Relationship between frequency and angular velocity, $\omega = 2\pi f = 2\pi/T$
- Angular acceleration, $\alpha = \frac{\Delta\omega}{\Delta t}$ [rad/s^2]
- We can use essentially the same kinematic equations as before to describe circular motion, we just need to replace linear quantities with angular quantities.

Everything that we've learned so far about circular motion also applies to rotational motion (i.e., the rigid body rotation of an object around some axis). An example of rotational motion is a bicycle wheel spinning around its axle. The fundamental difference between circular motion and rotation motion is that in rotational motion, all parts of an object DO NOT move at the same *linear* speed/velocity. They do have the same angular velocity.

8.1 Relating angular quantities to linear quantities

Often we'll want to be able to convert between angular quantities and linear quantities. This derivation requires calculus, so I will just give you the results and we can focus on interpreting the equations.

For all of the following we will assume that the radius is constant.

Position:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Or, in vector notation:

$$\boxed{\vec{x} = \langle r\cos\theta, r\sin\theta\rangle}$$

[Diagram of position vector.]

Velocity:

$$\boxed{\vec{v} = \langle -r\omega\sin\theta, r\omega\cos\theta\rangle}$$

The speed is the magnitude of the velocity:

$$v = |\vec{v}| = \sqrt{(-r\omega \sin\theta)^2 + (r\omega\cos\theta)^2} = \sqrt{r^2\omega^2(\sin^2\theta + \cos^2\theta)}$$

$$\boxed{|\vec{v}| = |\omega| r} \text{ or, in shorter notation } \boxed{v_t = \omega r}$$

[Insert diagram of velocity vector.]

Acceleration:

The acceleration consists of two parts: tangential and centripetal acceleration.

Tangential acceleration

If an object is traveling in a circle and it is speeding up or slowing down, then it is experiencing tangential acceleration. We already saw that the magnitude of the tangential velocity is given by

$$v_t = \omega r$$

The magnitude of the tangential acceleration is simply the change in tangential velocity divided by time,

$$a_t = \frac{\Delta v_t}{\Delta t} = \frac{\Delta \omega}{\Delta t} r = \alpha r$$

[Diagram of tangential acceleration vector.]

Centripetal acceleration

If an object is traveling in a circle with constant velocity, it still experiences an acceleration. Why? Recall that acceleration is a *vector* that represents the rate of change of the velocity *vector*. Centripetal acceleration causes the velocity vector to change direction. It must be perpendicular to the direction of motion, otherwise the object would change *speed*.

[Diagram of centripetal acceleration vector.]

For simplicity, use v to represent $|\vec{v}|$. The diagram contains two similar triangles, from which we can deduce that

$$\frac{d}{r} = \frac{\Delta v}{v}$$

The distance d is approximately given by

$$d \approx v\Delta t$$

which means that

$$\frac{v\Delta t}{r} \approx \frac{\Delta v}{v}$$

This becomes an equality as $\theta \to 0$. Rearranging,

$$\frac{v^2}{r} = \frac{\Delta v}{\Delta t} = a_c$$

The centripetal acceleration is non-zero as long as the object is moving in a circle. The faster the object is moving and the tighter the circle, the greater the acceleration must be.

[Diagram showing direction of the acceleration vectors, and pointing out that $\vec{a}_c$ always points inward and that $\vec{a}_c \perp \vec{a}_t$.]

8.2 Circular vs. rotational motion

When discussing an object that is undergoing circular motion, we generally assume that the object is small relative to the radius of its orbit. We are therefore assuming that the entire object is traveling with the same velocity and acceleration.

[Diagram of circular motion.]

For an object undergoing rotational motion, the different parts of the object are experiencing different velocities and accelerations, depending on the distance from the axis of rotation.

[Diagram of rotational motion.]

8.3 Example #1: bicycle chain

The wheels on a road bike are around 0.70 m in diameter. In order to travel 9 m/s (20 mph), how fast must the chain move? Consider two different sprockets, one with a diameter of 7 cm and another with a diameter of 14 cm. It is easiest to view this problem from the moving reference frame.

[Diagram of wheel and sprocket.]

The entire wheel must rotate with the same angular velocity. During one rotation, a point on the edge of the wheel travels a distance of $2\pi R$, whereas a point on the sprocket travels $2\pi r$. The velocity of the edge of the wheel is

$$v_w = \frac{2\pi R}{T} = \omega R$$

as we've already seen. The velocity of the point on the sprocket is

$$v_a = \frac{2\pi r}{T} = \omega r$$

Solving both for ω and setting them equal to each other,

$$\omega = \frac{v_a}{r} = \frac{v_w}{R}$$

$$v_a = v_w \frac{r}{R}$$

Plugging in values, we get 0.9 m/s for the small sprocket and 1.8 m/s for the large sprocket. The large sprocket is better for accelerating, but it is difficult to move fast with it because the chain must travel much faster than when it is on the small sprocket.

8.4 Example #2: Earth's centripetal acceleration

Calculate the centripetal acceleration of the Earth due to its motion around the Sun, and compare to gravitational acceleration. Assume a circular orbit.

$$a_c = \frac{v^2}{r}$$

Note that

$$v = \omega r$$

and so

$$a_c = \omega^2 r$$

The radius is one astronomical unit,

$$1\ \mathrm{AU} = 1.496 \times 10^{11}\ \mathrm{m}$$

And recall that $\omega = 2\pi f = 2\pi/T$. The period of the Earth's orbit is 86400 s $\times$ 365.25 d $=$ 31557600 s, which means that $\omega = 1.99 \times 10^{-7}$ rad/s.

Putting this together, we find that

$$a_c = 0.0059\ \mathrm{m/s}^2$$

which is really small. You don't sense this motion since it is several orders of magnitude smaller than g.

8.5 Example #3: disk in a hard drive

The disk in a hard drive in desktop computer rotates at 7200 rpm. And it takes something like 4 ms (4×10^{-3} s) to reach this frequency. The disk has a diameter of 13 cm. What is the tangential acceleration of the outer edge of the disk when it turns on? What is the centripetal acceleration of the outer edge of the disk? What is the maximum net acceleration of a point on the outer edge of the disk?

Given:
$f = 7200$ rpm $= 120$ rev/s
$r = 0.13$ m

Solution:
Tangential acceleration is given by

$$a_t = \alpha r = \frac{\Delta\omega}{\Delta t} r$$

$$\omega = 2\pi f = 750\ \mathrm{rad/s}$$

This means that

$$a_t = 12000\ \mathrm{m\ s}^{-2}$$

The centripetal acceleration at full speed is

$$a_c = \omega^2 r = \frac{v^2}{r} = 37000\ \mathrm{m\ s}^{-2}$$

The maximum possible acceleration would be when the tangential acceleration is still high and the disk is approaching 7200 rpm.

$$a_{\max} = \sqrt{a_c^2 + a_t^2} = 39000 \text{ m s}^{-2}$$

9 NEWTON'S LAWS

Objectives:

- Newton's Laws
- Demonstration of forces
- Free-body diagrams

9.1 Background

I spent the first few weeks discussing *kinematics*, which is the branch of physics that describes the motion of objects. We've talked about linear motion, two-dimensional motion, and circular/rotational motion. Today I'm going to start talking about *dynamics*, which is the study of causes of motion and changes in motion. Kinematics and dynamics make up what we refer to as mechanics. We're now moving into the latter half of the 1600S. We're going to start being able to address much more interesting questions.

We'll use the term *force* to describe the agent that causes motion — or as we'll see in a minute, changes in motion. We usually write force as $\vec{F}$. Force is a vector, just like displacement, velocity, and acceleration. Therefore it has a magnitude and direction, and can be split into vector components: $\vec{F} = \langle F_x, F_y \rangle$.

There are several different types of forces. We'll split them into two types: contact forces and body forces.

Contact force	Body force
friction/drag	gravity
tension	electric
normal	magnetic
spring	

[Sketches of the forces.]

Newton's Laws describe how forces affect the motion of objects.

9.2 Newton's First Law
The velocity of an object remains constant unless it is acted upon by an *external* force. In other words,

$$\sum \vec{F} = 0 \Rightarrow \vec{a} = 0.$$

The first law is a formalization of Galileo's work on kinematics. Although this will look like a special case of the second law, it is really defining the reference frames for which we can apply the second law. The second law is only valid for inertial (i.e., non-accelerating) references. A reference frame is inertial if an object travels at constant velocity if the sum of the forces is 0.

9.3 Newton's Second Law
In an inertial reference frame, the acceleration of an object is parallel and proportional to the net force and inversely proportional to the object's mass.

$$\sum \vec{F} = m\vec{a}$$

From this, we see that force has units of [kg·m/s^2], which we refer to as a newton [N]. This Law is based on the work of Galileo, Kepler, Brahe, and others.

This makes sense. Try pushing a penny and a piano. It takes much less effort to push a penny. Why? Because the penny has less mass. But what is mass???

9.4 Newton's Third Law
For every action, there is an equal and opposite reaction. In other words, the force exerted on object 2 by object 1 is equal and opposite to the force exerted on object 1 by object 2.

$$\vec{F}_{12} = -\vec{F}_{21}$$

This is the most confusing of the three laws. A couple of examples:

- A hammer hitting a nail. The hammer exerts a force on the nail, and the nail exerts a force on the hammer. This is why hammers sometimes break.
- A bug hitting a windshield. Which experiences a larger force — the bug, or the windshield? They actually experience the same force, but the bug has a much smaller mass than the car, and so it undergoes an extreme acceleration!

9.5 Demonstration of forces

- Have student hold onto a spring while I pull on the other side. They should agree that they feel a pulling force.
- Hang a mass from the spring. The spring stretches, so it must be exerting a force on the mass.
- Hang a mass from the string. Is the string exerting a force on the mass? If I tied a rope around your waist and pulled, would you feel a force? Molecular bonds are essentially springs. The string does stretch a little bit, and that's what causes tension.
- Place a book on the table. What forces are being exerted on the book? Is there actually a normal force acting upward on the book?

- Place the book on top of a compression spring. The spring changes length, so it must be exerting a force on the book.
- So we really can think of the normal force as being due to molecular bonds being compressed. Stand on table with laser beam pointing at the wall. Although they don't see the table deflect, they do see the position of the laser change. This is evidence that the table experienced compression.

9.6 Free-body diagrams

The key to solving problems using Newton's Laws is to draw free-body diagrams. The general method will be:

- Identify relevant forces in a free-body diagram.
- Use Newton's Second Law to set up 1 or 2 equations (motion in 1-D or 2-D). To do this, I always define forces as being positive, and then insert negative signs in Newton's Second Law if the force points in the negative x- or y-directions.
- Solve for unknown terms. Sometimes this involves solving for acceleration and then applying kinematics to describe an object's motion.

9.7 Example problems

Let's go through some examples. We'll draw free-body diagrams and identify the equations that we would use, but won't do anything more (yet).

Example #1: A person is standing on the floor

[Diagram of person standing on floor.]

Forces: Gravitational force pointing downward, and the normal force is pointing upward.

Equations:

$$\sum F_x = 0$$

$$\sum F_y = F_n - F_g = ma_y = 0$$

For this particular problem, $F_n = F_g$. In other words, the normal force equals the weight of the person.

What if the person is on an elevator that is accelerating upward. Then what is the normal force?

[Diagram of person in an elevator.]

Equations:

$$\sum F_y = F_n - F_g = ma_y$$
$$F_n = F_g + ma_y$$

Since the acceleration is positive, the normal force is greater than the person's weight. (If the acceleration was negative, then the normal force would be less than the person's weight.) This force, which is what the person feels on the bottom of their feet, is referred to as the apparent weight.

Example #2: A box is sliding down a frictionless inclined plane
[Diagram of box on ramp.]

Forces: $\vec{F}_g$ and $\vec{F}_n$. Which way are they pointing? What is $\vec{a}$?

Equations:

$$\sum F_x = F_g \sin\theta = ma_x$$
$$\sum F_y = F_n - F_g \cos\theta = ma_y$$

Example #3: A box is pulled up a ramp
A box is pulled up a ramp by a falling mass that is connected by a string. For this problem, let's just identify the forces on the diagram.

Forces: $\vec{F}_g$, $\vec{F}_n$, $\vec{F}_k$, $\vec{F}_t$
To solve this type of problem, we need to balance forces on both objects.

[Diagram of box on ramp, connected to falling mass.]

Example #4: Forces as vectors
You are given two forces (draw vectors). Find the third force that would hold the object in equilibrium. The key to this problem is that for an object in equilibrium, $\sum \vec{F} = 0$.

10 TYPES OF FORCES I

Objectives:

- Newton's Laws
- Gravitational force
- Centripetal force
- Normal force
- Tensional force

10.1 Newton's Laws

Last class I introduced Newton's Laws:

1. if $\sum \vec{F} = 0 \Rightarrow \vec{a} = 0$
2. $\sum \vec{F} = m\vec{a}$
3. $\vec{F}_{ab} = -\vec{F}_{ba}$

We then discussed how to use them to solve problems. The general approach is:

1. Draw a free-body diagram.
2. Insert forces into Newton's first or second law.
3. Do some algebra with one or more equations.
4. If necessary, use results to solve for an object's motion.

We went through some conceptual examples, but didn't fully solve them because we didn't have a good understanding of the different types of forces. During the next couple of classes we will carefully go through some of the more common forces.

10.2 Gravitational force

Prior to Newton, it was known that

1. Objects near the Earth's surface fall with constant acceleration (Galileo)
2. Planetary orbits obey $T^2 = cR^3$ (Kepler's third law)

Newton supposed that all object's must obey the same physical laws, regardless of whether they were located on Earth or in space, and he reasoned that planetary orbits are a result of gravity. He was able to come up with an equation that could explain both of these phenomenon:

$$\boxed{F_g = \frac{Gm_1m_2}{r^2}},$$

where F_g is the magnitude of the gravitational force, $G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2, m_1 and m_2 are the masses of two objects, and r is the distance between the objects' centers of mass. The gravitational force is directed along the line joining the two objects.

For objects near the Earth's surface, $m_1 = m_{earth} = 5.972 \times 10^{24}$ kg and $r \approx 6,371$ km. Therefore,

$$\frac{Gm_1}{r^2} \approx 9.81 \text{ m/s}^2 = g,$$

and so

$$F_g = mg,$$

where we've now defined m as being the mass of an object near the Earth's surface.

Let's consider an object in free-fall. If we add up the forces acting on the object, we have

$$\sum F_y = -F_g = ma_y.$$

But since $F_g = mg$,

$$-mg = ma_y \Rightarrow \boxed{a_y = -g},$$

and so all objects fall with with constant (as long as we ignore the effects of air resistance). So Newton's Law of Gravitation explains Galileo's observations. It also explains Kepler's Law of Harmony. We can show this crudely by assuming that planets travel in circular orbits with constant speed. (We would need to use calculus to take into account the fact that planets have elliptical orbits.)

In order for a planet to travel in a circle, it must have some centripetal acceleration, a_c, that points toward the center of the orbit. (Planets actually travel in ellipses, and you can do this more carefully by considering the motion along an ellipse, but this is easier and arrives at the same result.) From Newton's Second Law, we know that this acceleration must be caused by a force that points toward the center of the circle; we'll call that force a "centripetal force". If the centripetal force is due to gravity, then

$$F_g = ma_c.$$

We saw previously that $a_c = \omega^2 r$, and that angular frequency is related to orbital period through $\omega = 2\pi/T$. Therefore, $a_c = (2\pi)^2 r/T$ and as a result,

$$F_g = \frac{m(2\pi)^2 r}{T^2}.$$

Substituting in for F_g,

$$\frac{Gm_1 m}{r^2} = \frac{m(2\pi)^2 r}{T^2}.$$

Cancelling like terms and re-arranging,

$$\boxed{T^2 = \left(\frac{(2\pi)^2}{Gm_1}\right) r^3 = kr^3}.$$

To reiterate, near the Earth's surface, use $F_g = mg$, and farther away use $F_g = \frac{Gm_1 m_2}{r^2}$.

10.3 Normal forces

If a an object is sitting on the ground, there will be a gravitational force acting downward on the object and a normal force that keeps the object from moving through the ground. The normal force is always oriented normal (i.e., orthogonal) to the surface and it adjusts its magnitude to keep the object from moving through the surface.

[Insert free-body diagram of box sitting on the ground.]

You can think of this force like a stiff spring under compression. The molecules in the ground are being compressed, and so they respond by pushing upward. The normal force prevents the object from moving through ground. In this simple example, we have

$$\sum F_y = F_n - F_g = 0$$

$$F_n = F_g = mg$$

(Note that it is not generally true that $F_n = mg$.)

10.4 Tensional forces

Tensional forces are similar to normal forces in the sense that they are a response to other forces. You can think of tension as being like a stiff spring under extension. The molecular bonds in the rope or string are stretched just a little bit, and they are trying to return to their equilibrium lengths.

[Insert diagram of mass hanging from a string.]

In the above diagram, there are two forces acting on the hanging mass: gravity and the tensional force from the string. According to Newton's second law,

$$\sum F_y = F_t - F_g = ma_y = 0,$$

so

$$F_t = F_g = mg.$$

The larger the mass, the greater the tensional force (i.e., the tensional force adjusts its magnitude — until the string breaks!). We will often use the "massless string approximation", which allows us to set the tensional force constant throughout a string. To see why we can do this, let's consider the same problem as before, but now let's calculate the tensional force at the top of the string where it connects to the ceiling.

[Modified diagram of mass hanging from a string.]

Like before, we have

$$\sum F_y = F_t - F_g = ma_y = 0,$$

and so

$$F_t = F_g = mg.$$

However, the mass m now includes the mass of the object *and* the mass of the string. In other words, $m = m_o + m_s$. Thus,

$$F_t = (m_o + m_s)g.$$

How important is m_s? If $m_o = 100$ kg and $m_s = 1$ kg, then the tension at the top of the rope will be 1% larger than the tension at the bottom of the rope. For a lot of problems, that 1% difference is irrelevant (especially considering errors associated with lab experiments) and so we can ignore the mass of the rope.

10.5 Example problems

Example #1: Box pulled along a frictionless surface

A box is pulled along a frictionless surface with a rope. The rope makes an angle θ with horizontal. Derive expressions for the normal force and the horizontal acceleration.

[Insert diagram.]

$$\sum F_x = F_t \cos\theta = ma_x$$

$$\boxed{a_x = \frac{F_t}{m}\cos\theta}$$

$$\sum F_y = F_n + F_t \sin\theta - F_g = ma_y = 0$$

$$\boxed{F_n = F_g - F_t \sin\theta}$$

The normal force doesn't always balance gravity. It simply prevents the box from passing through the ground. In this case, tension in the rope decreases the normal force. Furthermore, two objects can be in contact but have $F_n = 0$. In this problem, if $F_g = F_t \sin\theta$, then $F_n = 0$. A normal force cannot cause an acceleration by itself, and thus $F_n \leq 0$.

Example #2: Atwood machine

Calculate the acceleration and time to fall for a mass suspended over a pulley.

[Diagram of Atwood machine.]

From Newton's second law, we can write down

$$T - m_1 g = m_1 a_1$$

and

$$T - m_2 g = m_2 a_2$$

We have two equations and three unknowns. However, we also know that $a_1 = -a_2$, and therefore

$$T - m_1 g = -m_1 a_2$$

With some algebra, we arrive at

$$\boxed{a_2 = \left(\frac{m_1 - m_2}{m_2 - m_1}\right) g}$$

Use kinematics to determine fall time.

$$\Delta y = v_i \Delta t + \frac{1}{2} a \Delta t^2$$

If initial velocity is zero, then

$$\Delta t = \sqrt{\frac{2\Delta y}{a}}$$

Example #3: Pulley systems
[Diagram of 2:1 pulley]

If $a \approx 0$, then we can write down

$$T = m_2 g$$

and

$$2T = m_1 g$$

which means that

$$2m_2 = m_1$$

In other words, mass m_2 can hold twice its weight.

What happens if we change the angle of the string? [Diagram with pulleys separated by some distance.]

We still have

$$T = m_2 g$$

but now we have

$$2T\cos\theta = m_1 g$$

which means that

$$2m_2\cos\theta = m_1$$

If $\theta > 0$, then we we need more mass at m_2 to balance m_1.

How can we get more leverage? Add additional pulleys. [Diagram of 3:1 pulley.]

Again, we still have

$$T = m_2 g$$

but now we have

$$3T = m_1 g$$

and therefore

$$3m_2 = m_1$$

Mass m_2 can support three times its weight.

Example #4: Positioning an automobile engine
A automobile engine has a weight of 3150 N. The engine is being positioned above an engine compartment. To position the engine, a worker is using a rope. Find the tension in the supporting cable and in the positioning rope. The cable is 10° from vertical, and the rope is 80° from vertical.

[Insert diagram.]

To solve this, think about the forces that are acting on the ring that connects the cable, the rope, and the engine.

$$\sum F_x = -T_1\sin\theta_1 + T_2\sin\theta_2 = 0$$

$$\sum F_y = T_1\cos\theta_1 - T_2\cos\theta_2 - F_g = 0$$

From the first equation,

$$T_1 = T_2 \frac{\sin\theta_2}{\sin\theta_1}$$

Inserting this into the second equation,

$$T_2 \frac{\sin\theta_2}{\sin\theta_1} \cos\theta_1 - T_2 \cos\theta_2 - F_g = 0$$

$$T_2 \left(\frac{\sin\theta_2}{\sin\theta_1} \cos\theta_1 - \cos\theta_2 \right) = F_g$$

$$T_2 = \frac{F_g}{\frac{\sin\theta_2}{\sin\theta_1} \cos\theta_1 - \cos\theta_2} \Rightarrow \boxed{T_2 = 547 \text{ N}}$$

Plugging this result back into the expression for T_1 gives

$$\boxed{T_1 = 3.10 \times 10^3 \text{ N}}$$

11 TYPES OF FORCES II

Objectives:

- Kinetic friction
- Static friction
- Drag

Last class:

- Gravitational force: $F_g = \frac{Gm_1m_2}{r^2}$, but use $F_g = mg$ near the Earth's surface.
- Normal force: $F_n > 0$, points perpendicular to surface.
- Tensional force: F_t, points in direction of string. Massless string approximation allows us to assume that the tensional force is constant through the string.

11.1 Kinetic friction

The normal force is very important when dealing with friction, which is known through experiments.

Consider this problem of a box sliding down a ramp:

[Diagram of box on a ramp.]

We can solve for both the normal and frictional forces.

Summing the forces in the y-direction:

$$\sum F_y = F_n - F_g \cos\theta = ma_y = 0$$

$$F_n = F_g \cos\theta$$

Summing the forces in the x-direction:

$$\sum F_x = F_g \sin\theta - f_k = ma_x \neq 0$$

$$F_k = m(a_x - g\sin\theta)$$

This means that we can measure F_n and F_k experimentally. If we do lots of experiments, we find that F_k is proportional to F_n.

[Diagram of F_k vs. F_n; plots as straight line with $y-$intercept of 0.]

This means that

$$\boxed{F_k = \mu_k F_n}$$

where μ_k is the coefficient of kinetic friction. It is a constant for a given system (combination of surfaces) and is determined experimentally. This is just a model for friction; its actually quite a bit more complicated and is still an active area of research.

What makes something have a high coefficient of friction?

11.2 Static friction

We should also consider the case in which the block isn't moving at all. This means that friction balances whatever forces would tend to cause acceleration (in this case, down the ramp). In our example, this would mean that

$$\sum F_x = F_g \sin\theta - F_s = 0$$

I've set $ma_x = 0$ and switched F_k to F_s to refer to static friction. Thus,

$$F_s = F_g \sin\theta.$$

The block will be in static equilibrium until a certain threshold is exceeded. That threshold is determined experimentally to be

$$\boxed{\max F_s = \mu_s F_n}$$

If you want to solve a problem in which a box isn't sliding initially, do the following:

1. Calculate the force that will tend to accelerate the box.
2. Does this force exceed $\max F_s$? If no, then the frictional force F_s will be . If yes, then the box will start sliding and you will have frictional force $F_k = \mu_k F_n$.

In general, $\mu_s > \mu_k$. In other words, its easier to keep an object moving than it is to get it moving in the first place.

11.3 Example problems involving friction

Example #1: pull box across a horizontal surface

You pull a box with a rope across a horizontal surface. The box is initially at rest, but you are able to overcome the maximum static friction. Plot the frictional force as a function of time. Let $m = 100$ kg, $\mu_s = 0.3$, and $\mu_k = 0.2$.

[Diagram of box being pulled by a rope.]

Before and after the box starts moving,

$$\sum F_y = F_n - F_g = 0 \Rightarrow F_n = F_g = mg$$

Before the box starts moving,

$$\sum F_x = F_t - F_s = 0 \Rightarrow F_s = F_t$$

Let's assume that we gradually increase F_t until the box starts moving. This occurs once $F_t \geq \max F_s = \mu_s F_n = \mu_s mg = 0.3 \times 100 \text{ kg} \times 9.81 \text{ m/s}^2 \approx 300 \text{ N}$. Once the box starts moving, we switch to kinetic friction: $F_k = \mu_k mg = 0.2 \times 100 \text{ kg} \times 9.81 \text{ m/s}^2 \approx 200 \text{ N}$.

[Plot of results.]

Example #2: Calculate the coefficient of friction of a block sliding over the table
Do this as a demonstration.

What do we need?

Forces on sliding block:

$$\sum F_x = F_t - F_k = m_1 a_x$$

$$F_k = F_t - m_1 a_x$$

$$\sum F_y = F_n - F_g = 0 \Rightarrow F_n = F_g$$

Forces on falling block:

$$\sum F_y = F_t - F_g = m_2 a_y$$
$$F_t = m_2 a_y + m_2 g$$

F_t on the falling block is the same as F_t on the sliding block. We have an additional constraint, which is that $a_x = -a_y = a$.

So this means that

$$F_k = m_2 a_y + m_2 g - m_1 a_x = m_2 g - (m_1 + m_2)a = \mu_k F_n = \mu_k m_1 g$$

$$\mu_k = \frac{m_2 g - (m_1 + m_2)a}{m_1 g}$$

We need to measure the acceleration.

$$\Delta x = v_i \Delta t + \frac{1}{2} a \Delta t^2 \Rightarrow a = \frac{2\Delta x}{\Delta t^2}$$

We can just as easily measure the vertical displacement.

11.4 Drag

I just want to briefly touch on drag since it is something that we are all familiar with, and you are probably also familiar with the concept of terminal velocity. Now that we have a basic understanding of forces, we can discuss where the idea of terminal velocity comes from.

Drag is a really complicated subject because it deals with fluid dynamics and turbulence — turbulence is still considered to be a pretty big mystery.

From experiments, though, we have a pretty simple expression that describes the drag force under certain circumstances:

1. The object is between a few millimeters and few meters in diameter.
2. The object's speed is less than a few hundred meters per second (100 m/s = 223 mph).
3. The object is moving through air near the Earth's surface.

Under these circumstances, we can write that

$$\boxed{F_d = \frac{1}{2} C_D \rho A v^2}$$

where F_d is the drag force, C_D is a drag coefficient that depends on the object's shape, $\rho = 1.22\ \text{kg/m}^3$ is the density of air at sea level, A is the cross-sectional area of the object, and v is the object's speed. The drag force points opposite the direction of motion. For many objects, the drag coefficient is approximately 1/2. We'll assume that is the case, and so

$$\boxed{F_d = \frac{1}{4} \rho A v^2}.$$

The drag force increases with the square of the speed of the object!

Let's consider an object that is falling straight down.

[Diagram of falling object.]

There are two forces acting on the object, F_g and F_d. They point in opposite directions. From Newton's second law,

$$\sum F_y = F_d - F_g = ma_y.$$

As the object accelerates, the drag force will become increasingly large and eventually F_d and F_g will cancel out (i.e., the object will stop accelerating, but it will not stop moving). When this happens

$$F_d - F_g = 0 \Rightarrow F_d = F_g.$$

Substituting in for F_d and F_g gives

$$\frac{1}{4}\rho A v^2 = mg$$

$$\boxed{v_{\text{term}} = \sqrt{\frac{4mg}{\rho A}}}$$

What does this equation tell us?

1. If two different objects are the same size and shape, the heavier object will fall faster.
2. Cross-sectional area increases the drag and decreases the terminal velocity.
3. If we removed the Earth's atmosphere, the terminal velocity would go to ∞.

Unfortunately, we can't really do anything more with drag without using calculus and differential equations. Let's see why by returning to the original force balance equation:

$$\sum F_y = F_d - F_g = ma_y$$

$$\frac{1}{4}\rho A v^2 - mg = ma_y$$

We don't have the tools to solve this equation — the acceleration is not constant — so we can't immediately compute the velocity as a function of time.

12 TYPES OF FORCES III

Objectives:

- Spring force

We have covered the following forces so far:

- Gravitational force: $F_g = \frac{Gm_1m_2}{r^2}$, but use $F_g = mg$ near the Earth's surface.
- Normal force: $F_n > 0$, points perpendicular to surface.
- Tensional force: F_t, points in direction of string. Massless string approximation allows us to assume that the tensional force is constant through the string.
- Frictional force: $F_k = \mu_k F_n$ for moving objects, where μ_k is the coefficient of kinetic friction, and $\max F_s = \mu_s F_n$ is the maximum force that can hold an object in equilibrium, where μ_s is the coefficient of static friction. If the force exerted on the object is less than $\max F_s$, the frictional force will exactly balance the sum of the other forces. Both μ_k and μ_s are dimensionless and determined experimentally.
- Drag force: $F_d = \frac{1}{2}C_D\rho A v^2$, for objects ranging in size from a few mm to a few meters in size, traveling less than about 100 m/s, and traveling through air near the Earth's surface.

Today I will introduce the spring force, which is our last force for the semester.

12.1 Hooke's "Law"

When I started talking about types of forces, I used a spring to demonstrate the idea of a force and as an analogy for tensional forces and normal forces — they can both be thought of as really stiff springs.

Introduce spring force by demonstrating how the restoring force depends on the displacement.

What do we know about springs? If you pull a spring, it will pull back in the opposite direction. If you push a spring, it will push back in the opposite direction. In other words, springs provide "restoring forces"; they always try to return to the equilibrium length (i.e., the length they would be if there is no force being exerted on the spring). Furthermore, the magnitude of that force depends on how far the spring has been stretched or compressed. This was first shown by Robert Hooke, and curiously, published as a latin anagram: "ceiiinosssttuv". If you unscramble that, you come up with "Ut tensio, sic vis", meaning "As the extension, so the force". The way we write this nowadays is

$$\boxed{F_{sp} = -k\Delta x}$$

where k is an empirical spring constant and Δx is the displacement of the spring from equilibrium. The spring force points in the direction of the spring. This expression is referred to as Hooke's Law, though its not really a law because it breaks down if you stretch the spring too much. The notation can be a little bit confusing. Because the spring force can point either direction (depending on whether the spring is stretched or compressed), you need to use this formula to figure out the direction of the force.

[Diagrams showing a spring being stretched or compressed, and discuss in terms of whether the force is positive or negative.]

Hooke's Law tells us that the force applied to a spring is linearly related to the displacement of the spring. If we double the force, we double the displacement. As an example, let's consider a mass hanging from a spring.

[Diagram of a mass hanging from a spring.]

If the system is in equilibrium, then the mass is not accelerating, and therefore

$$\sum F_y = F_s - F_g = 0$$

$$-k\Delta y - mg = 0 \Rightarrow \Delta y = \frac{-mg}{k} < 0$$

Why is $\Delta y < 0$? Because the spring is being stretched in the negative y-direction.

[Diagram showing change in position of the end of the spring.]

Demo: If I double the mass, the displacement of the spring doubles (as long as my masses aren't too large!).

12.2 Example problems

Example #1: Scale used to weigh fish

A scale used to weigh fish consists of a spring hung from a ceiling. The spring's equilibrium (unstretched length) is 30 cm. When a 4.0 kg fish is suspended from the spring, it stretches to a length of 42 cm.

a) What is the spring constant?
b) What is the length of the spring if an 8.0 kg fish is hung from the scale?

$$\sum F_y = F_s - F_g = 0$$
$$-k\Delta y - mg = 0$$
$$k = \frac{mg}{-\Delta y} \Rightarrow \boxed{k = 330 \text{ N/m}}$$

Rearranging,

$$\Delta y = \frac{-mg}{k} = -0.24 \text{ m}$$

The spring has been stretched downward by 24 cm. Its new length is therefore $\boxed{54 \text{ cm}}$.

Example #2: A toy train uses a spring

A toy train uses a spring to pull a 2.0 kg block across a horizontal surface. The train is motorized and moves forward at 5.0 cm/s. The spring constant has been measured to be 50 N/m, and the coefficient of static friction between the block and the surface is $\mu_s = 0.60$. The spring is at its equilibrium length at $t = 0$. Assume that at $t = 0$, the train instantaneously accelerates from rest to 5.0 cm/s, and then moves forward at a constant rate. When does the block slip?

[Diagram of the problem.]

$$\sum F_y = F_n - F_g = ma_y = 0 \Rightarrow F_n = F_g = mg$$

$$\sum F_x = F_{sp} - F_s = ma_x$$

Up until the time at which the block starts sliding, $a_x = 0$. We want to find the time at which $F_{sp} = \max F_s$.

$$F_{sp} - \max F_s = 0 \Rightarrow F_{sp} = \max F_s$$

What is the spring force? The spring is being stretch by the train, so the force exerted by the spring on the train is $F_{sp} = -k\Delta x$. As the train moves to the right, the force pulling back on the train increases. (This means that the force exerted by the trains motor must also increase with time.) If the force exerted by the spring on the train is $-k\Delta x$, then the force exerted on the block is $k\Delta x$ (Newton's third law). Therefore,

$$k\Delta x = \max F_s = \mu_s F_n = \mu_s mg$$

$$\Delta x = \frac{\mu_s mg}{k} = 0.235 \text{ m}$$

How long does it take the train to go that distance?

$$\Delta x = v\Delta t \Rightarrow \Delta t = \frac{\Delta x}{v} = 4.7 \text{ s}$$

Example #3: Mass on a spring traveling in a circle
An unstretched spring has a length of 0.1 m and a spring constant of $k = 50$ N/m. A 0.5 kg mass is attached to the spring, and it is spun in a horizontal circle with a constant angular velocity of $\omega = 2\pi$ rad/s, or $f = 1$ Hz. Assume that it is sliding on a frictionless surface. How much does the spring stretch?

[Diagram of rotating mass.]

This is a centripetal acceleration problem. First we sum the forces in the radial direction.

$$\sum F_r = ma_c = m\omega^2 r$$

$$F_{sp} = m\omega^2 r$$

The spring force is $k\Delta r$. Note that we have dropped the negative sign, which is because the radius is measured outward from the center of the circle but we are summing the forces that point toward the center of the circle.

$$k\Delta r = m\omega^2 r$$

$$k(r_f - r_i) = m\omega^2 r_f$$

$$r_f(k - m\omega^2) = kr_i$$

$$r_f = \frac{k}{k - m\omega^2} r_i = 0.165 \text{ m}$$

which means that the change in length is 0.065 m or 6.5 cm.

There is (sort of) a problem with this derivation. What happens if ω gets large? r_f starts to be come quite large, and eventually negative. This is a problem with our model and not with the derivation; Hooke's "Law" is only valid for small displacements.

13 CENTRIPETAL FORCE

Objectives:

- Centripetal force
- Orbital velocity
- Apparent weight

13.1 Background

We spent the last few lectures talking about different types of forces, and discussing how they affect linear motion. In the next two classes I'll talk about how forces cause circular and rotational motion by talking about the centripetal force (today) and torque (next class). The centripetal force is not really a new type of force — rather, it is any force that causes an object to rotate in a circle. It can be a tensional force, as in the case of tetherball, or gravity in the case of a satellite orbiting the Earth.

Demo: Why doesn't the object move in a straight line? Identify the centripetal force. (1) Bucket full of water. (2) Object on the end of a string. (3) Object sitting on a rotating disk.

We've already seen that if an object is moving in a circle, the object's acceleration must point toward the center of the circle. We referred to this as the centripetal acceleration, and saw that

$$a_c = \frac{v^2}{r} = \omega^2 r$$

A centripetal force, then, is any force that produces a centripetal acceleration. So

$$F_c = ma_c = m\frac{v^2}{r} = m\omega^2 r$$

The one thing that makes centripetal forces difficult to deal with is that when we add up forces, we have to add them in up in the "radial" direction.

13.2 Example #1: Satellite orbiting the Earth

One simple example of a centripetal force is a satellite orbiting the Earth.

[Insert diagram.]

If the satellite is far enough from the Earth we don't have to worry about drag (i.e., the satellite is not passing through the Earth's atmosphere). The only force acting on the satellite is gravity. Recall that

$$F_g = \frac{Gm_1 m_2}{r^2}$$

In order for the satellite to stay in a circular orbit, this force must point toward the center of the circle (it does) and equal the centripetal force ($F_g = F_c$). So this gives

$$\sum F_r = \frac{Gm_1m_2}{r^2} = m_2\frac{v^2}{r}$$

Rearranging, we find the orbital velocity:

$$\boxed{v = \sqrt{\frac{Gm_1}{r}}}$$

(Note that if we are close to the Earth's surface, and air resistance is ignored, then $v = \sqrt{gr}$.)

13.3 Example #2: Roller coaster and apparent weight

As a roller coaster car crosses the top of a 40-m-diameter loop-the-loop, its apparent weight is the same as its true weight. What is the car's speed at the top? How does the apparent weight at the bottom of the loop-de-loop (assuming that the speed remains constant) compare to its true weight?

True weight: $F_g = mg$
Apparent weight: F_n; this is the force that you feel pushing against you. If you were on a free-falling elevator, your apparent weight would be 0 because you wouldn't feel the elevator pushing upward.

At the top of the loop-de-loop, F_g and F_n are both pointing toward the center of the circle. Adding them up,

$$\sum F_r = F_g + F_n = ma_c = m\frac{v^2}{r}$$

Since the apparent weight is the same as the true weight, $F_g = F_n$, and so

$$2F_g = m\frac{v^2}{r}$$

$$2mg = m\frac{v^2}{r}$$

Rearranging,

$$\boxed{v = \sqrt{2gr} = 20 \text{ m/s}}$$

At the bottom of the loop-the-loop,

$$\sum F_r = F_n - F_g = ma_c = v\frac{v^2}{r}$$

$$F_n = F_g + m\frac{v^2}{r} = mg + m\frac{v^2}{r}$$

$$\frac{v^2}{r} = 2F_g$$

So, this means that

$$\boxed{F_n = F_g + 2F_g = 3F_g}$$

What did we implicitly assume here? That the velocity is the same at the top and at the bottom of the roller coaster. The velocity should actually be a bit higher at the bottom, and so $F_n > 3F_g$ as the cart goes around the bottom.

13.4 Example #3: Apparent weight at north pole vs. equator

A 75 kg person weighs themselves at the north pole and at the equator. Which scale reading is higher, and by how much? Assume that the Earth is a perfect sphere.

At the north pole, the person just spins in a circle. There is no centripetal acceleration. Therefore, summing the forces gives

$$\sum F_r = F_g - F_n = ma_c = 0$$
$$F_n = F_g = ma = 735.8 \text{ N}$$

At the equator, the person travels with in a circle, and so there is a centripetal force.

$$\sum F_r = F_g - F_n = m\omega^2 r$$

$$F_n = F_g - m\omega^2 r$$

The radius of the Earth is about $r = 6.37 \times 10^6$ m. It takes the Earth 86400 s to rotate once about its axis, and so $\omega = (2\pi)/(86400 \text{ s})$. As a result, the apparent weight is about 2.5 N lower at the equator than at the pole.

14 TORQUE I

Objectives:

- Torque
- Moment of inertia
- Newton's 2$^{\text{nd}}$ Law

14.1 Background

We previously saw how forces can produce circular motion. Essentially, you need a force that produces a centripetal acceleration. This force can be any type of force or combination of forces that points toward the center of a circle.

$$\sum F_r = \frac{mv^2}{r} = m\omega^2 r$$

Forces can also cause rigid bodies to rotate around some axis of rotation. We will use the term *torque* to refer to the ability of a force to cause a rotation. We will define torque as

$$\boxed{\tau = rF_\perp}$$

where τ is torque and has units of N·m, r is the radial distance from the axis of rotation to the point at which the force is being applied, and $F_\perp$ is the component of the force that is perpendicular to r. Following standard conventions, $\tau > 0$ for counter-clockwise rotation. You can also add torques in much the same way as you add forces.

[Demo with mass hanging at different points from a meter stick.]

14.2 Examples of torque

Example #1: You use a wrench to loosen a bolt

[Diagram of wrench and forces.]

If you apply force F at a distance r from the center of the bolt, and $r \perp F$, then you are exerting a torque $\tau = rF > 0$. If you exert the force at an angle $\theta = 45°$ relative to the radial axis, then $\tau = rF \sin\theta$ (in this case, $r \perp F \sin\theta$). If you double the radius, then $\tau = 2rF$.

Example #2: Two forces act on a rod

Two forces act on a rod that pivots about its center. Force F_2 is perpendicular to the rod and is a distance l from the axis of rotation. Force F_1 has a magnitude of 20 N and is oriented at a 45° angle to the rod. The force is acting on the other side of the axis of rotation and at a distance of $2l$ from the axis. What should F_2 be so that $\tau_{net} = 0$?

[Diagram of forces acting on a rod.]

$$\sum \tau = 2lF_1 \sin\theta - lF_1 = 0$$

$$F_2 = 2F_1 \sin\theta = 28 \text{ N}$$

Example #3: Rank forces in order of smallest to largest torque
Various forces act on a rod. The rod can pivot around one of its ends. Rank the forces in order of smallest to largest torque.

[Insert diagram.]

14.3 Torque and rotational motion
Torque affects the rotational motion of an object. To see how, let's start with a really simple system consisting of a point mass, m, attached to a massless rod of length l. The rod can rotate around its end.

[Diagram of point mass at the end of a rod.]

A force is exerted on the mass; the force is perpendicular to the rod. It therefore produces a tangential acceleration of the mass.

$$F_\perp = ma_t$$

Recalling that $a_t = \alpha r$,

$$F_\perp = m\alpha r.$$

Since $\tau = rF_\perp$,

$$F_\perp = \frac{\tau}{r} = m\alpha r \Rightarrow \boxed{\tau = mr^2\alpha}$$

What about a system of particles, or a solid object?

[Insert diagrams showing forces acting on translating and rotating objects.]

From this we have $\tau_1 = r_1F_1 = m_1{r_1}^2\alpha$, $\tau_2 = m_2{r_2}^2\alpha$, ... Because this is rigid body rotation, all points on the body have the same angular acceleration. The net torque is $\tau_{net} = \tau_1 + \tau_2 + \tau_3 + ...$

$$\tau_{net} = \alpha \sum_{i=1}^{N} m_i {r_i}^2$$

The summation can only be calculated analytically for simple objects. So instead we replace the summation with

$$I = \sum_{i=1}^{N} m_i {r_i}^2,$$

where I is the moment of inertia, giving

$$\tau_{net} = I\alpha$$

or equivalently,

$$\boxed{\sum \tau = I\alpha}$$

Does this look at all familiar? It is Newton's Second Law for rotation. The moment of inertia is a rotational equivalent to mass; it is an object's resistance to rotation. The moment of inertia can be calculated for some objects, for others it must be measured. Some that can be calculated:

Solid cylinder: $I = \frac{1}{2}MR^2$
Thin-walled cylinder: $I = MR^2$
Solid sphere: $I = \frac{2}{5}MR^2$
Spherical shell: $I = \frac{2}{3}MR^2$

Note that they all have a dependence on mass and radius squared.

14.4 Examples of rotational motion

Example #4: torque on rotating disk

A 0.2 kg, 0.2-m-diameter disk is spun around its central axis by a motor. What torque must the motor supply to take the disk from 0 to 1800 rpm in 4.0 s?

$$\tau_{net} = \sum \tau = I\alpha$$
$$\alpha = \frac{\Delta\omega}{\Delta t}$$

Given:
$\omega_i = 0$
$\omega_f = 1800 \text{ rpm} \times \frac{1 \text{ min}}{60 \text{ s}} \times \frac{2\pi}{1 \text{ rev}} = 188.5 \text{ rad/s}$
$\Delta t = 4.0$ s

So, from this $\boxed{\alpha = 47.1 \text{ rad/s}^2}$.

The moment of inertia can be calculated for a disk: $I = \frac{1}{2}mr^2 = 1 \times 10^{-3} \text{ kg} \cdot \text{m}^2$.

Inserting into the expression for τ, we get that $\boxed{\tau_{net} = 0.05 \text{ N} \cdot \text{m}}$.

Example #5: falling meter stick

[Video of falling meter stick with coins on top of it.]

The end of the meter stick accelerates downward faster than 9.81 m/s^2. To see where the break occurs, let's consider the torques on the meter stick at the instant that it starts to fall.

$$\sum \tau = \frac{1}{2}Lmg = I\alpha$$

For a stick rotating around its end, $I = mL^2/3$, and so

$$\frac{1}{2}Lmg = \frac{1}{3}mL^2\alpha$$

which means that the angular acceleration is

$$\alpha = \frac{3}{2}\frac{g}{L}$$

However, the tangential acceleration is not the same everywhere. Where is $a_t > g$? Recall that $a_t = \alpha r$. Set $a_t = g$ and solve for r.

$$\frac{3}{2}\frac{g}{L} = \frac{a_t}{r} = \frac{g}{r}$$

Solving for r,

$$r = \frac{2}{3}L$$

or two-thirds of the way down the meter stick.

15 TORQUE II

Objectives:

- Pulleys
- Rolling motion

15.1 Background

Last class I introduced the idea of torque.

$$\tau = rF_{\perp}$$

Forces generate torques, and the magnitude of the torque depends on (1) the distance from the axis of rotation to the point at which the force is acting and (2) the component of the force that is perpendicular to the lever arm. Torques are positive for counter-clockwise rotation.

We also saw that torques cause angular acceleration, such that

$$\sum \tau = I\alpha$$

where I is the moment of inertia. I can be thought of as an object's resistance to angular acceleration. This equation is Newton's Second Law for rotation.

Until now we've (implicitly) ignored objects' moments of inertia. Today I'd like to go through a couple of more involved problems that involve the moment of inertia.

15.2 Demo of coins on a falling meter stick

Elements of this problem:

- Gravitational torque: $-\frac{1}{2}Lmg = I\alpha$
- Moment of inertia: $I = \frac{1}{3}mL^2$
- α is the same everywhere, but a_y isn't!

15.3 Example #1: Block pulled by a string

A block is pulled by a string across a horizontal, frictionless plane. The other end of the string is suspended over a pulley and attached to a hanging mass. We will not assume that the pulley is massless but we will assume that the pulley is frictionless. Because the pulley has a moment of inertia, the tension in the two parts of the string are not necessarily equal. What is the acceleration of the block?

[Insert diagram.]

Start by summing the forces acting on mass m_1.

$$\sum F_x = T_1 = m_1 a_x \tag{1}$$

We want to find a_x. This is one equation with two unknowns. We need at least one more equation.

Now let's sum the forces acting on mass m_2.

$$\sum F_y = T_2 - F_{g,2} = m_2 a_y$$

Note that a_x and a_y have the same magnitude; $a_x > 0$ and $a_y < 0$, so $a_y = -a_x$. Therefore,

$$T_2 - m_2 g = -m_2 a_x \tag{2}$$

We've introduced another equation, but also one more unknown. We now have two equations and three unknowns.

We can come up with a third equation by summing the torques on the pulley.

$$\sum \tau = rT_1 - rT_2 = I\alpha \Rightarrow T_2 - T_1 = I\frac{\alpha}{r}$$

This equation has given us yet one more unknown, α. But we saw earlier that $a_t = \alpha r$, so $\alpha = a_t/r$. We need to be careful with the signs here. We have defined our torques to be positive for counter-clockwise rotation. We have also defined a_x to be positive if it causes clockwise rotation. Therefore, we want to set $\alpha = -a_x/r$. Thus,

$$T_1 - T_2 = -I\frac{a_x}{r^2} \tag{3}$$

We have three equations and three unknowns. Let's solve the first two equations for T_1 and T_2, respectively, and insert them into the third equation.

$$T_1 = m_1 a_x$$

$$T_2 = m_2 g - m_2 a_x$$

And so

$$m_1 a_x - m_2 g + m_2 a_x = -I\frac{a_x}{r^2}$$

We want to solve this for a_x.

$$-m_2 g = -m_2 a_x - m_1 a_x - I\frac{a_x}{r^2} = -\left(m_2 + m_1 + \frac{I}{r^2}\right) a_x$$

Finally, we arrive at

$$a_x = \frac{m_2 g}{m_2 + m_1 + \frac{I}{r^2}}$$

The moment of inertia of a disk is $I = (mr^2)/2$. If we treat the pulley as being a uniform disk of mass m_p, we can simplify the above equation to

$$\boxed{a_x = \frac{m_2 g}{m_2 + m_1 + \frac{m_p}{2}}} \tag{4}$$

Aside:

If you look back in your notes, you'll see that when we ignored the mass of the pulley, we arrived at

$$a_x = \frac{m_2 g}{m_2 + m_1}$$

Furthermore, if we plug this back into the original force balance equations, we see that

$$T_1 = \frac{m_1 m_2 g}{m_2 + m_1}$$

and

$$T_2 = m_2 g - \frac{m_2^2 g}{m_2 + m_1} = \frac{m_2^2 g + m_1 m_2 g}{m_2 + m_1} - \frac{m_2^2 g}{m_2 + m_1} = \frac{m_1 m_2 g}{m_2 + m_1}$$

In other words, if $m_p = 0$ then $T_1 = T_2$.

How much error do we introduce by assuming that $m_p = 0$? Let's plug in some numbers to find out.

$m_1 = 0.2$ kg
$m_2 = 0.1$ kg
$m_p = 0.01$ kg

a_x(with pulley's mass) = 3.22 m/s^2
a_x(without pulley's mass) = 3.27 m/s^2

That's an error of less than 2%.

15.4 Example #2: Ball rolling down a hill

Let's consider a ball rolling down a ramp without slipping. First, recall the rolling constraint. Note also that rolling motion can be thought of as a combination of linear translational motion and rotational motion around the object's center of mass.

[Insert diagram of rolling object.]

[Insert diagram.]

$$\sum F_x = F_g \sin\theta - F_f = ma \tag{5}$$

How do we deal with the frictional force? The ball isn't slipping, yet it is still moving... One way to do this is to consider a reference frame that is moving with the ball, and then calculate the torques around its center.

$$\sum \tau = -rF_f = I\alpha \tag{6}$$

We have seen that $\alpha = a_t/r$. In this case, $a > 0$ implies $\alpha < 0$, so $\alpha = -a/r$ and therefore

$$-rF_f = -I\frac{a}{r} \Rightarrow F_f = I\frac{a}{r^2}$$

Inserting this into the above equation gives

$$F_g \sin\theta - I\frac{a}{r^2} = ma \Rightarrow a\left(\frac{I}{r^2} + m\right) = F_g \sin\theta = mg\sin\theta$$

$$a = \frac{mg\sin\theta}{\frac{I}{r^2} + m}$$

For a solid, uniform sphere, $I = (2/5)mr^2$. Thus,

$$a = \frac{mg\sin\theta}{\frac{(2/5)mr^2}{r^2} + m} = \frac{mg\sin\theta}{(2/5)m + m} = \frac{mg\sin\theta}{(7/5)m}$$

$$\boxed{a = \frac{5}{7}g\sin\theta}$$

Alternative approach
The other way to solve this problem, if you're not comfortable thinking about the rotation of an object in a moving reference frame, is to sum the torques around the point where the ball is in contact with the ramp. Essentially, the gravitational force is not directly above this point and so the ball is constantly "falling". In this approach, the torque is entirely due to the component of the gravitational force that points down the ramp.

$$\sum \tau = -rF_g\sin\theta = I\alpha = -I\frac{a}{r}$$

The moment of inertia of a ball around its center of mass is $(2/5)mr^2$. Because the ball is not rotating around its center of mass in this reference frame, we need to use the parallel-axis theorem, so that

$$I = \frac{2}{5}mr^2 + mr^2 = \frac{7}{5}mr^2$$

Plugging this in, we find that

$$-rmg\sin\theta = -\frac{7}{5}mr^2\frac{a}{r} \Rightarrow \boxed{a = \frac{5}{7}g\sin\theta}$$

Kinematics
If the ball starts at rest, how long will it take the ball to reach the bottom of the ramp and what will its speed be?

Recall:

$$v_f^2 - v_i^2 = 2a\Delta x$$

$$v_f^2 = 2\frac{5}{7}g\sin\theta\frac{H}{\sin\theta} = \frac{10}{7}gH \Rightarrow \boxed{v_f = \sqrt{\frac{10}{7}gH}}$$

$$\Delta x = v_i\Delta t + \frac{1}{2}a\Delta t^2 = \frac{1}{2}a\Delta t^2$$
$$\frac{H}{\sin\theta} = \frac{1}{2}\frac{5}{7}g\sin\theta\Delta t^2$$
$$\Delta t^2 = \frac{5}{14}\frac{g}{H}\sin^2\theta$$

$$\boxed{\Delta t = \sqrt{\frac{5g}{14H}}\sin\theta}$$

16 STATIC EQUILIBRIUM

Objectives:

- Static equilbrium
- Stability
- Newton's first law

16.1 Background

We've spent the last couple of lectures discussing torque, which has many important consequences for physiology.

Demo: If you stand with your toes against a wall, you can't stand on your tip-toes. Why?

Demo: If you stand with your back against a wall, you can't pick up something off the floor a little ways in front of you without falling over. Why?

Other examples:

- Trees have to support large forces and torques (due to the weight of the brances) This affects the development of trees. Apparently tree trunks are structurally different than branches.
- Muscles in your body exert torque on your limbs whenever you move.

I'd like to spend the next couple of lectures discussing equilibrium, stability, and elasticity — topics that we've briefly touched on. I won't really be introducing much new material.

What do we know about object's that are at rest?

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum \tau = 0$$

These equations tell us that an object is fixed in space relative to some reference frame that may or may not be moving.

We have already done problems where $\sum \vec{F} = 0$. Now we'll also enforce that the objects don't rotate. We've already seen that $\tau = rF_{\perp}$. But if the object isn't rotating, what do we choose as our axis of rotation? Turns out that it doesn't matter, we pick any point that we want! We just have to be careful with the sign of the torques.

16.2 Example #1: Board lying across two scales

A 64-kg person stands on a 2-m long board that is lying across two scales. Assume that the board is light (i.e., massless) and rigid. What are the readings on the scales when the person stands 0.5 m from one of the boards (and 1.5 m from the other)?

[Insert diagram.]

We need to add the forces and torques acting on the board.

The force-balance equations give us one equation and two unknowns:

$$\sum F_x = 0$$

$$\sum F_y = F_{n,1} + F_{n,2} - F_g = 0$$

The torque balance gives us one additional equation. We are free to pick a convenient rotation axis. Let's pick a point where one of the unknown forces is acting.

$$\sum \tau = F_{n,1} \cdot 0 + F_{n,2} \cdot 2 \text{ m} - F_g \cdot 1.5 \text{ m} = 0$$

We now have three equations and three unknowns (F_g can be readily calculated). From the torque balance, we can find that

$$F_{n,2} = \frac{3}{4} F_g = \frac{3}{4} mg \Rightarrow \boxed{F_{n,2} = 470 \text{ N}}$$

.

From the force balance,

$$F_{n,1} + \frac{3}{4} F_g - F_g = 0 \Rightarrow F_{n,1} = \frac{1}{4} F_g \Rightarrow \boxed{F_{n,1} = 160 \text{ N}}$$

What if we had picked a different rotation axis? Let's try using the point where $F_{n,2}$ makes contact with the board. In that case,

$$\sum \tau = -F_{n,1} \cdot 2 \text{ m} + F_g \cdot 0.5 \text{ m} = 0$$

$$F_{n,1} = \frac{1}{4} F_g \Rightarrow \boxed{F_{n,1} = 160 \text{ N}}$$

We can actually use the torque balance equation multiple times to come up with additional equations — this problem can be solved with summing the forces in the $x-$ or $y-$directions.

16.3 Demo: balancing meter stick

Use fingers to hold meter stick horizontally. Bring them together, and they will arrive at the stick's center of mass. Why? The normal force, and therefore frictional force, is greatest for the finger that is closest to the center of mass of the meter stick.

16.4 Example #2: ladder leaning against a wall

A 3-m long ladder leans against a frictionless wall. The coefficient of static friction between the floor and the ladder is $\mu_s = 0.2$. At what angle does the ladder start to slide?

[Insert diagram.]

Identify forces and sum the forces and torques.

$$\sum F_x = F_w - F_s = 0 \Rightarrow F_s = F_w$$

$$\sum F_y = F_n - F_g = 0 \Rightarrow F_n = F_g$$

We want to know at what angle does F_w exceed $\max F_s$. Let's play with these equations for a minute before dealing with torque. We know that

$$\max F_s = \mu_s F_n = \mu_s F_g$$

So we need to know when F_w exceeds $\mu_s F_g$.

When can calculate F_w by summing the torques. We can pick any axis of rotation. What would be a convenient one here?

$$\sum \tau = -F_w l \cos\theta + \frac{1}{2} F_g l \sin\theta = 0$$

$$F_w = \frac{1}{2} F_g \tan\theta$$

When is

$$\frac{1}{2} F_g \tan\theta > \mu_s F_g?$$

$$\tan\theta > 2\mu_s$$

$$\theta > \tan^{-1}(2\mu_s) = 21.8^\circ$$

We could also use this to calculate the coefficient of static friction (demo).

16.5 Example #3: person standing near the end of a table

How close can a 70-kg person stand to the end of a 56-kg table before it tips over?

[Insert diagram of the table.]

$$\sum F_x = 0$$
$$\sum F_y = F_{n,1} + F_{n,2} - F_{g,t} - F_{g,p} = 0$$
$$\sum \tau = F_{g,t} l_1 - F_{n,1} l_2 - F_{g,p} L = 0$$

Rearranging the torque balance gives

$$F_{n,1} = \frac{F_{g,t} l_1 - F_{g,p} L}{l_2} = 0$$

(When $F_{n,1} = 0$, the table is just starting to tip.) Solving for L

$$L = \frac{F_{g,t}}{2F_{g,p}} = \frac{m_t}{2m_p} = 0.4 \text{ m}$$

L is the distance from the leg of the table. The person can therefore stand 0.15 m from the end of the table.

17 EQUILIBRIUM AND ELASTICITY

Objectives:

- Stability
- Elastic deformation

Last class we discussed problems involving static equilibrium. When an object is in static equilibrium,

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum \tau = 0$$

When calculating torques, you are free to pick any axis of rotation (the book refers to this as a "pivot point") that you'd like. Torque can be caused by any force, including gravity.

17.1 Stability

Gravitational torque is especially important in the context of stability and balance. The gravitational torque is calculated by applying the gravitational force at the object's center of mass. OK, but what is meant by an object's center of mass?

Center of mass is the "unique point where the weighted relative position of the distributed mass sums to zero". There is a gravitational force acting downward on each point within an object. Adding up the torques produced by each of these forces is equivalent to applying the total gravitational force to the object's center of mass.

We can find the angle at which an object will tip over by calculating the torque as a function of the tilt angle. When $\tau = 0$ the object is either completely stable or meta-stable. If it is meta-stable, tilting it a little bit one way or the other will cause it to topple.

The threshold for determining which direction an object will tip occurs when $\sum \tau = 0$. When we think about torque due to gravity, keep in mind that gravity acts on an object's center of mass.

[Insert diagrams of boxes tipping over.]

Consider the stability of a car.

[Insert diagram of a car with width W and center-of-mass height H.]

When the car is just about to roll, the center of mass of the car is positioned directly above the wheels. In other words, the torque acting about that point is zero. The critical angle, θ_c, at which this happens is related to W and H:

$$\tan\theta_c = \frac{W/2}{H} \Rightarrow \theta_c = \tan^{-1}\frac{W}{2H}$$

With a wider or shorter car, you have to tilt the car to a larger angle before it rolls over.

A related exercise is the ladder leaning against the wall (see previous lecture notes).

17.2 Elasticity

So far in our analysis of static equilibrium we've assumed that objects maintain their shape. In reality, all objects stretch, compress, and deform. We've already talked about this a little bit in the context of normal forces and tensional forces, which are a result of compression and tension occurring at the molecular scale. These forces are "restoring forces" that try to restore the objects to their equilibrium position. Materials that have restoring forces are called "elastic".

We can model elasticity of many materials using Hooke's Law. Recall that

$$F_{sp} = -k\Delta x$$

where k is a spring constant that is unique to a spring. We will replace k with

$$k = \frac{YA}{L}$$

where Y is Young's modulus (a material property), A is the cross-sectional area of the object, and L is the length of the object. An object with a large Young's modulus is resistant to changes in length.

When writing the elastic force, we often replace Δx with ΔL. So

$$F = -\frac{YA}{L}\Delta L \Rightarrow \frac{F}{A} = -Y\frac{\Delta L}{L}$$

F/A is the stress and $\Delta L/L$ is the strain (i.e., fractional change in length).

What does this equation tell us? What if we apply a large force to a thin rope? What if we construct a thicker rope out of the same material?

This equation also tells us that we should think about normal forces (and tensional forces) as being applied everywhere that two surfaces are in contact. However, instead of adding up a whole bunch of little forces, it is much easier to calculate the total force and apply it at the center of where two object's come in contact.

If we were to make a plot of F/A vs. $\Delta L/L$, we would find a linear relationship for small ΔL. This is the range for which Hooke's Law is valid. Beyond that the material will remain elastic for a little bit more deformation until it reaches the elastic limit. Beyond that the material behaves plastically — it changes length will very little change in force — until it reaches the breaking point, or tensile/compressive strength.

[Insert diagram.]

Interestingly, some materials stretch a lot before breaking while others can only stretch a little bit. For example, steel and spider silk have similar tensile strengths (break at the same stress), but silk undergoes much more extension than steel.

[Insert diagram.]

Like torque, elasticity is also a very important concept in biology — for example, it comes up in the context of extension and compression of bones.

17.3 Example #1: Elevator cable stretch

A 2300 kg car of a high-speed elevator is supported by six 1.27-cm-diameter cables. Young's moduls for the cables is 10^{10} N/m^2. When the elevator is on the bottom floor, the cables rise 90 m up the shaft to the motor.

On a busy morning, the elevator is on the bottom floor and fills up with 20 people who have a total mass of 1500 kg. The elevator accelerates upward at 2.3 m/s^2 until it reaches the cruising speed. (1) How much do the cables stretch due to the weight of the car? (2) How much additional stretch occur when the passengers are in the car? (3) And what is the total stretch of the cables when the elevator is accelerating? (In all cases, ignore the weight of the cables.)

[Insert diagram of elevator.]

(0) Stress-strain relationship

$$\frac{F}{A} = Y\frac{\Delta L}{L}$$
$$\Delta L = \frac{FL}{YA}$$

The length is $L = 90$ m.

The area is six times the area of each cable

$$A = 6\pi r^2 = 7.62 \times 10^{-4}\ \text{m}^2$$

(1) Stretch due to car

$$\sum F = F_t - F_g = 0$$
$$F_t = m_{car}g = 22500\ \text{N}$$

$$\boxed{\Delta L = 2.7\ \text{cm}}$$

(2) Stretch due to car and passengers

$$\sum F = F_t - F_g = 0$$
$$F_t = m_{car}g = 37200\ \text{N}$$

$$\boxed{\Delta L = 4.4\ \text{cm}}$$

The cables stretch by 1.7 cm when the passengers board the elevator.

(3) Elevator accelerating upward

$$\sum F = F_t - F_g = ma$$
$$F_t = m(g + a) = 46000\ \text{N}$$

$$\boxed{\Delta L = 5.4\ \text{cm}}$$

The cables stretch by an additional 1 cm.

17.4 Example #2: Plank supported by a rope

A 100-kg, 3.5-m-long plank is supported on its end by a 7.0-mm-diameter rope with a tensile strength of 6.0×10^7 N/m^2. How far along the plank, measured from the pivot, can an 800-kg piece of machinery be moved along the plank before the rope snaps?

[Insert diagram.]

We want to find at what point does $F_t/A = 6.0 \times 10^7$ N/m^2. The cross-sectional area is $A = \pi r^2 = 3.85 \times 10^{-5}$ m^2, so we need to determine when $F_t = 2310$ N.

$$\sum F_y = F_n - F_{g,p} - F_{g,m} + F_t = 0$$

F_t and F_n are unknown, so we need one another equation. We want to remove F_n from the first equation. The way to do this is to sum the torques around the point where the rope is holding up the plank. Let's call the length of the blank L, and the distance from the pivot to the machinery l.

$$\sum \tau = F_{g,p}\frac{L}{2} + F_{g,m}(L-l) - F_n L = 0$$

$$F_n L = F_{g,p}\frac{L}{2} + F_{g,m}(L-l)$$

$$F_n = \frac{1}{2}F_{g,p} + F_{g,m}\left(1 - \frac{l}{L}\right)$$

Let's plug the result into the force balance equation.

$$\frac{1}{2}F_{g,p} + F_{g,m}\left(1 - \frac{l}{L}\right) - F_{g,p} - F_{g,m} + F_t = 0$$

$$-\frac{1}{2}F_{g,p} - F_{g,m}\frac{l}{L} + F_t = 0$$

Solve this for l; we want to find for what l does $F_t = 2310$ N.

$$F_{g,m}\frac{l}{L} = F_t - \frac{1}{2}F_{g,p}$$

$$l = \left(F_t - \frac{1}{2}F_{g,p}\right)\frac{L}{F_{g,m}}$$

Now we can plug in values, and we find that

$$\boxed{l = 0.81 \text{ m}}$$

18 IMPULSE AND MOMENTUM

Objectives:

- Definition of impulse
- Relationship to momentum
- Impulse approximation
- System vs. environment

We have already made a great deal of progress in being able to describe the motion of an object that is subjected to forces. However, there are certain things that are quite difficult to analyze (e.g., collisions) using the ideas that we have developed so far. To more easily understand complex interactions, we often use conservation laws such as the conservation of momentum and the conservation of energy. We will also see that that many of the problems that we've already addressed can be solved much more quickly using conservation of momentum or energy — with the trade-off being that we have to use more abstract concepts.

We will spend the next few weeks discussing these conservation laws.

[Review class flow chart.]

18.1 Demos

Demo #1: Collision of two carts on a track.
Demo #2: Tennis ball and basketball falling; the tennis ball shoots up much higher than expected.

In these two demos, we can describe and predict some of the motion of the objects using the Newton's laws and the concept of forces, but we can't predict the outcome of the collisions.

What are the forces acting on an object during a collision? Let's make a sketch of the the force exerted by a wall as ball bounces off of the wall.

[Insert diagram. Force is zero, grows in magnitude, then returns back to zero as the ball leaves the wall.]

18.2 Impulse

The exact shape of the force-time graph may be quite complicated and unknown. To simplify the situation, we will define the impulse, J, such that

$$\boxed{J = F_{avg}\Delta t}$$

What this tells us is that a large force (think rapid acceleration) applied for a short time interval is equivalent to a small force (think slow acceleration) applied for a long time interval. Graphically, what is the impulse? How does it relate to the force-time graph? Its the area under the force curve. (For those of you taking calculus, impulse is the integral of force with respect to time.)

Impulse can be positive or negative because the average force can be positive or negative. What is it in our example of the ball hitting the wall?

Impulse is actually a vector quantity, so

$$\boxed{\vec{J} = \vec{F}_{avg}\Delta t}$$

18.3 Momentum

What we really want is a way to relate impulse to an object's change in velocity.

What is the $\vec{F}_{avg}$? (Think Newton's Laws...)

$$\vec{F}_{avg} = m\vec{a}_{avg} = m\frac{\Delta\vec{v}}{\Delta t}$$

So, this means that

$$\vec{F}_{avg}\Delta t = m\Delta\vec{v} = m\vec{v}_f - m\vec{v}_i$$

We'll now define a new term, which we'll call the momentum, to be

$$\boxed{\vec{p} = m\vec{v}}$$

Putting this all together, we have

$$\vec{F}_{avg}\Delta t = \vec{p}_f - \vec{p}_i \Rightarrow \boxed{\vec{J} = \Delta\vec{p}}$$

This last expression is the impulse-momentum theorem.

An impulsive force (one that is "short-lived") changes an objects momentum. If the mass of the object is unaffected by the impulsive force, then the impulse changes the object's velocity.

18.4 Example #1: Snowball sticks to wall
A student throws a 120-g snowball at 7.5 m/s at the side of a building, where it hits and sticks. What is the magnitude of the average force on the wall if the duration of the collision is 0.15 s?

Given:
$m = 120$ g
$v_i = 7.5$ m/s
$v_f = 0$ m/s
$\Delta t = 0.15$ s

Solve:

$$J = F_{avg}\Delta t = \Delta p = m\Delta v$$

$$F_{avg} = m\frac{\Delta v}{\Delta t} = 6.0 \text{ N}$$

What would happen if we increased Δt? The force would decrease — Δt saves lives!

18.5 Example #2: Tennis ball bouncing off of a wall
A 60-g tennis ball with an initial speed of 32 m/s hits a wall and rebounds with the same speed (but in the opposite direction). The force history is known: the force rises linearly from 0 to F_{max} in 2 ms, remains constant for 2 ms, and then decreases linearly to 0 in another 2 ms.

[Insert diagram of force history.]

What is the value of F_{max} during the collision?

Hint: impulse is the area under the force curve. So this means that

$$J = F_{max} \cdot 4 \text{ ms} = F_{max} \cdot 0.004 \text{ s}$$

We also know that the change in momentum is

$$\Delta p = m\Delta v = mv_f - mv_i = -mv_i - mv_i = -2mv_i$$

$$J = \Delta p \Rightarrow F_{max} \cdot 0.004 \text{ s} = -2mv_i \Rightarrow F_{max} = 960 \text{ N}$$

Note: because F_{max} is positive, according to the graph, then $v_i < 0$.

18.6 System vs. environment

Now let's analyze the collision of two objects in a little more detail. To do this, we need to define a few terms.

System: the collection of objects whose motion we want to analyze

Environment: all objects external to the system

Internal forces: forces between objects within the system

External forces: forces between the environment and the system

First, let's analyze the case of two carts moving on a horizontal track. If the system consists of the carts, then the net external force on the system is zero. During the collision, the force from cart 1 acting on cart 2 is equal and opposite to the force from cart 2 acting on cart 1.

$$F_{12} = -F_{21}$$

This means that

$$J_{12} = (F_{12})_{\text{avg}}\,\Delta t = -\,(F_{21})_{\text{avg}}\,\Delta t$$

And so

$$\Delta p_2 = -\Delta p_1$$

$$\Delta p_1 + \Delta p_2 = 0$$

$$(p_{1,f} + p_{2,f}) - (p_{1,i} + p_{2,i}) = 0$$

Now we'll define the *total momentum* as being the sum of the momentum of all objects in the system.

$$P = \sum_j p_j$$

Therefore,

$$P_f - P_i = \Delta P = 0$$

This is the *Law of Conservation of Momentum.* It states that the total momentum of the system is constant as long as there is no *net* external force acting on the system. Note that the momentum of individual objects can change.

What do we do if the net external force is not equal to zero???

If the forces acting during the collision are much greater than the net external force, then we can use the *impulse approximation.* During the short time interval of the collision, we can ignore the external forces.

19 CONSERVATION OF MOMENTUM

Objectives:

- Practice problems

Last class:

- Impulse: $\vec{J} = \vec{F}_{avg}\Delta t$ [Insert diagram]
- Impulse approximation: during a collision there may be other forces acting on an object other than the normal force from the collision. During that short time interval, the other forces may be relatively small compared to the impulsive force, and so we can ignore them.
- Momentum: $\vec{p} = m\vec{v}$
- Impulse-momentum theorem (for single particles): $\vec{J} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i$; derived with Newton's Second Law
- Total momentum (for a system of particles): $\vec{P} = \vec{p}_1 + \vec{p}_2 + ... = \sum_i^N \vec{p}_i$
- Conservation of momentum: $\Delta\vec{P} = 0$, or equivalently, $\vec{P}_f = \vec{P}_i$; derived with Newton's Third Law

I'm not going to introduce any more material today; instead I'd like to focus on understanding these concepts through examples. The first two examples will use impulses and the impulse-momentum theorem. Then we'll do some examples with conservation of momentum.

19.1 Example #1: Impulse of ball hitting the ground

A 0.5-kg ball is dropped from rest at a height of 1.2 m. The ball rebounds to a height of 0.7 m. What impulse was applied to the ball? If the collision lasted 0.01 s, what was the average force exerted on the ball by the ground?

Here is an example where will have to use the impulse approximation because gravity is always acting on the ball, including during the collision.

Four parts:
(1) Find the momentum before the collision. What do we need to calculate the momentum?

$$v_f^{\,2} - v_i^{\,2} = 2a\Delta y$$

The initial velocity is zero, and $a = -g = -9.81$ m/s^2. So

$$v_f = -\sqrt{-2g\Delta y}$$

(note that I had to add a negative sign) and so

$$p_1 = mv_f = m\sqrt{-2g\Delta y} = -2.43 \text{ kg} \cdot \text{m/s}$$

This is the momentum immediately before the ball hits the ground.

(2) Find the momentum immediately after the collision. What do we need to calculate the momentum?

$$v_f^{\,2} - v_i^{\,2} = 2a\Delta y$$

Here, $v_f = 0$, so

$$v_i = \sqrt{2g\Delta y}$$

and so

$$p_2 = m\sqrt{2g\Delta y} = 1.85 \text{ kg} \cdot \text{m/s}$$

This is the momentum immediately after the ball leaves the ground.

(3) Calculate the impulse.

$$J = \Delta p = p_2 - p_1 = 4.28 \text{ kg} \cdot \text{m/s}$$

(4) Find the average force.

$$J = F_{avg}\Delta t \Rightarrow F_{avg} = \frac{J}{\Delta t} = 428 \text{ N}$$

How does this compare to the force of gravity acting on the ball?

$$F_g = mg = 4.91 \text{ N}$$

19.2 Example #2: Baseball struck by a bat

A 150-g baseball is travelling horizontally through the air at 35 m/s when it is struck by a baseball bat. Immediately after being hit, the ball travels at a speed of 55 m/s at an angle of 25° from horizontal. What is the impulse delivered to the ball?

[Insert diagram.]

HINT: You need to take into account the fact that impulse and momentum are vectors.

$$\vec{p}_i = \langle p_{x,i}, p_{y,i} \rangle = \langle mv_{x,i}, 0 \rangle = \langle 5.25 \text{ kg} \cdot \text{m/s}, 0 \rangle$$

$$\vec{p}_f = \langle p_{x,f}, p_{y,f} \rangle = \langle -mv_f \cos\theta, mv_f \sin\theta \rangle = \langle -7.48 \text{ kg} \cdot \text{m/s}, 3.48 \text{ kg} \cdot \text{m/s} \rangle$$

$$\vec{J} = \Delta\vec{p} = \langle -12.7 \text{ kg} \cdot \text{m/s}, 3.5 \text{ kg} \cdot \text{m/s} \rangle$$

$$|\vec{J}| = 13.2 \text{ kg} \cdot \text{m/s}$$

19.3 Example #3: Person running on a cart

A person is standing on a long wheeled cart. The person and the cart are both initially stationary. The person weighs 80 kg and the cart is 500 kg. The person starts running until reaching a speed of 8 m/s to the right. What are the speed and direction of the cart?

[Insert diagram.]

$$\Delta P = P_f - P_i = 0$$

$$P_i = m_p v_{p,i} + m_c v_{c,i} = 0$$

So this means that

$$P_f = m_p v_{p,g} + m_c v_{c,g} = 0$$

We are given the velocity of the person relative to the cart, but we wrote P_i with respect to the ground. We need to use relative motion.

$$v_{p,g} = v_{p,c} + v_{c,g}$$

Plugging this is in and rearranging,

$$v_{c,g} = \frac{-m_p}{m_c + m_p} v_{p,c} = -1.10 \text{ m/s}$$

19.4 Example #3: Ice skaters push off of each other

Two skaters, standing at rest at the center of a rink, push off of each other. The skaters weigh 50 and 75 kg, respectively. The rink is 60 m long. It takes 20 s for the heavier skater to reach the edge of the rink. How long does it take the lighter skater?

[Insert diagram.]

Set-up: Once the skaters start moving, the net force on each skater is zero. They will travel at constant velocity. Because they are initially at rest, their initial momentum is zero, and therefore the final total momentum must also be zero.

$$v_1 = \frac{\Delta x_1}{\Delta t_1} = -15 \text{ m/s}$$

$$P_f = p_{1,f} + p_{2,f} = 0 \Rightarrow m_1 v_1 + m_2 v_2 = 0 \Rightarrow v_2 = -\frac{m_1 v_1}{m_2} = 2.25 \text{ m/s}$$

$$\Delta t_2 = \frac{\Delta x_2}{v_2} = 13.3 \text{ s}$$

Note: The most difficult part of solving conservation of momentum problems is often identifying the system. In this particular case, during the initial collision the net force on each skater is nonzero and so the momentum of each skater changes. However, if we define the system to be both skaters, then the net force on the system is zero (i.e., there are no external forces acting on the system).

19.5 Example #4: Spaceship explodes into three parts

A spaceship of mass 2.0×10^6 kg is cruising at a speed of 5.0×10^6 m/s when it explodes into three parts. One section, with mass 5.0×10^5 kg, shots straight backward at a speed of 2.0×10^6 m/s. The second section, with mass 8.0×10^5 kg, shoots off at a $90°$ angle at a speed of 1.0×10^6 m/s. What are the direction and speed of the third piece?

This is a conservation of momentum problem. Recall that momentum is a vector quantity.

[Insert diagram.]

Initial momentum:

$$\vec{P}_i = m\vec{v}_i = \langle 10^{13} \text{ kg} \cdot \text{m/s}, 0 \rangle$$

Final momentum:

$$\vec{P}_f = \vec{p}_{1,f} + \vec{p}_{2,f} + \vec{p}_{3,f} = \vec{P}_i$$

$$\vec{p}_{1,f} = \langle -10^{12} \text{ kg} \cdot \text{m/s}, 0 \rangle$$

$$\vec{p}_{2,f} = \langle 0.8 \times 10^{11} \text{ kg} \cdot \text{m/s} \rangle$$

So, this means that we have

$$\langle 10^{13} \text{ kg} \cdot \text{m/s}, 0 \rangle = \langle -10^{12} \text{ kg} \cdot \text{m/s}, 0 \rangle + \langle 0, 8 \times 10^{11} \text{ kg} \cdot \text{m/s} \rangle + \langle p_{3,x}, p_{3,y} \rangle$$

And so

$$p_{3,x} = 1.1 \times 10^{13} \text{ kg} \cdot \text{m/s}$$

$$p_{3,y} = -8.0 \times 10^{11} \text{ kg} \cdot \text{m/s}$$

To find the velocity, we need to know that the mass of the third piece is $m_3 = 7 \times 10^5$ kg.

$$\vec{v}_3 = \frac{\vec{p}_3}{m_3} = \langle 1.67 \times 10^7 \text{ m/s}, -1.1 \times 10^6 \text{ m/s} \rangle$$

20 ANGULAR MOMENTUM

Objectives:

- Angular momentum
- Conservation of angular momentum

20.1 Background

We've been treating momentum as a linear (but two-dimensional quantity). The momentum $\vec{p}$ of a particle spinning in a circle is not conserved, even if it moves at constant speed, because its velocity vector is continuously changing direction as a result of some external force. However, there is a quantity called the angular momentum that is conserved.

Today I'll talk about angular momentum and conservation of angular momentum, and also try to give you some more insights into the relationship between torque and force.

Recall the definition of torque,

$$\tau = rF\sin\theta$$

We saw that by summing the torques acting on an object,

$$\sum \tau = I\alpha.$$

The angular acceleration is the rate of change of angular velocity,

$$\alpha = \frac{\Delta\omega}{\Delta t}.$$

To account for the possibility that α can change with time, we can write the average torque as

$$\tau_{avg} = I\frac{\Delta\omega}{\Delta t}.$$

Rearranging,

$$\tau_{avg}\Delta t = I\Delta\omega.$$

This is analogous to what we had for linear motion:

$$\vec{F}_{avg}\Delta t = m\Delta\vec{v}.$$

So let's define angular momentum as

$$\boxed{L = I\omega}.$$

This equation tells us that, just like with linear momentum, an object can have a large angular momentum if it is spinning fast or has a large moment of inertia. What gives an object a large moment of inertia?

Putting this altogether,

$$\boxed{\tau_{avg}\Delta t = \Delta L}$$

which is the rotational equivalent of

$$F_{avg}\Delta t = \Delta p$$

If the net torque acting on a $s$ystem of objects is zero, then

$$\tau_{avg}\Delta t = 0 = I\Delta\omega = \Delta L.$$

This equation says that the change in angular momentum is zero if the net external torque acting on the system is zero. This is exactly analogous to the law of conservation of momentum, and we call this the law of conservation of angular momentum. Another way to write this is

$$\boxed{I_f \omega_f = I_i \omega_i}.$$

20.2 Example #1: Figure skater

A common example of the conservation of angular momentum that you've all seen is that of a figure skater spinning in circles. When they hold their arms and legs out they spin slowly, and when they pull them in tight they spin more quickly. To analyze this situation, let's treat a figure skater's hands as point masses and ignore the rest of their body. (This allows us to work with analytical expressions for the moment of inertia.)

A skater spins around on the tips of his blades while holding 5.0-kg weights in each hand. He begins with his arms straight out from his body and his hands 140 cm apart. While spinning at 2 rev/s, he pulls the weights in and holds them 50 cm apart against his shoulder. If we neglect the mass of the skater, how fast is he spinning when he pulls the weights in?

[Insert diagram of skater from above.]

There are no external torques acting on the skater (we're ignoring the friction between the skates and the ice, and any air resistance). This means that the skater's rotation follows the law of conservation of angular momentum, and so

$$I_i \omega_i = I_f \omega_f.$$

For a point mass moving in a circle, the moment of inertia is mr^2. Since we have two point masses, $I = 2mr^2$. Plugging this into the above equation yields

$$2m{r_i}^2\omega_i = 2m{r_f}^2\omega_f.$$

We want to solve for ω_f. So

$$\omega_f = \frac{{r_i}^2}{{r_f}^2}\omega_i = 16 \text{ rev/s} = 32\pi \text{ rad/s}.$$

20.3 Example #2: ???

Example: Pick another example...

20.4 Demos with rotating stand

If there are no external forces, then $L_i = L_f$. Discuss vector notation for $\vec{L}$.

1. Rotate in place and stand rotates the other direction. Think about how to define the system.

   Initially, $L_i = 0$, which means that $L_f = 0$. If I rotate counterclockwise, the stand has to rotate the opposite direction.

   [Diagram indicating momentum vectors.]

2. Stand with rotating wheel, and cause it to rotate the other direction. Why does this happen? It again has to do with conservation of angular momentum. I change the direction that it rotates, so the system has to respond by rotating the opposite direction.

   [Diagram indicating momentum vectors.]

3. Gyroscopic effect: Rotating wheel that doesn't fall. Why not?

   If I spin the wheel, there is angular momentum that is oriented along the wheel's axis. Once I start hanging this from a string, there is also a gravitational torque that is trying to cause downward rotation. Recall that

   $$\tau = I\alpha$$

   Let's keep this simple and think of α as a constant. Therefore,

   $$\tau = I\frac{\Delta\omega}{\Delta t}.$$

   Rearranging,

   $$\tau\Delta t = I\Delta\omega = \Delta L.$$

   So in other words, torque causes a change in angular momentum. The change in angular momentum caused by the gravitational torque points perpendicular to the wheel's angular momentum. After some amount of time, Δt, the new angular momentum becomes

$$L_{new} = L + \Delta L$$

[Draw vectors showing how this changes momentum vector. This means that the wheel must change direction.]

What if the wheel isn't spinning very fast? Then ΔL is quite a bit larger than L, and so $L_{new} \approx \Delta L$. The wheel doesn't rotate around the vertical axis very much, it just falls downward.

21 INTRODUCTION TO ENERGY

Objectives:

- Definitions
- Work
- Kinetic energy
- Gravitational potential energy

21.1 Work-energy theorem

Who has heard of energy? What is energy? What kinds of energy have you heard of?

Don't worry, you're not the only one that doesn't understand energy. Here is a quote from Richard Feynman, a Nobel prize winner in physics and one of the more colorful characters in physics.

> It is important to realize that in physics today, we have no knowledge of what energy is. We do not have a picture that energy comes in little blobs of a definite amount. It is not that way. However, there are formulas for calculating some numerical quantity…It is an abstract thing in that it does not tell us the mechanism or the reasons for the various formulas. (Richard Feynman)

Energy is

- an indirectly observed scalar (not vector!) quantity
- the ability of one system to do work on another system

The concept of energy is extremely powerful because, like momentum, it allows us to ignore some complex interactions. Even more importantly, energy is what allows us to bridge together different fields of science. To me, this is where physics starts to become really exciting. Although energy is a rather abstract idea, it is also extremely important.

There are many different types of energy:

- kinetic energy, K
- potential (stored) energy, U (we'll discuss a couple of types of potential energy)
- chemical energy, E_{chem}
- thermal energy, E_{th}
- ...

We are going to define the total energy of a system as the sum of all of these energies,

$$E = K + U + E_{chem} + E_{th} + ...$$

Energy can be transformed from one form to another (within the system). Can you think of some examples?

- The chemical energy stored in wood can be transformed into thermal energy.
- The chemical energy of food is converted into kinetic energy when you move.
- A falling object loses potential energy and gains kinetic energy.

Some transformations are "easier" than others. Its "easy" to create thermal energy, but difficult to turn thermal energy into potential energy or kinetic energy.

Energy transformation: Changes in energy within a system.
Energy transfer: Energy exchange between the system and the environment.
Environment: Everything not in the system...

Just like when we talked about momentum, we can define the system in any way that is convenient.

Energy transfer occurs as "work" (i.e., a mechanical transfer of energy) or "heat" (i.e., a thermodynamic transfer of energy due to a temperature difference).

If there is no heat being transferred into the system, then the work done on a system changes the energy of the system

$$\Delta E = W$$

This means that

$$\Delta E = \Delta K + \Delta U + \Delta E_{chem} + \Delta E_{th} + ... = W.$$

This is referred to as the work-energy theorem.

If our system is isolated (i.e., there is no work being done to or by the system), $W = 0$ and

$$\Delta E = 0.$$

Any idea what this is referred to? Its the conservation of energy, one of the most powerful concepts in classical mechanics.

OK, so this is all nice and dandy, but what are K, U, E_{chem}, E_{th}, W, ...?

21.2 Work

Let's first discuss work, the mechanical transfer of energy by an external force. We'll start by considering a constant external force, and we'll define work as

$$W = Fd,$$

where F is the external force applied to the system and d is the displacement of the system. F has to be parallel to d; if it isn't, we need to find the component of F that is parallel to d. Work has units of joules, with 1 J = 1 N · m. Don't confuse this with torque, even though the units are the same!

By the work-energy theorem, this means that

$$\Delta E = Fd.$$

21.2.1 Example #1: Pushing a book across a table

You push your physics book across the table at a constant rate. How much work do you do to the book?

First, how much force are you exerting?

$$\sum F_x = F - F_k = ma = 0$$

$$F = F_k = \mu_k F_n$$

$$\sum F_y = F_n - F_g = 0$$

$$F_n = F_g = mg$$

and therefore

$$F = \mu_k mg$$

$$W = Fd = \mu_k mgd$$

If the book weighs 1 kg, $\mu_k = 0.3$, and $d = 1$ m, then $\boxed{W = 2.9 \text{ J}}$.

You then drop the book into the garbage. How much work does gravity do to the book?

$$W_g = -F_g \Delta y = -mg\Delta y$$

We need to be careful here. From the definition of work, we need the force that points in the direction of displacement. If our coordinate system points up, then F_g points in the negative direction and $\Delta y < 0$. If the book weighs 1 kg and it falls a distance of 1 m, the work done by gravity is

$$\boxed{W_g = 9.81 \text{ J}}.$$

21.2.2 Example #2: Pulling a crate

You pull a crate with a rope at a constant velocity for 3 m. The tension in the rope is 70 N. How much work is done to the crate by the rope, and where does this energy go?

[Insert diagram.]

$$W = F_t \cos\theta \cdot d = 182 \text{ J}$$

As we'll see, the kinetic energy and potential energy of the crate is unchanged; the 182 J goes into thermal energy from friction.

21.3 Kinetic Energy

Let's consider what happens if you move an object with constant force and no friction or drag. For example, we could be using a rope to pull a box across a frictionless surface.

$$\sum F_x = F_{ext} = ma_x \Rightarrow a_x$$

The work that is done by force F_{ext} is

$$W = F_{ext}\Delta x = ma\Delta x$$

Recall from kinematics that

$$2a\Delta x = {v_f}^2 - {v_i}^2$$

Plugging this in and rearranging, we find that

$$W = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2$$

Let's define the kinetic energy (energy of motion) as

$$\boxed{K = \frac{1}{2}mv^2}$$

Note that v is the *speed*, not the velocity vector.

In this particular example,

$$W = \Delta K.$$

We can do a similar analysis for angular motion, and we find that

$$\boxed{K_{rot} = \frac{1}{2}I\omega^2}$$

The total kinetic energy of an object is

$$K_{total} = K_{trans} + K_{rot}.$$

21.4 Gravitational Potential Energy

What if instead we had moved the object vertically some distance Δy at a constant speed with some force F_{ext}?

$$\sum F_y = F_{ext} - F_g = ma_y$$

Since the object is moving at a constant speed, $a_y = 0$ and therefore

$$F_{ext} = F_g = mg.$$

The work done on the object is

$$W = F_{ext}\Delta y = mg\Delta y = \Delta E.$$

We will call

$$\boxed{\Delta U_g = mg\Delta y}$$

the change in gravitational potential energy (or often, just potential energy). Potential energy is always relative to some reference state, and so we always have to talk about changes in potential energy. Notice also that the path that we take doesn't affect the solution.

Potential energy is a stored energy; forces that store energy are called conservative.

What if we hadn't required that $a_y = 0$?

$$\sum F_y = F_{ext} - F_g = ma_y$$
$$F_{ext} = mg + ma_y$$
$$W = F_{ext}\Delta y = mg\Delta y + ma_y\Delta y$$

Using results from above,

$$W = mg\Delta y + \frac{1}{2}m\left({v_f}^2 - {v_i}^2\right)$$
$$W = \Delta U_g + \Delta K$$

21.4.1 Example #3: Ball dropped from rest

A ball is dropped from a height of 1 m. What is its speed right before it hits the ground?

The old method, using forces and kinematics:

$$\sum F_y = -F_g = ma_y \Rightarrow a_y = -g$$
$${v_f}^2 - {v_i}^2 = 2a_y\Delta y \Rightarrow {v_f}^2 = -2g\Delta y \Rightarrow \boxed{v_f = \sqrt{-2g\Delta y}}$$

The new method, using conservation of energy (define system to include ball and the Earth):

$$\Delta E = \Delta K + \Delta U_g = 0$$
$$\frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2 + mg\Delta y = 0 \Rightarrow \boxed{v_f = \sqrt{-2g\Delta y}}$$

21.4.2 Example #4: Box sliding down a frictionless ramp

A box slides down a frictionless ramp. What is its speed at the end of the ramp?

The old method, using forces and kinematics:

$$\sum F_x = F_g \sin\theta = ma_x \Rightarrow a_y = -g\sin\theta$$

Then, same as with the previous example,

$$v_f = \sqrt{-2g\sin\theta\Delta y},$$

but $\Delta y = -H/\sin\theta$, so

$$\boxed{v_f = \sqrt{-2gH}}$$

The new method, using conservation of energy (define system to include the ramp and the box):

$$\Delta K + \Delta U_g = 0 = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2 + mg\Delta y$$

$H = -\Delta y$ and $v_i = 0$, so

$$\boxed{v_f = \sqrt{-2gH}}$$

22 THERMAL ENERGY

Objectives:

- Friction
- Inelastic collisions
- Elastic collisions

22.1 Background

Recall from last class:

- $W = F_{ext} \cdot d = \Delta E$; note that F_{ext} and d should point in the same direction. This is the work-energy theorem.
- If $F_{ext} = 0 \Rightarrow \Delta E = 0$. This is called the conservation of energy. Conservation of energy, as with conservation of momentum, is especially useful when we aren't interested in time scales.
- Total energy of a system is $E = K + U + E_{th} + ...$
- One way to think of energy is the ability of a system to do work on another system
- An external force applied to a system does work on the system. It therefore increases the total energy of the system.
- When the system does work on the environment, the total energy of the system decreases.
- Many different types of energy: K, U, E_{th}, ...
- Kinetic energy: $K = \frac{1}{2}mv^2$
- Gravitational potential energy: $\Delta U_g = mg\Delta y$

In lab you witnessed changes in thermal energy due to friction and due to collisions.

[Insert graph of kinetic energy vs. time for inelastic collision.]

The kinetic energy of the cart(s) was not conserved because it was getting transformed into thermal energy.

22.2 Thermal Energy Due to Friction

We'll start with a simple system: a box is pulled across a horizontal, rough surface. What is the work done on the box by friction?

$$\sum F_x = F_t = ma_x$$

The only external force acting on the system is the tension from the rope, which has to equal the frictional force between the box and the floor (otherwise the box wouldn't move with constant speed). Defining the work is therefore easy:

$$W = F_t \Delta x = F_k \Delta x$$

This work causes molecules at the box's surface to vibrate (i.e., heat up), so we refer to this as thermal energy.

$$\boxed{\Delta E_{th} = F_k \Delta x = \mu_k F_n \Delta x}$$

(In the case of your experiments, its actually a bit more complicated because thermal energy was created on the spinning axles. The idea is basically the same though.)

22.2.1 Example #1: Sledding down a hill

You sled down a 3-m tall hill. The hill is frictionless, but there is some soft snow at the bottom that has a coefficient of kinetic friction of $\mu_k = 0.05$. The ground at the bottom of the hill is horizontal. How far will you slide before coming to rest?

[Insert diagram of hill.]

How would we have solved this previously? Its fairly involved, but we could do this using forces and kinematics. But its super easy using conservation of energy.

$$\Delta E = \Delta K + \Delta U_g + \Delta E_{th} = 0$$

. You start at rest and finish at rest, so $\Delta K =$. This equation therefore becomes

$$mg\Delta y + F_k \Delta x = 0,$$

where Δx is the distance that you travel after reaching the bottom of the hill. Recall that the frictional force is given by

$$F_k = \mu_k F_n$$

In this problem, it easy to see that $F_n = F_g = mg$ when you are sliding on horizontal ground. So, this means that

$$mg\Delta y + \mu_k mg\Delta x = 0$$

$$\Delta y + \mu_k \Delta x = 0 \Rightarrow \boxed{\Delta x = \frac{-\Delta y}{\mu_k} = 60 \text{ m}}$$

Note that the solution does not depend on the angle or length of the hill!

22.3 Thermal energy of inelastic collisions

Extreme case: An object collides inelastically with a stationary object (e.g., wall, floor, etc). All of the kinetic energy is converted to thermal energy, so $\Delta E_{th} = K_i$, where K_i is the kinetic energy before the collision.

In the case that both objects move, the result is a bit different. What did you see in lab?

[Insert diagrams of velocity vs. time, momentum vs. time, and kinetic energy vs. time.]

You should've seen that momentum was conserved, but kinetic energy was not. And because we know that the object's stick together, its easy to determine how much thermal energy is generated during the collision. (Is it easy? How would we do it?)

Start with conservation of momentum. We'll consider the specific case where one of the carts is initially stationary. (We could do the more general problem, but it gets messy.)

$$P_i = m_1 v_i = (m_1 + m_2) v_f = P_f$$

$$v_f = \frac{m_1}{m_1 + m_2} v_i$$

Now we want to figure out how much thermal energy is generated, which we do by using conservation of energy.

$$\Delta E = 0 = \Delta K + \Delta E_{th}$$

$$\Delta E_{th} = -\Delta K = -(K_f - K_i) = K_i - K_f$$

Recalling that for each object in a system

$$K = \frac{1}{2} m v^2$$

we find that

$$K_i = \frac{1}{2} m_1 {v_i}^2$$

$$K_f = \frac{1}{2} (m_1 + m_2) {v_f}^2$$

Now let's plug v_f into the K_f,

$$K_f = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2} v_i\right)^2 = \frac{1}{2}\frac{{m_1}^2}{m_1 + m_2}{v_i}^2$$

Oh, but notice that this is the same as

$$K_f = K_i \left(\frac{m_1}{m_1 + m_2} \right)$$

Okay, now taking $K_i - K_f$,

$$\Delta E_{th} = K_i - K_i \left(\frac{m_1}{m_1 + m_2} \right) = \boxed{K_i \left(1 - \frac{m_1}{m_1 + m_2} \right)}$$

- What if m_2 is really small (e.g., a car hitting a bug)? Almost none of the kinetic energy is transformed into thermal energy.
- What if $m_1 \approx m_2$, as with the carts hitting each other? $\Delta E_{th} = \frac{1}{2} K_i$; half of the energy is transformed into thermal energy.
- What if m_2 is extremely large (e.g., a car hitting a building)? All of the kinetic energy is transformed into thermal energy.

22.4 Thermal Energy of Perfectly Elastic Collisions

Perfectly elastic collisions involve no change in kinetic energy (i.e., kinetic energy is conserved). We can use $P_i = P_f$ and $K_i = K_f$ to solve for the motion of objects that collide elastically.

22.5 Thermal Energy of Elastic Collisions

Elastic collisions are often described using a coefficient of restitution.

$$v_f = C_r v_i$$

where C_r is empirically determined and $0 \leq C_r \leq 1$.

From this, we can ask how much heat is generated during a collision?

Again, we'll consider a simple case to keep the algebra relatively simple. Let's consider a cart moving on a frictionless track that collides with a wall. Define the cart and the track/ground as being the system, so that there are no external forces.

$$0 = \Delta E = \Delta K + \Delta E_{th}$$

Here, ΔE_{th} refers to the thermal energy generated by the collision. So,

$$0 = \frac{1}{2} m {v_f}^2 - \frac{1}{2} m {v_i}^2 + \Delta E_{th}$$

$$0 = \frac{1}{2} m (C_r v_i)^2 - \frac{1}{2} m {v_i}^2 + \Delta E_{th}$$

$$0 = \frac{1}{2} m {C_r}^2 {v_i}^2 - \frac{1}{2} m {v_i}^2 + \Delta E_{th}$$

$$0 = \frac{1}{2} m {v_i}^2 ({C_r}^2 - 1) + \Delta E_{th}$$

$$\Delta E_{th} = \frac{1}{2} m {v_i}^2 (1 - {C_r}^2) = K_i (1 - {C_r}^2)$$

22.5.1 Example #2: Ball bouncing on the ground

A ball is dropped from a height of 1 m. The coefficient of restitution between the ball and the floor is 0.8. How high does the ball bounce?

$$\Delta E = 0 = \Delta K + \Delta U_g + \Delta E_{th}$$

Initially and at the peak, $K = 0$ so $\Delta K = 0$. So

$$0 = \Delta U_g + \Delta E_{th} = mg\Delta y + K_b(1 - {C_r}^2),$$

where K_b is the kinetic energy right before the collision. If we take the ground as our reference state, then $K_b = U_i = mgy_i$ (all of the energy goes into kinetic energy). Therefore,

$$0 = mg\Delta y + mgy_i(1 - {C_r}^2) = y_f - y_i + y_i - y_i{C_r}^2 = y_f - y_i{C_r}^2$$

And so

$$y_f = y_i{C_r}^2 = 1\text{ m} \cdot (0.8)^2 = 0.64\text{ m}$$

(Notice that this equation here is one simple way to figure out an object's coefficient of restitution.)

23 ELASTIC POTENTIAL ENERGY

Objectives:

- Work-energy exercises
- Elastic potential energy: energy associated with extension/compression of springs

23.1 Background

Recall from last class:

- Work: $W = Fd = \Delta E = \Delta K + \Delta U_g + \Delta E_{th} + ...$
- Translational kinetic energy: $K_{trans} = \frac{1}{2}mv^2$
- Rotational kinetic energy: $K_{rot} = \frac{1}{2}I\omega^2$
- Gravitational potential energy: $\Delta U_g = mg\Delta y$; always from some reference state
- Thermal energy due to friction: $\Delta E_{th} = F_k d$

Today I'm going to introduce one more type of energy, but before doing so let's do a couple of exercises.

23.2 Example #1: Object rolling down a ramp

Let's look at the velocity of an object *rolling* down a ramp. Define the system as the being the object and the ramp, so there are no external forces. We will assume that the object starts at rest.

$$\Delta E = 0 = \Delta K + \Delta U_g$$

In this problem, there is no sliding, and so $\Delta E_{th} = 0$. In this problem we have to worry about rotational kinetic energy.

$$0 = \Delta K_{trans} + \Delta K_{rot} + \Delta U_g = \frac{1}{2}m{v_f}^2 + \frac{1}{2}I{\omega_f}^2 + mg\Delta y$$

We have seen previously that

$$v_t = \omega r.$$

In this problem, $v > 0$, but $\omega < 0$. This means that

$$\omega = \frac{-v}{r}.$$

Plugging this into the energy balance equation gives

$$0 = \frac{1}{2}m{v_f}^2 + \frac{1}{2}I\left(\frac{v_f}{r}\right)^2 + mg\Delta y = {v_f}^2\left(m + \frac{I}{r^2}\right) + 2mg\Delta y$$

Rearranging,

$${v_f}^2 = \frac{-2mg\Delta y}{m + \frac{I}{r^2}}$$

or

$$\boxed{v_f = \sqrt{\frac{-2mg\Delta y}{m + \frac{I}{r^2}}}}$$

We did essentially the same exercise earlier in the semester, and during lab. Let's write this in terms of acceleration by using the kinematic equations for constant acceleration.

$$v_f{}^2 - v_i{}^2 = 2a\Delta x \Rightarrow a = \frac{v_f{}^2}{2\Delta x}$$

$$a = \frac{1}{2\Delta x}\frac{-2mg\Delta y}{m+\frac{I}{r^2}} = \frac{2mg(-\Delta y/\Delta x)}{m+\frac{I}{r^2}}$$

Notice that $-\Delta y/\Delta x = \sin\theta$, so this gives

$$\boxed{a = \frac{2mg\sin\theta}{m+\frac{I}{r^2}}}$$

This is exactly the same solution as what we arrived at previously.

23.3 Example #2: Tennis ball bouncing off of a basketball
Mass of tennis ball = 0.075 kg Mass of basketball = 0.50 kg

[Insert diagram.]

Assume: perfectly elastic collisions ($K_i = K_f$) and that $v_{2i} \approx -v_{1i}$.

From conservation of momentum:

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

and so

$$m_1v_{1i} - m_2v_{1i} = m_1v_{1f} + m_2v_{2f}$$

or equivalently,

$$\boxed{(m_1 - m_2)v_{1i} = m_1v_{1f} + m_2v_{2f}}$$

We are trying to relate v_{1f} to v_{1i}. Need another equation in order to remove v_{2f}. Since we are assuming that the collision is *perfectly elastic*, we can set $\Delta K_1 + \Delta K_2 = 0$. That is,

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^{+}\frac{1}{2}m_2v_{2f}^2$$

Make same substitution as before and multiply by two, so that

$$\boxed{(m_1 + m_2)v_{1i}^2 = m_1v_{1f}^2 + m_2v_{2f}^2}$$

Now do a bunch of messy algebra to arrive at

$$\boxed{v_{1f} = \left(\frac{m_1 - 3m_2}{m_1 + m_2}\right) v_{1i}}$$

For the masses given,

$$v_{1f} \approx -2.5 v_{1i}$$

23.4 Elastic potential energy

We need to define one more type of energy (for now), which is elastic potential energy. Springs store energy. If we compress or extend the spring at a constant rate, we can find how much energy is stored in the spring. We'll apply an external force to the spring, so that

$$W = \Delta E = \Delta U_s$$

W is the work needed to compress or extend the spring, and is $W = F_{avg}\Delta x$, where Δx is the displacement of the spring. The force that we apply to compress or extend the spring has to balance the spring force (recall that $F_s = -k\Delta x$).

[Insert diagram of applied force, varying linearly with x.]

From this figure, we can see that the average force is

$$F_{avg} = \frac{1}{2} k \Delta x$$

and so the elastic potential is

$$\Delta U_s = \frac{1}{2} k \Delta x^2$$

We typically define $U_s = 0$ when the spring is in its equilibrium length, so we can write

$$\boxed{U_s = \frac{1}{2} k \Delta x^2}$$

Note that this is always positive. The spring stores energy whether it is in extension or compression.

23.4.1 Example #3: How much energy stored in a spring?

How far must you stretch a spring to store 200 J if $k = 1000$ N/m?

$$U_s = \frac{1}{2} k \Delta x^2$$

$$\Delta x = \sqrt{\frac{2U_s}{k}} = 0.63 \text{ m}$$

23.4.2 Example #4: Spring shoots marble off a table

A spring is clamped to a table. You compress the spring a distance of 0.2 m, and use the spring to shoot a marble horizontally. The marble, which has a mass of 0.02 kg, travels a distance of 5 m (horizontally) and 1.5 m (vertically). What is the spring constant?

There are (at least) a couple of ways to solve this, both making use of $\Delta E = 0$.

Most direct way: let t_i be when the spring is fully compressed, and t_f be the time when the marble hits the ground. Conservation of energy tells us that:

$$0 = \Delta E = \Delta K + \Delta U_g + \Delta U_s = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2 + mg\Delta y - \frac{1}{2}k\Delta x^2$$

We know that $v_i = 0$, so this reduces to

$$0 = \frac{1}{2}m{v_f}^2 + mg\Delta y + \frac{1}{2}k\Delta x^2$$

What is v_f? It is the final speed, so we need to find

$$v_f = \sqrt{{v_{x,f}}^2 + {v_{y,f}}^2}.$$

We can figure out ${v_{x,f}}^2$ and ${v_{y,f}}^2$ using kinematics. First let's find that it takes the marble to fall to the ground.

$$\Delta y = v_{y,i}\Delta t + \frac{1}{2}a_y\Delta t^2 \Rightarrow \boxed{\Delta t^2 = \frac{-2\Delta y}{g}}$$

So

$${v_{x,f}}^2 = \left(\frac{L}{\Delta t}\right)^2 \Rightarrow \boxed{{v_{x,f}}^2 = \frac{gL^2}{-2\Delta y}}$$

How about ${v_{y,f}}^2$?

$$v_{y,f} - v_{y,i} = a_y\Delta t$$

$$v_{y,f}^2 = g^2\Delta t^2 = g^2\frac{-2\Delta y}{g} \Rightarrow \boxed{{v_{y,f}}^2 = -2g\Delta y}$$

This means that

$$\boxed{{v_f}^2 = \frac{gL^2}{-2\Delta y} - 2g\Delta y}$$

Now, plugging this into the conservation of energy equation,

$$0 = \frac{1}{2}m\left(\frac{gL^2}{-2\Delta y} - 2g\Delta y\right) + mg\Delta y - \frac{1}{2}k\Delta x^2$$

$$0 = \frac{mgL^2}{-4\Delta y} - mg\Delta y + mg\Delta y - \frac{1}{2}k\Delta x^2$$

$$0 = \frac{mgL^2}{-4\Delta y} - \frac{1}{2}k\Delta x^2$$

$$k = \frac{mgL^2}{-2\Delta y\Delta x^2} = \boxed{48.9 \text{ N/m}^2}$$

24 THERMODYNAMICS I

Objectives:

- First Law of Thermodynamics — Conservation of Energy
- Second Law of Thermodynamics — Entropy
- Latent heat
- Specific heat

24.1 Background

The last few lectures we have been discussing the idea of energy and work, and we've focussed primarily on mechanical energy. Recall the work-energy theorem:

$$W = \Delta E = \Delta K + \Delta U_g + \Delta E_{th} + ...$$

But what is thermal energy? Its energy associated with motion of particles (i.e., atoms, molecules, electrons, protons) within a material. For basic friction, we saw that

$$\Delta E_{th} = F_k \Delta x$$

As the thermal energy increases, more molecules vibrate and so the temperature of the system increases.

The thermal energy can also change if the molecular vibrations cause molecules outside of the system to vibrate — this is a transfer of kinetic energy at the molecular scale, and it only happens when there is a temperature difference between the system and the environment. Thermal energy that is exchanged between a system and the environment due to a temperature difference is referred to as *heat*.

How is heat transferred between two objects? In general terms... Take two objects at different temperatures. This means that their molecules are vibrating at different rates. Place the objects next to each other, and there are essentially millions of collisions occurring between the objects. Elastic collisions transfer energy from the warm object to the cold object until the kinetic energy of the molecules is the same in both objects.

[Insert diagram.]

24.2 First Law of Thermodynamics

Let's modify the work-energy equation to account for heat.

$$W + Q = \Delta E,$$

where Q is the heat exchanged between the system and the environment. If $W, Q > 0$, energy goes into the system. If $Q, W < 0$, energy leaves the system. This equation is the first law of thermodynamics. It states that energy cannot be created or destroyed.

Consider a system in which $\Delta E = \Delta E_{th}$ (no other forms of energy are changing). Let's say you pump air into a tire. What happens?

- You're doing work ($W > 0$), so $\Delta E_{th} > 0$ and the temperature increases (due to an increase in air pressure).

- Eventually heat is lost to the environment, $\Delta E_{th} < 0$, and so $Q < 0$.

Let's now consider a couple of common applications of the first law of thermodynamics.

(1) Heat generally flows from warm reservoirs to cold reservoirs. Heat pumps work against this natural flow — refrigerators use heat pumps. In a really simple model of a refrigerator, you have a cold reservoir (where you store food), a hot reservoir (the room), and a heat pump that runs on electricity. If we think of the heat pump as being the system, there is work being done on the system by electricity, and heat flowing into and out of the system.

[Diagram of a heat pump.]

$$W_{in} + Q_c + Q_h = \Delta E$$

If the energy of the heat pump is constant, then

$$W_{in} + Q_c + Q_h = 0$$

and so

$$Q_h = W_{in} + Q_c$$

This tells us that more heat flows out of the heat pump than flows in.

(2) Heat engines are basically the opposite of heat pumps. They follow the natural flow of heat — and take advantage of it to do work.

[Diagram of a heat engine.]

$$Q_h - Q_c - W_{out} = 0$$

$$W_{out} = Q_h - Q_c$$

The right hand side of this equation depends on temperature differences (we'll get to that later). So the larger the temperature difference, the more work that the engine can do.

The efficiency of an engine is

$$e = \frac{\text{what you get}}{\text{what you paid}} = \frac{Q_h - Q_c}{Q_h}$$

From the second law of thermodynamics, which I'll discuss in a minute,

$$e = \frac{T_h - T_c}{T_h} = 1 - \frac{T_c}{T_h}$$

.

This is the maximum efficiency of an engine. Note that $e < 1$ and e increases as the temperature difference increases.

24.3 Second Law of Thermodynamics

We already saw that heat naturally flows from hot to cold.

[Insert diagram. Molecules in a hot object have higher kinetic energy than molecules in a cold object.]

Hot molecules are more likely to increase the energy of the cold molecules... Sometimes its easier to think of this in terms of two gases that are initially separated. The hot molecules move faster and are therefore more likely to cross the initial boundary and warm up other molecules.

Its possible, but highly unlikely, that the system would evolve the other way (that the cold molecules would transfer energy to the warm molecules and cause them to vibrate more quickly). We use the term *entropy* to describe this unlikeliness.

The Second Law of Thermodynamics tells us that the entropy of an *isolated system* never decreases. (The Second Law can be shown empirically, or derived from statistical mechanics...) As a result, order goes to disorder and kinetic and potential energies are "lost" to thermal energy.

This explains why engines can't be 100% efficient. If all incoming heat is converted to work, the entropy of the system would decrease.

24.4 Specific and Latent Heat
What happens when heat is transferred to or away from an object?

There are basically two possibilities. The object changes temperature or it changes phase (e.g., from a solid to a liquid).

(1) The **specific heat**, c, is the amount of heat required to raise 1 kg of a substance by 1 K. So the heat required to change the temperature of an object by ΔT is

$$\boxed{Q = mc\Delta T}$$

Specific heat is a material property and can vary widely. For mercury $c = 140$ J/(kg K), whereas for water $c = 4190$ J/(kg K). It actually takes a lot of heat to warm up/cool down water. This is important for biology (e.g., ocean temperatures are fairly constant).

This equation also tells us that it takes more heat to warm up/cool down large objects. This is why large animals can survive harsh winters more easily than small animals.

(Note that the specific heat is often treated as a constant, but it doesn't have to be. For seawater, it depends on salinity and temperature.)

(2) **Latent heat** or heat of transformation is the amount of heat needed to change the phase of a substance. Essentially, when a material changes phase, heat goes into breaking bonds.

- When a material changes from solid to liquid, or vice-versa, we refer to the latent heat of fusion, L_f.
- When a material changes from liquid to gas, or vice-versa, we refer to the latent heat of vaporization, L_v.
- L_f and L_v are material properties and have units of J/kg.

The heat required to change phase is simply the mass times the latent heat.

$$\boxed{Q_f = \pm mL_f}$$

$$\boxed{Q_v = \pm mL_v}$$

$+ \Rightarrow$ heat must be added to melt or vaporize.

$- \Rightarrow$ heat must be removed to freeze or condense.

Note that $L_v >> L_f$. Melting only requires breaking enough bonds than an object can start to flow. Vaporization requires enough energy to break all bonds and send molecules flying through the air.

Maybe not surprisingly, water also has really high L_f and L_v, which is important for biology. It doesn't easily freeze or evaporate.

24.4.1 Example #1: Vaporization of mercury

How much heat is need to change 20 g of mercury at 20°C to to vapor at its boiling point?

Given:
$c = 140$ J/(kg K)
$T_m = 357$°C
$L_v = 2.96 \times 10^5$ J/kg

(a) $Q_T = mc\Delta T = 943$ J

(b) $Q_f = mL_f = 5920$ J

(c) $Q_{total} = Q_T + Q_f = 6863$ J

25 THERMODYNAMICS II

Objectives:

- Review from last class
- Practice problems

25.1 Laws of thermodynamics; specific heat and latent heat

Last class I introduced the first and second laws of thermodynamics and discussed what happens when heat is transferred into or out of a system.

- First Law of Thermodynamics: $W + Q = \Delta E$, where Q is *heat* that is transferred into or out of the system (which can occur in several ways). Think of heat as the transfer of thermal energy.
- Second Law of Thermodynamics:
  - Entropy (disorder) of a *closed* system can never decrease; in other words, order tends toward disorder.
  - Heat flows from hot objects to cold objects, which causes the thermal energy of the objects to change.
  - Mechanical energy (kinetic, gravitational potential, etc) tends to turn into thermal energy. This transformation is not reversible.
- When heat enters or exits a system, the objects in the system change temperature and/or phase.
- The heat need to produce a temperature change is $Q = mc\Delta T$, where c is the specific heat (a material property). Rearranging, this is also $\Delta T = Q/(mc)$, which might be more intuitive.
- The needed to change the phase of a material is $Q_f = \pm mL_f$ (for solid to liquid or vice-versa) and $Q_v = \pm mL_v$ (for liquid to gas or vice-versa). L_f and L_v are the latent heats of fusion and vaporization (also material properties). $L_v >> L_f$.

I'd like to spend some time today thinking about these ideas by way of example problems.

25.2 Example #1: Melting snowpack

Consider a melting snowpack in spring. The snow is below 0°C but the air is warmer than that. The surface snow melts first, and the water percolates downward. It freezes when it comes into contact with cold snow and in doing so releases latent heat to the surrounding snow pack. This causes the surrounding snowpack to warm up to a uniform temperature (basically, 0°C), and then melting can happen much more quickly. Wet snow avalanches happen occur in spring when the entire snowpack warms up and becomes saturated. The process can be exacerbated by rainfall.

[Insert diagram of spring snowpack.]

25.3 Example #2: Place ice in water
You place 50 g of ice at -10°C into 200 g of water at 20°C. What happens? Does the ice melt? What is the final temperature of the system? Again will we will assume that the glass is perfectly insulated so that we don't have to worry about energy exchanges with the environment.

Given:
$c_i = 2220$ J/(kg K)
$c_w = 4200$ J/(kg K)
$L_f = 334 \times 10^3$ J/kg

What happens:
(1) Amount of heat that the water can lose: $Q_{water} = m_w c_w \Delta T_w = 16800$ J
(2) Amount of heat needed to warm up the ice to 0°C: $Q_i = m_i c_i \Delta T_i = 1110$ J
(3) Amount of heat needed to melt all of the ice: $Q_f = mL_f = 16700$

The sum of (2) and (3) is greater than the amount of heat that the water can lose. This means that not all of the ice melts and that the final temperature will be 0°C. How much ice melts?

After warming up the ice, the water can still release 15700 J before it starts to freeze. From (3),

$$m = \frac{Q_f}{L_f} = 47 \text{ g}$$

There are 3 g of ice remaining.

As you solve this type of problem, you need to ask a series of "if, then" questions.

25.4 Example #3: Place hot metal in water
Consider what happens when hot steel is placed in water in a perfectly insulated cup with no gas (i.e., any void space is essentially a vacuum). The Second Law of Thermodynamics tells us that heat is transferred from the hot metal to the water until the temperatures are uniform. What is the final temperature of the system?

Let: $m_s = 0.05$ kg, $m_w = 0.2$ kg, $T_s = 100°$ C, $T_w = 20$°C, $c_s = 420$ J/(kg·K), and $c_w = 4200$ J/(kg·K)

We can solve for this using the First Law of Thermodynamics, i.e.,

$$Q + W = \Delta E$$

There are no external forces acting on the system, so $W = 0$, and because this is an isolated system, $\Delta E = 0$. This means that $Q = 0$ (which I guess we already knew because the system is insulated).

However, there is heat flowing from the metal to the water. The metal cools down and the water warms up. So let's call Q_m the amount of heat released by the metal, and Q_w is the amount of heat transferred into the water.

$$Q = Q_s + Q_w = 0$$

$$m_s c_s (T_f - T_s) + m_w c_w (T_f - T_w) = 0$$

Here, T_s and T_w are the initial temperatures of the steel and the water. Solving for T_f:

$$T_f = \frac{m_s c_s T_s + m_w c_w T_w}{m_s c_s + m_w c_w} = 294 \text{ K} = 22°\text{C}$$

Alternatively, you could measure the temperatures and masses, and if you know the specific heat of water you can figure out the specific heat of the metal (and maybe identify the metal).

25.5 Bomb calorimeter

Similar ideas are used in calorimetry.

[Insert diagram of a bomb calorimeter.]

Essentially, you burn the sample and observe the water heating up. The heat released by the sample equals the heat absorbed by the water. 1 food calorie is the amount of heat needed to increase 1 kg of water by 1 K.

$$1 \text{ food calorie} = 1 \text{ kcal} = 1 \text{ Calorie} = 4200 \text{ J}$$

25.6 Climbing Mt. Juneau

We've spent the last couple of classes discussing the First and Second Laws of Thermodynamics, and what happens when heat is transferred into or out of a system or object (change in temperature or change in phase).

One of the super useful things about thermodynamics is that it connects different branches of science. For example, we can ask questions like, "How many Snickers bars does it take me to climb Mt. Juneau?" and "During the climb, how much would my body temperature change if I don't have an efficient way of transferring heat away from my body?". Let's say that I weigh 75 kg and the mountain is 1000 m tall.

How would we go about addressing these questions? What do we need to know to address these questions?

- The amount of energy it takes to climb Mt. Juneau.
- The amount of (chemical) energy stored in a Snickers bar.

- The efficiency of the human body.

Overview:

If no heat loss, $W + Q = 0 = \Delta U_g + \Delta E_{ch} + \Delta E_{th}$

If heat loss, $Q = \Delta U_g + \Delta E_{ch}$

The amount of energy that I need is $\Delta U_g = mg\Delta y \approx 7.5 \times 10^5$ J

There are about 250 kcal in a Snickers bar, or 1.05×10^6 J.

So, from conservation of energy,

$$W + Q = 0 = \Delta E = \Delta U_g + \Delta E_{ch} + \Delta E_{th}$$

Let's set $\Delta E_{ch} = n\Delta E_{Snickers}$, where n is the number of of Snickers bars that I eat. The change in chemical energy depends on how many Snickers bars are consumed. We can think of chemical energy as stored energy. In this case, the change in chemical energy is a negative number, indicating that I am using up stored energy.

If my body is 100% efficient, then no energy is "lost" to thermal energy, and so

$$0 = \Delta U_g + n\Delta E_{Snickers}$$

Therefore

$$n = \frac{-\Delta U_g}{\Delta E_{Snickers}} = 0.72 \text{ Snickers}$$

Hmm... I think I'm going to be pretty hungry by the time that I get to the top. In reality, the human body is closer to 25% efficiency. (You can figure out a person's efficiency by doing experiments similar to the one that was used to figure out how much energy is in food. Feed somebody some food and watch their body temperature change. Okay, its a little more complicated than that, but you get the idea.)

This means that 75% of the energy that I get from the Snickers (from chemical energy) will be transformed into thermal energy. So let's set

$$\Delta E_{th} = -\frac{3}{4}n\Delta E_{Snickers}.$$

Our energy balance is

$$0 = \Delta U_g + \Delta E_{chem} + \Delta E_{th}$$

$$0 = \Delta U_g + n\Delta E_{Snickers} - \frac{3}{4}n\Delta E_{Snickers} = \Delta U_g + \frac{1}{4}n\Delta E_{Snickers}$$

Solving for n gives

$$n = \frac{-4\Delta U_g}{\Delta E_{Snickers}} = \boxed{2.86 \text{ Snickers}}$$

That sounds better!

How much thermal energy is generated?

$$\Delta E_{th} = \boxed{2.25 \times 10^6 \text{ J}}$$

If I don't have an efficient way to get rid of this, my body temperature will change.

$$Q = mc\Delta T$$

Or

$$\Delta T = \frac{Q}{mc}$$

Usually we use Q to refer to heat that is transferred into or out of a system, but we can also use it to describe heat transfer between components within a system...

For mammals, $c \approx 3400$ J/(kg $\cdot$ K).

Using my mass and the change in thermal energy,

$$\boxed{\Delta T = 8.8 \text{ K!!!}}$$

We better figure out a way to get rid of the excess thermal energy. (Unless its winter and we want to keep that thermal energy....)

The way that our bodies do that is by sweating. Sweat transfers some of the heat to the outside of our bodies. The sweat then evaporates. This evaporation occurs below the boiling point. Why? Because some of the hotter than average molecules (which have high energy) manage to escape. This helps to keep us cool. We can calculate how much sweat must evaporate to keep the body temperature constant.

During the climb up Mt. Juneau, I generated 2.25×10^6 J of thermal energy. We want to get rid of all of that via evaporation.

$$Q = mL_v$$

$$m = \frac{Q}{L_v}$$

The latent heat of vaporization for water is 2.26×10^6 J/kg. This means that I need to produce about 1 kg (1 L) of sweat. To stay hydrated I would need to drink that same amount. Also sounds quite reasonable.

26 HEAT TRANSFER

Objectives:

- Conduction
- Advection/convection
- Electromagnetic radiation
- Evaporative cooling

26.1 Background

Recall:

- First Law of Thermodynamics: $W + Q = \Delta E$
- Heat transfer into or out of a system changes the temperature and/or phase of the material in the system

Demo: fireproof balloon

What are some mechanisms for heat transfer?

Demos: melting ice; convection tube

Three basic types of heat transfer, and all depend on temperature differences:

- conduction
- convection/advection
- electromagnetic radiation

26.2 Conduction

When there is a temperature difference across an object, energy is transferred from the warm side (w/ fast molecules) to the cool side (w/ slow molecules). (We already saw this.)

[Insert diagram; rod with fire on one end and ice on the other.

The *rate* at which heat is transferred depends on the object's composition, length, cross-sectional area, and the temperature difference.

$$\frac{Q}{\Delta t} = \left(\frac{kA}{L}\right)\Delta T,$$

where k is the thermal conductivity (a material property) and has units of W/(m K).

Copper: $k = 400$ W/(m·K)
Wood: $k = 0.2$ W/(m·K)

This is why you typically use a wood spoon when cooking, and its also why pots often have copper bottoms. Aluminum, iron, and steel also have high thermal conductivities, but not as high.

Notice that the rate of heat transfer depends on ΔT. What happens when two objects equilibrate?

[Sketch of $\Delta Q/\Delta t$ of two objects equilibrating.]

26.2.1 Example #1: Heat transfer through a floor.
We can think about the rate at which heat travels through a floor. Let's say that the room temperature is 19.6°C, the temperature below the floor is 16.2°C, and the floor has an area of 22 m^2 and is 0.018 m thick.

The rate of heat transfer is

$$\frac{Q}{\Delta t} = \left(\frac{kA}{L}\right)\Delta T = 830 \text{ J/s} = 830 \text{ W}$$

(Rate at which energy is changing is the power, and has units of watts.)

26.3 Advection/convection
Advection → transport of heat by a moving material
Convection → vertical transport of heat (e.g., heat a pot of water from below)

Advection and convection are especially important for fluids and gases. In order to fully understand this method of heat transport, you have to model both fluid flow and thermodynamics. And advection and convection are often linked to conduction.

For example, consider warm water flowing through a pipe.

[Diagram of pipe.]

The warm water is carrying heat; this is advection. If the water is warmer than the air on the outside of the pipe, then heat will conduct through the pipe and radiate into the environment.

26.4 Electromagnetic radiation
(We'll talk about this in much more detail next semester.) Basically what you need to know now is that object's emit electromagnetic waves. The energy that the waves carry, as well as the wavelength of the waves, depends on the material properties, the surface area, and the temperature.

$$\frac{Q}{\Delta t} = e\sigma A T^4$$

(We find this equation by modifying some other equations that we haven't seen yet...)

e is the emissivity (unitless); it is the ratio of energy radiated to energy absorbed. $e = 0.97$ for humans
σ is the Stefan-Boltzmann constant; $\sigma = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4}$
A is the surface area
T is temperature in Kelvin

Because the environment is also radiating based on its temperature, the net radiation from an object is

$$\frac{Q_{net}}{\Delta t} = e\sigma A(T^4 - T_0^4),$$

where T_0 is the temperature of the environment. If $T > T_0$, the object cools; if $T < T_0$, the object warms up.

There is an important feedback here that comes into play when thinking about the temperature of a planet. If you increase the temperature of the planet, you also increase the rate at which it radiates energy outward (and a small change in temperature results in a big change in radiation) which counteracts the warming of the planet. This helps to stabilize a planet's temperature.

26.5 Evaporative cooling
A fourth way that you can transfer energy out of a system is by evaporation. Let's discuss this by way of an example.

A 68 kg woman cycles at a constant 15 km/h. All of the metabolic energy that does not go to forward propulsion is converted to thermal energy in her body. If the only way her body has to keep cool is by evaporation, how many kilograms of water must she lose to perspiration each hour to keep her body temperature constant?

The metabolic power of a 68 kg cyclist going 15 km/h is 480 W (480 J/s). This is the rate at which the cyclist is using energy and is essentially determined experimentally. 75% of this is transformed into thermal energy. So 360 W of thermal energy is produced; this much must be released by evaporation to keep the cyclists temperature constant. So

$$\frac{Q}{\Delta t} = \frac{mL_v}{\Delta t}$$

$$\frac{m}{\Delta t} = \frac{Q}{\Delta t}\frac{1}{L_v} = 1.5 \times 10^{-4} \text{ kg/s} = 0.54 \text{ kg/hr}$$

This is about half a liter per hour. Seems reasonable.

When you perspire, warm water (sweat) moves to the outside of your body. It then evaporates; evaporation requires heat, so it reduces the thermal energy of the sweat that is left behind...

26.6 Refrigerators

How do refrigerators work? By making use of the thermodynamics of gases and liquids to get heat to flow from a cold reservoir to a warm reservoir. In particular, they utilize electromagnetic radiation, advection, conduction, and latent and specific heats.

[Insert diagram.]

Steps:

- Vapor flows through a compressor, which increases the temperature of the gas.
- The vapor then flows through a condenser (coils on the back of a refrigerator). The vapor in the condensor is warmer than surrounding air, so energy is radiated into the air. $Q < 0$, so $\Delta E < 0$. The temperature of the vapor decreases (specific heat) and eventually the vapor condenses into liquid (latent heat).
- The liquid then flows through an expansion valve; the associated decrease in pressure results in some of the liquid turning into vapor. This results in a mixture of liquid and vapor at low pressure and low temperature.
- The mixture travels through the evaporator, where it comes in contact with “warm” air from the cold reservoir (conduction). The warm air causes the remaining liquid to evaporate. $Q > 0$ so $\Delta E > 0$.
- The cycle repeats.

Need to carefully select your refrigerant for your particular application. For example, you want:

- The boiling point to be below the target temperature so that air from the cold reservoir can turn liquid into vapor.
- A high latent heat of vaporization to release a lot of energy in condenser.

27 THERMODYNAMICS OF GASES

Objectives:

- Ideal gas law
- Connection to thermodynamics
  - Isovolumetric processes
  - Isobaric processes
  - Isothermal processes
  - Adiabatic processes

So far this semester we've focused exclusively on solids. We're going to spend a fair bit of time over the next few weeks talking about fluids and gases. As a way to segue into those topics, I'd like to talk about the thermodynamics of gases.

27.1 Ideal Gas Law

In describing how refrigerators work, I made use of the fact that the temperature of a gas changes when it is compressed or expanded. The easiest way to see this is through the ideal gas law:

$$PV = nRT$$

P is pressure [1 N/m^2 = 1 Pa]
V is volume [m^3]
n is the number of moles of gas in a container
R is the gas constant (8.31 J/(mol·K))
T is temperature [K]

Although originally discovered experimentally, the ideal gas law can be derived by considering elastic collisions of (numerous) particles within a closed container. The ideal gas law works well for gases that are monatomic (e.g., He, Ne, Ar) and have high temperature and low pressure. It neglects molecular size and intermolecular interactions.

What is pressure and where does it come from? Essentially, the molecules in a gas are constantly colliding with the walls of the container. This means that there are contact forces between the molecules and the container. If you add up the forces from all of the collisions that are occurring at any one instant, and divide by the surface area, you get the pressure:

$$P = \frac{F}{A}$$

or equivalently, the force due to the gas pressure is

$$F = PA$$

Often we are interested in the pressure difference between the inside and outside of a container.

$$F_{net} = F_2 - F_1 = P_2A - P_1A = (P_2 - P_1)A = \Delta PA$$

It is this pressure difference that creates "suction".

Atmospheric pressure is actually quite large — about 101.3 kPa at sea level.

27.1.1 Demo: Pressure and the Magdeburg hemisphere

Pump air out of Magdeburg hemispheres, essentially creating a vacuum. By pumping air out, we are removing molecules and therefore reducing the number of collisions with the container, and so the pressure goes to 0. What is the force that is needed to overcome air pressure and pull apart the hemisphere?

Pressure acts perpendicular to a surface. Here, we are really only concerned with the component of the pressure force that is parallel to the direction of the applied forces. So the area of interest is the cross-sectional area of the sphere.

Let's say that the radius of the hemisphere is 4 cm. The cross-sectional area is therefore $\pi(0.05)^2 = 0.008$ m^2. This means that the force (from the atmospheric pressure) acting on either side of the hemisphere is $F = PA = 800$ N. That force acts on both sides; you would need to apply a force of 1600 N to be able to open the hemisphere. (For comparison, the gravitational force acting on me is about 750 N.)

27.1.2 Moles

What is n? Its the number of moles of gas in a particle.

$$n = \frac{N}{N_a}$$

where N is the number of basic particles in a gas and N_a is Avogadro's number (6.02×10^{23} mol^{-1}). (This is based on the number of basic particles in 12 g of ^{12}C, which is equal to 1 mole...)

If we deal with closed containers, then n will be constant. So let's rearrange the ideal gas law:

$$\frac{PV}{T} = nR = \text{constant for closed containers}$$

If we do something to the container (e.g., change its shape or heat it up), then the pressure, volume, and temperature have to adjust their values so that

$$\frac{P_f V_f}{T_f} = \frac{P_i V_i}{T_i}$$

27.1.3 How gases change with time

Often we are interested in knowing how a gas changes with time. There are three end-member possibilities:

(1) Constant volume (isovolumetric or isometric or isochoric): increase temperature, pressure increases

(2) Constant pressure (isobaric): increase volume, temperature increases

(3) Constant temperature (isothermal): increase volume, decrease pressure

[Insert PV diagram showing what these processes look like.]

27.2 Connection to thermodynamics

Recall that

$$Q + W = \Delta E$$

(1) For gases, the specific heat depends on whether a heat transfer is occurring to a system that is at constant pressure or constant volume. c_p is commonly about 50% greater than c_v.

(2) Isovolumetric processes involve heat transfer into or out of a system.

(3) Isobaric and isothermal processes do work on the environment (or the environment does work on the gas) because the volume of the container changes.

27.2.1 Isovolumetric processes

In an isovolumetric process, no work is done on the gas, and so

$$Q = \Delta E$$

I haven't shown this, but from statistical mechanics we know that the thermal energy of an ideal gas is proportional to temperature:

$$\Delta E_{th} = \frac{3}{2} nR\Delta T$$

For isochoric processes acting on a container at rest, we only have to worry about heat transfer into the container.

$$Q = \Delta E_{th} = \frac{3}{2} nR\Delta T$$

This looks a lot like the equation that we saw for specific heat ($Q = mc\Delta T$). For processes occurring at constant volume, the molar specific heat for monatomic gases is $c_v = 3R/2$ (12.5 J/(mol·K)), and we write $Q = nc_v\Delta T$.

The temperature change is proportional to the amount of heat transfer.

$$\Delta T = \frac{2Q}{3nR} = T_2 - T_1$$

And for ischoric processes, we have

$$\frac{P_1}{T_1} = \frac{P_2}{T_2}$$

With some algebra, we arrive at

$$\Delta P = \frac{P_1}{T_1}\frac{2Q}{3nR}$$

In other words, the change in pressure depends on the heat transfer and the initial pressure and temperature.

27.2.2 Isobaric processes

In isobaric processes, the volume can change in response to pressure differentials. Let's think about the work done by the gas on the environment. First,

$$W_{gas} = -W$$

so the first law of thermodynamics becomes

$$Q - W_{gas} = \Delta E$$

We can figure out W_{gas} for a constant pressure (isobaric) process by considering the expansion of a piston.

$$W_{gas} = F_{gas}\Delta x = P_{gas}A_{piston}\Delta x = P\Delta V$$

Notice that the work done by the gas is the area under the PV graph (this is always true). This gives

$$Q - P\Delta V = \Delta E = \Delta E_{th}$$

In cases where Q is not equal to zero,

$$Q = \frac{3}{2}nR\Delta T + P\Delta V = \frac{3}{2}nR\Delta T + nR\Delta T = \frac{5}{2}nR\Delta T$$

Again, this looks a lot like the equation for specific heat, but this is for constant pressure. Here, $c_p = 5R/2$ (20.8 J/(mol·K)) for monatomic gases and we write $Q = nc_p\Delta T$.

27.2.3 Isothermal processes

In isothermal processes, the temperature remains constant. So

$$Q - W_{gas} = \Delta E_{th} = 0$$

This means that

$$W_{gas} = Q$$

In other words, in order for the temperature to remain constant, the area under the PV curve must equal the heat transfer into the system. And because temperature is constant, $P \propto 1/V$ (because $P_iV_i = P_fV_f$).

Using calculus, we can show that

$$\Delta V = V_i\left(e^{\frac{W_{gas}}{nrT}} - 1\right)$$

In other words, the change in volume depends exponentially on the work that is done by the gas.

27.2.4 Adiabatic processes

An adiabatic process is one which there is no heat transfer. Since $Q = 0$, $W = \Delta E_{th}$. If you do work on the gas (you compress it), then the thermal energy (i.e., temperature) will increase. Conversely, if the gas does work on the environment then the gas will expand and cool.

[Diagram of adiabatic processes.]

27.2.5 Conceptual question: Do heat and work depend on the path?

In the following diagram, the gas moves from an initial state (P_i, V_i) to a final state (P_f, V_f). Are the work and heat along each of these paths the same, and if not, how are they different?

[Diagram of gas transformation from (P_i, V_i) to (P_f, V_f).]

We now that the initial and final temperatures must be the same, and therefore ΔE_{th} is the same along both paths. This means that

$$W_1 + Q_1 = W_2 + Q_2$$

No work is done when the volume is constant, only when the volume is increasing. Therefore, recalling the work done by the environment for isobaric processes,

$$-P_f \Delta V + Q_1 = -P_i \Delta V + Q_2$$

From this its clear that

$$W_1 < W_2$$

which must also mean that

$$Q_1 > Q_2$$

28 INTRODUCTION TO FLUIDS

Objectives:

- Hydrostatics
  - Density
  - Pressure

28.1 Review of thermodynamics of gases

Today I'd like to start talking about fluids, but before doing that I think we should do an example related to the last lecture.

Important ideas from last time:

- Ideal gas law: $PV = nRT$. For closed containers, we can saw that $\frac{PV}{T} = \text{constant}$.
- Pressure at sea level: $P = 101.325$ kPa; this is equivalent to 101.325 kN acting on 1 m^2 or 10.1325 N on 1 cm^2
- Change in thermal energy of a gas is given by $\Delta E_{th} = \frac{3}{2}nR\Delta T$
- Work and heat (transfer) can change the total energy of a gas
- Isochoric or isovolumetric process: heat transferred into a gas container with no change in volume
- Isobaric process: pressure is constant, but volume and temperature can change due to work or heat transfer
- Isothermal process: no change in temperature (or thermal energy), so work done by a gas must balance with heat transfer into the gas
- The specific heat of a gas depends on whether the volume or pressure change...

28.1.1 Example #1: expandable cube

An expandable cube, initially 20 cm on each side, contains 3.0 g of helium at 20°C. 1000 J of heat is transferred into this gas. What are (a) the final pressure if the process is at a constant volume and (b) the final volume if the process is at constant pressure?

Also given:
3 g of He is 0.75 mol
$C_v = 12.5$ J/(mol·K)
$C_p = 20.8$ J/(mol·K)

(a) Because this is at constant volume, and because

$$\frac{PV}{T} = \text{constant},$$

we have

$$\frac{P}{T} = \text{constant} \Rightarrow \frac{P_i}{T_i} = \frac{P_f}{T_f}.$$

We want to solve for the final pressure, so

$$P_f = \frac{T_f}{T_i}P_i.$$

We know that $T_i = 293$ K. What are T_f and P_i? Heating the gas will change its temperature

$$Q = nC_v\Delta T \Rightarrow \Delta T = \frac{Q}{nC_v} = 107 \text{ K},$$

so $T_f = 400$ K. The initial pressure we can find using the ideal gas law.

$$P_iV_i = nRT_i \Rightarrow P_i = \frac{nRT_i}{V_i} = 228 \text{ kPa}$$

Plugging this into the equation for P_f gives

$$\boxed{P_f = 311 \text{ kPa}}$$

(b) Because this is constant pressure, we have

$$\frac{PV}{T} = \text{constant} \Rightarrow \frac{V}{T} = \text{constant}$$

This means that

$$\frac{V_i}{T_i} = \frac{V_f}{T_f} \Rightarrow V_f = \frac{T_f}{T_i}V_i$$

The initial volume and temperature are given. We just need to calculate T_f.

$$Q = nC_p\Delta T \Rightarrow \Delta T = \frac{Q}{nC_p} = 64.1 \text{ K},$$

so $T_f = 357$ K. Consequently,

$$\boxed{V_f = 0.00975 \text{ m}^3 = 9.75 \text{ L}}$$

.

28.2 Hydrostatics

That's all that I have to say (for now, anyway) about thermodynamics and gases. I'd like to now turn our attention to fluids. We'll first focus on hydrostatics — fluids at rest.

What is a fluid? Its a substance that flows and takes the shape of a container.

[Diagram of container containing a fluid.]

In a fluid, molecules are weakly bonded but can slide past each other.

28.3 Density
An important parameter for describing fluids is density.

$$\rho = \frac{m}{V}$$

Sometimes ρ depends on pressure, sometimes it doesn't...

Examples:
sea water: ~ 1030 kg/m^3; it depends on salinity, temperature, and pressure

fresh water: ~ 1000 kg/m^3; it depends on temperature

- at 20°C, $\rho = 998.2071$ kg/m^3
- at 4°C, $\rho = 999.9720$ kg/m^3
- at 0°C, $\rho = 998.8395$ kg/m^3

ice: ~ 917 kg/m^3 (if bubble free, otherwise < 917 kg/m^3)

firn: ~ 800 kg/m^3

fresh snow: 100–300 kg/m^3

Water is weird. For most substances, the solid form is more dense than the liquid form. If this were also true for water, life wouldn't exist — at least not as we know it. Its also weird that the densest fresh water is 4°C.

Ice that forms in a water body floats at the surface, and it insulates the water from cold air. The water at depth remains liquid.

[Draw diagram.]

Its very difficult to form thick lake ice or sea ice. "Multi-year" ice in the Arctic Ocean is just a few meters thick.

One important process involving water is the overturning of lakes during spring and fall, which is due to the fact that the densest water is at 4°C.

[Draw diagrams showing overturning lakes and temperature profiles.]

28.4 Hydrostatic pressure

Why does dense material sink? To answer that, we need to think a bit about pressure.

[Insert diagram with a column of water in a container.]

Let's assume that the column is in hydrostatic equilibrium.

There is a force from the overlying water and air that acts on downward on the column of water. This force is equal to

$$F_{down} = P_0 A + mg = \rho V g = \rho h A g$$

From Newton's Third Law, there is a force pushing back on the column with an equal and opposite force. If, as before, we define pressure as $P = F/A$, then the force upward force exerted by the parcel of water is

$$F = PA = P_0 A + \rho h A g$$

Dividing by A gives

$$\boxed{P = P_0 + \rho g h}$$

The hydrostatic fluid pressure depends on depth.

This pressure acts equally in all directions. If that wasn't the case, the fluid parcel would change shape. (One way to check would be to put a pressure sensor in water – basically a device with a spring that responds to the force from the water – and rotate it about in the water column.) This tells us that in a static fluid, water pressure is constant along horizontal lines and the water surface rises to the same level everywhere.

I always think its nice to have a reference value for physical units. A convenient one for pressure is to think about the water pressure below 1 m of water.

$$P = \rho g h = 1000 \text{ kg/m}^3 \times 9.81 \text{ m/s}^2 \times 1 \text{ m} \approx 10^4 \text{ Pa}$$

Note that the water pressure below 10 m of water is 10^5 Pa, which is basically the same as the atmospheric pressure at sea level. In other words, 10 m of water is equivalent to several kilometers of air.

Gas pressure also varies with elevation for the same reason as fluid pressure. However, since the density of gases are so low, the pressure variations are small over short distances. In a closed container, we treat gas pressure as constant.

28.4.1 Pressure demos

- Bottle with pin holes at different elevations to show that pressure varies with depth
- Pipette to illustrate water pressure in a vacuum

$$P_0 = \rho g d + P_{bubble}$$

$$P_{bubble} = P_0 - \rho g d$$

If $d = 10$ cm and $\rho = 1000$ kg/m^3, then $P_{bubble} = 100.344$ kPa

29 HYDROSTATICS

Objectives:

- Archimedes' principle
- Using pressure

Recall from last class:

- Density: $\rho = \frac{m}{V}$
- Hydrostatic pressure: $P = \rho g h$, where h is depth. Units are in Pa. Pressure acts in all directions.
- Convenient reference: The pressure beneath 1 m of water is $\sim 10^4$ Pa, or 10% of atmospheric pressure. They are approximately equal at 10 m depth.

29.1 Archimedes' principle

From these ideas, we can talk about why objects sink or float.

Let's look at the forces acting on an object in water.

[Diagram of object submerged in water.]

$$\sum F_y = F_{\text{below}} - F_g - F_{\text{above}} = F_{net} = ma_y$$

The forces from below and above the object are due to pressure (recall that $F = PA$).

$$\rho_w g(h+L)A - \rho_w g h A - \rho_o g L A = m a_y$$

$$\rho_w g L A - \rho_o g L A = \rho_o L A a_y$$

The upward force is referred to as the buoyant force. Notice that it is equal to the weight of the displaced fluid. This is referred to as "Archimedes' Principle". In other words,

$$F_b = \rho_f g V_d,$$

where ρ_f is the density of the fluid and V_d is the volume of displaced water.

If the buoyant force exceeds the object's weight, the object will rise; if it is lower than the object's weight, the object will sink.

Dividing by LA and re-arranging, we arrive at

$$\frac{(\rho_w - \rho_o)}{\rho_o} g = a_y$$

An object that is more dense than water will sink, and the rate at which it sinks depends on the density differences. (If this we an object falling through air, $\rho_o >> \rho_w$ and so $a_y = -g$.

29.1.1 Demo: Cartesian diver

By squeezing the bottle, you are increasing the pressure, which causes water to flow into the Cartesian diver. The average density of the diver increases, cause it to sink. If you release the pressure the diver will rise.

29.2 Demo with objects sinking/floating in water

How does buoyancy change their acceleration?

29.3 Icebergs

I'm sure you've all heard that 90% of an iceberg is below the water surface. We can explain that using Archimedes' principle.

We'll consider an iceberg floating at rest at the water surface, and ask how much of the iceberg sticks out of the water.

[Insert diagram.]

$$\sum F_y = F_b - F_g = 0$$

$$F_b = m_d g = \rho_w V_d g,$$

where V_d is the volume of displaced water.

$$F_g = m_i g = \rho_i V_i g$$

Combining these gives

$$\rho_w V_d g - \rho_i V_i g = 0$$

Dividing by g and re-arranging gives

$$\frac{V_d}{V_i} = \frac{\rho_i}{\rho_w} \approx 0.9$$

90% of the iceberg is submerged.

29.4 Measuring gas pressure

Fluids are often used to measure gas pressure (such as atmospheric pressure)

29.4.1 Manometer

[Insert diagram.]

In a fluid that is in hydrostatic equilibrium, the pressure is constant along horizontal lines. (If this wasn't the case the fluid would move upward or downwardin a manometer.)

So in a manometer, this means that

$$P_1 = P_2$$

where

$$P_1 = P_{gas}$$

and

$$P_2 = P_o + \rho g h$$

Therefore, the gas pressure is

$$P_{gas} = P_o + \rho g h.$$

For this to work though, we need to know the atmospheric pressure. For that, we need a barometer.

29.4.2 Barometer

[Insert diagram of a barometer.]

In a barometer,

$$P_o = \rho g h,$$

where ρ is the fluid density and h is the height of the column.

You could use any fluid, but its best to use really dense fluids (that way the barometer can be small). If you use water (with a density of 1000 kg/m^3), you would need a column that is

$$h = \frac{P_o}{\rho g} = 10 \text{ m}$$

Mercury has a density of 13,534 kg/m^3. The column of mercury that you need is only 0.76 m tall — this is about 30 in of mercury...

29.5 Example #1: Ranking exercise

Example: Rank in order, from largest to smallest, the densities of blocks a, b, and c.

[Insert diagram.]

Example #2: Sphere submerged in water

A sphere completely submerged in water is tethered to the bottom with a string. The tension in the string is one-third the weight of the sphere. What is the density of the sphere?

[Insert diagram.]

$$\sum F_y = F_b - F_g - F_t = ma_y = 0$$
$$F_t = \frac{1}{3}F_g$$
$$F_b - \frac{4}{3}F_g = 0$$
$$\rho_w gV - \frac{4}{3}\rho gV = 0$$
$$\rho = \frac{3}{4}\rho_w = \boxed{750 \text{ kg/m}^3}$$

29.6 Introduction to fluid dynamics

That's basically all that I have to say about hydrostatics. Its more interesting to talk about fluid dynamics, or fluid motion. To do this, we are going to consider an ideal fluid. We will assume three things:

1. The fluid is incompressible. This is a good approximation for many fluids, especially water.
2. The fluid flow is steady (laminar) and non-turbulent. [Insert diagram.] The transition from laminar to turbulent flow depends on the fluid properties, the flow speed, and the geometry of the flow.
3. The fluid is non-viscous. Viscosity is a fluid's resistance to flow; think of it as internal friction. Water has a low viscosity, syrup has a high viscosity.

Because the fluid flow is assumed incompressible, we will have a convenient equation that describes mass continuity.

The amount of fluid that enters the upstream end of a pipe in time Δt has to equal the amount of fluid that pass through the downstream end of a pipe.

Volume V_1 enters the pipe in time Δt; volume V_2 exits the pipe in the same amount of time. Due to incompressibility, $V_1 = V_2$.

$$V_1 = A_1 \Delta x_1 = A_1 v_1 \Delta t$$

$$V_2 = A_2 \Delta x_2 = A_2 v_2 \Delta t$$

Because we are dealing with an ideal fluid, the velocity across each cross-section is constant. And so

$$\boxed{A_1 v_1 = A_2 v_2}$$

If $A_2 < A_1$, as is the case for the diagram that I've drawn, then the fluid velocity increases.

We'll define a new term, called the flux, or volume flux, as

$$Q = \frac{V}{\Delta t} = Av$$

The flux is constant in a tube.

30 FLUID DYNAMICS

Objectives:

- Ideal fluids
- Flux continuity equation
- Bernoulli's equation

30.1 Pressure

- Fluids: material that flows
  - Gases: highly compressible fluids
  - Liquids: weakly compressible fluids with well-defined surfaces
- Pressure = force / area; depends on weight of overlying fluid
  - For liquids in hydrostatic equilibrium: $P = P_{atm} + \rho g h$ in an open container, or $P = P_{gas} + \rho g h$ in a closed container; assume that density is constant
  - For gases (especially in closed containers): often assume constant pressure. Because the density is low, changes in pressure are small over short distances. If considering large distances, have to account for changes in density.
  - Pressure is constant along horizontal lines for a fluid that is in hydrostatic equilibrium.

30.2 Ideal fluids

We will assume three things:

1. The fluid is incompressible. This is a good approximation for many liquids, especially water, and is not even a bad approximation for gases.
2. The fluid flow is steady (laminar) and non-turbulent. [Insert diagram.]

   The transition from laminar to turbulent flow depends on the fluid properties, the flow speed, and the geometry of the flow. [Discuss water coming from faucet.]
3. The fluid is non-viscous. Viscosity is a fluid's resistance to flow; think of it as internal friction. Water has a low viscosity, syrup has a high viscosity. (Later we'll take into account viscosity.)

30.3 Flux continuity

These assumptions lead to a nice continuity equation for fluid flow through a *pipe*:

$$Q = vA = \text{constant}$$

where Q is the flux [m^3/s], v is the velocity, and A is the cross-sectional area. Note that for ideal fluids, the flow velocity does not vary across the channel.

[Insert diagram of a pipe showing that $Q_1 = Q_2$.]

The continuity equation is different for open channel flow, because the depth of the channel can also change. In that case,

$$\frac{\Delta V}{\Delta t} = Q_{in} - Q_{out}$$

30.3.1 Example #1: Water flow through variable diameter pipe

Water enters a pipe with a diameter of 1 cm at 4 m/s. The pipe expands to 2 cm, then shrinks to 0.5 cm. (a) What is the flux? (b) What are the water speeds in each section of pipe?

(a) Flux:

$$Q = Av = \pi r^2 v = \pi(0.05 \text{ m})^2 \times 4 \text{ m/s} = 0.03 \text{ m}^3/s$$

(b) Speed at points 2 and 3:

$$v_2 = \frac{A_1 v_1}{A_2} = 1 \text{ m/s}$$

$$v_3 = \frac{A_1 v_1}{A_3} = 16 \text{ m/s}$$

30.4 Bernoulli's Equation

Ok, that's great, but what causes fluid motion? Fluid flow is constant through straight stretches of pipe, but accelerates or decelerates as the pipe diameter varies.

For an ideal fluid in a pipe, there are no external forces acting on the fluid — only pressure.

Consider a small section of fluid in a pipe:

[Insert diagram.]

$$\sum F_x = P_l A - P_r A = A(P_l - P_r) = A\Delta P = ma_x$$

ΔP represents the pressure difference along the pipe. If $\Delta P > 0$, the water accelerates to the right. The larger the pressure difference, the greater the acceleration. This means that pressure is high where the fluid is moving slowly, and low where the fluid is moving quickly. As a result, pressure is low in narrow sections of pipe. This is referred to as the Bernoulli effect.

[Demo of Bernoulli effect.]

OK, so pressure gradients (differences) are one thing that causes fluid to flow. Gravity also causes fluids to flow. Pressure gradients and gravity both represent forces; we want to relate these things to fluid flow in a single equation. To do this, we'll use the work-energy theorem.

Recall:

$$W = \Delta E = \Delta K + \Delta U_g$$

Here we are assuming that the fluid is inviscid, so there is no change in thermal energy.

We're going to look at what happens to a volume of water as it flows through a pipe that changes elevation and has a change in area.

[Insert diagram.]

As the slug of water moves through the pipe, there a loss of volume on the left and a gain in volume on the right.

$$\Delta V_1 = A_1 \Delta x_1$$

or

$$\Delta V_2 = A_2 \Delta x_2$$

Due to incompressibility, $\Delta V_1 = \Delta V_2 = \Delta V$.

The change in kinetic energy is

$$\boxed{\Delta K = \frac{1}{2}\rho \Delta V ({v_2}^2 - {v_1}^2)}$$

The change in potential energy is

$$\boxed{\Delta U_g = mg\Delta h = \rho \Delta V (h_2 - h_1)}$$

Ok, so we have ΔK and ΔU_g. How much work was done to the volume of water? Work done by pressure on the left was

$$W_1 = F_1 \Delta x_1 = P_1 A_1 \Delta x_1 = P_1 \Delta V$$

Work done by pressure on the right is the opposite, so

$$W_2 = -F_2\Delta x_2 = -P_2 A_2 \Delta x_2 = -P_2 \Delta V$$

The net work is

$$\boxed{W = W_1 + W_2 = (P_1 - P_2)\Delta V}$$

Putting this all together,

$$(P_1 - P_2)\Delta V = \frac{1}{2}\rho\Delta V({v_2}^2 - {v_1}^2) + \rho\Delta V(h_2 - h_1)$$

Dividing by ΔV and re-arranging,

$$P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g h_2 = \text{constant}$$

This is referred to as Bernoulli's Equation.

This quantity is constant along a streamline (a path that a particle takes) in an ideal fluid.

Let's think about what this means by looking at the fluid flow through a pipe that has changes in elevation and diameter.

[Insert diagram.]

30.4.1 Example #2: Water flow from a reservoir

Water flows from a reservoir, through an intake tube, and down to a turbine. The intake tube has a diameter of 100 cm and is 50 m below the reservoir surface. The water drops 200 m to the turbine; water flows into the turbine through a 50 cm diameter nozzle. (a) What is the water speed into the turbine? (b) By how much does the inlet pressure differ from hydrostatic pressure?

(a) There is no flow at the surface of the reservoir, and the elevation at $y_3 = 0$.

$$P_{\text{atm}} + \rho g y_1 = P_{\text{atm}} + \frac{1}{2}\rho_g v_3^2$$

$$v_3 = \sqrt{2gy_1} = 70 \text{ m/s}$$

Note that the speed decreases as the reservoir drains.

(b) The inlet pressure differs from hydrostatic pressure because the water is flowing.

$$v_2 A_2 = v_3 A_3$$

$$v_2 \pi r_2^2 = v_3 \pi r_3^2$$

$$v_2 = v_3 \frac{r_3^2}{r_2^2} = v_3 \left(\frac{r_3}{r_2}\right)^2 = \frac{v_3}{4} = \frac{\sqrt{2gy_1}}{4}$$

Now we use Bernoulli's equation:

$$P_{\text{atm}} + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

$$P_2 = P_{\text{atm}} + \rho g(y_1 - y_2) - \frac{1}{2}\rho v_2^2$$

Notice that

$$P_{\text{static}} = P_{\text{atm}} + \rho g(y_1 - y_2)$$

so the difference in pressure from hydrostatic pressure is simply

$$\frac{1}{2}\rho v_2^2 = \frac{1}{2}\rho \frac{2gy_1}{16} = \frac{\rho g y_1}{16} = 153000 \text{ Pa} \approx 1.5 \text{ atm}$$

The hydrostatic pressure at this depth is 5918000 Pa, or about 6 atm.

31 FLUID DYNAMICS II

Objectives:

- Apply ideas from last class
    1. Ideal fluid: incompressible, laminar, inviscid
    2. Flux continuity: $Q = vA =$ constant
    3. Bernoulli's equation: $P + \frac{1}{2}\rho v^2 + \rho g h =$ constant
- Viscosity

31.1 Demo using Venturi tube

One really useful application of Bernoulli's equation is that it can be used to measure fluid flow through pipes.

[Insert diagram.]

If we look at the streamline that passes through the center of this tube,

$$P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g y_2$$

But points 1 and 2 are at the same elevation, so $y_1 = y_2$, and so

$$P_1 + \frac{1}{2}\rho {v_1}^2 = P_2 + \frac{1}{2}\rho {v_2}^2$$

We can re-arrange this so that

$$P_2 - P_1 = \frac{1}{2}\rho({v_2}^2 - {v_1}^2)$$

From flux continuity,

$$v_1 A_1 = v_2 A_2$$

so

$$v_2 = v_1 \left(\frac{A_1}{A_2}\right)$$

Plugging this into Bernoulli's equation gives

$$P_2 - P_1 = \frac{1}{2}\rho \left({v_1}^2 - \left(\frac{A_1}{A_2}\right)^2 {v_1}^2\right) = \frac{1}{2}\rho {v_1}^2 \left(1 - \left(\frac{A_1}{A_2}\right)^2\right)$$

What is $P_2 - P_1$? The air pressure doesn't change much with elevation (in part because there is no flow in the vertical direction). This means that

$$P_1 + \rho g h = P_2$$

where h is the difference in height of the columns of water. Rearranging,

$$P_2 - P_1 = \rho g h$$

Inserting this into Bernoulli's equation,

$$\rho g h = \frac{1}{2}\rho {v_2}^2 \left(1 - \left(\frac{A_2}{A_1}\right)^2\right)$$

Dividing by ρ and solving for v_2 gives

$$\boxed{v_1 = \sqrt{\frac{2gh}{1 - \left(\frac{A_1}{A_2}\right)^2}}}$$

For this problem, $h = 0.1$ m, $\rho = 1000$ kg/m^3, $r_1 = 0.01$ m, $r_2 = 0.03$ m, and therefore $A_1/A_2 = 1/9$.

And this means that

$$\boxed{v_1 = 1.4 \text{ m/s}}$$

What is the flux?

$$Q = v_1 A_1 = v_1 \pi r_1^2 = 0.00044 \text{ m}^3/\text{s}$$

Human lungs hold about 6 L of air, or 0.006 m^3. With this flux, it would take 13.5 s to empty your lungs ($\Delta t = V_{lung}/Q$).

31.2 Viscosity

Let's see what happens if we relax the condition that the fluid is inviscid, but maintain the other conditions.

[Insert diagram of fluid flow through a pipe with friction.]

From conservation of energy:

$$W = \Delta K + \Delta U_g + \Delta E_{th}$$

Following the same steps as when we derived Bernoulli's equation,

$$(P_1 - P_2)\Delta V = \frac{1}{2}\rho\Delta V({v_2}^2 - {v_1}^2) + \rho g(h_2 - h_1) + \Delta E_{th}$$

If we consider a horizontal pipe with uniform diameter, $h_1 = h_2$ and flux continuity requires that $v_1 = v_2$. Therefore,

$$\Delta E_{th} = (P_1 - P_2)\Delta V$$

Friction within the fluid, and between the fluid and walls of the pipe, cause $\Delta E_{th} > 0$. This means that $P_2 < P_1$.

Experiments show that the pressure difference needed to drive laminar flow of a viscous fluid through a pipe is

$$P_1 - P_2 = 8\pi\eta\frac{Lv_{avg}}{A}$$

where η is the fluid viscosity and has units of Pa·s. The viscosity of water is about 10^{-3} Pa·s, while the viscosity of honey is 20–600 Pa·s. This equation is referred to as Poiseuille's Equation. It tells us that the average velocity of fluid flowing through a section of pipe is

$$v_{avg} = \frac{(P_1 - P_2)}{L}\frac{A}{8\pi\eta}$$

In other words, you need a pressure gradient to drive flow, large pipes provide less resistance, and fluids with high viscosity flow slowly. How do you keep the velocity high?

Examples:

1. Alaska pipeline: Use pump stations to create large pressure gradients.
2. Clogged arteries: Use blood thinner to reduce η, or scraping out arteries to increase A.

We can also relate this to the amount of thermal energy that is produced in a section of fluid.

$$\Delta E_{th} = 8\pi\eta\frac{Lv_{avg}}{A}\Delta V$$

where $\Delta V = A\Delta x$, so

$$\Delta E_{th} = (P_1 - P_2)\Delta V = 8\pi\eta L v_{avg}\Delta x$$

The amount of thermal energy that is generated in the fluid is proportional to the viscosity, length of pipe, velocity, and displacement of the fluid.

So I've now just opened a can of worms... Viscosity is generally highly temperature dependent, and density is sometimes temperature dependent (so assumption of incompressibility is questionable).

31.2.1 Example #1: Application of Poiseuille's equation

A stiff, 10-cm long tube with an inner diameter of 3.0 mm is attached to a small hole in the side of a tall beaker. The tube sticks out horizontally and is open at one end. The beaker is filled with with 20°C water ($\eta = 10^{-3}$ Pa · s) to a level 45 cm above the hole, and is continually topped off to maintain that level. What is the volume flow rate (flux) through the tube?

The pressure difference between the ends of the tube is

$$P_1 - P_2 = (P_{atm} + \rho gh) - (P_{atm}) = \rho gh$$

Poiseuille's equation tells us how the pressure difference relates to the flow of the water.

$$P_1 - P_2 = 8\pi\eta\frac{v_{avg}L}{A} = \rho gh$$

Solving for v_{avg}

$$v_{avg} = \frac{\rho ghA}{8\pi\eta L} =$$

and multiplying by A:

$$Q = \frac{\rho ghA^2}{8\pi\eta L} = \frac{\rho gh\pi^2 r^4}{8\pi\eta L} \Rightarrow \boxed{Q = 8.8 \times 10^{-5}\ \mathrm{m^3/s}}$$

32 OSCILLATIONS

Objectives:

- simple harmonic motion
- linear restoring forces: spring, bobbing icebergs, pendulums
- equations of motion

We've now discussed motion of solids and fluids. We'll now talk about a type of motion that is common to both: oscillations and waves (waves are a type of oscillation). We'll start by studying simple harmonic motion.

32.1 Springs

Consider the forces acting on a mass that is on a frictionless, horizontal surface and is connected to a spring.

[Insert digram.]

$$\sum F_x = F_{sp} = ma$$

$$\sum F_x = -kx = ma$$

Here, I have define $x = 0$ as the location of the end of the spring when it is in equilibrium. Acceleration is the rate of change velocity, and velocity is the rate of change of position. In calculus, you would express a_x as a second derivative.

$$-kx = m\frac{d^2x}{dt^2}$$

This is a second-order, ordinary differential equation. This is the sort of equation that you would learn to solve in MATH 302: Differential Equations.

If you stretch the spring to an initial displacement x_i and release it from rest, the solution to this differential equation is

$$x = x_i \cos\left(\sqrt{\frac{k}{m}}t\right)$$

[Sketch of $x(t)$.]

From this graph, its apparent that the period of oscillation is

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

and the frequency is

$$\boxed{f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

The velocity will be high when the displacement is 0, and vice-versa. This suggests that the velocity varies as a sine function, with the same frequency:

$$v = v_{max}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Let's use conservation of energy to find v_{max}.

$$0 = \Delta K + \Delta U_s = \frac{1}{2}m\left(v_f^2 - v_i^2\right) + \frac{1}{2}k\left(x_f^2 - x_i^2\right)$$

When $x = x_i$, $v = 0$ and when $v = v_{max}$, $x = 0$. Therefore,

$$mv_{max}^2 = kx_i^2 \Rightarrow \boxed{v_{max} = x_i\sqrt{\frac{k}{m}}}$$

Since the acceleration is slope of velocity, we expect it to vary as a cosine function. We can show this quite easily using Newton's second law.

$$\sum F_x = -kx = ma$$

$$a = -\frac{k}{m}x = -x_i\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right)$$

Demo: The period of oscillation is large for large masses and small spring constants.

Any system that oscillates sinusoidally, such as a spring, is referred to as a *simple harmonic oscillator*. Simple harmonic motion occurs when you have a *linear restoring force*.

32.2 Simple pendulums
Pendulums are also (approximately) simple harmonic oscillators.

[Insert diagram.]

The force-balance along the trajectory of the pendulum is

$$\sum F = -F_g \sin\theta = ma_s = m\frac{d^2 s}{dt^2}$$

If θ is "small", then $\sin\theta \approx \theta$ and therefore

$$-mg\theta = m\frac{d^2 s}{dt^2}$$

The angle θ is related to the arc length by $s = L\theta$, so

$$-mg\theta = mL\frac{d^2\theta}{dt^2}$$

and therefore the gravitational force is a *linear restoring force*. Simplifying,

$$-g\theta = L\frac{d^2\theta}{dt^2}$$

which should look kind of familiar. If we start at rest at $\theta = \theta_i$, then the solution is

$$\theta = \theta_i \cos\left(\sqrt{\frac{g}{L}}t\right)$$

and the period of oscillation is

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

Note that $\sqrt{g} \approx \pi$, so a simple rule of thumb is that

$$T = 2\sqrt{L}$$

In lab I asked you to try to find a mathematical relationship between period and length by plotting period vs. length using a log-log scale. Taking the logarithms of both sides of the equation,

$$\log T = \log 2 + \log\sqrt{L} = \log 2 + \frac{1}{2}\log L$$

$$T' = 0.3 + 0.5L'$$

32.2.1 Biological application

Biological application: Part of walking involves letting your legs swing like pendulums under the influence of gravity. The more you "use" gravity, the less work that your muscles have to do. This is accomplished by moving your legs at their "natural frequency". For example, if your leg length is $L = 0.75$ m, T=1.74 s. It should be easy for you to take two steps in 1.7 s. Moving it more or less quickly takes extra work from your muscles.

Thus the speed that animals walk depends on the length of their legs.

[Diagram of swinging leg, with angles and distances indicated.]

$$v = \frac{\Delta x}{\Delta t}$$

Let $\Delta t = T$, where T is the amount of time it takes to make two steps (one with each foot). The distance traveled is $4L \sin\theta \approx 4L\theta$. Therefore,

$$v \approx \frac{4L\theta}{2\pi\sqrt{L/g}} = \frac{2\theta}{\pi}\sqrt{\frac{g}{L}} \approx 2\theta\sqrt{L}$$

[Sketch v vs L.]

If $\theta = 10°$ and $L = 0.75$ m, then

$$\boxed{v = 0.30 \text{ m/s} = 1.09 \text{ km/h}}$$

32.3 Physical pendulums

How is the situation different for a physical pendulum (an pendulum that has weight distributed along its length)? We can address this using torque.

[Diagram of a physical pendulum.]

$$\sum \tau = -mgl \sin\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

Again use the small angle approximation,

$$-mgl\theta = I\frac{d^2\theta}{dt^2}$$

which is very similar to the equations we wrote down previously. The solution is

$$\theta = \theta_i \cos\left(\sqrt{\frac{mgl}{I}}t\right)$$

where

$$\boxed{T = 2\pi\sqrt{\frac{I}{mgl}}}$$

Recall that the moment of inertia of a point mass is $I = mL^2$ and note that its center of mass is a distance of $l = L$ from the axis of rotation. Thus,

$$T = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}$$

A better estimate for a human leg would be a cylinder that rotates around its end. The moment of inertia of cylinder rotating around its end is $I = (1/3)mL^2$ and the center of mass is located at $l = L/2$.

The natural period of oscillation is

$$T = 2\pi\sqrt{\frac{(1/3)mL^2}{mgL/2}} = 2\pi\sqrt{\frac{2L}{3g}}$$

Again using $L = 0.75$ m, we find that the period is $T = 1.42$ s.

Multiply v from previous calculation by $\sqrt{3/2}$ to find walking speed, which gives $v = 1.33$ km/h.

33 DAMPED AND DRIVEN OSCILLATIONS

Objectives:

- Simple harmonic motion
- Damped oscillations
- Driven oscillations

33.1 Simple harmonic motion

Last class:

- Oscillations caused by restoring forces (e.g., stretch a spring, it pulls back)
- Simple harmonic motion (i.e., sinusoidal oscillations) are caused by *linear* restoring forces
- Examples of linear restoring forces: masses on springs and pendulums (for small angles)
- Derived equations of motion for simple harmonic motion

Displacement as a function of time:

$$x(t) = A\cos(2\pi f t) = A\cos\left(2\pi\frac{t}{T}\right)$$

Here A is the amplitude of the oscillations.

Velocity is the instantaneous slope of position. Graphically, we saw that this means that

$$v(t) = -v_{max}\sin(2\pi f t)$$

Accleration is the slope of velocity. Graphically, we saw that this means that

$$a(t) = -a_{max}\cos(2\pi f t)$$

33.1.1 Springs

- Amplitude is $A = x_i$
- From conservation of energy: $v_{max} = x_i\sqrt{\frac{k}{m}}$
- From Newton's Second Law: $a_{max} = x_i\frac{k}{m}$
- $T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$

33.1.2 Pendulums

For pendulums, we use angular position instead of linear position and use the small angle approximation. Thus, the amplitude is θ_i, which is the initial angular position. The equations work out to be identical (with different variables) to the equation for an oscillating spring. We found that

- for a simple pendulum, $T = 2\pi\sqrt{\frac{L}{g}} \approx 2\sqrt{L}$
- for a physical pendulum, $T = 2\pi\sqrt{\frac{I}{mgl}}$, where I is the moment of inertia around the axis of rotation and d is the distance from the axis of rotation to the center of mass of the pendulum

33.1.3 Example #1: Pendulum on Mars

The first astronauts to visit Mars are each allowed to take along some personal items to remind them of home. One astronaut takes along a grandfather clock, which, on Earth, has a pendulum that takes 1 second per swing, each swing corresponding to one tick of the clock. When the clock is set up on Mars, will it run faster or slower? How much faster or slower?

$$T = 2\pi\sqrt{L}g$$

Gravitational acceleration is lower on Mars (3.711 m/s^2), so the pendulum will run slower. To find how much slower, we need to determine the length of the pendulum.

$$L = g\left(\frac{T}{2\pi}\right)^2 = 0.25 \text{ m}$$

$$T_{Mars} = 2\pi\sqrt{\frac{0.25 \text{ m}}{3.711 \text{ m/s}^2}} = 1.63 \text{ s}$$

The clock runs 63% slower. So when 24 hours have passed, the clock will only think that 14.7 hours have passed.

33.1.4 Example #2: Oscillating spring

A 204 g block is suspended from a vertical spring, causing the spring to stretch by 20 cm. The block is then pulled down an additional 10 cm and released. What is the speed of the block when it is 5.0 cm above the equilibrium position?

$$v = v_{max}\sin(2\pi f t)$$

(Be careful with sign here, and explain why its different than what we've done before.)

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$v_{max} = x_i\sqrt{\frac{k}{m}}$$

This problem involves several steps. We need to find v_{max}, f, and t to solve for the speed. However, f and v_{max} both depend on k, so we also need to figure that out. Let's start with that.

In equilibrium,

$$\sum F_y = F_{sp} - F_g = -k\Delta y - mg = 0$$

$$k = \frac{-mg}{\Delta y}$$

Since $\Delta y = -0.1$ m and $m = 0.204$ kg, we find that $k = 20$ N/m. This means that $f = 1.58$ Hz and $v_{max} = 0.99$ m/s.

Now we need to figure out t when $y = 0.05$ m.

$$y = y_i\cos(2\pi f t)$$

$$t = \frac{1}{2\pi f}\cos^{-1}\left(\frac{y}{y_i}\right) = 0.11 \text{ s}$$

Now plug this all back into the equation for v

$$\boxed{v = 0.86 \text{ m/s}}$$

33.2 Damped oscillations

We've so far only discussed simple harmonic oscillators in the absence of external forces. External forces can cause oscillations to be damped or to grow. In damped oscillations, mechanical energy is converted to thermal energy. In driven oscillations, an external force does work on the system.

For a pendulum, the main energy loss is due to air resistance, which depends on speed. Let's include this drag force in the force-balance equation. Summing the forces in the direction of the pendulum's motion,

$$\sum F = -F_g \sin\theta - F_d = ma_t$$

The drag force is proportional to the pendulum's velocity

$$F_d = cv_t$$

where c is a constant that depends on the density of the fluid (air) and the cross-sectional area and shape of the mass hanging from the pendulum. Inserting this into the above equation,

$$-mg\sin\theta - cv_t = ma_t$$

Note that v_t and a_t are the tangential velocity and acceleration. Recall that $v = \omega r = \omega L$ and $a_t = \alpha r = \alpha L$. Thus, our force balance equation becomes

$$-mg\sin\theta - cL\omega = mL\alpha$$

Using the small-angle approximation, this reduces to

$$-mg\theta - cL\omega = mL\alpha$$

We need differential equations to solve this.

Note that angular velocity is the rate of change of angular position, and angular acceleration is the rate of change of angular velocity. This gives

$$-mg\theta - cL\frac{d\theta}{dt} = mL\frac{d^2\theta}{dt^2}$$

which is a second-order ordinary differential equation. This one has the solution

$$\boxed{\theta(t) = \theta_i e^{-\frac{c}{2m}t}\cos\left(\sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}\,t\right)}$$

I admit, this is a mess! But it tells us two things:

(1) The amplitude decays exponentially with time according because of the term $e^{-\frac{c}{2m}t}$. The e-folding time, which is the time at which the amplitude is $1/e$ of the original amplitude, is $t = 2m/c$.

For a metal sphere with a diameter of 0.01 m, $t \approx 400$ s. This is why pendulums can oscillate for a long time. You can increase the e-folding time by increasing the mass of the pendulum, so that air resistance has less of an impact on its motion.

(2) The natural frequency of the pendulum is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L} - \frac{c^2}{4m^2}}$$

Including drag decreases the frequency ever so slightly. For a 2-cm diameter steel ball, $c^2/(4m^2) \approx 2.7 \times 10^{-9}$ s^{-2}. If $L = 1$ m, then $g/L \approx 10$ s^{-2}. So the drag has only a very minor effect on the pendulum's frequency.

[Diagram of a damped oscillation.]

33.2.1 Example #3: Damped pendulum

A 500 g mass on a string oscillates as a pendulum. The pendulum's energy decays to 50% of its initial value in 30 s. What is the value of the damping constant?

This question is essentially asking when does $e^{-\frac{c}{2m}t} = 0.5$? In this case, we aren't told anything about the drag coefficient, so we have to think of this more generally. Instead, this could be written as $e^{-\frac{t}{\tau}} = 0.5$, where τ is the damping constant or e-folding time. Solving for τ:

$$\tau = -\frac{t}{\ln 0.5} = -\frac{30 \text{ s}}{\ln 0.5} = 43.2 \text{ s}$$

33.3 Driven oscillations and resonance

Oscillating systems have a natural frequency when left alone (as we've already seen). What happens if you subject a system to a periodic, driving frequency?

If the natural frequency matches the natural frequency of the system, you get large oscillations (especially if damping is small) — this is called resonance. Modeling this is more difficult than modeling damped oscillations, which required differential equations, so let's just look at it conceptually.

[Frequency-response curve.]

33.3.1 Demo: Pendulums of different lengths on a rod

33.3.2 Example #4: Cars on springs

Cars ride on springs; they therefore have a natural frequency. For a particular car, the natural frequency is 2 Hz. The car is driving 20 mph over a washboard road with bumps every 10 ft. What is the driving frequency? How does it compare to the natural frequency?

$$v = \frac{\Delta x}{\Delta t} \Rightarrow T = \frac{\Delta x}{v} = \frac{10 \text{ ft}}{20 \text{ mph}} = 0.35 \text{ s}$$

Which means that

$$f = \frac{1}{T} = 2.9 \text{ Hz}$$

This is just above the natural frequency. Driving faster will reduce the amplitude of the oscillations.

34 INTRODUCTION TO WAVES

Objectives:

- Wave model
  - Oscillations of fixed points
  - Snapshots in time
  - Wave speed

34.1 Background

Waves are a type of oscillation. There are many types of waves:

- sound waves – gas
- elastic waves – solid
- water waves – liquid
- electromagnetic waves – don't require a medium...

These can be categorized as two different types of waves.

(1) Mechanical waves:

- Motion of a substance caused by a disturbance to the substance. Mechanical waves include sound, elastic, and water waves.
- Wave speed, v, depends on material properties.
- No net motion of material. Waves transmit energy, not particles. (Demonstrate with a slinky.)

(2) Electromagnetic waves:

- Waves of an electromagnetic field. Includes visible light, radio waves, x-rays, ...
- Not due to the motion of a substance. EM waves can travel through vacuums!

The details of wave behavior differ, but we can discuss basic properties of waves with a "wave model".

The most basic waves are transverse (wave on a string) or longitudinal (sound wave). For transverse waves, the particle motion is perpendicular to the wave direction, whereas for longitudinal waves the particle motion is parallel to the wave direction.

Waves can travel as a single pulse or a series of pulses. When waves are produced by a simple harmonic oscillator, the waves are sinusoidal.

[Diagram of a sinusoidal wave; show how its position changes at some later time.]

We can describe the wave displacement, y, by

$$y(x,t) = A\cos\left(2\pi\frac{x}{\lambda} \pm 2\pi\frac{t}{T}\right)$$

The displacement depends on both location and time. x/λ describes the wave shape. The $+$ indicates that the wave is travelling to the left, the $-$ indicates that it is travelling to the right. Note that the wave model uses three dimensions (x, y, and t)... Also note that this is written in terms of a transverse wave, but it also applies to longitudinal waves. The vertical displacement can just be replaced with a horizontal displacement relative to an object's initial position.

Let's analyze this equation in a bit of detail.

[Sketches of the waves.]

34.2 Oscillations of fixed points

First, consider fixed points along the wave, e.g., $x = 0$. Then

$$y(0,t) = A\cos\left(\pm 2\pi\frac{t}{T}\right)$$

Each particle in the waves oscillates sinusoidally with amplitude A.

What if $x = \lambda/2$?

$$y\left(\frac{\lambda}{2},t\right) = A\cos\left(2\pi\frac{\lambda/2}{\lambda} \pm 2\pi\frac{t}{T}\right) = A\cos\left(\pi \pm 2\pi\frac{t}{T}\right)$$

The π at the beginning is just a "phase" shift. To see how, let's recall an identity from trigonometry:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Here, $\alpha = \pi$; $\cos\pi = -1$ and $\sin\pi = 0$. This means that

$$y\left(\frac{\lambda}{2}, t\right) = -A\cos\left(\pm 2\pi\frac{t}{T}\right)$$

which is what we would expect. If we looked at $x = \lambda$, we would find that

$$y(\lambda, t) = A\cos\left(\pm 2\pi\frac{t}{T}\right)$$

which is the same as for $x = 0$. This is good!

[Diagram showing how the points oscillate in time.]

34.3 Snapshots in time

Now let's take a look at snapshots in time for a wave that is travelling to the right (use negative sign in the equation). If $t = 0$, this means that

$$y(x, 0) = A\cos\left(2\pi\frac{x}{\lambda}\right)$$

When $t = T/4$, this gives

$$y\left(x, \frac{T}{4}\right) = A\cos\left(2\pi\frac{x}{\lambda} - \frac{\pi}{2}\right),$$

which represents a 90° phase shift to the right. To see this, we can again use the angle sum and difference identity. In this case, $\beta = \pi/2$, so $\cos\beta = 0$ and $\sin\beta = 1$. This means that

$$y\left(x, \frac{T}{4}\right) = A\sin\left(2\pi\frac{x}{\lambda}\right)$$

[Diagram showing a wave at different points in time.]

34.4 Wave speed

The wave model characterizes the speed at which a wave propagates to the left or right.

$$v = \frac{\Delta x}{\Delta t} = \frac{\lambda}{T} \Rightarrow \lambda = vT$$

Plugging this into the wave model gives

$$y(x,t) = A\cos\left(2\pi\frac{x}{vT} \pm 2\pi\frac{t}{T}\right) = A\cos\left(2\pi\frac{1}{T}\left(\frac{x}{v} \pm t\right)\right) = A\cos\left(2\pi f\left(\frac{x}{v} \pm t\right)\right)$$

This clearly shows that the angular frequency is the same if you are talking about a peak traveling a distance λ in time T or if you are talking about a fixed point going through one oscillation.

For some systems, the wave speed is independent of frequency. If this is the case, if you know the frequency or period you can calculate the wavelength, and vice-versa.

high frequency waves = short wavelength = high pitch (sound) or color blue (EM waves)

low frequency waves = long wavelength = low pitch (sound) or color red (EM waves)

34.4.1 Wave on a string

From a force balance analysis:

$$v = \sqrt{\frac{F_t}{\mu}},$$

where F_t is the tensional force and μ is the linear density (mass / length). Waves travel quickly through taut strings that don't have much mass.

34.4.2 Wave travelling through an ideal gas

From thermodynamics and analysis of ideal gas law:

$$v_{sound} = \sqrt{\frac{\gamma RT}{M}}$$

where $\gamma = c_p/c_v$ is the adiabatic index and is often close to one, $R = 8.31$ J/(mol $\cdot$ K) is the gas constant, T is temperature in Kelvin, and M is the molar mass (kg/mol of the gas). The speed of sound basically depends on temperature.

For non-ideal gases, v_{sound} has a slight dependency on density, and therefore pressure, but we won't worry about that.

34.4.3 Electromagnetic waves

Predicted by Maxwell, and has not been proven otherwise. In a vacuum,

$$v = c \approx 3 \times 10^8 \text{ m/s} = \text{constant}$$

In other materials,

$$v = \frac{c}{n},$$

where $n \geq 1$ is the index of refraction. It essentially has to do with interactions between the electromagnetic wave and electrons in the material.

35 TRAVELING WAVES, DOPPLER EFFECT, AND SHOCK WAVES

Objectives:

- Review description of traveling waves
- Doppler effect
- Shock waves

35.1 Review of traveling waves

- Wave model for waves produced by simple harmonic oscillator: $\Delta y(x,t) = A\cos\left(2\pi\frac{x}{\lambda} \pm 2\pi\frac{t}{T}\right)$
- Waves may be transverse or longitudinal
- Wave speed: $v = \frac{\lambda}{T} = \lambda f$
  - Wave on a string: $v = \sqrt{\frac{F_t}{\mu}}$, where F_t is the tensional force and μ is the linear density (mass / length).
  - Sound wave in an ideal gas: $v_{sound} = \sqrt{\frac{\gamma RT}{M}}$, where $\gamma = c_p/c_v$ is the adiabatic index and is often close to one, $R = 8.31$ J/(mol $\cdot$ K) is the gas constant, T is temperature in Kelvin, and M is the molar mass (kg/mol of the gas). The speed of sound basically depends on temperature. For non-ideal gases, v_{sound} has a slight dependency on density, and therefore pressure, but we won't worry about that.
  - Electromagnetic waves in a vacuum: $v = c \approx 3 \times 10^8$ m/s = constant; for travel through a material (e.g., glass or water), $v = \frac{c}{n}$, where $n \geq 1$ is the index of refraction. It essentially has to do with interactions between the electromagnetic wave and electrons in the material.
- Note that the wave model describes how the displacement caused by a wave varies in time and space. It does not determine what controls the the wave speed, frequency, wavelength, and amplitude of specific types of waves. Those are things that are unique to a system.

For waves that have a constant velocity, if you know the frequency or period you can easily calculate the wavelength, and vice-versa.

high frequency waves = short wavelength = high pitch (sound) or color blue (EM waves)

low frequency waves = long wavelength = low pitch (sound) or color red (EM waves)

[Demo: sound chirp with audacity]

The human ear can detect sounds between about 20 Hz and 20 kHz. Frequencies less than 20 Hz are referred to as infrasound. These waves travel large distances, and are used for detecting nuclear explosions, studying volcanoes and earthquakes, ... Frequencies greater than 20 kHz are referred to as ultrasound. Their short wavelengths make them useful for imaging soft tissue/fetuses.

35.1.1 Example #1: Find wave frequency

A sinusoidal wave travels with speed 200 m/s. Its wavelength is 4.0 m. What is its frequency?

$$v = \frac{\lambda}{T} = \lambda f$$

$$f = \frac{v}{\lambda} = 50 \text{ Hz}$$

35.1.2 Example #2: Analysis of wave model

A traveling wave has displacement given by $y(x,t) = (2.0 \text{ cm}) \times \cos(2\pi x - 4\pi t)$, where x is measured in cm and t in s.

(a) Draw a snapshot graph of this wave at t=0 s.

(b) On the same set of axes, use a dotted line to show the snapshot graph of the wave at t=1/8 s.

(c) What is the speed of the wave?

[Snapshot graph.]

Discuss phase shift using sum and difference identity.

$$v = \frac{\lambda}{T}$$

$$\frac{2\pi}{\lambda} = 2\pi \Rightarrow \lambda = 1 \text{ cm}$$

$$\frac{2\pi}{T} = 4\pi \Rightarrow T = 2 \text{ s}$$

$$\boxed{v = 0.5 \text{ cm/s}}$$

35.1.3 Example #3: Wave on spider silk

A female orb spider has a mass of 0.50 g. She is suspended from a tree branch by a 1.1 m length of 0.0020-mm-diameter silk. Spider silk has a density of 1300 kg/m^3. If you tap the branch and send a vibration down the thread, how long does it take to reach the spider?

$$v = \sqrt{\frac{F_t}{\mu}}$$

The tensional force is found from Newton's first law

$$\sum F_y = F_t - F_g = 0 \Rightarrow F_t = mg,$$

and the linear density is

$$\mu = \frac{m_s}{L} = \frac{\rho V}{L} = \frac{\rho\pi(d/2)^2 L}{L} = \frac{1}{4}\rho\pi d^2$$

Putting this together,

$$v = \sqrt{\frac{4mg}{\rho\pi d^2}} = \frac{2}{d}\sqrt{\frac{mg}{\rho\pi}} = 1.10 \times 10^3 \text{ m/s}$$

The time that it takes to travel a distance L is

$$\Delta t = \frac{L}{v} = 10^{-3} \text{ s} = 1.0 \text{ ms}$$

35.2 Doppler effect

Many wave sources emit spherical waves (this is difficult to visualize). The sound that is heard depends on whether or not the source is moving. This is referred to as the Doppler Effect.

[Diagram of stationary source and equally-spaced spherical waves.]

[Diagram of moving source, showing that frequency depends on position relative to the source.]

If the wave is travelling to the right at speed v_o, and the source is travelling at v_s, the observed wavelength is

$$\lambda = v_o T - v_s T = (v_o - v_s)T_o = \frac{v_o - v_s}{f_o}.$$

But you perceive the wavelength as

$$\lambda = \frac{v_o}{f}$$

Setting these equal gives

$$\frac{v_o}{f} = \frac{v_o - v_s}{f_o}$$

Taking the inverse

$$\frac{f}{v_o} = \frac{f_o}{v_o - v_s}$$

Multiplying by v_o gives

$$\boxed{f = \frac{f_o}{1 - v_s/v_o}}$$

This is for an object moving toward you. For an object moving away from you, the result is

$$\boxed{f = \frac{f_o}{1 + v_s/v_o}}$$

If the object is moving toward you and $v_s = v_0$, the equation blows up. This is when you get a shock wave ("sonic boom" for sound waves). If $v_s > v_0$ you get a negative frequency, which doesn't make sense.

[Demo: Animations of sound waves illustrating the Doppler effect and shock waves]

[Demo: Whirling tuning fork.]

35.2.1 Example #4: Doppler effect on a bicycle

A whistle you use to call your hunting dog has a frequency of 21 kHz, but your dog is ignoring it. You suspect the whistle may not be working, but you can't hear sounds above 20 kHz. To test it, you ask a friend to blow the whistle, then you hop on your bicycle. In which direction should you ride (toward or away from your friend) and at what minimum speed to know if the whistle is working?

You need to ride away from your friend to reduce the frequency, and you need to use the equation

$$f = \frac{f_o}{1 + v_s/v_o}$$

Solving for v_s:

$$v_s = \left(\frac{f_o}{f} - 1\right) v_o$$

$v_o = 343$ m/s, $f_o = 21$ kHz, $f = 20$ kHz

$$v_s = 17.1 \text{ m/s}$$

36 WAVE SUPERPOSITION AND STANDING WAVES

Objectives:

- Wave superposition
- Power spectrum
- Standing waves

36.1 Wave superposition

Waves are unique in that they can pass through each other. When they do so, they can constructively or destructively interfere.

[Diagrams of constructive and destructive interference of a wave travelling on a string.]

This is referred to as wave superposition. Wave superposition has several important implications for sound (and other waves).

1. Sound intensity/quality in a room. Sound waves are not one-dimensional.

   [Diagram of sound waves coming from speakers in a room.]

   Where do you get constructive interference? Destructive interference? The answer depends on wavelengths and room geometry (when you account for echos). This is something that we'll see again next semester when we talk about the fact that light is a type of wave.

2. Sound waves are generally not a simple sinusoid with one frequency. Instead, they are a combination of many sinusoidal functions with different frequencies.

So for example, the displacement at one location due to several waves travelling from the same location and in the same direction is

$$y(x,t) = y_1 + y_2 + y_3 + ... = A_1 \cos\left(2\pi\left(\frac{x}{\lambda_1} - \frac{t}{T_1}\right)\right) + A_2 \cos\left(2\pi\left(\frac{x}{\lambda_2} - \frac{t}{T_2}\right)\right) + A_3 \cos\left(2\pi\left(\frac{x}{\lambda_3} - \frac{t}{T_3}\right)\right) + ...$$

We can replace λ_i in each term with

$$\lambda_i = vT_i,$$

where $v = 343$ m/s is the speed of sound (a constant).

$$y(x,t) = A_1 \cos\left(2\pi\left(\frac{x}{vT_1} - \frac{t}{T_1}\right)\right) + A_2 \cos\left(2\pi\left(\frac{x}{vT_2} - \frac{t}{T_2}\right)\right) + A_3 \cos\left(2\pi\left(\frac{x}{vT_3} - \frac{t}{T_3}\right)\right) + ...$$

Now replacing $1/T_i = f_i$, and factoring out f_i, this gives

$$y(x,t) = A_1 \cos\left(2\pi f_1\left(\frac{x}{v} - t\right)\right) + A_2 \cos\left(2\pi f_2\left(\frac{x}{v} - t\right)\right) + A_3 \cos\left(2\pi f_3\left(\frac{x}{v} - t\right)\right) + ...$$

Each of these terms has a unique amplitude, A, and frequency, f. Another way to visualize this is to plot a power spectrum (which you did in lab).

[Diagram of power spectrum.]

[Diagram showing what the waves indicated in the power spectrum actually look like.]

[Diagram to show result of adding waves together.]

3. Beats occur when two waves with similar frequencies interfere with each other.

   [Diagram of beats – sometimes constructive interference, sometimes destructive interference.]

   Add them together and get quiet and loud periods.

   [Diagram showing resultant wave.]

   You still have constant sound – this isn't like the beating of a drum.

   No need to show this, but

$$A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t) = 2A \cos\left(2\pi \frac{f_1 + f_2}{2} t\right) \cos\left(2\pi \frac{f_1 - f_2}{2} t\right)$$

   If f_1 and f_2 are close, then the frequency of the second cosine is often too low to be perceived as pitch. Therefore the pitch that you hear is determined by the frequency of the first cosine function: $1/2(f_1 + f_2) \approx f_1$. And the frequency of volume modulation is approximately

$$f_{beat} = |f_1 - f_2|$$

   Musicians use beats to tune their instruments. You remove tune the instrument by changing its frequency so as to remove any beats from being produced...

36.2 Standing waves

Standing waves, which are waves in which the peaks and troughs don't move, can be described by adding together to travelling waves.

[Insert diagram of a guitar string, fundamental mode oscillation].

When you pluck a string, there are other vibrational modes that are present.

[Diagram of second and third harmonics.]

The wavelength of each mode is

$$\lambda_m = \frac{2L}{m}$$

where L is the length of the string and m is the mode.

$$v = \lambda_m f_m$$

$$f_m = \frac{v}{\lambda_m} = m\frac{v}{2L}$$

So this means that $f_1 = v/(2L)$, and $f_2 = 2f_1$, $f_3 = 3f_1$, etc. Note that second harmonic is the same as the first overtone.

Describing standing waves by adding or subtracting together travelling waves. For the case of the oscillating string, we want to subtract the travelling waves:

$$y(x,t) = A\cos\left(2\pi\frac{x}{\lambda} - 2\pi\frac{t}{T}\right) - A\cos\left(2\pi\frac{x}{\lambda} + 2\pi\frac{t}{T}\right)$$

A trig identity that you may or may not know:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Using this to expand the above expression, we get

$$y(x,t) = A\left[\cos\left(2\pi\frac{x}{\lambda}\right)\cos\left(2\pi\frac{t}{T}\right) + \sin\left(2\pi\frac{x}{\lambda}\right)\sin\left(2\pi\frac{x}{\lambda}\right) - \cos\left(2\pi\frac{x}{\lambda}\right)\cos\left(2\pi\frac{t}{T}\right) + \sin\left(2\pi\frac{x}{\lambda}\right)\sin\left(2\pi\frac{x}{\lambda}\right)\right]$$

$$\boxed{y(x,t) = 2A\sin\left(2\pi\frac{x}{\lambda}\right)\sin\left(2\pi\frac{t}{T}\right)}$$

The first sine describes the how the oscillation varies in space, and the second sine describes how it varies in time. Let's think about it in terms of the first harmonic. The displacement is always zero at $x = 0$ and $x = L = \lambda/2$, and the displacement is always largest at $x = L/2 = \lambda/4$.

37 STANDING WAVES AND MUSIC

Objectives:

- Standing waves on strings
- Standing waves in open-open and open-closed tubes
- Musical notes

37.1 Wave superposition

Last class we saw that waves can constructively or destructively interfere

Wave superposition:

$$\Delta y(x,t) = \Delta y_1(x,t) + \Delta y_2(x,t)$$

where

$$\Delta y_i(x,t) = A_i \cos\left(2\pi \frac{x}{\lambda_i} \pm 2\pi \frac{t}{T_i}\right)$$

37.2 Standing waves

Standing waves are waves in which the peaks don't travel. Places with no displacement are called nodes, and places of maximum displacement are referred to as anti-nodes. Standing waves can be thought of as two waves of equal frequency, amplitude, and speed, travelling in opposite directions. Using wave superposition, we showed that the equation for a standing wave on a string that is fixed on both ends is:

$$\Delta y(x,t) = A \sin\left(2\pi \frac{x}{\lambda}\right) \sin\left(2\pi \frac{t}{T}\right)$$

where $0 \leq x \leq L$

37.2.1 Strings

[Video demo of standing wave on a string.]

[Diagrams showing various standing waves and associated wavelengths.]

Only certain wavelengths/frequencies are possible. For strings that are fixed on both ends, the possible solutions are

$$\lambda_m = \frac{2L}{m}$$

where $m = 1, 2, 3, \ldots$ are the harmonics. This corresponds to frequencies of

$$f_m = m\left(\frac{v}{2L}\right) = mf_1$$

where v is the wave speed.

37.2.2 Open-open tubes

Standing waves can also form in tubes. These standing waves are longitudinal waves resulting from compression and rarefaction of air. Here, we describe the waves in terms of variations in pressure. Our constraint is that the pressure on the open end of tubes is (approximately) equal to atmospheric pressure.

[Diagrams for open-open tube, with pressure fixed on both ends.]

The same possible wavelengths are possible as for a string that is fixed on both ends, and the pressure could be described as

$$\Delta P(x,t) = A\sin\left(2\pi\frac{x}{\lambda}\right)\sin\left(2\pi\frac{t}{T}\right)$$

where $0 \leq x \leq L$.

37.2.3 Open-closed tubes

The situation is a little different for open-closed tubes because the pressure on the closed end of the tube can differ from atmospheric pressure.

[Diagrams for open-open tube, with pressure fixed on end.]

For open-closed tubes, we have a different set of harmonics than we had for open-open tubes:

$$\lambda_m = \frac{4L}{m}$$

where $m = 1, 3, 5, \ldots$. Note that open-closed tubes only have odd-numbered modes. This corresponds to frequencies of

$$f_m = m\left(\frac{v}{4L}\right) = mf_1$$

It turns out that you can use the same wave model equation to describe the standing waves in this situation, but with a different relationship for λ.

37.2.4 Wave speed

Recall:

- For waves on strings,

$$v = \sqrt{\frac{F_t}{\mu}}$$

  and for sound waves in air

$$v = \sqrt{\frac{\gamma RT}{M}}$$

37.3 Musical notes

I'd like to finish our discussion of waves by talking a bit about musical notes.

Simples notes:

- note1.wav: $f_1 = 220$ Hz (note "A")
- note2.wav: $f_1 = 440$ Hz (note "A", one octave higher)
- note3.wav: $f_1 = 880$ Hz (note "A", two octaves higher)

Beats:

- beats1.wav: $f_1 = 400$ Hz; $f_2 = 410$ Hz; $f_{\text{beat}} = 10$ Hz $\Rightarrow T = 0.1$ s $\Rightarrow$ can't hear beats
- beats2.wav: $f_1 = 400$ Hz; $f_2 = 401$ Hz; $f_{\text{beat}} = 1$ Hz $\Rightarrow T = 1$ s
- beats3.wav: $f_1 = 400$ Hz; $f_2 = 400.5$ Hz; $f_{\text{beat}} = 0.5$ Hz $\Rightarrow T = 2$ s
- beats4.wav: $f_1 = 100$ Hz; $f_2 = 101$ Hz; $f_{\text{beat}} = 1$ Hz $\Rightarrow T = 2$ s

Harmonics:

- harmonic_A.wav: $f_1 = 220$ Hz (dropped an octave), $f_2 = 440$ Hz, $f_3 = 660$ Hz, $f_4 = 880$ Hz, $f_5 = 1100$ Hz
- harmonic_Csharp.wav: $f_1 = 277.2$ Hz, $f_2 = 554.4$ Hz, $f_3 = 831.6$ Hz, $f_4 = 1108.8$ Hz, $f_5 = 1386$ Hz
- harmonic_D.wav: $f_1 = 293.33$ Hz (dropped an octave), $f_2 = 586.67$ Hz, $f_3 = 880$ Hz, $f_4 = 1173.33$ Hz, $f_5 = 1466.67$ Hz
- harmonic_E.wav: $f_1 = 330$ Hz, $f_2 = 660$ Hz, $f_3 = 990$ Hz, $f_4 = 1320$ Hz, $f_5 = 1650$ Hz

Consonance, dissonance, and chords

- consonance_fifth_AE.wav: combine A and E; some harmonics match and sounds good

$$(f_1)_E = 3/2 * (f_1)_A$$

- consonance_fourth_AD.wav: combine A and D; some harmonics match and sounds good

$$(f_1)_D = 4/3 * (f_1)_A$$

- triad_chord.wav: root, third, and fifth: A, $C^{\#}$, and E
- dissonance_AAsharp.wav: simultaneously play two adjacent notes (A and $A^{\#}$); feels unsettled or unresolved. Dissonance is not necessarily a bad thing; its basically beats, but the beat frequency is below hearing threshold (i.e., in the infrasound). Infrasound has been used in movies to produce an unsettled feeling

This has just been a brief intro to music theory. But, basically, the following things determine notes:

- natural frequencies of vibration
  - tube length
  - open-open vs open-closed
- number of overtones/harmonics
- "preciseness" of overtones
- timbre: fundamental frequency might not be most significant harmonic; how sounds decay with time;