MATH475 – Assignment 1, chapter 1 + 2
Due Tuesday September 19 at 10pm on Crowdmark

Problem 1: Find the general solution $u(x, y, z)$ in $\mathbb{R}^3$ for the following PDEs:

a) $u_x = 0$

b) $u_{xy} = 0$

c) $u_{xyz} = 0$

Problem 2: Let $\epsilon > 0$ and consider the domain $\Omega_\epsilon = \{\mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}|| > \epsilon\}$. Show that the function

$$\Phi(\mathbf{x}) = \frac{1}{2\pi} \ln ||\mathbf{x}||$$

is a solution to the Laplace equation $\Delta u = \nabla^2 u = 0$ in Ω_ϵ.

Problem 3: Let $\Omega = \{(x, t) \in \mathbb{R}^2 \mid t > 0\}$ and consider the following initial value problem:

$$\begin{cases} u_t = u_{xxxx} & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \quad \text{(i.e.: } u(x,0) = g(x)) \end{cases}$$

a) Verify that the functions $u_n(x, t) = \frac{1}{n} e^{n^4 t} \sin(nx)$, where n is any positive integer, all satisfy the PDE in Ω.

b) Is this initial value problem well-posed in the sense of Definition 1.5.1 from the textbook?

Problem 4: Supose that all solutions of $\langle a, b\rangle \cdot \nabla u = 0$ satisfy $u(1, 2) = u(3, 6)$ where $a, b \in \mathbb{R}$, what is the ratio b/a?

Problem 5: Consider a fluid in motion whose variations in density are modeled by the transport equation $\rho_t + \sin(t)\rho_x = 0$.

a) Compute $\rho(x, t)$ for $x \in \mathbb{R}$ and $t > 0$ given the initial condition $\rho(x, 0) = \sin(x)$.

b) Write down the trajectory $X(t)$ of a fluid particle that passes through $x = 2$ at $t = 2\pi$.

Problem 6: Find an implicit representation for the characteristics of the PDE $yu_x + xu_y = 0$ in $\mathbb{R}^2$. Does the auxiliary condition $u(x, 0) = x$ yield a well-posed problem for this PDE in $\mathbb{R}^2$? Which well-posedness condition is not satisfied?

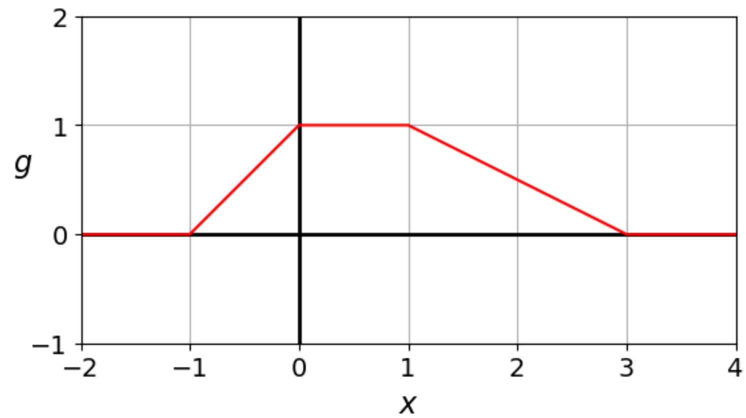
Problem 7: Find a first order linear PDE whose solutions are constant along characteristics of the form $x^2 + y^2 = C$ for $C > 0$. Does the auxiliary condition $u(x, 0) = x$ yield a well-posed problem for this PDE in $\mathbb{R}^2 \setminus \{(0, 0)\}$? Which well-posedness condition is not satisfied?

Problem 8: Solve the following initial value problem in the half-domain where $x_3 > 1$:

$$\begin{cases} \mathbf{x} \cdot \nabla u = u \\ u(x_1, x_2, 1) = x_1 + x_2 \end{cases}$$

Problem 9: Solve the inhomogeneous Burgers' equation $u_t + uu_x = f$ with initial condition $u(x, 0) = -x$ and $f \in \mathbb{R}$. At what time $t_{\text{shock}} > 0$ does the solution first becomes discontinuous? How does the source term f modify the solution to the homogeneous problem?

Problem 10: Consider Burgers' equation $u_t + uu_x = 0$ with initial condition $g(x, 0)$ shown below:



Sketch the characteristics in the upper x-t plane and draw the solution $u(x, t_{\text{shock}})$ where $t_{\text{shock}} > 0$ is the time when u first becomes discontinuous.

Problem 11: *This problem requires reading section 2.5.5 from the textbook, but no calculation.* Sketch the level curves (contours of constant value) of the solutions to the Eikonal boundary value problem

$$\begin{cases} ||\nabla u|| = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is

a) the open unit disk centered at the origin.

b) the square defined by $(-1, 1)^2$.

c) the L-shaped domain defined by $(-1, 1)^2 \setminus [0, 1)^2$.

Problem 12: Fix step sizes $\Delta x, \Delta t > 0$ and let U be a discretized solution to the following initial value problem for $t > 0$:

$$\begin{cases} u_t + cu_x = 0 \\ u(x, 0) = g(x) \end{cases}$$

Keeping the notation where $U_j^n = u(j\Delta x, n\Delta t)$, write a centered finite difference scheme for U_j^{n+1} that is fourth order accurate in space (i.e. $O(\Delta x^4)$).

Hint: Form a linear combination of the x-Taylor expansions of u for $j \pm 1$ and *$j \pm 2$ and determine what coefficients will cancel the $\Delta x^0, \Delta x^2$* and *Δx^3 terms and only leave u_x and higher order terms.*