Homological Algebra IV

Maps

$$I \xrightarrow{F} J \rightsquigarrow C_*(I) \xrightarrow{DF} C_*(J), \quad (DF)_{ij} = \delta_{F(i),j}$$

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$$\Gamma \xrightarrow{f} \check{\hat{\Gamma}} = \begin{cases} \Gamma_0 \xrightarrow{f_0} \check{\hat{\Gamma}}_0 \\ \Gamma_1 \xrightarrow{f_1} \check{\hat{\Gamma}}_1 \end{cases} \text{so that} \begin{cases} f(s(e)) = s(f(e)) & (0) \\ f(t(e)) = t(f(e)) & (1) \end{cases}$$

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$$\begin{aligned} C_*(\Gamma) &\xrightarrow{f_*} C_*(\hat{\check{\Gamma}}) \\ v &\longmapsto f_0(v) \\ e &\longmapsto f_1(e) \end{aligned}$$

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$$C_*(\Gamma) \xrightarrow{f_*} C_*(\hat{\Gamma})$$

$$v \longmapsto f_0(v)$$

$$e \longmapsto f_1(e)$$

$$(0) + (1)$$

$$\Downarrow$$

$$\star \quad f_* \circ \partial = \partial \circ f_* \quad \star$$

<u>Def$^{\underline{n}}$</u> A map of chain complexes

$$V \xrightarrow{f} W$$

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is the data of linear maps

$$V_i \xrightarrow{f_i} W_i \quad \forall i$$

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so that:

$$f \circ \partial_V = \partial_W \circ f$$

the boundary of
f(blah)
"
f(boundary of blah)

$\underline{\text{Def}^{\underline{n}}}$ A map of chain complexes

continuous linear map

$$V \xrightarrow{f} W$$

is the data of linear maps

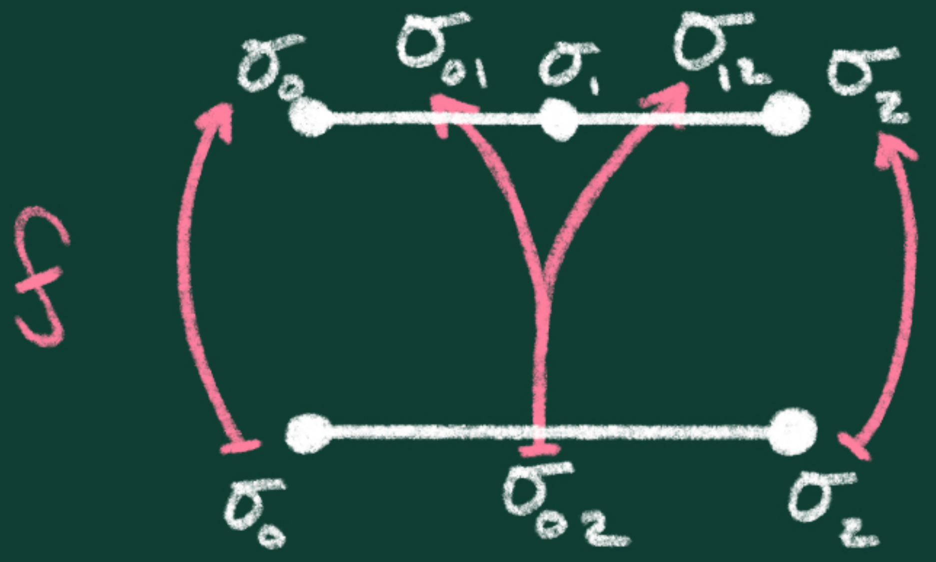
$$V_i \xrightarrow{f_i} W_i \quad \forall i$$

so that:

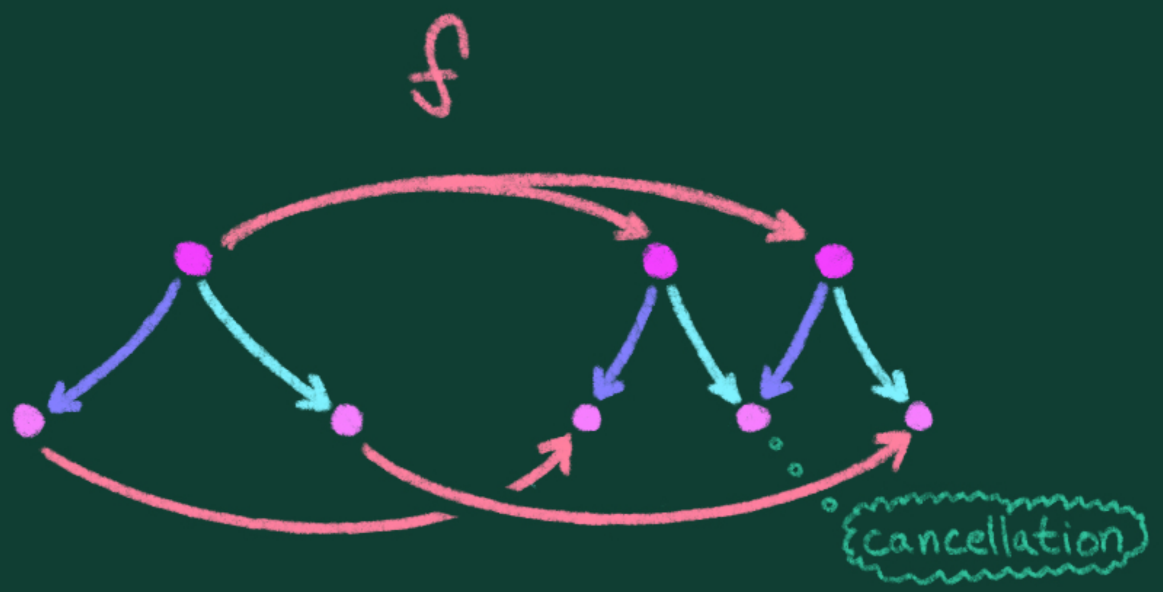
continuity condition

$$f \circ \partial_V = \partial_W \circ f$$

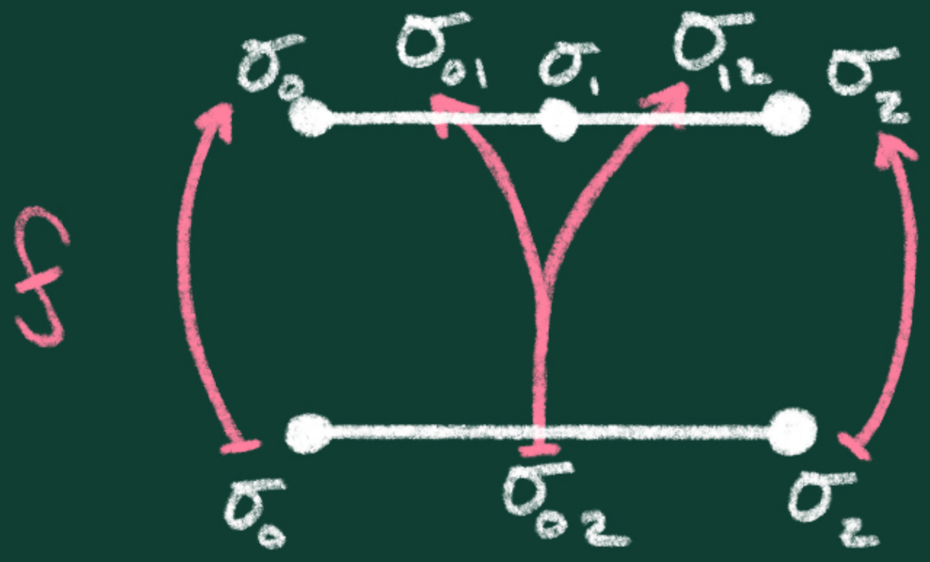
the boundary of f(blah)
"
f(boundary of blah)



f
σ_0
σ_{01}
σ_1
σ_{12}
σ_2
σ_0
σ_{02}
σ_2



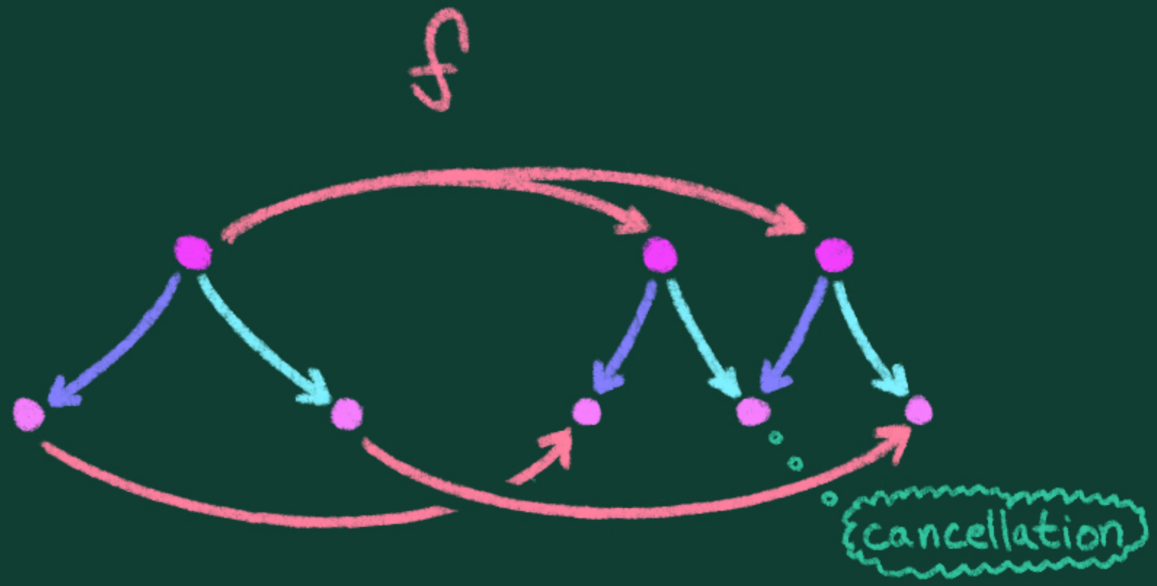
f
cancellation

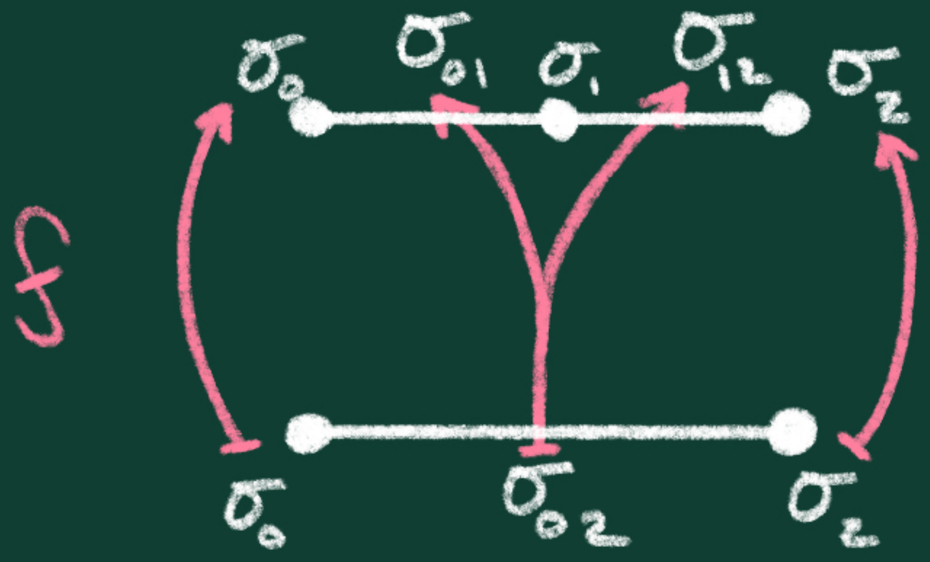


$$
\begin{aligned}
f(\partial \sigma_{02}) &= f(\sigma_2 - \sigma_0) \\
&= \hat{\sigma}_2 - \hat{\sigma}_0 \\
&= (\hat{\sigma}_1 - \hat{\sigma}_0) + (\hat{\sigma}_2 - \hat{\sigma}_1)
\end{aligned}
$$

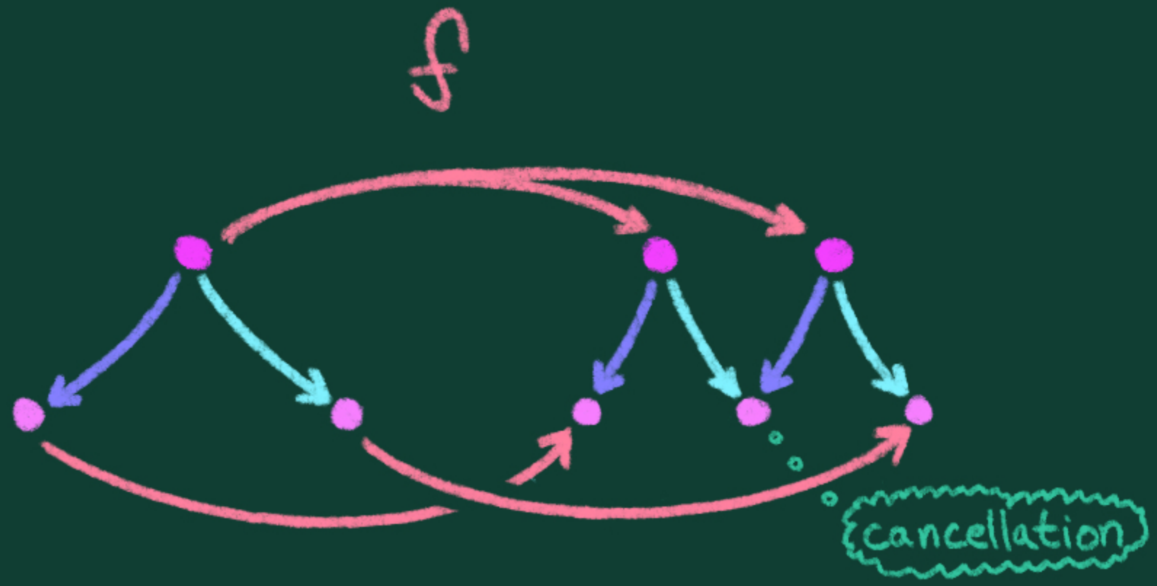
$$\parallel$$

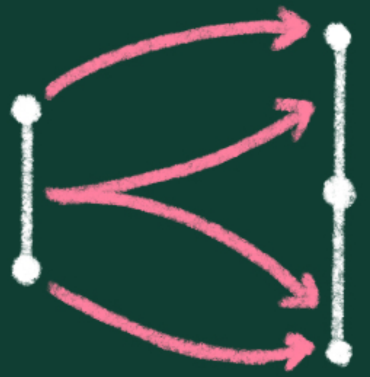
$$\partial f(\sigma_{02}) = \partial(\hat{\sigma}_{01} + \hat{\sigma}_{12})$$





$$
\begin{array}{ccl}
f(\partial \sigma_{02}) & = & f(\sigma_2 - \sigma_0) \\
& = & \tilde{\sigma}_2 - \tilde{\sigma}_0 \\
\Vert & & \\
& = & (\tilde{\sigma}_1 - \tilde{\sigma}_0) + (\tilde{\sigma}_2 - \tilde{\sigma}_1) \\
\partial f(\sigma_{02}) & = & \partial(\tilde{\sigma}_{01} + \tilde{\sigma}_{12})
\end{array}
$$





Note: f is <u>not</u> a map of graphs

$$\{C_*(\bullet \longrightarrow \bullet) \longrightarrow \mathcal{V}\} \cong \{v, w \in \mathcal{V}_0 \;\&\; \gamma \in \mathcal{V}_1 \mid \partial\gamma = v - w\}$$

$$\{C_*(\bullet \longrightarrow \bullet) \longrightarrow Y\} \cong \{v, w \in Y_0 \ \&\ \gamma \in Y_1 \mid \partial\gamma = v - w\}$$

"path" from v to w in Y

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"path" from v to w in Y

$$\{C_*(\bullet\!\rightarrow\!\bullet\!\rightarrow\!\bullet) \longrightarrow Y\} \cong \left\{ \begin{array}{l} v_0, v_1, v_2 \in Y_0,\ \gamma_0, \gamma_1 \in Y_1 \quad \Big| \\ \partial\gamma_0 = v_1 - v_0 \ \&\ \partial\gamma_1 = v_2 - v_1 \end{array} \right\}$$

$$\left\{ C_*(\bullet \to \bullet) \longrightarrow \Gamma \right\} \cong \left\{ v, w \in \Gamma_0 \ \& \ \gamma \in \Gamma_1 \mid \partial \gamma = v - w \right\}$$

"path" from v to w in Γ

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$$C_*(\bullet \to \bullet) \longrightarrow C_*(\bullet \to \bullet \to \bullet) \longrightarrow \Gamma$$

$$\{ C_*(\bullet \longrightarrow \bullet) \longrightarrow Y \} \simeq \{ v, w \in Y_0 \ \& \ \gamma \in Y_1 \mid \partial \gamma = v - w \}$$

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$$\begin{tikzcd} C_*(\bullet \to \bullet) \arrow[r] \arrow[rr, bend right, "{(v_0, v_2, \gamma_0 + \gamma_1)}"'] & C_*(\bullet \to \bullet \to \bullet) \arrow[r] & Y \end{tikzcd}$$

$$\{C_*(\bullet \rightarrow \bullet) \to V\} \simeq \{v, w \in V_0 \;\&\; \gamma \in V_1 \mid \partial\gamma = v - w\}$$

"path" from v to w in V

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geometric interpretation of linear structure

extends to $\uparrow$ dim

sum $\sim$ concatenation

$$\mathbb{R}[k]_i = \begin{cases} \mathbb{R} & i = k \\ 0 & i \neq k \end{cases}$$

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$$\{\mathbb{R}[k] \longrightarrow V\} \simeq \{v \in V_k \mid \partial v = 0\}$$

$$\mathbb{R}[k]_k \xrightarrow{v} V_k \longmapsto v(1) \in V_k$$

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$\mathbb{R}[k]$ "corepresents" the data of a deg k cycle

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we call such a map a *cycle* of degree k

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$$\{V \longrightarrow \mathbb{R}[k]\} \simeq \{\varphi \in V_k^\vee \mid \varphi(v + \partial \check{v}) = \varphi(v)\}$$

a map $V \xrightarrow{\varphi} \mathbb{R}[k]$ is a cocycle of degree k

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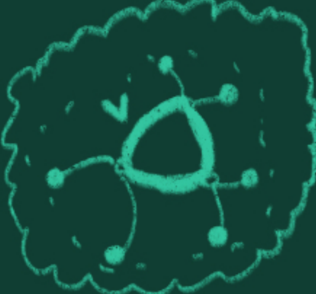
$$C_*(\Gamma) \xrightarrow{\varphi} \mathbb{R}[0]$$

w/ $\partial(e) = v_1 - v_0$

$$\mathbb{R}[K]_i = \begin{cases} \mathbb{R} & i = K \\ 0 & i \neq K \end{cases}$$

$\mathbb{R}[K]$ "corepresents" the data of a deg K cycle

$$\{\mathbb{R}[K] \longrightarrow V\} \simeq \{v \in V_K \mid \partial v = 0\}$$



$$\mathbb{R}[K]_K \xrightarrow{v} V_K \longmapsto v(1) \in V_K$$

we call such a map a <u>cycle</u> of degree K

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$C_*(\Gamma) \xrightarrow{\varphi} \mathbb{R}[0]$

w/ $\partial(e) = v_1 - v_0$

$$\Longrightarrow \varphi(v_1) = \varphi(v_0)$$

$\varphi(v_0) = \varphi(v_1)$ when v_0 & v_1 can be "connected" "locally constant" path γ $\rightsquigarrow$ $\int_\gamma$: v.f in deg 1 $\ni A \longmapsto \int_\gamma A$

$$C^\infty(\mathbb{R}^3) \xrightarrow{\nabla} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\times} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\mathbb{R}^3)$$

$$\underset{A}{\mathfrak{X}(\mathbb{R}^3)}$$

path γ $\rightsquigarrow$ $\int_\gamma : \Omega^\bullet(\mathbb{R}^3) \longrightarrow \mathbb{R}[-1]$

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$$\underset{\text{=}}{\;}\qquad\qquad \underset{A}{\;}$$

path $\gamma \rightsquigarrow \int_\gamma : \Omega^\bullet(\mathbb{R}^3) \longrightarrow \mathbb{R}[-1]$

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When is $\int_\gamma$ chain map?

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When is $\int_\gamma$ chain map?

$$\int_\gamma (A + df)$$

$$= \int_\gamma A$$

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When is $\int_\gamma$ chain map?

$$\int_\gamma (A + df) = \int_\gamma A + \int_{\partial\gamma} f$$

$$= \int_\gamma A$$ when γ is a closed loop

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abelian gauge field "photon"

When is $\int_\gamma$ chain map?

$$\int_\gamma (A + df) = \int_\gamma A + \int_{\partial\gamma} f$$

$$= \int_\gamma A \quad \text{when } \gamma \text{ is a closed loop}$$

Wilson loop observable

$$C^\infty(\mathbb{R}^3) \xrightarrow{\nabla} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\times} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\mathbb{R}^3)$$

"

A

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When is $\int_\gamma$ chain map?

$$\int_\gamma (A + \underbrace{df}) = \int_\gamma A + \int_{\partial\gamma} f$$

gauge transformation

$$= \int_\gamma A$$

when γ is a closed loop

Wilson loop observable

$C^\infty(\mathbb{R}^3) \xrightarrow{\nabla} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\times} \mathfrak{X}(\mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\mathbb{R}^3)$

$\quad\ \ \|\qquad\qquad\quad A\in$

path γ $\rightsquigarrow$ $\int_\gamma : \Omega^\bullet(\mathbb{R}^3) \longrightarrow \mathbb{R}[-1]$

v.f in deg 1 $\quad A \longmapsto \int_\gamma A$

abelian gauge field "photon"

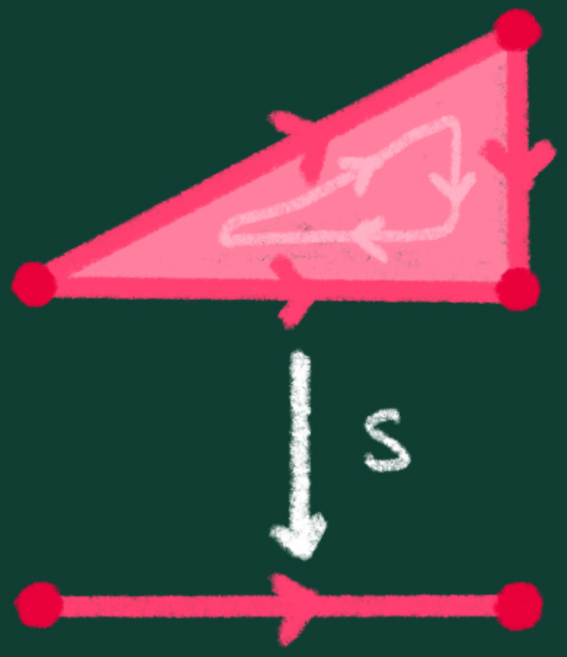
When is $\int_\gamma$ chain map?

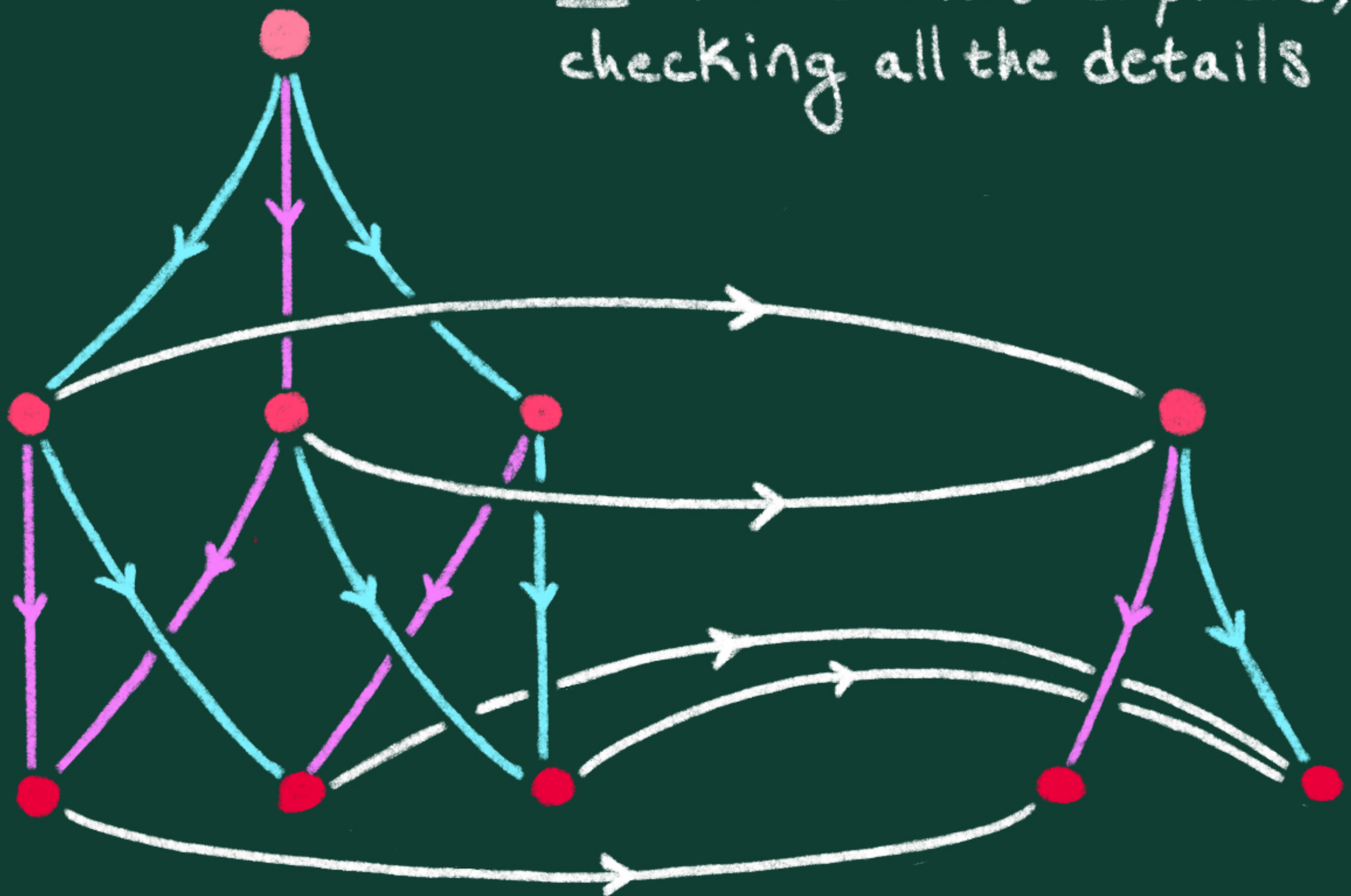
$$\int_\gamma (A + \underbrace{df}_{\text{gauge transformation}}) = \int_\gamma A + \int_{\partial\gamma} f$$

loop part of Wilson loop
$\Rightarrow$ gauge invariance
$\Rightarrow$ physically meaningful

$= \int_\gamma A$ when γ is a closed loop

Ex: Make this explicit, checking all the details





$$C_*(\Delta^2) \xrightarrow{s_*} C_*(\Delta^1)$$