ENGR727-001

ENGR 727-001 ADVANCED MECHANICS OF MATERIALS: HOMEWORK 4

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Problems:

- ✓ Problem 1
- ✓ Problem 2
- ✓ Problem 3
- ✓ Problem 4
- ✓ Problem 5
- ✓ Problem 6
- ✓ Problem 7
- ✓ Problem 8
- ❏ Problem 9
- ✓ Problem 10
- ✓ Problem 11
- ✓ Problem 12

1. PROBLEM 1

A square, glass block in the side of a skyscraper is loaded so that the block is in a state of plane strain ($\epsilon_{zx} = \epsilon_{zy} = \epsilon_{zz} = 0$). (a) Determine the displacements for the block for the deformations shown and the strain components for the xy-coordinate axes. (b) Determine the strain components for the XY-axes.

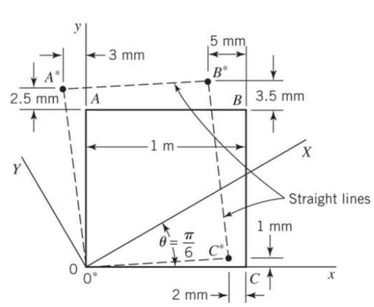

FIGURE 1-1. ADAPTED FROM ASSIGNMENT INSTRUCTIONS.

— Problem Statement

SOLUTION

The displacement for the block are as shown in Fig. 1-1. The strain components along the xy-axes can be found from building the strain state in matrix notation:

$$[\epsilon] = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \tag{1}$$

Equations $u(x, y)$ and $v(x, y)$ can be found from solving a linear system of equations for which there are four equations and four unknowns:

$$u(x, y) := \begin{bmatrix} 1 & x_1 & y_1 & xy_1 \\ 1 & x_2 & y_2 & xy_2 \\ 1 & x_3 & y_3 & xy_3 \\ 1 & x_4 & y_4 & xy_4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \Bigg[$$
$$d(x, y) := \begin{bmatrix} 1 & x_1 & y_1 & xy_1 \\ 1 & x_2 & y_2 & xy_2 \\ 1 & x_3 & y_3 & xy_3 \\ 1 & x_4 & y_4 & xy_4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \Bigg[$$

Solving Eq. 1-2 with input data from the appropriate points lends to Eq. 1-3.

$$u(x, y) = 6.50521303491303e - 19 * x * y$$
$$v(x, y) = -2.38524477946811e - 18 * x *$$

This means that Eq. 1-1 can now be solved to reveal the strain state of the glass block: $[\epsilon_{xy}] = \begin{bmatrix} -2m & -2m \\ -2m & 2.50m \end{bmatrix}$.

To find the strain components projected onto the XY-axes, which is at some angle offset from the xy-axes, Eq. 1-4 must be implemented.

$$\begin{aligned} \epsilon_{x'} &= \epsilon_x \cos^2(\theta) + \epsilon_y \sin^2(\theta) + \gamma_{xy} \sin(\theta)\cos(\theta) \\ \gamma_{x'y'} &= -(\epsilon_x - \epsilon_y)\sin(2\theta) + \gamma_{xy}\cos(2\theta) \end{aligned} \quad (4)$$

, wherein $\epsilon_{y'}$ can be found by replacing θ with $\theta + \frac{\pi}{2}$ in the equation for $\epsilon_{x'}$. This yields $[\epsilon_{XY}] = \begin{bmatrix} -491.03u & 2.90m \\ -2.90m & 991.03u \end{bmatrix}$

2. PROBLEM 2

A square plate, 1 m on a long side, is loaded in a state of plane strain and is deformed as shown. (a) Write expressions for the u and v displacements for any point on the plate. (b) Determine the components of **Green Strain** in the plate. (c) Determine the total **Green Strain** at point B for a line element in the direction of line OB. (d) For point B, compare the components of strain from part (b) to the components of strain for **Small-Displacement Theory**. (e) Compare the strain determined in part (c) to the corresponding strain using **Small-Displacement Theory**.

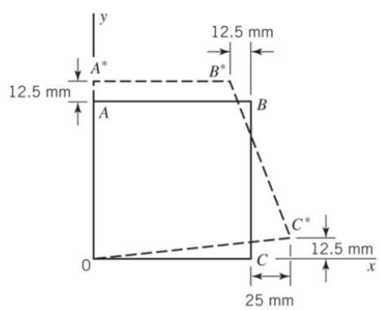

FIGURE 2-1. ADAPTED FROM ASSIGNMENT INSTRUCTIONS.

— Problem Statement

SOLUTION

As demonstrated in Sec. 1, expressions for displacement for any point on the plate may be found by solving Eq. 1-2. This yields the following expressions:

$$\begin{aligned} u(x,y) &= -0.0375 * x * y + 0.025 * x - 1.38777878078145e-17 * y + 1.00184002094058e-17 \\ v(x,y) &= -0.0125 * x * y + 0.0125 * x + 0.0125 * y + 2.57711895321568e-18 \end{aligned}$$

The strain state of the plate may be found by solving Eq. 2-5 with the developed displacement equations.

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \end{aligned} \quad (5)$$

The strain state, then, becomes: $\epsilon_{Green} = \begin{bmatrix} -12.42m & -13.70a \\ -13.70a & 12.58m \end{bmatrix}$. The percent error between ϵ_{Green} and $\epsilon_{Small} = \begin{bmatrix} -12.50m & -13.88a \\ -13.88a & 12.50m \end{bmatrix}$ is 0.0062 %. Projecting this strain state along the line OB can be found by the *direction of cosines* for this $< 1, 1 >$ vector ($l = 0.7071$ and $m = 0.7071$): $(\epsilon_{Green})_{OB} = \begin{bmatrix} -8.78m & -9.69a \\ -9.69a & 8.89m \end{bmatrix}$. The percent error between ϵ_{Small} and $(\epsilon_{Green})_{OB}$ is 0.2876 %.

3. PROBLEM 3

Solve Problem 2.3 from the textbook.

— Problem Statement

PROBLEM 2.3

A displacement field in a body is given by

$$u = c(x^2 + 10), \quad v = 2cyz, \quad w = c(-xy$$

where $c = 10^{-4}$. Determine the state of strain on an element positioned at $(0, 2, 1)$.

SOLUTION

The strain state can be found from determining the strain state at the point in the displacement field:

$$[\epsilon] = \begin{bmatrix} \epsilon_x = \frac{\partial u}{\partial x} & \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \gamma_{xz} = \\ 0 & \epsilon_y = \frac{\partial v}{\partial y} & \gamma_{yz} = \\ 0 & 0 & \epsilon_z = \end{bmatrix}$$

When Eq. 3-6 is plugged into Eq. 3-7, this takes the form

$$[\epsilon] = \begin{bmatrix} 2x & 0 & -y \\ 0 & 2z & 2y - x \\ 0 & 0 & 2z \end{bmatrix} \times 10^{-4}$$

which further yields [eq-5-strain_p] for $\epsilon(x = 0, y = 2, z = 1)$.

$$[\epsilon] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \times 10^{-4} \quad (8)$$

ANSWER

The strain tensor at point $(0,2,1)$, $\epsilon = \begin{bmatrix} 0 & 0 & -200u \\ 0 & 200u & 400u \\ 0 & 0 & 200u \end{bmatrix}$.

4. PROBLEM 4

Solve Problem 2.4 from the textbook.

— Problem Statement

PROBLEM 2.4

The displacement field and strain distribution in a member have the form

$$\begin{aligned} u &= a_0 x^2 y^2 + a_1 x y^2 + a_2 x^2 y \\ v &= b_0 x^2 y + b_1 x y \\ \gamma_{xy} &= c_0 x^2 y + c_1 x y + c_2 x^2 + c_3 y^2 \end{aligned} \tag{9}$$

Which relationships connecting the constraints (a's, b's, and c's) make the foregoing expressions (Eq. 4-9) possible?

ANSWER

These equations assume *plane-strain*; therefore, the constants can be found from the contribution of the respective derivatives of each displacement field equation.

5. PROBLEM 5

Solve Problem 2.9 from the textbook.

— Problem Statement

PROBLEM 2.9

A $100\,mm$ by $150\,mm$ rectangular plate $QABC$ is deformed into the shape shown by the dashed lines in Fig. 5-1. All dimensions shown in the figure are in millimeters. Determine at point Q (a) the strain components ϵ_x, ϵ_y, and γ_{xy} and (b) the principal strains and the direction of the principal axes.

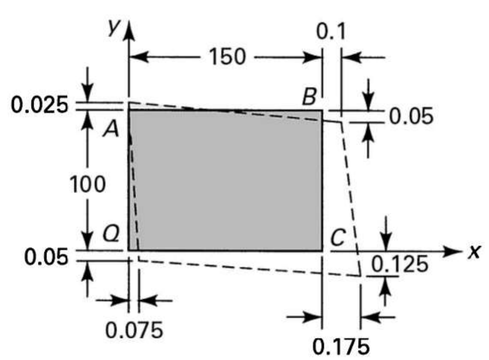

Figure P2.9.

FIGURE 5-1. ADAPTED FROM [1].

SOLUTION

Following a procedure similar to that displayed in Sec. 1 yields the strain state: $\begin{bmatrix} -12.50m & -13.88a \\ -13.88a & 12.50m \end{bmatrix}$ which yields the principal strains: $\epsilon_1, \epsilon_2, \epsilon_3 = 12.50m, 12.50m, 0$ in the $\theta_p = 277.56a\ rad$ direction.

6. PROBLEM 6

Solve Problem 2.12 from the textbook.

— Problem Statement

PROBLEM 2.12

A thin, rectangular plate $a = 20\ mm \times b = 12\ mm$ (Fig. 6-1) is acted upon by a stress distribution resulting in the uniform strains $\epsilon_x = 300\mu$, $\epsilon_y = 500\mu$, and $\gamma_{xy} = 200\mu$. Determine the changes in length of diagonals QB and AC.

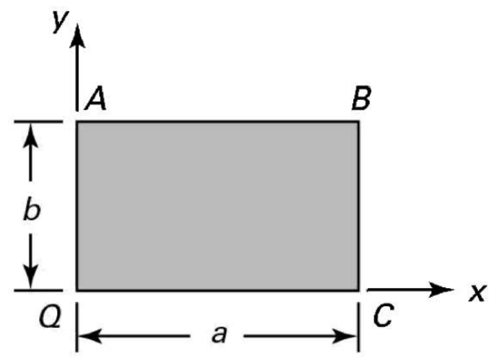

Figure P2.11.

FIGURE 6-1. ADAPTED FROM [1].

SOLUTION

Because the plate undergoes a displacement that yields a symmetric strain state, the change in length for diagonals AC and QB are equal and found by finding the difference of the hypotenuse: e.g. $a' = (1+\epsilon_x)a_0 = 20.01$. Therefore, $\Delta l_{AC} = 8.23m\ m$ and $\Delta l_{QB} = 8.23m\ m$.

7. PROBLEM 7

Solve Problem 2.22 from the textbook.

— Problem Statement

PROBLEM 2.22

Solve Problem 2.21 [1] for a state of strain given by

$$\begin{bmatrix} 400 & 100 & 0 \\ 100 & 0 & -200 \\ 0 & -200 & 600 \end{bmatrix} \mu \tag{10}$$

Problem 2.21 asks to determine (a) the strain invariants; (b) the normal strain in the x' direction which is directed at an angle $\theta = 30\,^\circ$ from the x-axis; (c) the principal strains ϵ_1, ϵ_2, and ϵ_3; and, (d) the maximum shear strain.

ANSWERS

The strain invariants of given strain state (Eq. 7-10):

$$
\begin{aligned}
J_1 &= p_x + p_y + p_z = 1m \\
J_2 &= p_x p_y + p_x p_z + p_y p_z \\
&\quad - p_{xy}^2 - p_{yz}^2 - p_{xz}^2 \\
J_2 &= 190n \\
J_3 &= \|\mathbf{p}\| = 0.0
\end{aligned}
$$

The normal strain along x', which is $\theta = 30\,^\circ$ up from the x-axis, is $\epsilon_{x'} = 556.21u$.

The principal strains come from solving $\epsilon_p^3 - J_1\epsilon_p^2 + J_2\epsilon_p - J_3 = 0$:

- $\epsilon_1 = 1.15m$
- $\epsilon_2 = 231.47u$
- $\epsilon_3 = 82.74u$

The maximum principal strain (magnitude and direction), $\epsilon_1 = 1.15m \angle -360.04y\,rad$. The magnitude of the shear strain is the average of the principal strains (from **Mohr's Circle**): $\gamma_{max} = 82.74u$.

8. PROBLEM 8

Solve Problem 2.24 from the textbook.

— Problem Statement
PROBLEM 2.24

At a point in a loaded frame, the strain with respect to the coordinate set xyz is

$$
\begin{bmatrix} -300 & -583 & -300 \\ -583 & 200 & -67 \\ -300 & -67 & -200 \end{bmatrix} \mu \qquad (11)
$$

Determine (a) the magnitudes and directions of the principal strains and (b) the maxmimum shear strains.

ANSWERS

The principal strains come from solving $\epsilon_p^3 - J_1\epsilon_p^2 + J_2\epsilon_p - J_3 = 0$: The strain invariants of given strain state (Eq. 8-11):

$$
\begin{aligned}
J_1 &= p_x + p_y + p_z = -300u \\
J_2 &= p_x p_y + p_x p_z + p_y p_z \\
&\quad - p_{xy}^2 - p_{yz}^2 - p_{xz}^2 \\
J_2 &= -474.38n \\
J_3 &= \|\mathbf{p}\| = 0.0
\end{aligned}
$$

- $\epsilon_1 = 710.18u$
- $\epsilon_2 = 710.18u$
- $\epsilon_3 = 79.09u$

The maximum principal strain (magnitude and direction), $\epsilon_1 = 710.18u \angle -1.30\,rad$. The magnitude of the shear strain is the average of the principal strains (from **Mohr's Circle**): $\gamma_{max} = 79.09u$.

9. PROBLEM 9

Solve Problem 2.28 from the textbook.

— Problem Statement
PROBLEM 2.28

A $16\,mm \times 16\,mm$ square $ABCD$ is sketched on a plate before loading. Subsequent to loading, the square becomes the rhombus illustrated in Fig. 9-1. Determine the (a) modulus of elasticity, (b) Poisson's Ratio, and (c) the shear modulus of elasticity.

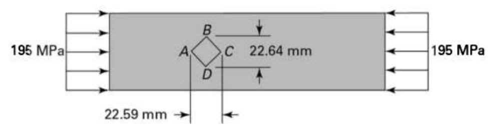

Figure P2.28.

FIGURE 9-1. ADAPTED FROM [1].

10. PROBLEM 10

Solve Problem 2.52 from the textbook.

— Problem Statement
PROBLEM 2.52

The distribution of stress in a structural member is given (in megapascals) by Eqs. (d) of Example 1.2 of Chapter 1 (Eq. 10-12). Calculate the strains at the specified point $Q(\frac{3}{4}, \frac{1}{4}, \frac{1}{2})$ for $E = 200\,GPa$ and $\nu = 0.25$.

$$
\begin{aligned}
\sigma_x &= -x^3 + y^2, \quad & \tau_{xy} &= 5z + 2y^2 \\
\sigma_y &= 2x^2 + \frac{1}{2}y^2, \quad & \tau_{xz} &= xz^3 + x^2 y \qquad (12 \\
\sigma_z &= 4y^2 - z^3, \quad & \tau_{yz} &= 0
\end{aligned}
$$

SOLUTION

The strain state can be found from determining the strain state at the point in the displacement field:

$$
[\epsilon] = \begin{bmatrix} \epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] & & \\ 0 & & \epsilon_y = \frac{1}{E} \\ 0 & & \end{bmatrix}
$$

When Eq. 10-12 and point Q are plugged into Eq. 10-13, this yields [eq-5-strain_p] for $\epsilon(x = 0, y = 2, z = 1)$.

$$[\epsilon] = \begin{bmatrix} -3.40u & 32.81u & 2.93u \\ 0 & 6.07u & 0 \\ 0 & 0 & -371.09n \end{bmatrix} \tag{14}$$

ANSWER

The strain tensor at point $(0, 2, 1)$, $[\epsilon] = \begin{bmatrix} -3.40u & 32.81u & 2.93u \\ 0 & 6.07u & 0 \\ 0 & 0 & -371.09n \end{bmatrix}$.

11. PROBLEM 11

Solve Problem 2.53 from the textbook.

— Problem Statement

PROBLEM 2.53

An aluminum alloy plate ($E = 70\,GPa$, $\nu = \frac{1}{3}$) of dimensions $a = 300\,mm$, $b = 400\,mm$, and thickness $t = 10\,mm$ is subjected to biaxial stresses as shown in Fig. 11-1. Calculate the change in (a) the length AB and (b) the volume of the plate.

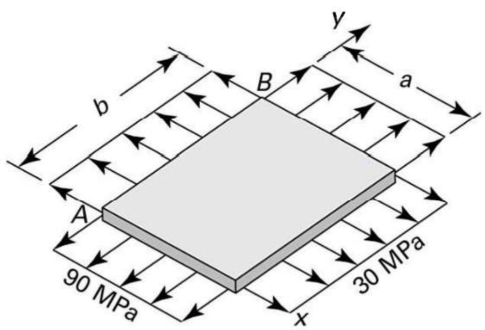

Figure P2.53.

FIGURE 11-1. ADAPTED FROM [1].

SOLUTION

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \\ \epsilon_y &= \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned} \tag{15}$$

Using the provided stress tensor, the strain state may be found by solving Eq. 11-15 which yields: $[\epsilon] = \begin{bmatrix} 0 & -428.57u \\ -142.86u & 1.14m \end{bmatrix}$. Therefore, $\Delta b = \epsilon_y b = 457.14m\ mm$ and $\Delta V = (\epsilon_x + \epsilon_y)V_0 = 685.71\ mm^3$.

12. PROBLEM 12

Solve Problem 2.54 from the textbook.

— Problem Statement

PROBLEM 2.54

The steel, rectangular parallelepiped ($E = 200\,GPa$ and $\nu = 0.3$) shown in Fig. 12-1 has dimensions $a = 250\,mm$, $b = 200\,mm$, and $c = 150\,mm$. It is subjected to triaxial stresses $\sigma_x = -60\,MPa$, $\sigma_y = -50\,MPa$, and $\sigma_z = -40\,MPa$ acting on the x, y, and z faces. Determine (a) the changes Δa, Δb, and Δc in the dimensions of the block, and (b) the change ΔV in the volume.

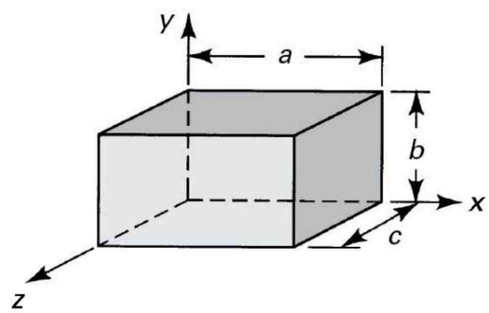

Figure P2.54.

FIGURE 12-1. ADAPTED FROM [1].

SOLUTION

Following a similar procedure as in Sec. 11, the changes in length for each side and the final volume of the parallelepiped are: $\Delta a = 0.006\,mm$, $\Delta b = 0.04\,mm$, $\Delta c = 0.069\,mm$, and $V_f = [1 + (\epsilon_x - 2\nu\epsilon_x)]dxdydz = V_0 + \Delta V = -2.25k\ mm^3$.

BIBLIOGRAPHY

[1] A. Ugural and S. Fenster, *Advanced Mechanics of Materials and Applied Elasticity (International Series in the Physical and Chemical Engineering Sciences)*, Sixth. Pearson, 2019.