# TOPIC 1: RELATIONS

Normally relation deals with matching of elements from the first set called DOMAIN with the element of the second set called RANGE.

# Relations

A relation "R" is the rule that connects or links the elements of one set with the elements of the other set.
Some examples of relations are listed below:

1. "Is a brother of "
2. "Is a sister of "
3. "Is a husband of "
4. "Is equal to "
5. "Is greater than "
6. "Is less than "

Normally relations between two sets are indicated by an arrow coming from one element of the first set going to the element of the other set.
Relations Between Two Sets
Find relations between two sets
The relation can be denoted as:
R = {(a, b): a is an element of the first set, b is an element of the second set}
Consider the following table

| X | -3 | 0.5 | 1 | 2 | 5 | 6 |
|---|---|---|---|---|---|---|
| Y | -6 | 1 | 2 | 4 | 10 | 12 |

This is the relation which can be written as a set of ordered pairs {(-3, -6), (0.5, 1), (1, 2), (2, 4), (5, 10), (6, 12)}. The table shows that the relation satisfies the equation y=2x. The relation R defining the set of all ordered pairs (x, y) such that y = 2x can be written symbolically as:
R = {(x, y): y = 2X}.
Relations Between Members in a Set
Find relations between members in a set
Which of the following ordered pairs belong to the relation {(x, y): y>x}?
(1, 2), (2, 1), (-3, 4), (-3, -5), (2, 2), (-8, 0), (-8, -3).
Solution.
(1, 2), (-3, 4), (-8, 0), (8,-3).
Relations Pictorially
Demonstrate relations pictorially
For example the relation " is greater than " involving numbers 1,2,3,4,5 and 6 where 1,3 and 5 belong to set A and 2,4 and 6 belong to set B can be indicate as follows:-

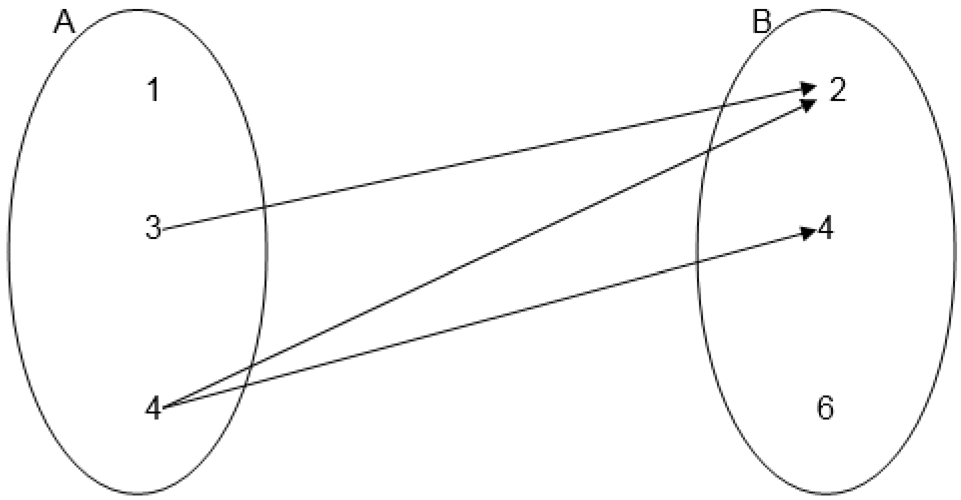



**Figure1.**

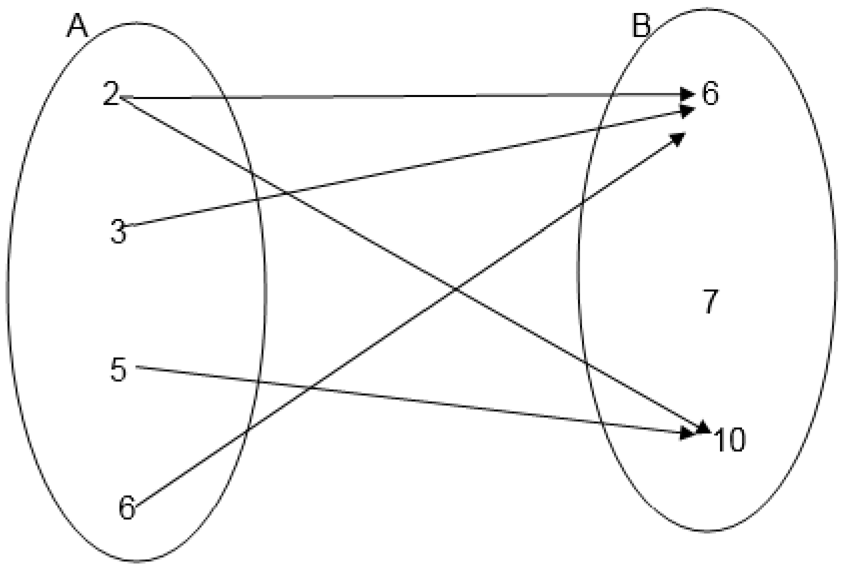
This kind of relation representation is referred to as **pictorial representation.**
Relations can also be defined in terms of ordered pairs (a,b) for which a is related to b and a is an element of set A while b is an element of set B.

That is R = $\left\{ (a, b):\ a \in A,\ b \in B \text{ and a is related to b} \right\}$

The symbol ∈ means belongs to or is a member of.

For example the relation " is a factor of " for numbers 2,3,5,6,7 and 10 where 2,3,5 and 6 belong to set A and 6,7 and 10 belong to set B can be illustrated as follows:-

*Using a pictorial representation,*



Also as a set of ordered pairs as

R= $\{(2, 6), (3, 6), (6, 6), (2,10)$ and $(5,10)\}$

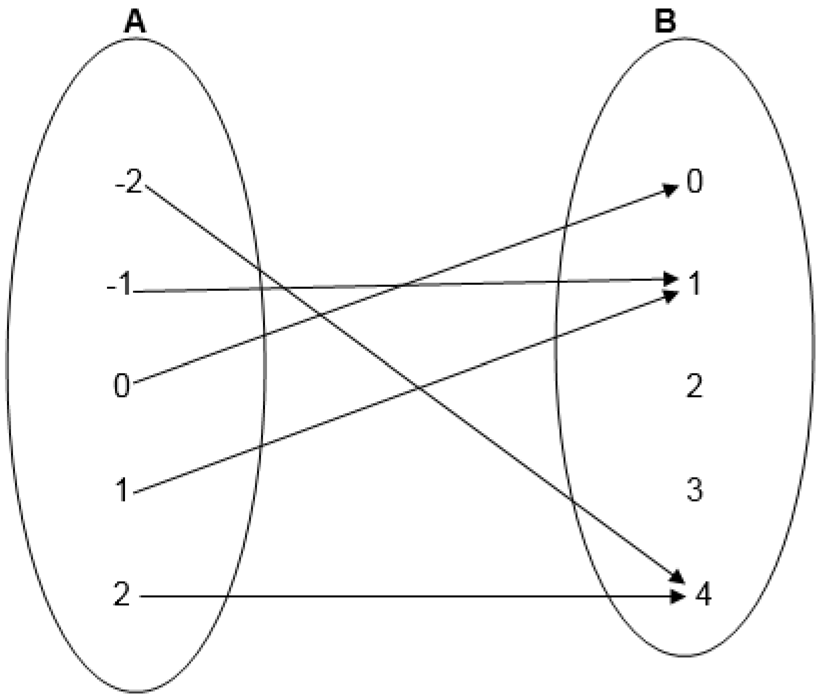
Example 1

<!-- [if !supportLists]-->1. Draw an arrow diagram to illustrate the relation which connects each element of set A with its square.

Let A = $\{-2,-1,0,1,2\}$

B = $\{0, 1, 2, 3, 4\}$

Solution




Example 2

**Using the information given in example 1, write down the relation in set notation of ordered pairs. List the elements of ordered pairs.**

**Solution:**

R = $\{(a, b): a \in A, b \in B$ and $b=a^2\}$

R= $\{(-2, 4), (-1,1), (0,0), (1,1), (2,4)\}$

Example 3

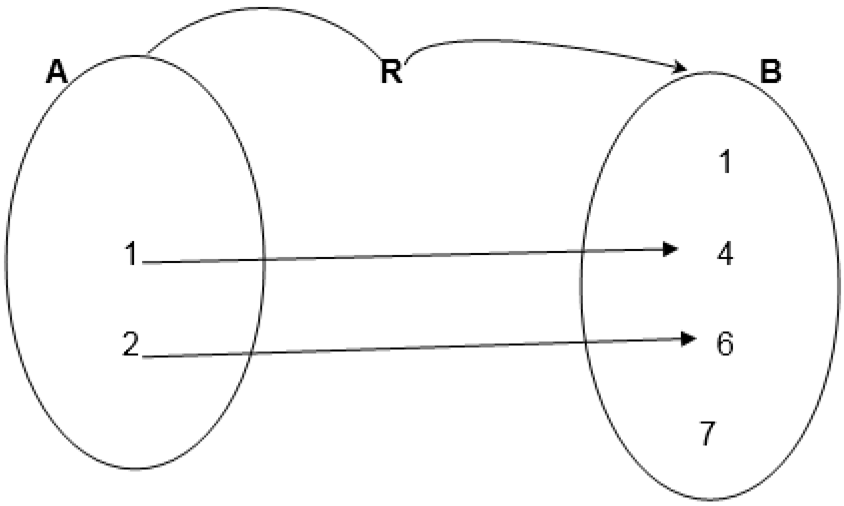
As we,

Let R = $\left\{ (a, b): a \in A,\ b \in B \text{ and } b = 2a \right\}$

Where A= $\{1, 2, 3\}$ and B = $\{1, 4, 6, 7\}$

Use a pictorial diagram to illustrate R.

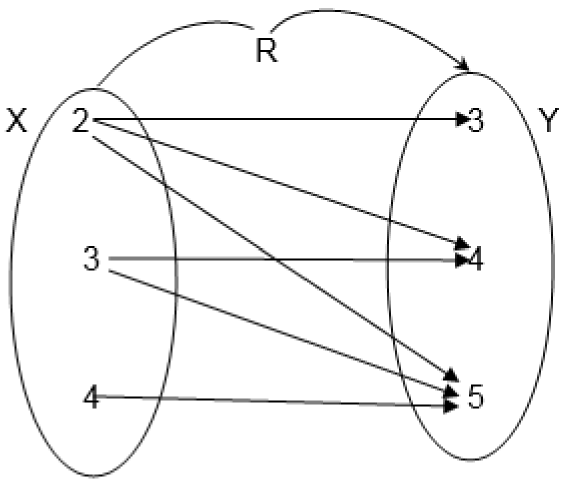
Solution;




Example 4

Let X= **{**2, 3, 4 **}** and Y= {3 ,4, 5}

Draw an arrow diagram to illustrate the relation " is less than"




Exercise 1

Let P= {Tanzania, China, Burundi, Nigeria}

Draw a pictorial diagram between P and itself to show the relation

"Has a larger population than"

**2**. Let A = 9,10,14,12 and B = 2,5,7,9 Draw an arrow diagram between A and B to illustrate the relation " is a multiple of"

**3**.Let A = mass, Length, time and

B = {Centimeters, Seconds, Hours, Kilograms, Tones}
Use the set notation of ordered pairs to illustrate the relation "Can be measured in"
**4**. A group people contain the following; Paul Koko, Alice Juma, Paul Hassan and Musa Koko. Let F be the set of all first names, and S the set of all second names.
Draw an arrow diagram to show the connection between F and S
5. Let R={ (x, y)**:** y=x+2}
Where x∈A and A ={ -1,0,1,2}
and y∈B, List all members of set B
Exercise 2
1. Let the relation be defined

as $R = \left\{ \begin{array}{l} (x,y): x \in X, y \in Y \\ \text{and } y-x+4=0 \end{array} \right\}$

for which x is an integer less than 10 but greater than 2, then the following ordered pair does not belong to R

(a) ( 3,-1) (b) (10,6), (c) (5,1), (d) (4,0) ( )

2. Let A={ 3,4,6,9}

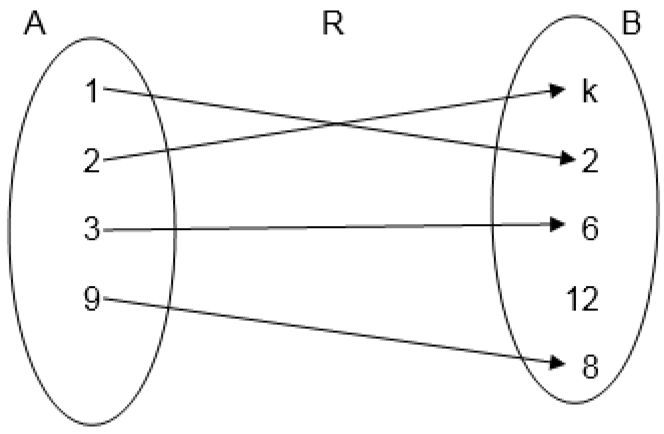
If we draw an arrow diagram between A and itself to show the relation "is a multiple of"

How many arrows are counted?

(a) 4 arrows (b) 8 arrows

(b) 6 arrows (d) 12 arrows ( )

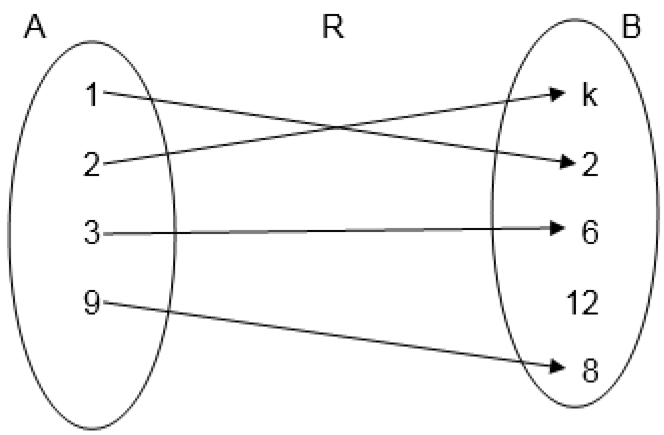
Consider the following pictorial diagram representing a relation R.




From the figure above, the value of k is

(a) 4 (b) -4 (c) 12 (d) 3 ( )

Let the relation R be defined as




From the figure above, the value of k is

(a) 4 (b) -4 (c) 12 (d) 3 ( )

A relation R on sets a and B where A = 1,2,3,4,5 and B = 7,8,9,10,11,12 is defined as " is a factor of "

How many elements from set a are connected to 12 which is an element of set B?

(a) 1, (b) 2, (c) 3, (d) 4 ( )

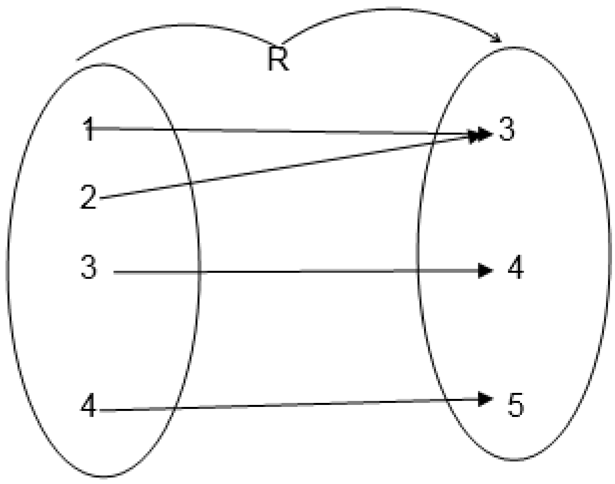
# Graph of a Relation

A Graph of a Relation Represented by a Linear Inequality

Draw a graph of a relation represented by a linear inequality

Given a relation between two sets of numbers, a graph of the relation is obtained by plotting all the ordered pairs of numbers which occur in the relation

Consider the following relation



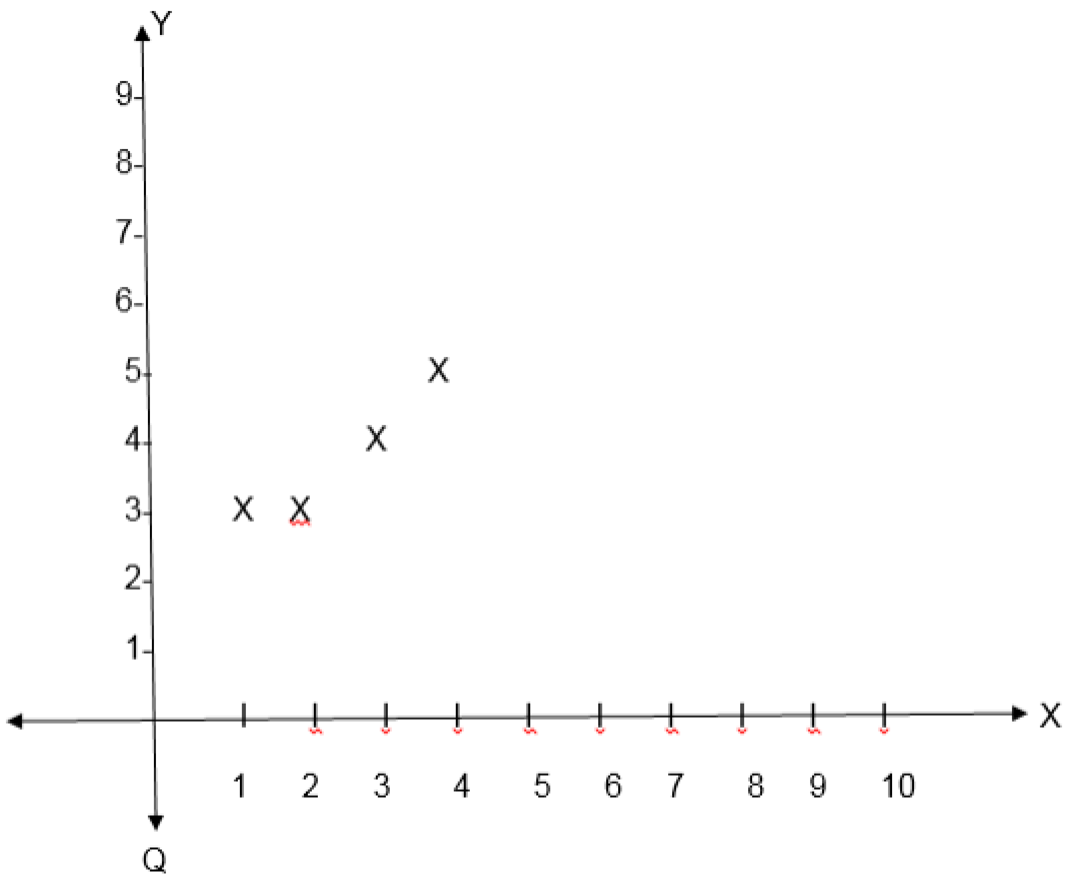

The above relation can be written as a set ordered Pairs

as R = { (1,3) , (2,3) , (3,4) , (4,5) }

So 1 is related to 3 , 2 is related to 3 and so on , there fore

( 1,3) , ( 2,3), (3,4), (4,5) are all on the graph

The graph of R is shown the following diagram( x-y plane).




Example 5

**Solved:**

1. Let P = $\{2,3,4,5\}$ and Q = $\{1,2, 3, 4, 5, 6\}$

draw a graph to illustrate the relation " is a factor of"

**Solution**

The relation "is a factor of " can be written as a set of ordered pairs as

R= $\{(2,2)\ , (2,4), (2,6), (3,3), (3,6), (4,4), (5,5)\}$

**Note** that some relations have graphs representing special figures like straight lines or curves.
Example 6
Draw the graph for the relation R= {(x, y): y = 2x +1} Where both x and y are real numbers.
**Solution**
The equation y = 2x +1 represents a straight line, this line passes throng uncountable points. To draw its graph we must have at least two points through which the line passes.

Now let x = 0, y = 2 x 0 +1= 1

(x ,y ) = (0 ,1 )

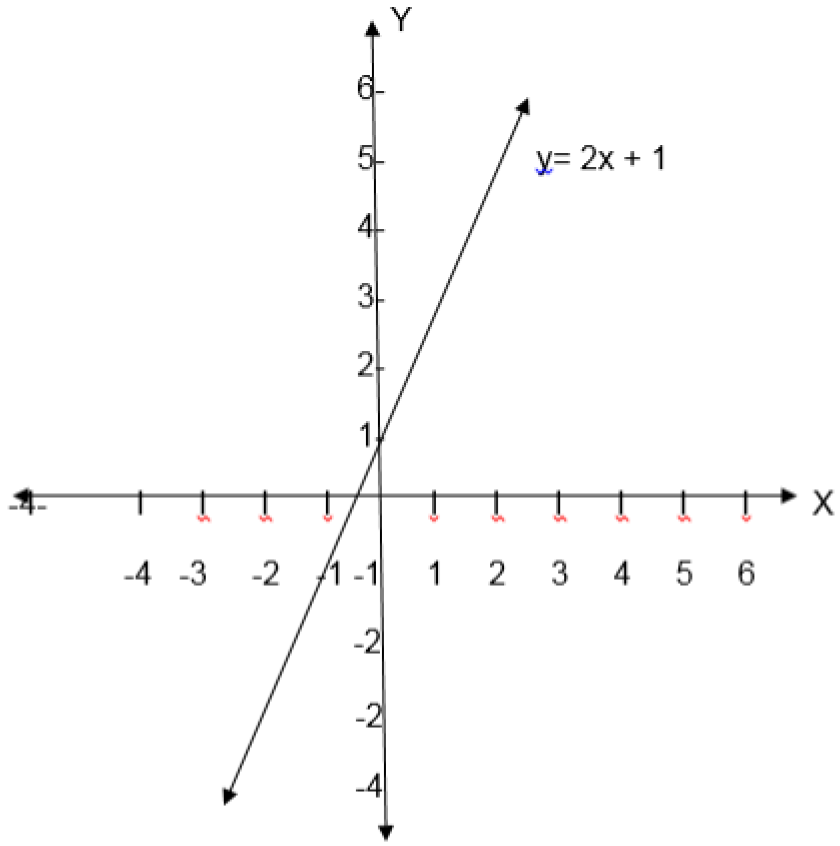
Also y =0

0 = 2x+ 1

X = - ½

(x , y) =( -1/2,0)

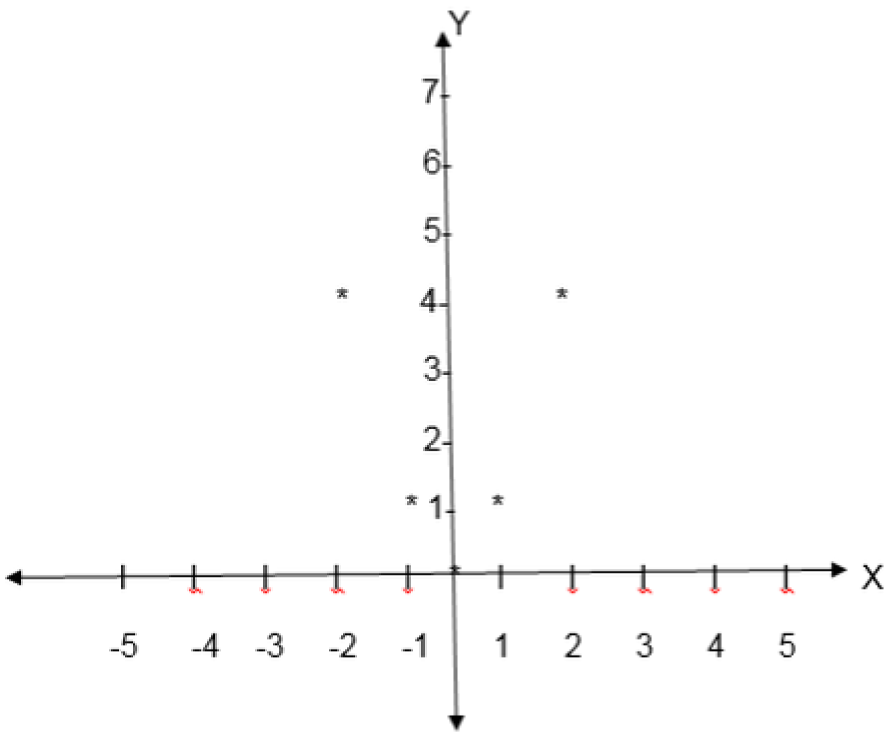
Graph;




Example 7
Let A = {-2,-1,0, 1, 2 } and B ={0,1,2,3,4}
Let the relation R be $y= x^{2,}$ where $x \in A$ and $y \in B$. Draw the graph of R
**Solution**

$R = \{ (-2,4), (-1,1), (0,0), (1,1), (2,4) \}$

Graph:




NB: When the relation is given by an equation such as y = f (x), the domain is the set containing x-values satisfying the equation and the range is the set of y-values satisfying the given equation.
Exercise 3
Test Yourself:

1. Let $p = \{ x: 0 \leq x \leq 1 \}$ and $Q = \{ y: 1 \leq y \leq 3 \}$

   Draw the graph of the relation given by $y = 2x + 1$

2. The relation R is given by $y = 2x + 3$, write R as the set of ordered pairs where x is an integer such that $-1 < x < 11$.

3. Let $A = \{1, 5, 16, 20\}$ and $B = \{4, 10, 17, 19\}$

   (a) Draw an arrow diagram to show the relation "is less than ".

   (b) Draw the graph of this relation.

**Quiz.**

1. The relation whose graph is a straight line passing through (1,1) and (2,3) is

(a) $x = 2y - 1$ (b) $y = x + 2$

(c) $y = 2x - 1$ (d) $x = 2y$ ( )

2. One of the paints through which the graph of the relation $x - y = 5$ passes is

(a) (0, 5) (b) (2,3) (c) (3,2) (d) (0, -5) ( )

3. Given that A = $\{x: 3 \leq x \leq 5\}$ and B= $\{y: -6 \leq y \leq 5\}$, an element from set A is

mapped onto an element in set B by the relation " is less than" if A and B are sets of integers, what is the greatest integer in set A can be mapped onto an element from set B ?

(a) 3 (b) 4 (c) 5 (d) -6 ( )

4. Let R = $\{(x, y): y = x^2 + 1\}$ what is the domain of R?

(a) $\{x : x > 1\}$ (b) $\{x : x$ is any real number$\}$

(c) = $\{x : x \geq 0\}$ (d) $\{x : x < 1\}$ ( )

5. What is the range of the relation R = $\{(x,y): y = x^2\}$

(a) $\{y: y < 0\}$ (b) $\{y: y > 0\}$

(b) $\{y: y$ is any real number$\}$ (d) $\{y: y \geq 0\}$ ( )

# Domain and Range of a Relation

The Domain of Relation

State the domain of relation

**Domain**: The domain of a function is the set of all possible input values (often the "x" variable), which produce a valid output from a particular function. It is the set of all real numbers for which a function is mathematically defined.

The Range of a Relation

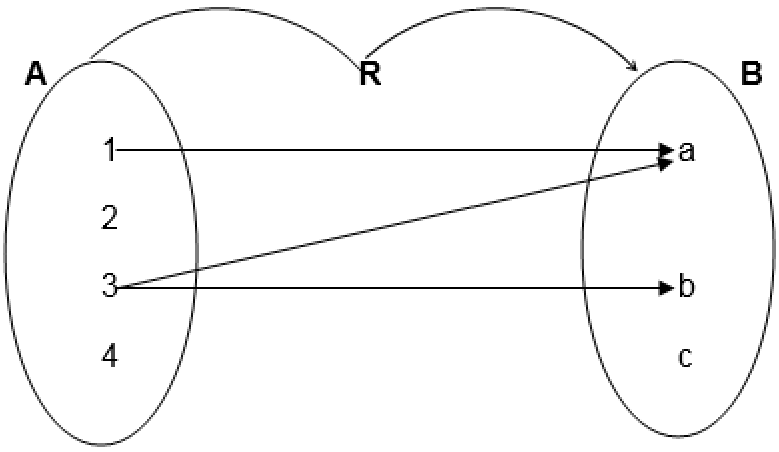
State the range of a relation

**Range**: The range is the set of all possible output values (usually the variable y, or sometimes expressed as f(x)), which result from using a particular function.

If R is the relation on two sets A and B such that set A is an independent set while B is the dependent set, then set A is the Domain while B is the Co-domain or Range.

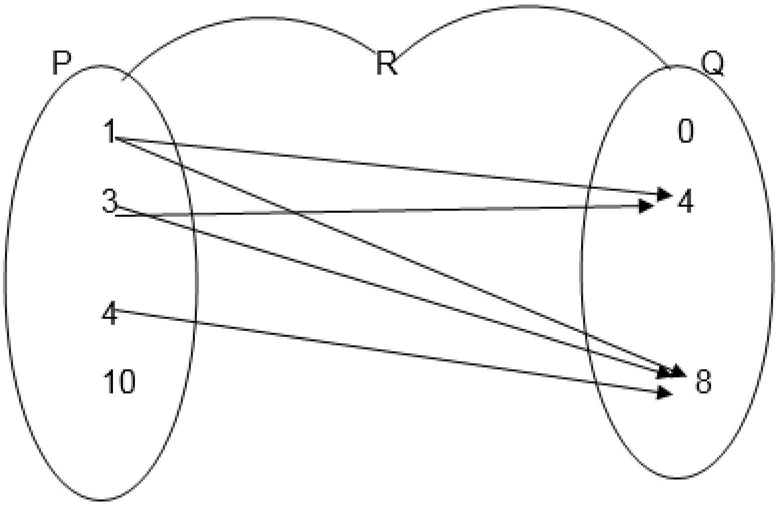
Note that each member of set A must be mapped to at least one element of set B and each member of set B must be an image of at least one element in set A.

Consider the following relation




For the above relation, the domain is $\{1,3\}$ while the range is $\{a,b\}$

Example 8
Let P = 1,3,4,10 and Q = 0,4,8
Find the domain and range of the relation R:" is less than"




From the pictorial representation of the relation R above, the

Domain is {1,3,4} and the Range is {4,8}

Example 9
As we,

Let $R = \left\{ (x, y): y = x+1 \text{ and } -2 < x \leq 8 \right\}$

WhereR is the relation and both x and y are integers.

State the domain and range of R

**Solution**

Domain $= \left\{ x: -2 < x \leq 8 \right\}$

$= \left\{ -1, 0, 1, 2,3,4,5,6,7,8 \right\}$

Range $= \left\{ y: y = x + 1 \right\}$

$= \left\{ 0,1,2,3,4,5,6,7,8,9 \right\}$

$\therefore$Domain $= \left\{ 1,0 ,1,2,3,4,5,6,7,8 \right\}$

and Range $= \left\{ 0,1,2,3,4,5,6,7,8,9 \right\}$

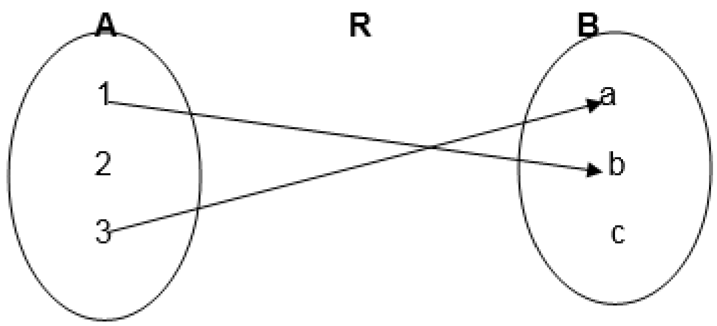
Exercise 4

1. Let A = { 3,5,7,9 } and B = {1,4,6,8 } , find the domain and range of the relation “is greater than on sets A and B

2. Let Z = { Triangle, quadrateral, pentagon, hexagon }

and W = {1,2,3,4,5}. Find the domain and range of the relation between Z and W that connects each polygon with the number of its sides.

3. State the domain and range of the following relation.




4. Let X ={3, 4, 5, 6} and
Y ={2, 4, 6, 8}

Draw the pictorial diagram to illustrate the relation “is less than or equal to‘ and state its domain and range

**Inequalities:**

The equations involving the signs < , ≤, > or $^3$ are called inequalities

Eg. x<3 x is less than 3

x>3 x is greater than 3

x≤ 2 x is less or equal to 2

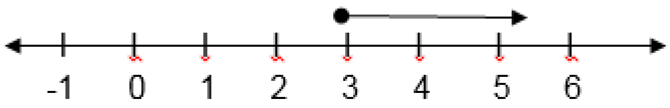
x$^3$ 2 x is greater or equal to 2

x > y x is greater or than y etc

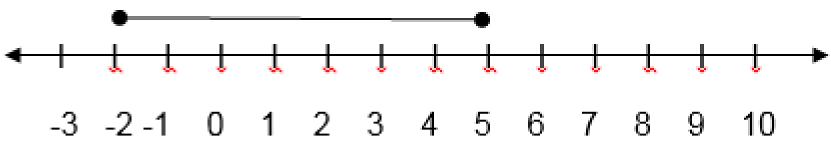
Inequalities can be shown on a number line as in the following

Examples;

1. $x \geq 3$




2. $-2 \leq x \leq 5$




Inequalities involving two variables:

If the inequality involves two variables it is treated as an equation and its graph is drawn in such a way that a dotted line is used for > and < signs while normal lines are used for those involving ≤ and ≥.

The line drawn separates the x-y plane into two parts/regions

The region satisfying the given inequality is shaded and before shading it must be tested by choosing one point lying in any of the two regions,

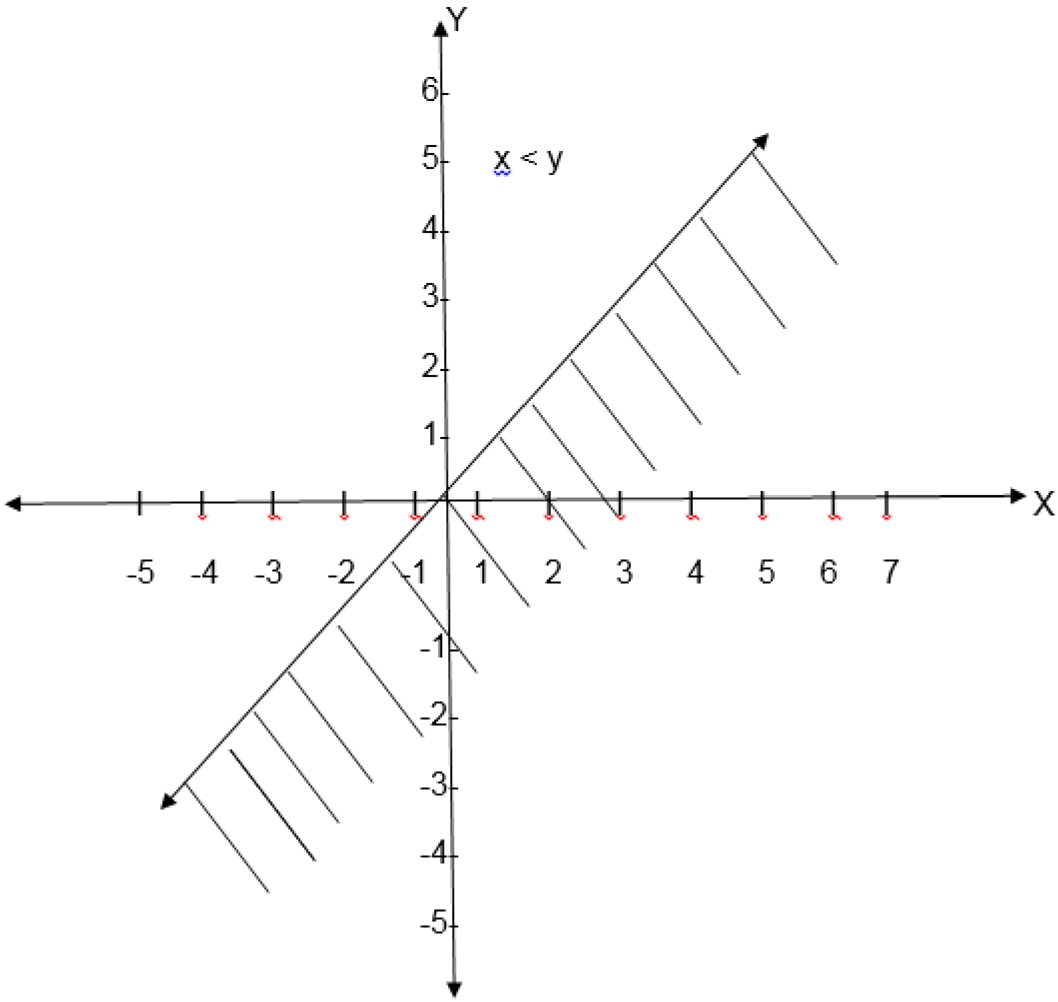
Example 10

1. Draw the graph of the relation R = {(x, y)**:** x>y}

**Solution:**

x>y is the line x =y but a dotted line is used.

Graph



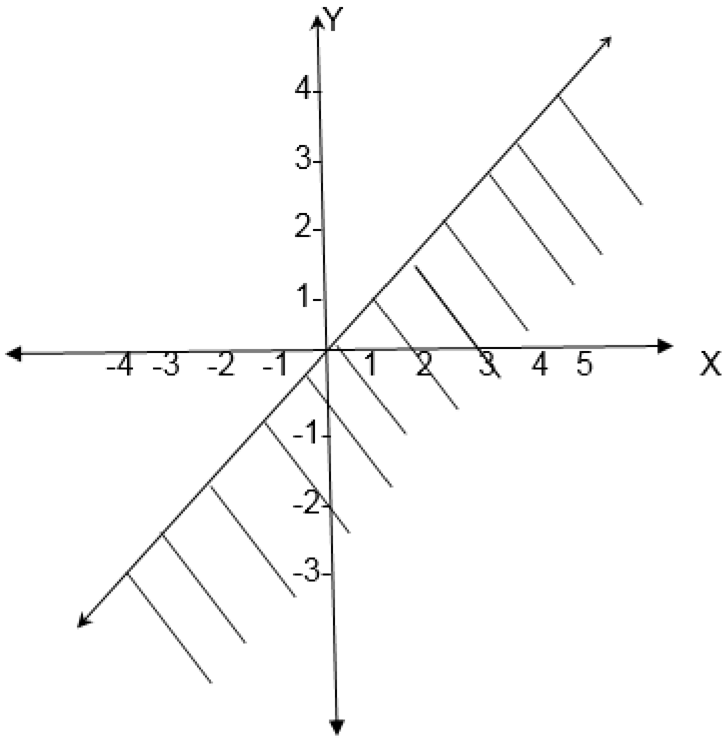

If you draw a graph of the relation R = {(x,y ) **:** x < y} , the same line is draw but shading is done on the upper part of the line.

Exercise 5

1. Draw the graph of the relation R = {(x,y )**:** x + y > 0}

2 .Draw the graph of the relation R = {( x ,y ) **:** x – y ³ -2}

3. Write down the inequality for the relation given by the following graph



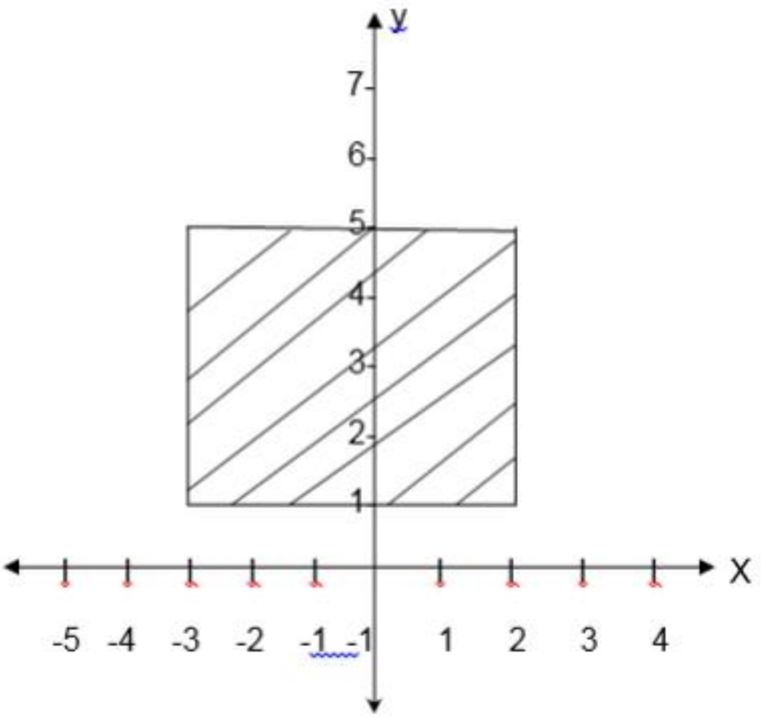

4. Draw a graph of the inequality for the relation x >-2 and shade the required region.

**Domain and Range from the graph**

***Definition**:* Domain is the set of all x values that satisfy the given equation or inequality. Similarly Range is the set of all y value satisfying the given equation or inequality

Example 11

1. Consider the following graph and state its domain and range.

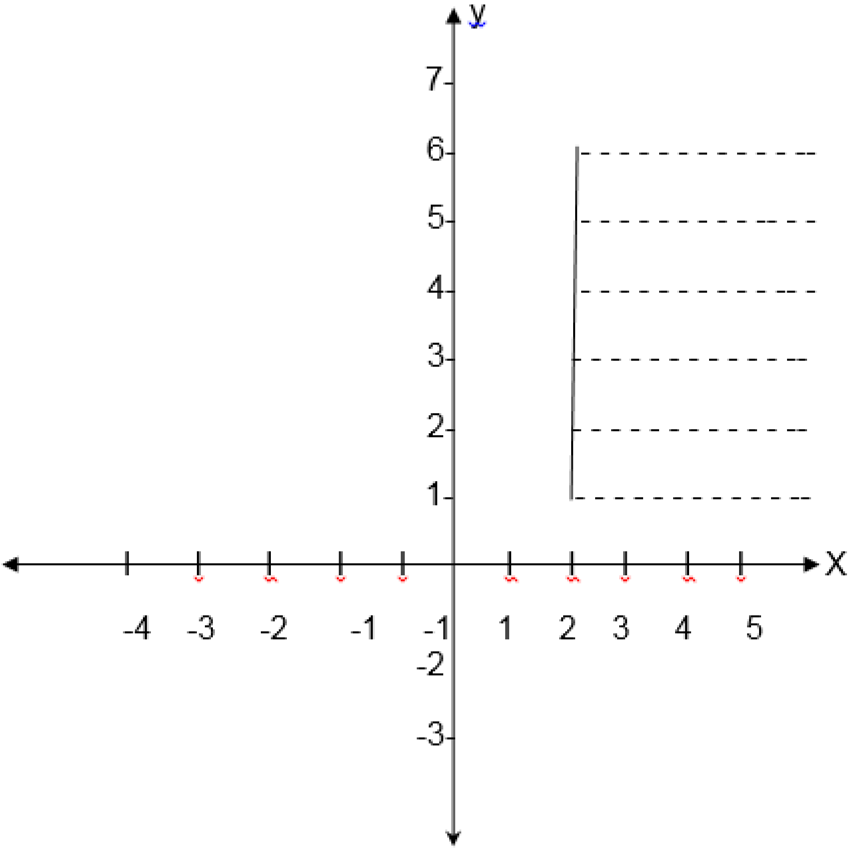



**Solution**

$$\text{Domain} = \{x: -3 \le x \le 2\}$$

$$\text{Range} = \{y: 1 \le y \le 5\}$$

Example 12
State the domain and range of the relation whose graph is given below.




**Solution;**

$$\text{Domain} = \{x : x \ge 2\}$$

$$\text{Range} = \{y : 1 \le y \le 6\}$$

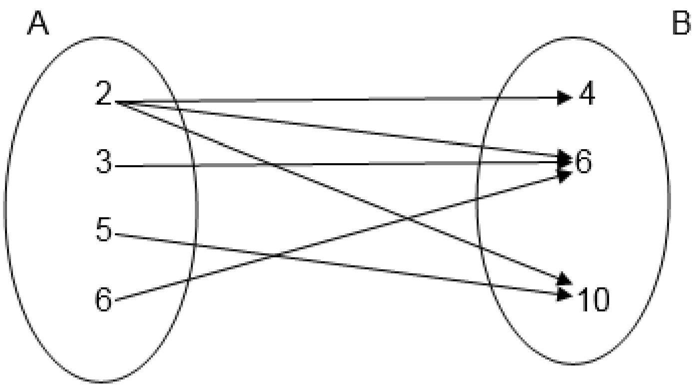
# Inverse of a Relation

The Inverse of a Relation Pictorially
Explain the Inverse of a relation pictorially
If there is a relation between two sets A and B interchanging A and B gives the inverse of the relation.
If R is the relation, then its inverse is denoted by $R^{-1}$

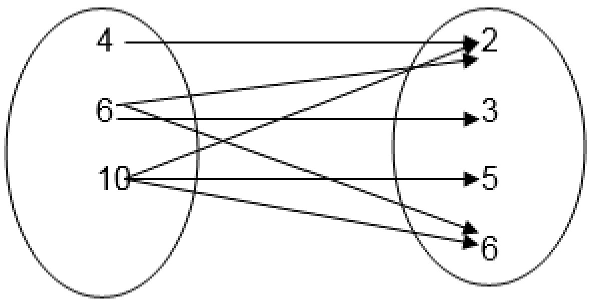
- If the relation is shown by an arrow diagram then reversing the direction of the arrow gives its inverse
- If the relation is given by ordered pair ( x, y) , then inter changing the variables gives inverse of the relation, that is (y,x) is the inverse of the relation. So domain of R = Range of R -1 and range of R = domain of $R^{-1}$

Example 13
1.

. let A = $\{2,3,5,6\}$ and B = $\{4,6,10\}$ the relation " is the factor of " is shown below




The inverse of this relation is " is a multiple of "




Inverse of a Relation
Find inverse of a relation
Example 14
Find the inverse of the relation R ={ ( x, y)**:**$x+ 3\ ^3\ y$}
**Solution**
$R^{-1}$ is obtained by inter changing the variables x and y.

$$X + 3 \geq y$$

$$Y + 3 \geq x$$

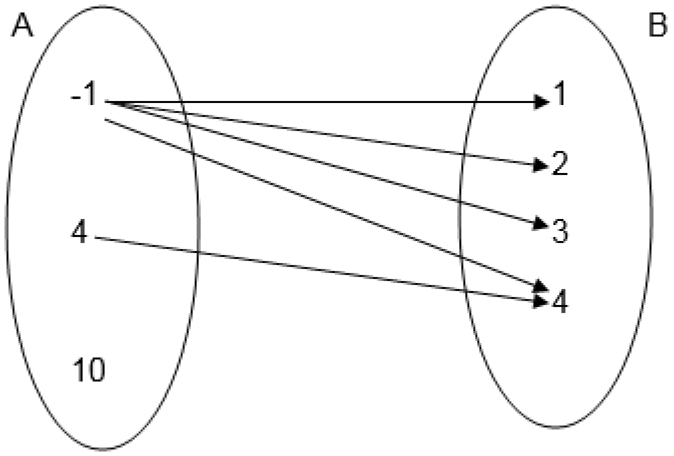
$$y \geq x - 3$$

$$\therefore R^{-1} = \{ (x, y) : y \geq x - 3 \}$$

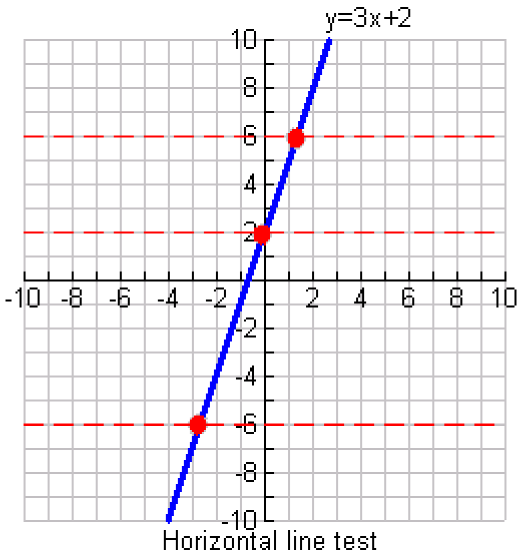
$$\text{Or} \quad R^{-1} = \{ (x, y) : x - 3 \leq y \}$$

Example 15
Find the inverse of the relation
R ={ ( x , y )**:** y = 2x }
***Solution***
R ={( x , y )**:** y = 2x }
After interchanging the variable x and y, the equation
y = 2x becomes x = 2y

or y = ½ x
so $R^{-1}$ = ( x, y ) **:** y = ½ x
Exercise 6
1 .Let A = 3,4,5 and B '= 1,4,7 find the inverse of the reaction " is less than " which maps an element from set A on to the element in set B
2 .Find the inverse of the relation R = {( x ,y ) : y > x – 1}
3 .Find the inverse of the following relation represented in pictorial diagram




4 .State the domain and range for the relation given in question 3 above
5. State the domain and range of the inverse of the relation given in question 1 above.
A Graph of the Inverse of a Relation
Draw a graph of the inverse of a relation
Use the**horizontal line test**to determine if a function has an*inverse function*.
If ANY horizontal line intersects your original function in ONLY ONE location, your function has an inverse which is also a function.
The function*y*= 3*x*+ 2, shown at the right, HAS an*inverse function*because it passes the horizontal line test.



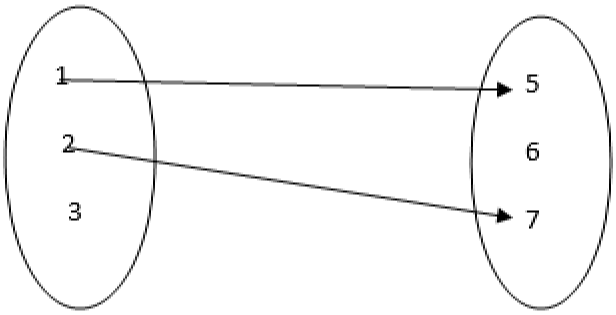

# TOPIC 2: FUNCTIONS

Normally relation deals with matching of elements from the first set called DOMAIN with the element of the second set called RANGE.
Definitions:

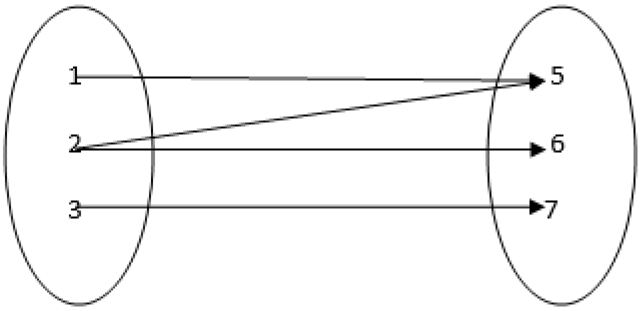
A function is a relation with a property that for each element in the domain there is only one corresponding element in the range or co- domain
Therefore functions are relations but not all relations are functions

Representation of a Function
The Concept of a Functions Pictorially
Explain the concept of a functions pictorially
Example 1
Which of the following relation are functions?

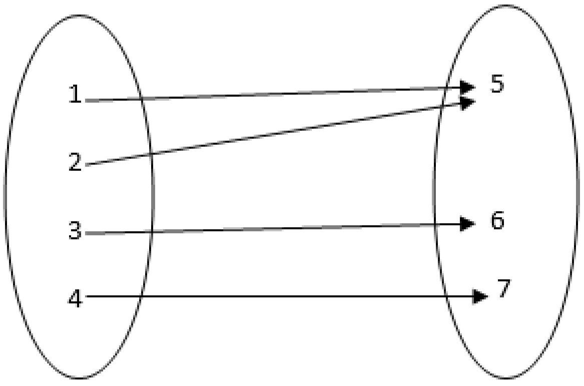
(a)




(b)




(c)



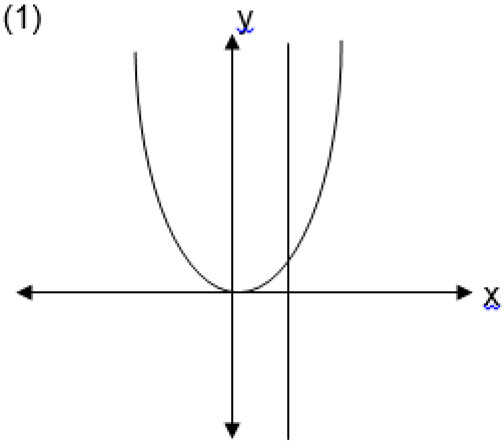

**Solution**

a. It is not a function since 3 and 6 remain unmapped.
b. It is not a function because 2 has two images ( 5 and 6)
c. It is a function because each of 1, 2, 3 and 4 is connected to exactly one of 5, 6 or 7.

Functions

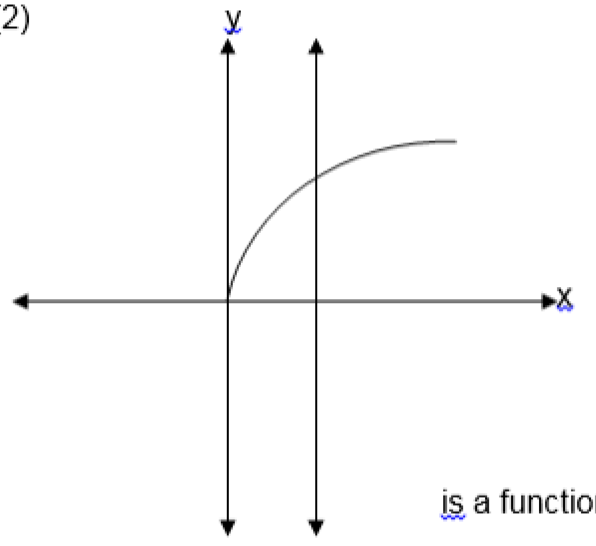
Identify functions
TESTING FOR FUNCTIONS;
If a line parallel to the y-axis is drawn and it passes through two or more points on the graph of the relation then the relation is not a function.
If it passes through only one point then the relation is a function
Example 2
Identify each of the following graphs as functions or not.

(1)



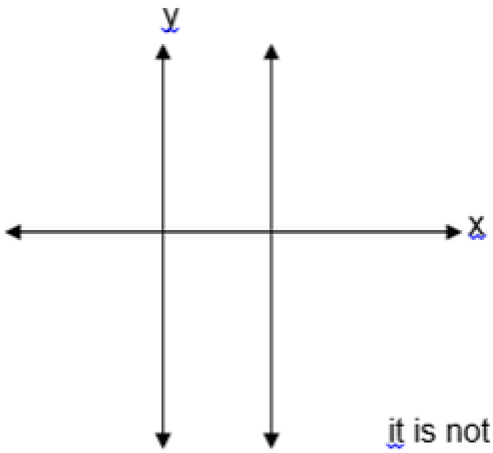

is a function.

(2)



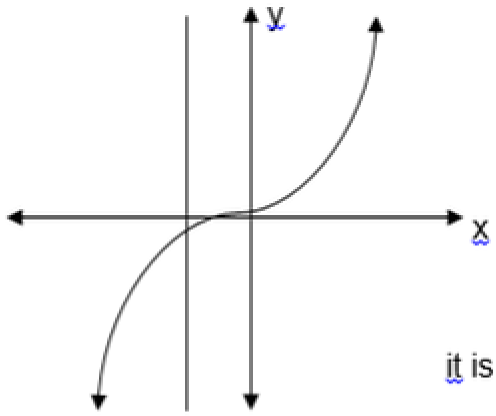

is a function.

(3)



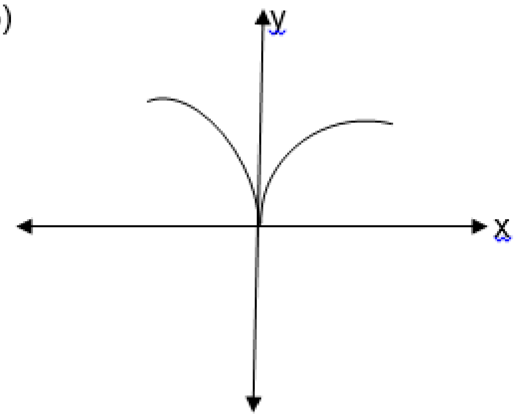

it is not a function.

(4)

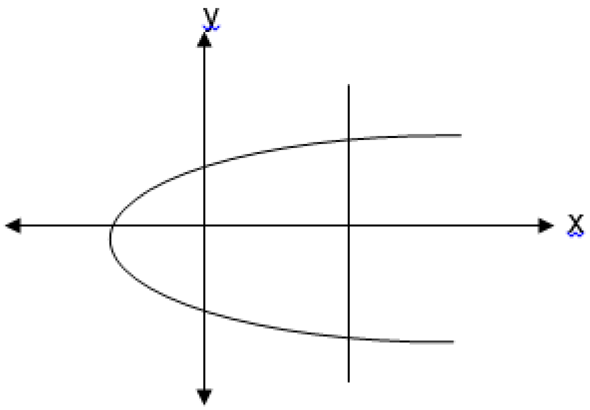



it is a function

(5)



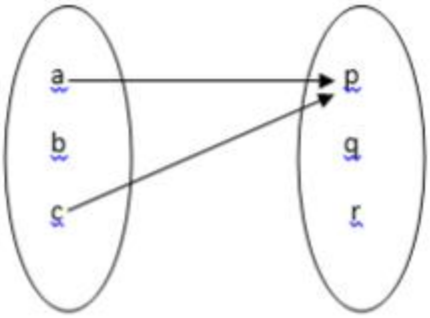
(6)



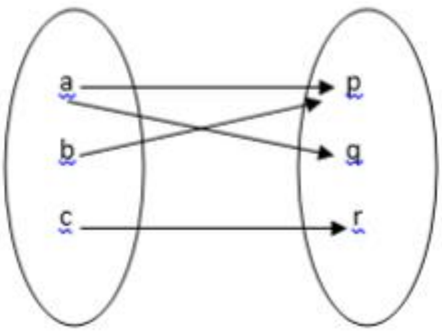

Exercise 1

1. Which of the following relations are functions?

(a)




(b)




2. Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
and B ={ 2, 3, 5, 7 }
Draw an arrow diagram to illustrate the relation " is a multiple of ' is it a function ? why?
3. let A = {1,-1 ,2,-2} and
B = {1, 2, 3, 4 } which of the following relations are functions ?

a. $\{ (x, y) : x < y \}$

b. $\{ (x, y) : x > y \}$

c. { ( x , y ) : $y = x^2$}

# Domain and Range of a Function

The Domain of a Function

State the domain of a function

If y = f (x),that is y is a function of x ,then domain is a set of x values that satisfy the equation y = f (x).

The Range of Function

State the range of function

If y = f (x),that is y is a function of x , then therange is a set of y value satisfying the equation y = f (x).

Example 3

1. Let f(x) = 3x – 5 for all value of x such that -2 £ x £ 3 find its range

**Solution**

f (x) = y = 3x -5

When x = -2

f(-2) = y = 3x(-2)-5 = -11 , so (x,y)=(-2,-11)

f(3) =y= 3x3-5 = 4, so when x = 3 , y = 4

Therefore y is found in between – 11 and 4

Range ={ y**:** - 11 £ y £ 4}

Example 4

If f (x) = $x^2$ – 3, state the domain and range of f (x)

Solutions;

Domain = all real numbers

Range:

f(x) = y = $x^2 - 3$

Make x the subject

y+ 3 = $x^2$

$\sqrt{y + 3} = x$

Since there is no square root of negative number(s) then y + 3 must be greater than or equal to zero.

i.e $y + 3 \geq 0$

$y \geq -3$

$\therefore$ Range= $\{y: y \geq -3\}$

Exercise 2

1. For each of the following functions, state the domain and range

a. f(x) = 2x + 7 for 2 £ x £ 5

b. f(x) = x – 1 for -4 £ x £ 6

c. f(x) = 5 - 3x such that -2 £ f(x) < 8

2. for each of the following functions state the domain and range

a. f(x) = $x^2$

b. f(x) = $x^2+2$

c. f(x) = 2x + 1

d. f(x) = $1 - x^2$

Exercise 3

1.The range of the function
f(x) 3 – 2x for 0 ³0 x £7 is;

a. y: -18£ y £3

b. y: -3£ y £18

c. y: 3 £y £18

d. y: -18 £ y £-3

2. The range of the function
f(x)=2x+1 is y: -3£ y £17 what is the domain of this function?

a. x**:** - 3£ x £17

b. x**:** - 2£ x £8

c. x: -17 £ x £3

3.Which of the following relations represents a function:

a. R = (x, y) : y = for x ≥0

b. R= (x, y) : $y^2$ = x-2 for x ≥0

c. R= (x, y) : y = for x ≥0 and y ≥0

d. R= (x, y) : x = 7 for all values of y

4.Which of the following relations is a function:

a. R = (x, y): -2 £ x £6, 3 £ y<8 and x<y, Where both x and y are integers

b. R= (x, y): -2 £ x £6, 3 £ y<8 and x<y, Where both x and y are integers

c. R= (x,y): y = √(x+2) for x ≥-2.

d. R = (x, y): y=√(2-x) for x ≤2 and y ≤0

5.Let f (x) = $x^2$ + 1. Which of the following is true?

a. f (-2) < f (0)

b. f (3)> f (-4)

c. f (-5) = f (5)

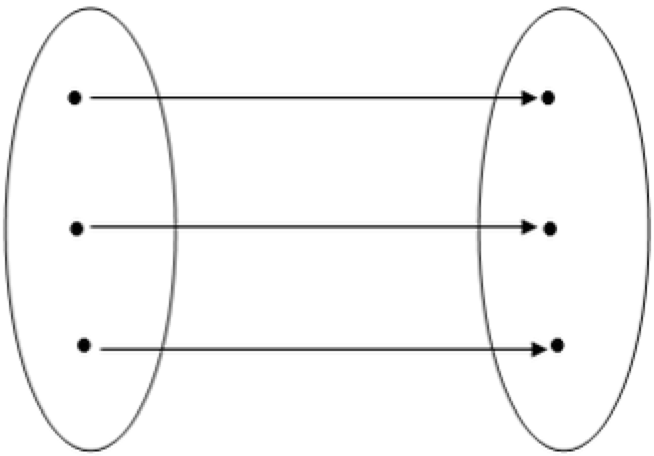
d. The function crosses , y – axis at 1

***One to one and many to one functions:***

***One to functions;***

A one to one function is a function in which one element from the domain is mapped to exactly one element in the range:

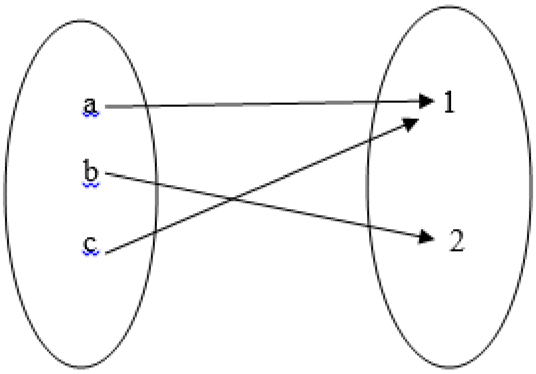
That is if a ≠b then f (a) ≠f (b)



One to one function

**Many to one function;**
This is another type of function with a property that two or more elements from the domain can have one image (the same image).

i.e $f(a) = f(b)$ but $a \neq b$




Examples of one to one functions

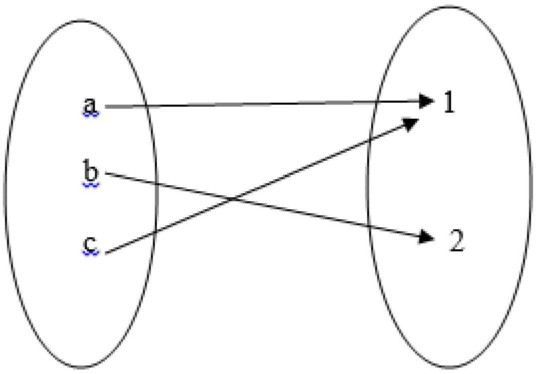
1. $f(x) = 3x + 2$
2. $f(x) = x + 6$
3. $f(x) = x^3 + 1$ etc

Examples of many to one function

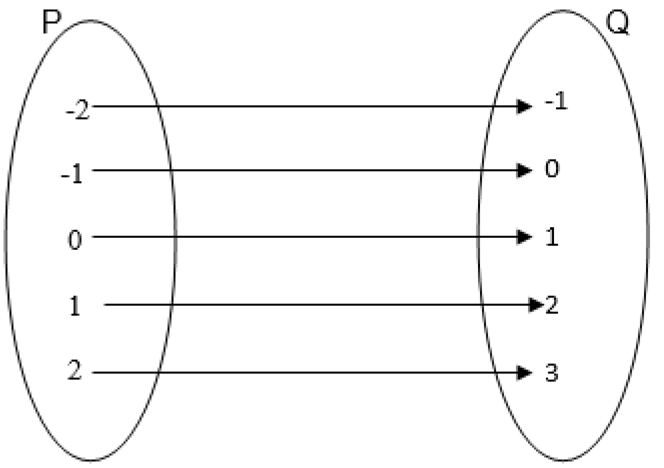
1. $f(x) = x^2 + 1$
2. $f(x) = x^4 - 2$ etc

NB. All functions with odd degrees are one to one function and all functions with even degrees are many to one functions.
Example 5
Let A = -2, -1, 0, 1, 2 and B = 0, 1, 4 and the function f mapping each element from set A to those of B is defined as $f(x)=x^2$.Is f one to one function?

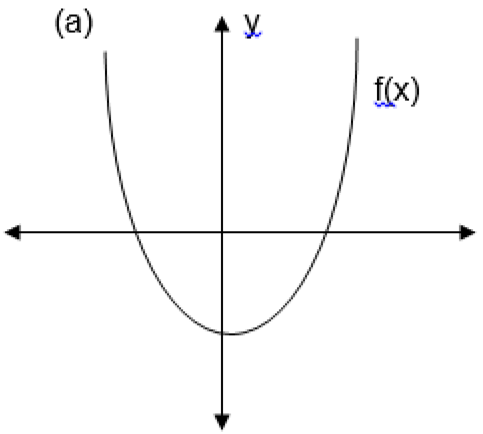
i.e $f(a) = f(b)$ but $a \neq b$

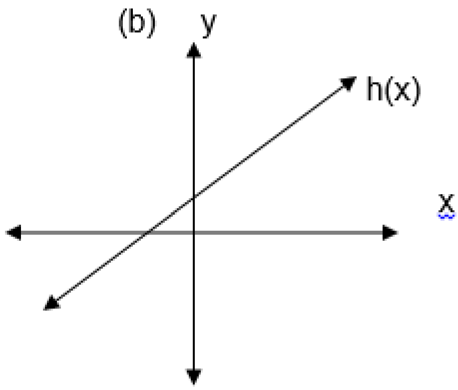



Example 6
Let P = {-2, -1, 0,1,2} and
Q = {-1, 0, 1, 2, 3}
$g(x) = x + 1$, is g one to one function?
**Solution:**



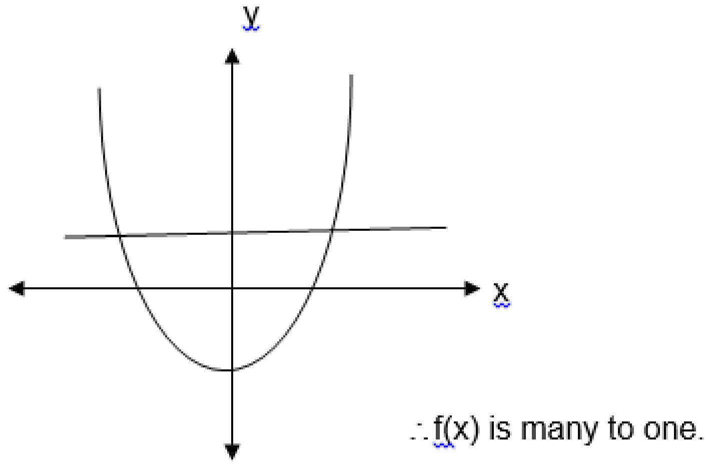

g (x) is one to one function because every element in P has only one image in Q
NB: In example 1, f(x) is not a one to one function because -2 and 2 in A have the same image in B, that is 4 is the image of both 2 and -2.
Also 1 is the image of both 1 nd -1.
Example 7
State whether or not if the following graphs represent a one to one function:



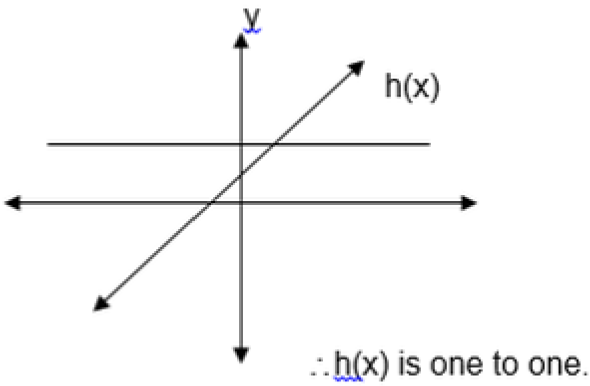




**Solution:**
Draw a line parallel to the x axis and see if it crosses the graph at more than one points. If it does, then, the function is many to one and if it crosses at only one point then the graph represents a one to one function.

(a)




(b)




# Graphic Function

Graphs of Functions

Draw graphs of functions

Many functions are given as equations, this being the case, drawing a graph of the equation is obtaining the graph of the equation which defines the function.

Note that, you can draw a graph of a function if you know the limits of its independent variables as well as dependent variables. i.e you must know the domain and range of the given function.

Example 8

Draw the graph of the following functions

a. $f(x) = 3x - 1$

b. $g(x) = x^2 - 2x - 1$

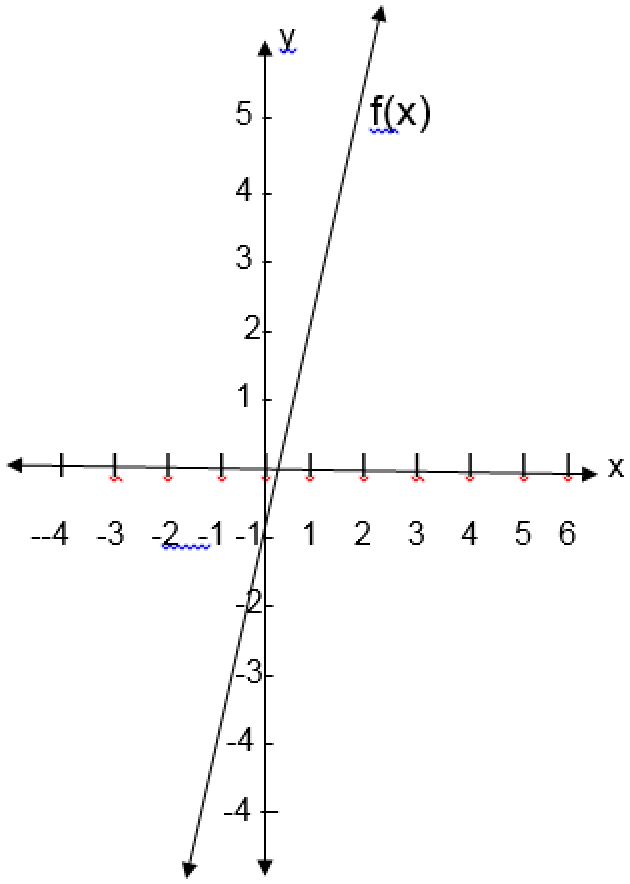
c. $h(x) = x^3$

**Solution**

$f(x) = 3x - 1$

The domain and range of f are the sets of all real numbers
$f(x) = y = 3x - 1$
So $y = 3x - 1$
Table of value :

| | | | | |
|---|---|---|---|---|
| X | -1 | 0 | 2 | 1/3 |
| Y | -4 | -1 | 5 | 0 |




$g(x) = x^2 -2x -1$
$y=x^2-2-1$
a=-1, b=-2 1 and c=-1

The turning point T.P = $\left(\frac{-b}{2a}, \frac{4ac - \mathrm{b}^2}{4a}\right)$= $\left(\frac{2}{2}, \frac{-4-4}{4}\right)$**= (1,-2)**

T. p = (1, -2)

Y-Intercept is -1

So (x, y) = (0, -1)

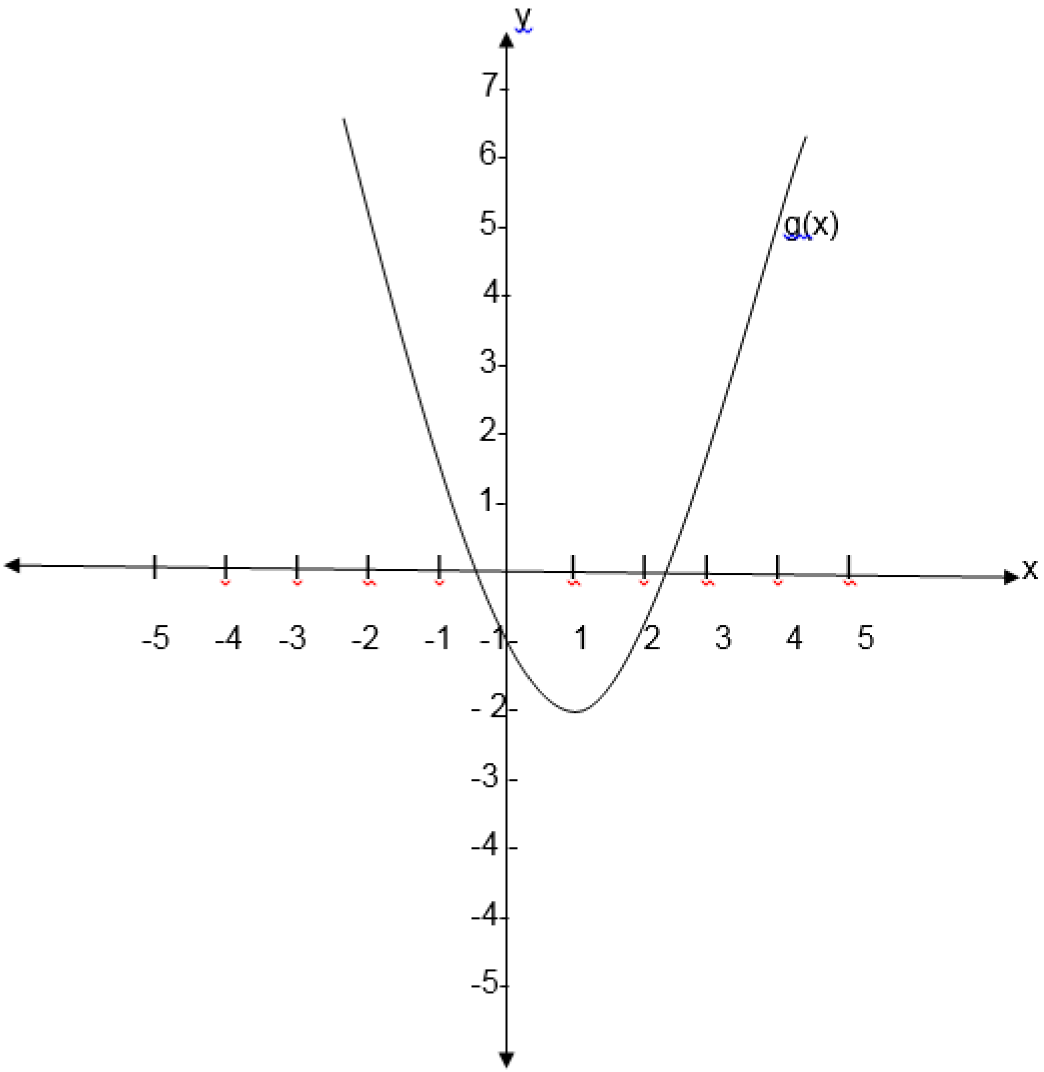
x – Intercepts are (1$\pm\sqrt{2}$) which is obtained by solving for x when y = 0

So (x, y) = [(1$\pm\sqrt{2}$), 0]

Other value are tabulated as follows

| X | -1 | 1 | 3 |
|---|---|---|---|
| Y | 2 | -2 | 2 |

Graph:



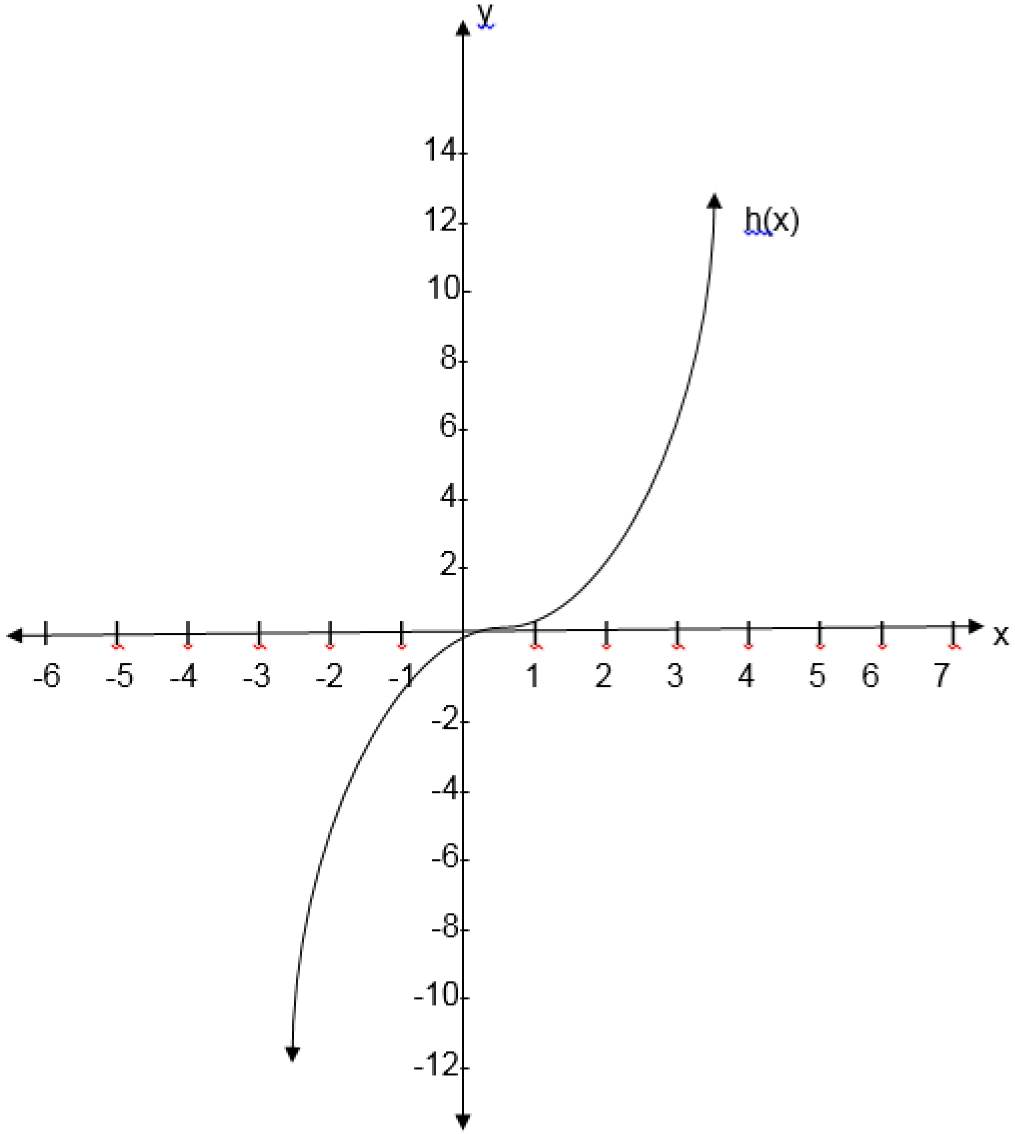

for $h(x) = x^3$

**Solution:**

Table of Values:

| X | -2 | -1 | 0- | 1 | 2 |
|---|---|---|---|---|---|
| H(x) =y | -8 | -1 | 0 | 1 | 8 |

Graph:




The first graph is the graph of linear function, the second one is called the graph of a quadratic function and the last graph is for cubic function.

Example 9

Draw a graph of the function:

$f(x) = -1 + 6x - x^2$

**Solution:**

a=-1, b=6, c=-1

The turning point T.P = $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right) = \left(\frac{-6}{-2}, \frac{4-36}{-4}\right) = (3,8)$

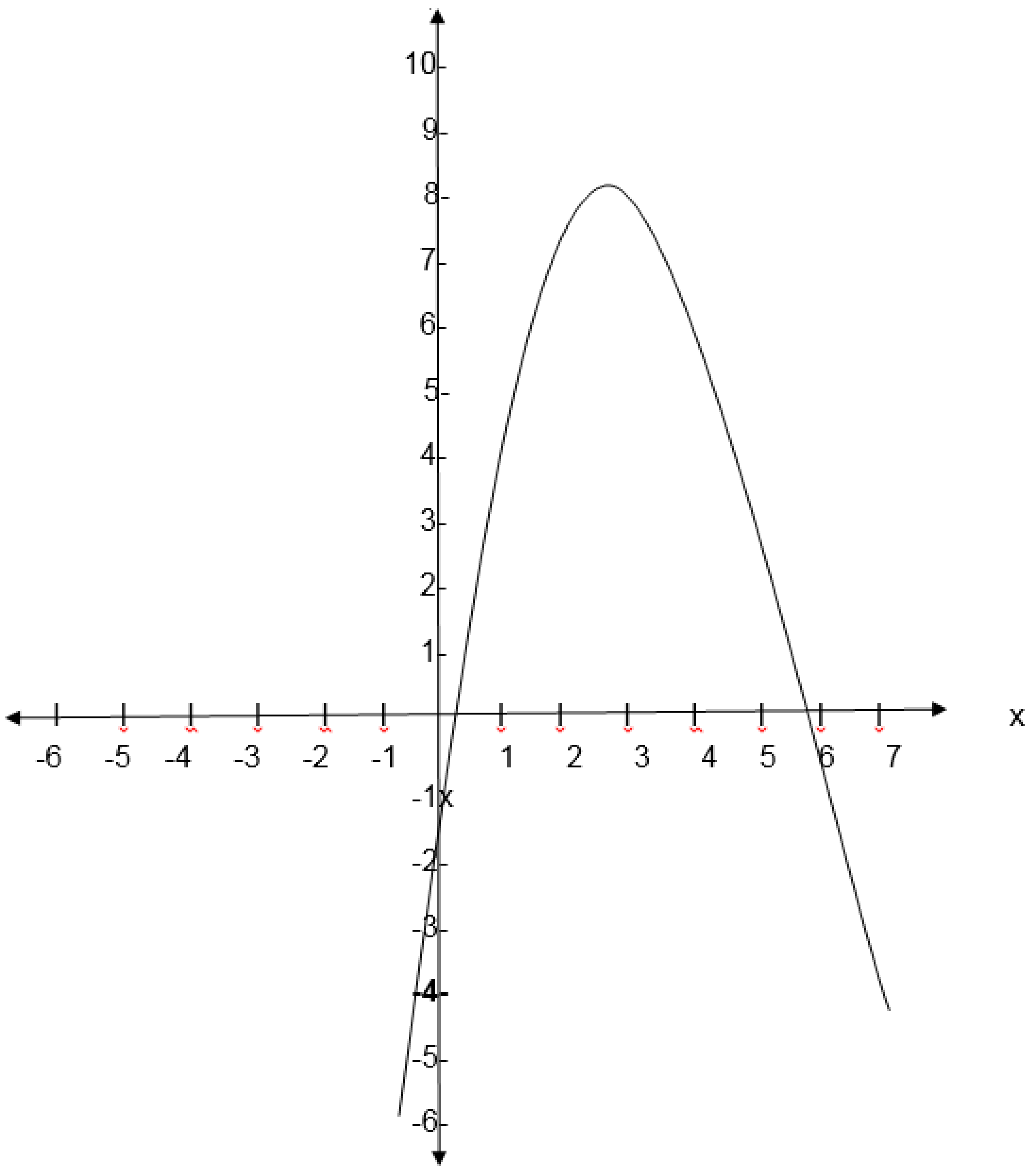
(X, y) = (3, 8)

Y-intercept = -1

(X, y)= (0,-1)

Other values

| | | | | | | |
|---|---|---|---|---|---|---|
| X | -4 | -1 | 1 | 4 | 6 | 7 |
| Y | -41 | -8 | 4 | 7 | -1 | -8 |




Exercise 4

1.Which of the following are one to one function?

a. $f(x) = 3x – x^2$

b. $g (x) = x-1$

c. $k(x) =x^3+1$

d. $f(x) =x+x^2+x^3$

e. $k(x)=x^4$

2. Draw the graph of the following functions:

a. $f(x) = 3x – x2$

b. $h (x) = x+1$

c. $g(x) =x^{3}- x^{2}+3$

3. At what values of x does the graph of the function $f(x) = x^2+x-6$ cross thex- axis?

a. x=-3 and x=7

b. x=8 and x=-6

c. x=-3 and x=2

d. x=4 and x=-1

4. Which of the following function is one to one function?

a. $f(x)=x^2+2$

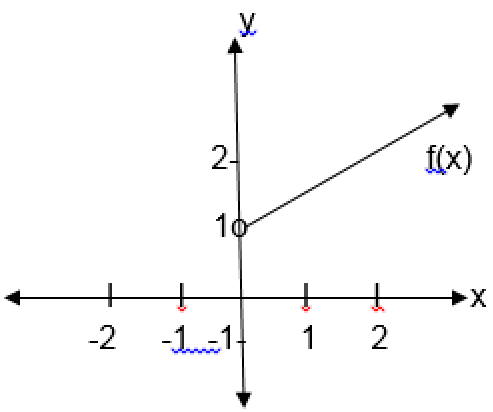
b. $f(x) =x^4-x^2$

c. $f(x)=x^5-7$

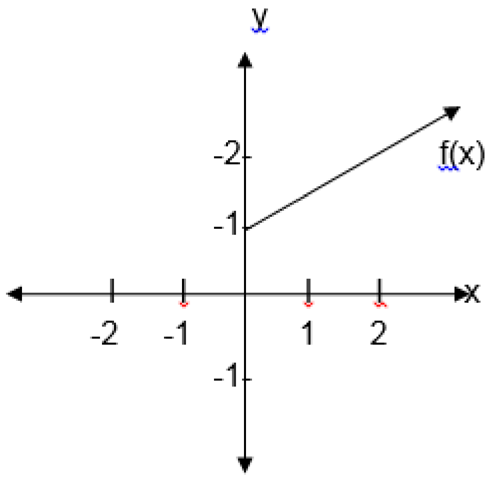
d. $f(x)=x^2+x+2$

Functions with more than one part.
Some functions consist of more than one part. When drawing their graphs draw the parts separately.
If the graph includes an end point, indicate it with a solid dot if it does not include the end point indicate it with a hollow dot.
E.g. draw the graphs of the functions
(a) F(x) x+1 for x>0
(b) f(x)=x+1for $x^{3}0$

(a)



(b)




Example 10
Solved.

1. Let $f(x) = \left\{ \begin{array}{lll} -2 & \text{if} & x<0 \\ 0 & \text{if} & x=0 \\ 2 & \text{if} & x>0 \end{array} \right\}$

(a)Find f(-8)

(b) Find f (16)

(c) Sketch its graph
(d) State the domain and range of f

**Solution:**

(a) f (-8 means f(x=-8)

But-8<0 and it is stated that when x<0

f(x) =-2

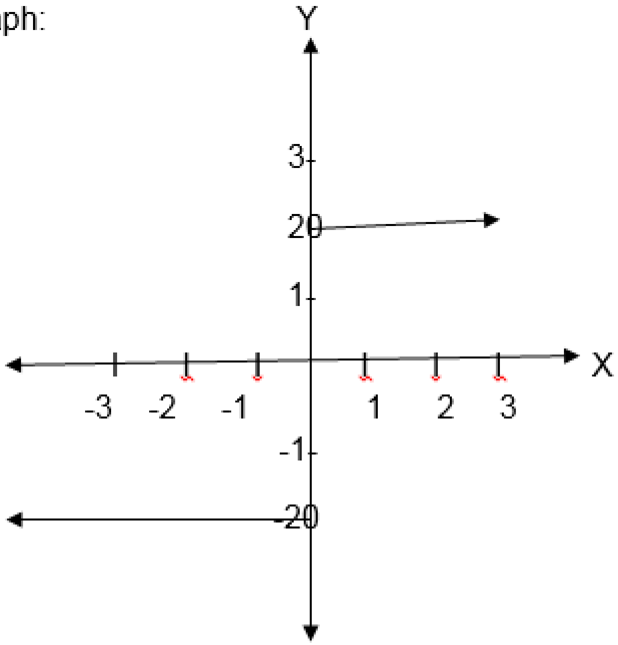
Therefore f (-8) =f(x<0) =-2

$\therefore$ F (-8) =-2

(b)Since 16>0 and f(x)=2 if x>0, then f(16)=2

$\therefore$ f(16)=2.

Graph:




(d)Domain = {All read numbers}

Range ={-2, 0, 2}

Exercise 5

Sketch the graph of each of the following functions and for each case state the domain and range.

(a) F(x) $\begin{cases} -1 \text{ if} & x<0 \\ 0 \text{ if} & x=0 \\ 2 \text{ if} & x>0 \end{cases}$

(b) F(x) $\begin{cases} x+1 \text{ if} & x \geq 1 \\ 2 \text{ if} & x+1<0 \end{cases}$

(c) F(x) = $\begin{cases} -3 \text{ for} & x<0 \\ 1 \text{ for} & 0=0 \leq x \leq 1 \\ 4 \text{ for} & x>2 \end{cases}$

**Absolute value functions (Modulus functions)**

The absolute function is defined

As $f(x) = |x| \; = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

So if x is positive, $|x|$ remains unchanged and if x is negative, $|x|$ changes to –x which also becomes positive. Hence x is always grater or equal to Zero,

i.e. $|x|$≥0 for all real values of x.

E.g. x = -2, $|x|$ = $|-2|$=2 So x=-2 then $|x|$=2

And when x=2, $|x|$ =$|2|$ =2

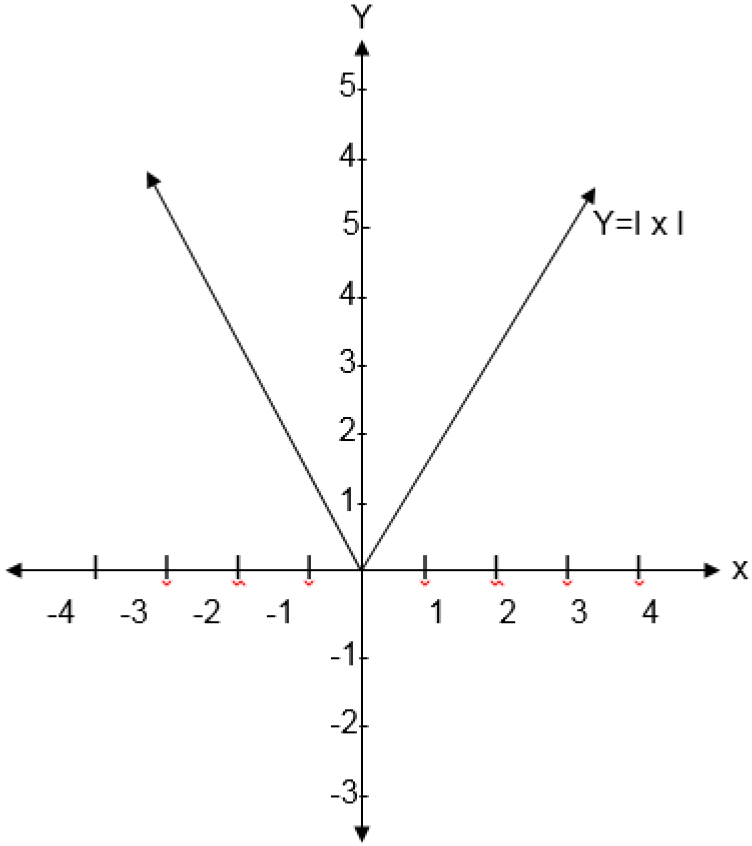
So x=2 then $|x|$=2

We can obtain the graph of f(x) =$|x|$ as follows;

Table of values

| X | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| I x I | 3 | 2 | 1 | 0 | 1 | 2 | 3 |

Graph




Example 11

Solve the following <!--[endif]-->

let f(x) $=|x+2|$,

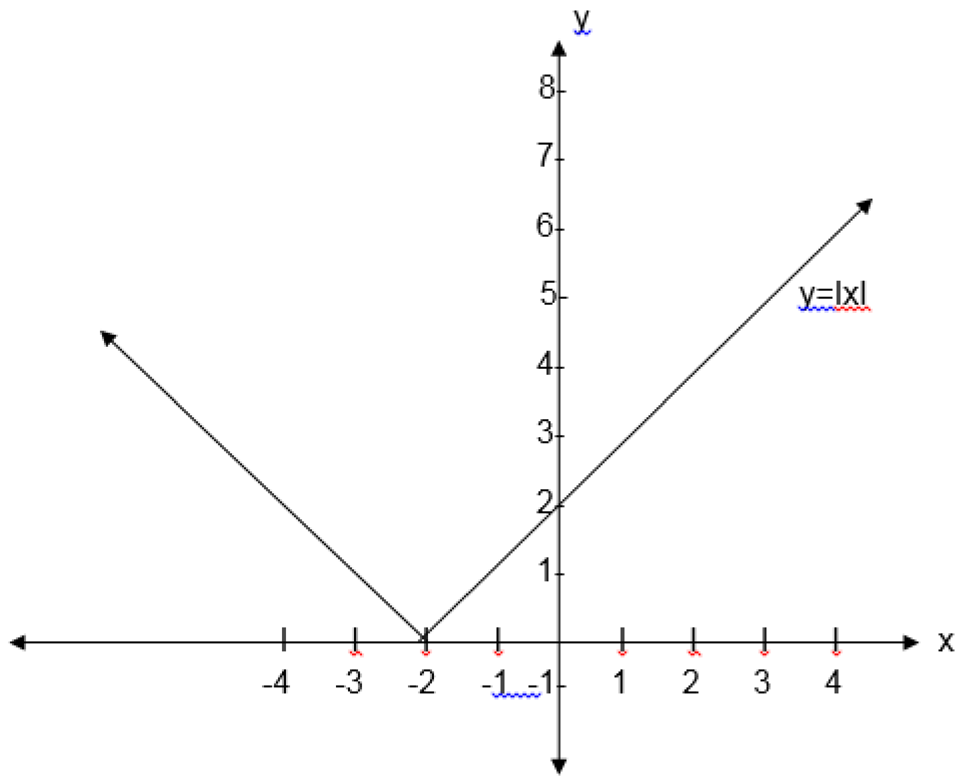
(a) sketch the graph of f(x)

(b) State its range.

Solution

table of values.

| X | -4 | -3 | -2 | 0 | 1 | 4 |
|---|---|---|---|---|---|---|
| y =f(x) | 2 | 1 | 0 | 2 | 3 | 6 |




(b) Range = $\{y: y \geq 0\}$

When sketching a modulus function it is helpful to find first the minimum or maximum value of y.

Suppose the function is given as

$y = a|x - b| + c$

The vertex in this case is the point where x – b = 0 i.e. at x = b

The value of y becomes

y= ax0+ c = c

So (x, y) = (b, c) which is the vertex of the function

**Step functions:**

For any number x, $[x]$ is the value of x when rounded down to the integer below or equal to it.

So $[x]$ is the greatest integer which is less than or equal to x.

I.e. if n ≤ x ≤ n +1, then [x] = n

For example [1.5] = 1, [2.73] =2

[5, 3/8] =5, [6] =6, [-5] = -5

[-3.2] = -4 and [-2$\frac{2}{7}$] = -2

When drawing the graph of y = [x] the graph is horizontal between integers.

Example 12
Draw the graph of

y = [x + ½]

***Solution***

If [x] = -4, then -4 ≤ x < -3.5

[x] = -3, then -3 ≤ x< -2.5

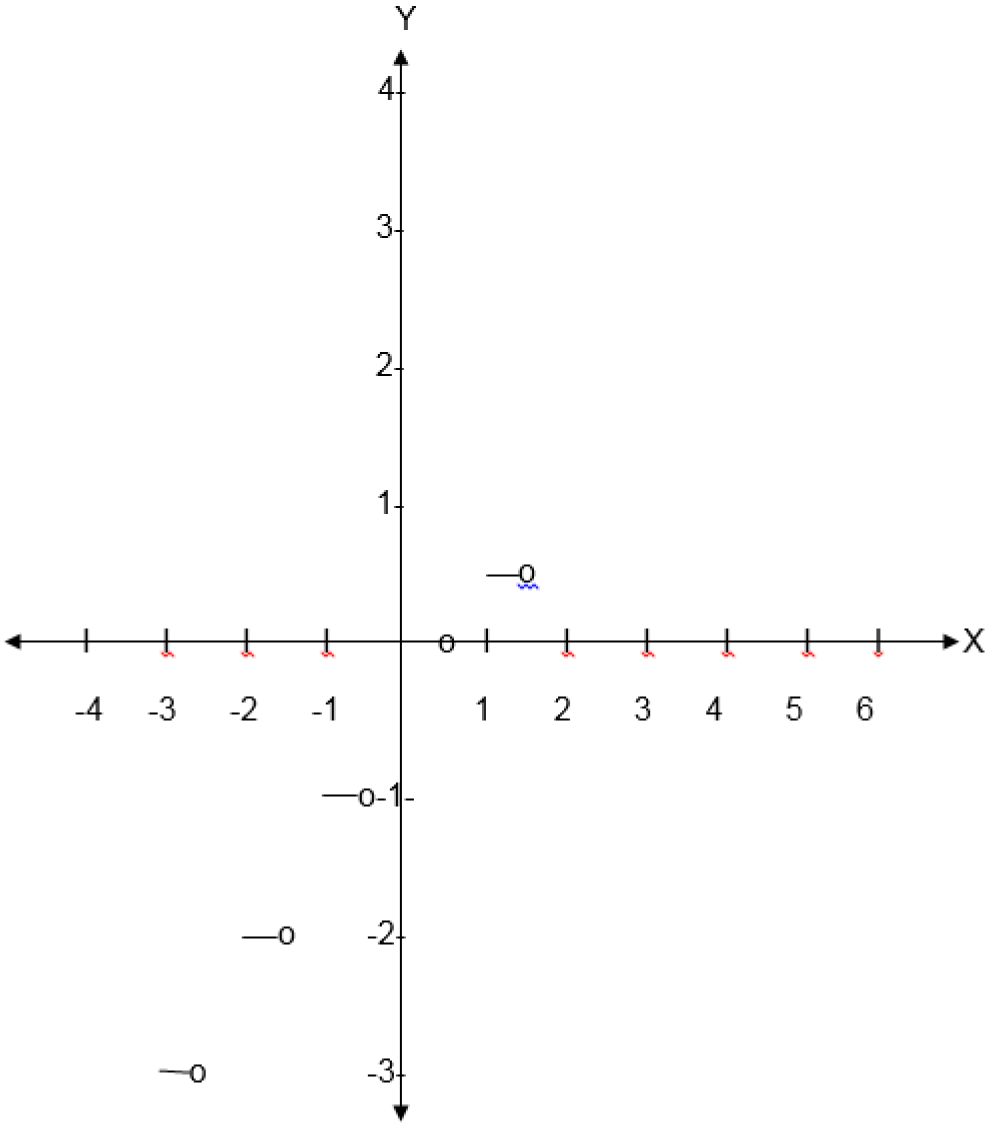
[x] = -2, then -2 ≤ x < -1.5

[x] = -1, then -1 ≤ x < - 0.5

[x] = 0, then 0 ≤ x < 0.5

[x] = 1, then 1≤ x< 1.5

Graph




Note that the graph obtained is like steps such functions are called steps functions

Exercise 6

1. Draw the graph of

(a) $F(x) = |x + 2| + 3$ (b) $f(x) = 2 - |x|$

2. Draw the graph of $f(x) = 3|x - 2| + 5$ hence state the domain and range of f(x)

3. Given that $f(x) = [x]$

Find (a) f (-6.5) (b) f (12.01)

4. $F(x) = 4 – [x]$

Find (a) f (-2.6) (b) f (3.3)

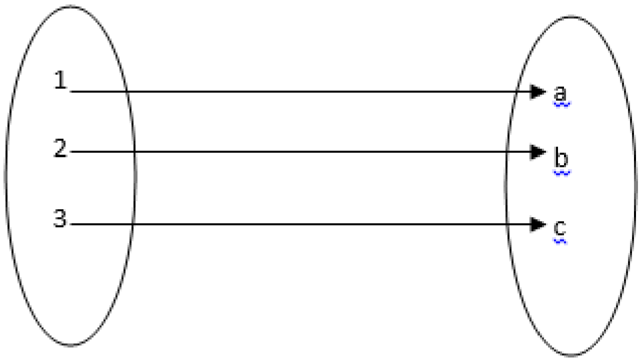
5. Draw the graph of the function $f(x) = [x] + 3$ and state its domain and range

6. Sketch the graph of the functions (a) $y = [2x]$ (b) $y = [x – 3]$

# Inverse of a Function

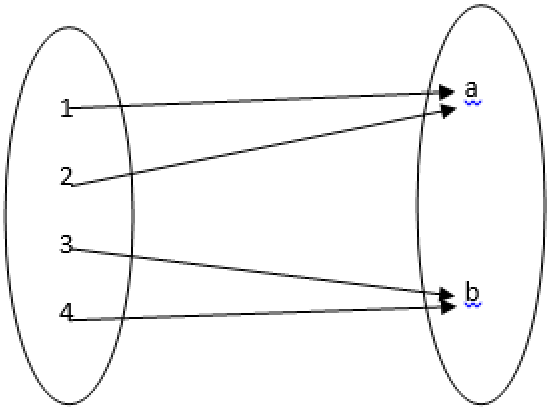
The Inverse of a Function
Explain the inverse of a function
In the discussion about relation we defined the inverse of relation.
It is true that the inverse of the relation is also a relation.
Similarly because a function is also relation then every function has its inverse
The Inverse of a Function Pictorially
Show the inverse of a function pictorially
According to the definition of function the inverse of a function is also a function if and only if the function is one to one

(a)



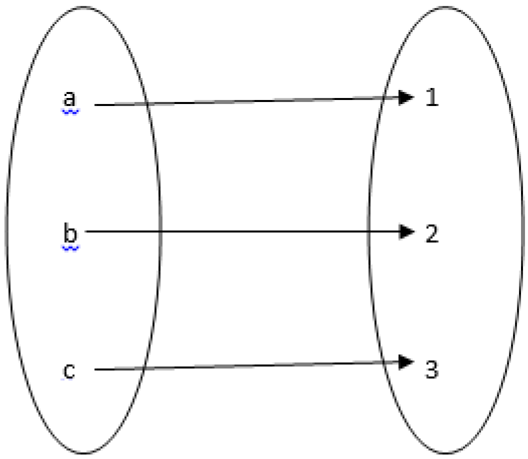

The inverse is also a function

(b)



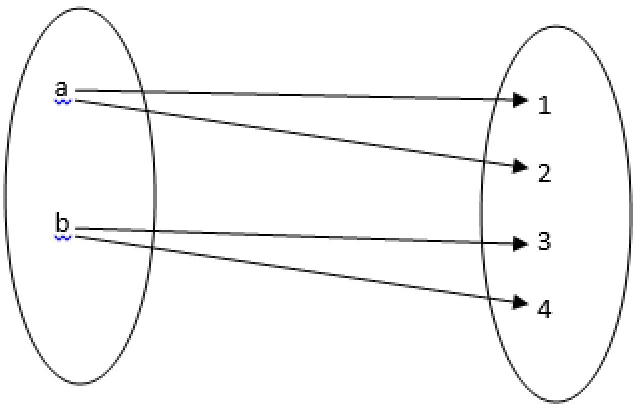

The inverse is not a function.

The inverse of a function given in (a) is shown below;




Its inverse is also a function

And the inverse of the function given in (b) is




The inverse is NOT a function.

The Inverse of a Function

Find the inverse of a function

If the function f is one to one function given by an equation, then its inverse is denoted by $f^{-1}$ which is obtained by inter changing the variables x and y then making y the subject of the formula.

I.e. If $y=f(x)$, then $x = f^{-1}(y)$

Example 13

1. Find the inverse of each of the following functions;

a. $F(x) = 3x-6$

b. $F(x) = x^3$

***Solution:***

(a) $f(x) = 3x-6$

Inverse: $y = 3x-6$

Then $x = 3y-6$

$x+6 = 3y$

$y = \frac{x+6}{3}$

$\therefore\ f^{-1}(x) = \frac{x+6}{3}$

(b) $f(x) = y = x^3$

$y = x^3$

Inverse: $x = y^3$

Then $y = \sqrt[3]{x}$

$\therefore f^{-1}(x) = \sqrt[3]{x}$

A Graph of the Inverse of a Function

Draw a graph of the inverse of a function

Example 14

find the inverse of the function $f(x) = x-5$ and then sketch the graph of $f^{-1}(x)$ , also state the domain and range of $f^{-1}(x)$.

solution:

$F(x) = y = x-5$

$y = x-5$

Inverse: $x = y-5$

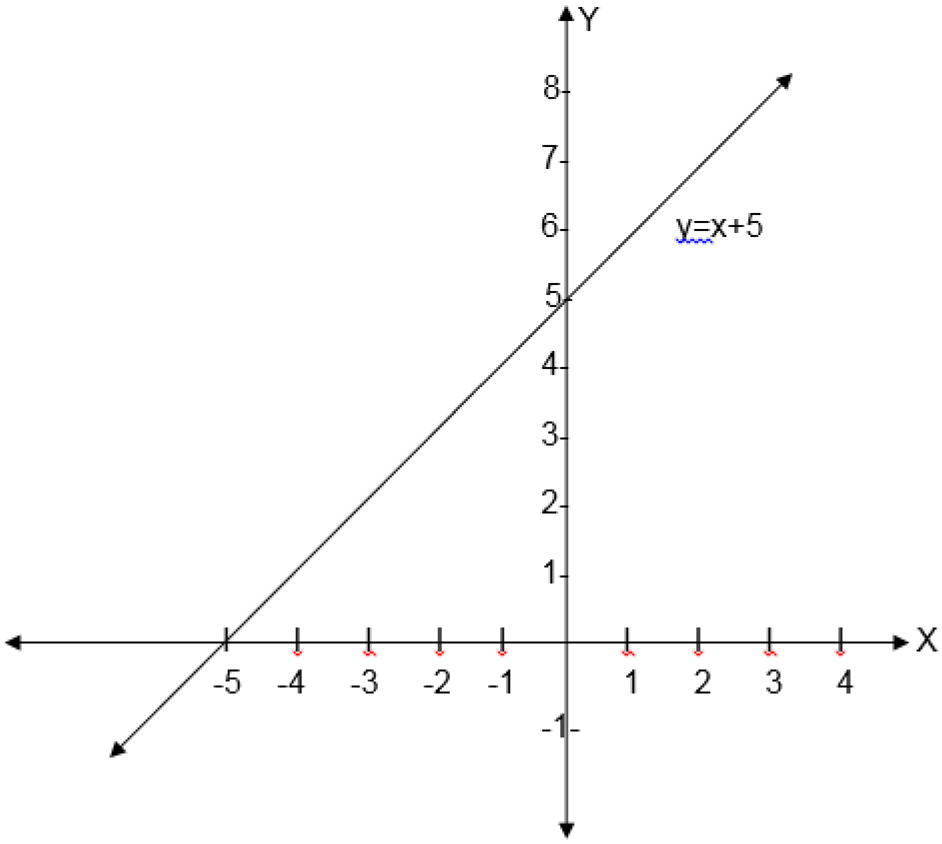
$y = x+5$

$\therefore f^{-1}(x) = x+5$

Graph of f-1 (x):

Table of values

| | | |
|---|---|---|
| X | 0 | -5 |
| Y | 5 | 0 |




Domain = {All real numbers}
Range = {All real numbers}
***NB:*** if a function f takes a domain A to a range B, then the inverse $f^{-1}$ takes B back to A. Hence the domain of $f^{-1}$ is the range of f, and the range of $f^{-1}$ is the domain of f.
The Domain and Range of Inverse of Functions
State the domain and range of inverse of functions
Example 15
Solve;

Let $f(x) = 3x - 2$ for $\{0 \le x \le 5\}$

Where x is a real number.

Find the domain and range of $f^{-1}$

***Solutions:***

The function takes all values of y between

$f(0) = -2$ and $f(5) = 13$

So range $= \{y: -2 \le y \le 13\}$

The domain of f is the range of $f^{-1}$ and the range of f is the domain of $f^{-1}$

$\therefore$ The domain of $f^{-1} = \{-2 \le x \le 13\}$

And range $= \{0 \le y \le 5\}$

Exercise 7

1.Find the inverse of each of the following functions:

(a) $F(x) = 3x^2+8$

(b) $F(x) = x-9$

2. given that $f(x) = 7x-4$ find $f^{-1}(8)$

3. given that $f(x) = \frac{3x+4}{5}$ for $-3 \leq x \leq 8$

Find $f^{-1}(x)$ and state the domain and range of $f^{-1}(x)$

4. Let $f(x) = x^3-1$

Evaluate (1) $f^{-1}(6)$ (ii) $f^{-1}(k)$

Exercise 8

1. given that $f(x) = x^2-2^{[x]}+3$, what is the value of f (-4)?

(a) 15 (b) 11 (c) 27 (d)-5 ( )

2. Let $f(x) = \begin{cases} x, & if\ x \leq 0 \\ x-2, & if\ x > 0 \end{cases}$ What is a value of f(6)?

(a) 4 (b) 6 (c)-8 (d)8 ( )

3. given that $f(x) = |x-5|-7$, what is the minimum value of f(x)?

(a) 7 (b) 0 (c) 5 (d) -7 ( )

4. Let f(x) =2x –i/3 what is the value of x such that f-1 (x) =2?

(a) $\frac{4}{3}$ (b) $\frac{3}{2}$ (c) $\frac{11}{3}$ (d) $\frac{8}{3}$ ( )

5.The function whose inverse is $f^{-1}(x) = 3x+5$ is given by;

(a) $f(x) = \frac{1}{3}(x-5)$ (b) $f(x) = \frac{3}{5x}$ ( )

(b) (c) $f(x) = \frac{1}{5}(x+3)$ (d) $f(x) = 5x-3$

# TOPIC 3: STATISTICS

## Mean

Calculating the Mean from a Set of Data, Frequency Distribution Tables and Histogram
Calculate the mean from a set of data, frequency distribution tables and histogram
Measures of central tendency:
After collecting the data, organizing it and illustrating it by means of diagrams, there is a need to calculate certain statistical measures to describe the data more precisely. There are various types of measures of central tendency – the arithmetic mean (or simply the mean), the median, and the mode. Once the measures of central tendency are found, it is easier to compare two or more sets of data.
The arithmetic mean
When people are asked to find the measure of central tendency of some numbers, they usually find the total of the numbers, and then divide this total by however many numbers there are. This type of measure of central tendency is the arithmetic mean. If the n values are x1+x2+x3 ………+xn then the arithmetic mean is = x1 + x2+ ….. + xn/n

$$\text{Therefore } \boxed{\bar{x} = \frac{x_1 + x_2 + \ldots.. + x_n}{n}}$$

Example 1
The masses of some parcels are 5kg, 8kg, 20kg and 15kg. Find the mean mass of the parcels.
**Solution**
Total mass = (5 + 8 + 20 + 15) kg = 48kg
The number of parcels = 4
The mean mass = 48kg ÷ 4 = 12kg
The arithmetic mean used as measure of central tendency can be misleading as can be seen in the following example.
Example 2
John and Mussa played for the local cricket team. In the last six batting innings, they scored the following number of runs.John: 64, 0, 1, 2, 4, 1;Mussa: 15, 20, 13, 11 , 10, 3.Find the mean score of each player. Which player would you rather have in your team? Give a reason.
**Solution**
John's mean = (64 + 0 + 1 + 2 + 4 + 1) ÷ 6 = 12
Mussa's mean = (15 + 20 + 13 + 11 + 10 + 3) ÷ 6 = 12
Each player has the same mean score. However, observing the individual scores suggests that they are different types of player. If you are looking for a steady reliable player, you would probably choose Mussa.
Often it is possible to use the mean of one set of numbers to find the mean of another set of related numbers.
Suppose a number a is added to or subtracted from all the data. Then a is added to or subtracted from the mean.
Suppose the n values are $x!$+$x!$ + $x!$ .........+$x!$. Multiply each by a, and we obtain $ax!$+$ax!$ + $ax!$ .........+$ax!$. So we see that the mean has been multiplied by a.
Interpreting the Mean Obtained from a Set Data, Frequency Distribution Tables and Histogram
Interpret the mean obtained from a set data, frequency distribution tables and histogram
Measures of central tendency from frequency tables
If the data has already been put into a frequency table, the calculation of the measures of central tendency is slightly easier.
Exercise 1

Juma rolled a six- sided die 50 times. The scores he obtained are summarized in the following table.Calculate the mean score

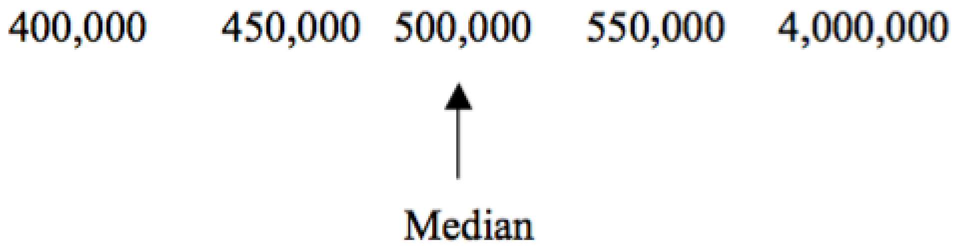
| | | | | | | |
|---|---|---|---|---|---|---|
| Score (x) | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency (f) | 8 | 10 | 7 | 5 | 12 | 8 |

**Solution**

10 scores of 2 give a total 10 x 2 = 20
8 scores of 1 gives a total 8 x 1 = 8
And so on, giving a total score of
8 x 1 +10 x 2+7 x 3 + 5 x 4 + 12 x 5 + 8 x 6 = 177
The total frequency = 8 + 10 + 7 + 5 + 12 + 8 = 50
The mean score = 177 ÷ 50 = 3.34

# Medium

The Concept of Median
Explain the concept of median
Mr.Samwel owns a small factory. He earns about 4,000,000/- from it each year. He employs 4 people. They earn 550,000/-, 500,000/-, 450,000/- and 400, 000/-.The mean income of these five people is(4,000,000 + 550,000 + 500,000 + 450,000 + 400,000 ÷ 5 = 1,180,000/-
If you said to one the employees that they earned about 1,180,000/- each year they would disagree with you. In this type of situation when one of the values is different from the others (as in Example 2), the mean is not the best measure of central tendency to use.Arrange the incomes in increasing order of size as follows:




The value that appears in the middle is called the median. In this case the value of 500,000/- is a much better idea of the average wage earned by the employees. The median is not affected by isolated values (sometimes called rogue values) that are much larger or smaller than the rest of the data.
If the data consists of an even number of values, find the mean of two middle values as shown in the next example.
The Medium from a Set of Data
Calculate the medium from a set of data
Example 3
Find the median of the numbers: 12, 23, 10, 8, 22, 14, 30, and 18.

**Solution**

Arranging in increasing order of size, we get 8 10 12 14 18 22 23 30
Median = (14 + 18) ÷ = 16
The Median using Frequency Distribution Tables and Cumulative Curve
Find the median using frequency distribution tables and cumulative curve
Example 4
Juma rolled a six- sided die 50 times. The scores he obtained are summarized in the following table. Calculate the modianl score

**Solution**

here are 50 items of data, so if you arrange them in order of size, the positions are1 .................... 25 and 26 ................. 50. The median will be the average of the 25th and 26th number.
In the table there are 8 scores of 1, followed by 10 scores of 2. This gives you 8 + 10 = 18 numbers. These are then followed by 7 scores of 3. This gives 18 + 7 = 25 numbers. It follows that the 25th number is a 3. The 26th number must be the first number in the next group, which is a 4.

The median is then = (3 + 4) ÷ 2 = 3.5
The Median Obtained from the Data
Interpret the median obtained from the data
Exercise 2

1. The times of five athletes in the 100 m were: 12.5 s, 12.9s, 14.8s, 15.0s, 25.2s. Find the median time. Why is the median a better measure of central tendency to use than the mean?
2. Iddi has 6 maths tests during a school term. His marks are recorded below. Find the mean and the median mark. Explain why the median is a better measure of central tendency than the mean 73 78 82 0 75 86
3. The table below gives the percentage prevalence of HIV infection in female blood donors for the years 1992 to 2003. Find the mean and median of these figures.

| 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5.9 | 6.2 | 4.8 | 9.4 | 8.2 | 11.6 | 11.8 | 12.6 | 13.3 | 13.7 | 12.3 | 11.9 |

# Mode

The Concept of Mode
Explain the concept of mode
The mode is value that occurs most often in a set of data.This is another measure of central tendency. It is possible for data to have more than one mode.
Data with two modes are said to be bi – modal.Why mode? The mode is often important to know. For example:

a. If you ran a shoe shop you would want to know the most popular size.
b. If you ran a restaurant you would want to know what type of food is ordered most.

The Mode
Calculate the mode
Example 5
State the mode for the following sets of numbers:

a. 0, 0, 1, 1, 1, 2, 2, 3, 4, 5, 5
b. 58, 57, 60, 59, 50, 56, 62
c. 5, 10, 10, 10, 15, 15, 20, 20, 20, 25

**Solution**

a. 1 occurs most (3 times): The mode is 1
b. All the numbers appear once: There is no mode.
c. There are three 10s and three 20s: Modes are 10 and 20.

Exercise 3

1. Ten pupils were asked how many brothers or sisters they had. The results are recorded below. Find the mode number 0, 1 , 1, 2, 5, 0, 1 3 , 1 and 4.
2. Eight motorists were asked how many times they had taken the driving test before they passed. The results are recorded below. Find the mode number. 14213141
3. Give examples of where the mode is a better measure of central tendency than either the mean or the median.
4. Find the mode of these sets of numbers.

a. 0, 1, 1, 3, 4, 5, 5, 5, 6, 7, 8
b. 3, 8, 4, 3, 8, 4, 3, 8, 8, 3, 3, 4
c. 5, 12, 6, 5, 11, 12, 5, 5, 8, 12, 7, 12

d. 3, 6, 2, 8, 2, 1, 9, 12, 15

Finding the Mode using Frequency Distribution and a Histogram

Find the mode using frequency distribution and a histogram

Grouped data

Suppose a set of data consists of many different values, such as heights of people measured to the nearest centimeter. Then the data is grouped, for example into 160 – 165 cm, and so on. If the data has been grouped together in classes, then unless you have a list of all the individual values, you only know approximately what each value is. For this reason, you can only estimate the mean and the median. Also, if all the values are different, you do not have a single value as the mode. Instead you have a modal class, as shown in the example below.

Data grouped in classes can be illustrated by a histogram.Suppose one of the intervals is from 10 to 19, where data has been rounded to the nearest whole number. The class limits are 10 and 19. The data in this interval could be as low as 9.5 or as high as 19.5. These are the class boundaries. The width of the interval is the difference between the class boundaries, in this case it is 10.

The histogram consists of rectangles between the class boundaries, with height corresponding to the frequency. The area of each rectangle is proportional to the frequency.

Example 6

The examination results (rounded to the nearest whole number %) are given for a group of students.

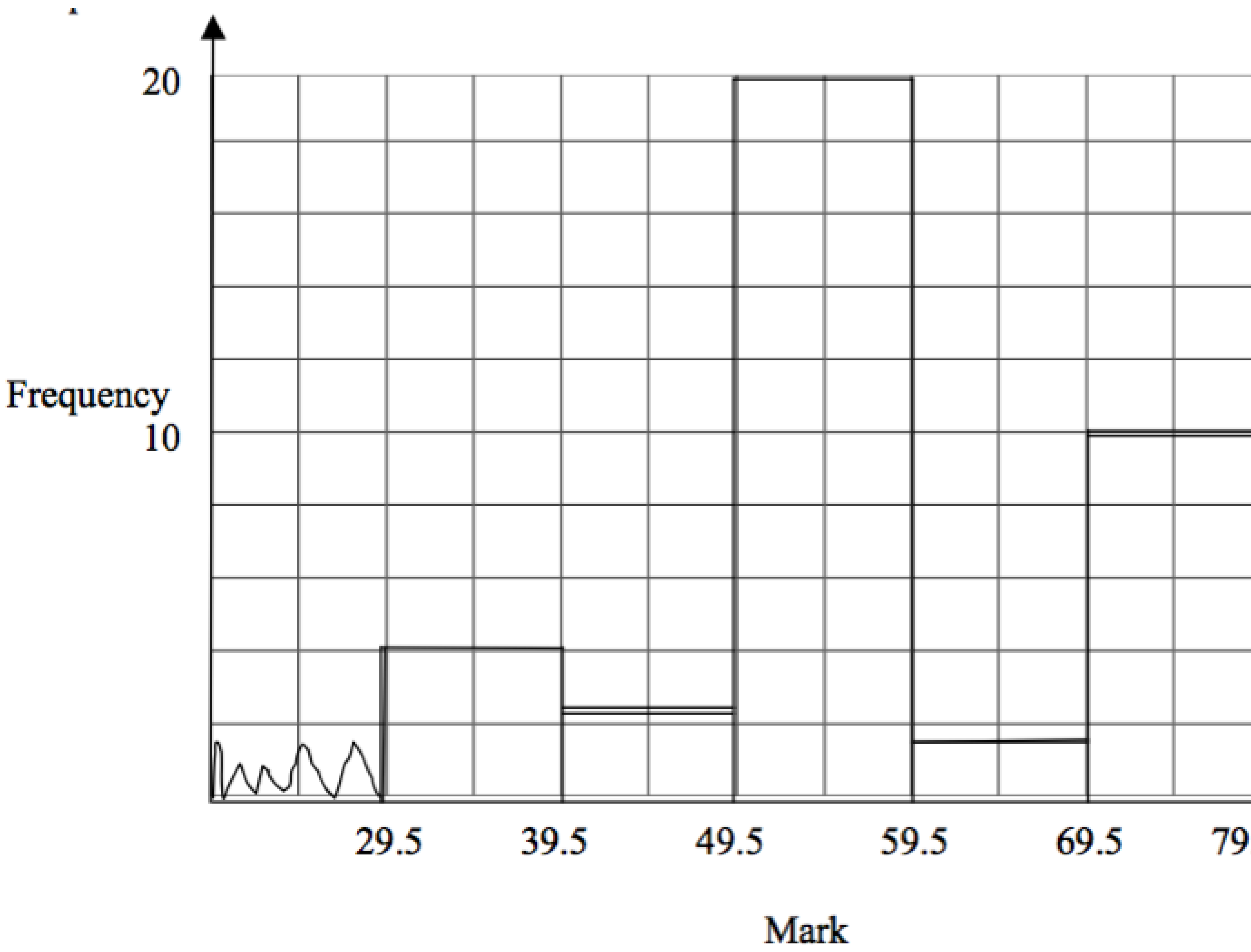
| Mark (%) | 30 – 39 | 40 -49 | 50 – 59 | 60 - 69 | 70 - 79 |
|---|---|---|---|---|---|
| Frequency | 5 | 3 | 20 | 2 | 10 |

a. Draw a histogram

b. state the modal class

**Solution**

For a histogram, the horizontal axis is for the data values, and the vertical axis is for the frequencies. So label the horizontal axis with the marks from 30 to 80. To indicate that the axis does not start at 0 put a zig – zag to the left of 30. Label the vertical axis with frequencies from 0 to 20. The first interval has limits 30 and 39. The class boundaries are 29.5 and 39.5. It has a frequency of 5. So draw a box covering the interval, and with height 5. Repeat with the other intervals




Interpreting the Mode Obtained from the Data

Interpret the mode obtained from the data

Example 7

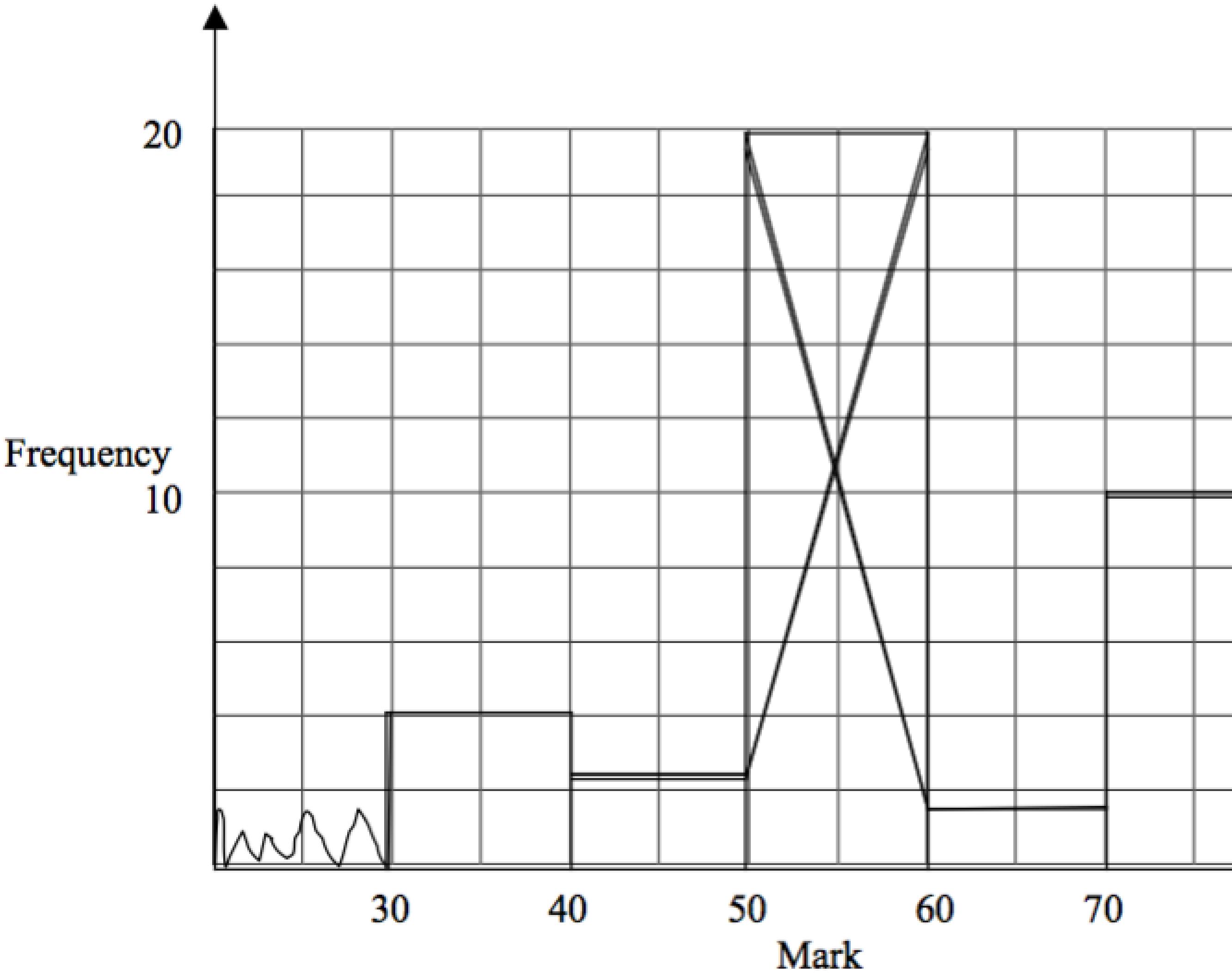
The examination results (rounded to the nearest whole number %) are given for a group of students.

| Mark (%) | 30 – 39 | 40 -49 | 50 – 59 | 60 - 69 | 70 - 79 |
|---|---|---|---|---|---|
| Frequency | 5 | 3 | 20 | 2 | 10 |

Estimate the mode

**Solution**

To estimate the mode, there are two methods.




**By drawing:**Use the histogram of the first part.Then proceed as follow;

- Step 1: Draw a straight line from the top left hand corner of the rectangle of the modal class, to the top left hand corner of the rectangle of the class to the right of the modal class.
- Step 2: Draw a line from the top right hand corner of the rectangle of the modal class,to the top right of the modal class to the left of the modal class.
- Step 3: Find where these two lines intersect. This gives the mode as 54 on the horizontal axis.

**By calculation:**Let

- fM = frequency of the modal group
- fR = frequency of the group to the right of the modal group
- fL = frequency of the group to the left of the modal group
- W = width of the modal group
- L = lower class boundary of the modal group

$$\text{The mode} = L + \frac{(f_M - f_L)W}{(2f_M - f_R - f_L)}$$

In this example $f_M = 20$. $f_L = 3$, $F_R = 2$, $L = 49.5$, $W = 10$

$$\text{Mode} = 49.5 + \frac{(20 - 3) \times 10}{(2 \times 20 - 2 - 3)}$$

$$= 49.5 + \frac{170}{35} = 54.4$$

$\therefore$ The mode is 54.4

# TOPIC 4: RATES AND VARIATIONS

## Rates

A rate is found by dividing one quantity by another.
Rates of Quantities of Different Kinds
Relate rates of quantities of different kinds
For example a rate of pay consists of the money paid divided by the time worked. If a man receives 1,000 shilling for two hours work, his rate of pay 1000 ÷ 2 = 500 shillings per hour. From the above example, we find out that

$$\text{Rate (R)} = \frac{\text{Amount of money (M)}}{\text{Time (T)}}$$

$$\boxed{R = \frac{M}{T}}$$

From $R = \frac{M}{T}$

$M = R \times T$ and $T = \frac{M}{R}$

Example 1

1. A man is paid 6,000/= for 8 hours work.

a. What is his rate of pay?

b. At this rate, how much would he receive for 20 hours work?

c. At this rate, how long must he work to receive 30,000 shillings?

***Solution*:**

(a) $R = \dfrac{M}{T}$

$M = 6{,}000/=$

$T = 8$ hours

$R = \dfrac{6{,}000}{8}$ per hour

$R = 750/-$ Per hour

$\therefore$ His rate of pay is 750/- per hour.

(b) From $R = \dfrac{M}{T}$, $M = R \times T$

$T = 20$ hours

$R = 750/-$ per hour

$M = ?$

$M = \times T = \dfrac{750 \times 20 \text{ hours}}{1\text{hour}}$

$M = 15{,}000/-$

$\therefore$ For 20 hours of work the many would receive 15,000/-

(c) $R = 750$ Per hour

$M = 30{,}000/-$

$T = ?$

From $R = \dfrac{M}{T}$

$T = \dfrac{M}{R}$

$T = \dfrac{30{,}000}{750}$ hours

$T = 40$ hours

$\therefore$To receive 30,000/-, he must work for 40 hours.

Quantities of the Same Kind

Relate quantities of the same kind

Example 2

A student is growing plants she measures the rate at which two of them are growing. Plant A grew 5cm in 10 days, and plant B grew 8cm in 12 days. Which plant is growing more quickly?

(c) $R = 750$ Per hour

$M = 30{,}000/-$

$T = ?$

From $R = \dfrac{M}{T}$

$T = \dfrac{M}{R}$

$T = \dfrac{30{,}000}{750}$ hours

$T = 40$ hours

$\therefore$To receive 30,000/-, he must work for 40 hours.

Exercise 1

1. A woman is paid 12,000/= for 8 hours work.

(c) $R = 750$ Per hour

$M = 30{,}000/-$

$T = ?$

From $R = \frac{M}{T}$

$T = \frac{M}{R}$

$T = \frac{30{,}000}{750}$ hours

$T = 40$ hours

$\therefore$To receive 30,000/-, he must work for 40 hours.

Converting Tanzanian Currency into other Currencies

Convert Tanzanian currency into other currencies

Different countries have different currencies. Normally money is changed from one currency to another using what is called a Rate of Exchange.

This makes trade and travel between countries convenient.

Conversion of money is done by multiplying or dividing by the rate of exchange.

Eg. If at a certain time there are 1,100 shillings to each UK pound (£), to go from £to shillings, multiply by 1,100, and to go from shillings to £divide by 1,100.

**NB:** The rate of exchange between two countries varies from time to time.

Example 3

Suppose the current rate of change between the Tanzanian shillings and the Euro is 650 Tsh per Euro.

a. A tourist changes 200 euros to Tsh. How much does he get?

b. A business woman changes 2,080,000 Tsh to euros. How much does she get?

(c) $R = 750$ Per hour

$M = 30{,}000/-$

$T = ?$

From $R = \frac{M}{T}$

$T = \frac{M}{R}$

$T = \frac{30{,}000}{750}$ hours

$T = 40$ hours

$\therefore$To receive 30,000/-, he must work for 40 hours.

Exercise 2

At a certain time there are 600 Tsh to one US dollar ($).

(c) $R = 750$ Per hour

$M = 30{,}000/-$

$T = ?$

From $R = \frac{M}{T}$

$T = \frac{M}{R}$

$T = \frac{30{,}000}{750}$ hours

$T = 40$ hours

$\therefore$To receive 30,000/-, he must work for 40 hours.

# Variations

The Concept of Direct Variation

Explain the concept of direct variation

Some quantities are connected in such a way that they increase and decrease together at the same rate. Afar example if one quantity is doubled the other quantity is also doubled. These quantities are Directly Proportional or Vary Directly.

Eg. If a car is driven at a constant speed, the distance it goes is directly proportional to the time taken. Also the amount of maize you buy is directly proportional to the amount of money you spend.

If y is directly proportional to x, we write $y \propto x$.

This expression is not an equation. To change it to an equation, introduce a constant (k) called the constant of proportionality.

So $y \propto x$, then $y = kx$ for some constant k.

This means $\boxed{\frac{y}{x} = k.}$

Problems on Direct Variations

Solve problems on direct variations

Example 4

1. Suppose different weights are hung from a wire. The extension of the wire is proportional to the weight hanging.

Suppose a weight of 2kg gives an extension of 5cm.

Find an equation giving the extension e cm in terms of weight w kg. Find the weight for an extension of 3cm.

**Solution:**

From the statement above, $e \propto w$

So $e = kw$, or $k = \frac{e}{w}$

But e = 5cm when w = 2kg

$k = \frac{5}{2} = 2.5$

$k = 2.5$

$\therefore \frac{e}{w} = 2.5$ or $e = 2.5w$

When e = 3cm, w =?

Again $e = w \times 2.5$

$3 = 2.5 \times w$

$W = \frac{3}{2.5} = 1.2$ kg

W = 1.2 kg

∴ A weight of 1.2kg gives an extension of 3cm.

Example 5

Given that y is proportional to x such that, when x = 40, y = 5. Find an equation giving y in terms of x and use it to find (a) y when x = 15 (b) x when y = 20.

***Solution:***

$y \propto x$

$y = kx$

$k = \frac{y}{x}$

When x = 40, y = 5

$k = \frac{5}{40} = \frac{1}{8} x$

$k = \frac{1}{8}$

∴The equation is $y = \frac{1}{8} x$

Exercise 3

The variables m and n are directly proportional to each other such that when m = 3, n= 12.

(a) Find an equation giving m in terms of n.
(b) Find m when n = 18
(c) Find n when m = 15.

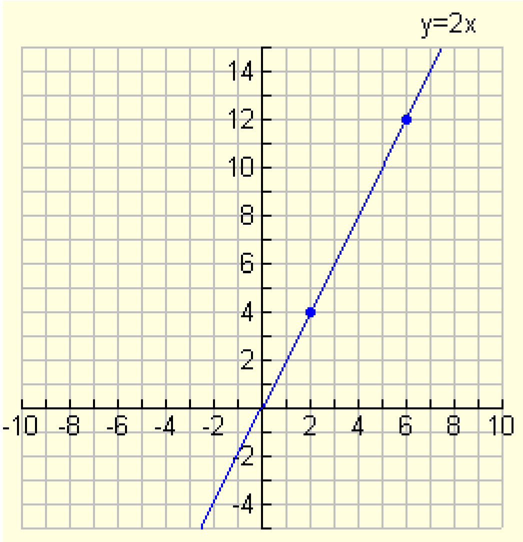
2. The mass m kg of a piece of metal is proportional to its volume $Vm^3$. The mass of $0.2m^3$ is 210 kg.
   (a) Find the equation connecting m and v.
   (b) Find the volume of 14kg of the metal.
   (c) Draw the graph of m against v.

Graphs of Direct Variation

Draw graphs of direct variation

Example 6

The linear equation graph at the right shows that as the*x*value increases, so does the*y*value increase for the coordinates that lie on this line.




This is a graph of**direct variation**

The Concept of Inverse Variation

Explain the concept of inverse variation

In some cases one quantity increase at the same rate as another decrease. For example, if the first quantity is doubled, the second quantity is halved.

In this case the quantities vary inversely, or they are inversely proportional. e.g. The number of men employed to dig a field is inversely proportional to the time it takes. Also the time to travel a journey is inversely proportional to the speed. We use the same symbol (∝) for proportionality.

e.g. If y is inversely proportional to x, we write $y \propto \frac{1}{x}$,

Then $\boxed{y = \frac{k}{x} \text{ or } xy = k}$

Problems on Inverse Variations

Solve problems on inverse variations

Example 7

1. Suppose a mass of a gas is kept at a constant temperature. The volume of the gas is inversely proportional to its pressure.

If the volume is $0.8m^3$ when the pressure when the is $250kg/m^3$, find the formula giving the volume $vm^3$ in terms of the pressure P $kg/m^2$. What is the volume when the pressure is increased to $1{,}000kg/m^2$?

***Solution:***

$v \propto \frac{1}{p}$, so $v = \frac{k}{p}$ or $vp = k$

When $v = 0.8$

$p = 250$

$k = 0.8 \times 250 = 200$

So $vp = 200$

or $v = \frac{200}{p}$

Again $p = 1{,}000$

$v = ?$

$v = \frac{200}{p} = \frac{200}{1000}$

$v = \frac{2}{10} = 0.2$

$\therefore$ The volume is $0.2m^3$.

Example 8

Given that y is inversely proportional to x, such that x = 8 when y = 15. Find the formula connecting x and y by expressing y in terms of x and use it to find (a) y when x = 10, (b) x when y = 3

***Solution;***

$y \propto \frac{1}{x}$,

$xy = k$

When $x = 8$, $y = 15$

So $k = 8 \times 15 = 120$

$xy = 120$

$y = \frac{120}{x}$

(a) y when x = 10

$y = \frac{120}{10}$

$y = 12$

$\therefore y = 12$ when $x = 10$

(b) x when y = 3

From $xy = 120$, $x = \frac{120}{y} = \frac{120}{3}$

$x = 40$.

Therefore $x = 40$ when $y = 3$.

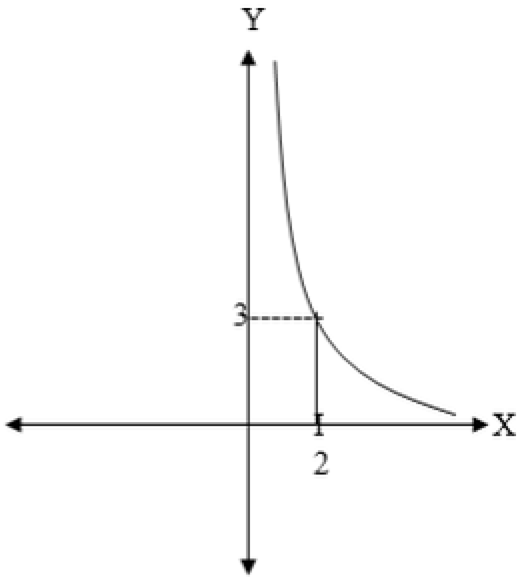
Exercise 4

The quantities p and q are inversely proportional to each other such that when q = 20, p=1.2

a. Find the equation giving p in terms of q
b. Find q when p = 0.5
c. Find p when q = 160

Given that y is inversely proportional to x such that when y = 6, x =7. Find the equation connecting x and y by expressing x in terms of y and hence find x when y = 36

The number of workers needed to repair a road is inversely proportional to the time taken. If 12 workers can finish the repair in 10 days, how long will 30 workers take?
Graphs Relating Inverse Variations
Draw graphs relating inverse variations
The graph of y against x is shown for which y = 3 when x =2




This is an example of a reciprocal graph.

**Proportion to powers:**
Sometimes a quantity is proportional to a power of another quantity. For example the area A of a circle is proportional to the square of its radius r,
So $A \propto r^2$ or $A= kr^2$

The constant of the above equation is $\pi$ whose value is $\frac{22}{7}$ or 3.142

Therefore $A= \pi r^2$.
Sometimes a quantity is inversely proportional to a power of another quantity. E.g. Newton's law of gravity states that the force of attraction (F) between two bodies is inversely proportional to the square of the distance d between them, so $F \propto \frac{1}{d^2}$ ,

Example 9
1. The mass of spheres of a certain metal is proportional to the cube of their radii. A sphere of radius 10cm has mass 42kg. Find the formula giving the mass m kg in terms of radius r cm. Find the radius of the sphere with mass 5.25 kg.

***Solution:***

$m \propto r^3$

So $m = kr^3$

To find k, put the values of m and r in the equation.

$42 = k \times 10^3$

$k = \frac{42}{10^3} = 0.042$

$m = 0.042r^3$

When $m = 5.25$

$r = ?$

We have $m = 0.042\, r^3$

$5.25 = 0.042r^3$

$r^3 = \frac{5.25}{0.042}$

$r = \sqrt[3]{5.25 \div 0.042} = 5$

$\therefore$ The radius of the sphere is 5cm

Example 10

Given that M is proportional to the square of N and when N = 0.3, M = 2.7. Find the equation giving M in terms of N, and hence find the value of:

a. M when N = 1.5

b. N when M = 0.3

***Solution:***

$M \propto N^2$

$M = kN^2$

$k = \frac{M}{N^2}$

When $M = 2.7$, $N = 0.3$

So $k = \frac{2.7}{(0.3)^2} = 30$

Therefore $M = 30N^2$

(a) M=? when N = 1.5

From $M = 30N^2$

$M = 30 \times (1.5)^2$

$= 67.5$

$\therefore M = 67.5$

(b) N=? When M = 0.3

From $M = 30N^2$

$N^2 = \frac{M}{30}$

$N^2 = \frac{0.3}{30} = 0.01$

$N = \sqrt{0.01} = 0.1$

$\therefore N = 0.1$

Joint Variation in Solving Problems

Use joint variation in solving problems

If a quantity varies as the product of two other quantities then it varies jointly with them. eg. If $y = 3vu^2$, then y varies jointly with v and $u^2$.

Also if $p = \frac{100q}{r}$ then p varies jointly with q and the inverse of r.

Example 11

1. Suppose a mass of a gas with volume $Vm^3$ is under pressure P $kg/m^2$ and has absolute temperature $T^0$. The volume of the gas varies jointly with its absolute temperature and inversely with its pressure. At a temperature of 300 k and pressure of $80kg/m^2$, the volume is $0.5m^3$. Find the formula for the volume in terms of T and P.

***Solution:***

V varies as the product of T and $\frac{1}{P}$,

So $V = \frac{kT}{P}$

$\frac{VP}{T} = k$

But V = 0.5, T = 300 and P = 80

So $k = \frac{0.5 \times 80}{300}$

$k = \frac{40}{300} = \frac{2}{15}$

$k = \frac{2}{15}$

$V = \frac{kT}{P} = \frac{2}{15}\frac{T}{P}$

$\therefore V = \frac{2}{15}\frac{T}{P}$

Example 12

m varies jointly with p and q such that when p = 12 and q = 5 then m= 15. Find m in terms of p and q and hence find m when P = 3 and q = 28

***Solution:***

$m \propto pq$

$m = kpq$

$k = \frac{m}{pq}$

But p = 12, q = 5 and m = 15

So $k = \frac{15}{12 \times 5} = \frac{1}{4}$

Then $m = \frac{pq}{4}$

From $m = \frac{pq}{4}$ , when p = 3 and q = 28,

m= ¼ x 3 x 28 = 21

∴ When p = 3 and q = 28, m = 21

Exercise 5

1. M is inversely proportional to the cube of N, when N =2 then M = 20.
   a. Find an equation giving M in terms of N.
   b. Find M when N = 4
   c. Find N when M = 5.
2. P is inversely proportional to the square root of Q. When Q = 16 then P =5.
   a. Find an equation connecting P and Q expressing P in terms of Q.
   b. Find P when Q = 9
3. When a body is moving rapidly through the air, the air resistance R newtons is proportional to the square of the velocity Vm/s, At a velocity of 50m/s, the air resistance is 20N.
   a. Find R in terms of V
   b. Find the resistance at 100m/s.
4. B varies jointly with A and the inverse of C. When A = 3 and C = 12 then B = 20.
   a. Find B in terms of A and C.
   b. Find B when A = 8 and C= 2
5. The mass m kg of a solid wooden cylinder varies with the height h (m) and with the square of the radius r (m). If v = 0.2 and h = 1.4, then M = 150. Find m in terms of h and r.

Joint variation leading to areas and volumes

Many formulas for areas and volumes involve joint variation. For example the volume of a cylinder is given by $v = \pi r^2 h$.

So the volume varies jointly with the height and the square of the radius. i.e $v \propto r^2 h$.

Example 13

1. A cylinder has radius 3cm and volume 10cm3. If the radius of the base is increased to 4cm without altering the height of the cylinder what effect does this have on the volume?

***Solution:***

The volume of the cylinder is v = base area x height

$v = Ah = \pi r^2 h$.

Where $\pi$ is a constant and the height is constant, so v is proportional to the square of r.

$v \propto \pi r^2$

If r changes from 3 to 4, then it is multiplied by 4/3, and hence v is multiplied by $(4/3)^2 = \frac{16}{9}$

$\therefore$The new volume is $(16/9) \times 10 = \frac{160}{9}$ cm$^3$

Alternatively, as $v \propto r^2$,

$\frac{v}{r^2}$ is constant, if the new volume is v, then $\frac{10}{3^3} = \frac{v}{4^2}$

Hence $v = \frac{4^2 \times 10}{3^2} = \frac{160}{9}$ cm$^3$

$\therefore v = \frac{160}{9}$ cm$^3$

Example 14

A pyramid has a square base. If the height decreases by 10% but the volume remains constant, what must the side of the base increase by? (i.e What increase in the side will of set the decrease in the height?).

***Solution:***

$v = \frac{1}{3} r^2 h$, where r is the side of the base and h is the height.

The volume varies jointly with h and the square of r.

$V \propto hr^2$, so $\frac{v}{hr^2}$ is constant

Let the height change from $h_1$ to $h_2$ and r from $r_1$, to $r_2$ while v remains constant.

So $\frac{v}{\mathrm{h1(r1)^2}} = \frac{v}{\mathrm{h2(r2)^2}}$

If the height (h) decreases by 10% then the new height is $h - \frac{1}{10}h = (1 - \frac{1}{10})h = \frac{9}{10}h$

So if h changes from $h_1$ to $h_2$, then $h_2 = \frac{9}{10}h_1$

$$\frac{v}{\mathrm{h1(r1)^2}} = \frac{v}{\frac{9}{10}\mathrm{h1(r2)^2}}$$

or $\frac{1}{\mathrm{(r1)^2}} = \frac{10}{9\mathrm{(r2)^2}}$ which means $\frac{r_2}{r_1} = \sqrt{\frac{10}{9}} = 1.054$

So $r_2 = 1.054 r_1$ which corresponding to an increase of 5.4%.

$\therefore$ The side of the base must increase by 5.4%.

Exercise 6

1. A box has a square base of side 5cm. The volume of the box is 56cm$^3$. If the sides increase by 10%, without the height changing, what is the new volume of the box?

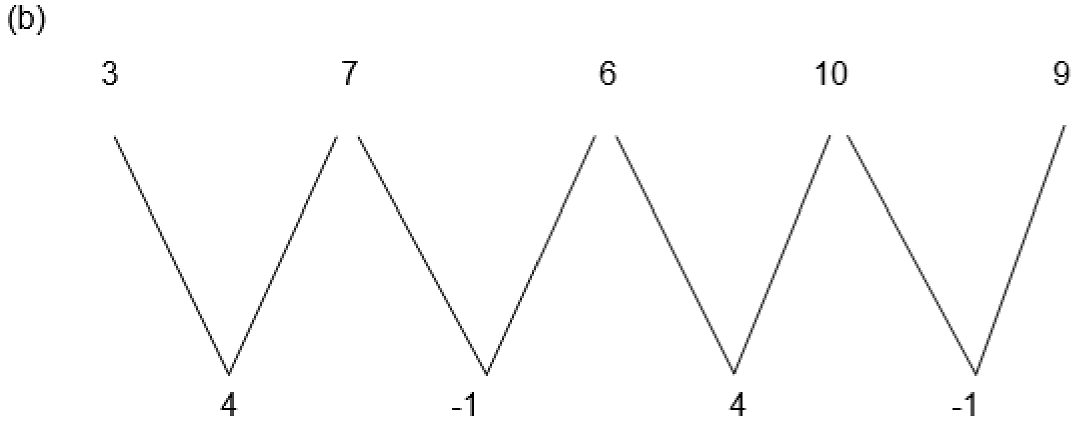
2. A cone has volume 30cm$^3$. If the radius increases by 10% and the height by 5%, what is the new volume of the cone?
3. A water tank holds 1,000 liters, and is in the shape of cuboids. The lengths of the sides of the base are enlarged by a scale factor of 1.4 without altering the height. What volume will the tank now hold?
4. The height of a cylinder is reduced by 20%. What percentage change is needed in the radius, if the volume remains constant?

# TOPIC 5: SEQUENCE AND SERIES

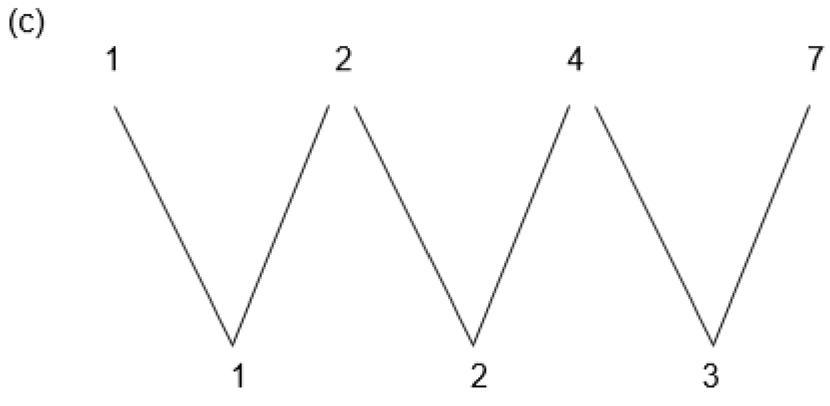
Sequences
The Concept of Sequence
Explain the concept of sequence
A Sequence is the arrangement of numbers or is a list of numbers following a clear pattern such that one number and the next are separated by comma (,).
Example: $a_1$, $a_2$, $a_3$, $a_4$ ……………………..
NB: Each number found in a Series or Sequence is called a ***term***.
Example 1
Find the next three terms in the following sequences.

a. 5, 8, 11, 14, 17,………………………………
b. 3, 7, 6, 10, 9, …………………………………
c. 1, 2, 4, 7, ………………………………………
d. 2, 9, 20, 35, …………………………………

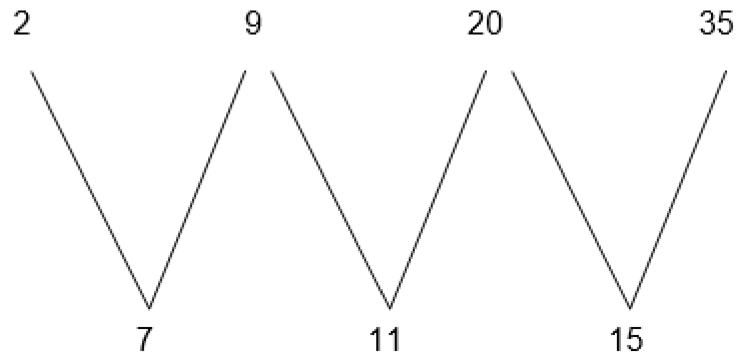
***Solution:***
(a)You can see that each term is less to the next by 3.
So next three terms are (17+3),(17+3+3) and 17+3+3x3)
Which are 20, 23, and 26




Alternately add 4 and subs tract 1. The sequence then extends to 13, 12, 16




We see that the difference is increasing by 1 each time. So the next three terms are 11, 16 and 22.




The differences are increased by 4 each time, so the next three terms are 54, 77 and 104.

Example 2

Write down the first three terms in the sequences where the n$^{th}$ term is given by the formulae.

(a) 4n-3 (b) $\frac{n^2+1}{2}$ (c) $\frac{2}{n+1}$

**Solution**:

(a) 4n-3.

n=1, 4n-3 = 4×1-3=1

n=2, 4n-3 = 4×2-3=5

n=3, 4n-3 = 4× 3-3=9

∴The Sequence is 1, 5, 9, …………………

(b) $\frac{n^2+1}{2}$

n=1, $\frac{n^2+1}{2} = \frac{1^2+1}{2} = 1$

n=2, $\frac{n^2+1}{2} = \frac{2^2+1}{2} = \mathbf{\frac{5}{2}}$

n=3, $\frac{n^2+1}{2} = \frac{3^2+1}{2} = 5$

∴ The sequence is 1, $\frac{5}{2}$, 5 ,……………

(c) $\frac{2}{n+1}$

n=1, $\frac{2}{n+1} = \frac{2}{1+1} = 1$

n=2, $\frac{2}{n+1} = \frac{2}{2+1} = \frac{2}{3}$

n=3, $\frac{2}{n+1} = \frac{2}{3+1} = \frac{1}{2}$

$\therefore$ The sequence is 1, $\frac{2}{3}$, $\frac{1}{2}$, .......

Example 3
The $k^{th}$ term of a series is $k^2 + 4$
Find the sum of the first four terms in the series
***Solution:***
k=1, $k^2+4=1^2+4=5$
k=2, $k^2+4=2^2+4=8$
k=3, $k^2+4=3^2+4=13$
k=4, $k^2+4=4^2+4=20$
So the series is 5+8+13+20 and its sum is **46**
Example 4
Find the $n^{th}$ term of the following sequences:

(a) 2, 3, 4, 5, 6, ......................................

(b) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, ..............

***Solution:***

(a) First term = 1+1=2

Second term = 2+1=3

Third term = 3+1=4

Fourth term = 4+1=5 and so on

So the $n^{th}$ is n+1.

(b) The numerators of fractions are 1, 2, 3, 4 ……………………… and the fractions are 2, 3, 4, 5 ………………..

So the $n^{th}$ term for numerator is n and that of denominator is n+1,

Combining them gives the $n^{th}$ term of the fractions.

So the $n^{th}$ term is $\frac{\text{numerator nth term}}{\text{Denominator nth term}}$ which is $\frac{n}{n+1}$

$\therefore$ The $n^{th}$ term is $\frac{n}{n+1}$

Exercise 1

1. Write down the next three terms in the following sequences

(a) 1,5,9,13,17, ……………………

(b) 27, 24, 21,18, …………………..

(c) 5, 8, 9, 12, 13, ……………………

(d) 1, 8, 27, 64, …………………….

(e) 1, $\frac{-1}{2}$, $\frac{1}{4}$, $\frac{-1}{8}$, ……………………..

2. Find the first three terms in the sequence:

a. 5n+2
b. 1-3k
c. $n^2+n+1$
d. $2^n$

3. Find the sum of the first four terms of the series where the $k^{th}$ term is given by:

a. 5k+3
b. $k^3-1$
c. $2^k$

4. Find the $n^{th}$ term of these sequences:

(a) 2, 4, 5, 6, ……………………………………

(b) 4, 5, 6, 7, ……………………………………

(c) 10, 20, 30, 40,………………………………

(d) 2, 4, 6, 8,……………………………………..

(e) $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, ……………………………...

(f) $2\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{1}{4}$, $5\frac{1}{5}$, ……………………………...

An Arithmetic Progression (AP) and Geometric Progression (GP)
Identify an arithmetic progression (AP) and geometric progression (GP)
When the series or sequence is such that between two consecutive terms there is a difference which is fixed, then the series or sequence is called an arithmetic progression (A.P)
The fixed difference (number) between two consecutive terms is called the common difference (d)
Example 5
In the sequence 4, 7, 19, 13, 16 there is a common difference which is
7-4=10-7=13-10=16-13=3.
So the common difference (d)=3.
Note that in arithmetic progression (A.P) the difference between two successive terms is always the same.
Sometimes numbers may be decreasing instead of increasing, the arithmetic sequence or series while terms decrease have a negative number as a common difference.
Example 6
The common difference of the sequence 6, 4, 0, -2, …………………… is
4-6=2-4=0-2=-2-0=-2
So the common difference is **-2.**
In general if $A_1$, $A_2$, $A_3$, $A_4$, ……………………… $A_n$ are the terms of the arithmetic sequence , then the common difference is ;

$$\boxed{d=A_2-A_1=A_3-A_2=A_4-A_3=\ldots\ldots\ldots\ldots}$$

Example 7
For each of the following sequences, find the common difference and write the next two terms.

(a) -2, 5, 12, 19, 26, ………………………………………

(b) $4\frac{1}{2}$, $3\frac{1}{4}$, $2\frac{3}{4}$, …..

Solution:

(a) d =5-(-2)=12-5=19-12 d=7

$\therefore$ The next term to 26 is 26+7=33 and the next to 33 is 33+7=40.

So the next two terms are 33 and 40.

$d = 3\frac{1}{4} - 4\frac{1}{2} = 2 - 3\frac{3}{4},$

or $d = \frac{3}{4} - 2$

So $d = -\frac{5}{4},$

$\therefore$ The common difference is $-\frac{5}{4},$

The next two terms are: $\left(\frac{3}{4} + \left(\frac{-5}{4}\right)\right)$ and $\left(\frac{3}{4} + \left(\frac{-5}{4}\right)\right) + \left(\frac{-5}{4}\right)$

Which are $\frac{-1}{2}$ and $\frac{-7}{4}$

Exercise 2

1. Find the common difference for each of the following sequence:
   - a. 11, 14, 17, 20, …………………………………
   - b. 2, 4, 6, 8, 10, ……………………………………
   - c. 0.1, 0.11, 0.111, 0.1111 , …… … … … … …
   - d. y, y+3, y+6, y+9, y+12, … …… … … … ……

2. State whetherthe following sequence are arithmetic or not:
   - a. 2, 5, 8, 11, 14, ………….. …………….. ……
   - b. 1, 3, 4, 6, 7, 9, 10, ………………………………
   - c. y, y + x, y+2x, y+3x, … ………… ……

3. The temperatureat a mid day is $3^0$c, and it falls by $2^0$c each hour. Find the temperature at the end of the next four hours.

Geometric Progression (G.P).

When the series or Sequence is such that between two consecutive terms there is a ration which is fixed, then the series or sequence is called a geometric progression (G.P)

The fixed ratio(number) between two successive terms is called the common ratio (r).

Example 8

In 2, 4, 8, 16, 32, … … …… … … …….

There is a common ration which is

$\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = \frac{32}{16} =$ …………………………………

So the common ratio in this case is (r) =2

Note that like in arithmetic progression (A.P), in geometric progression (G.P) the common ratio does not change.

Also the terms may be decreasing instead of increasing, the geometric sequence or series whose terms decrease have a positive common ratio which is less than 1 for the progression with positive terms.

Eg. In the sequence 8, 4, 2, 1…. The common ratio is r = $\frac{4}{8}$ or $\frac{2}{4}$ or $\frac{1}{2}$

So the common ratio is $\frac{1}{2}$.

Generally if $G_1, G_2, G_3, G_4, \ldots\ldots G_n$ are the terms geometric sequence then the common ratio is

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = = \frac{G_4}{G3} = \frac{G_n}{Gn-1}$$

Example 9

For each of the following sequence find the common ratio.

(a) 3, 6, 12, 24, ………………………………

(b) $1\ \frac{1}{2},= \frac{1}{4},= \frac{1}{8}$ ……………………………………

***Solution;***

(a) 3, 6,12, 24, ………………………………….

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = \frac{G_4}{G3}$$

$$r = \frac{6}{3} = \frac{12}{6} = \frac{24}{12} = 2$$

∴ The common ratio (r) = 2

(b) 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8},$ ………………………………….

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = \frac{G_4}{G3}$$

$$r = \left(\frac{1}{2} \div 1\right) \text{or} \left(\frac{1}{4} \div \frac{1}{2}\right) \text{ or } \left(\frac{1}{8} \div \frac{1}{4}\right)$$

$$r = \frac{1}{2}$$

∴ The common ratio (r) = $\frac{1}{2}$

Example 10

For the following geometric sequences, find the common ratio and write down the next two terms:

(a) 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$..............................................

(b) 10, -5, 2.5, -1.25 ..........................................

***Solution:***

(a) Since 2,3, $6\frac{3}{4}$ ..... is a geometric sequence, it has a common ratio (r) which is found as follows

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = \frac{G_4}{G3}$$

$$r=\frac{3}{2}, = \frac{4\frac{1}{2}}{3} = \frac{6\frac{1}{2}}{4\frac{1}{2}}$$

$$r= \frac{3}{2}$$

$\therefore$ The common ratio (r) = $\frac{3}{2}$, or 1.5

The next term is found by multiplying the term considered to be the last term by the common ratio.

So the next term to $6\frac{3}{4}$ is $6\frac{3}{4} \times 1.5 = \frac{81}{8}$

and the next term to $\frac{81}{8}$ is $\frac{81}{8} \times 1.5 = \frac{243}{16}$,

$\therefore$ The next two terms of the sequence are $\frac{81}{8}$ $and$ $\frac{243}{16}$,

(b) 10,-5, 2.5, -1.25, .......................................

$$r =\frac{-5}{10} = \frac{2.5}{-5} = \frac{1.25}{2.5}$$

$\therefore$ The common ratio (r) $=-\frac{1}{2}$

The next two terms are:

$\left(-1.25 \times (\frac{-1}{2})\right)$ and $\left(-1.25 \times (\frac{-1}{2})\right) \times \left(\frac{-1}{2})\right)$

Which are $\frac{5}{8}$, and $\frac{-5}{16}$

$\therefore$ The next two terms are $\frac{5}{8}$, and $\frac{-5}{16}$

Exercise 3

1. Which of the following sequences are geometric

a. 1, 2, 4, 8, 16, ……………………………………
b. 2, 6, 18, 54, 162, …………………………………
c. 1, -1,1,-1,1, ………………………………………
d. $x^2$, $2x^3$, 4x4, $8x^3$…………………………………
e. 1, 2, 4, 7, 10, ………………………………………
f. 0.1, 0.2, 0.3, 0.4, 0.5, ……………………………
g. 3, 6, 9, 12,15, ……………………………………….

2. Find thecommon difference for each of the following geometric progressions (G.P)

(a) 1, -0.5, 0.25, -0.125, …………………………………

(b) 4x, $4x^2$, $4x^3$, $4x^4$, …………………………………

(c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},$…………………………………………

(d) 1.1, 2.2, 4.4, 8.8, … … ….

3. Find thenext term of the sequence 2, 10, 50, 500,………………….

4. The populationof a town is decreasing so that every year the population declines by a quarter. If the population is originally 100,000. What will it be after 5 years?

The General Term of an AP

Find the general term of an AP

If $A_1$, $A_2$, $A_3$, …………………$A_n$ are the terms of an arithmetic sequence, then there is a common difference d which is given by

$d = A_2 – A_1 = A_3 – A_2 = A_n – A_n – 1$

that is $d = A_2 – A_1$ or

$d = A_3 – A_2$ or

$d = A_4 – A_3$ or

|

|

|

$d = A_n – A_{n-1}$

So $d = A_2 – A_1$ , $d= A_3 – A_2$, ……………………………………………

Means $A_2 = A_1 + d$

$A_3 = A_2 + d$

But $A_2 = A_1 + d$

So, $A_3 = [A_1 + d] + d = A_1 + 2d$

But . $A_3 = A_1 + 2d$ which means

$A_4 = [A_1 + 2d] + d$

$= A_1 + 3d$
Putting into consideration this pattern, it is true that
$A_5 = A_1 + 4d$
$A_6 = A_1 + 5d$
$A_n = A_1 + (n-1)d$
Where $A_n$ is the $n^{th}$ term
The $n^{th}$ term of the sequence with first term $A_1$ and common difference d is given by

$$\boxed{A_n=A_1+(n-1)d}$$

Example 11
Find theformula for the $n^{th}$ term of the sequence 8 , 9.5, 11, 12.5, 14, 15.5,……

***Solution***

First term $(A_1) = 8$

Common difference (d) = $A_2-A_1=A_3-A_2$

Or $A_6-A_5$

$d=9.5-8$

or $d=15.5-14$

$d=1.5$

But $A_n=A_1+ (n-1) d$

$A_n= 8+ (n-1) x1.5$

$A_n=8+1.5n-1.5$

$A_n=6.5+1.5n$

$\therefore A_n=1.5n+6.5$

Note that the $n^{th}$ term gives every term in the sequence,
For example when n=3, you have $A_3$=1.5x3+6.5=11
So $A_3$=11 where 11 is given in the sequence above having the third position.
Therefore $A_n$ shows the position of the term in sequence and of $A_1+(n-1)d$ gives the value of the term for any positive integer.
Example 12
The $5^{th}$ term of an arithmetic sequence is 11, and the $8^{th}$ term is 26. Find the first five terms.

***Solution:***

Given that

$A_5=11$ and $A_8 = 26$

From

$A_n=A_1+ (n-1) d$

$11=A_1+ (5-1) d$

$11=A_1+4d$ and

$26=A_1+ (8-1) d$

$26=A_1+7d$

So $A_1+4d=11$ …………….. (1)

$A_1+7d=26$………………….(2)

Solving for $A_1$ and d simultaneously gives

$d=5$ and $A_1=-9$

But $A_n=A1+ (n-1) d$

$A_n= -9+ (n-1) x5$

$A_n=-9+5n-5$

$A_n=5n-14$

$A_n=5n-14$

So $n=1$

$A_1 = 5x1-14=-9$

$n = 2$

$A_2 = 5x2-14=-4$

$n = 3$

$A_3 = 5x3-14=1$

$n = 4$

$A_4 = 5x4-14=16$

$n = 5$

$A_5 = 5x5-14=11$

$\therefore$ The first five terms are -9,-4, 1, 6, and 11.

Example 13

The 8th term of an arithmetic sequence is 9 greater than the 5th term, and the 10th term is 10 times the 2nd term. Find

a. The common difference (d)

b. 20th term.

**Solution:**

Let $A_1$, $A_2$, $A_3$,……………………… $A_n$

Be the terms of the given sequence $A_8$ > A5 by 9 means

$A_8 - A_5= 9$ and

$A_{10}$ is 10 times the second term means

$A_{10}=10A_2$

But from $A_n=A_1+ (n-1) d$,

$A_8=A_1+7d$

$A_5=A_1+4d$

$A_{10}=A_1+9d$

and $A_2=A_1+d$

So $A_1+7d-(A_1+4d) =9$

$3d=9$

$d=3$

Also, $A_{10}=10A_2$ means $A_1+9d=10(A_1+d)$

$A_1+9d=10A_1+10d$

$9A_1= -d$

$A_1= \frac{-d}{9}$

But d=3

$A_1= \frac{-3}{9}$

$A_1= \frac{-1}{3}$

Therefore the common difference (d) is 3,

To get the 20$^{th}$ term, use the n$^{th}$ term

I.e. $A_n=A_1+ (n-1) d$

$A_{20}=A_1+ (20-1) d$

$A_{20}= -\frac{1}{3} + 19 \times 3 = \frac{170}{3}$

$\therefore$ The 20$^{th}$ term is $\frac{170}{3}$

The General Term of GP

Find the general term of GP

If $G_1$, $G_2$, $G_3$,……………..$G_n$ are the terms of a geometric sequence, then they have a common ratio (r) which is given by

$$r= \frac{G_2}{G1} = \frac{G_3}{G2} = = \frac{G_4}{G3} = \frac{G_n}{Gn-1}$$

This means $G_2=G_1r$, $G_3=G_2r$, $G_4=G_3r$ and $G_n=G_{n-1}r$

So $G_3= (G_1r)\ r$

Where $G_1r =G_2$

$G_3=G_1r^2$

$G_4= (G_1r^2)\ r$

Where $G_1r^2=G_3$

$G_4=G_1r^3$

Following this pattern, you find that $G_5=G_1r^4$, $G_6=G_1r^5$, $G_7=G_1r^6$ etc

We have seen that

$G_1=G_1=G_1r^0$

$G_2=G_1r^1=G_1r$

$G_3=G_1r^2$

$G_4=G_1r^3$

$G_5=G_1r^4$

$G_6=G_1r^5$

|

|

|

|

$G_n=G_1r^{n-1}$

Where $G_n$ is the $n^{th}$ term.

$\therefore$ $\boxed{G_n=G_1r^{n-1}}$

Example 14

Find the formula for the $n^{th}$ term of each of the following geometric sequence.

a. 2, 6, 18, 54 , ………………………………

b. 4,-2, 1, -0.5, 0.25 …………………………

***Solution:***

(a) 2, 6, 18, 54 ,………………………………….

(b) From the sequence 2,6,18,54

(c) $G_1=2, G_2=6, G_3=18$ and $G_4=54$

The common ratio (r) $=\frac{G_2}{G1} = \frac{G_3}{G2}$

So r = $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$

$r=3$

But $G_n = G_1 r^{n-1}$

$G_n = 2(3)^{n-1}$

$G_n = 2x\ (3^n)\ x\ (3)^{-1}$

$G_n = 2\times (3^n) \times \frac{1}{3}$

$G_n = \frac{2}{3} \times 3^n$

$\therefore$ The $n^{th}$ term is $G_n = \frac{2}{3} \times 3^n$ or $G_n = 2\times (3^{n-1})$

(c) 4, -2, 1, -0.5, 0.25,……………….

$G_1=4$

$G_2=-2$

$\frac{G_2}{G1} = \frac{G_3}{G2} = \frac{-2}{4} = \frac{1}{-2} = r$

$r=\frac{1}{-2}$

but $G_n = G_1 r^{n-1}$

$G_n = 4\times \left(\frac{-1}{2}\right)^{n-1}$

$\therefore$ The nth term is $G_n = 4\times \left(\frac{-1}{2}\right)^{n-1}$

Example 15

Considering that,

A geometric sequence is given by 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, …………………….

Find the $13^{th}$ term of this sequence.

***Solution;***

Given the sequence3, 1, $\frac{1}{3}$, $\frac{1}{9}$, …………………….

$G_1=3$

$G_2=1$

$G_2=\frac{1}{3}$

$G_4=\frac{1}{9}$, $G_{13}=?$

$r=\frac{G_2}{G1}=\frac{G_3}{G2}=\frac{1}{3}=\frac{\frac{1}{3}}{1}$

$r=\frac{1}{3}$

from $G_n=G_1r^{n-1}$

$G_{13}$ is found when n=13

So $G_{13}=3\left(\frac{1}{3}\right)^{13-1}$

$G_{13}=3\left(\frac{1}{3}\right)^{12}$

$= 3\times\frac{1}{3}\times\left(\frac{1}{3}\right)^{11}$

$=\left(\frac{1}{3}\right)^{11}=3^{-11}$

$\therefore$ The $13^{th}$ term is $\left(\frac{1}{3}\right)^{11}$ or $3^{-11}$

Exercise 4

1. In the arithmetic sequence, the $17^{th}$ term is 30 and $9^{th}$ term is 42 find the first three terms.
2. In the Arithmeticsequence the third term 12 and the $9^{th}$ term 24. Find the $n^{th}$ term of the sequence and use it to find the $15^{th}$ term.
3. Find the$15^{th}$ term of the sequence 5, 10, 20, 40 ,…………………………
4. A population isincreasing and every year it is multiplied by 1.03. If it starts off at 10,000,000, what will it be after n years?
5. The first termof the geometric sequence is 7 and the common ratio is 4. What is the $9^{th}$ term of this sequence?

# Series

The Formula for a Sum of an Arithmetic Progression

Derive the formula for a sum of an arithmetic progression

When the terms are separated by addition (+) sign, there we have what we call a series.

Example: 2+4+6+8+……………………………

Is a series with the first term ($A_1$) 2 and common difference (d) 2

It is possible to establish a formula for the sum of the first n terms of the arithmetic progression.

Let $S_n$ denote the sum of the first n terms of the arithmetic series.

Consider the sum of the first 5, terms of arithmetic progression (AP) whose first term is 1 and whose common difference (d) is 1.

So $S_5 = A_1+A_2+A_3+A_4+A_5$

$S_5=1+2+3+4+5$ ………………………….. (1)

The first case is the sum of five terms which are increasing from 1 up to 5 while the second case shows the same sum but the terms are decreasing from 5 to 1.

If you add (1) and (2) together, you find that

$S_5+S_5=(1+5) + (2+4) +(3+3) + (4+2) + (5+1)$

$2S_5 = 6+6+6+6+6$

$2S_5=30$

Dividing by 2 each side gives

$S_5=\frac{30}{2} =15$

$\therefore 1+2+3+4+5=15$

Now for any number of terms (n)

$S_n=A_1+A_2+A_3+\ldots\ldots\ldots\ldots+A_{n-1}+A_n$

Which is the same as

$S_n=A_n + A_{n-1}+A_{n-2}+\ldots\ldots A_2+A_1$

So $2S_n= (A_1+A_n) + (A_2+A_{n-1}) + (A_3+A_{n-2}) + (A_{n-1}+A_2) + (A_n+A_1)$

Also we can write

$S_n=A_1+ (A_1+d) + (A_1+2d) + \ldots\ldots\ldots\ldots\ldots\ldots+A_n$

And $S_n = A_n+ (A_n-d) + (A_n-2d)+\ldots\ldots\ldots\ldots\ldots\ldots+A_1$

Which means

$2S_n = (A_1+A_n) + (A_1+d+A_n-d) + (A_1+2d+A_n-2d) +\ldots\ldots+ (A_n+A_1)$

$2 S_n = (A_1+A_n) + (A_1+A_n) + (A_1+A_n) +\ldots\ldots\ldots\ldots\ldots$ n times

$2S_n=n\times (A_1+A_n)$

$S_n= \frac{n}{2} (A_1+A_n)$

but $A_n = A_1+ (n-1) d$

$S_n = \frac{n}{2} (A1 + A1 + (n-1)d)$

$S_n= \frac{n}{2} \left(2A_1 + (n-1) d\right)$

$\therefore$The sum of n terms in arithmetic progression is

$$\boxed{S_n= \frac{n}{2} [ 2A_1+ (n-1) d]}$$

Example 16

Findthe sum of the first 20 terms of the series

$1+ \ 1\frac{1}{2} + 2\frac{1}{2} + \ \dots\dots\dots\dots$

***Solution:***

$S_n = \frac{n}{2}(2A_1 + (n-1)d\ )$

From the series above

$A_1$=1, d= $\frac{1}{2}$ and n=20

Therefore $S_{20} = \frac{20}{2}\left(2 \times 1 + (20-1)\frac{1}{2}\right)$

$S_{20} = 10\left(2 + \frac{19}{2}\right)$

$= 10 \times \frac{23}{2} = 115$

$\therefore S_{20} = 115$

Example 17

Find the sum of the series 4+7+10+13+…………….+304

Solution:

To use the formula for summation of n terms, you must know how many terms are there, i.e finding the value of n;

Now

$A_1$=4, d=3 and $A_n$ = 304 n=?

$A_n = A_1 + (n-1)d$

304 = 4+ (n-1)x3

304 = 4+3n-3

304=3n+1

304=3n

$n = \frac{303}{3} = 101$

Therefore we are required to find the sum of the 101 terms of the given series.

But $S_n = \frac{n}{2}((2A1 + (n-1)d)$

$S_{101} = \frac{101}{2}(2 \times 4 + (101-1)x3)$

$S_{101} = \frac{101}{2}(8 + 300)$

$= \frac{101}{2}(308)$

$S_{101} = 101 \times 154 = 15{,}554$

$\therefore$The sum of 4+7+10+13+ …………………………+304 is **15,554.**

Example 18

How many terms of the series 1+3+5+7+………………. are needed to make the sum of 169?

**Solution:**

$S_n=169$

$A_1=1$

$d=2$

$n=?$

From $S_n=\frac{n}{2}(2A_1+(n-1)d)$

$169=\frac{n}{2}(2\times 1+(n-1)\times 2)$

$169=\frac{n}{2}\times(2n)$

$169=n^2$

$n=\sqrt{169}=13$

∴13 terms are required.

Exercise 5

1. Find thesum of the first 20 terms of the series
   a. 2+5+8+11+……………………
   b. 19+16+13+10+7+……………
2. Find thenumber of terms and the sum of the series:
   a. 1+3+5+7+ ……………………………………
   b. 40+37+34+31+…………………+-257
3. The sumof the first 10 terms of an arithmetic progression (A.P) is 40, and the sum of the next 10 terms is 80. Find the sum of the first five terms of the series.
4. One dayFrola spends 40 minutes of her home work. The length of time she spends increase by 4 minutes each day. Find the total length of time she spends after eight days.

The Arithmetic Mean

Calculate the arithmetic mean

Remember that the arithmetic mean (M) of n numbers is found by adding them and then dividing the sum by n, e.g the arithmetic mean of a,b,c and d is

$$M=\frac{a+b+c+d}{4}$$

The Formula for the Sum of a Geometric Progression

Derive the formula for the sum of a geometric progression

Geometric series are the series that can be written as

$G_1+G_2+G_3+$ ………………………..$G_n$

Example: 2+4+8+16+ ………………………..$+G_n$

Or 1+3+9+27+81+……………………

Suppose we want to find the sum of 1+3+9+9+27+81+…………………

$S_5$=!+3+9+27+81……………………….(1)

If we multiply $s_n$ by the common ratio(r), we have.

$rS_5 = r(1+3+9+27+81)$

but r=3

So $3S_5$= 3+9+27+81+243………………..(2)

Subtracting (1) from (2) gives

$3S_5 - S_5 = (3+9+27+81+243)-(1+3+9+27+81) = 243-1$

$2\ S_5 = 242$

$S_5 = \frac{242}{2} = 121$

In general, for a series with terms $G_1+G_2+G_3+$ …………………………$G_n$, and common ratio($r \neq 1$),

$S_n = G_1+G_1r+G_1r^2+$…………………..$+G_1r^{n-1}$

$rS_n = G_1r+ G_1r^2+ G_1r^3+$……………….$+G_1r^n$

Now $rS_n - S_n = G_1r^n - G_1$

$rS_n - S_n = G_1\ (r^n-1)$

$S_n(r-1) = G_1\ (r^n-1)$

$S_n = \frac{G_1(r^n-1)}{r-1}$

$\therefore \boxed{S_n = \frac{G_1(r^n-1)}{r-1}}$

**NB; If $-1 < r < 1$, it is easier to use the formula in the form of**

$$S_n = \frac{G_1(1-r^n)}{1-r}$$

Example 19

1. Find the sum of the geometric series 2+4+8+ ………………..+2048

*Solution:*

$G_1=2$, $r=2$

$G_n=2048$, n?

$S_n=?$

From $G_n=G_1r^{n-1}$

$2048 = 2\times(2^{n-1})$

$2048= 2\times2^n\times \frac{1}{2}$

$2^{11}=2^n$

$n=11$

Also $S_n = G_1 \frac{(r^n-1)}{(r-1)}$

$$S_{11} = 2 \times \frac{(2^{11}-1)}{(2-1)}$$

$=2(2^{11}-1)$

$=2(2048-1)$

$=4094$

∴The sum is **4094**

Example 20

Find the sum of the first 8 terms of the series 5+20+80+320+ ……………

**Solution**

$G_1=5$, r $=\frac{G2}{G1}=\frac{20}{5}=4$

r=4

$S_8$=?

From $S_n = \frac{G1(r^n-1)}{r-1}$

$S_8=5\times\frac{(4^8-1)}{4-1}$

$S_8=\frac{5\times(65{,}536-1)}{3}$

$=\frac{5\times65{,}536}{3}$

$S_8$ =109,225

∴The sum is **109,225**

Exercise 6

1. For each ofthe following series, find the number of terms and hence the sum of the series.
   a. 1+3+9+…………………+729
   b. 1-2+4-8+………………+1,024
2. Find the sum of the first 10 terms of the series

   (a) 4+2+1+½ +………………………………..

   (b) $4-2+1-\frac{1}{2}+$……………………………………
3. Masanja sets off on a long Journey. The first day he walks 30km, but the distance he walks each day is 10% less than on the previous day. Find the total distance he has walked after 12 days.

The Geometric Mean

Calculate the geometric mean

The Geometric mean (GM) of n positive numbers is found by taking the $n^{th}$ root of their product.

Example 21

The Geometric mean of a, b, c and d is

GM= $\sqrt[4]{abcd}$

Therefore the arithmetic mean of 3 and 5 is

M=$\frac{3+5}{2}$ =$\frac{8}{2}$=4

∴M=4

But the geometric mean of 3 and 5 is

G.M $\sqrt{3x5}$

G.M= $\sqrt{15}$ = 3.87

∴The geometric mean of 3 and 5 is G.M=3.87

The arithmetic mean and geometric mean can be used to check that a sequence is an arithmetic or geometric respectively.

FACTS:

1. If a, b and c are consecutive three term of arithmetic progression (A.P), then b is the arithmetic mean of a and c
2. If a, b and c are three consecutive terms of geometric progression (G.P), then b is the geometric mean (G.M) of a and c.

**Proof:**

(1) Suppose a, b and c are the three consecutive terms of an arithmetic progression then

d=b-a or d=c-b

So. b-a=c-b

b+b=a+c

2b=a+c

b=$\frac{a+c}{2}$ as required.

(ii) Let a, b and c be the consecutive terms of geometric progression (G.P)

Then $r=\frac{b}{a}$ or $r = \frac{c}{b}$

So $\frac{b}{a} = \frac{c}{b}$

bxb=axc

$b^2$=axc

$b=\sqrt{axc}$ as required.

Example 22

Find the arithmetic and geometric means of

(a) 3 and 12 (b) 5,8,14

Solution

(a) Arithmetic mean

$$M= \frac{3+12}{2} = \frac{15}{2} = 7.5$$

M=7.5

$$G.M=\sqrt{3 \times 12} = \sqrt{36} =6$$

G.M=6

∴The arithmetic and Geometric means of 3 and 12 are 7.5 and 6 respectively.

$$\text{(b) } M= \frac{5+8+14}{2} = \frac{27}{3} = 9$$

$$G.M= 3\sqrt{5x8x14} = 3\sqrt{560} = 8.24$$

∴The arithmetic and geometric mean of 5, 8 and 14 are 9 and 8.24 respectively.

Exercise 7

1. Find the arithmetic and geometric means of the following;

   a. $x_1, x_3$
   b. 4x, 9x
   c. 4a, 25a.

2. The arithmeticmean and geometric mean of two numbers are 7.5 and 6 respectively. Find the two numbers.

# Compound Interest

Compound Interest using Formula

Calculate compound interest using formula

Suppose money is invested or borrowed. At the end of a year, interest is calculated. Suppose this interest is added to the original principal, and at the end of the next year interest is added to the new principal. This process may be continued for a number of years.

This process is called *COMPOUND INTEREST.*

When money is invested at a compound interest, the amount of money increase as a geometric sequence.

Example 23

Ibrahimu invested 20,000/= at 6% compound interest. How much was there after 5 years?

***Solution:***

Increasing by 6% is equivalent to multiplying by $(1+\frac{6}{100}) = 1.06$

So after one year the amount is

$20,000 \times 1.06$

After two years: $20,000 \times 1.06 \times 1.06$

$= 20,000 \times (1.06)^2$

After 3 years: $20,000 \times (1.06)^3$

After 4 years: $20,000 \times (1.06)^4$

After 5 years: $20,000 \times (1.06)^5$

$\approx 26,765.$

$\therefore$ After 5 years, the amount will approximately be 26,765/=

Now let the principal be P, the rate R% and the time in years be n.

The interest in one year is $\frac{PR}{100}$

The amount after one year is

$$P + \frac{PR}{100} = P(1 + \frac{R}{100})$$

So the new principal is P$(1 + \frac{R}{100})$

So the interest in the second year is $\frac{PR}{100}$ = P$\left(1 + \frac{R}{100}\right) R$

So after two years the principal is = P $\left(1 + \frac{R}{100}\right) \left(1 + \frac{R}{100}\right)$

Which is P$\left(1 + \frac{R}{100}\right)^2$ and so on.

Therefore n years the principal is A =P$\left(1 + \frac{R}{100}\right)^n$

Example 24

At the beginning of each year Martha invests 10,000/= at 5% compound interest. How much does she have at the end of the $10^{th}$ year?

***Solution:***

She has made 10 different investments each giving different amount of interest.

The $1^{st}$ investment has had 10 years of interest, hence it is 10,000 x $(1.05)^{10}$

The $2^{nd}$ investment has had 9years of interest. So it is 10,000 x $(1.05)^9$

The $3^{rd}$ investment has had 8 years of interest. Hence it is 10,000 x $(1.05)^8$

Following this pattern,

The $10^{th}$ investment has had 1 year of interest. Hence it is 10,000x1.05.

The sum of all these amounts is given by;

$10{,}000 \times 1.05^{10} + 10{,}000 \times 1.05^9 + 10{,}000 \times 1.05^8 + \ldots\ldots\ldots\ldots\ldots + 10{,}000 \times 1.05$

This geometric series with first term 10,000 x 1.05 and common ratio 1.05.

Hence the sum is $\frac{10{,}500(1.05^{10} - 1)}{1.05 - 1}$ = 132,068.

$\therefore$She has 132,000/= (nearest 100/-)

Exercise 8

1. Find thetotal amount of the following savings if they earn compound interest.
   a. 100,000/= for 2years at 6% p.a
   b. 250,000/= for 3 years at 4.5% p.a
   c. 400,000/= for 20years at 5.5% p.a
2. A populationis increasing at 2% if it starts at 10,000,000 what will it be after 20 years.
3. At the beginning of each year 600,000/= is invested at 6% compound interest.Find the total value of the investment at the end of the $15^{th}$ year.

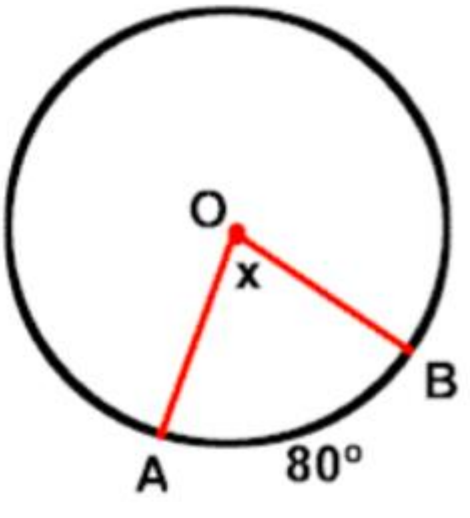
# TOPIC 6: CIRCLES

# Definition of Terms

Circle, Chord, Radius, Diameter, Circumference, Arc, Sector, Centre and Segment of a Circle
Define circle, chord, radius, diameter, circumference, arc, sector, centre and segment of a circle
A **circle:** is the locus or the set of all points equidistant from a fixed point called the center.
**Arc:** a curved line that is part of the circumference of a circle
**Chord:** a line segment within a circle that touches 2 points on the circle.
**Circumference:** The distance around the circle.
**Diameter:** The longest distance from one end of a circle to the other.
**Origin:** the center of the circle
**Pi($\pi$):**A number, 3.141592..., equal to (the circumference) / (the diameter) of any circle.
**Radius:** distance from center of circle to any point on it.
**Sector:** is like a slice of pie (a circle wedge).
**Tangent of circle:** a line perpendicular to the radius that touches ONLY one point on the circle.
**NB: Diameter = 2 x radius of circle**
Circumference of Circle = **PI x diameter** = 2 PI x radius

# Central Angle

The Formula for the Length of an Arc
Derive the formula for the length of an arc
Circumference of Circle = PI x diameter = 2 PI x radius where PI =$\pi$= 3.141592...
The Central Angle
Calculate the central angle
A central angle is an angle formed by two intersecting radii such that its vertex is at thecenter of the circle.
<AOB is a central angle. Its intercepted arc is the minor arc from A to B.

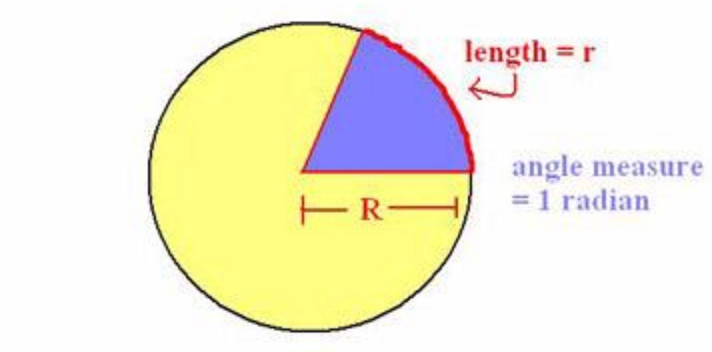


Central Angle = Intercepted Arc

$$m\measuredangle AOB = m\overset{\frown}{AB}$$

$m<AOB = 80°$

The Concept of Radian Measure
Explain the concept of radian measure
Radians are the standard mathematical way to measure angles. One radian is equal to the angle created by taking the radius of a circle and stretching it along the edge of the circle.
The radian is a pure mathematical measurement and therefore is preferred by mathematicians over degree measures. For use in everyday work, the degree is easier to work with, but for purely mathematical pursuits, the radian gives better results. You probably will never see radian measures used in construction or surveying, but it is a common unit in mathematics and physics.

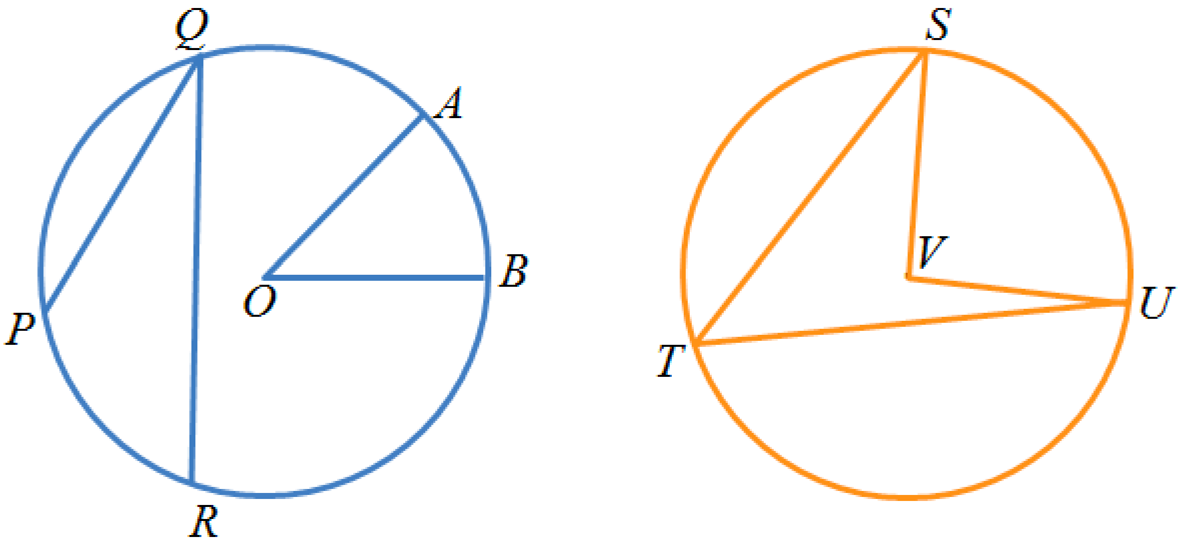



Radians to Degree and Vice Versa
Convert radians to degree and vice versa
The unit used to describe the measurement of an angle that is most familiar is thedegree. To convert radians to degrees or degrees to radians, the following relationship can be used.
angle in degrees = angle in radians * (180/pi)
So, 180 degrees = pi radians
Example 1
Convert 45 degrees to radians
**Solution**
45 = 57.32*radians
radians = 45/57.32
radians = 0.785
Most often when writing degree measure in radians, pi is not calculated in, so for this problem, the more accurate answer would beradians = 45 pi/180 = pi/4
Example 2
Convert pi/3 radians to degree
degrees = (pi/3) * (180/pi)
degrees = 180/3 = 60°

# Angles Properties

Circle Theorems of Inscribed Angles
Prove circle theorems of inscribed angles
An**inscribed**angle is formed when two secant lines intersect on a circle. It can also be formed using a secant line and a tangent line intersecting on a circle. A**central angle**, on the other hand, is an angle whose vertex is the center of the circle and whose sides pass through a pair of points on the circle, therefore subtending an arc.In this post, we explore the relationship between inscribed angles and

central angles having the same subtended arc. The angle of the subtended arc is the same as the measure of the central angle (by definition).




In the first circle,is a central angle subtended by arc. Angleis an inscribed angle subtended by arc. In the second circle,is an inscribed angle andis a central angle. Both angles are subtending arc.

What can you say about the two angles subtending the same arc? Draw several cases of central angles and inscribed angles subtending the same arc and measure them. Use a dynamic geometry software if necessary. Are your observations the same?

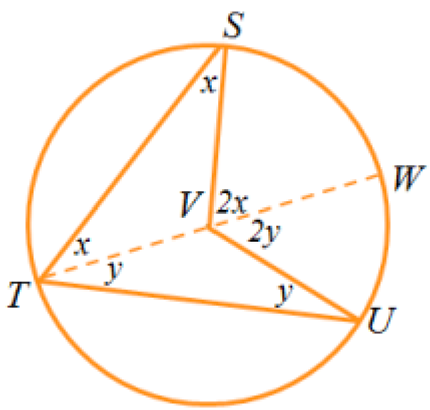
In the discussion below, we prove one of the three cases of the relationship between a central angle and an inscribed angle subtending the same arc.

**Theorem**

The measure of an angle inscribed in a circle is half the measure of the arc it intercepts. Note that this is equivalent to the measure of the inscribed angle is half the measure of the central angle if they intercept the same arc.

**Proof**

Letbe an inscribed angle andbe a central angle both subtending arcas shown in the figure. Draw line. This forms two isosceles trianglesandsince two of their sides are radii of the circle.




In triangle, if we let the measure ofbe, then angleis also. By theexterior angle theorem, the measure of angle. This is also similar to triangle. If we let angle, it follows thatis equal to 2*y*. In effect, the measure of the inscribed angleand the measure of central anglewhich is what we want to prove.

The Circle Theorems in Solving Related Problems

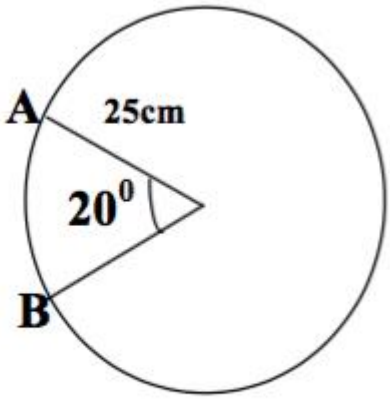
Apply the circle theorems in solving related problems

Example 3

An arc subtends an angle of 200 at the center of the circle of radius25cm.Find the length of this arc.

**Solution**

r =25cm, $\boldsymbol{\theta}$=20°




$$\textbf{The length of the arc AB (l) is given by } l = \frac{\pi r\theta}{180^\circ}$$

$$l = \frac{3.142\times25\times20}{180}cm = 8.73cm$$

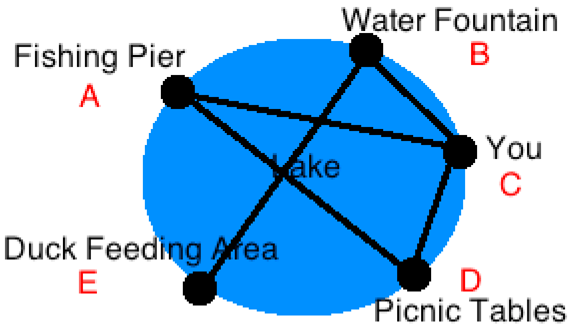
The length of the arc is 8.73cm.
Example 4
An arc of length 5cm subtends 50° at the center of the circle, what is theradius of the circle?
l=5cm, $\boldsymbol{\theta}$=50°, r=?

$$\text{Again from } l = \frac{\pi r\theta}{180^\circ},\ r = \frac{180^\circ l}{\pi\theta} = \frac{180^\circ\times5cm}{3.142\times50} = 5.7cm$$

$\therefore$ **The radius is 5.7cm.**

# Chord Properties of a Circle

Chord Properties of a Circle
Identify chord properties of a circle
Imagine that you are on one side of a perfectly circular lake and looking across to a fishing pier on the other side. The chord is the line going across the circle from point A (you) to point B (the fishing pier). The circle outlining the lake's perimeter is called thecircumference. Achord of a circleis a line that connects two points on a circle's circumference.
To illustrate further, let's look at several points of reference on the same circular lake from before. If each point of reference (i.e. duck feeding area, picnic tables, you, water fountain, and fishing pier) were directly on this lake's circumference, then each line connecting a point to another point on the circle would be chords.



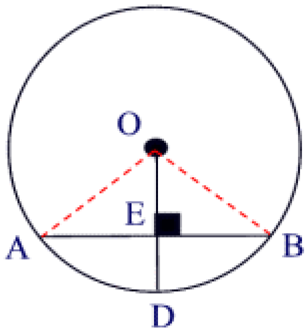

- The line between the fishing pier and you is now chord AC
- The line between the water fountain and duck feeding area is now chord BE
- The line between you and the picnic tables is chord CD

If we had a chord that went directly through the center of a circle, it would be called adiameter. If we had a line that did not stop at the circle's circumference and instead extended into infinity, it would no longer be a chord; it would be called asecant.

The Theorem on the Perpendicular Bisector to a Chord

Prove the theorem on the perpendicular bisector to a chord.

**Proof of Theorem**

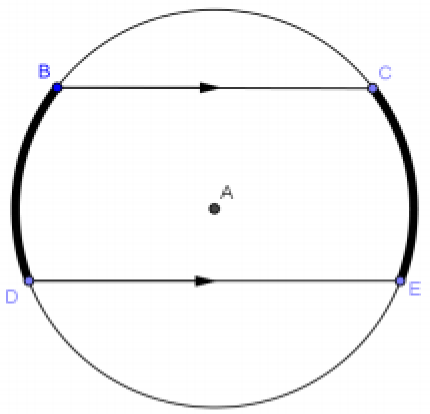



Given: $\odot O$, $\overline{OD} \perp \overline{AB}$

Prove: $\overline{OD}$ bisects $\overline{AB}$

| Statements | | Reasons | |
|---|---|---|---|
| 1. | $\odot O$, $\overline{OD} \perp \overline{AB}$ | 1. | Given |
| 2. | Draw $\overline{OA}$, $\overline{OB}$ | 2. | Two points determine exactly one line. |
| 3. | $\sphericalangle OEA$, $\sphericalangle OEB$ are right angles | 3. | Perpendicular lines meet to form right angles. |
| 4. | $\triangle OEA$, $\triangle OEB$ are right triangles | 4. | A right triangle contains one right angle. |
| 5. | $\overline{OA} \cong \overline{OB}$ | 5. | Radii in a circle are congruent. |
| 6. | $\overline{OE} \cong \overline{OE}$ | 6. | Reflexive Property - A segment is congruent to itself. |
| 7. | $\triangle AOE \cong \triangle BOE$ | 7. | HL - If the hypotenuse and leg of one right triangle are congruent to the corresponding parts of another triangle, the triangles are congruent. |
| 8. | $\overline{AE} = \overline{BE}$ | 8. | CPCTC - Corresponding parts of congruent triangles are congruent. |
| 9. | $E$ is the midpoint of $\overline{AB}$ | 9. | Midpoint of a line segment is the point on that line segment that divides the segment two congruent segments. |
| 10. | $\overline{OD}$ bisects $\overline{AB}$ | 10. | Bisector of a line segment is any line (or subset of a line) that intersects the segment at its midpoint. |

The Theorem on Parallel Chords
Prove the theorem on parallel chords
Parallel chords in the same circle always cut congruent arcs. Parallel chords intercept congruent arcs.

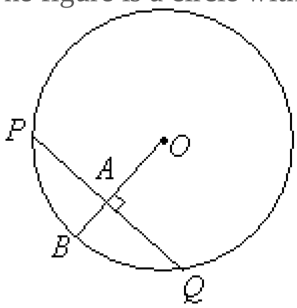



- Construct a diameter perpendicular to the parallel chords.
- What does this diameter do to each chord? *The diameter bisects each chord.*
- Reflect across the diameter (or fold on the diameter). What happens to the endpoints? *The reflection takes the endpoints on one side to the endpoints on the other side. It, therefore, takes arc to arc. Distances from the center are preserved.*
- What have we proven? *Arcs between parallel chords are congruent.*

The Theorems on Chords in Solving Related Problems
Apply the theorems on chords in solving related problems

Example 5

The figure is a circle with centre $O$. Given $PQ$ = 12 cm. Find the length of $PA$.



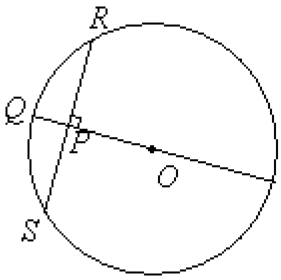

***Solution:***

The radius $OB$ is perpendicular to $PQ$. So, $OB$ is a perpendicular bisector of $PQ$.

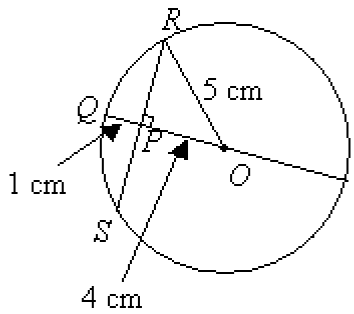
$$PA = \frac{1}{2} \times PQ$$
$$= \frac{1}{2} \times 12$$
$$= 6 \text{ cm}$$

Example 6

The figure is a circle with centre $O$ and diameter 10 cm. $PQ$ = 1 cm. Find the length of $RS$.




***Solution:***



$$OR = \frac{1}{2} \times 10$$
$$= 5 \text{ cm}$$

$OP = OQ - PQ$

$= 5$ cm $- 1$ cm $= 4$ cm

Using Pythagoras' theorem,

$$OR^2 = OP^2 + RP^2$$
$$\Rightarrow 5^2 = 4^2 + RP^2$$
$$\Rightarrow RP = \sqrt{25-16} = 3 \text{ cm}$$

Since $OQ$ is a radius that is perpendicular to the chord $RS$, it divides the chord into two equal parts.

$RS = 2RP = 2 \times 3 = 6$ cm

# Tangent Properties

A Tangent to a Circle

Describe a tangent to a circle

Tangent is a line which touches a circle. The point where the line touches the circle is called the point of contact. A tangent is perpendicular to the radius at the point of contact.

Tangent Properties of a Circle

Identify tangent properties of a circle

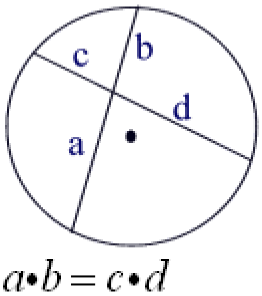
A tangent to a circle is perpendicular to the radius at the point of tangency. A common tangent is a line that is a tangent to each of two circles. A common external tangent does not intersect the segment that joins the centers of the circles. A common internal tangent intersects the segment that joins the centers of the circles.

Tangent Theorems

Prove tangent theorems

Theorem 1

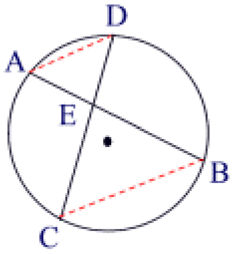
If two chords intersect in a circle, the product of the lengths of the segments of one chord equal the product of the segments of the other.




$$a \bullet b = c \bullet d$$

**Intersecting Chords Rule:** (segment piece)×(segment piece) =(segment piece)×(segment piece)

**Theorem Proof:**

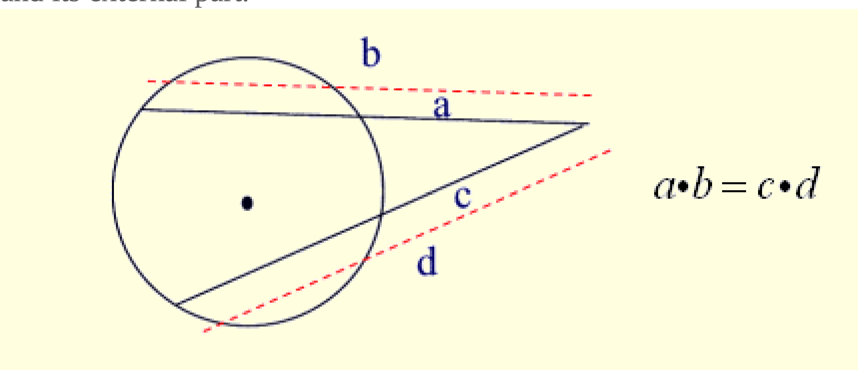



Given: Chords $\overline{AB}$ and $\overline{CD}$

Prove: $AE \cdot EB = CE \cdot ED$

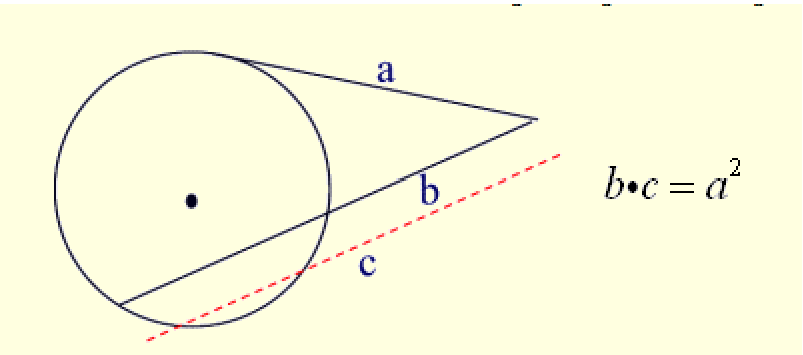
| Statements | | Reasons | |
|---|---|---|---|
| 1. | Chords $\overline{AB}$ and $\overline{CD}$ | 1. | Given |
| 2. | Draw $\overline{AC}$, $\overline{BD}$ | 2. | Two points determine only one line. |
| 3. | $\measuredangle A \cong \measuredangle C$; $\measuredangle B \cong \measuredangle D$ | 3. | If two inscribed angles intercept the same arc, the angles are congruent. |
| 4. | $\triangle ADE \sim \triangle CBE$ | 4. | AA - If two angles of one triangle are congruent to the corresponding angles of another triangle, the triangles are similar. |
| 5. | $\dfrac{AE}{CE} = \dfrac{ED}{EB}$ | 5. | Corresponding sides of similar triangles are in proportion. |
| 6. | $AE \cdot EB = CE \cdot ED$ | 6. | In a proportion, the product of the means equals the product of the extremes. |

**Theorem 2:**
If two secant segments are drawn to a circle from the same external point, the product of the length of one secant segment and its external part is equal to the product of the length of the other secant segment and its external part.




**Secant-Secant Rule:** (whole secant)×(external part) =(whole secant)×(external part)

**Theorem 3:**
If a secant segment and tangent segment are drawn to a circle from the same external point, the product of the length of the secant segment and its external part equals the square of the length of the tangent segment.

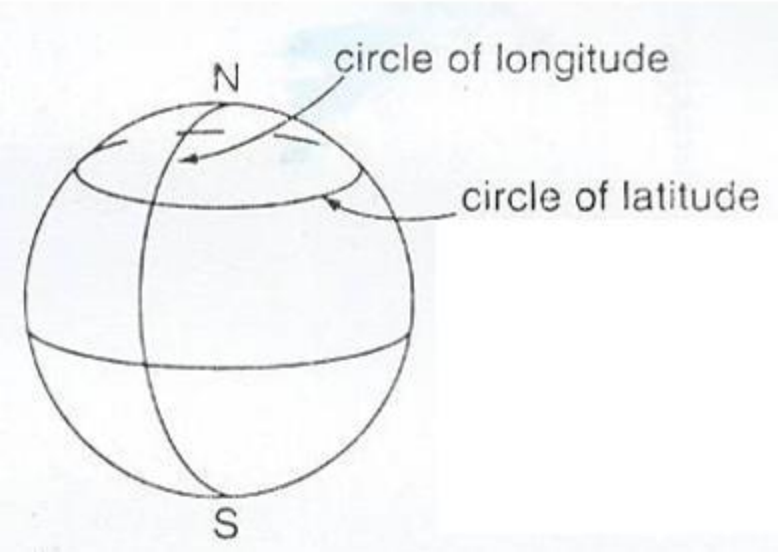



**Secant-Tangent Rule:**(whole secant)×(external part) =(tangent)$^2$
Theorems Relating to Tangent to a Circle in Solving Problems
Apply theorems relating to tangent to a circle in solving problems
Example 7
Two common tangents to a circle form a minor arc with a central angle of 140 degrees. Find the angle formed between the tangents.
**Solution**
Two tangents and two radii form a figure with 360°. If y is the angle formed between the tangents then y + 2(90) + 140° = 360°
y = 40°.
The angle formed between tangents is 40 degrees.

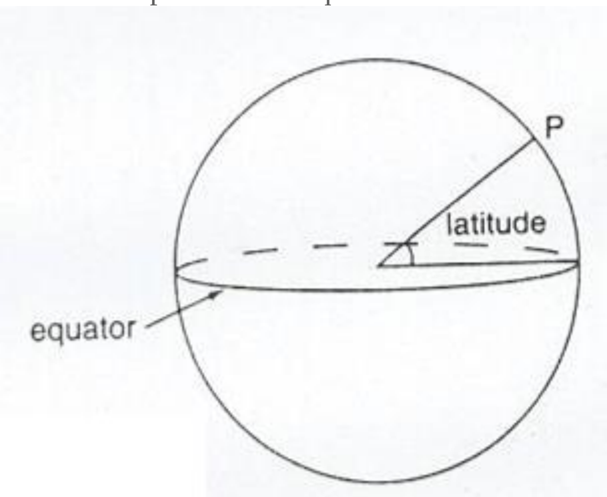
# TOPIC 7: THE EARTH AS THE SPHERE

# Features and Location of Places

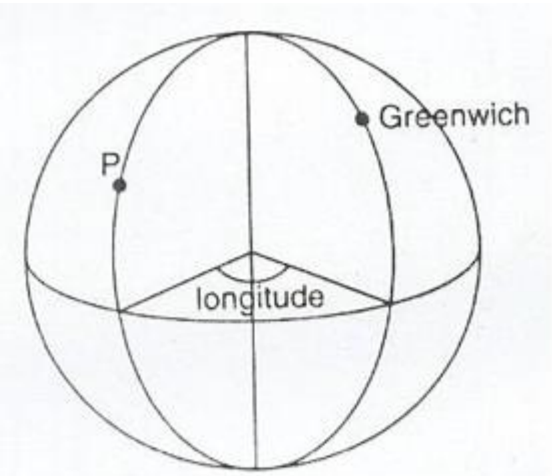
The Equator, Great Circle, Small Circles, Meridian, Latitudes and Longitudes
Describe the equator, great circle, small circles, meridian, latitudes and longitudes
**Definition of latitude and longitude**
The Earth is not a perfect sphere, as it is slightly flatter at the north and south poles than at the equator. But for most purposes we assume that it is a sphere.
The position of any point on earth is located by circles round the earth, as follows:
The earth rotates about its axis, which stretches from the north to the South Pole.
Circles round the Earth perpendicular to the axis are circles of **Latitude** and Circles round the Earth which go through the poles are circles of ***Longitude*** or ***meridians*.**
Consider the following diagram



Normally ***Latitude*** is defined relative to the equator, which is the circle of latitude round the middle of the Earth while ***Longitude*** is defined relative to the circle of longitude which passes through Greenwich in London (**Greenwich meridian).**
The latitude of a position tells us how far north or south of the equator it is while the longitude of a position shows us how far east or west of the Greenwich meridian it is.
**Latitude;** If we draw a line from the centre of the Earth to any position *P* , then the angle between this line and the plane of the equator is the latitude of P.




**Longitude:** This is the angle between the plane through the circle of any Longitude P and the plane of the Greenwich meridian .

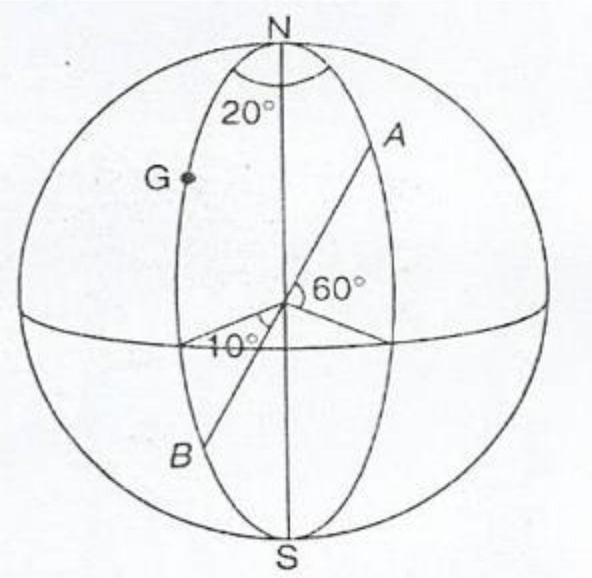



***Latitude*** can be either North or South of the equator while **Longitude** can be either East or West of Greenwich.
When locating the latitude and longitude of a place we write the latitude first then longitude.
Example 1
Dar es Salaam has latitude 7°S (i.e. 7° south of the equator) and longitude 39°E (i.e. 39° east of the Greenwich meridian). So Dar es Salaam is at (7°S, 39°E).
***NB***; Greenwich itself has latitude 51°N ( i.e. 51° north of the equator)and longitude 0° (by definition). Johannesburg has latitude 26°S (i.e. 26 south of the equator) and longitude 28°E(i.e. 28°east of the Greenwich meridian), therefore Johannesburg is at (26°S, 28°E).The north pole has latitude 90°S but its longitude is not defined. ( Every circle of longitude goes through the north pole).The south pole has latitude 90°s. Its longitude is not defined. ***So all points on the equator (such as Nanyuki in Kenya) have latitude 0°***
**Ranges; Latitude** varies between 90°S (at the south pole) to 90°N (at the north pole).
**Ranges; Latitude** varies between 90°S (at the south pole) to 90°N (at the north pole). **Longitude**varies between 180°E and 180°W. These are the longitudes on the opposite side of the Earth from Greenwich.
***GREAT AND SMALL CIRCLES:*** There is an essential difference between latitude and longitude. Circles of longitude all have equal circumference. Circles of latitude get smaller as they approach the

poles. The centre of a circle of longitude is at the centre of the earth. They are called ***great circles***.For circles of latitude, only the equator itself is a great circle. Circles of latitude are called ***small circles.***

Example 2

Find the latitudes and longitudes of A and B on the diagram below;




***Solution;***

The point A is 60° above equator, and 20° east of Greenwich.

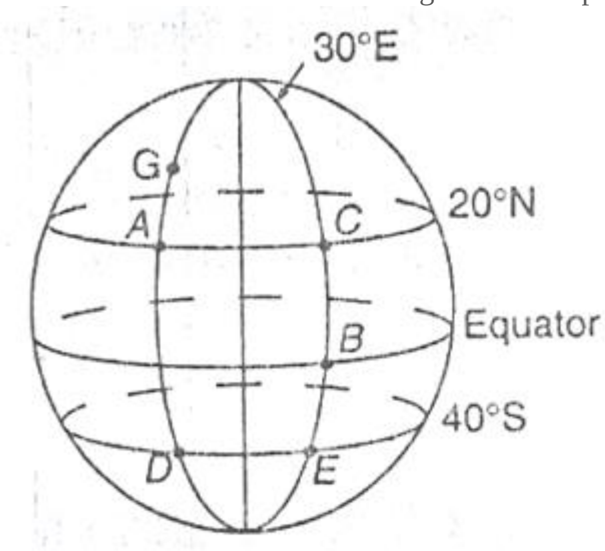
So the point A is at (60°N, 20°E)

The point B is 10° below the equator, and on the Greenwich meridian.

So the point B is at (10°S, 0°).

Exercise 1

1. Write down the latitude and longitude of the places shown on figures below:




2. Copy the diagram show on the figure above and mark these points:

   a. (10°N, 30°E)
   b. (20°N, 20°W)
   c. (0°, 20°W)

3. Obtain a globe, and on it identify the following places.

   a. (40°S, 30°E)
   b. (50°S,20°W)
   c. (10°N,40°W)

d. (40°N, 30°E)

e. (80°N,10°E)

f. (0°,0°)

***Difference between angles of latitude or longitude***

Suppose two places have the same longitude but different latitudes. Then they are north and south of each other.

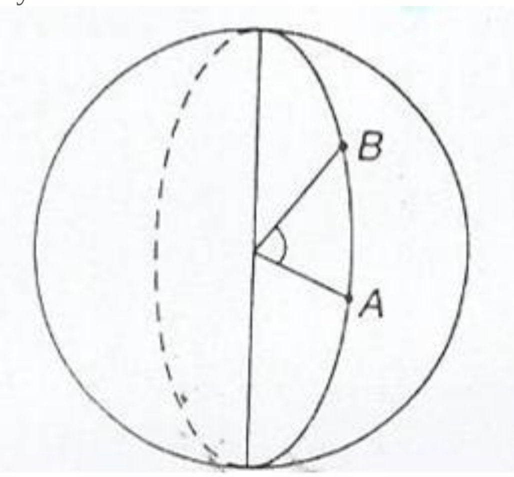
In finding the difference between the latitudes take account of whether they are on the same side of the equator or not.

- If both points are south of the equator subtract the latitudes
- If both points are the north of the equator subtract the latitudes
- If one point is south of the equator and the other north then *add* the latitudes

Similarly, suppose two places have the same latitudes but different longitudes:

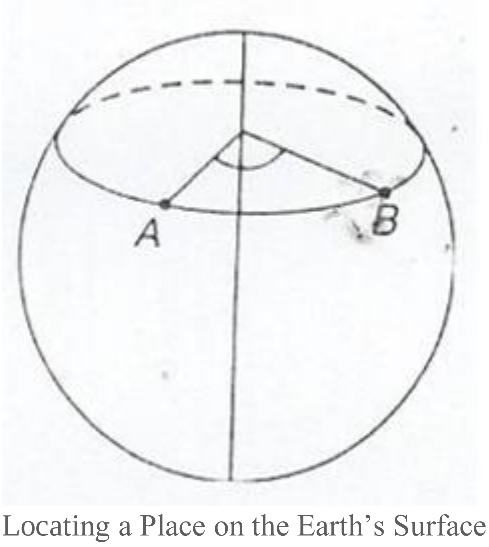
- If both points are east of Greenwich subtract the Longitudes
- If both points are west of Greenwich subtract the Longitudes
- If one point is east of Greenwich and the other west then *add* the longitudes

Suppose places *A* and *B* are on the same longitude, then the difference in latitude is the**angle**subtended by *AB* at the centre of the earth.




Suppose places A and *B* are on the same latitude.

Then the difference in longitude is the **angle** subtended by *AB* on the earth's axis.




Locating a Place on the Earth's Surface

Locate a place on the Earth's surface

Example 3

Three places on longitude 30°E are Alexandria (in Egypt) at (31°N, 30°E), Kigali (in Rwanda) at (2°S, 30°) and Pietermaritzburg (in South Africa) at (30°S, 30°E).
Find the difference in latitude between

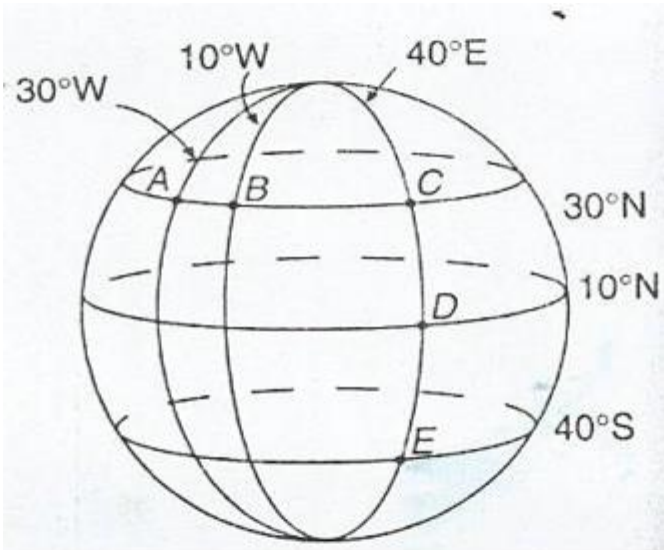
a. Kigali and Pietermaritzburg

b. Kigali and Alexandria

***Solution***
(a)Both Towns are south of the equator. So subtract the latitudes. 30 – 2 = 28
Therefore the difference is $28^0$
(b) Kigali is south of the equator, and Alexandria is north, so add the latitudes
31 + 2 = 33
*The difference is* $33^0$
Example 4
A plane starts at Chileka airport (in Malawi) which is at (16°S, 35°E). It flies west for 50°. What is its new latitude and longitude?
***Solution***
Since it flies west, then subtract 35° from 50°. This gives 15°
The new longitude is now west of Greenwich, hence the plane is at (16°S, 15°W).
Exercise 2
1. In the diagram shown in the following figure find,

a. The difference in longitude between A and B

b. The difference in longitude between D and E



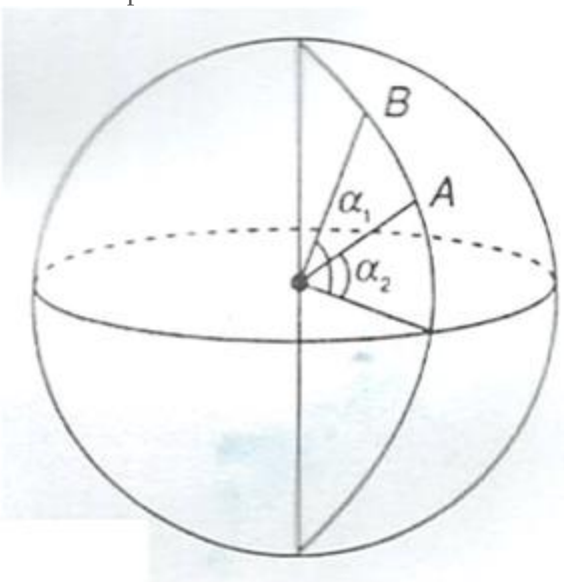
2. Find the difference in latitude between the following pairs of places
   (a) Iringa (8°S, 36°E) and Gendi (4°S, 36°E)
   (b) Zanzibar ((6°S, 39°E) and Chiungutwa (11°S,39°E)
3. Find the difference in longitude between the following pairs of places
   a. Ibadan (Nigeria), (7°N,4°E) and Makurdi (Nigeria), (8°S,36°E)
   b. Ibadan and Kumasi (Ghana) (7°N, 2°W)
4. The following is a list of places. Find pairs of places that have the same latitude or the same longitude. For each pair with the same longitude, find the difference in latitude.

| | | |
|---|---|---|
| A (30°S, 20°E) | B (10°S, 50°E) | C (20°S, 40°W) |
| D (20°N, 40°W) | E (10°N, 50°W) | F (100°S, 50°E) |
| G (30°S, 40°E) | H (20°N, 10°E) | I (20°N, 40°E) |
| J (10°N, 30°E) | K (30°N, 40°W) | L (10°S, 10°E) |

5. A plane starts at (20°N, 27°E) and flies south for 42°. What is its new latitude and longitude?
6. A plane starts at (51°N, 31°E) and flies west for 45°. What is its new latitude and longitude?

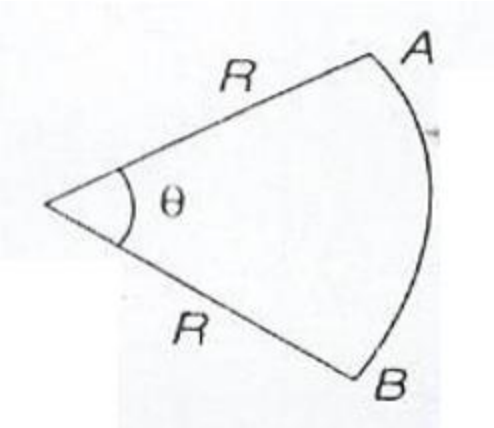
# Distances along Great Circles

Distances along Great Circles
Calculate distances along great circles
Take two places X and Y on the same line of longitude, i.e. one place is due north of the other. Suppose X is due north of Y. When travelling north from Y to X, you travel along part of a circle of longitude that is you travel along an arc of the circle.
The diagram below shows two points A and B on the same circle of longitude.



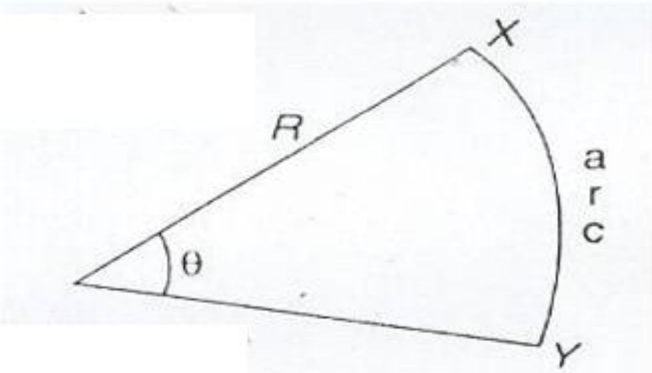

The difference between their latitudes is $\theta = \alpha_1 - \alpha_2$

In the figure above, the sector containing the arc AB subtending∅, is shown.
Recall the formula for the length of arc.




If an arc subtends $\theta$ at the Centre of the circle of radius R, then

Arc length = $\frac{2\pi R\theta}{360}$




In the figure above R is the radius of the Earth and $\theta$ is the difference between the latitudes of X and Y. Taking R to be 6,400 km, the formula becomes

$$\text{Distance} = \frac{2\pi \times 6{,}400 \times \theta}{360} = 111.7\theta$$

**NB**: Remember, to find the difference in latitudes, take account of whether the places are north or south of the equator. If they are all found in south or north, then subtract the latitudes. If one is south and the other North then add the latitudes.

***Nautical miles***

Distances are also measured in ***nautical miles***. One **nautical** mile (nm) corresponds to one minute of latitude. Let the difference in latitudes between two places be $\theta$.

The number of minutes in $\theta$ is $60\theta$, since 1 degree has 60 minutes.

Hence the distance between the places, along the arc of longitude, is $60\theta$ nm.

So $60\theta$ nm $= \frac{2\pi R\theta}{360}$ Kilometers

If we divide by $60\theta$, and take R = 6,400,

then 1nm = $\frac{2\pi \times 6{,}400}{360 \times 60}$ = 1.862 km.

***1knot is speed of 1 nautical mile per hour.***

Navigation Related Problems

Solve navigation related problems

Example 5

Find the distance between Alexandria (31°N, 30°E) and Kigali (2°S, 30°E)

*Solution*

Note that both places are on the same longitude. The difference in latitude is 33°.

Use the formula

Distance = $\frac{2\pi\times6{,}400\times33}{360}$ = 3,690

*Therefore the distance is* ***3,690 km.***

The difference in latitude is 33°. Hence the difference in minutes is $33 \times 60' = 1{,}980'$.

This is the distance in nautical miles. The distance is 1,980 nm.

***Note:*** $1{,}980 \times 1.862 = 3{,}690$, to 3 significant figures. Hence the two answers are the same.

Example 6

A plane starts at ($20^0$S, $30^0$E), and flies north for 4000 km. Find its new latitude and longitude.

***Solution:***

The plane flies north, hence its longitude is unchanged. The plane starts south of the equator, and flying north. It may cross the equator, and so end up north of the equator. In this case the latitude south of the equator will be negative.

Suppose the plane has flown along $x^0$ of latitude. Then using the formula for arc length

$$4{,}000 = \frac{2\pi Rx}{360}$$

$$x = \frac{4{,}000\times360}{2\pi R} = 35.8$$

So subtracting 35.8° from 20°S implies 20°-35.8° = -15.8°

A negative latitude south is equivalent to a latitude north. Hence the new latitude is 15.8° N therefore rounding the answer to the nearest degree the new position is (16°N, 30°E).

Example 7

A plane flies north from (10°S, 30°E) to ((27°N, 30°E) taking a time of 3 hours.

Find its speed, giving your answer I both knots and kilometers per hour.

***Solution***

The plane is flying along a line of longitude.

Its change in latitude is 10°+27° = 37°

The number of minutes is 60 x 37 = 2,220', so it has flown 2,220 nautical miles.

To find the speed in knots, divide 2,220 nautical miles by 3, ***the speed is 740 knots***.

Recall that 1 nautical mile is 1.862 km. So 1 knot (1 nm per hour) is equal to 1.862 km/hr.

Multiplying 740 by 1.862 gives the speed, ***so the speed is 1,378 km/hr***

Exercise 3

Consider the following Questions.

1. Find the distance in kms between the following places.
   (a) Bangalore (13°N, 78°E) and Agra (27°N, 78°E)
   (b) Asmara (15°N, 39°E) and Mombasa (4°S, 39°E)
2. A plane starts as ((30°N, 40°W) and flies 300 km north. What is its new position?
3. A ship starts at (30°N, 150°W) and sails south for 4,700 km. what is its new position?

4. A plane starts at (35°N, 90°W) and flies south for 5,590 km. What is its new position?
5. A plane flies at 600km/h. How long does it take to go from (32°N,51°E) to (5°S,51°E)
6. A ship sails at 20km/h. How long does it take to go from (31°N, 150°W) to (22°S,150°W)?
7. Find the distances in nautical miles between the places in question 1.
8. A plane starts at (10°N, 18°E) and flies north for 1,200 nm. Find its new position.
9. A ship sails south at 10.4 knots. if it starts at (15°N, 150°W), find its position after 40 hrs.
10. A Plane flies south from (12°N, 36°E) to (7°S, 36°E), taking 2½ hours. Find its speed, giving your answer in both knots and kms per hr.
11. A ship sailed north from (45°S, 28°W) to (39°S, 28°W), taking 30hrs. What was its speed? Give your answer in both knots and kms per hr.

# Distances along Small Circles

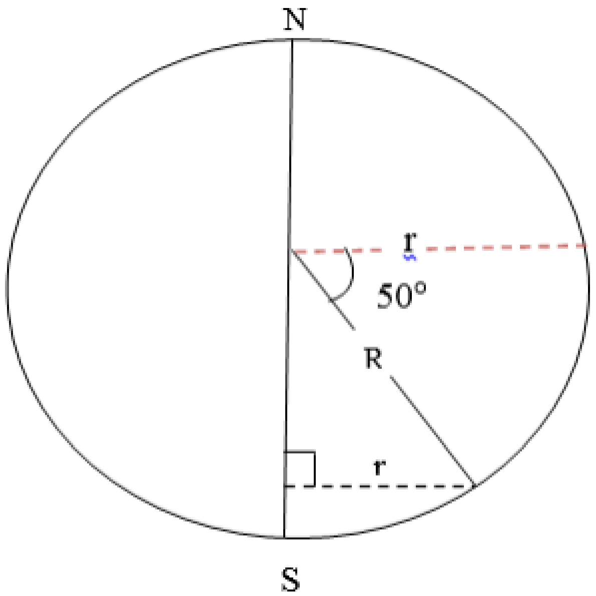
Distance along Small Circles

Calculate distance along small circles

Suppose P and Q are places west or east of each other, i.e they lie on the same circle of latitude. Then when you travel due east or west from P to Q you travel along an arc of the circle of latitude.

The situation here is slightly different from that of the previous section. While circles of longitude all have the same length, circles of latitude get smaller as they get nearer the poles.

Consider the circle of latitude 50°S. Let its radius be r km.




Then taking a point P on the circle, and letting R be radius of the earth:

$$Cos 50° = \frac{r}{R} \quad \text{or} \quad r = R \times \cos 50°$$

In general, the circle of latitude $\alpha$ has radius R$\cos\alpha$ .This is true for latitudes both north and south of the equator . Now the distance along a circle of latitude can be found .Suppose the difference between longitudes is $\theta$ .Then the length of arc going from P to Q is

$$\text{Arc length} = \frac{2\pi r\theta}{360} = \frac{2\pi(R\cos\alpha)\theta}{360}$$

But in kilometers $\frac{2\pi R}{360}$ = 111.7. Hence the arc length is ***1117.7*** $(\cos\alpha)\theta$ ***km***

**Nautical miles**

Generally if two points on the same latitude $\alpha$ have difference in longitude $\theta$, then in nautical miles the distance along the circle of latitude is $60\theta\cos\alpha$ *nm*

Example 8

Find the distance in km and nm along a circle of latitude between (20°N, 30°E) and (20°N, 40°W).

***Solution:***

Both places are on latitude 20°N. The difference in longitude is 70°. Use the formula for distance.

Distance = 111.7 cos20° x 70°. Hence the distance in nautical miles is 60 x 70 x cos20°

The distance is ***3,950 nm.***

Example 9

A ship starts at (40°S, 30°W) and sails due west for 1,000 km. Find its new latitude and longitude.

***Solution***:

Because it sails due west, the latitude remains unchanged. Suppose it has sailed through x° of longitude. Use formula.

1,000=111.7 x cos40° × x°

Hence $x = \frac{1{,}000}{111.7\cos 40°} = 12$

Add 12° to the longitude, obtaining 42°.

Therefore the new position is *(40°S, 42°W).*

Example 10

A ship sails west from (20°S, 15°E) to (20°S, 23°E), taking 37 hours. Find speed, in knots and in kms per hr.

***Solution:***

The difference in longitude between the two points is 8°. Hence the distance, in nautical miles, is.

60 x 8 x cos20° = 451 nm

Divide by 37 to obtain the speed

The speed is ***12.2 knots.***

To obtain the speed in kms per hr, multiply by 1.862.

The speed is ***22.7 km/hr.***

Exercise 4

Consider the following Questions.

1. Find the distance in km between these places.
   a. Johannesburg (26°S, 28°W) and Maputo (26°S,33°W)
   b. Washington (38°N,77°W) and San Francisco (38°N,122°W)
2. A plane flies due east from (30°S,13°E) for 1,000km. What is its new position?
3. A plane flies due east from (45°N, 5°W) for 1,200 km. what is its new position?
4. A plane flies due east from (23°S,12°W) for 1,650km. What is its new position?
5. A ship sails due west from (30°S,12°E) for 2,800km, what is its new position?
6. Fins the distances in nautical miles between the places of question 1.
7. A plane starts at (40°S,30°E) and flies east for 600nm. Find its new position.
8. A plane starts at (20°N,10°W) and flies east for 720nm. Find its new position.
9. A ship starts at (40°S, 140°W), It sails north for 12 hrs, then east for 20hrs. If its speed was constant 20knots, find the latitude and longitude of its final position.
10. A ship starts at (30°N, 45°W), and sails north to (34°N,45°W). It then sails east to (34°N,41°W). if the journey took 30 hrs, find the speed, given that it was constant.
11. A ship can sail at 24km/hr. it starts at (30°S, 60°E). It sails due south for 8 hrs, then due east 2 hr. find the latitude and longitude of its final position.

***Navigation***

*Suppose a ship is sailing in a sea current, or that a plane is flying in a wind. Then the course set the ship or plane is not the direction that it will move in. the actual direction and speed can be found either by scale or by the use of Pythagoras's theorem and trigonometry.*
*Draw the line representing the motion of the ship relative to the water. At the end of this line draw a line representing the current. Draw the third side of the triangle. This side, shown with a double – headed arrow, is the actual course of ship.*

Example 11

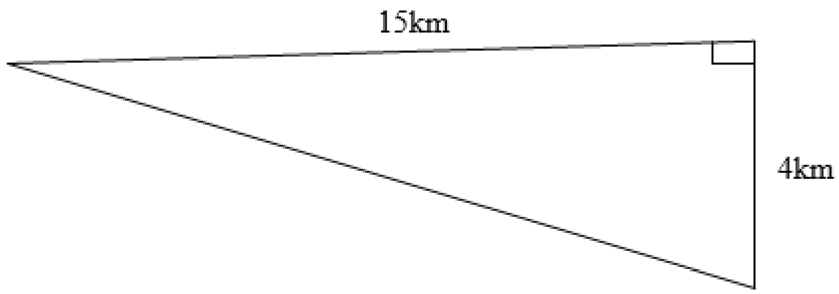
A ship sets course due east. In still water the ship can sail at 15km/hr. There is a current following due south of 4kkm/hr. use a scale drawing to find.

a. The speed of the ship

b. The bearing of the sip.

***Solution:***

In one hour the ship sails 15km east relative to the water. Draw a horizontal line of length 15cm. In one hour the current pulls the ship 4km south. At the end of the horizontal line, draw a vertical line of length 4cm.




(a) Measure the third side of triangle as 15.5cm.
The speed of the ship is ***15.5 km/hr.***

(b) Measure the angle between the course of the ship and east as 15°. Add this to 90°
The ship is sailing along a bearing of 105°

**Note**: These results can also be found by Pythagoras' theorem and trigonometry.
Thus the speed is $\sqrt{15^2 + 4^2}$, and the angle with east is tan $tan^{-1}\left(\frac{4}{15}\right)$

Example 12

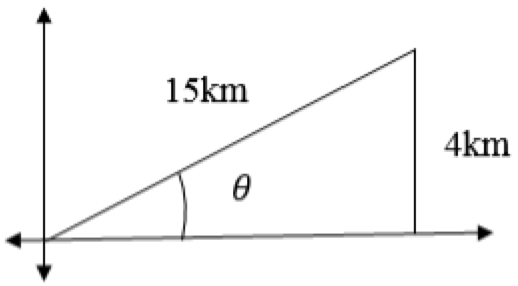
The ship of example 10 needs to travel due east. Calculate the following.

a. What course should be set?

b. How long will the ship take to cover 120km?

***Solution***

The ship needs to set a course slightly north of east, consider the following diagram.




(a) By trigonometry:

Sin $\theta = \frac{4}{15}$ , hence $\theta = 15.46°$ , Substract this from 90°

A course of ***074.5 º***should be set.

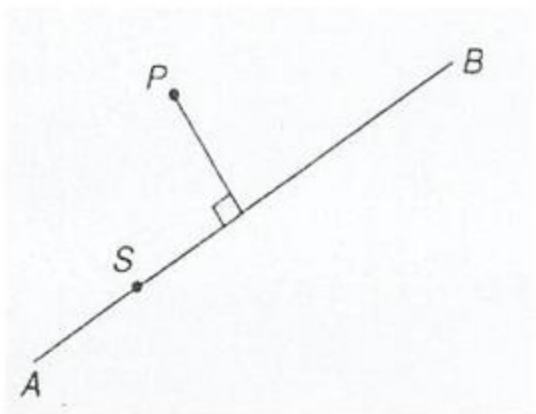
(b) Using Pythagoras' theorem in triangle,

Actual speed of the ship = $\sqrt{15^2 - 4^2}$ = 14.46 km/hr

Now divide 120by this speed.

The ship will take ***8.3 hrs***.

***Note:*** With no current, the journey would take 8hrs. The journey takes slightly longer when there is a current.Suppose a ship or a plane does not directly reach a position. We can still find how close the ship or plane is to the position.



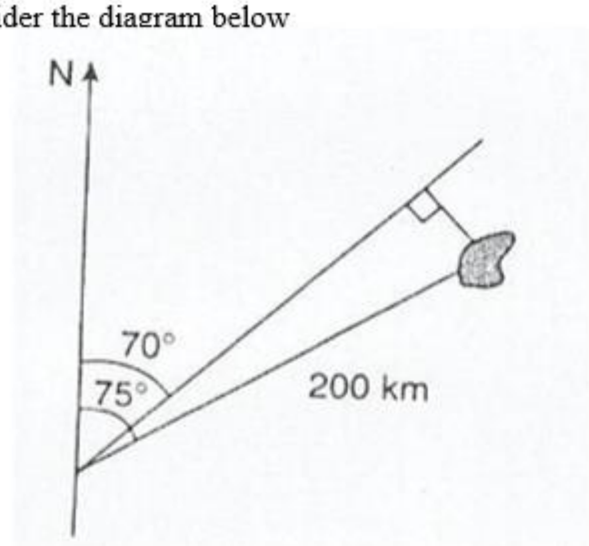

In the diagram above, the ship S (or plane) is travelling on a straight line AB.
The shortest distance to point P is when SP is perpendicular to AB.

Example 13

A small island is 200km away on a bearing of 075°. A ship sails on a bearing of 070°.Find the closest that the ship is to the island.

**Solution:**

Consider the diagram below




The angle between the directions of the island and the path of the ship is:

$75° - 70° = 5°$

When a ship is closest to the island, the line from the island to the ship is at 90° to the path of the ship, hence, by trigonometry,

$d=200 \times \sin 5° = 17.4km.$

The closet distance is ***17.4km.***

Exercise 5

1. Find the difference in longitude between Cape Town (34°S, 18°E) and Buenos Aires (34°S, 58°W)
2. A ship startsat (15°N, 30°W) and sails south for 2,500km. Where does it end up?
3. Find thedistance in km along circle of latitude between cape Town and Buenos aires (see question 2)
4. A plane starts at (37°S, 23°W) and flies east for 1,500 km. where does it end up?
5. Find thedistance in nau

# TOPIC 8: ACCOUNTS

# Double Entry

The Meaning of Double Entry
Explain the meaning of double entry
Businesses need to keep records of their transactions. The process of keeping record is called **bookkeeping**. The simplest form of bookkeeping is single entry. Every transaction is recorded once. This is unreliable because:

- If an arithmetic mistake is made, it is very difficult to find and correct it
- If a transaction is omitted, it is difficult to find it

Now, a more reliable method of bookkeeping is ***double entry***.
Different Types of Ledger
Explain different types of ledger
Businesses record their accounts in books called ledgers. A ledger is a main book that contains various accounts.
There are three main ledgers:

- The sales ledger
- The purchases ledger
- The general ledger

-The ***sales ledger:*** Is the ledger that records the accounts of debtors. The ledger is also known as the ***debtors' ledger.*** A ***debtor*** is a person who owes money to the business, that is a person to whom the business sold goods on credit. So when the business has a new customer, it will open an account in the sales ledger for that customer.
***-The purchases ledger:*** Is the ledger that records the accounts of creditors. This ledger is also known as the ***creditors' ledger.*** A ***creditor*** is a person whom the business owes money, that is a person from whom the business bought goods on credit. When the business has a new supplier, it will open an account in the purchases ledger for that supplier.
***-The general ledger***: Is a ledger that records all accounts other than debtors' and creditors' accountants. Examples of accounts recorded all accounts other than debtors' and creditors' accounts are fixed assets and expense.
-Double entry: Is a bookkeeping system whereby every transaction is recorded twice in the ledger. It is recorded on the left as **debit (DR),** and on the right as **credit (CR).** Every transaction involves the giving and receiving of a benefit.
A Ledger
Construct a ledger
Suppose that a company takes 50,000/- from the bank to pay wages.

- The bank account gives the benefit, and so is credited 50,000/-
- The wages account receives the benefit, and so is debited 50,000/-

Suppose the company buys assets worth 100,000/- from ABC Limited.

- ABC Limited has given the benefit, and so is credited 100,000/-
- The fixed assets account has received the benefit, and so is debited 100,000/-

When transactions are written in a ledger, they are said to be posted to the ledger.
Posting Entries in the Ledger
Post entries in the ledger
Example 1
The transactions shown in the table below belong to XYZ Traders; Post them to the relevant ledgers.

| | |
|---|---|
| June 1: Started business with capital | 50,000 |
| June 2: Bought furniture for cash | 10,000 |
| June 4: Purchased goods for cash | 2,000 |
| June 5: Bought goods from John on credit | 4,000 |
| June 5: Sold goods for cash | 3,000 |
| June 6: Paid rent | 500 |
| June 7: Sold goods to Pwagu on credit | 5,000 |
| June 8: Bought goods from Masatu on credit | 6,000 |
| June 9: Paid electricity bill in cash | 1,000 |
| June 10: Sold goods to Alyoce on credit | 8,000 |
| June 11: Received cash from Pwagu | 2,000 |
| June 12: Withdrew cash for personal use | 7,000 |
| June 15: Paid cash to Masatu | 3,000 |

***Solution***

In the first transaction, money is taken from the capital account and placed cash account. Hence the capital account is credited and he cash account is debited. In the' Particular' column, write the other account involved in the transaction.

In the second transaction, furniture is bought for cash. So the cash account is credited and the furniture account is debited.

Pwagu and Aloyce are customers, so they each have accounts in the sales ledger, John and Masatu are suppliers, so they each have accounts in the purchases ledger.

Other items are capital, cash, furniture, purchases, sales, rent, electricity and drawing. They all have accounts in the general ledger.

GENERAL LEDGER

Capital account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| | | | | June 1 | Cash | | 50,000 |

Cash account

| DR | | | | | CR | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 1 | Capital | | 50,000 | June 2 | Furniture | | 10,000 |
| June 5 | Sales | | 3,000 | June 4 | Purchases | | 2,000 |
| June 11 | Pwagu | | 2,000 | June 6 | Rent | | 500 |
| | | | | June 9 | Electricity | | 1,000 |
| | | | | June 12 | Drawings | | 7,000 |
| | | | | June 15 | Masau | | 3,000 |

Furniture account

| DR | | | | | CR | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 2 | Cash | | 10,000 | | | | |

Purchases account

| DR | | | | | CR | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 4 | Cash | | 2,000 | | | | |
| June 5 | John | | 4,000 | | | | |
| June 8 | Masatu | | 6,000 | | | | |

Sales account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| | | | | June 5 | Cash | | 3,000 |
| | | | | June 7 | Pwagu | | 5,000 |
| | | | | June 10 | Aloyce | | 8,000 |

Rent account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 6 | Cash | | 500 | | | | |

Electricity account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 9 | Cash | | 1,000 | | | | |

Drawing account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 12 | Cash | | 7,000 | | | | |

PURCHASES LEDGER

John account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| | | | | June 5 | Purchases | | 4,000 |

Masatu account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 15 | Cash | | 3,000 | June 8 | Purchases | | 6,000 |

**Sales ledger**

Pwagu account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 7 | Sales | | 5,000 | June 11 | Cash | | 2,000 |

Aloyce account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 10 | Sales | | 8,000 | | | | |

***Note:*** Check that for each transaction there are two equal entries, one for debit and one for credit, for instance:
June 11: Received cash from Pwagu, 2,000
-Pwagu's account is credited 2,000
-The cash account is debited 2,000
This is what is meant by double entry.
Closing the Simple Accounts
Close the simple accounts
Closing the accounts is the process of balancing the accounts. This involves determining the totals of the debits and credits, and finding the difference between the two sides. The difference is ***the balancing figure***, which is placed in the side that is less. This makes the two sides equal.
Example 2
Consider the following account from Example 1. Close this account.
Cash account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 1 | Capital | | 50,000 | June 2 | Furniture | | 10,000 |
| June 5 | Sales | | 3,000 | June 4 | Purchases | | 2,000 |
| June 11 | Pwagu | | 2,000 | June 6 | Rent | | 500 |
| | | | | June 9 | Electricity | | 1,000 |
| | | | | June 12 | Drawings | | 7,000 |
| | | | | June 15 | Masatu | | 3,000 |

***Solution***
This account is closed as follows:
The total debit is 55,000/-
The total credit is 23,500/-
The difference is 31, 500/-
The credit side is less by 31, 500/-. Place this amount in the credit side using the words' balance c/d' (c/d means carried down) in the 'Particular' and 'Folio' columns. This is shown in the following table.

Cash account:

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 1 | Capital | | 50,000 | June 2 | Furniture | | 10,000 |
| June 5 | Sales | | 3,000 | June 4 | Purchases | | 2,000 |
| June 11 | Pwagu | | 2,000 | June 6 | Rent | | 500 |
| | | | | June 9 | Electricity | | 1,000 |
| | | | | June 12 | Drawings | | 7,000 |
| | | | | June 15 | Masatu | | 3,000 |
| | | | | June 30 | Balance | c/d | 31,500 |
| | | | 55,000 | | | | 55,000 |
| July 1 | Balance | b/d | 31,500 | | | | |

The balance c/d shows the amount that stands on the account on the closing date. It appears as balance b/d (b/d means brought down) on the opening date of the next trading period, on the other side of the ledger.

Exercise 1

1. What is a ledger? Give an explanation of three ledgers you know, with an example of accounts kept in each ledger.
2. For each of the following transactions, name the ledger it would be posted to, and whether this would be as credit or debit.
    1. a lorry bought for cash
    2. goods sold to Mr. Sabaya for cash

After

| | | |
|---|---|---|
| June 1: | Commenced business with | 30,000/- |
| June 2: | Bought furniture | 5,000/- |
| June 2: | Purchased goods for cash | 8,000/- |
| June 3: | Bought motor vehicle | 10,000/- |
| June 5: | Bought goods from Mambo on credit | 15,000/- |
| June 7: | Sold goods for cash | 40,000/- |
| June 8: | Paid rent in cash | 2,000/- |
| June 10: | Sold goods for cash | 13,000/- |
| June 11: | Bought goods for cash | 10,000/- |
| June 13: | Paid insurance in cash | 4,000/- |
| June 14: | Sold goods on credit to Abuu | 20,000/- |
| June 15: | Paid cash to Mambo | 5,000/- |
| June 20: | Sold goods for cash | 12,000/- |
| June 21: | Paid salaries in cash | 15,000/- |
| June 25: | Received cash from Abuu | 8,000/- |

# Trial Balance

The Concept of Trial Balance

Explain the concept of trial balance

***Trial balance*** is a statement which shows the balances of accounts extracted from the ledger. At the end of each trading period, the accounts in the ledger are closed, that is the balance of each account is determined. These balances are then shown in the trial balance.

Below is the format of a trial balance.

TRIAL BALANCE as at 30 June 2005

| Account | DR | CR |
|---|---|---|
| | | |

Accounts with debit balances are posted in the DR column and those with credit balances in the CR column.

***Functions of trial balance***

The trial balance serves the following two major roles:

-It checks the arithmetical accuracy of the ledger. The double entry system requires posting equal amounts to debits and credits. Therefore the trial balance is expected to balance if the arithmetic was correct. If there is a difference in the totals of the debit and credit columns of the trial balance, then some errors were made.

-It simplifies the preparation of the final accounts. The trial balance contains all the accounts extracted from the ledgers. This makes it easy to post the accounts to the final accounts.

Construction of Trial Balance

Construct trial balance

Look again at Example 1 of XYZ Traders. The accounts, after being closed, appear as follows:

Capital account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 30 | Balance | c/d | 50,000 | June 1 | Cash | | 50,000 |
| | | | 50,000 | | | | 50,000 |
| | | | | July 1 | Balance | b/d | 50,000 |

Cash account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 1 | Capital | | 50,000 | June 2 | Furniture | | 10,000 |
| June 5 | Sales | | 3,000 | June 4 | Purchases | | 2,000 |
| June 11 | Pwagu | | 2,000 | June 6 | Rent | | 500 |
| | | | | June 9 | Electricity | | 1,000 |
| | | | | June 12 | Drawings | | 7,000 |
| | | | | June 15 | Masatu | | 3,000 |
| | | | | June 30 | Balance | c/d | 31,500 |
| | | | 55,000 | | | | 55,000 |
| July 1 | Balance | b/d | 31,500 | | | | |

Purchases account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 4 | Cash | | 2,000 | June 30 | Balance | c/d | 12,000 |
| June 5 | John | | 4,000 | | | | |
| Jun 8 | Masatu | | 6,000 | | | | |
| | | | 12,000 | | | | 12,000 |
| July 1 | Balance | b/d | 12,000 | | | | |

Sales account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 30 | Balance | c/d | 16,000 | June 5 | Cash | | 3,000 |
| | | | | June 7 | Pwagu | | 5,000 |
| | | | | June 10 | Alyoce | | 8,000 |
| | | | 16,000 | | | | 16,000 |
| | | | | July 1 | Balance | b/d | 16,000 |

Rent account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Account |
| June 6 | Cash | | 500 | June 30 | Balance | c/d | 500 |
| | | | 500 | | | | 500 |
| July 1 | Balance | b/d | 500 | | | | |

Electricity account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Account |
| June 9 | Cash | | 1,000 | June 30 | Balance | c/d | 1,000 |
| | | | 1,000 | | | | 1,000 |
| July 1 | Balance | b/d | 1,000 | | | | |

**And other accounts closing.**

***NB.*** The balance b/d determines whether the account has a debit or credit balance.

Example 3

Construct the trial balance for XYZ Traders of Example 1.

***Solution***
***XYZ TRADERS** TRIAL BALANCE* as at 30 June 2005

| **Account** | **DR** | **CR** |
|---|---|---|
| Capital | | 50,000 |
| Cash | 31,500 | |
| Furniture | 10,000 | |
| Purchases | 12,000 | |
| Sales | | 16,000 |
| Rent | 500 | |
| Electricity | 1,000 | |
| Drawings | 7,000 | |
| Creditors: John | | 4,000 |
| Masatu | | 3,000 |
| Debtors: Pwagu | 3,000 | |
| Aloyce | 8,000 | |
| | 73,000 | 73,000 |

The two columns balance. This confirms that the accounts are correct.

Exercise 2

1. Why is trial balance referred to as statement of arithmetical accuracy?
2. Trial balance is statement and not part of double entry. explain why?
3. The following balances were extracted from the ledgers of Doka traders on 30 June 2005.

| | |
|---|---|
| Capital | 50,000 |
| Furniture | 35,000 |
| Motor vehicle | 45,000 |
| Sales | 75,000 |
| Purchases | 44,000 |
| Creditors | 76,000 |
| Debtors | 45,000 |
| Insurance | 13,000 |
| Cash | 1,000 |
| Discount Received | 7,000 |
| Discount allowed | 4,000 |
| Drawings | 16,000 |
| Electricity | 7,000 |

Prepare a trial balance from the given balances.

Debit Balances and Credit Balances

Post debit balances and credit balances

**DEBIT BALANCES:**

Cash account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 1 | Capital | | 50,000 | June 2 | Furniture | | 10,000 |
| June 5 | Sales | | 3,000 | June 4 | Purchases | | 2,000 |
| June 11 | Pwagu | | 2,000 | June 6 | Rent | | 500 |
| | | | | June 9 | Electricity | | 1,000 |
| | | | | June 12 | Drawings | | 7,000 |
| | | | | June 15 | Masatu | | 3,000 |
| | | | | June 30 | Balance | c/d | 31,500 |
| | | | 55,000 | | | | 55,000 |
| July 1 | Balance | b/d | 31,500 | | | | |

Furniture account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 2 | Cash | | 10,000 | June 30 | Balance | c/d | 10,000 |
| | | | 10,000 | | | | 10,000 |
| July 1 | Balance | b/d | 10,000 | | | | |

Purchases account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| **Date** | **Particular** | **Folio** | **Amount** | **Date** | **Particular** | **Folio** | **Amount** |
| June 4 | Cash | | 2,000 | June 30 | Balance | c/d | 12,000 |
| June 5 | John | | 4,000 | | | | |
| Jun 8 | Masatu | | 6,000 | | | | |
| | | | 12,000 | | | | 12,000 |
| July 1 | Balance | b/d | 12,000 | | | | |

**CREDIT BALANCES:**

John account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 30 | Balance | c/d | 4,000 | June 5 | Purchases | | 4,000 |
| | | | 4,000 | | | | 4,000 |
| | | | | July 1 | Balance | b/d | 4,000 |

Masatu account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 15 | Cash | | 3,000 | June 8 | Purchases | | 6,000 |
| June 30 | Balance | c/d | 3,000 | | | | |
| | | | 6,000 | | | | 6,000 |
| | | | | July 1 | Balance | b/d | 3,000 |

Sales account

| DR | | | | CR | | | |
|---|---|---|---|---|---|---|---|
| Date | Particular | Folio | Amount | Date | Particular | Folio | Amount |
| June 30 | Balance | c/d | 16,000 | June 5 | Cash | | 3,000 |
| | | | | June 7 | Pwagu | | 5,000 |
| | | | | June 10 | Alyoce | | 8,000 |
| | | | 16,000 | | | | 16,000 |
| | | | | July 1 | Balance | b/d | 16,000 |

Checking the Balances
Check the balances
Activity 1
Check the balances

# Trading Profit and Loss

Gross Profit/Loss using Trading Account
Ascertain gross profit/loss using trading account
The trading and profit and loss A/C is an account that is composed of two accounts, the trading A/C, and the profit and loss A/C.
***The trading A/C*** is used to determine the gross profit of the goods sold:
Gross profit = sales – cost of goods sold
***The profit and loss account (A/C)*** is the part of the account that determines the net profit loss:
Net profit = gross profit – expenses
Net loss = expenses – gross profit
***-In the profit and loss A/C, the gross profit and other revenues are credited to the account while the operating expenses are debited.***
***-We have net profit if the credit is greater than the debit and net loss if the debit is greater than the credit.***
Example 4
The following is the trial balance of FMHN Trading Co. as at 31 December 2004. Prepare the trading and profit and loss accounts for the year 2004.
***FMHN TRADING CO.***
***TRIAL BALANCE*** as at 31 December 2004

| Account | DR | CR |
|---|---|---|
| Capital | | 41,500 |
| Machines | 4,550 | - |
| Motor vehicle | 5,520 | - |
| Stock | 24,970 | - |
| Debtors | 20,960 | - |
| Creditors | - | 16,275 |
| Cash | 755 | - |
| Purchases | 71, 630 | - |
| Sales | - | 90,370 |
| Salaries | 8,417 | - |
| Office expenses | 1,370 | |
| Discount allowed | 563 | |
| Drawings | 9,500 | |
| | 148,235 | 148,235 |

The following information was available at 31 December 2004:

(i) stock at 31 December 2004: 27, 340/-

(ii) office expenses owing: 110/-

(iii) Depreciation to be charged as motor vehicle 20% and machines 10%.

**Solution**

First, adjust the trial balance to include the three items above.

(i) Add the stock and the purchases, to obtain 96,600.
Subtract the stock at the end of the period to obtain 69,260.
This gives the cost of the stock sold.

(ii) Add 110 to the office expenses account, increasing it to 1,480.

(iii) Motor: 20% of 5,520 is1104.
Machines: 10% of 4,550 is 455.
Include these under depreciation.

Set out the account as shown below.

The gross profit of 21,110 is found to be subtracting the cost of the stock (69, 260) from the sales (90,370).

The net profit of 9,091 is found by subtracting the costs of the firm (salaries, office expenses, discount, and depreciation from the gross profit.

*FMHN TRADING CO.*

*TRADING AND PROFIT AND LOSS A/C* For the year ending 31st December 2004

| | | | |
|---|---|---|---|
| Opening stock | 24,970 | | |
| Purchases | 71,630 | Sales | 90,370 |
| Goods available for sale | 96,600 | | |
| Less: Closing stock | 27,340 | | |
| Cost of goods sold | 69,260 | | |
| Gross profit c/d | 21,110 | | |
| | 90,370 | 90,370 | |
| Salaries | 8,417 | Gross profit b/d 21,110 | |
| Office expense 1,370 | | | |
| Add: Owing 110 | 1,480 | | |
| Discount allowed | 563 | | |
| Depreciation: Motor vehicle | 1,104 | | |
| Machines | 455 | | |
| Net profit | 9,091 | | |
| | 21,110 | 21,110 | |

Net Profit/Loss Account

Ascertain net profit/loss account

When net loss is recorded, the profit and loss A/C appears as shown in the following example.

PROFIT AND LOSS A/C

For the year ending ……………………

| | | | |
|---|---|---|---|
| Salaries | 1,500 | Gross profit b/d | 12,500 |
| Electricity | 3,400 | Discount received | 1,500 |
| Rent | 800 | | |
| Advertising | 2,700 | | |
| Discount allowed | 4,000 | Net loss | 4,000 |
| Office expenses | 5,600 | | 18,000 |
| | 18,000 | | |

Exercise 3

1. Explain the function of trading A/C.

2. The following is the trial balance of KOKU Traders as at 30th June 1990.

| | DR | CR |
|---|---|---|
| Capital | - | |
| Land and Buildings | 10,000 | 35,748 |
| Advertisement | 254 | |
| Motor van | 1,854 | |
| Furniture and Equipment | 2,200 | |
| Debtors | 9,772 | |
| Creditors | - | 16,810 |
| Cash at hand | 7,584 | |
| Cash in hand | 704 | |
| Stock 1st July | 24, 050 | |
| Insurance | 384 | |
| Bad debts | 52 | |
| Sales | - | 147,668 |
| Purchases | 136,970 | |
| Postage & Telephone | 344 | |
| Electricity | 384 | |
| Wages | 5,674 | |
| | 200, 226 | |
| | | 200,226 |

# Balance Sheet

A Balance Sheet

Construct a balance sheet

A balance sheet is a statement which shows the financial position of a business at a particular date. It shows the ***assets*** on one side and ***liabilities*** on the other.

Assets are divided into two: **fixed assets** and **current assets.**

-***Fixed assets*** are possessions of the business that assist the business in its operations, and benefit the business for more than one accounting period.

-***Current assets*** are assets of the business used in generating income during the accounting period.

***Liabilities*** are also grouped into two: ***long term liabilities***, which are payable in more than one accounting period and current liabilities, which are payable within the accounting period.

The following is the format of balance sheet showing the common items of the balance sheet.

XYZ TRADERS
BALANCE SHEET as at 31st December 2004

| | | | | | |
|---|---|---|---|---|---|
| LONG TERM LIABILITIES | | | FIXED | | |
| Capital | xxx | | Motor vehicle | xxx | |
| **Add**: Net profit | xx | | **Less**: Depreciation | xx | xxx |
| | xxx | | | | |
| **Less:** Drawings xxx | | xxxx | Furniture | xxx | |
| | | | **Less:** Depreciation | xx | xxx |
| Loan | | xxxx | | | |
| CURRENT LIABILITIES | | | CURRENT ASSETS | | |
| Creditors | | xxx | Stock | | xxx |
| Owing expenses | | xxx | Debtors | | xxx |
| | | xxxx | Prepaid | | xxx |
| | | | Bank | | xxx |
| | | | Cash | | xxxx |

***NB:*** Net loss is subtracted from the capital. Stock appearing on the balance sheet is the stock at the date of the balance sheet.

Posting Entries in Balance Sheets
Post entries in balance sheets
Example 5
Considering FMHN Trading Co. from above example, the balance sheet will be as

FMHN TRADING CO.
BALANCE SHEET as at 31st December 2004

| | | | | | |
|---|---|---|---|---|---|
| **LONG TERM LIABILITIES** | | | **FIXED ASSESTS** | | |
| Capital | 41,590 | | Machines | 4,550 | |
| **Add:** Net profit | 9,091 | | **Less:** Depreciation | 455 | 4,095 |
| | 50,681 | | | | |
| **Less:** Drawings | 9,500 | 41,181 | Motor vehicle | 5,520 | |
| | | | **Less:** depreciation | 1,104 | 4,416 |
| **CURRENT LIABILITIES** | | | **CURRENT ASSETS** | | |
| Creditors | 16,275 | | Stock | | 27,340 |
| Office expenses owing | 110 | | Debtors | | 20,960 |
| | 57,566 | | Cash | | 755 |
| | | | | | 57,566 |

Generally, from the balance sheet, we get the equation:

$$\boxed{\textit{Assets} = \textit{Capital} + \textit{Liabilities}}$$

This equation is called "***The balance sheet equation***".

Interpreting Information from the Balance Sheet
Interpret information from the balance sheet
From the balance sheet, useful information concerning the business can be extracted. The interpretation then depends on the use of the information.

The following are some of the useful information provided by the balance sheet.

1. **Capital:** The capital available at the data of the balance sheet is shown after adjusting the previous capital.
2. **Working capital:** Working capital or circulating capital is given by:
   Current assets – current liabilities
3. **Liquidity ratio:** The ratio measures the ability of the business to repay current liabilities out of the current assets. The ratio can be shown in two ways:

(i) Current assets $= \dfrac{\text{current assets}}{\text{Current liabilities}}$

(ii) Quick ratio $= \dfrac{\text{current assets} - \text{stock}}{\text{Current liabilities}}$

The quick ratio measures the ability of the business to pay current liabilities out of current assets excluding stock which is considered less liquid.

From the balance sheet of FMHN Trading Co. we can find the following.

1. Capital at 31st December 2004 is 41. 181/-
2. Working capital is given by:
   Current assets – current liabilities = 49.055 – 16, 385
   = 32, 670/-
3. (i) current ratio $= \dfrac{\text{current assets}}{\text{Current liabilities}}$

   $= \dfrac{49{,}055}{16{,}385} = 2.9 \approx 3$

   (ii) Quick ratio $= \dfrac{\text{current assets} - \text{stock}}{\text{Current liabilities}}$

   $= \dfrac{49{,}055 - 27{,}340}{16{,}385}$

   $= \dfrac{21{,}715}{16{,}385} = 1.3$

***Note:*** The liquidity ratio is favorable if it is greater than or equal to 1.

Exercise 4

1. Prepare the balance sheet for the balances given in the table below.

| | |
|---|---|
| Capital | 45,000 |
| Drawings | 28,000 |
| Creditors | 4,300 |
| Stock | 5,000 |
| Debtors | 8,000 |
| Premises | 16,000 |
| Motor vehicle | 8,000 |
| Bank | 4,000 |
| Cash | 9,000 |
| Net profit | 2,700 |
| Bank overdraft | 6,000 |

2. From the balance sheet constructed in question 2, determine the following.

a. Working capital

b. Quick ratio

c. Current ratio

***Summary***

*-In double entry bookkeeping, every transaction is recorded twice, once as credit and once as debit*

*Debit receives the benefit.*

*Credit provides the benefit.*

*-A trial balance finds the totals of the debits and credits in all the accounts. The total debit should equal the total credit. If they are not equal, a mistake has been made.*

*-The trading and profit and loss account determines there total profit or loss over a period.*

*-The balance sheet shows the financial position of business at a particular date.*

Exercise 5

1. Explainan advantage of double entry bookkeeping over single entry bookkeeping.
2. Givethree accounts that would be kept in the general ledger.
3. Define the quick liquidity ratio.
4. Xis acustomer of company PQR. Below are the transaction made by X over a month.

| | |
|---|---|
| Nov 3rd | Sold 34,000/- of goods to X on credits |
| Nov 5th | Sold 27,000/- - of goods to X on credit |
| Nov 10th | Received 55,000/- - cash from X |
| Nov 14th | Sold 29,000/- - of goods to X on credit |
| Nov 20th | Sold 44,000/- - of goods to X on credit |
| Nov 25th | Received 74,000/- - cash from X |

***(a)*** Post these transaction to X's account, the cash account and the sales account.

***(b)*** Close X'S account

5. The following table shows the closing balances of company PQR of question 4.

| Account | DR | CR |
|---|---|---|
| Capital | | 100,000 |
| Cash | 98,000 | |
| Purchases | 254,000 | |
| Sales | | 179,000 |
| Supplier A | | 88,000 |
| Supplier B | | 102,000 |
| Customer X | 5,000 | |
| Customer Y | 12,000 | |

6. Below is a trial balance for Nyati Ltd. The closing stock was 1, 750,000/-, and the van was depreciated at 25%. Set up the trading and profit and loss account.

| Account | DR | CR |
|---|---|---|
| Capital | | 61,000 |
| Van | 660,000 | |
| Stock | 1,220,000 | |
| Debtors | 155,000 | |
| Creditors | | 160,000 |
| Cash | 86,000 | |
| Purchases | 2,100,000 | |
| Sales | | 5,800,000 |
| Office expenses and salaries | 1,800,000 | |
| | 6,021, 000 | 6,021,000 |

7. Complete the balance sheet below for Nyati Ltd. of question 6.

NYATI LTD.

| | |
|---|---|
| LONG TERM LIABLITIES | FIXED ASSETS |
| Capital | Van |
| **Add:** Net profit | **Less**: Depreciation |
| | |
| CURRENT LIABLITIES | CURRENT ASSETS |
| Creditors | Stock |
| Expenses | Cash |

8. Use the balance sheet constructed in question 7 to find the following.

a. capital

b. working capital

c. current liquidity

d. quick liquidity rate