Lecture 5 Homework John Marangola 3/11/21

1). Let C_i be the event that the i^{th} cat is correct, A_i be the event that the attacker reveal's the ith cat, for $i = 1, 2, 3$.

Suppose the user selects cat 1,

$$P(A_3 | C_1) = \frac{1}{2}$$

$$P(A_3 | C_2) = 1$$

$P(A_3 | C_3) = 0$ as this would result in a premature termination of the game not in the interests of the attacker.

To compute the probability of cats 1 and 2 being correct if the attacker reveals cat 3:

$$P(C_k | A_j) = \frac{P(A_j | C_k) p(C_k)}{P(A_j)} \quad \text{(by Bayes theorem)}$$

$$p(A_j) = P(A_j | C_1) p(C_1) + p(A_j | C_2) p(C_2) + p(A_j | C_3) p(C_3)$$

$$p(A_j) = \sum_{i=1}^{3} P(A_j | C_i) p(C_i), \quad \text{therefore}$$

$$P(C_k | A_j) = \frac{p(A_j | C_k) p(C_k)}{\sum_{i=1}^{3} P(A_j | C_i) p(C_i)}$$

thus, the probability that the correct cat is 2 (assuming cat 3 is revealed) is given by:

$$P(C_2 | A_3) = \frac{P(A_3 | C_2) P(C_2)}{\sum_{i=1}^{3} P(A_3 | C_i) p(C_i)} =$$

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Using the conditional probabilities computed above, it follows that

$$P(C_2|A_3) = \frac{(1)(\frac{1}{3})}{(\frac{1}{2})(\frac{1}{3}) + (1)(\frac{1}{3}) + 0(\frac{1}{3})} = \frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} =$$

$$\frac{\frac{1}{3}}{\frac{1}{6}+\frac{2}{6}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}$$

Likewise,

$$p(C_1|A_3) = \frac{P(A_3|C_1)p(C_1)}{\sum_{i=1}^{3} p(A_3|C_i)p(C_i)} \quad \text{where } p(A_3) = \frac{1}{2},$$

thus $$p(C_1|A_3) = \frac{(\frac{1}{2})(\frac{1}{3})}{(\frac{1}{2})} = \frac{2}{1}\cdot\frac{1}{6} = \boxed{\frac{1}{3}}.$$

$P(C_2|A_3) > p(C_1|A_3)$, therefore the user has a significantly greater chance of winning if they switch their initial guess after a cat has been revealed by the attacker. It is in the best interests of the user to switch their guess.

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```python
"""
Author: John Marangola
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        Randomness and Computation
        Lecture 5 Written Homework
"""
import random
import math

def simulate(n):
    """Monte Carlo Simulation for n trials of the Monte Python Problem
    Args:
        n (int): number of trials
    """
    switch_correct, stay_correct, nswitch = 0, 0, 0 # Variables to keep track of the results of simulation
    for j in range(n):
        winning_cat = random.randint(1, 3) # Attacker randomly generates correct cat
        user_initial_choice = random.randint(1, 3) # Users initial guess
        potential_reveals = set()
        for i in range(1, 4):
            if i == user_initial_choice or i == winning_cat: # Cannot reveal correct cat or users initial choice--the attacker likes to play with their food
                continue
            potential_reveals.add(i)
        # Randomly choosing which Cat the attacker decides to reveal (iff there exists a choice)
        revealed = random.choice(list(potential_reveals))
        # Randomly simulate the user's decision to switch
        switch = random.randint(0, 1)
        if switch:
            # Determine the user's switched guess
            if len(potential_reveals) > 1: switch_to = list(potential_reveals.difference(set((revealed,))))[0]
            else:  switch_to = winning_cat
            nswitch += 1
            switch_correct += (winning_cat == switch_to)
        else:
            stay_correct += (winning_cat == user_initial_choice)
    # Output results
    print(f"Total trials: {n}")
    print(f"Number of times user switched: {nswitch}, Number of times user stayed: {n-nswitch}")
    print(f"Proportion of correct switches: {switch_correct/nswitch}\nProportion of correct stays: {stay_correct/(n-nswitch)}")

if __name__ == "__main__":
    # Simulate n times
    n = 100000      You, 3 hours ago • Homework 3 finished product with comments
    simulate(n)
```

```
Total trials: 100000
Number of times user switched: 50079, Number of times user stayed: 49921
Proportion of correct switches: 0.666187423870285
Proportion of correct stays: 0.33336671941667834
(base) johnmarangola@Johns-MacBook-Pro-2 Randomness %
```

3': virtualenv) 0 0 Live Share

3).

(a). let X be a random variable corresponding to the number of aces a bridge partner has. Let E_1 be the event that $X \geq 1$, and E_2 be the event that $X \geq 2$

$$P(X \geq 2 \mid X \geq 1) = \underbrace{P(E_2 \mid E_1)}$$

The number of hands with 2 or more aces:

$$\binom{48}{11}\binom{4}{2} + \binom{48}{10}\binom{4}{3} + \binom{48}{9}\binom{4}{4}$$

$$= \sum_{i=0}^{3}\binom{48}{11-i}\binom{4}{2+i} \quad \text{equivalatly}$$

The number of hands with at least 1 ace:

$$= \sum_{j=1}^{4}\binom{48}{13-j}\binom{4}{j}$$

$$\text{Thus } P(E_2 \mid E_1) = \frac{\sum_{i=0}^{3}\binom{48}{11-i}\binom{4}{2+i}}{\sum_{j=1}^{4}\binom{48}{13-j}\binom{4}{j}} \Leftrightarrow \frac{\sum_{i=0}^{3}\binom{48}{11-i}\binom{4}{2+i}}{\binom{52}{13}-\binom{48}{13}}$$

$$\boxed{\approx 0.3696}$$

B). $\dfrac{\binom{3}{1}\binom{48}{11}}{\binom{51}{12}} = P(2 \overset{\text{(exactly)}}{\text{Aces}} \mid \text{Ace of spades})$

$$= \frac{\binom{3}{1}\binom{48}{11}}{\binom{51}{12}} + \frac{\binom{3}{2}\binom{48}{10}}{\binom{51}{12}} + \frac{\binom{3}{3}\binom{48}{9}}{\binom{51}{12}} = 0.87$$

$= p(2 \text{ Aces} \mid \text{Ace of sp}) + p(3A \mid \text{AoS}) + p(4A \mid \text{Aos})$

$$= \sum_{i=1}^{3} \frac{\binom{3}{i}\binom{48}{12-i}}{\binom{51}{12}} = 0.87$$

C). This result is certainly not intuitive, it was very suprising to me when I computed it initially. Upon further thought however, it is somewhat more intuitive when you take into account the given condition that she has an ace of spades as this is a small subset of ~~all~~ the set of all possible hands which means that the probability of this result within this subset (ie. given this condition) will be very different.