The Art of Infinite Reckoning

$$2^{\aleph_0} = \aleph_0$$

We provide a set and define a ring onto (surjective) the 2-adic numbers and thus with a cardinality of $2^{\aleph_0}$. We show that this set has a non-trivial exception to Cantor's diagonal argument, implying $2^{\aleph_0} = \aleph_0$. Then we reconstruct a series of subsets of strings in this ring, each set with a cardinality of $\aleph_0$. Thus, we have found a set that has both a cardinality of $\aleph_0$ and $2^{\aleph_0}$ thus proving $2^{\aleph_0} = \aleph_0$. This result implies P=PSPACE.

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Introduction

I assume in writing this paper that the reader has a basic understanding of formal language theory, set theory and an understanding of Cantor's diagonal argument. There are numerous resources to help the reader understand these subjects available freely online and at university libraries. It is not in the scope of this writing to provide these necessary prerequisites.

I understand that it is unheard of for any mathematician to disprove an accepted theorem after over a century of study. However, I ask that the reader approach this work with an open mind to the possibility that we have not had the necessary techniques of formal language available until after the establishment of Cantor's theorem when applied to infinities in order to find exception to his argument.

Essentially, this is a work on formal language, providing us with a foundation for a type of context sensitive grammar that should allow for highly efficient computing.

This writing is an accumulation of more than 10 years of independent study in mathematics after a eureka moment of insight into recursive infinity after reading "Gödel, Escher, Bach" by Douglas Hofstadter.

The title of this paper is in homage to "The Hindu Art of Reckoning" by al-Khwarizmi, which introduced the Arabic numerals and the concept of 0 to the West. It is, additionally, an ominous pun.

I dedicate this writing to Mr. Pat Scenna, an inspiring teacher who encouraged me to think creatively about the possible impossibles in mathematics.

1.1 Natural Transfinite Number

1.1.1 Given the Von Neumann transfinite ordinal ω in base *n* and some positive or neutral z_x(mod *n*) we define any positive Natural Transfinite Number ***N*** as some integer ([Wik16], [And]):

$$N \geq \sum_{x=0}^{x \to \omega \to \infty} 10^x (z_{x+1}); z_{\omega+1} = 1$$

1.1.2 Similarly, we define a base *n* negative Natural Transfinite Number (NTN) for some negative or neutral z_x(mod *n*), ***N*** as some integer:

$$N \leq \sum_{x=0}^{x \to \omega \to \infty} 10^x (z_{x+1}); z_{\omega+1} = -1$$

1.1.3 We say that this is a sum unbounded past omega. Also, by placing any values for any z, in base 2, we can intuitively represent Natural Transfinite Numbers by replacing any z with a 0 or a 1. Furthermore, we can represent an infinite sequence of 0 or 1 by new symbols, ***0*** and ***1***, respectively. We can also represent their combinations, [01], [10], [***01***], etc. where for [10], every odd x, z_x is 0, and every even x, z_x is 1 and where for [110] every 3x+1 z_x is 0 and all other z through infinity x are 1. This can be seen as an actualized p-adic binary expansion. N.B. that p-adic numbers are potential expansions of the rational numbers and defined differently than Natural Transfinite Number. The difference in notational conventions is of particular note. ([Wik17])

2.1 The class of computable transfinite number.

2.1.1 Formally, let Σ be an alphabet with at least 6 elements and the empty string ϵ, let Σ_1* be the set of finite strings over Σ and let Σ_1*′ be the set of finite strings over a subset of at least two elements of Σ. Let Σ_2* be the set of countably infinite length strings over Σ. A language of the class of computable transfinite languages **T** has a subset Σ_3* of Σ_1* such that a subset ***S*** of Σ_2* is non-injective and surjective to the set Σ_3*. For any language *L* in the class **T**, the set of symbols Σ* over *L*(**T**), Σ* = Σ_1*′∪ Σ_3*.

2.1.2 As such, **T** is a class of languages that comprise of finite strings that allow for a grammar that can represent at least some infinite strings in a finite, computable manner.

*2.2 The Grammar for a Specific Language L_T in **T**.*

2.2.1 Informally, to help understand a grammar of the language class **T**, let's consider a language L_T as the language consisting of the alphabet over $\Sigma=\{ \epsilon, 0, 1, \boldsymbol{0}, \boldsymbol{1}, [\ ,]\ \}$, such that $\Sigma^{*\prime}=\{ 0, 1 \}$, the infinite sequence ...0,0,0,0,0... maps to $\boldsymbol{0}$, the infinite sequence, ...1, 1, 1, 1, 1... maps to $\boldsymbol{1}$, the sequence ...0,1,0,1,0,1,0,1... maps to [01], the sequence ...0,1,1,0,1,1,0,1,1,0,1,1... maps to [011], et al. for the set of all functions of infinite length described by 0 and 1 surjective with Σ^*.

2.2.2 Formally, the Grammar $L_T(G)$ is a tuple, $\{V, \Sigma, R, S\}$.
$S \in V$,
$V \cap \Sigma = \varnothing$,
$V=\{S, T, U\}$
$\Sigma=\{ \epsilon, 0, 1, \boldsymbol{0}, \boldsymbol{1}, [,] \}$,
and R is a set of rules, each in the form $A \longrightarrow w$ for some string $w \in (V \cup \Sigma)$ when $A \in V$.
$R=\{S\to\epsilon$, $S\to A$, $S\to B$, $S\to C$, $S\to D$, $A\to 0A$, $A\to 1A$, $A\to 0C$, $A\to 1B$, $A\to 1$, $A\to 0$, $B\to \boldsymbol{0}C$, $C\to \boldsymbol{1}B$, $C\to \boldsymbol{1}$, $B\to \boldsymbol{0}$, $B\to 0$, $C\to 1$, $B\to 0A$, $C\to 1A$, $D\to A[U]A$, $D\to A[U]$, $D\to [U]A$, $D\to B[U]B$, $D\to B[U]$, $D\to [U]B$, $D\to C[U]C$, $D\to C[U]$, $D\to [U]C$, $D\to [U]$, $D\to B[U]C$, $D\to C[U]B$, $U\to 01B$, $U\to 10C$, $U\to 01$, $U\to 10$, $U\to \boldsymbol{1}0$, $U\to 1\boldsymbol{0}$, $U\to 0\boldsymbol{1}$, $U\to \boldsymbol{0}1$, $U\to \boldsymbol{1}0C$, $U\to 1\boldsymbol{0}C$, $U\to 0\boldsymbol{1}B$, $U\to \boldsymbol{0}1B$, $D\to [U]A[U]$, $D\to [U]B[U]$, $D\to [U]C[U]$, $D\to [U][U]$, $D\to A[U]A[U]A$, $D\to B[U]B[U]B$, $D\to C[U]C[U]C$, $D\to A[U]B[U]C$, $D\to B[U]A[U]C$, $D\to C[U]A[U]B$, $D\to C[U]B[U]A$, $D\to A[U]C[U]B$, $D\to B[U]C[U]A$, $D\to C[U]C[U]A$, $D\to C[U]C[U]B$, $D\to B[U]B[U]A$, $D\to B[U]B[U]C$, $D\to A[U]A[U]B$, $D\to A[U]A[U]C$, $D\to A[U]C[U]C$, $D\to A[U]B[U]B$, $D\to B[U]A[U]A$, $D\to B[U]C[U]C$, $D\to A[U]C[U]A$, $D\to A[U]B[U]A$, $D\to B[U]A[U]B$, $D\to B[U]C[U]B$, $D\to C[U]A[U]C$, $D\to C[U]B[U]C\}$

2.2.3 This grammar is context sensitive and essentially non-contracting, thus, L_T is a context sensitive language. ([Wik11])

2.2.4 We define ρ as the set of all well formed strings over L_T.

2.2.5 Let $\Sigma_3{}^*= \Sigma$ where all grammars generated by the nonterminals B, C, D and U that cannot be generated exclusively by the nonterminal A have an onto mapping to a string of infinite length generated by A. Let us also be able to represent a string in ρ by continued fractions to find the i^{th} digit of such infinite length string such that there exists a natural surjection between Natural Transfinite Number in ρ and the 2-adic expansions. Since the set of 2-adic numbers is Cauchy complete, their cardinality is $2^{\aleph_0}$. Since ρ is surjective to the 2-adic numbers, it's cardinality must be at least $2^{\aleph_0}$.

2.3 Addition and Multiplication in L_T.

2.3.1 The set ρ of all well formed strings in L_T is a ring with two binary operations, the functions $+ : \rho \times \rho \rightarrow \rho$ and $\cdot : \rho \times \rho \rightarrow \rho$ where × is the Cartesian product.

2.3.2 The operations are closed in ρ. For all a, b in ρ, the result a+b is in ρ and the result a · b is in ρ. The operations are commutative in ρ; for all *a*, *b* in ρ, the equation $a + b = b + a$ holds and for all *a*, *b*, *c* in ρ, the equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds.

2.3.3 There is an additive identity. i.e. $a + 0 = a$, for any member of ρ. Also, ***0***=0 and $a + S(b) = S(a + b)$ for all *a*, *b*; define S(***1***) = ***0*** ; S(0) = 1, where S(a) is a successor function.

2.3.4 There exists an additive inverse; For each *a* in ρ, there exists an element *b* in ρ such that $a + b = b + a = 0$. Please N.B. that the additive inverse is not always negative, for example: ***10*** + ***0***1***0=0*** and ***1***0+10=***0***.

2.3.5 The operations are associative; For all *a*, *b*, *c* in ρ, the equation $(a + b) + c = a + (b + c)$ holds.

2.3.6 There is a multiplicative identity; there exists an element *1* in ρ, such that for all elements *a* in ρ, the equation $1 \cdot a = a \cdot 1 = a$ holds.

2.3.7 The operations are distributive; for all *a*, *b* and *c* in ρ, the equation $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ holds and for all *a*, *b* and *c* in ρ, the equation $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ holds.

2.3.8 Thus, the set is an Abelian group under addition and a monoid under multiplication and ρ is a ring. ([Wik10])

2.3.9 In order to define string equivalencies, let us now define a set ρ` where we can include expanded strings of ρ by Kleene star expansions of internal strings brackets with or without their brackets, i.e. '(xy)*([(xy)*])*(xy)*' and an expansion rule such that when where there exists a ***0*** in a string, there can be a Kleene star expansion of '0' either to the left or the right or both of ***0*** such that '0****0***0*' is a valid string and similar for ***1*** such that '1****1***1*' is a valid string. Furthermore, the expansions are equal to strings from which they expanded. Since these are Kleene star expansions in a countably infinite set, the cardinality of the set ρ` is the same as ρ. ([Lew], [Wik15])

2.3.10 For example, the well formed string ***01***[001]***0***110 can be expanded to 0***0***01***1***11[001][001]00***0***110; ***01***[001]***0***110=0***0***01***1***11[001][001]00***0***110. Both ***01***[001]***0***110 and 0***0***01***1***11[001][001]00***0***110 are well formed strings in ρ`. This allows us to properly construct the generating functions for the proof.

2.3.11 We define division as the inverse of multiplication.

3.1 Cantor's Diagonal Argument is Insufficient for Proof

3.1.1 We consider a method f(M) over a set M yielding a domain f(M)⟶g(M) where either g(M) ∈ M or g(M) ∉ M insufficient for proof when there exists an alternative method f(M\`) consistent with the axiom set of M, but over the set M\` that also allows for f(M\`)⟶$g$(M\`); g(M\`) ∈ M\` iff g(M) ∉ M or f(M\`)⟶$g$(M\`); g(M\`) ∉ M\` iff g(M) ∈ M. We define left-concatenation of string 'a' over string 'b' as 'a'+'b' = 'ab'. We define right-concatenation of string 'a' over string 'b' as 'b' +'a' = 'ba'. Let str(f(x_i)) be the cumulative string of the function f(x_i) right concatenated on f(x_{i-1}) right concatenated on f(x_{i-2})... right concatenated on f(x_{i-i}). Let str(g(x_i)) be the cumulative string of the function g(x_i) right concatenated on g(x_{i-1}) right concatenated on g(x_{i-2})... right concatenated on g(x_{i-i}). Let f(M) be Cantor's Diagonal Method and M be arbitrary binary strings over $\mathbb{N}_{bin}$ such that there is a case for the power set $\mathbf{P}(\mathbb{N}_{bin}) > \mathbb{N}_{bin}$ and $\mathbf{P}(\mathbb{N}_{bin})=2^{\aleph_0}$. Let g(M)=str(g(x_i)) where x_i is the last element of the last string in a list of total order in M. It is well established that g(M) ∉ M through Cantor's method, f(M)⟶g(M). Let f(M\`) be a diagonal method on the set ρ\`, for every unique x in ρ\`. Since every unique element in ρ\` can be mapped to every unique element in ρ, via definition 2.3.9, and ρ has a cardinality of $\aleph_0$ via Proof 2 in this article, we can generate a single list (*fig. 1*) of Natural Transfinite Numbers in ρ\` of cardinality $\aleph_0$. We can also, through 2.3.9 create equivalent lists of NTNs as necessary for proof.

3.1.2 The following proof argues that Cantor's diagonal method is insufficient for proof by providing a non-trivial exception to his method when applied to ρ\`.

Proof 1

Claim: Cantor's diagonal method f(M) is insufficient for proof. Method: by exception.

1. By definition, a method f(M) is insufficient for proof when f(M) ⟶ g(M); p(M) ∉ M iff f(M\`) ⟶ $g$(M\`); g(M\`) ∈ M\`. For some sets M and M\` consistent with some axiom set. In this instance, assume consistency with Zermelo-Fraenkel set theory.
2. Let f(M) be Cantor's diagonal method on a set of arbitrary binary strings, with the strings bijective over the natural numbers.
3. Let f(M\`) be Cantor's diagonal method on unique elements in ρ\`.
4. Given a list of unique strings in ρ\` (*fig. 1*) or their equivalencies via definition 2.3.9, ordered top to bottom from least to greatest (such that there is a total order to the set), we can construct $g$(M\`) through f(M\`) where $f$(M\`) is a consistent diagonal method on a list of unique strings in ρ\` or an equivalent string.
5. Recall that ***0*** represents an unbounded set of of 0's on a string in L_T through a subset ***S*** of Σ_2* that is non-injective and surjective to the set Σ_3* (section 2.1 of this paper) and through definition 2.3.9.
6. Thus, the first symbol generated in g(M\`) can be 1 when $f$(M\`) is 0 and if f(M) is ***0***, then by definition 2.3.9, f(M) can also be 0 and g(M\`) is 1. For the sake of consistency, we should do this for all ***0*** and for all ***1*** when necessary. When forming $g$(M\`) with an unbounded sequence in Σ_2*, we can use an equivalency of this expansion via definition 2.3.9 for g(M\`).

7. The first symbol of the first string is ***0***.
8. ***0*** $\longrightarrow$0***0*** via definition 2.3.9 $g(0) \longrightarrow 1$.
9. The second symbol of the second string is 0 as the first symbol of the second string is ***0*** and ***0***$\longrightarrow$***0***0 via definition 2.3.9.
10. The first ω strings in M\` or the equivalencies of M\` in this order, least to greatest, are such that $f(x)=0$ for the i^{th} x in i^{th} the string where $i \leq \omega$. While not obvious, the straightforward proof of this is left to the reader, but assume a mathematically inductive process on steps 10 and 11 of this proof. Thus, $g(i)=1$ for any $i \leq \omega$.
11. For all $g(x)$ where $i \leq \omega$, $g(x_i)=1$ for the first ω strings and $i=\omega$. By definition 2.3.9 and step 5 of this proof, str($g(x_\omega)$)=***1***
12. Exhaustion is defined as a change in output due to no other alternatives in a process.
13. When $i=\omega+1$, $f(x_i)$ must be $\neg 0$ by exhaustion. Since 1***0*** < ***1***, then $f(x_{\omega+1})=1$ and str($g(x_{\omega+1})$)=***1***0.
14. As we repeat this process when $i=\omega^2$ we find str($g(x_i)$)=[***1***0].
15. And by exhaustion, $i=\omega^2+1$; str($g(x_i)$)=[***1***0]1.
16. By exhaustion, $i=\omega^2+\omega$; str($g(x_i)$)=[***1***0]1***0***.
17. As we repeat this process, $i=\omega^2+\omega^2$; $g(x_i)$=[***1***0][1***0***] when $i=\omega^2+\omega^2$; x_i=***1***. And we have exhausted the list in total. Thus g(M\`)=[***1***0][1***0***]. (N.B. The cardinality of a set of size $\omega^2+\omega^2 = 2\aleph_0{}^2 = \aleph_0$)
18. If g(M\`)=[***1***0][1***0***] or an expansion of [***1***0][1***0***] is in M\` through definition 2.3.9, then f(M) must be insufficient for proof.
19. All elements in M\` and their equivalencies are in ρ\` by definition (step 4).
20. [***1***0][1***0***] is in ρ through the grammar L_T found in section 2.2.
21. ρ is a subset of ρ\`.
22. [***1***0][1***0***] is in M\`
23. f(M) is Cantor's diagonal method by definition.
24. f(M) is insufficient for proof when $f(M) \longrightarrow g(M)$; $g(M) \notin M$ iff $f(M\grave{}) \longrightarrow g(M\grave{})$; $g(M\grave{}) \in M\grave{}$
25. Since g(M\`)=[***1***0][1***0***]; $g(M\grave{}) \in M\grave{}$
26. ∴ Cantor's diagonal method is insufficient for proof ∎

Proof 2

Claim: $2^{\aleph_0} = \aleph_0$.

Method: We provide a series of sets and their unions, each with a cardinality of $\aleph_0$ such that the set ρ, proven to have a cardinality $\geq 2^{\aleph_0}$ also has a cardinality of $\aleph_0$.

1. It is obvious ρ has a cardinality $2^{\aleph_0}$ since there is a natural bijection with the 2-adic numbers and the p-adic numbers are Cauchy complete for any prime number p. Rigorous proof of this is left to the reader as an exercise. N.B. that arbitrary binary strings of infinite length are also allowed in the grammar producing ρ.
2. There exists a generating function for a set A_{0a} = {***0***, ***0***1, ***0***10...}; $f_A(x) = \sum_{n\geq 0}^{\infty} a_n x^n$ ([Wil], [Wik5])

3. There exists a generating function for a set A_{0b} = {***01***, ***01***0, ***01***01, ***01***010, ***01***001, ***01***000, ***01***0111...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{01} - a_n x^n$

4. The cardinality of A_{0a} and A_{0b} and their union, A_0 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$. [Ruc]

5. There exists a generating function for a set A_{1a} = {***01*****0**, ***01*****0**1, ***01*****0**10, ***01*****0**11...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{010} + a_n x^n$

5. There exists a generating function for a set A_{1b} = {***01*****01**, ***01*****01**0, ***01*****01**01...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{0101} - a_n x^n$

6. The cardinality of each A_{1a} and A_{1b} and their union, A_1 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.

7. We define left-concatenation of string 'a' over string 'b' as 'a'+'b' = 'ab'. We define right-concatenation of string 'a' over string 'b' as 'b' +'a' = 'ba'.

8. There exists a function to determine any set A_x such that the union A_{xa} with A_{xb} is A_x such that the first element of every set in A_{xa} can be found by right-concatenating '1***0***' over '***0***' exactly x times and the first element of every set in A_{xb} can be found by right-concatenating '1***0***' over '***0***' exactly x times and then right-concatenate '***1***' once.

9. We define the set A_ω as the union of the set of sets $\{A_0, A_1, A_2, \ldots A_x, A_{x+1}, \ldots\}$.

10. The cardinality of the set A_ω is $2\aleph_0^2 = \aleph_0$ since we are adding: $2\aleph_0 + 2\aleph_0$, $\aleph_0$ times.

11. There exists a generating function for a set B_{0a} = {***1***, ***1***0, ***1***01, ***1***00...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{1} - a_n x^n$

12. There exists a generating function for a set B_{0b} = {***10***, ***10***1, ***10***10, ***10***11, ***10***100...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{10} + a_n x^n$

12. The cardinality of the sets B_{0a} and B_{0b} and their union, B_0 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.

13. There exists a generating function for a set B_{1a} = {***10*****1**, ***10*****1**0, ***10*****1**01, ***10*****1**00...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{101} - a_n x^n$

14. There exists a generating function for a set B_{1b} = {***10*****10**, ***10*****10**1, ***10*****10**10...}; $f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{1010} + a_n x^n$

15. The cardinality of these sets and their union is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.

16. There exists a function to determine any set B_x such that the union B_{xa} with B_{xb} is B_x such that the first element of every set in B_{xa} can be found by right-concatenating '0***1***' over '***1***'

exactly x times and the first element of every set in B_{xb} can be found by right-concatenating '0***1***' over '***1***' exactly x times and then right-concatenate '***0***' once.

17. We define the set B_{ω} as the union of the set of sets $\{B_0, B_1, B_2, \ldots B_x, B_{x+1}, \ldots\}$. The cardinality of the set B_{ω} is $2\aleph_0^2 = \aleph_0$ since we are adding: $2\aleph_0 + 2\aleph_0$, $\aleph_0$ times.
18. There exists a generating function for a set Γ_{0a} = {***010***, ***010***1, ***010***10...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{010} + a_n x^n$$
19. There exists a generating function for a set Γ_{0b} = {***0101***, ***0101***0, ***0101***01...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{0101} - a_n x^n$$
20. The cardinality of Γ_{0a} and Γ_{0b} and their union, Γ_0 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.
21. There exists a generating function for a set Γ_{1a} = {***01010***, ***01010***1, ***01010***10...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{01010} + a_n x^n$$
22. There exists a generating function for a set Γ_{1b} = {***010101***, ***010101***0, ***010101***01...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{010101} - a_n x^n$$
23. The cardinality of Γ_{1a} and Γ_{1b} and their union, Γ_1 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.
24. There exists a function to determine any set Γ_x such that the union Γ_{xa} with Γ_{xb} is Γ_x such that the first element of every set in Γ_{xa} can be found by right-concatenating '***10***' over '***0***' exactly x times and the first element of every set in Γ_{xb} can be found by right-concatenating '***01***' over '***0***' exactly x times and then right-concatenate '***1***' once.
25. We define the set Γ_{ω} as the union of the set of sets $\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots \Gamma_x, \Gamma_{x+1}, \ldots\}$. The cardinality of the set Γ_{ω} is $2\aleph_0^2 = \aleph_0$ since we are adding: $2\aleph_0 + 2\aleph_0$, $\aleph_0$ times.
26. There exists a generating function for a set Δ_{0a} = {***1010***, ***1010***1, ***1010***10...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{1010} + a_n x^n$$
27. There exists a generating function for a set Δ_{0b} = {***10101***, ***10101***0, ***10101***01...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{10101} - a_n x^n$$
28. The cardinality of Δ_{0a} and Δ_{0b} and their union, Δ_0 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.
29. There exists a generating function for a set Δ_{1a} = {***101010***, ***101010***1, ***101010***10...}; $$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{101010} + a_n x^n$$

30. There exists a generating function for a set Δ_{1b} = {***1010101***, ***1010101***0, ***1010101***01...};

$$f_A(x) = \sum_{n\geq 0}^{\infty} \boldsymbol{1010101} - a_n x^n$$

31. The cardinality of Δ_{1a} and Δ_{1b} and their union, Δ_1 is $\aleph_0$ since each set has a one to one correspondence with the natural numbers and $\aleph_0 + \aleph_0 = 2\aleph_0 = \aleph_0$.
32. There exists a function to determine any set Δ_x such that the union Δ_{xa} with Δ_{xb} is Δ_x such that the first element of every set in Δ_{xa} can be found by right-concatenating '***01***' over '***1***' exactly x times and then right-concatenate '***0***' once. and the first element of every set in Δ_{xb} can be found by right-concatenating '***01***' over '***1***' exactly x times.
33. We define the set Δ_ω as the union of the set of sets $\{\Delta_0, \Delta_1, \Delta_2, \ldots \Delta_x, \Delta_{x+1}, \ldots\}$. The cardinality of the set Δ_ω is $2\aleph_0^2 = \aleph_0$ since we are adding: $2\aleph_0 + 2\aleph_0$, $\aleph_0$ times.
34. $\mathbb{N}_{bin}$ is the set of natural numbers written in binary and has a cardinality of $\aleph_0$ by definition.
35. We define the set E_ω as the set $A_\omega \cup B_\omega \cup \Gamma_\omega \cup \Delta_\omega \cup \mathbb{N}_{bin}$. E_ω has a cardinality of $\aleph_0$ since $\aleph_0 + \aleph_0 + \aleph_0 + \aleph_0 + \aleph_0 = \aleph_0$.
36. Consider the set Z of the unary expansions of '0' along with the empty string such that Z={ϵ, 0, 00, 000, 0000...}. Z has the cardinality $\aleph_0$.
37. Each element in Z is left-concatenated over each element in E_ω to form the set H_ω.
38. The cardinality of H_ω is $\aleph_0^2 = \aleph_0$, since we are adding: $\aleph_0 + \aleph_0$, $\aleph_0$ times.
39. Θ_ω is the set of all strings in H_ω, but with brackets over each string with the exception of strings of less than length 2. Please note that brackets do not occur in the middle of any string such that [00***1***] is in Θ_ω but [00***1***]0 is not in Θ_ω. We include the empty string ϵ in Θ_ω.
40. There is a direct one to one correspondence of H_ω with E_ω such that the cardinality of H_ω is $\aleph_0$.
41. K_r is the set of all strings in H_ω right-concatenated over all strings in Θ_ω.
42. The cardinality of K_r is $\aleph_0^2 = \aleph_0$.
43. K_l is the set of all strings in E_ω left-concatenated over all strings in Θ_ω.
44. The cardinality of K_l is $\aleph_0^2 = \aleph_0$.
45. K_c is the set of all strings in Δ_ω right-concatenated over all strings in K_l.
46. The cardinality of K_c is $\aleph_0^2 = \aleph_0$.
47. K_z is the set of all strings in K_c right concatenated over K_r.
48. The cardinality of K_z is $\aleph_0^2 = \aleph_0$.
49. K_ω is defined as $K_l \cup K_r \cup K_c \cup K_z \cup \Theta_\omega$.
50. K_ω has a cardinality $\aleph_0 + \aleph_0 + \aleph_0 + \aleph_0 + \aleph_0 = \aleph_0$.
51. K_ω is exactly the set that can be generated by the grammar L_T(G).
52. $K_\omega = \rho$
53. $\rho = 2^{\aleph_0}$ by bijection with the 2-adic numbers
54. $K_\omega = 4(2(2\aleph_0 + 2\aleph_0 + 2\aleph_0)^2 + 2\aleph_0 + \aleph_0)^4 + (2(2\aleph_0 + 2\aleph_0)^2 + 2\aleph_0 + \aleph_0)^2 = \aleph_0$
55. $\therefore 2^{\aleph_0} = \aleph_0$ ∎

4.1 A brief note on Cohen's proof of the independence of the Continuum Hypothesis with ZFC.

Cohen's proof holds only in so much where Cantor's theorem holds for infinite universes and is correct so much as Cantor's theorem is correct, however, since as we have shown, Cantor's theorem is not correct when applied to hierarchies of infinities, so neither can Cohen's proof of the independence of the continuum hypothesis, despite Gödel's reading of Cohen's proof, which is also based on Cantor's work. In fact, what we have shown in this paper is fully consistent with ZF set theory, with or without the axiom of choice and Gödel's proof of the consistency of the CH with ZFC still holds. ([Cho], [Coh])

4.2 Further research

$2^{\aleph_0} = \aleph_0 \longrightarrow$ P=PSPACE ∎

Since the Löwenheim-Skolem theorem and all diagonal arguments in logic and computer science must collapse we should be actively searching for a fast algorithm for a PSPACE-complete problem and creating other context sensitive grammars similar to the one provided here for computer intelligence research. ([Wik6], [Wik7], [Wik8], [Wik9])

J Mark Inman
July 30, 2011
On the steps of 77 Massachusetts Ave.
Cambridge, MA
(revised Aug. 14, 2011)

$$
\left|\begin{array}{ccccccccc}
\boldsymbol{0} & 0 & . & . & . & 0 & 0 & 0 & 0 \\
\boldsymbol{0} & 0 & . & . & . & 0 & 0 & 0 & 1 \\
\boldsymbol{0} & 0 & . & . & . & 0 & 0 & 1 & 0 \\
\boldsymbol{0} & 0 & . & . & . & 0 & 0 & 1 & 1 \\
\boldsymbol{0} & 0 & . & . & . & 0 & 1 & 0 & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
\boldsymbol{0} & 0 & \boldsymbol{1} & . & . & . & 1 & 1 & 1 \\
\boldsymbol{0} & 1 & \boldsymbol{0} & . & . & . & 0 & 0 & 0 \\
\boldsymbol{0} & 1 & \boldsymbol{0} & . & . & . & 0 & 0 & 1 \\
\boldsymbol{0} & 1 & \boldsymbol{0} & . & . & . & 0 & 1 & 0 \\
\boldsymbol{0} & 1 & \boldsymbol{0} & . & . & . & 0 & 1 & 1 \\
\boldsymbol{0} & 1 & \boldsymbol{0} & . & . & . & 1 & 0 & 0 \\
\boldsymbol{0} & 1 & \boldsymbol{0} & . & . & . & 1 & 0 & 1 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
\boldsymbol{0} & 1 & \boldsymbol{0} & 1 & . & . & . & 0 & 1 \\
[\boldsymbol{0}1] & [\boldsymbol{0}1] & [\boldsymbol{0}1] & [\boldsymbol{0}1] & . & . & . & 1 & 0 \\
[0\boldsymbol{0}1] & [0\boldsymbol{0}1] & [\boldsymbol{0}1] & [\boldsymbol{0}1] & . & . & . & 1 & 1 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
[\boldsymbol{0}10] & [\boldsymbol{0}10] & [\boldsymbol{0}10] & [\boldsymbol{0}10] & [\boldsymbol{0}10] & [\boldsymbol{0}10] & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
[01\boldsymbol{0}] & [01\boldsymbol{0}] & [01\boldsymbol{0}] & [01\boldsymbol{0}] & [01\boldsymbol{0}] & [01\boldsymbol{0}] & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
[\boldsymbol{1}0] & [\boldsymbol{1}0] & [\boldsymbol{1}0] & [\boldsymbol{1}0] & [\boldsymbol{1}0] & [\boldsymbol{1}0] & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
\boldsymbol{1} & 0 & \boldsymbol{1} & . & . & . & 1 & 1 & 1 \\
\boldsymbol{1} & 1 & \boldsymbol{0} & . & . & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
\boldsymbol{1} & 1 & 1 & . & . & . & 0 & 1 & 1 \\
\boldsymbol{1} & 1 & 1 & . & . & . & 1 & 0 & 0 \\
\boldsymbol{1} & 1 & 1 & . & . & . & 1 & 0 & 1 \\
\boldsymbol{1} & 1 & 1 & . & . & . & 1 & 1 & 0 \\
\boldsymbol{1} & 1 & 1 & . & . & . & 1 & 1 & 1
\end{array}\right|
$$

fig. 1

A note on sources: It is academically dishonest to view relevant Wikipedia entries and then not site that Wikipedia article as a source just because it was a Wikipedia article. There is a plague of academic dishonesty by nearly all academic luddites afraid to admit a primary learning resource that they use on a daily basis. For this reason, I include the relevant Wikipedia articles in my bibliography, despite how it might "look" academically.

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