Part I: Bayesian inference using the integrated nested Laplace approximation (INLA)

Joaquín Martínez-Minaya, November 10, 2025

VAlencia BAyesian Research Group
Statistical Modeling Ecology Group
Grupo de Ingeniería Estadística Multivariante
jmarmin@eio.upv.es

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Motivation example

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Example: Scoring penalties Valencia C. F.

- Liga Santander is one of the famous league around the world. In this example, we use data of the last 10 seasons in order to know the chance of success (π) to score a penalty for **Valencia Club de Fútbol**.

Example: Scoring penalties Valencia C. F.

Response variable + Data

- $Y = \text{score/miss the penalty}$
- The model is generated by Y
- **Bernoulli** with parameter π, i.e., $Y \sim Ber(\pi)$
- **Likelihood**

$$p(\boldsymbol{y} \mid \pi) = \ell(\pi) \propto \pi^k (1-\pi)^{N-k}$$

k: times that a player score a penalty (30).

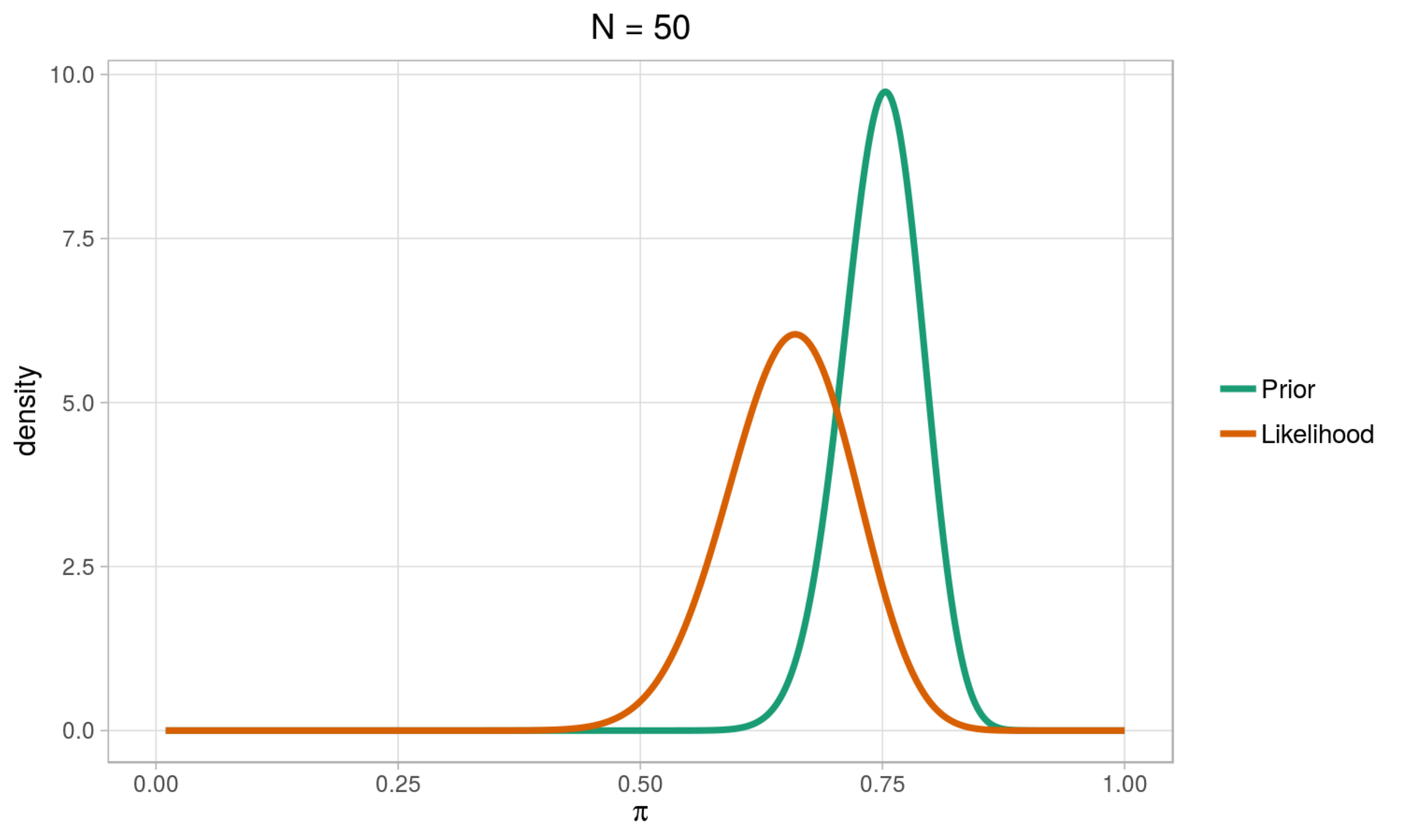
N: total penalties in 10 seasons (50 matches).

Prior knowledge about the parameter π

- **Beta distribution** seems adecuate to model a proportion π.
- After asking some experts, we end up with a 75 percentage chance to score a penalty.
- We express this uncertainty using percentiles $per_{90} = 0.80$ and $per_{50} = 0.75$.
- The corresponding values for a and b are $a = 83.46$ and $b = 28.05$.
- **Prior distribution**

$$p(\pi) \propto \pi^{a-1}(1-\pi)^{b-1}$$

Example. Likelihood *vs* Prior



Graphical Model

Likelihood

$$p(\boldsymbol{y} \mid \pi) \propto \pi^{k}(1-\pi)^{N-k}$$

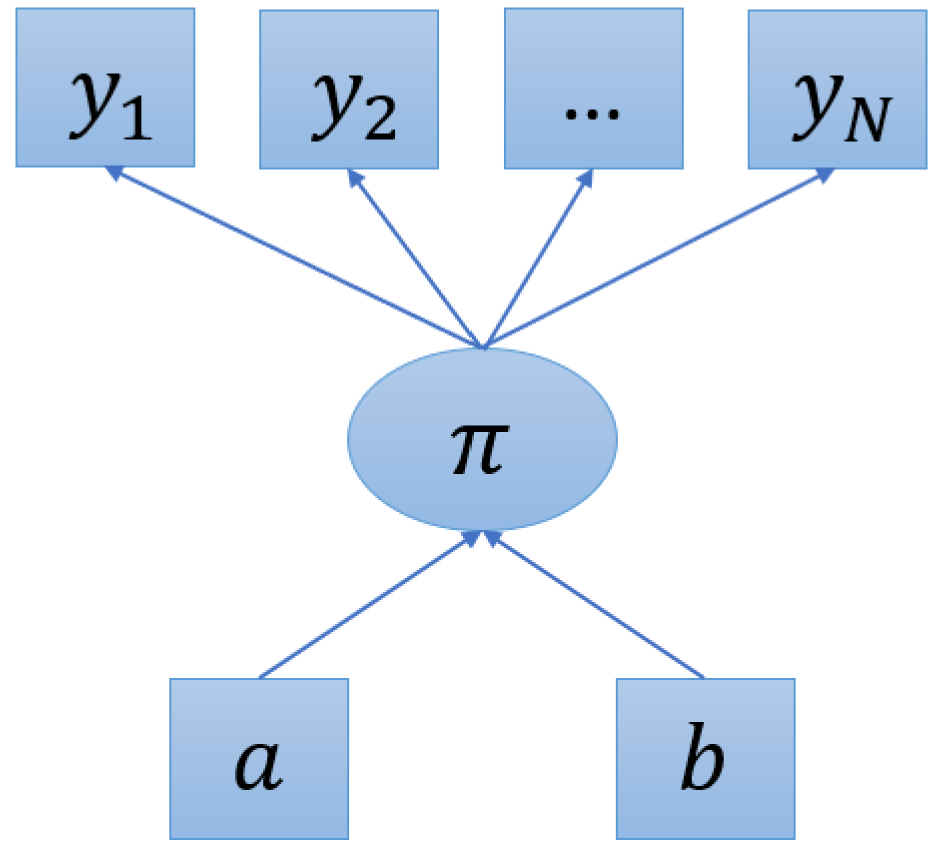
Prior distribution

$$p(\pi) \propto \pi^{a-1}(1-\pi)^{b-1}$$

$$\pi \sim \mathrm{Beta}(a, b)$$

Ellipses: variables

Squares: data



Posterior distribution. Bayesian learning process

Estimating the probability to score a penalty

Likelihood

$$p(\boldsymbol{y} \mid \pi) = \pi^k (1-\pi)^{N-k}$$

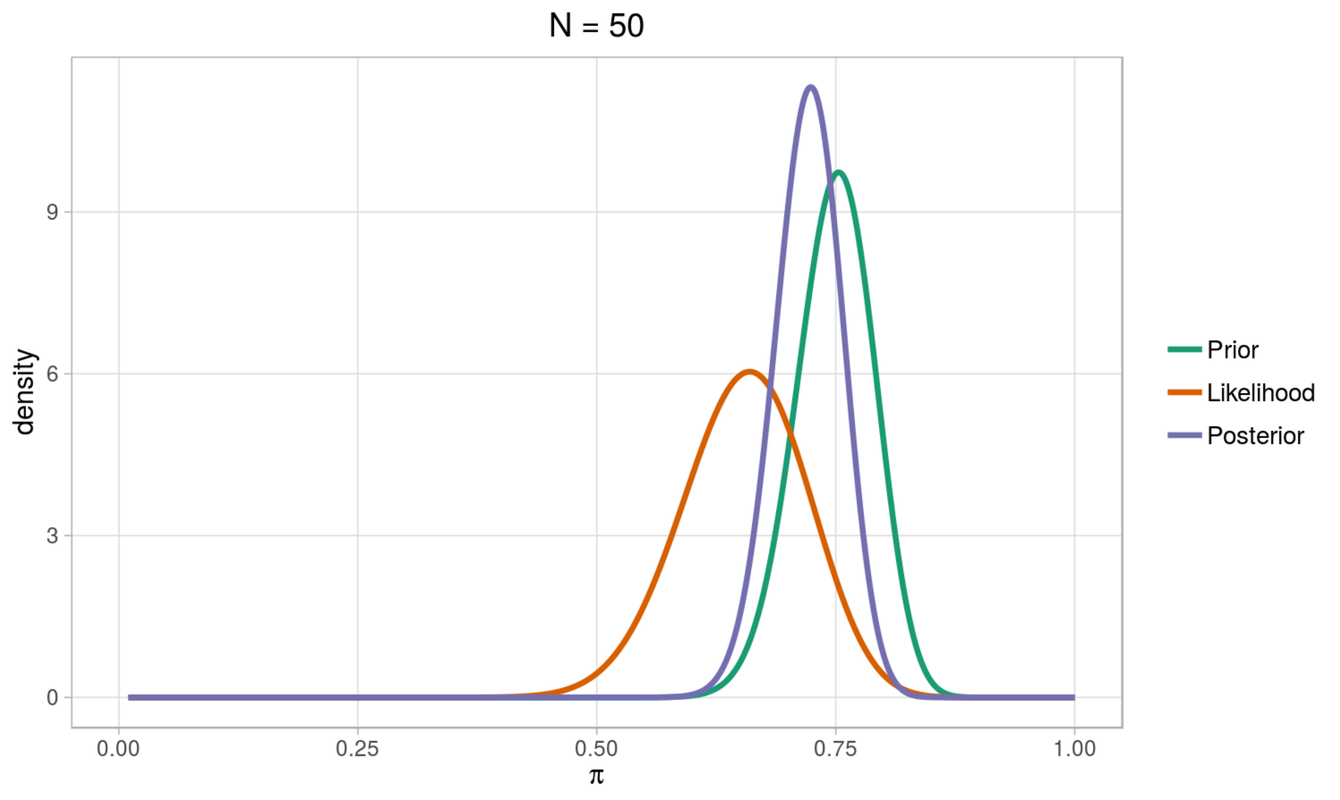
Prior distribution

$$p(\pi) = \pi^{a-1}(1-\pi)^{b-1}$$

Posterior distribution

$$\begin{aligned} p(\pi \mid \boldsymbol{y}) &\propto p(\boldsymbol{y} \mid \pi) \cdot p(\pi) \\ &\propto \pi^{k+a-1}(1-\pi)^{N-k+b-1} \end{aligned}$$

$$\pi \mid \boldsymbol{y} \sim \mathrm{Beta}(k+a, N-k+b)$$
$$\pi \mid \boldsymbol{y} \sim \mathrm{Beta}(30+83.46, 20+28.05)$$
$$\pi \mid \boldsymbol{y} \sim \mathrm{Beta}(113.46, 48.05)$$



Let's try to understand how a prior works:
https://minaya.shinyapps.io/Beta-Conjugate-Priors/

Describing results: point estimators, credible interval

- We obtain a **probability or a density function as a posterior**. So, we can deal with the complete distribution

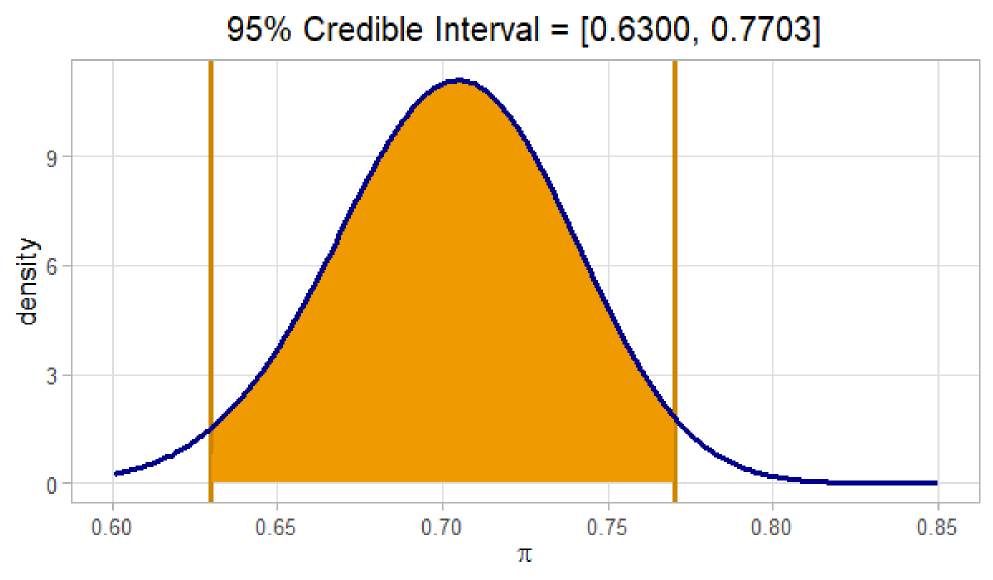
Point estimates

- Mean, median, mode

Credible intervals

- $100(1-\alpha)\%$ credibility interval (CI) for a parameter $\boldsymbol{\theta}$ is defined as the pair of values a and b such as : $\boldsymbol{p(\theta \leq a \mid y) = \alpha/2}$ and $\boldsymbol{p(\theta \geq b \mid y) = 1-\alpha/2}$

- The $\boldsymbol{IC_{95\%}(\pi) = (0.63; 0.77)}$

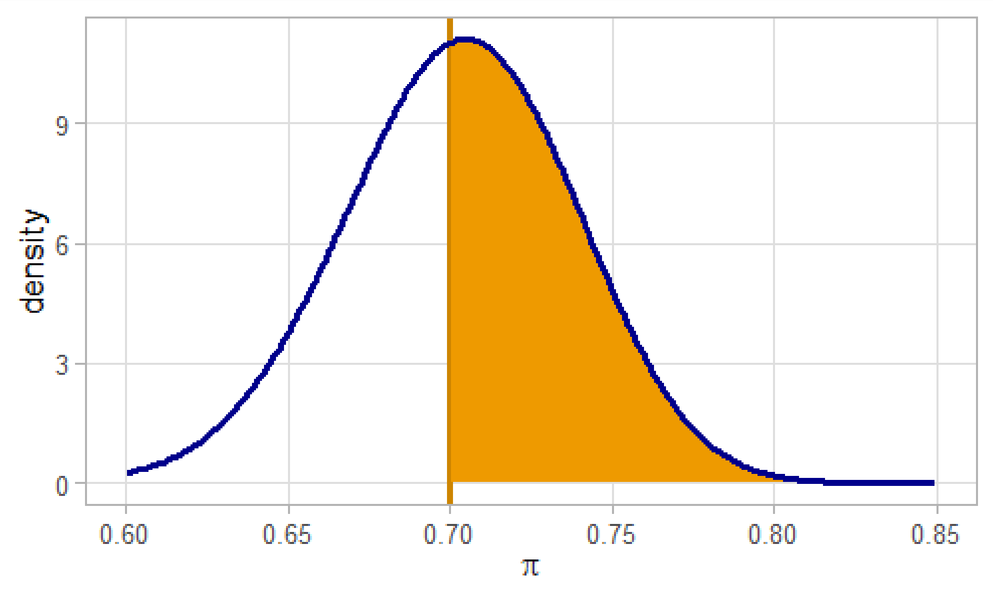


Credible interval *vs* Confidence interval

- **Frequentist approach**: a $100(1-\alpha)\%$ confidence interval is defined such that, if the data collection process is repeated again and again, then in the long run, $100(1-\alpha)$ **of the confidence intervals formed would contain the (fixed) unknown parameter value**.

- **Bayesian approach**: a $100(1-\alpha)\%$ credible interval will explicitly indicate **the posterior probability that θ lies within its boundaries**. So, we talk about credibility intervals.
  - The $IC_{95\%}(\pi) = (0.63; 0.77)$ means that the probability for π to be between 0.63 and 0.77 is 0.95.

- Bayesian inference can also provide any probability statements about parameters.
  - For example, we could compute $p(\pi > 0.7 \mid \boldsymbol{y}) = 0.5368$.



Different teams

- We consider same experiment in **10 different teams**
- How can we model this situation? and what can we conclude?
- More generally, how can we incorporate **random effects**?

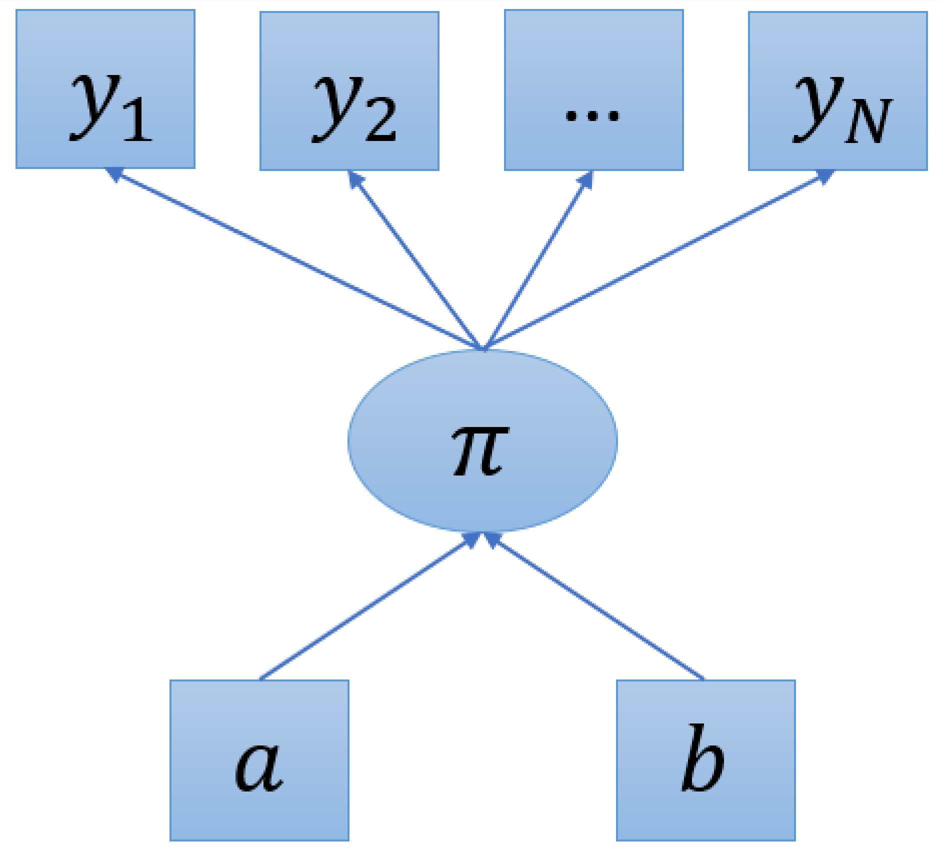
Three ways to do so

1. **All teams have the same characteristics.**

- Apply a **joint analysis** to all the teams.
- The probability of score a penalty (π) is the **same in all teams**.
- Observations are independent and identically distributed.

$$y_i \mid \pi \sim \text{Ber}(\pi)$$

$$\pi \sim \text{Beta}(a, b)\,, \ \text{with a and b fixed}$$



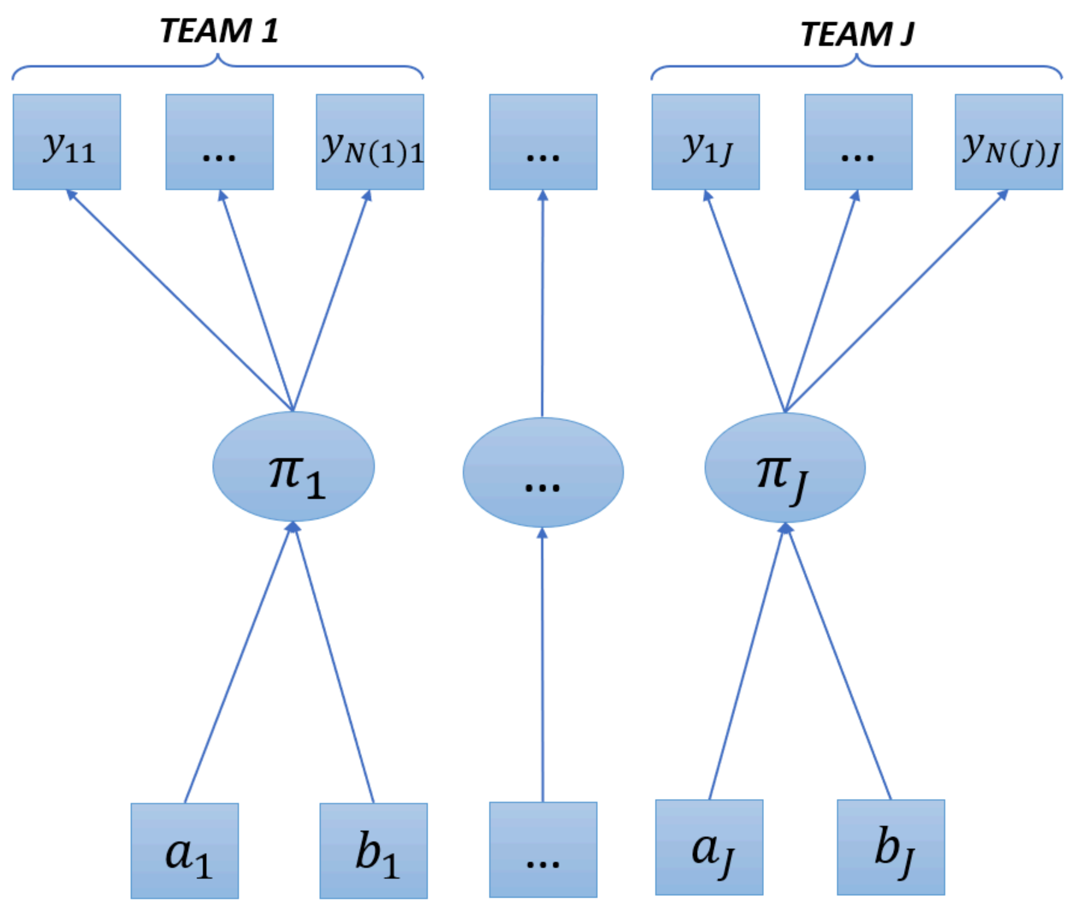
Three ways to do so

2. **Each team is different and has nothing in common with the others.**

- Apply an analysis to **each team separately**.
- Assume a **different proportion of presence** in each one: π_j, $j = 1, \ldots, J$. In this case, $J = 10$.
- Observations are independent but are **distributed differently in each team**.
- **Likelihood** is different for each team. For each j

$$y_{ij} \mid \pi_j \sim \text{Ber}(\pi_j)$$

$$\pi_j \sim \text{Beta}(a_j, b_j)\ , \ \text{with } a_j \text{ and } b_j \text{ fixed}$$



In view of the two possible modelings

- Is it reasonable to assume **the same proportion of presence** in all teams?

- There are reasons to suggest that **there is variability in those proportions**:

  - The teams do not behave the same way.
  - The observations of the same team are more similar among themselves than when they are from different teams.

- Is it reasonable to think that **there is no relationship between the proportions of presence** of the different teams?

Although not identical, **teams** are at least **similar**.

Three ways to do so

3. Consider a **hierarchical model**.

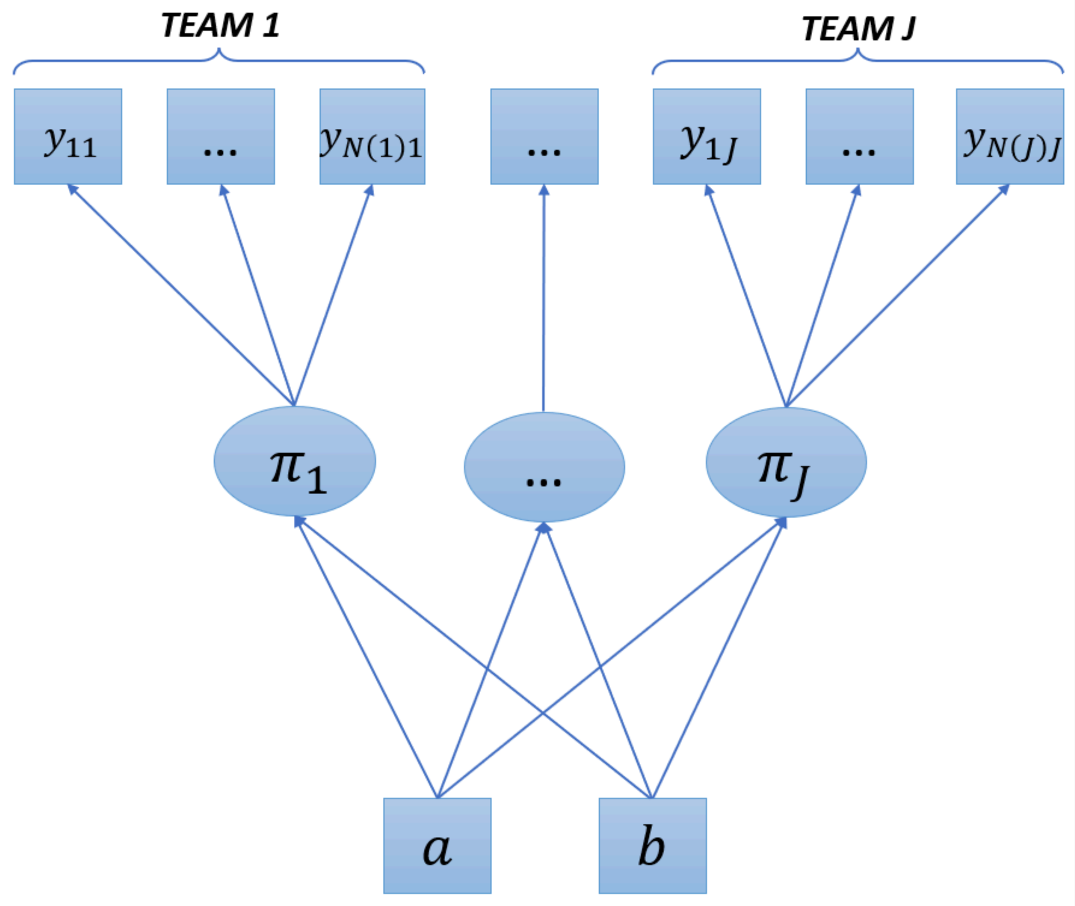
- The parametric vector $\boldsymbol{\pi} = \pi_1, \ldots, \pi_J$ is a **random sampling from a common distribution** that depends on a vector of **hyperparameters**, α and β, partial or totally unknown.

- The model

$$y_{ij} \mid \pi_j \sim \text{Ber}(\pi_j) \; , j = 1, \ldots, J$$

$$\pi_j \sim \text{Beta}(\alpha, \beta)$$

$$\alpha \sim p(a), \beta \sim p(b)$$



Numerical approaches

- When applying Bayesian Statistics, most of the usual models do not yield analytic expressions for neither the posterior nor the predictive posterior distributions.

- Most of the **complications that appear in the Bayesian methodology** come from the resolution of integrals that appear when applying the learning process:

    - The normalization constant of the posterior distribution,

    - moments and quantiles of the posterior,

    - credible regions, probabilities in the contrasts, etc.

Solutions:

- **Monte Carlo methods: MCMC**.

- **INLA**.

Outline

1. Why INLA?

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INLA as an alternative to MCMC

- MCMC is an asymptotically exact method whereas INLA is an **approximation**. Their error are frequently very similar, as has been shown in many simulation studies.

- INLA is a **fast alternative** to MCMC for the general class of latent Gaussian models (LGMs). Many familiar models can be re-cast to look like LGMs:

    - **generalized linear models**, **generalized additive models**, smoothing spline models,
    - state space models, semi-parametric regression, **random walk (first and second order)** models, longitudinal data models,
    - **spatial and spatiotemporal** models, log-Gaussian Cox processes and geostatistical and geoadditive models., etc.
- To understand INLA, we need to be familiar with:
    - Latent Gaussian models
    - Gaussian Markov Random Fields (GMRFs)
    - Laplace approximations

2. Elements to understand how INLA works

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Latent Gaussian model

Level 1 : likelihood

The first stage is formed by the **conditionally independent likelihood** function of data coming from a certain exponential family distribution:

$$p(\boldsymbol{y} \mid \boldsymbol{\theta}, \boldsymbol{\psi}_1) = \prod_{i=1}^{n} p(y_i \mid \eta_i(\boldsymbol{\theta}), \boldsymbol{\psi}_1)$$

- $\boldsymbol{y} = (y_1, \ldots, y_n)^T$ is the response vector, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)^T$ is the **latent field**,
- $\boldsymbol{\psi}_1$ is the hyperparameter vector of the exponential family distribution and
- $\eta_i(\boldsymbol{\theta})$ is the $\boldsymbol{i}$-th linear predictor that connects the data to the latent field.

Indeed each $\boldsymbol{\eta}_i$ can take a more general additive form:

$$\eta_i = \beta_0 + \sum_{j=1}^{J} \beta_k x_{ij} + \sum_{k=1}^{K} f^{(k)}(z_{ik})$$

Latent Gaussian model

Level 2: latent Gaussian field

- The second stage is formed by the **latent Gaussian field**, where we attribute a Gaussian distribution with mean $\boldsymbol{\mu}$ and precision matrix $Q(\boldsymbol{\psi}_2)$ to the latent field $\boldsymbol{\theta}$ conditioned on the hyperparameters $\boldsymbol{\psi}_2$, that is:

$$\boldsymbol{\theta} \mid \boldsymbol{\psi}_2 \sim \mathcal{N}(\mathbf{0}, Q^{-1}(\boldsymbol{\psi}_2))$$

- If we can assume conditional independence in $\boldsymbol{\theta}$, then this latent field is a **Gaussian Markov Random Field (GMRF)**.

Level 3: hyperparameters

- Finally, the third stage is formed by the **prior distribution** assigned to the hyperparameters:

$$\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) \sim p(\boldsymbol{\psi})$$

GLMM for Germinated seeds

- Breslow and Clayton (1993) present a dataset where they account for the proportion of seeds that germinated on each of 21 plates, arranged according to a 2 × 2 factorial layout by seed and type of root extract.

The variables are:

- **r**: number of germinated seeds per plate
- **n**: total number of seeds per plate
- **x1**: seed type (0: seed *O. aegyptiaco* 75; 1: seed *O. aegyptiaco* 73)
- **x2**: root extract (0: bean; 1: cucumber)
- **plate**: plate indicator. Dataset available in the **INLA** package.

r	n	x1	x2	plate
10	39	0	0	1
23	62	0	0	2
23	81	0	0	3
26	51	0	0	4
17	39	0	0	5
5	6	0	1	6

Example: mixed-effects model

- We assume the counts follow a conditionally independent **Binomial likelihood**:

$$y_i \mid \pi_i \sim \text{Binomial}(n_i, \pi_i), \quad i = 1, \ldots, 21$$

- We include **linear effects** for covariates $x1_i$ and $x2_i$ for each observation, and a **random effect** at the plate level, b_i:

$$\begin{aligned}
\eta_i &= \text{logit}(\pi_i) = \beta_0 + \beta_1 x1_i + \beta_2 x2_i + b_i \\
\beta_j &\sim \mathcal{N}(0, \tau_\beta^{-1}), \quad \tau_\beta \text{ known}, \; j = 0, 1, 2 \\
b_i &\sim \mathcal{N}(0, \tau_b^{-1})
\end{aligned}$$

So, in this case $\boldsymbol{\theta} = (\beta_0, \beta_1, \beta_2, b_1, \ldots, b_{21})$. A **Gaussian prior** is assigned for each element of the **latent field**, so that $\boldsymbol{\theta} \mid \boldsymbol{\psi}$ is **Gaussian distributed**.

- To assign the prior for $\boldsymbol{\psi} = (\tau_b)$:

$$\log(\tau_b) \sim \text{logGamma}(1, 5 \cdot 10^{-5})$$

Mixed Models for Measurement Agreement

- A study investigates agreement between devices measuring **respiratory rates** in **COPD patients** across **11 activities**, comparing a **chest-band device** to a **gold standard device** (Oxycon mobile).

- The dataset includes 21 subjects performing activities such as sitting, walking, and climbing stairs. The variables are:
  - **y**: respiratory rate (breaths per minute).
  - **device**: measurement device (**oxicon**, **chest_band**).
  - **replicate**: replicate measurements within each activity/subject.
  - **act**: activity type (11 levels, e.g., sitting, climbing stairs).

subj	y	replicate	act	device
1	38.19294	1	Sitting	oxicon
1	40.65189	2	Sitting	oxicon
1	36.00310	3	Sitting	oxicon

Gaussian Markov Random Fields (GMRFs)

- A GMRF is a random vector following a **multivariate normal distribution** with Markov properties.

$$\boldsymbol{i \neq j,\ \theta_i \mid \theta_{ij},}$$

  being $\boldsymbol{-ij}$ all elements other than $\boldsymbol{i}$ and $\boldsymbol{j}$.

- Rue et al. (2009) showed how conditional independence properties are encoded in the precision matrix, and how this can be exploited to improve computation involving these matrices.

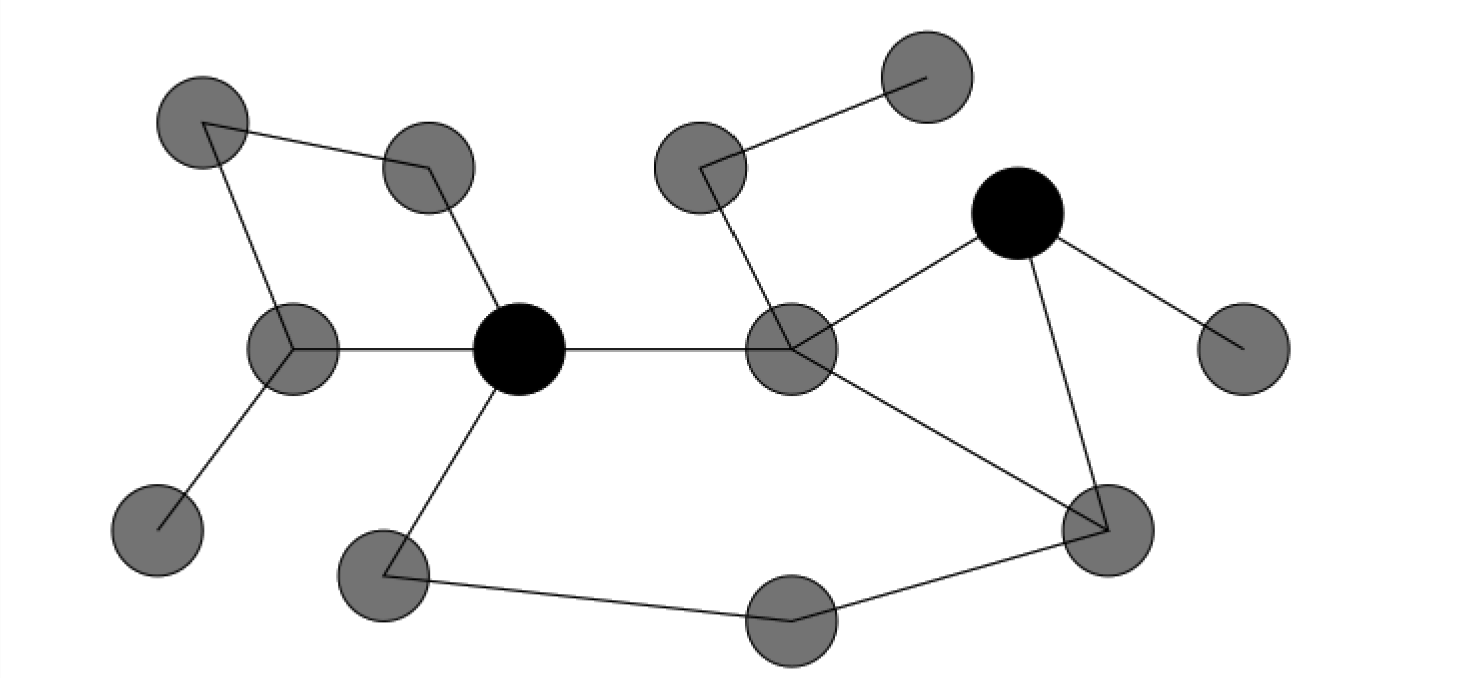
$$\boldsymbol{i \neq j,\ \theta_i \perp \theta_j \mid \theta_{ij},}$$

$$\theta_i \perp \theta_j \mid \boldsymbol{\theta}_{ij} \leftrightarrow \boldsymbol{Q}_{ij} = 0$$

- This Markov assumption in the GMRF results in a **sparse precision matrix**. This sparseness aids extremely fast computation.

The pairwise Markov property

The two black nodes are conditionally independent given the gray nodes



Example: precision matrix in AR1

Covariance matrix $(\boldsymbol{\Sigma})$

0.8730	0.6957	0.5201	0.3460	0.1728
0.6957	1.3931	1.0417	0.6929	0.3460
0.5201	1.0417	1.5659	1.0417	0.5201
0.3460	0.6929	1.0417	1.3931	0.6957
0.1728	0.3460	0.5201	0.6957	0.8730

Precision matrix $(\boldsymbol{Q})$

1.9025	-0.9500	0.0000	0.0000	0.0000
-0.9500	1.9025	-0.9500	0.0000	0.0000
0.0000	-0.9500	1.9025	-0.9500	0.0000
0.0000	0.0000	-0.9500	1.9025	-0.9500
0.0000	0.0000	0.0000	-0.9500	1.9025

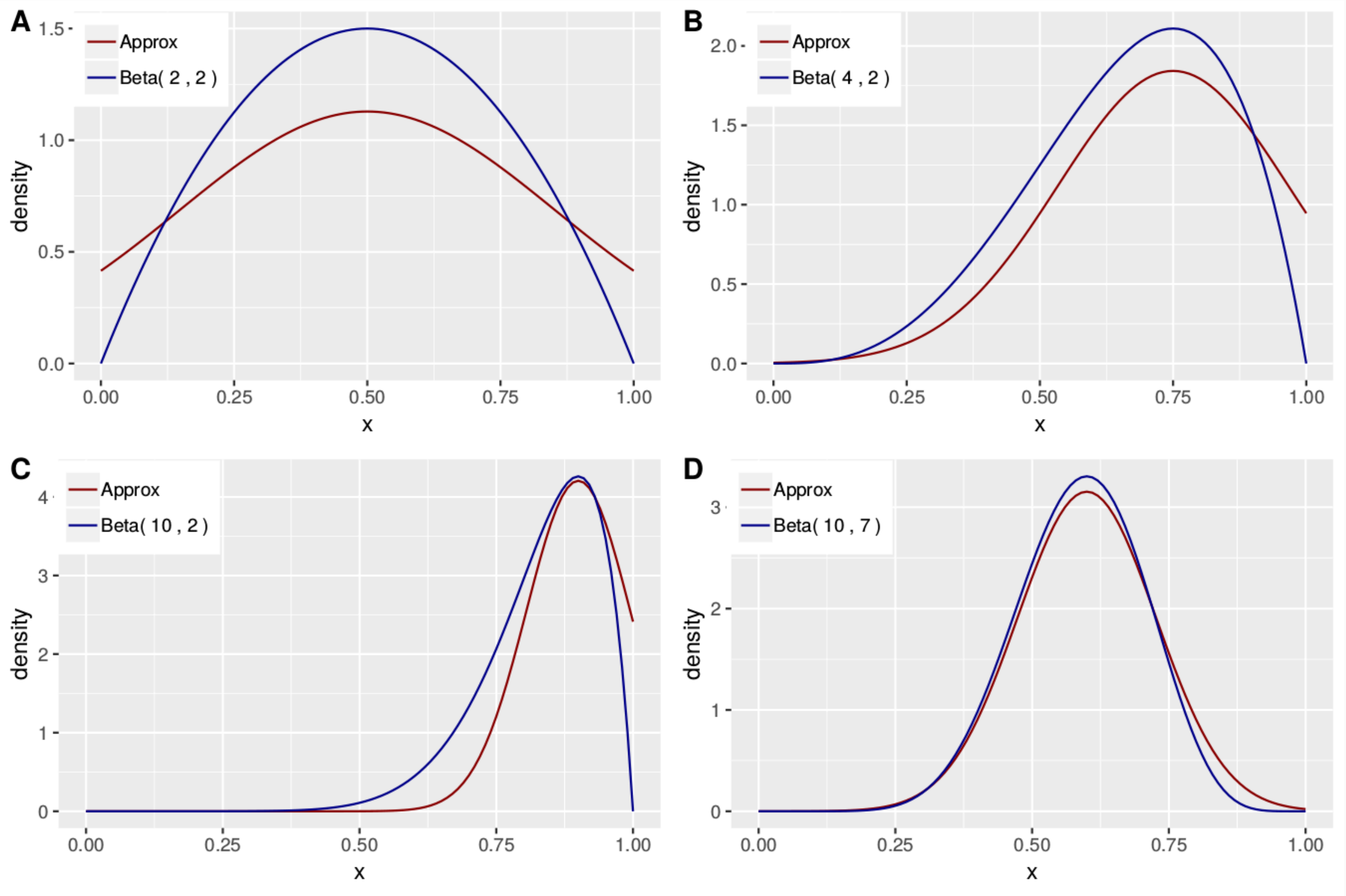
Laplace approximations

- **The Laplace approximation** is used to estimate any distribution $p(\theta)$ with a normal distribution.
- It uses the first three terms (quadratic function) **Taylor series expansion** around the mode θ^* of a function to approximate its log.
- Using the approximation, $p(\theta)$ can be approximated using a **Gaussian distribution** with mean the mode θ^* and variance the Fisher information, $\frac{-1}{\frac{d^2 \log(p(\theta^*))}{d\theta^2}}$.

$$p(\theta) \approx \mathcal{N}\left(\theta^*, \frac{-1}{\frac{d^2 \log(p(\theta^*))}{d\theta^2}}\right)$$

- It can be easily expanded to the multivariate case.

Example: approximating the beta distribution



3. Putting all the pieces together: INLA

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Marginals of the latent field and hyperparameters

$$p(\theta_i \mid \boldsymbol{y}) = \int p(\theta_i \mid \boldsymbol{\psi}, \boldsymbol{y}) \cdot p(\boldsymbol{\psi} \mid \boldsymbol{y}) d\boldsymbol{\psi} \ , \ i = 1, \ldots, n$$

$$p(\psi_j \mid \boldsymbol{y}) = \int p(\boldsymbol{\psi} \mid \boldsymbol{y}) d\boldsymbol{\psi}_{-j} \, , \ j = 1, \ldots, m$$

- As a result, we have to numerically approximate:

  1. The **joint posterior distribution of the hyperparmeters** $p(\boldsymbol{\psi} \mid \boldsymbol{y})$, needed to calculate the posterior hyperparameters marginals $p(\psi_j \mid \boldsymbol{y})$, and the posterior marginals of the latent field $p(\theta_i \mid \boldsymbol{y})$.

  2. The **marginals of the full conditional distribution** of $\boldsymbol{\theta}$, $p(\theta_i \mid \boldsymbol{\psi}, \boldsymbol{y})$, needed to compute the posterior marginals of the latent field $p(\theta_i \mid \boldsymbol{y})$.

Hyperparameters: joint posterior distribution

- The approximation is computed as follows

$$\tilde{p}(\boldsymbol{\psi} \mid \boldsymbol{y}) := \left.\frac{p(\boldsymbol{\theta}, \boldsymbol{\psi}|\boldsymbol{y})}{p_G(\boldsymbol{\theta} \mid \boldsymbol{\psi}, \boldsymbol{y})}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*(\boldsymbol{\psi})},$$

- where:
  - $p_G(\boldsymbol{\theta} \mid \boldsymbol{\psi}, \boldsymbol{y})$ is the Gaussian approximation to the full conditional of $\boldsymbol{\theta}$, $p(\boldsymbol{\theta} \mid \boldsymbol{\psi}, \boldsymbol{y})$ given by the **Laplace method**, and,
  - $\boldsymbol{\theta}^*(\boldsymbol{\psi})$ is the mode of the full conditional of $\boldsymbol{\theta}$ for a given $\boldsymbol{\psi}$.
  - Note: this approximation is exact if $p(\boldsymbol{\theta} \mid \boldsymbol{y}, \boldsymbol{\psi})$ is Gaussian.

Full posterior marginals for the latent field

Gaussian approximation

- Conditional posterior distributions $p(\theta_i \mid \boldsymbol{\psi}, \boldsymbol{y})$ are approximated directly as the marginals from $p_G(\boldsymbol{\theta} \mid \boldsymbol{\psi}, \boldsymbol{y})$.
- It is the **fastest to compute** but with possible **errors** in the location of the posterior mean.

Laplace approximation

- The vector $\boldsymbol{\theta}$ is rewriten as $\boldsymbol{\theta} = (\theta_i, \boldsymbol{\theta}_{-i})$, and the Laplace approximation is used for each element of the latent field

$$\tilde{p}(\theta_i \mid \boldsymbol{\psi}, \boldsymbol{y}) := \left. \frac{p(\boldsymbol{\theta}, \boldsymbol{\psi} | \boldsymbol{y})}{p_{LG}(\boldsymbol{\theta}_{-i} \mid \theta_i, \boldsymbol{\psi}, \boldsymbol{y})} \right|_{\boldsymbol{\theta}_{-i} = \boldsymbol{\theta}^*_{-i}(\theta_i, \boldsymbol{\psi})},$$

  where $p_{LG}(\boldsymbol{\theta}_{-i} \mid \theta_i, \boldsymbol{\psi}, \boldsymbol{y})$ is the Laplace Gaussian approximation to $p(\boldsymbol{\theta}_{-i} \mid \theta_i, \boldsymbol{\psi}, \boldsymbol{y})$ and $\boldsymbol{\theta}_{-i}$ is its mode.

- The **most accurate** but **time consuming**.

Full posterior marginals for the latent field

Simplified Laplace approximation

- Based on a Taylor's series expansion of third order.
- **Fast to compute** and usually **accurate enough**.

Final step: integration

- The INLA algorithm uses Newton-like methods to explore the joint posterior distribution for the hyperparameters $\tilde{\boldsymbol{p}}(\boldsymbol{\psi}|\boldsymbol{y})$ to find **suitable points** for the numerical integration.

- Posterior marginals for the **latent variables** $\tilde{\boldsymbol{p}}(\theta_i|\boldsymbol{y})$ are then computed via numerical integration as:

$$\tilde{p}(\theta_i \mid \boldsymbol{y}) = \int \tilde{p}(\theta_i \mid \boldsymbol{\psi}, \boldsymbol{y})\tilde{p}(\boldsymbol{\psi} \mid \boldsymbol{y})\mathrm{d}\boldsymbol{\psi} \approx \sum_{k=1}^{K} \tilde{p}(\theta_i \mid \boldsymbol{\psi}^{(k)}, \boldsymbol{y})\tilde{p}(\boldsymbol{\psi}^{(k)} \mid \boldsymbol{y})\Delta_k$$

- Posterior marginals for the **hyperparameters** ψ_j are approximated using the integrations points previously constructed.

The new avenue in INLA

- Posterior means of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ might be inaccurate based on the Gaussian assumption of the conditional posterior.

- **Variational Bayes correction** to Gaussian means by Van Niekerk and Rue (2021) can be used to efficiently calculate an improved mean for the marginal posteriors of the linear predictors, by improving the posterior means of the latent field.

- It is improved based on the following variational function:

$$E_{q(\boldsymbol{\theta}|\boldsymbol{y})}[-\log(p(\boldsymbol{\theta} \mid \boldsymbol{y}))] + KLD[q(\boldsymbol{\theta} \mid \boldsymbol{y}) \mid\mid p(\boldsymbol{\theta})]] \, ,$$

1. $q(.)$ a member of the variational class,

2. $p(\boldsymbol{\theta} \mid \boldsymbol{y})$ the posterior distribution of the latent field, and

3. $p(\boldsymbol{\theta})$ the posterior distribution of the latent field.

4. R-INLA

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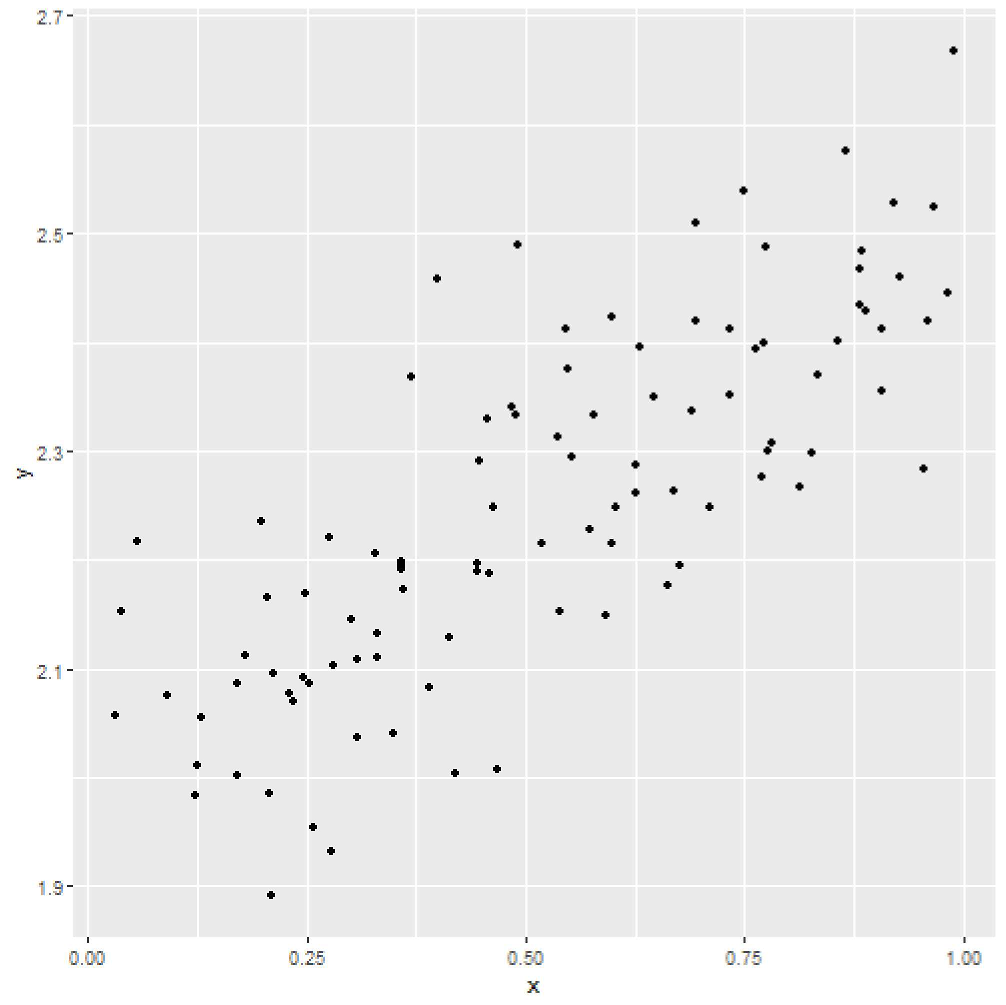
Data

y	x	id
2.109177	0.3077661	1
1.954976	0.2576725	2
2.294048	0.5523224	3
2.217938	0.0563832	4
2.007082	0.4685493	5
2.339932	0.4837707	6



Fitting the model using `R-INLA`

Defining the formula

```
formula <- y ~ 1 + x # 1 is refered to the intercept term
formula <- y ~ 1 + f(x, model = "linear")
```

Calling `R-INLA`

```
model1 <- inla(formula,
               family       = 'gaussian',
               data         = data,
               control.inla = list(strategy = 'simplified.laplace'))
```

Posterior distributions

Posterior distribution of the parameters

	mean	sd	0.025quant	0.5quant	0.975quant	mode	kld
(Intercept)	1.9946	0.0225	1.9503	1.9946	2.0389	1.9946	0
x	0.4935	0.0388	0.4174	0.4935	0.5697	0.4935	0

Posterior distributions of the hyperparameters

	mean	sd	0.025quant	0.5quant	0.975quant	mode
Precision for the Gaussian observations	99.4863	14.0682	73.8685	98.8272	128.9193	97.5018

Families

```r
inla.list.models(section = "likelihood")
```

```
## Section [likelihood]
##       0binomial                     New 0-inflated Binomial
##       0binomialS                    New 0-inflated Binomial Swap
##       0poisson                      New 0-inflated Poisson
##       0poissonS                     New 0-inflated Poisson Swap
##       agaussian                     The aggregated Gaussian likelihoood
##       bcgaussian                    The Box-Cox Gaussian likelihoood
##       bell                          The Bell likelihood
##       beta                          The Beta likelihood
##       betabinomial                  The Beta-Binomial likelihood
##       betabinomialna                The Beta-Binomial Normal approximation likelihood
##       bgev                          The blended Generalized Extreme Value likelihood
##       binomial                      The Binomial likelihood
##       binomialmix                   Binomial mixture
##       cbinomial                     The clustered Binomial likelihood
##       cennbinomial2                 The CenNegBinomial2 likelihood (similar to cenpoisson2)
```

Latent effects

```r
inla.list.models(section = "latent")
```

```
## Section [latent]
##      2diid                          (This model is obsolute)
##      ar                             Auto-regressive model of order p (AR(p))
##      ar1                            Auto-regressive model of order 1 (AR(1))
##      ar1c                           Auto-regressive model of order 1 w/covariates
##      besag                          The Besag area model (CAR-model)
##      besag2                         The shared Besag model
##      besagproper                    A proper version of the Besag model
##      besagproper2                   An alternative proper version of the Besag model
##      bym                            The BYM-model (Besag-York-Mollier model)
##      bym2                           The BYM-model with the PC priors
##      cgeneric                       Generic latent model specified using C
##      clinear                        Constrained linear effect
##      copy                           Create a copy of a model component
##      crw2                           Exact solution to the random walk of order 2
##      dmatern                        Dense Matern field
```

Hyperpriors

```
inla.list.models(section = "prior")
```

```
## Section [prior]
##       betacorrelation               Beta prior for the correlation
##       dirichlet                     Dirichlet prior
##       expression:                   A generic prior defined using expressions
##       flat                          A constant prior
##       gamma                         Gamma prior
##       gaussian                      Gaussian prior
##       invalid                       Void prior
##       jeffreystdf                   Jeffreys prior for the doc
##       laplace                       Laplace prior
##       linksnintercept               Skew-normal-link intercept-prior
##       logflat                       A constant prior for log(theta)
##       loggamma                      Log-Gamma prior
##       logiflat                      A constant prior for log(1/theta)
##       logitbeta                     Logit prior for a probability
##       logtgaussian                  Truncated Gaussian prior
```

5. Model Selection

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Model selection scores in `R-INLA`

- When use different covariates and random effects, we need some measures to select the best model:
  - **DIC**: deviance information criteria

$$DIC = 2 * E(D(\boldsymbol{\theta})) - D(E(\boldsymbol{\theta}))$$

  - **WAIC**: within-sample predictive score

$$WAIC = \sum_i var_{post}(log(p(y_i|\boldsymbol{\theta})))$$

  - **LCPO**: leave-one-out cross-validation score

$$CPO_i = p(y_i \mid y_{-i})$$

$$LCPO = -\overline{\log(CPO_i)}$$

6. Example

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Bayesian splines

- GLMM with independent random effect does not cover situations in which relationship between the response variable and the covariate is not linear.

- In INLA, we can do this by means of the **random walk** of order 1 and 2.

  - **First order Random Walk (RW1)**

$$\Delta x_j = x_j - x_{j+1} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{\tau}\right)$$

  - **Second order Random Walk (RW2)**

$$\Delta^2 x_i = x_i - 2x_{i+1} + x_{i+2} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{\tau}\right)$$

  - The prior for the hyperparameter τ is reparametrized in terms of their logarithm:

$$\log(\tau) \sim \text{logGamma}(1, 5 \cdot 10^{-5})\ .$$

Smoothing time series of binomial data

- The number of **occurrences of rainfall** over 1 mm in the Tokyo area for each calendar year during two years (1983-84) are registered.

- It is of interest to estimate the underlying probability π_t of rainfall for calendar day t which is, a priori, assumed to change gradually over time.

- For each day $t = 1, \dots, 366$ of the year we have the number of days that rained y_t and the number of days that were observed n_t.

Dataset

y	n	time
0	2	1
0	2	2
1	2	3
1	2	4
0	2	5
1	2	6

Smoothing time series of binomial data. The model

- A conditionally independent **binomial likelihood** function:

$$y_t \mid \pi_t \sim \text{Binomial}(n, \pi_t),\ t = 1, \dots, 366$$

  with (usual) logit link function $\pi_t = \frac{\exp(\eta_t)}{1+\exp(\eta_t)}$.

- We assume that (instead of a linear predictor), $\eta_t = f_t$, where f_t follows a circular **random walk** of second order (RW2) model with precision τ:

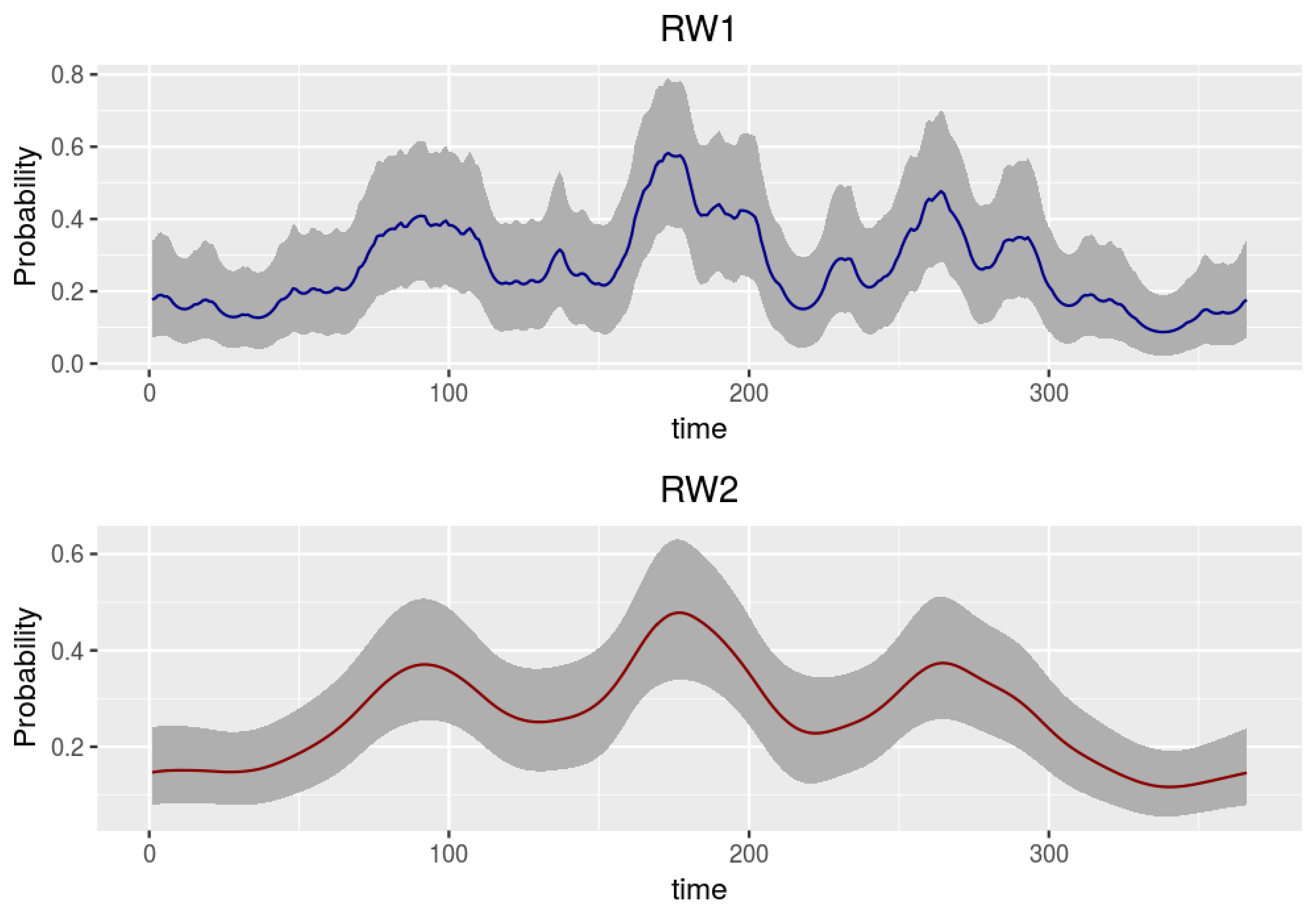
$$\Delta^2 f_i = f_i - 2f_{i+1} + f_{i+2} \sim \mathcal{N}(0, \tau^{-1}).$$

  The fact that we use a circular model here means that in this case f_1 is a neighbor of f_{366}. So, in this case $\boldsymbol{\theta} = (f_1, \dots, f_{366})$ and again $\boldsymbol{\theta}|\boldsymbol{\psi}$ is **Gaussian distributed**.

- To assing the prior of $\boldsymbol{\psi} = (\tau)$:

$$\log(\tau) \sim \text{logGamma}(1, 5 \cdot 10^{-5})\ .$$

Posterior distribution of the probability



7. References

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Part I: Bayesian inference using the integrated nested Laplace approximation (INLA)

Joaquín Martínez-Minaya, November 10, 2025
VAlencia BAyesian Research Group
Statistical Modeling Ecology Group
Grupo de Ingeniería Estadística Multivariante

jmarmin@eio.upv.es

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