POLITECNICO DI MILANO

STEN Doctoral Courses

Advanced Measurement Methods

Frequency Analysis

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Frequency Analysis

- **Introduction**
- **Dynamic characteristics of linear systems**
- **Fourier Transform**
- **Discrete Fourier transform (DFT)**
- **Fast Fourier transform (FFT)**
- **Power spectrum**
- **Autocorrelation**
- **Conclusions**

Constant parameter Linear Systems

An ideal system has constant parameters and is linear

A system has ***constant parameters*** if all fundamental properties of the system are invariant with respect to time.

The constant parameter assumption is reasonably valid for many measuring instrument

A system is ***linear*** if the response characteristics are additive and homogeneous

additive property $$f(x_1 + x_2) = f(x_1) + f(x_2)$$

homogeneous property $$f(cx) = cf(x)$$

The linearity assumption for real instruments is somewhat more critical, but it may be assumed, at least over some limited range of inputs.

Dynamic characteristics

The dynamic characteristics of a constant parameter linear system can be described by a function ***h(t)***, which is defined as the output of the system to a unit impulse applied a time τ before.

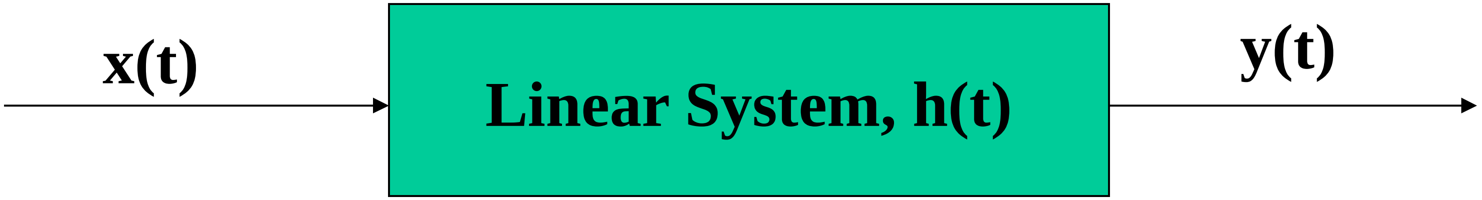
For any arbitrary input ***x(t)***, the system output ***y(t)*** is given by the ***convolution integral***

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

The value of the output is given as a weighted linear (infinite) sum over the entire history of the input.

To be ***physically realizable***, the system should respond only to past impulse and thus

$$h(\tau) = 0 \quad \text{for} \;\; \tau < 0$$



Dynamic characteristics

A constant parameter linear system can also be characterized by a transfer function ***H(s)*** defined as the ***Laplace transform*** of ***h(t).***

$$H(s) = \int_0^{\infty} h(\tau)\exp(-s\tau)\,d\tau \qquad s = \alpha + j\omega$$

A constant parameter linear system has the following properties:

Frequency preservation

For any sinusoidal input ***x(t)*** the output ***y(t)*** must also be sinusoidal with the same frequency as ***x(t)***

Stability

A system is stable if h(t) is absolutely integrable $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

Dynamic characteristics

If a constant parameter linear system is ***stable***, then the dynamic characteristics can be described by a ***frequency response function H(ω)***, which is defined as the ***Fourier transform*** of ***h(t).***

$$H(\omega) = \int_0^{\infty} h(\tau)\exp(-j\omega\tau)\,d\tau \qquad H(\omega) = |H(\omega)|\ e^{-j\phi(\omega)}$$

The frequency response function is a special case of the transfer function where $\alpha = 0$.

For physically realizable and stable systems, the frequency response function may replace the transfer function with no loss of useful information.

X(ω) = Fourier transform of x(t)

Y(ω) = Fourier transform of y(t)

$$Y(\omega) = H(\omega)\,X(\omega)$$

*The **convolution integral** reduces to the algebraic multiplication!!!*

Fourier transform

If one system described by $\boldsymbol{H_1(\omega)}$ is followed by a second system described by $\boldsymbol{H_2(\omega)}$, and there is no feedback between the two, then the overall system may be described by $\boldsymbol{H(\omega)}$

$$H(\omega) = H_1(\omega)\, H_2(\omega)$$

$$|H(\omega)| = |H_1(\omega)|\,|H_2(\omega)|$$

$$\Phi(\omega) = \Phi_1(\omega) + \Phi_2(\omega)$$

On cascading two system the ***gain factors*** *multiply and the* ***phase factors*** *add.*

Fourier transform

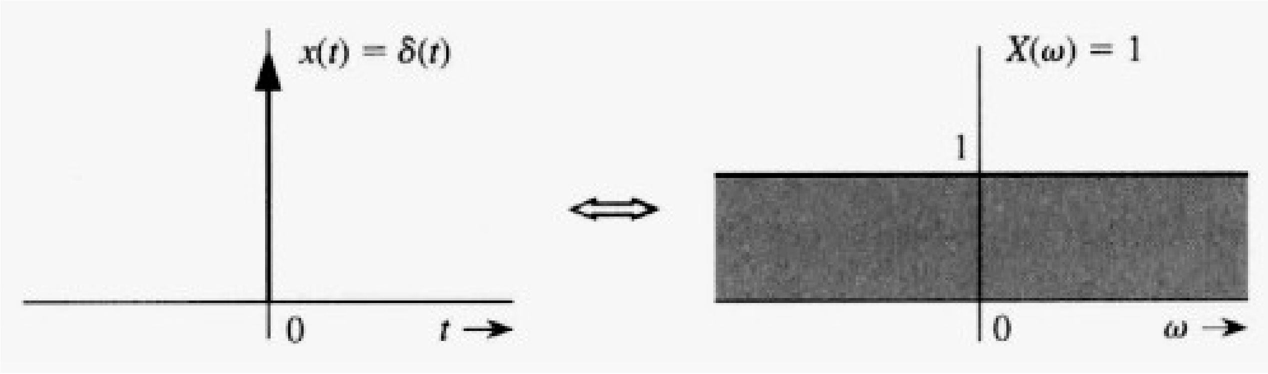
The *Fourier transform is a complex number*

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = F[y(t)] = A(\omega) + jB(\omega)$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t}\, d\omega = F^{-1}[Y(\omega)]$$ *inverse Fourier transform*

y(t) e Y(ω) are called: ***forward and inverse Fourier Transform***

Fourier transform of a unit impulse $$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = 1$$



Fourier transform

Differentiation

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp(j\omega t)d\omega$$

$$\frac{dy(t)}{dt} = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp(j\omega t)d\omega = F^{-1}[j\omega Y(\omega)] \longrightarrow F\left[\frac{dy(t)}{dt}\right] = j\omega Y(\omega)$$

differentiation = multiplication (x $j\omega$)

Integration: *Inverse operation*

$$\int_{-\infty}^{\infty} y(t)\,dt \leftrightarrow \frac{1}{j\omega} Y(\omega)$$

integration = division (: $j\omega$)

Second order systems

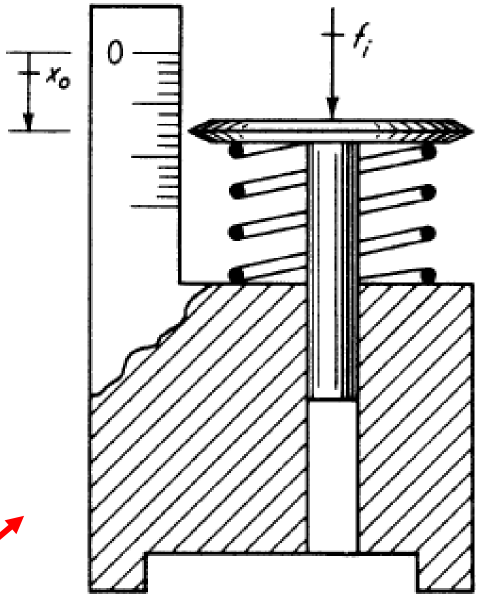
$$a_2 \frac{d^2 q_o}{dt^2} + a_1 \frac{dq_o}{dt} + a_0 q_o = b_0 q_i$$

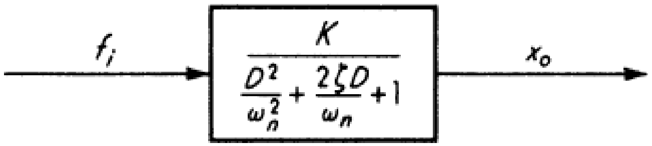
$$K = \frac{b_0}{a_0}$$ Static gain or sensitivity

$$\omega_n = \sqrt{\frac{a_0}{a_2}}$$ Undumped natural frequency

$$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$$ Damping ratio

Simple structure consisting of a mass, a spring and a damping (dashpot)





Second order systems

$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+k\,y=F$$

Equation of motion when a force F(t) is applied to a mass m and the output is the resulting displacement y(t)

$$K=\frac{1}{k}$$

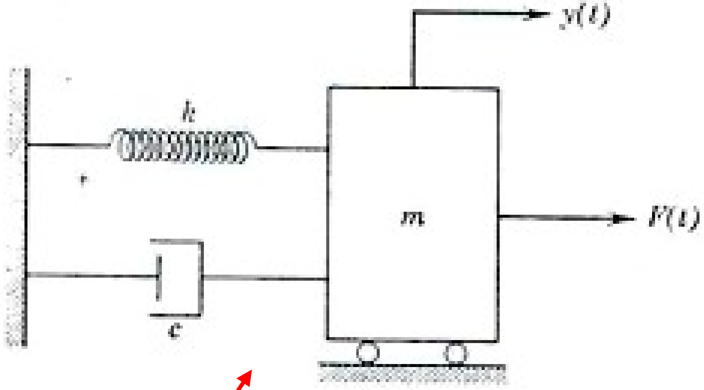
Static gain or sensitivity

$$\omega_n=\sqrt{\frac{k}{m}}$$

Undumped natural frequency

$$\zeta=\frac{c}{2\sqrt{k\,m}}$$

Damping ratio



Mechanical system consisting of a mass, a spring and a dashpot.

Second order systems

Taking the Fourier transform of both sides of the equation of motion, the following frequency response or transfer function is obtained:

$$H(\omega) = \frac{K}{\sqrt{1-\left(\dfrac{\omega}{\omega_n}\right)^2 + j2\varsigma\dfrac{\omega}{\omega_n}}}$$

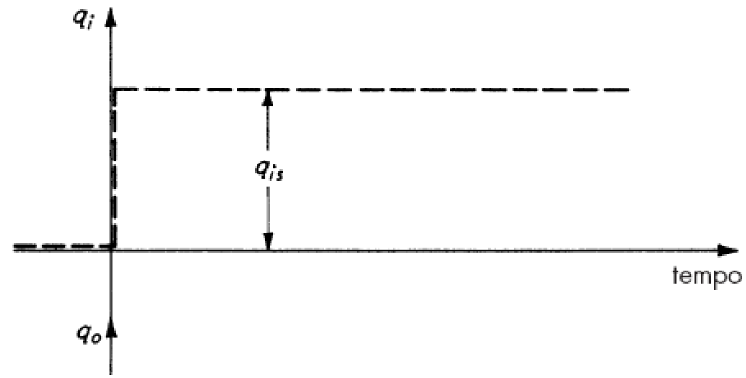
Which gives the following ***gain factor***:

$$|H(\omega)| = \frac{K}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]^2 + \left[j2\varsigma\dfrac{\omega}{\omega_n}\right]^2}}$$

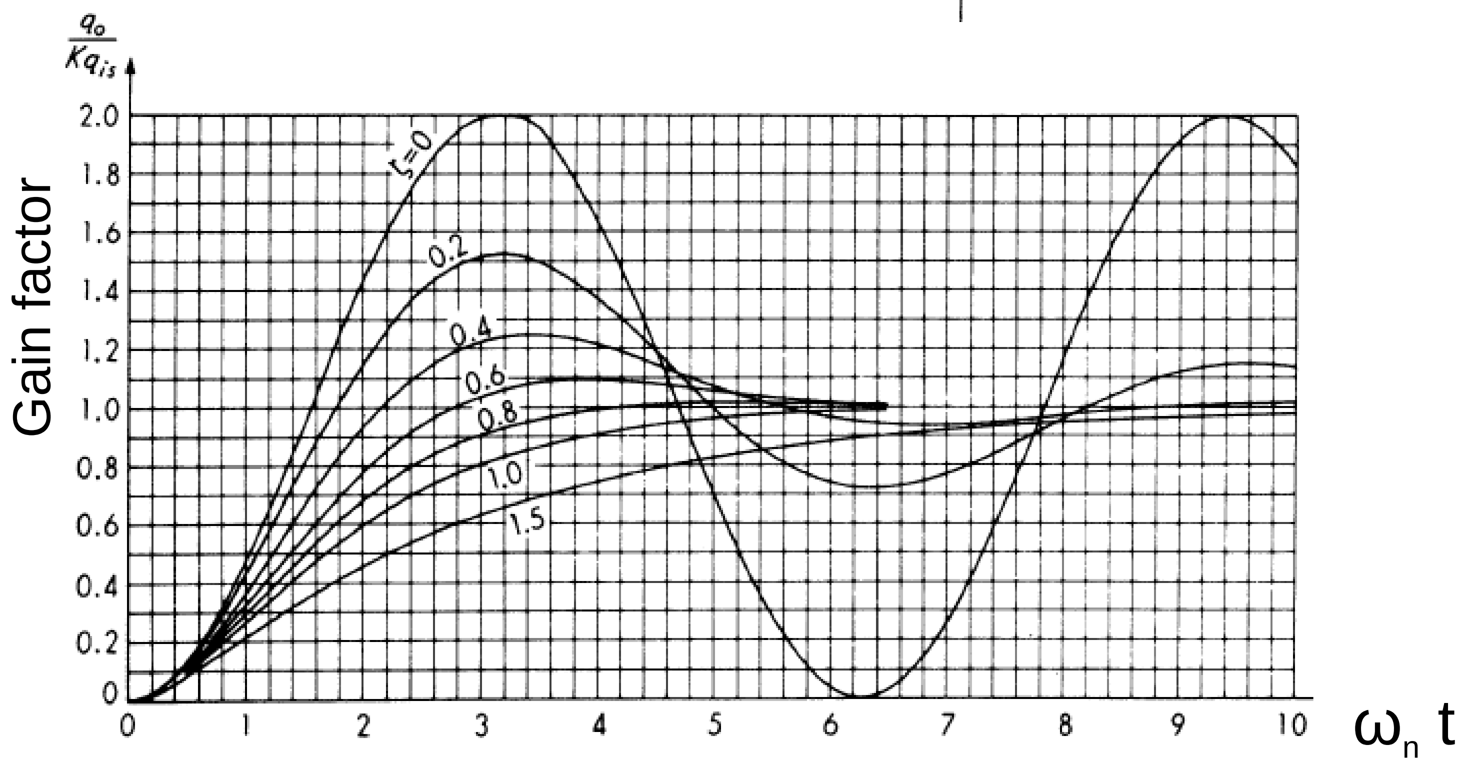
and ***phase factor***:

$$\Phi(\omega) = \tan^{-1}\left[\frac{2\varsigma\omega/\omega_n}{1-(\omega/\omega_n)^2}\right]$$

Second order systems



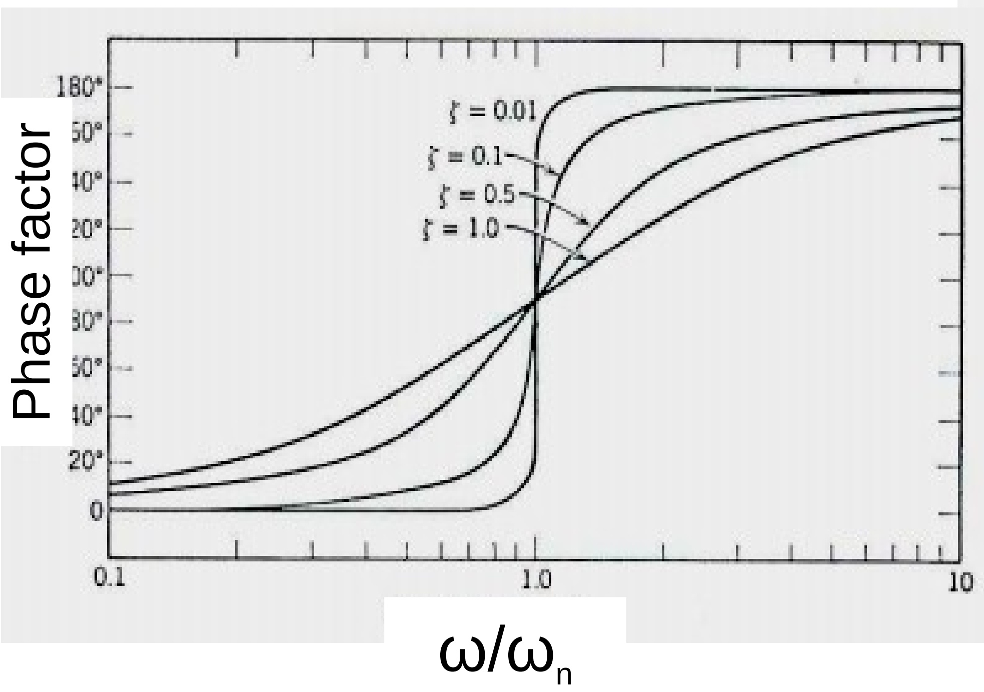
Frequency response to a step input

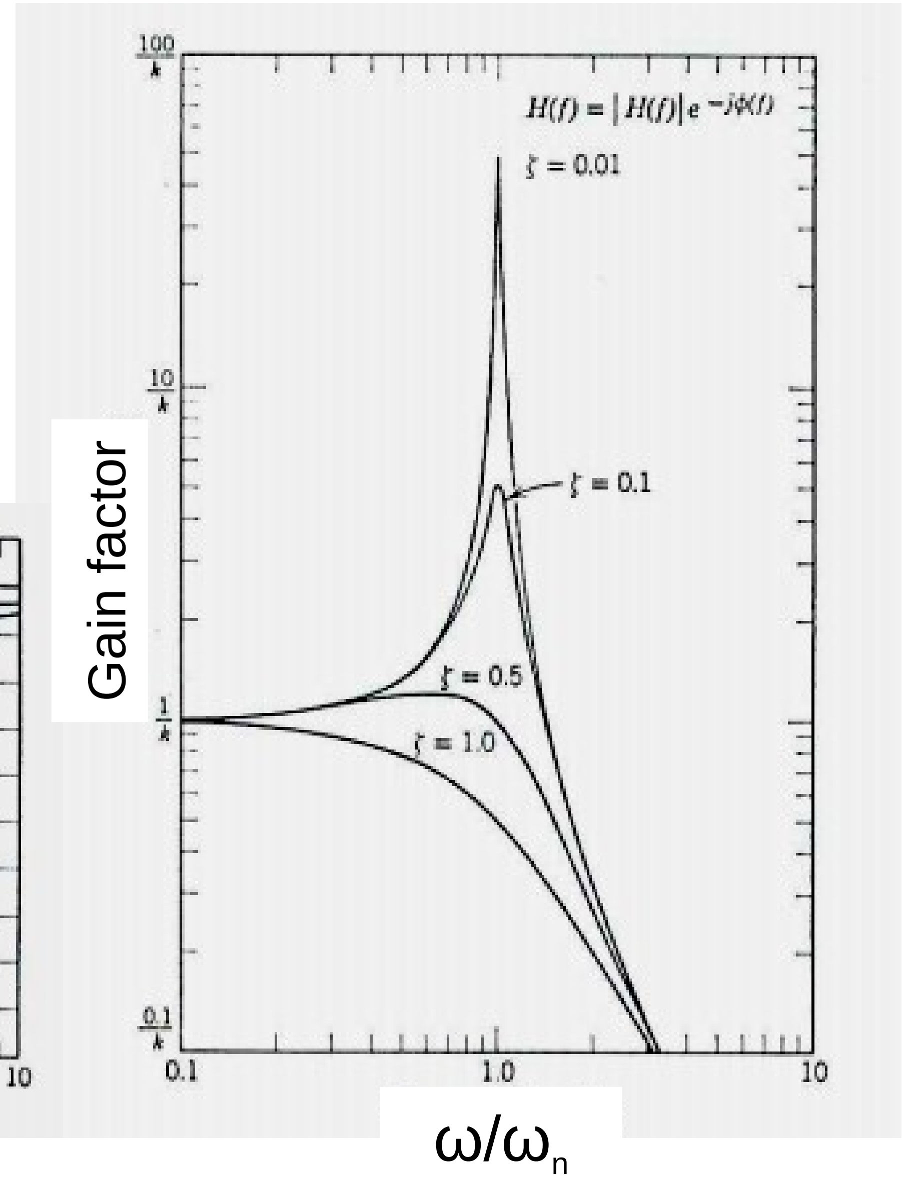


Second order systems

Frequency response to a sinusoidal input at frequency ω.

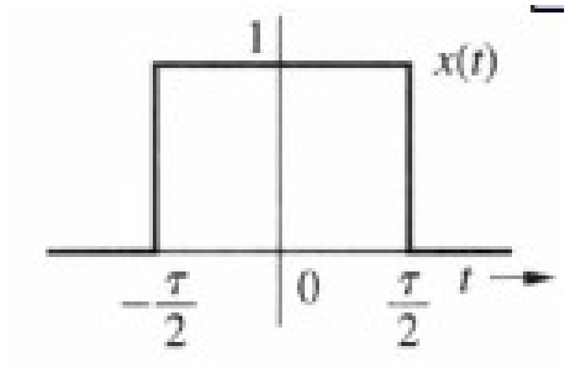
Finding the output for a sinusoidal input we can determine the frequency response from the amplitude change and phase shift between the output and input.

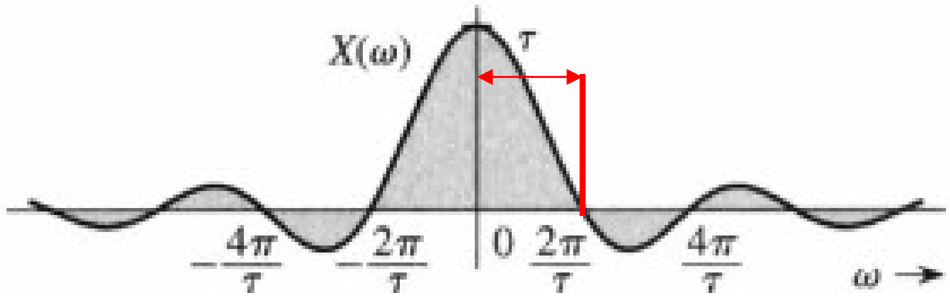




Unit impulse

Fourier Transform of a unit rectangular window of duration τ





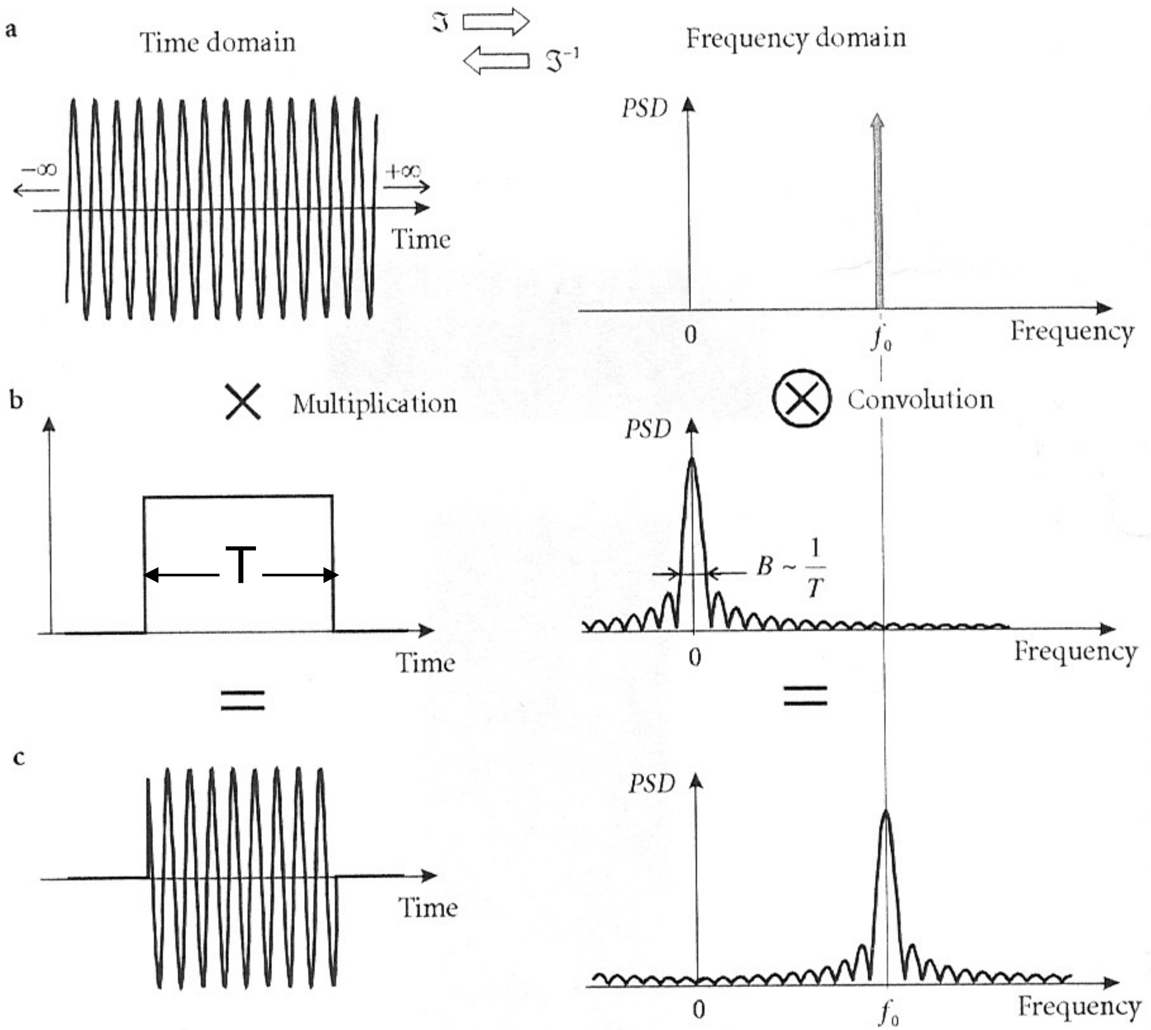
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-\tau/2}^{\tau/2} = \frac{\tau\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]}{j\frac{2\tau}{2}\omega} = \tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}}$$

Fourier transform of a rectangular window has a significant amplitude in a frequency range estending up to ω

$$\frac{\omega\tau}{2} = \pi \quad \Rightarrow \quad \omega = \frac{2\pi}{\tau}$$

Discrete Fourier Transform

Time domain ⟷ Frequency domain (Convolution)



A multiplication of two signals in the time domain is equivalent to a convolution in the frequency domain.

This explains spectral broadening (<u>leakage</u>) due to finite record lengths.

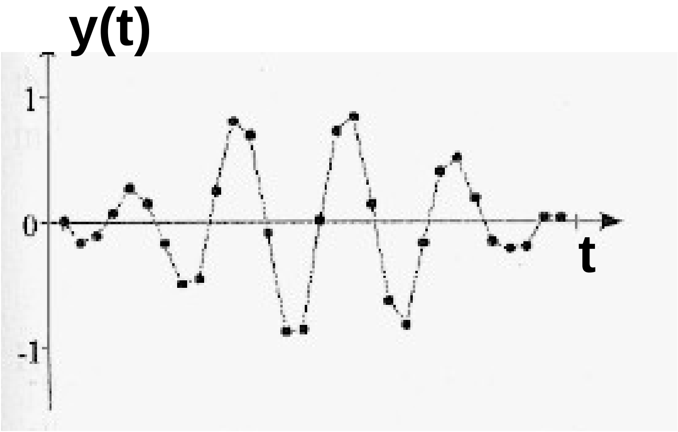
(a) Infinite sine function and related spectrum,

(b) rectangular function and related spectrum,

(c) finite sine function and related spectrum

Digital data processing

Sampling theorem (Shannon)



In practical applications data acquisition is limited to a finite time interval (0,T) and we define a finite-range Fourier transform

$$Y(\omega, T) = \int_0^T y(t)\, e^{-j\omega t}\, dt$$

and its discrete version

$$Y(\omega, T) = \Delta t \sum_0^{N-1} y_n\, e^{-j\omega\Delta t}$$

where $y_n = y(n\Delta t)$ *digital sampling of a continuous function*

$$f_s = \frac{1}{\Delta t} \geq 2 f_{max}$$ *sampling frequency*

$$f_{Ny} = \frac{1}{2\Delta t} = \frac{1}{2} f_s$$ ***Nyquist** cutoff frequency: maximum frequency to reconstruct the original signal*

DFT computation

DFT transforms N real numbers y_n (output of a measurement) in N complex numbers Y_k

$$Y_k = \frac{Y(\omega_k, T)}{\Delta t} = \sum_{0}^{N-1} y_n \exp\left[-j\frac{2\pi kn}{N}\right] \qquad k = 0,1,2,\ldots, N-1$$

Using the Eulero identity $$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$Y_k = F(y_n) = \sum_{n=0}^{N-1}\left[y_n \cos\left(\frac{2\pi kn}{N}\right) - jy_n \sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$y_n = F^{-1}(Y_k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[Y_k \cos\left(\frac{2\pi kn}{N}\right) + jY_k \sin\left(\frac{2\pi kn}{N}\right)\right]$$

Note that results are unique only out to k = N/2 since the Nyquist cutoff frequency occurs at this point.

Fast Fourier Transform (FFT)

DFT computation, using the integral definition, requires a large number of multipli-add operations ≈ **N^2**.

The introduction of algorithms for the fast computation of DFT sharply reduced the number of operation to **N logN** and increased the cost effectiveness of digital analysis.

These algorithms, usually referred as ***Fast Fourier Transform (FFT)*** procedures, are limited to digital records of length $N = 2^n$

Hence data sequences must either be truncated or have zeros added (*zero padding*) to obtain the required number of data points.

Example: $N = 2^{12} = 4096$

computation time of DFT $\approx N^2 \approx 16.8 \times 10^6$

computation time of FFT $\approx N \log N \approx 14.8 \times 10^3$

Spectral resolution

A commonly used technique with the FFT is that of ***zero padding***.

Without changing the spectral content of the signal, zero padding forces the FFT algorithm to estimate the spectrum at additional frequencies between zero and f_{max}, thus improving the resolution.

Consider a signal doubled in length by adding zeros. The transform becomes

$$X_k = \sum_{n=0}^{2N-1} x_n \exp\left(-j\frac{2\pi nk}{2N}\right) \qquad k = 0,1,\ldots,(2N-1)$$

Since $x_n = 0$ for n = N, N + 1, . . . , (2N −1), this can be written as

$$X_k = \sum_{n=0}^{N-1} x_n \exp\left(-j\frac{2\pi n(k/2)}{N}\right) \qquad k = 0,1,\ldots,(2N-1)$$

which is identical to the N-point transform for every other k value. However now X_k is also computed at intermediate k values and the interpolation of peak locations can be improved.

Power spectrum

The power spectral density function of a digitized data sets y_n is defined as

$$G_x(\omega) = \frac{2}{T}\left|Y(\omega, T)\right|^2 \qquad\qquad T = N\Delta t$$

And the discrete spectrum estimate becomes

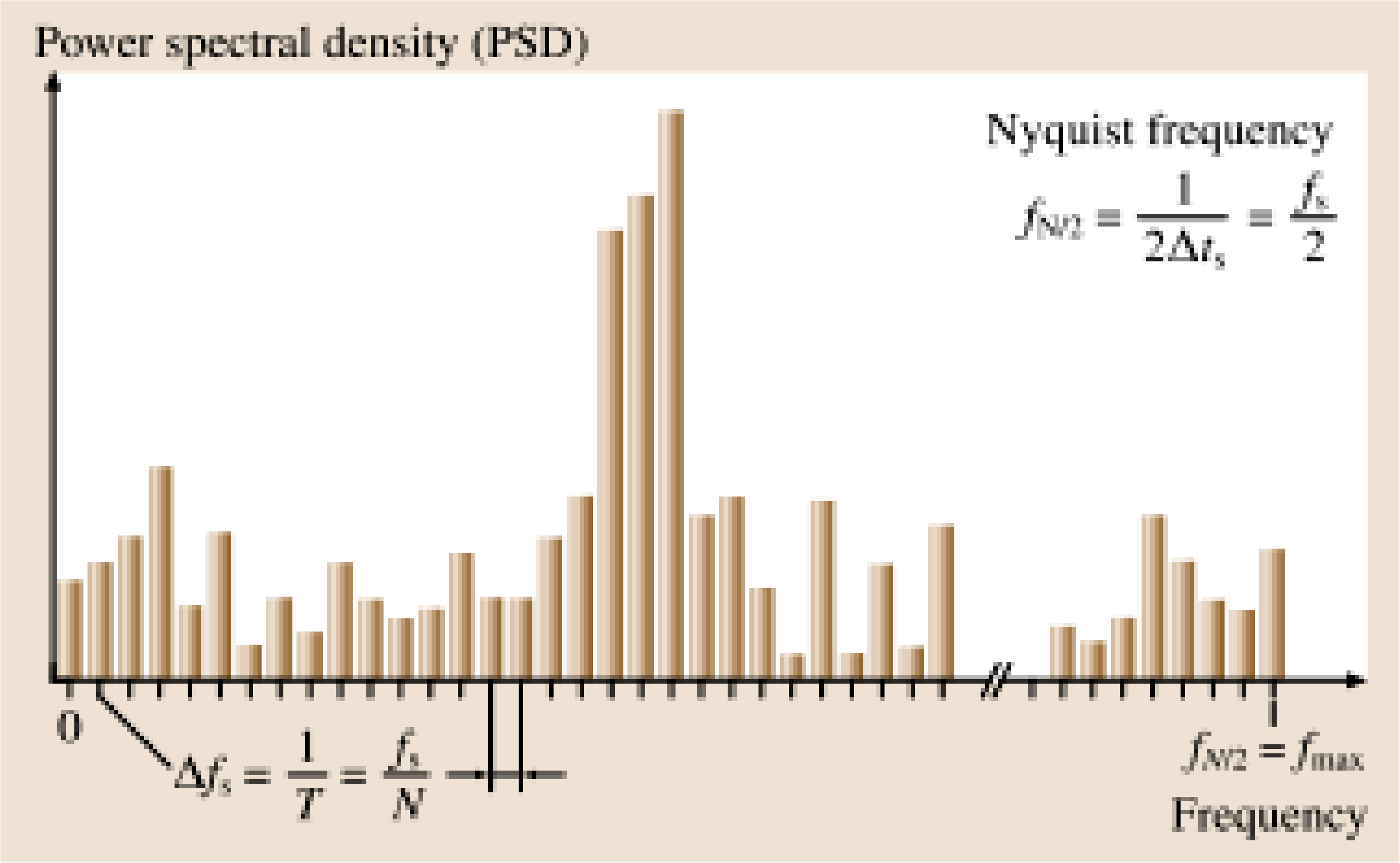
$$G_k = \frac{2}{N\Delta t}\left|Y(\omega_k, T)\right|^2 = \frac{2\Delta t}{N}\left|Y_k\right|^2$$

Parseval Theorem

$$\int_{-\infty}^{\infty}\left|y(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|Y(\omega)\right|^2 d\omega$$

Energy spectral density

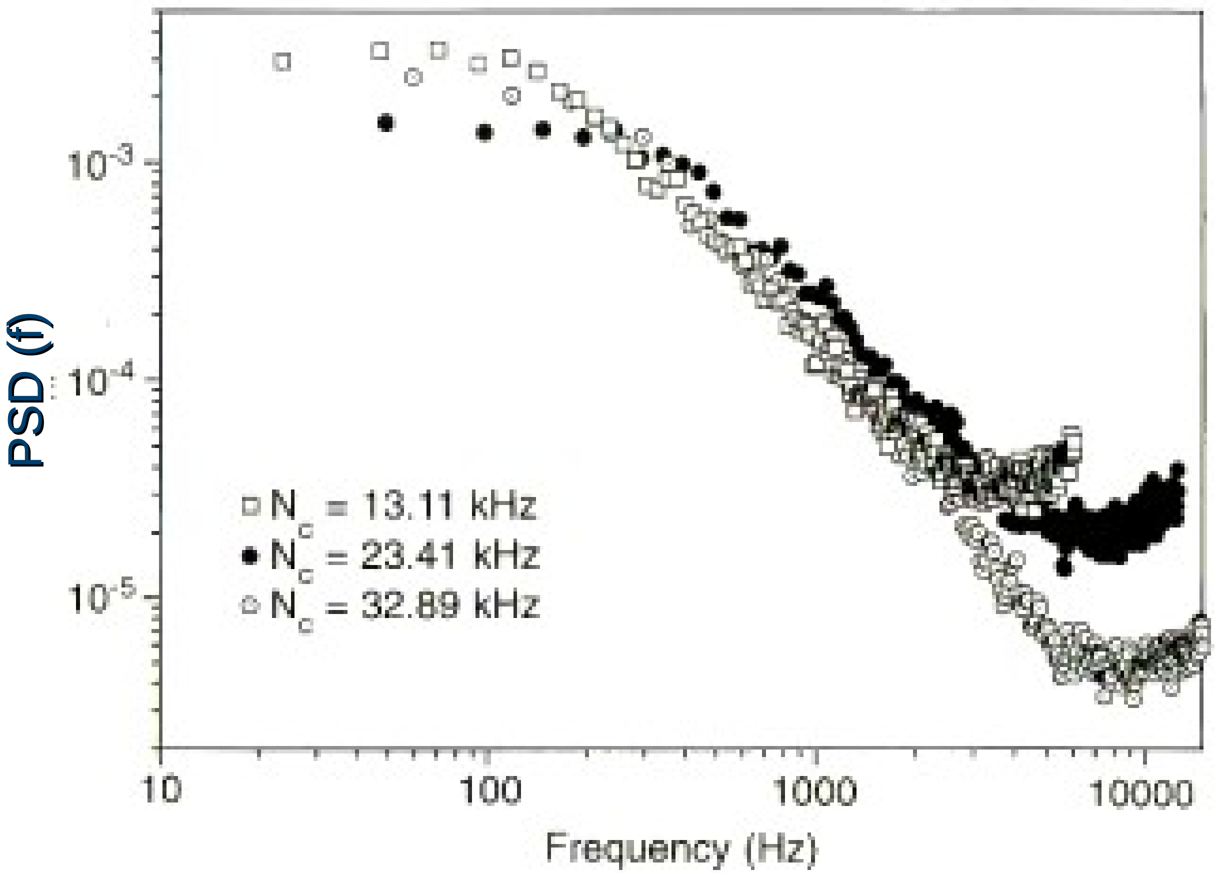
Power Spectral Density (PSD)



The power spectral density and the sampling parameters

Power Spectral Density (PSD)

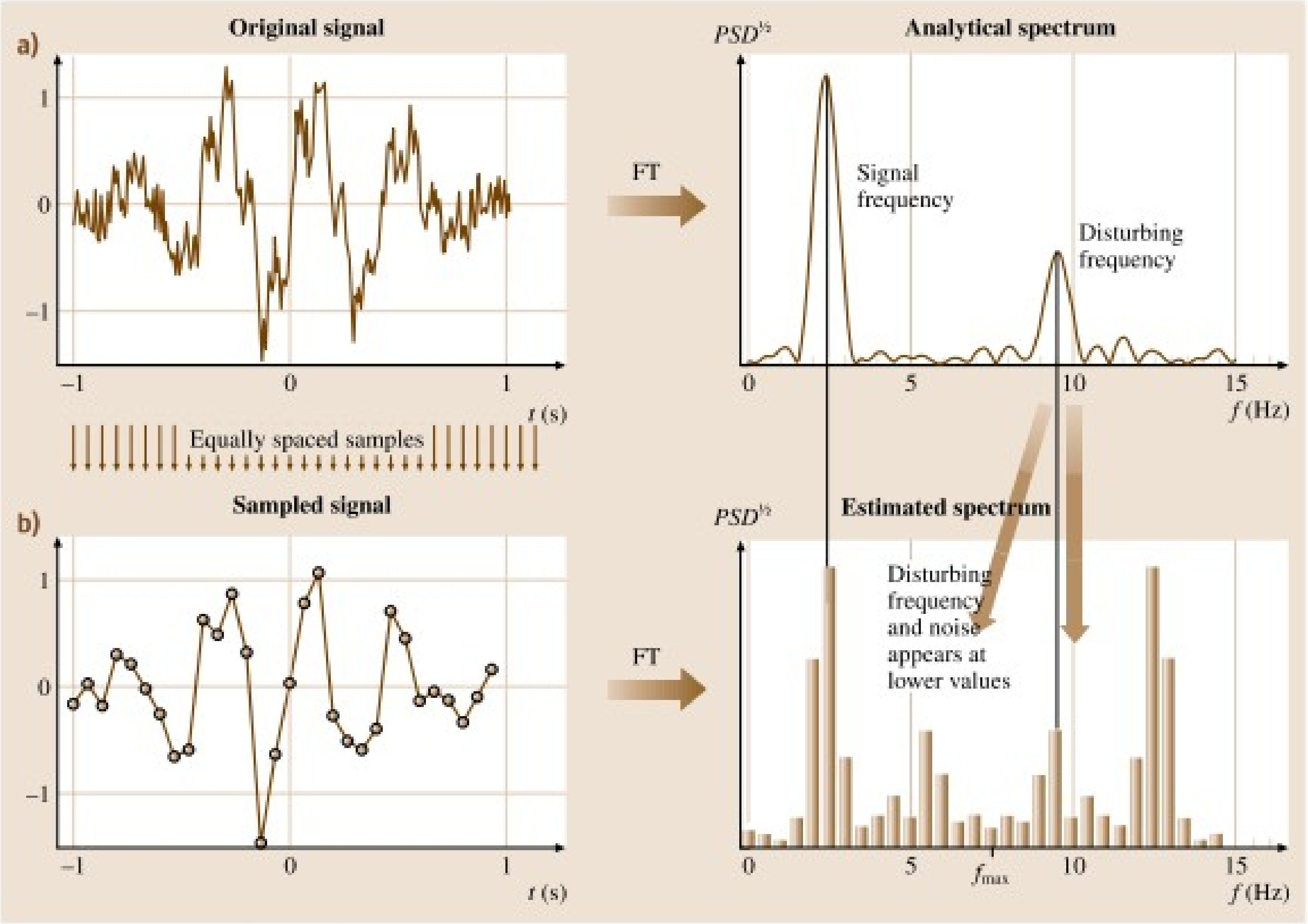
Example of PSD of velocity fluctuations in turbulent flows



Turbolent energy decreases with frequency due to viscous dissipation

Discrete Fourier Trasform (DFT)

Aliasing error in a spectrum is due to signal frequencies occurring above the Nyquist frequency.



(a) Original signal and spectrum,

(b) sampled signal and falsified spectrum.

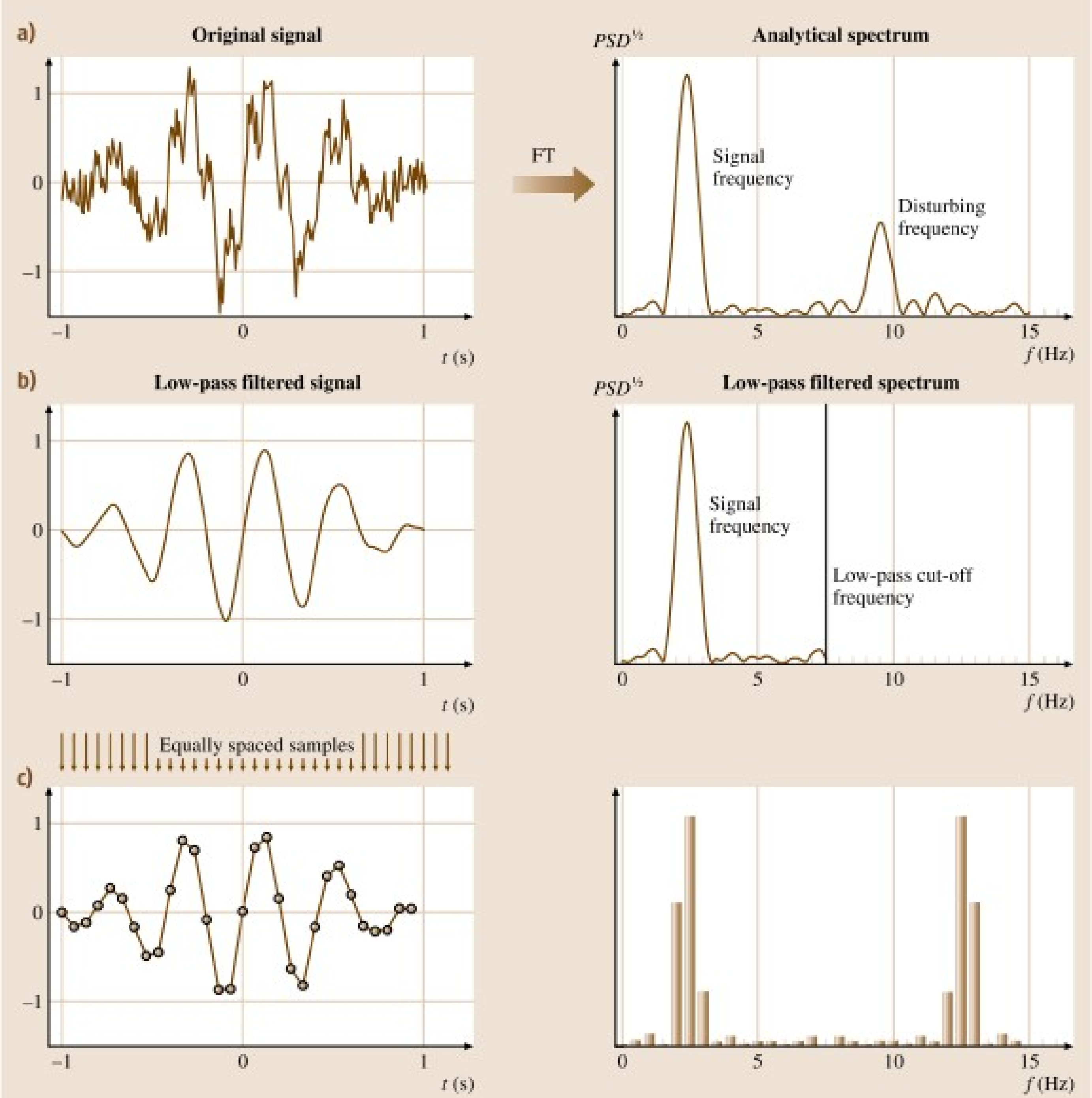
Aliasing error

Elimination of the aliasing error using a low-pass, anti-aliasing filter.

(a) Original signal and spectrum,

(b) low-pass filtered signal and spectrum,

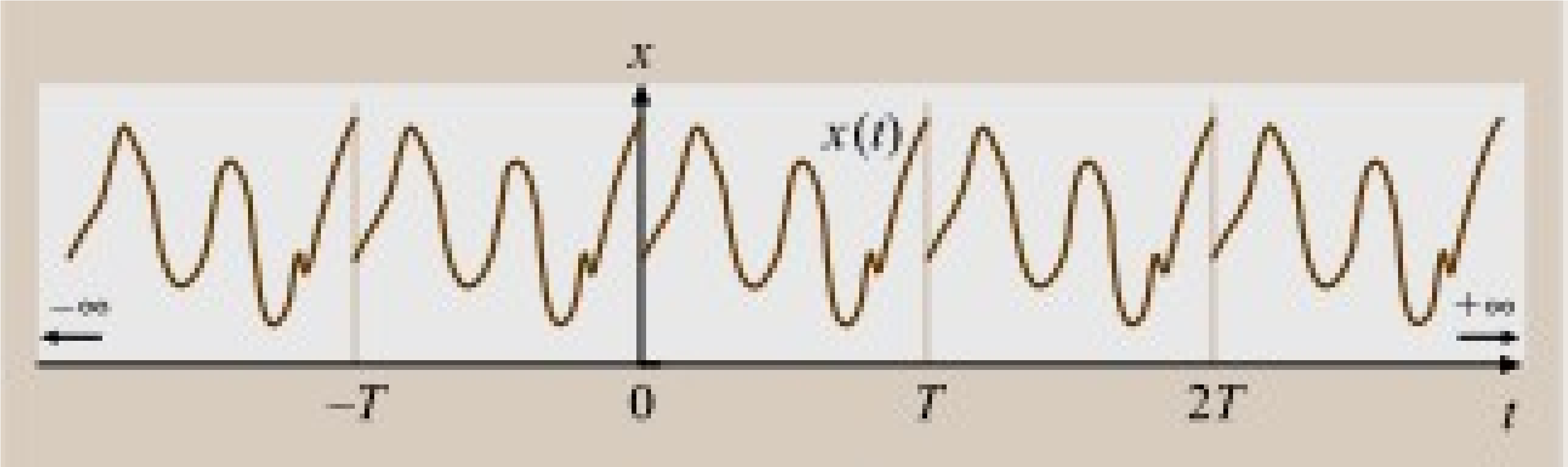
(c) sampled signal and non-aliased spectrum.



Discrete Fourier Transform (DFT)

If the beginning and end of the record do not merge smoothly into one another, sudden amplitude jumps are perceived, which give rise to additional frequency components in the spectrum.

These end effects are unimportant for records of long duration; however, they deserve attention with short records.



These effects are diminished by applying ***window functions*** in the time domain.

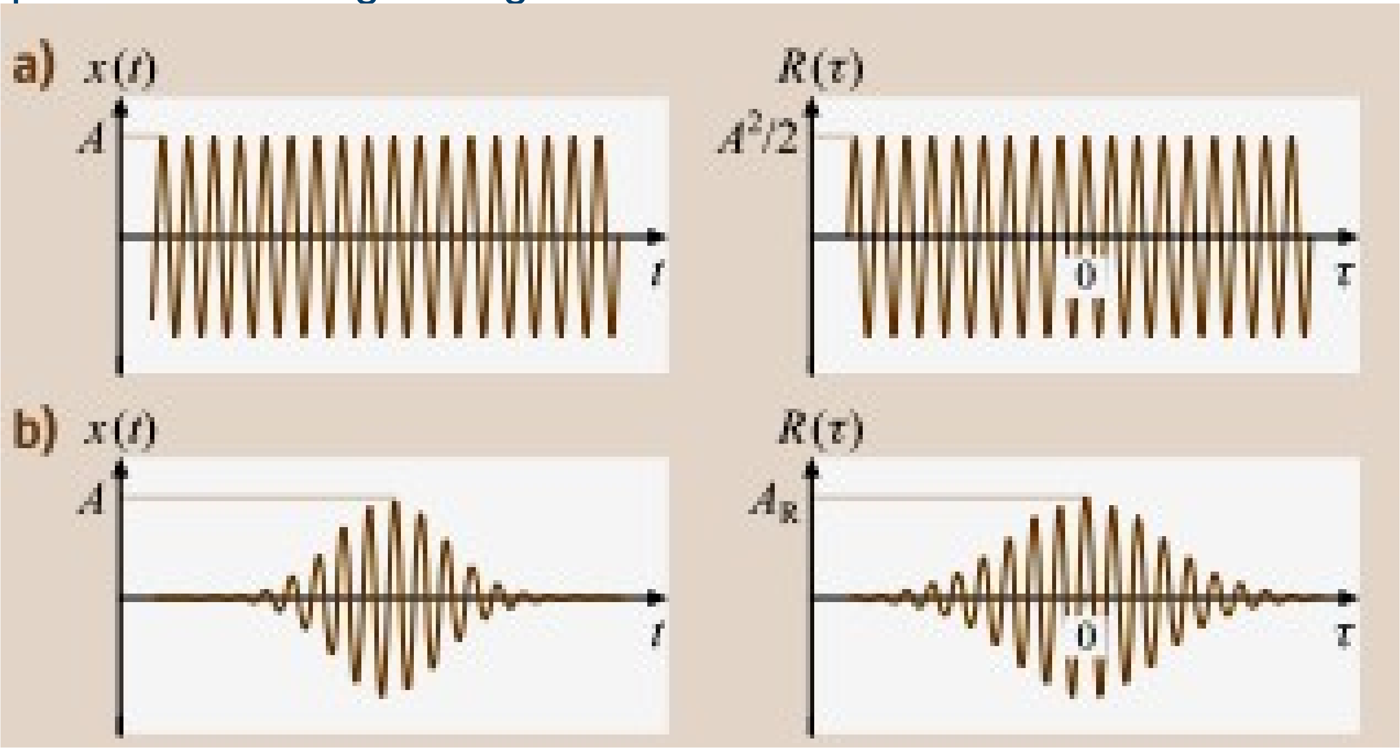
Window functions scale the input data amplitude and force a tapering to zero at the beginning and end of the signal.

Correlation Function

For N data values X_n (n = 1,2,…N) of a stationary process x(t), the (temporal) autocorrelation function is defined as

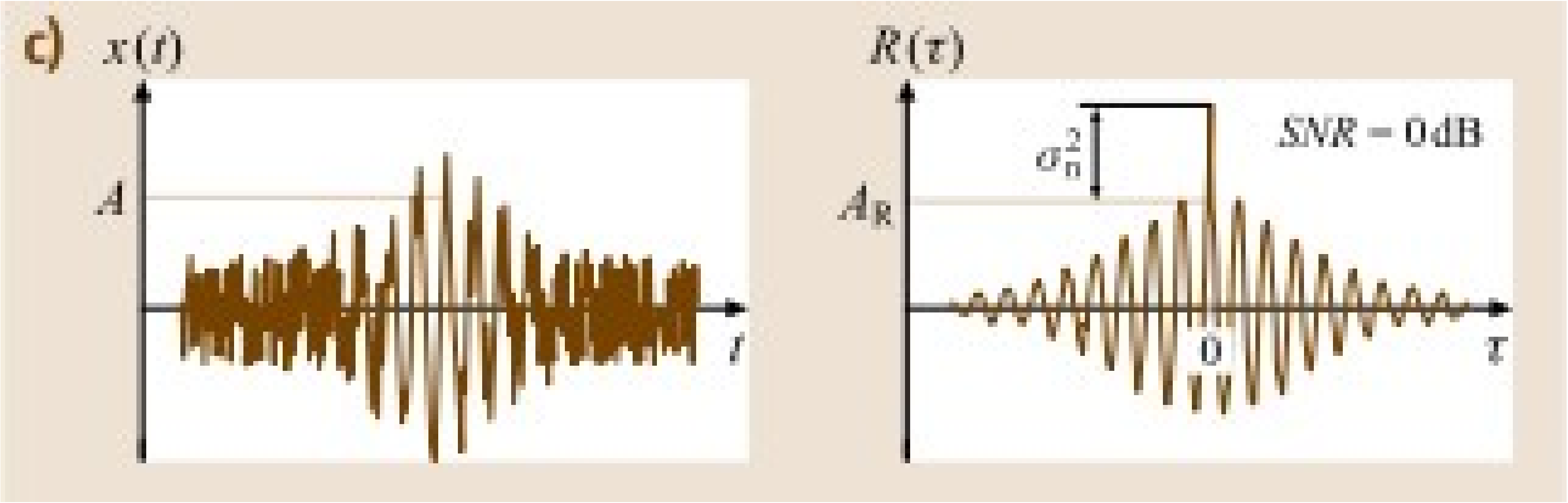
$$R_k = R(k\Delta\tau) = \frac{1}{N-k}\sum_{n=1}^{N-k} x_n x_{n+k} \qquad k = 0, 1, 2..., m$$

R_k is by definition symmetric about $\tau = 0$ and exhibits a periodicity at the same period as the original signal.



Correlation Function

In the autocorrelation function the contribution of signal noise can be found entirely in the first coefficient of the autocorrelation function, i. e., at $\tau = 0$. This is because the signal noise has no inherent time scale, meaning that it is completely random and not correlated with itself over any length of time.



The autocorrelation function is particularly interesting for signal processing, because it provides a means of separating the noise effects from the signal, thus, improving the estimation of signal frequency.

Correlation Function

The information available in spectral (frequency) domain is also available in the correlation (time) domain, since the autocorrelation function R(τ) forms a Fourier transform pair with the power spectral density (***Wiener–Khinchine relation***).

$$R_n = R(\tau = n\Delta\tau) = \frac{f_s}{N}\sum_{k=0}^{N-1} G_k \exp\left(+j\frac{2\pi nk}{N}\right) \qquad n = 0,1,\ldots,(N-1)$$

$$G_k = G(f = f_k) = \frac{1}{f_s}\sum_{n=0}^{N-1} R_n \exp\left(-j\frac{2\pi nk}{N}\right) \qquad k = 0,1,\ldots,(N-1)$$

The double use of FFT procedures to compute both the spectral density and the correlation functions can make this total operation more efficient.

This method exhibits a speed advantage (N log N compared to N^2) that increases with increasing data record length (with $N = 2^{13} = 8192$ will be 8 times faster.

Fourier Transform Table (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	

Fourier Transform Table (2)

No.	$x(t)$	$X(\omega)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos \omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$

Fourier Transform Table (3)

No.	$x(t)$	$X(\omega)$	
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\mathrm{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\,\mathrm{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\,\mathrm{sinc}\,(Wt)$	$\mathrm{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\,\mathrm{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\,\mathrm{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	