POLITECNICO DI MILANO

Basic concepts of Optics

by Andrea Lucchini, PhD

Bibliography

These slides are mainly based on the book **Introduction to Optics** by **Germain Chartier,** Springer, 1997, available in the e-books collection subscribed by Politecnico di Milano.

In particular, the first chapter, entitled **Order of Magnitude in Optics** presents a complete introduction to the basic concepts and applications.

Probably, the most complete and rigorous treatise on the subject is **Principles of Optics**, by **M. Born and E. Wolf**, Pergamon Press, Oxford, 1970.

What is light?

Germain Chartier, Introduction to Optics, 1997

The word ***light*** immediatly evokes *something*, which is immaterial, and which propagates in space at very high speed. This *something* may receive **two different descriptions:**

- *energy grains*, called photons;
- a wave.

Photons are characterized by:	Waves are characterized by:
• an individual energy W;	• a frequency ν;
• a momentum p.	• a wave vector **k**.
Planck relationship:	**De Broglie** relationship:
$W = h\nu$	$\mathbf{p} = \frac{h}{2\pi}\mathbf{k}$

h = 6.626 $\cdot 10^{-34}$ J s, Planck's constant.

What is light?

Optical photon energy is of the order of electron volts (eV).

$$1\ \text{eV} = 1.6\cdot10^{-19}\ \text{C} \cdot 1\ \text{V} = 1.6\cdot10^{-19}\ \text{J}$$

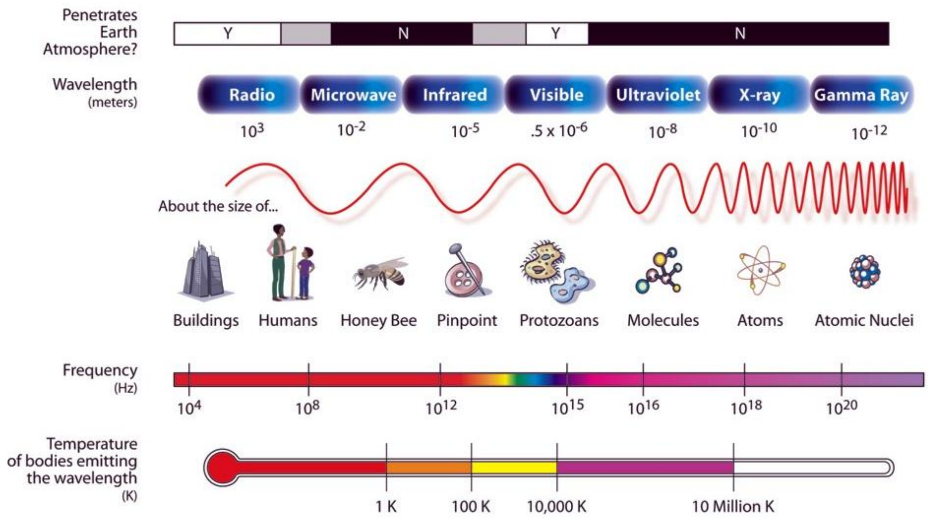
Optical frequencies are of the order of 10^{14} Hz.

In the following light will be described as a wave.

Wave concept is closely related to two main notions:

- Time-varying phenomenon;
- Propagating phenomenon.

Electromagnetic spectrum



Electromagnetic field - 1

Electric and magnetic field

The description of an electromagnetic field in any material, including vacuum, requires four basic vectors:

E electric field
D electric displacement vector or electric induction vector
B magnetic induction vector or magnetic vector
H magnetic field vector or magnetic excitation vector

Optical materials do not usually have any magnetic property; so ***H*** and ***B*** are strictly proportional and collinear

$\mathbf{B} = \mu_0 \mathbf{H}$

$\mu_0 = 4\pi \cdot 10^{-7}$ H·m^{-1}
(H=henry)

Electromagnetic field- 2

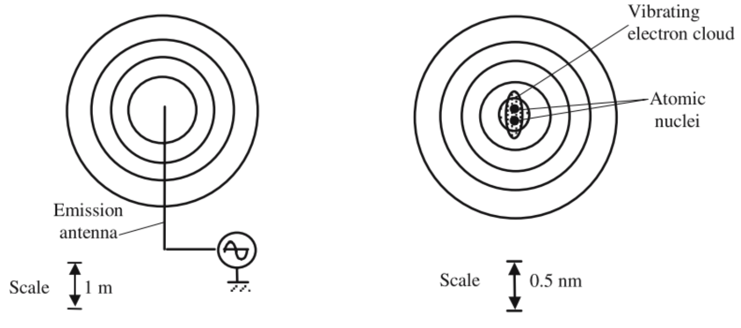
Except in the very special case of nonlinear optics, the relationship between ***E*** and ***D*** is linear

Isotropic media: ***E*** and ***D*** are simply proportional

$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E}$ $\varepsilon_0 = 10^{-9}/(36\pi)\cdot \mathrm{F\cdot m^{-1}}$ (F=farad)

ε = medium permittivity
ε_0 = vacuum permittivity
ε_r = relative permittivity (ε_r =1÷10)
For a given material ε_r varies with the color (i.e. frequency) of the waves (dispersion).
Anisotropic media: the relationship between E and D remains linear but it is described by a tensor (the vectors are no longer collinear).

Wave emission -1



(a) Scheme of radio emission. (b) Scheme of a molecule emitting light.

Most electromagnetic sources originate from vibrations of electric dipole: two electric charges with opposite signs set into a periodic motion.
In the surrounding space an electric field and a magnetic field are simultaneously generated, synchronously vibrating.

Wave emission - 2

- A luminous object is a piece of material, i.e., a collection of atoms.
- This collection receives energy and transforms it into electromagnetic waves. In most cases the energy is provided in to the atoms in a random way, during atomic collisions.

Phenomenological model of the wave emission process

- In an atom at rest electrons have equilibrium positions with regard to the nucleus.
- After being released from those positions, electrons go back to equilibrium by doing damped oscillations.
- The oscillation frequency is characteristic of an atom and of the special electron under consideration, being mainly determined by the electronic shell on which the electron is.

Wave emission - 3

Differential equation of the motion of an electron

$$\frac{d^2s}{dt^2} + \gamma\frac{ds}{dt} + \omega_a^2 s = \frac{e}{m} f(t)$$

- γ = damping coefficient;
- ω_a = eigenvalue of the angular frequency of the electron in the atom (absorption band of the material);
- e = electron charge;
- f(t) = electric field acting on the electron;
- ef(t) = electric force;
- m = electron mass.

Free regime: general solution of the differential equation obtained when the right-hand side of the equation is made equal to zero.

Forced regime: solution of the equation obtained when the right-hand side of the equation is made equal to ef(t)/m.

Wave emission - 4

The **free regime** is a damped oscillation

$$\frac{d^2s}{dt^2} + 2\gamma\frac{ds}{dt} + \omega_a^2 s = 0$$

$$s(t) = s_0 \exp(-\gamma t)\sin(\omega t - \varphi)$$

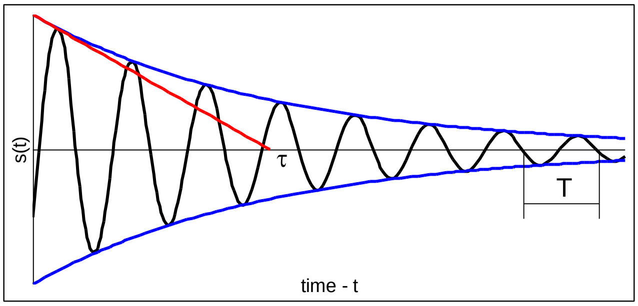
$$\omega = \sqrt{\omega_a^2 - \gamma^2}; \quad \omega_a^2 >> \gamma^2 \cong \omega_a \Rightarrow \omega \cong \omega_a$$

ξ_0 and φ are integration constants depending on initial conditions

$\tau=\gamma^{-1}$ damping time constant (infinite for a perfectly coherent source)

$T_a=2\pi/\omega_a$ oscillation period

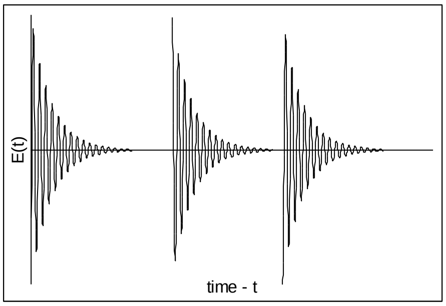
$T_a << \tau$



Wave emission - 5

Solution in the case of light sources

- At random times θ_p, the oscillator receives short burst of energy.
- The mean time interval between two consecutive bursts is longer than the damping time constants.
- The amplitude of the electromagnetic field emitted by the oscillating dipole follows the same time variation law as the motion of the electron.



$$\mathbf{E}(t) = \mathbf{E}_0 \sum_{i=1}^{n} H(t-\theta_i)\exp\left(-\frac{t-\theta_i}{\tau}\right)\cos(\omega_a t - \varphi_i)$$

Wave emission - 6

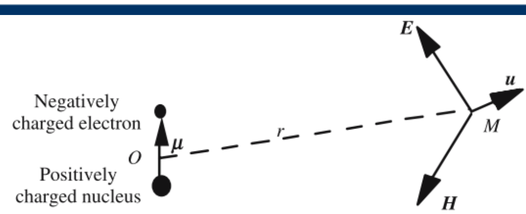
e = electron charge

d = vector joining the nucleus to the electron

μ=*e**d***, electric dipole moment (same time variations as ***d***)

c = light speed in vacuum

$\lambda = 2\pi c/\omega$ wavelength



For: OM >> λ

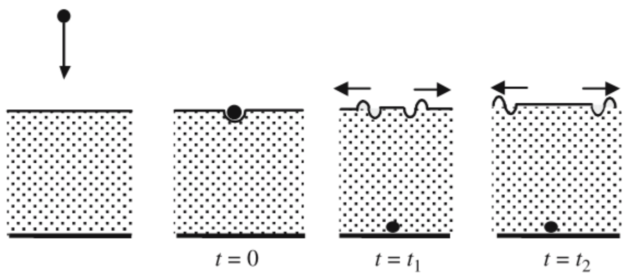
the radiated electromagnetic field is described by the two vectors:

E is orthogonal to OM and inside the plane Π defined by OM and μ

H is orthogonal to OM and also to the plane Π

$$\mathbf{E}(t) = \frac{\mathbf{E}_0}{r}\sin\left[\omega_a\left(t-\frac{r}{c}\right)\right] \qquad \mathbf{H}(t) = \frac{\mathbf{H}_0}{r}\sin\left[\omega_a\left(t-\frac{r}{c}\right)\right]$$

Wave propagation - 1



A visible case: propagation of waves along the free surface of a liquid.

- The surface is distorted (source).
- The distortion is transmitted to the neighboring points.
- The wave profile is quite similar to the initial distortion.

Wave propagation - 2

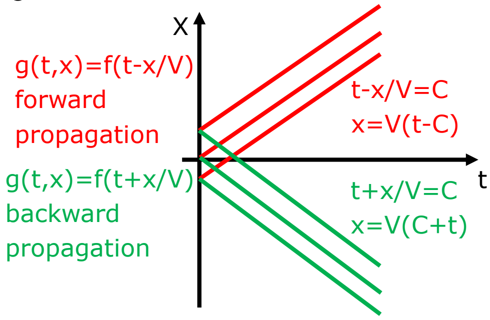
Mathematical description of propagation

Planar waves

$$g(t,x) = f\left(t \pm \frac{x}{V}\right)$$

$$\text{if}: \quad t_1 \pm \frac{x_1}{V} = t_2 \pm \frac{x_2}{V}$$

$$\Rightarrow \quad g(t_1,x_1) = g(t_2,x_2)$$



g has the same value for all the points located in a given plane orthogonal to the x axis, i.e. surface waves (wavefronts) are parallel planes.

When propagating inside a material the wave will give energy to the atoms

$$g(x,t) = e^{-\alpha x} f\left(t \pm \frac{x}{V}\right)$$

α = absorpition coefficient of the material [m^{-1}]

Wave propagation - 3

Spherical waves

$$g(r,t) = a(r) f\left(t \pm \frac{r}{V}\right)$$

$a(r)$ must be proportional to 1/r in order to keep constant the flux of energy across the different spheres centered at the focus

Spherical waves propagating inside an absorbing material

$$a(r) = \frac{e^{-\alpha r}}{r}$$

Sine waves (Harmonic waves)

A wave is said to be harmonic when the time variation is sinusoidal

Harmonic planar wave

$$g(u) = A\cos(u) = A\cos\left[\omega\left(t \pm \frac{x}{V}\right)\right]$$

Wave propagation - 4

Main parameters used to describe an harmonic planar wave.

frequency	ν	[Hz]	
Pulsation (angular frequancy)	ω	[rad·s^{-1}]	$\omega=2\pi\nu$
period	T	[s]	$T=1/\nu=2\pi/\omega$
speed of propagation	V	[m·s^{-1}]	
wavelength	λ	[m]	$\lambda=VT=V/\nu=2\pi V/\omega$
wave vector module (space pulsation)	k	[m^{-1}]	$k=2\pi/\lambda=\omega/V$
wave number (space frequency)	N	[m]	$N=1/\lambda=k/2\pi$

Since the propagation function f(u) is sinusoidal, to cope easily with a change of units, its argument u should not have any dimension

$$u=\omega\left(t-\frac{x}{V}\right)=\omega t-\frac{\omega}{V}x=\omega t-kx=2\pi\left(\frac{t}{T}-\frac{x}{\lambda}\right)$$

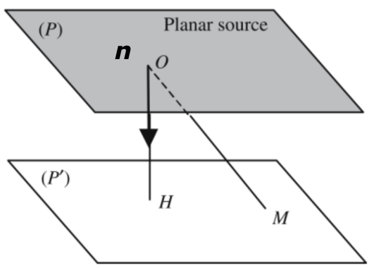
Wave vector of a planar harmonic wave: $\mathbf{k} = k\mathbf{n}_x = (2\pi/\lambda)\ \mathbf{n}_x = (\omega/V)\ \mathbf{n}_x$,

$\mathbf{n}_x$ = unit vector of the direction of propagation

Wave propagation - 5

Propagation from a planar source (planar wave)

n = unit vector normal to plane (P)



$$\eta=t\pm\frac{\mathbf{OM}\cdot\mathbf{n}}{V}=t\pm\frac{x_1n_1+x_2n_2+x_3n_3}{V}$$

$$G=G_0f(\eta)=G_0f\left(t\pm\frac{x_1n_1+x_2n_2+x_3n_3}{V}\right)$$

Wave propagation - 6

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The function G necessarily obeys a very specific partial differential equation, called the **wave equation**

$$\frac{\partial \eta}{\partial t} = 1 \qquad \frac{\partial^2 \eta}{\partial t^2} = 0$$

$$\frac{\partial \eta}{\partial x_i} = \pm \frac{n_i}{V} \qquad \frac{\partial^2 \eta}{\partial x_i^2} = 0 \qquad i=1,2,3$$

$$\frac{\partial G}{\partial t} = \frac{dG}{d\eta} \qquad \frac{\partial^2 G}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial t}\right) = \frac{d}{d\eta}\left(\frac{dG}{d\eta}\frac{\partial \eta}{\partial t}\right)\frac{\partial \eta}{\partial t} = \frac{d^2 G}{d\eta^2}$$

$$\frac{\partial G}{\partial x_i} = \pm \frac{n_i}{V}\frac{dG}{d\eta} \qquad \frac{\partial^2 G}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial G}{\partial x_i}\right) = \frac{d}{d\eta}\left(\frac{dG}{d\eta}\frac{\partial \eta}{\partial x_i}\right)\frac{\partial \eta}{\partial x_i} = \left(\frac{n_i}{V}\right)^2 \frac{d^2 G}{d\eta^2}$$

$$\sum_{i=1}^{3} \frac{\partial^2 G}{\partial x_i^2} - \sum_{i=1}^{3}\left(\frac{n_i}{V}\right)^2 \frac{\partial^2 G}{\partial t^2} = \nabla^2 G - \frac{1}{V^2}\frac{\partial^2 G}{\partial t^2} = 0 \qquad i=1,2,3$$

Wave propagation - 7

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Planar waves are an important family of solution, not the only one.

Consider a direction associated to some unit vector **n**, a planar wave is a function f(x, y, z, t) which, at a given time t, keeps the same value at any point M belonging to a plane (P) orthogonal to **n**.

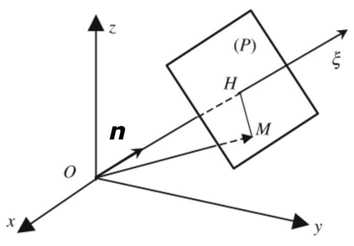
$$\xi = \mathbf{n}\cdot\mathbf{r} = \sum_{i=1}^{3} n_i x_i \qquad \frac{\partial \xi}{\partial x_i} = n_i \qquad \frac{\partial^2 \xi}{\partial x_i^2} = 0 \qquad i=1,2,3$$

$$\frac{\partial f}{\partial x_i} = n_i \frac{\partial f}{\partial \xi} \qquad \frac{\partial^2 f}{\partial x_i^2} = n_i^2 \frac{\partial^2 f}{\partial \xi^2}$$

$$\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2} = \sum_{i=1}^{3} n_i^2 \frac{\partial^2 f}{\partial \xi^2} = \frac{\partial^2 f}{\partial \xi^2}$$

$$\frac{\partial^2 f}{\partial \xi^2} - \frac{1}{V^2}\frac{\partial^2 f}{\partial t^2} = 0$$

$$f(t,\xi) = f_1\left(t - \frac{\xi}{V}\right) + f_2\left(t + \frac{\xi}{V}\right)$$



Wave propagation - 8

Propagation from a point source

spherical waves: solutions of the wave equation that keep a constant value at any point on a sphere (center O; radius r)

The symmetry suggests the **spherical coordinates** (x,y,z) ⇒ (r)

$$\sum_{i=1}^{3}\frac{\partial^2 f}{\partial x_i^2}=\frac{1}{r}\frac{\partial^2 (rf)}{\partial r^2}$$

Multiplying both sides by r and taking advantage of the fact that time and space derivatives commute, the problem can be cast in terms of the **auxiliary function g=rf**, which allows to use the results obtained for **planar waves**

$$\frac{1}{r}\frac{\partial^2 (rf)}{\partial r^2}-\frac{1}{V^2}\frac{\partial^2 f}{\partial t^2}=0$$

$$\frac{\partial^2 (rf)}{\partial r^2}-\frac{1}{V^2}\frac{\partial^2 (rf)}{\partial t^2}=0$$

g is the sum of two functions:

- a wave diverging from O: g=g(t-r/V)
- a wave converging toward O: g=g(t+r/V)

$$rf=f_1\left(t-\frac{r}{V}\right)+f_2\left(t+\frac{r}{V}\right)$$

$$f=\frac{1}{r}\left[f_1\left(t-\frac{r}{V}\right)+f_2\left(t+\frac{r}{V}\right)\right]$$

Electromagnetic waves - 1

From an electromagnetic point of view a system is characterized by:

- A medium, or different media, which can be:
  - absorbing or transparent;
  - isotropic or anisotropic;
  - homogeneous or inhomogeneous.
- Discontinuities across the surfaces separating the previous media:
  - separation between two dielectric materials (vacuum being considered as a dielectric material);
  - separation between a dielectric material and a metal (a metal being considered as a dielectric material for which ε is a complex number).
- A time variation law imposed on the electromagnetic field at some points of the system, these points are called *sources of radiation.*

Even in the simple case of an isotropic transparent medium, Electromagnetism is a very mathematical game which is played in a six-dimensional space. For a given system we must find, at any time and at any point, a six-component vector $\boldsymbol{EM}_{(x,y,z,t)}$ which is called the "electromagnetic vector." The representation, in the three-dimensional usual geometrical space of a six-component vector, is usually a difficult exercise; hopefully in this case the six components may be associated in two individual sets of three components; the electric vector $\boldsymbol{E}_{(x,y,z,t)}$ on the one hand and the magnetic vector $\boldsymbol{H}_{(x,y,z,t)}$ on the other hand,

$$\boldsymbol{EM}_{(x,y,z,t)}=\{\boldsymbol{E}_{(x,y,z,t)}\cap\boldsymbol{H}_{(x,y,z,t)}\}.$$

Electromagnetic waves - 2

To fulfill the laws of Electromagnetism $\boldsymbol{E}_{(x,y,z,t)}$ and $\boldsymbol{H}_{(x,y,z,t)}$ must satisfy:

- Four basic equations, called Maxwell's equations.
- Continuity (or discontinuity) conditions when crossing surfaces separating different media.
- Boundary conditions in the vicinity of the points where the fields have imposed values corresponding to the sources of radiation.

Because of the linearity of Maxwell's equations, any possible field in a given system may be written as

$$\boldsymbol{EM}_{(x,y,z,t)} = \sum_i \alpha_i \boldsymbol{EM}_{i(x,y,z)}.$$

The set of vectors $\boldsymbol{EM}_{i(x,y,z)}$ constitutes a complete, although not unique, basis which is characteristic of the system and allows the representation of any possible fields inside the system.

Electromagnetic waves - 3

The sources of radiation play a key role in the problem, eventually they fix the field which really exists. The presence of an electromagnetic field corresponds to some electromagnetic energy stored inside the system, it is at the places where the sources are located that this energy is transferred, this is the reason why the sources should always be connected to an external source of energy.

Let (x_S, y_S, z_S) be the coordinates of a radiation source located at point S, the field $\boldsymbol{EM}_{(x_S,y_S,z_S)}$ at point S cannot follow any arbitrary law. It must be possible to use the set of basis vectors $\boldsymbol{EM}_{i(x,y,z)}$ to obtain an expression of the electromagnetic field at point S. The following formula, which also defines the $\alpha_{i(t)}$ coefficients are finally obtained from

$$\boldsymbol{EM}_{(x_S,y_S,z_S,t)} = \sum_i \alpha_i \boldsymbol{EM}_{i(x_S,y_S,z_S,t)}.$$

Electromagnetic waves - 4

Main Operators Used in Electromagnetism.

$V(x, y, z)$ is a scalar.
$\boldsymbol{E}_{(x,y,z)} = E_{x(x,y,z)}\boldsymbol{x} + E_{y(x,y,z)}\boldsymbol{y} + E_{z(x,y,z)}\boldsymbol{z}$ is a vector.
$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is an orthogonal and normalized trihedral.

Nabla operator: $\nabla = \left[\boldsymbol{x}\frac{\partial}{\partial x} + \boldsymbol{y}\frac{\partial}{\partial y}\boldsymbol{z}\frac{\partial}{\partial z}\right].$

Gradient: $\mathrm{grad}(V_{(x,y,z)}) = \left[\boldsymbol{x}\frac{\partial V}{\partial x} + \boldsymbol{y}\frac{\partial V}{\partial y} + \boldsymbol{z}\frac{\partial V}{\partial z}\right] = \nabla V$ vector.

Divergence: $\mathrm{div}(\boldsymbol{E}) = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] = \nabla \boldsymbol{E}$ scalar.

Scalar Laplacian: $\Delta V = \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right] = \mathrm{div}[\mathrm{grad}(V)] = \nabla(\nabla \mathrm{V}) = \nabla^2 V.$

curl: $\mathrm{curl}(\boldsymbol{E}) = \boldsymbol{x}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right) + \boldsymbol{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \boldsymbol{z}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) = \nabla \wedge \boldsymbol{E}.$

$\mathrm{curl}(\boldsymbol{E})$ is a vector.

Vector Laplacian: $\Delta(\boldsymbol{E}) = \boldsymbol{x}\Delta(E_x) + \boldsymbol{y}\Delta(E_y) + \boldsymbol{z}\Delta(E_z) = \Delta^2\boldsymbol{E}$ vector.

The following important identity can be established:

$$\mathrm{curl}[\mathrm{curl}(\boldsymbol{E})] = \mathrm{grad}(\mathrm{div}\ \boldsymbol{E}) - \Delta\boldsymbol{E} = \nabla \wedge (\nabla \wedge \boldsymbol{E}) = \nabla(\nabla\boldsymbol{E}) - \nabla^2\boldsymbol{E}.$$

Electromagnetic waves - 5

Maxwell's equations are linear relations between the first time and space partial derivatives of the electric and magnetic field components. The following expressions correspond to the propagation of the usual light waves, i.e.

- propagation in nonmagnetic nondispersive transparent dielectric materials
- electric charge density within the propagation domain is equal to zero
- the only electric currents are displacement currents

They are quite acceptable as long as the frequencies remain outside the absorption bands of the of the material where the waves are propagated

Electromagnetic waves - 6

Maxwell's equations	
$\mathrm{curl}(\boldsymbol{E}) = -\mu_0 \dfrac{\partial \boldsymbol{H}}{\partial t}$,	$\nabla \wedge \boldsymbol{E} = -\mu_0 \dfrac{\partial \boldsymbol{H}}{\partial t}$,
$\mathrm{curl}(\boldsymbol{H}) = +\dfrac{\partial \boldsymbol{D}}{\partial t}$,	$\nabla \wedge \boldsymbol{H} = +\dfrac{\partial \boldsymbol{D}}{\partial t}$,
$\mathrm{div}(\boldsymbol{D}) = 0$,	$\nabla \boldsymbol{D} = 0$,
$\mathrm{div}(\boldsymbol{H}) = 0$,	$\nabla \boldsymbol{H} = 0$,
$\boldsymbol{D} = \varepsilon \boldsymbol{E}$.	$\boldsymbol{D} = \varepsilon \boldsymbol{E}$.

By simple manipulations it can be shown that

$$\nabla^2 \boldsymbol{E} - \varepsilon\mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \quad \text{and} \quad \nabla^2 \boldsymbol{H} - \varepsilon\mu_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$$

Any component EM_i of the electromagnetic field follows the wave equation

$$\frac{\partial^2 EM_i}{\partial x^2} + \frac{\partial^2 EM_i}{\partial y^2} + \frac{\partial^2 EM_i}{\partial z^2} - \varepsilon\mu_0 \frac{\partial^2 EM_i}{\partial t^2} = 0 \qquad V = \frac{1}{\sqrt{\varepsilon\mu_0}}$$

Electromagnetic waves - 7

- A harmonic planar wave is a wave for which both time and space variation laws are sinusoidal.
- For this category of waves, Maxwell's equations can be written as algebraic equations instead of partial derivative equations.
- The following complex notation for the electromagnetic field is introduced:

$$EM_i = A_i e^{j\omega(t - \boldsymbol{ur}/V)} = A_i e^{j\omega t} e^{-j\omega \boldsymbol{ur}/V} = A_i e^{j\omega t} e^{-j\omega \boldsymbol{kr}}$$

$$\boldsymbol{k} = \boldsymbol{u}\frac{\omega}{V} \quad \text{is the wave vector.}$$

- deriving once versus x (or y, or z) → multiplying by $-jk_x$ (or $-jk_y$, or $-jk_z$);
- deriving twice versus x (or y, or z) → multiplying by $-k_x^2$ (or $-k_y^2$, or $-k_z^2$);
- taking the divergence of a vector → scalar multiplication of the vector by $-j\boldsymbol{k}$;
- taking the Laplacian of a vector → multiplication by the number $-k^2$;
- taking the curl of a vector → vector multiplication by the vector $-j\boldsymbol{k}$.

Electromagnetic waves - 8

$$\frac{\partial}{\partial t} \rightarrow j\omega, \quad \nabla \rightarrow -j\boldsymbol{k} \quad \text{and} \quad \nabla^2 \rightarrow -k^2$$

Maxwell's equations (general case)	Maxwell's equations (planar waves)
$\nabla \wedge \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$,	$\boldsymbol{k} \wedge \boldsymbol{E} = \omega\mu_0 \boldsymbol{H}$,
$\nabla \wedge \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t}$,	$\boldsymbol{k} \wedge \boldsymbol{H} = -\omega\varepsilon\boldsymbol{E}$,
$\nabla \varepsilon \boldsymbol{E} = 0$,	$\boldsymbol{k}\boldsymbol{E} = 0$,
$\nabla \boldsymbol{H} = 0$.	$\boldsymbol{k}\boldsymbol{H} = 0$.

Electromagnetic waves - 9

Let us consider:

- a propagation medium having infinite extension, filled with a dielectric material (ε and μ_0)
- an angular frequency ω and a wave vector $\boldsymbol{k}$
- an electromagnetic field defined by an electric field $\boldsymbol{E_0}$ and a magnetic field $\boldsymbol{H_0}$

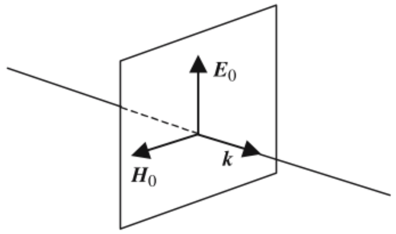
The planar waves defined by the formulas

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{j\omega t} e^{-j\boldsymbol{k}\boldsymbol{r}}$$
$$\boldsymbol{H} = \boldsymbol{H}_0 e^{j\omega t} e^{-j\boldsymbol{k}\boldsymbol{r}}$$

must fullfill the following conditions

$$\boldsymbol{E} = -\frac{1}{\omega\varepsilon}\boldsymbol{k} \wedge \boldsymbol{H}, \quad \boldsymbol{k} \wedge (\boldsymbol{k} \wedge \boldsymbol{E}) = +\omega\mu_0 \boldsymbol{k} \wedge \boldsymbol{H}$$
$$\boldsymbol{H} = +\frac{1}{\omega\mu_0}\boldsymbol{k} \wedge \boldsymbol{E}, \quad \boldsymbol{k} \wedge (\boldsymbol{k} \wedge \boldsymbol{H}) = -\omega\varepsilon \boldsymbol{k} \wedge \boldsymbol{H}.$$

Electromagnetic waves - 10



Respective positions of vectors $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ lying in the wave plane

$$k^2 = \varepsilon\mu_0\omega^2 \quad \rightarrow \quad k = \omega\sqrt{\varepsilon\mu_0} \quad \text{dispersion law}$$

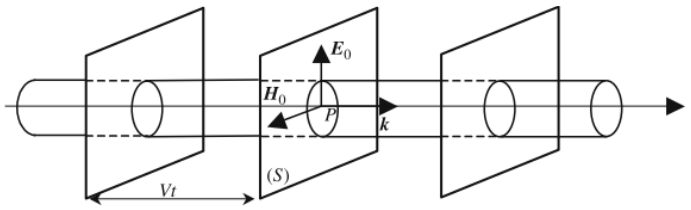
$$k = \omega/V \quad \rightarrow \quad \varepsilon\mu_0 V^2 = 1 \quad \rightarrow \quad V = \frac{1}{\sqrt{\varepsilon\mu_0}} \quad \textit{phase velocity}$$

$$Z = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon}} = \mu_0 V = \frac{1}{\varepsilon V} \qquad \textit{wave impedance}$$

$$n = \frac{c}{V} = \frac{\sqrt{\varepsilon\mu_0}}{\sqrt{\varepsilon_0\mu_0}} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \quad \rightarrow \quad n^2 = \varepsilon_r \quad \textit{Index of Refraction}$$

Electromagnetic waves -11

Energy transportation



The energy flux through the surface S is called the light intensity and it is equal to the flux of the Poynting vector

$$\boldsymbol{\Pi} = \boldsymbol{E} \wedge \boldsymbol{H}$$

Electromagnetic wave - 12

Electric field at a point P: $\mathbf{E} = E_0 \cos(\omega t - \varphi)\mathbf{i}$

Magnetic field at a point P: $\mathbf{H} = H_0 \cos(\omega t - \varphi)\mathbf{j}$

Electromagnetic energy density (J/m^3) at P:

$$u(t) = \frac{\varepsilon E^2 + \mu_0 H^2}{2}\cos^2(\omega t - \varphi)$$

$$u_{\text{stored}} = \langle u(t) \rangle = \frac{\varepsilon E_0^2 + \mu_0 H_0^2}{4} = \frac{\varepsilon E_0^2}{2} = \frac{\mu_0 H_0^2}{2}$$

Light intensity (W/m^2) at P:

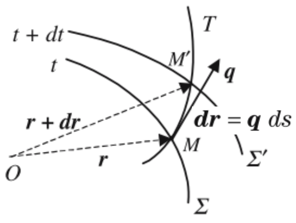
$$I = \frac{1}{S}\frac{1}{t}u_{\text{stored}} SVt = u_{\text{stored}} V = V\frac{\varepsilon E_0^2}{2} = V\frac{\mu_0 H_0^2}{2} = \frac{1}{2}\frac{E_0^2}{Z} = \frac{1}{2} Z H_0^2$$

Electromagnetic waves - 13

Harmonic wave: $\mathbf{EM}_i(\mathbf{r},t) = \mathbf{a}_i(\mathbf{r})\cos[\omega t - \mathbf{g}_i(\mathbf{r})]$

- equiamplitude surfaces along which $a_{(r)}$ keeps a constant value
- wave surfaces (wavefronts) along which the phase difference $g_{i(r)}$ remains constant

Let us imagine an observer moving along some trajectory T:
how should he move if he wants to see a constant instant phase (ωt-$g_{i(r)}$) ?



$$\omega\, dt = g_{(\boldsymbol{r}+d\boldsymbol{r})} - g_{(\boldsymbol{r})} = d\boldsymbol{r}\ \text{grad}(g_{(\boldsymbol{r})})$$

$$\frac{ds}{dt} = \frac{\omega}{\boldsymbol{q}\ \text{grad}(g_{(\boldsymbol{r})})}$$

The speed is minimum when q and grad(g(r)) are parallel

Electromagnetic waves - 14

Trajectories for which the speed is minimum are orthogonal to wave surfaces and are called **light rays**

$$V = \frac{\omega}{\|\mathrm{grad}(g_{(\boldsymbol{r})})\|}$$

$$\boldsymbol{r} = x\boldsymbol{x} + y\boldsymbol{y} + z\boldsymbol{z}, \quad \boldsymbol{k} = k_x\boldsymbol{x} + k_y\boldsymbol{y} + k_z\boldsymbol{z}$$

For a planar wave

$$g_{(\boldsymbol{r})} = \boldsymbol{kr} = k_x\boldsymbol{x} + k_y\boldsymbol{y} + k_z\boldsymbol{z} \quad \rightarrow \quad \mathrm{grad}(g_{(\boldsymbol{r})}) = k_x\boldsymbol{x} + k_y\boldsymbol{y} + k_z\boldsymbol{z},$$

$$\|\mathrm{grad}(g_{(\boldsymbol{r})})\| = (k_x^2 + k_y^2 + k_z^2)^{1/2} = k \quad \rightarrow \quad V = \frac{\omega}{\|\mathrm{grad}(g_{(\boldsymbol{r})})\|} = \frac{\omega}{k} = \frac{c}{n}$$

Geometrical optics - 1

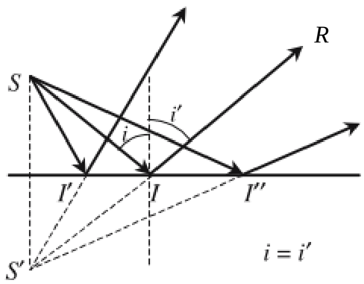
Fermat's principle - principle of the shortest optical path
17th century

The path followed by light going from point A to point B is such that the transit time is stationary

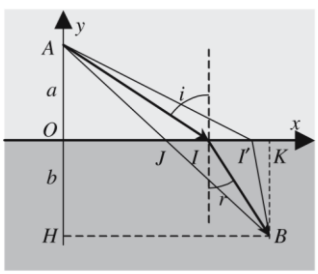
- The **principle of Fermat** has experimental roots.
- The **Snell-Descartes laws of reflection and refraction** can be mathematically demonstrated from the principle of Fermat.
- Starting from the Snell-Descartes laws it is possible to demonstrate the theorem of Fermat.
- The principle of Fermat, as well as the Snell-Descartes laws, are easily obtained from Maxwell's equations.

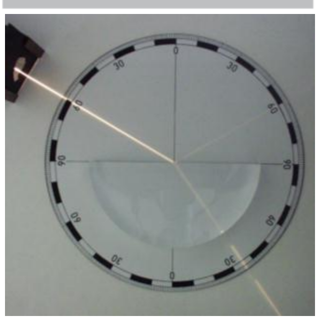
Geometrical optics - 2

Law of reflection



Geometrical optics - 3





Law of refraction

$$t = \frac{1}{c}\left[n_1\sqrt{x^2+a^2} + n_2\sqrt{(b'-x)^2+b^2}\right]$$

$OA = a$
$OH = b$
$OK = b'$
$OI = x$

$$\frac{dt}{dx} = \frac{1}{c}\left[\frac{n_1 x}{\sqrt{x^2+a^2}} - \frac{n_2(b'-x)}{\sqrt{(b'-x)^2+b^2}}\right] = 0$$

$$\frac{n_1 x}{\sqrt{x^2+a^2}} = \frac{n_2(b'-x)}{\sqrt{(b'-x)^2+b^2}}$$

$$\frac{dt}{dx} = \frac{1}{c}\left[\frac{n_1 x}{\sqrt{x^2+a^2}} - \frac{n_2(b'-x)}{\sqrt{(b'-x)^2+b^2}}\right] = 0$$

$$\sin(i) = \frac{x}{\sqrt{x^2+a^2}} \quad ; \quad \sin(r) = \frac{(b'-x)}{\sqrt{(b'-x)^2+b^2}}$$

$$n_1 \sin(i) = n_2 \sin(r)$$

Geometrical optics - 4

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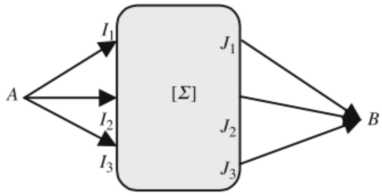
Stigmatism

If all the rays proceeding from a source A converge toward the same point B:

- the system is stigmatic for points A and B.
- the points A and B are conjugate across the optical system.

Optical system [Σ]:

- it is the limited by two interfaces.
- it is a succession of either continuously or discontinuously inhomogeneous mediums.
- Input and output mediums are homogeneous, before and after [Σ] light rays are rectilinear.
- [Σ] All the paths from A to B take the same time.



Geometrical optics - 5

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Stigmatism = perfect imaging = sharp imaging

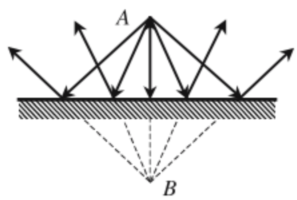
This notion is restricted to geometrical optics because diffraction forbids any electromagnetic wave to be focused inside a spot smaller than the wavelength.

Many optical systems can be considered as approximately stigmatic.

A planar mirror is stigmatic for any point

A planar mirror is strictly stigmatic. Any incident beam issued from A gives a reflected beam passing through point B symmetric to A with regard to the plane of the mirror.

B is the image of A.



Geometrical optics - 6

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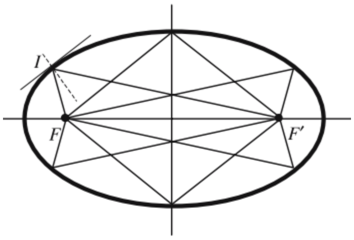
An elliptic mirror is stigmatic for its focus

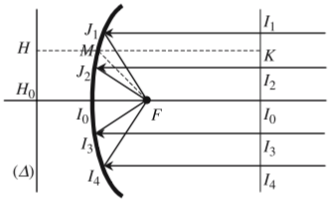
An ellipse is a set of points for which the sum of the distances to two given points, called focus, is constant. Any beam issued from one focus is reflected toward the other focus. It's a well-known property of an ellipse that the normal to the ellipse surface at point I is a bisector of the angle FIF'.



An parabolic mirror is stigmatic for its focus

A parabolic mirror is stigmatic for its focuses F, the other focus is at infinity. A parabola is a set of points located at equal distance from a point called the focus and a line (D) called the directrix.

MF = MH ⇒ the paths I_i J_i F, = I_0 H_0 ; i=1,2,3,…



Geometrical optics - 7

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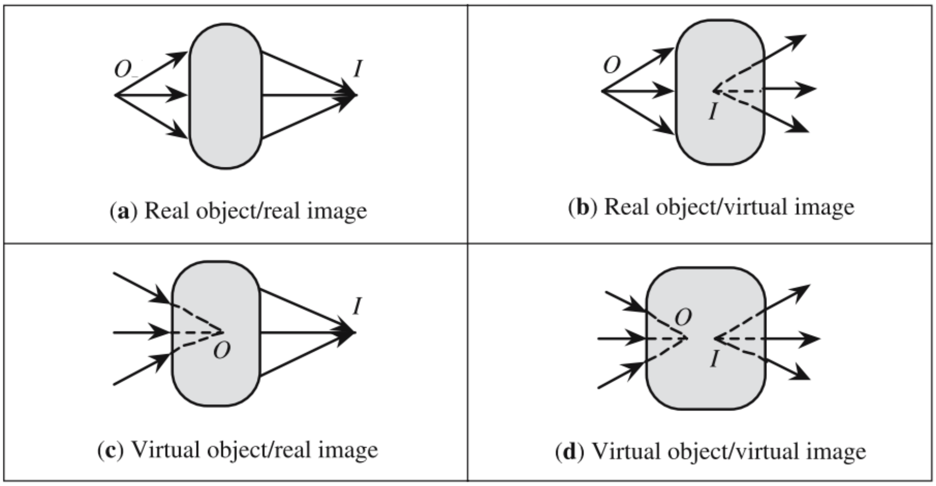
Formation of images

- An imaging instrument is a device limited by some input interface and some output interface.
- This instrument, receiving light rays from point sources (object), gives emerging beams which converge toward point images.
- According to the fact that the point source and the point image are before or after the input/output interfaces, the object and the image will be real or virtual.

Geometrical optics - 8



(**a**) Real object/real image

(**b**) Real object/virtual image

(**c**) Virtual object/real image

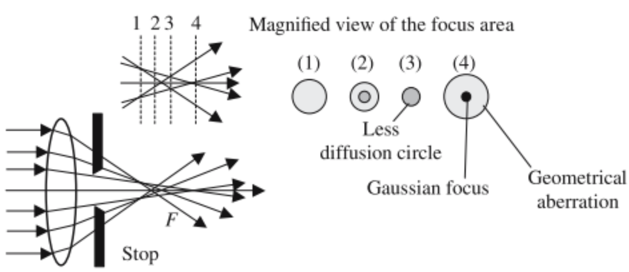
(**d**) Virtual object/virtual image

Geometrical optics - 9

Approximate imaging

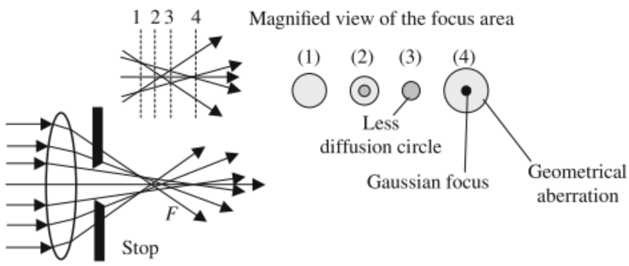
A beam of parallel rays is transformed by a lens into a beam of rays, which almost converge to the same point. **A lens is too converging at its periphery as compared to its center** (the marginal rays are bent too much).

A stop is to avoid them contributing to the formation of the image.



Geometrical optics - 10

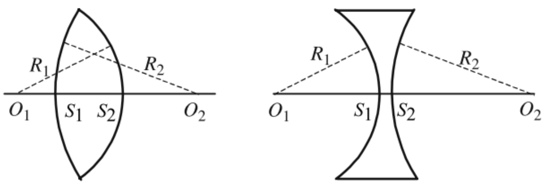
- Position 4 of the screen: a very bright spot is observed, the size of which is determined by diffraction phenomena (1.22 f λ/d, f focal length, λ wavelength, d lens diameter). The spot is surrounded by a less luminous circular halo which constitutes the geometrical aberrations.
- Position 3 of the screen: circle of less diffusion.
- If a stop is used, geometrical aberrations are limited.
- If a stop is not used the screen should be placed in position 3.



Geometrical optics - 11

(Thin) spherical lenses

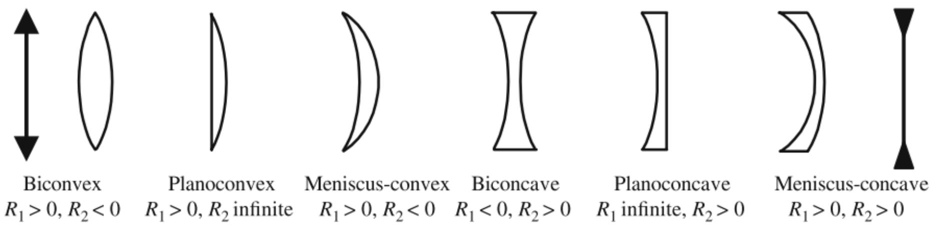
- Transparent material limited by two spherical interfaces.
- O_1 and O_2 centers.
- R_1 and R_2 radii of curvature.
- O_1O_2 axis of the lens.
- S_1 and S_2 summits of the lens.
- If the thickness of the lens is far smaller than the radius of curvature, S_1 and S_2 are assimilated to only one point S: **the center of the lens**.



Geometrical optics - 12

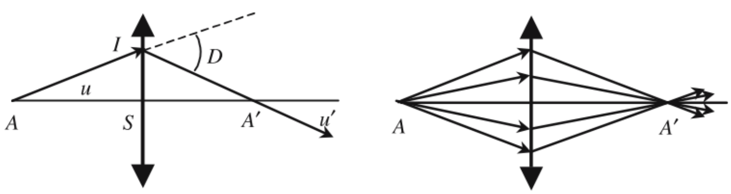
Different kinds of spherical lenses. Since the lenses are often thin, a symbolic representation is often used as shown on the left for converging lenses and on the right for diverging lenses.

The radii of curvature are algebraic quantities, the signs correspond to an orientation from left to right.



Geometrical optics - 13

The image processing in a thin spherical lens



- The angle of deviation *D* is proportional to the distance *SI*: $D = K\,SI$
- $K^{-1}=f$ is the focal length of the lens
- *K* is related to the index of refraction and to the radii of curvature

$$K = \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

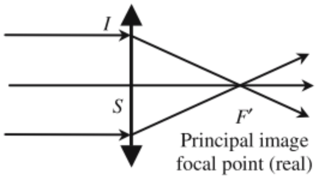
- To each point *A* of the axis is associated another point *A'* (bijection)

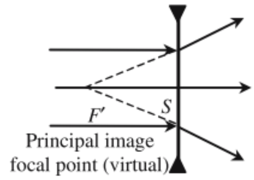
Geometrical optics - 14

A focal point is conjugated with a point that is at infinity

Principal image focal point:

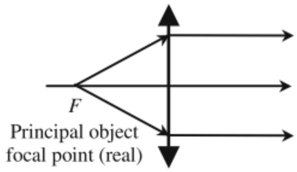
image of a point source at infinity in the direction of the axis

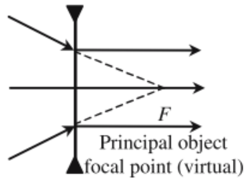




Principal object focal point:

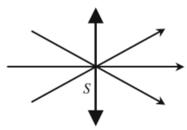
point having its image at infinity in the direction of the axis

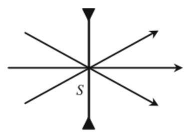




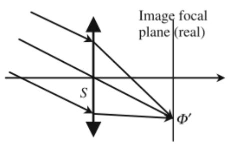
Geometrical optics - 15

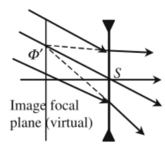
A ray which is directed toward the center of a thin lens is not bent.



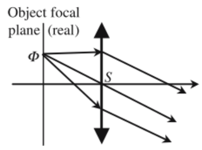


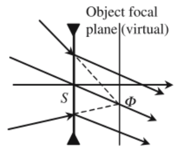
Secondary image focal point Φ‘
image of a point at infinity in the direction of the line Φ‘ S, joining the center and the focal point.





Secondary object focal point Φ
its image is at infinity in the direction of the line ΦS, joining the center and the focal point

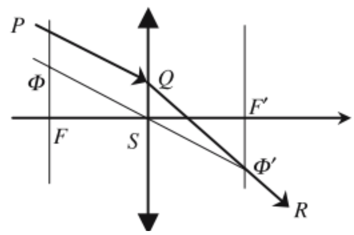




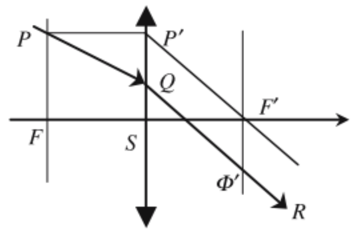
Geometrical optics - 16

Construction of an emerging ray associated to an incident ray

The emerging ray QR should pass by the image focal point F' associated to the direction of the incident ray PQ.



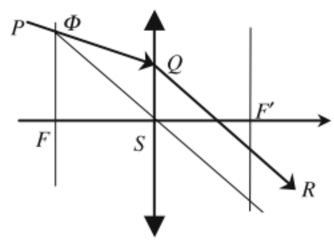
P belongs to the object focal plane. Considered as an incident ray PP' gives an emerging ray P'F': QR should be parallel to P'F'.



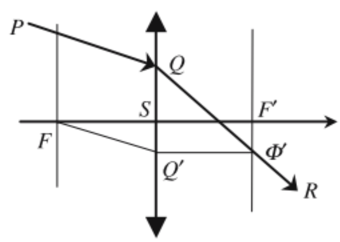
Geometrical optics - 17

Construction of an emerging ray associated to an incident ray

PQ intersects the object focal plane at Φ. Considered as an incident ray, ΦS is not bent by the lens. QR is parallel to ΦS.



FQ' is drawn parallel to PQ, considered as an incident ray it gives an emergent ray Q'Φ' parallel to the axis. QR intersects the image focal plane at Φ'.



Geometrical optics - 18

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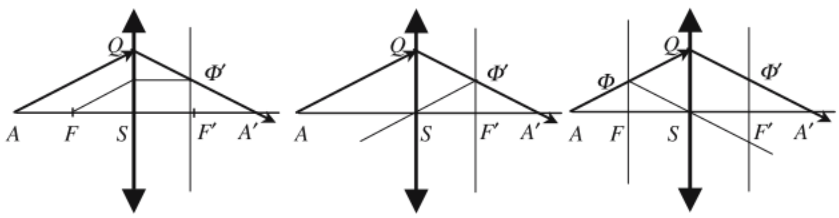
Construction of the Image of a Point Object A Belonging to the Axis

The image A' is at the intersection of the axis with an emergent ray associated to an incident ray AQ coming from A

Each of the previous methods can be used



Geometrical optics - 19

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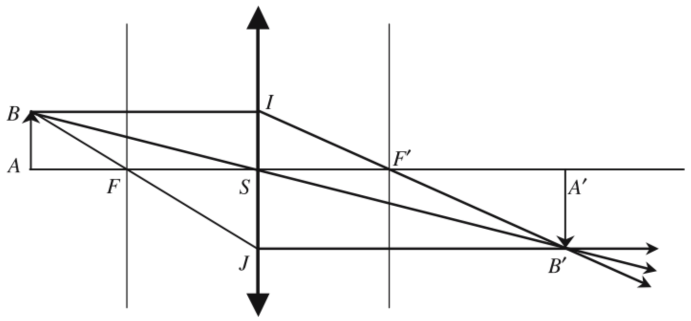
Construction of the Image of a Point Object Out of the Axis

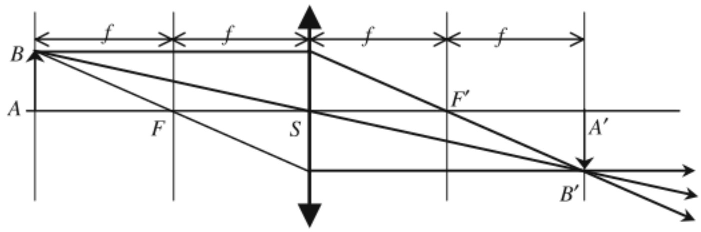
It can be proved that the image formed by a lens of a small object belonging to a plane normal to the axis will belong to another orthogonal plane. The two planes are said to be conjugated by the lens



Geometrical optics - 20

The 4-f arrangement.

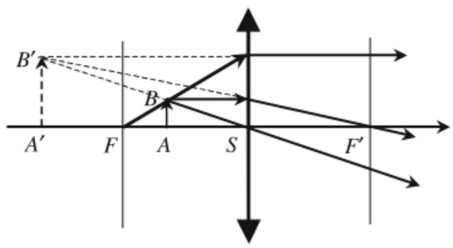
The object is placed at a distance equal to twice the focal length in front of a converging lens. The image is real and located at the same distance behind the lens, it has the same size as the object, but is inverted. In this disposition the object-image separation is the shortest.



Geometrical optics - 21

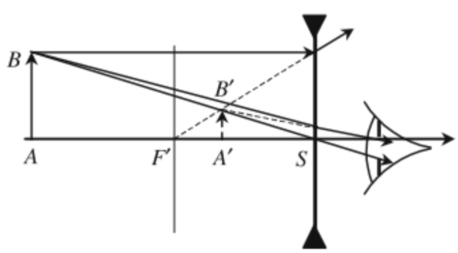
Lens for a long-sighted eye

Virtual image of a real object. A real object placed between the lens and its focal plane gives a virtual image.



Lens for a short-sighted eye

The real object AB is replaced by the virtual image A'B' located nearer to the eye.



Geometrical optics - 23

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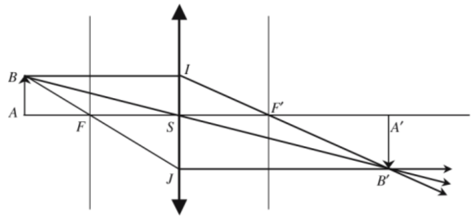
The **lenses equations** are formulas relating:

- the abscissas of two conjugate points;
- the ratio of the image to the object size (magnification).

Kinds of equations mainly differ in the choice of the origin:

- **Descartes' formula**: the summit S of the lens is taken as a common origin for both the object and the image space.
- **Newton's formula**: two different origins are chosen, namely the object focal point F for the object space, and the image focal point F' for the image space.

Conjugation equations are readily obtained from similarity of triangles SAB/SA'B', FAB/FSJ, F'IS/F'A'B'



Geometrical optics - 24

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Descartes' equations	*Newton's equations*
$\frac{1}{SA'} - \frac{1}{SA} = \frac{1}{SF'} = -\frac{1}{SF},$	$FAF'A' = SFSF' = -f^2.$
setting $SA = p$, $SA' = p'$, and $SF' = f$,	$\gamma = \frac{A'B'}{AB} = \frac{FS}{FA} = \frac{F'A'}{F'S}.$
$\frac{1}{p'} - \frac{1}{p} = \frac{1}{f'},\quad \gamma = \frac{A'B'}{AB} = \frac{SA'}{SA} = \frac{p'}{p}.$	(All quantities are algebraic.)

Perfect imaging cannot be obtained by refraction at a spherical interface.

Approximate imaging can be achieved using rays making a small angle with axis: **paraxial rays**.

In such conditions the Snell-Descartes law:

$$n \sin i = n' \sin i',$$

is replaced by the Kepler law:

$$ni = n'i'$$

Detection - 1

The simplest description of a detector is the following first-order linear differential equation:

$$\frac{dr}{dt}+\frac{1}{\tau_{\text{resp}}}r = KI(t)$$

r(t) is the output signal of the detector (response).

I(t) is the input signal of the detector (light intensity).

K and τ_{resp} are proportionality coefficients characteristic of the detector.

τ_{resp} is called the *rise time* and for most detectors it is of the order of *microseconds.*

Detection - 2

Response to a wave of constant amplitude

$$I = I_0 = \text{constant} \qquad\qquad r(t) = K\tau_{\text{resp}}I_0$$

Response to a slowly varying signal

$$I(t+\tau_{\text{resp}}) \approx I(t) \quad \text{as} \quad I(t+\tau_{\text{resp}}) \approx I(t)+\tau_{\text{resp}}\frac{dI}{dt},$$

$$\tau_{\text{resp}}\frac{dI}{dt} << I(t) \quad\rightarrow\quad \tau_{\text{resp}}\frac{dI}{dt} << I(t) \quad\rightarrow\quad r = K\tau_{\text{resp}}I(t)$$

For slow phenomena, the time variation of the response, exactly follows the variation of the intensity.

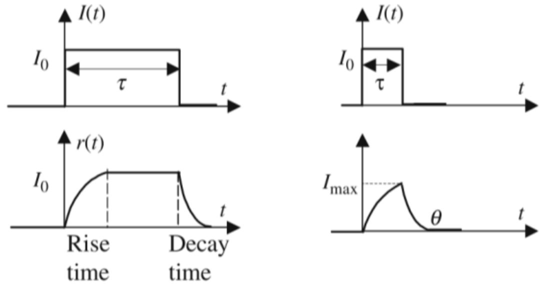
Detection - 3

Response to a rectangular signal

$$r(t) = I_0 K \tau_{\mathrm{resp}} (1 - e^{-t/\tau_{\mathrm{resp}}}) \quad \text{for } 0 \le t \le \tau,$$
$$r(t) = I_0 K \tau_{\mathrm{resp}} e^{-(t-\tau)/\tau_{\mathrm{resp}}} \quad \text{for } t > \tau.$$

signal

detector



A photodetector cannot discriminate two pulses which are separated by an interval shorter than its rise time

Detection - 4

Response to a sinusoidal excitation

$$I(t) = I_0 \cos \Omega t$$

$$r(t) = K \tau_{\mathrm{resp}} \,\mathrm{Re}\left\{ \frac{e^{j\Omega t}}{1 + j\Omega\tau} \right\}$$

The photodetector behaves like a filter:

- the response is still a sine of angular frequency Ω, but
- the amplitude decreases with frequency
- for typical light frequencies (10^{14} Hz >> τ_{resp}^{-1}) the response is the average intensity I_0

Interference - 1

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Interference is sometimes given the following paradoxical description:

$$\text{Light} + \text{light} \rightarrow \text{darkness.}$$

However, usual observation shows that most often the situation is

$$\text{Light} + \text{light} \rightarrow \text{twice as much light,}$$

When a phenomenon of interference occurs:

The intensity resulting from the superimposition of two beams is not equal to the addition of their individual intensities.

- Interference patterns can only be observed if a photodetector receives simultaneously several interfering beams.
- An interference phenomenon is revealed by time-averaging light intensities.
- No interference with no detection.

Interference - 2

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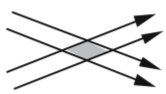
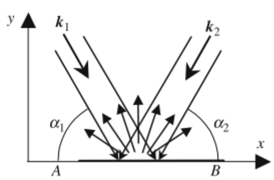
Two light beams cross inside a transparent material. The electrons of the atoms located in the shaded area are simultaneously submitted to the electromagnetic fields of the two beams. As the medium is perfectly transparent and homogeneous no light is diffused outside: no interference pattern can be seen.



Two light beams illuminate a sheet of paper. The electrons of the paper behave in the same way as those of the transparent medium, but they reemit light above the screen; this light then goes to a photodetector. It is when the detector averages the square of the light amplitude that a nonlinear interaction occurs and when the fringes are created. In fact the fringes only exist on the surface of the retina of the eye of the observer.



Interference - 3

Conditions for two light beams to produce interference

$$y_1 = a\cos(\omega_1 t + \varphi_1) \qquad y_2 = a\cos(\omega_2 t + \varphi_2)$$

The detector response is proportional to the average value of $(y_1 + y_2)^2$:

$$\left\langle a^2[\cos(\omega_1 t+\varphi_1)+\cos(\omega_2 t+\varphi_2)]^2\right\rangle$$
$$=\left\langle a^2\left\{1+\cos[(\omega_1-\omega_2)t+\varphi_1-\varphi_2)]+\cdots\right.\right.$$
$$\left.\left.+\frac{\cos(2\omega_1 t+2\varphi_1)+\cos(2\omega_2 t+2\varphi_2)+2\cos(\omega_1+\omega_2 t+\varphi_1+\varphi_2)}{2}\right\}\right\rangle.$$

- Due to the very high frequency, the terms containing $2\omega_1$, $2\omega_2$ and $\omega_1+\omega_2$ give a zero response of the detector.
- To give a contribution, ω_1-ω_2 must be such that $2\pi/(\omega_1-\omega_2) \cong \tau_{resp}$.

For example $\tau_{\text{resp}} = 10^{-6}\,\text{s} \rightarrow \dfrac{\Delta\omega}{\omega} = \dfrac{(\omega_1-\omega_2)}{\omega_1} \le 10^{-9}$

Interference - 4

- First condition: the two frequencies must be (almost) equal.
- Second condition: the two light sources must be coherent.

$$E_1 = a\cos(\omega t - \boldsymbol{k}_1\boldsymbol{OM} + \varphi_1) \qquad E_2 = a\cos(\omega t - \boldsymbol{k}_2\boldsymbol{OM} + \varphi_2)$$

$$\boldsymbol{k}_1 = \frac{2\pi}{\lambda}(\boldsymbol{x}\cos\alpha_1 - \boldsymbol{y}\sin\alpha_1) \qquad \boldsymbol{k}_2 = \frac{2\pi}{\lambda}(-\boldsymbol{x}\cos\alpha_2 - \boldsymbol{y}\sin\alpha_2)$$

The two sources are coherent: This means that their phase difference doesn't vary at random, for the sake of simplicity we will just consider the phase difference to be constant. On the plane Π, fringes can be observed with a periodic repartition governed by the term $(\pi x/\lambda)(\cos\alpha_1 + \cos\alpha_2)$, the fringe separation is then $\lambda/(\cos\alpha_1 + \cos\alpha_2)$.

$$I = \left\langle (E_1+E_2)^2\right\rangle = 4a^2\cos^2\left[\frac{\pi x}{\lambda}(\cos\alpha_1+\cos\alpha_2)+\frac{\varphi_1-\varphi_2}{2}\right]$$

Interference - 5

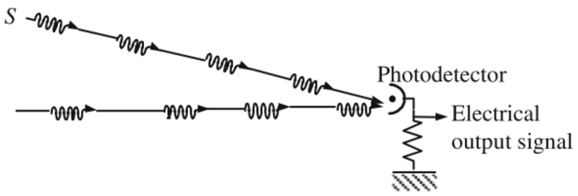
The two sources are not coherent: The phase difference ($\varphi_1 - \varphi_2$) cannot at all be considered as constant, but varies randomly with time. At a given time, ($\varphi_1 - \varphi_2$) has some value to which can be associated an interference fringe pattern; as time advances, different patterns follow one another at a pace that cannot be resolved by any photodetectors: the fringes are not clear anymore.

$$I = \left\langle (E_1 + E_2)^2 \right\rangle = \left\langle 2a^2 \cos^2\left[\frac{\pi x}{\lambda}(\cos\alpha_1 + \cos\alpha_2) - \frac{\varphi_1 - \varphi_2}{2}\right]\right\rangle$$

$$= a^2 \left\langle 1 + \cos\left[2\frac{\pi x}{\lambda}(\cos\alpha_1 + \cos\alpha_2) - (\varphi_1 - \varphi_2)\right]\right\rangle = a^2$$

Interference - 6

Coherence in time



Interference between waves coming from two different sources. The two sources emit wave packets of the same frequency but having no phase coherence. No interference can be seen by the detector for two reasons: first, it is very unlikely that two wave packets should arrive simultaneously and, second, even if this were the case, the phase difference would randomly vary from one coincidence to the next.

Interference - 7

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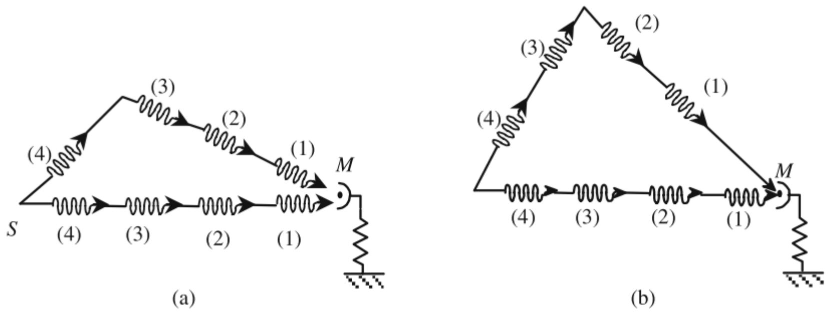
Coherence in space



Superposition of two waves coming from the same source after they have traveled independently. In (a) the two light paths are only slightly different, the two wave packets arriving at the detector come from the same initial one: the phase difference is not at random but is determined by the lengths of the two optical trajectories: interference is possible. In (b) the length difference is larger than the length of coherence of the source: even if two different wave packets meet on the detector they cannot interfere.

Diffraction - 1

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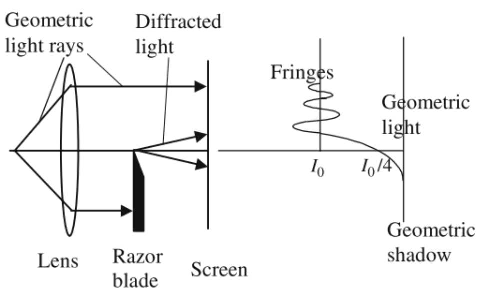
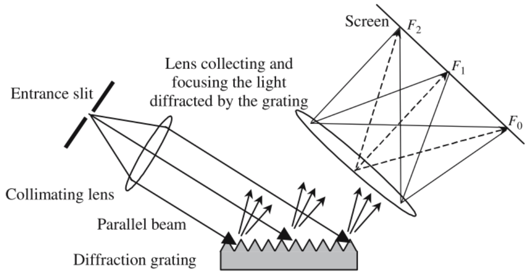


Diffraction by a sharp edge. A beam of parallel rays is partially hindered by an obstacle limited by a straight and very sharp boundary having a tiny radius of curvature (a razor blade). Some light is diffracted by the edge and interferes with the undiffracted part of the light: fringes can be seen on the observation screen.

- The illumination is equal to $I_0/4$ at the boundary between shadow and light, and goes smoothly to zero far enough into the shadow.
- In the clear part of the screen, the illumination has well-contrasted oscillations and goes to I_0 for points far from the edge.

Diffraction - 2

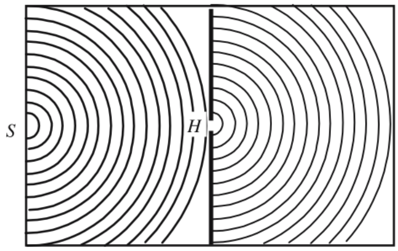
- Diffraction appears as a limitation of the rectilinear propagation of light.
- Diffraction is always followed by interference between the initial beam and the diffracted beams: the related fringes are called diffraction fringes.
- Diffraction is enhanced when the diffracting obstacles have a periodic repetition and it becomes quite spectacular if the periodicity has the same order of magnitude as the wavelength (diffraction grating).

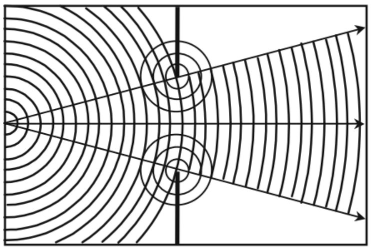


Diffraction - 3

Principle of Huygens-Fresnel (intuitive approach)

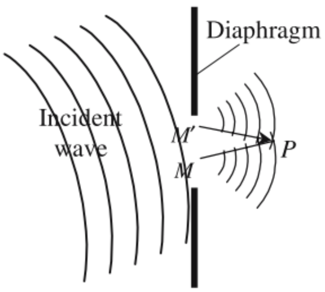
Each point of a medium (disturbed by passing wave) becomes source of disturbance which propagates from this point in all directions indiscriminately. The interference (addition) of all disturbances then results in a certain amplitude of detected wave (say in certain location at a screen).





As demonstrated by Helmholtz and Kirchhoff, this principle is consistent with the wave equation (slide 18) and can be deduced as a theorem.

Diffraction - 4



$F(M)$ = vibration created by the incident wave in M.
Elementary vibration sent from M to P:

$$dE = KF(M)\frac{e^{-jkPM}}{MP}d\sigma.$$

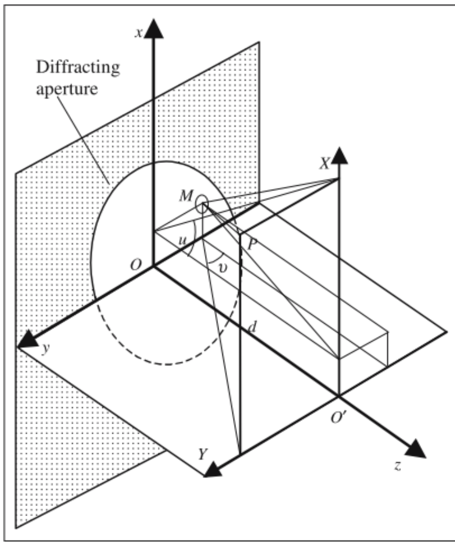
Global vibration in P:

$$E = \frac{j}{\lambda}\iint_{\text{diaphragm}} F(M)\frac{e^{-jkPM}}{MP}d\sigma.$$

- A mathematical formulation of the H-F principle is to place over the opening Σ of the diffracting diagram auxiliary sources.
- The complex amplitude of the auxiliary sources is proportional to that at the same place, but without diaphragm.
- The proportionality constant can be evaluated as j/λ

Diffraction - 5

Statement of a problem of diffraction



Fresnel

$$E_{(X,Y)} = \frac{j}{\lambda}\iint_{\text{diaphragm}} f_{(x,y)}\frac{e^{-jkPM}}{MP}dx\,dy,$$

$$E_{(X,Y)} \approx \frac{j}{\lambda d}\iint_{\text{diaphragm}} f_{(x,y)}e^{-jkPM}dx\,dy,$$

$$E_{(X,Y)} \approx \frac{j}{\lambda d}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{(x,y)}e^{-jkPM}dx\,dy.$$

Fraunhofer

Diffraction - 6

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Near field diffraction (Fresnel)

- The most general and the most difficult problem.
- The distance between the diffracting aperture and the point of observation is far larger than the wavelength, but comparable to the size of the aperture.

Far field diffraction (Fraunhofer)

- x,y << d or d $\rightarrow +\infty$
- Infinity can be brought to the focal plane of a lens.

Diffraction - 8

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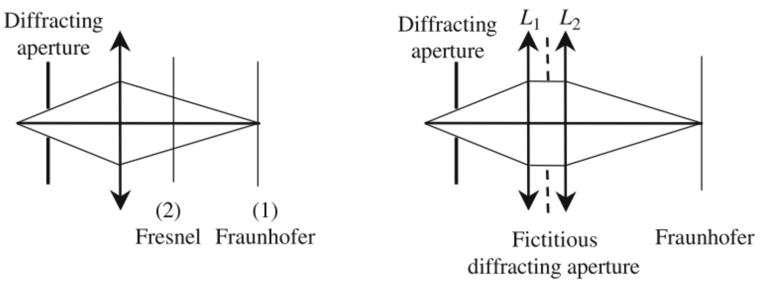


If the observation screen is in (2), we have Fresnel diffraction, while (1) corresponds to Fraunhofer diffraction. The lens of the left-hand figure (focal length f) is replaced by two separate lenses (focal lengths f_1 and f_2, such that $1/f_1 + 1/f_2 = 1/f$). The right-hand figure can then be considered as a far field diffraction arrangement through the fictitious diffracting aperture.

Diffraction - 7

Far field diffraction

$$F(X,Y)=\frac{j}{\lambda d}e^{-jkd}e^{-jk\frac{X^2+Y^2}{d}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)e^{-jk(ux+vy)}\,dx\,dy$$

$$F(k_x,k_y)=K\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)e^{-j(k_x x+k_y y)}\,dx\,dy,$$

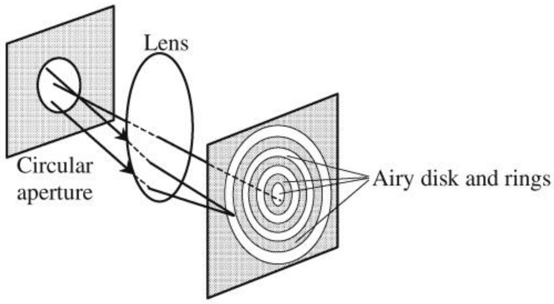
$$K=\frac{j}{\lambda d}e^{-jkd}e^{-jk\left(X^2+Y^2\right)/2d}$$

These expressions have a very noticeable interpretation:

- It is the decomposition of the diffracted field in planar harmonic waves.
- $F(k_x,k_y)$ is the two-dimensional Fourier transform of $f(x,y)$.
- The focal plane of a lens visualizes the Fourier transform of the incident wave.

Diffraction - 9

Resolving power of an imaging system

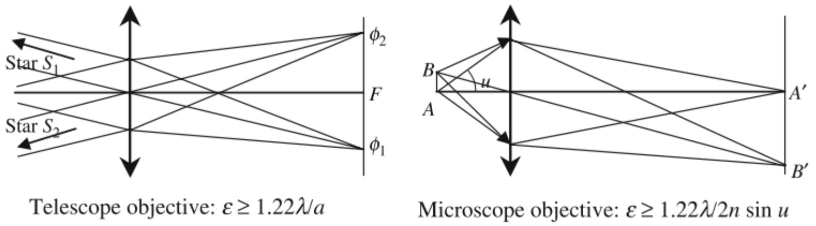


The diffraction pattern of a circular aperture, observed in the focal plane of a lens, is made of a disk surrounded by rings. The central disk is often called an "Airy disk."

$$\rho_{\text{Airy disk}}=f\frac{3.83}{\pi}\frac{\lambda}{2a}=1.22\frac{\lambda}{2a}f$$

Diffraction - 10

- Because of diffraction, the image of a point source is an Airy disk.
- The images of two neighboring points are seen separately id the center of one disk coincides with the first zero of the other (**Rayleigh criterion**).
- The distance between two points that are seen separately is the **limit of resolution** of the objective.



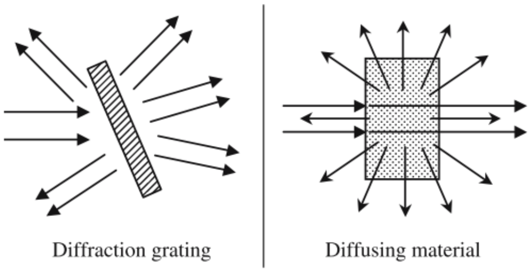
Resolving power of an objective. For the two images, (ϕ_1 and ϕ_2) or (A' and B'), to be separated when observed through an eyepiece, it is considered that their distance should be greater than the radius of the first dark Airy ring.

Diffraction gratings - 1

- The object under consideration is made of the collection of smaller (one to fifty times the wavelength) constituents or particles.
- The diffracted light is the result of interference of all the contributions arising from the single particles.

- *Diffusion*, where the spatial distribution of the particles is at random. Typical examples are the repartition of water drops in fog, or of nitrogen and oxygen molecules in the atmosphere.
- *Diffraction*, where the distribution of the particles is perfectly organized. The two most famous examples are the equidistant grooves of a grating and the arrangement of the atoms inside a perfect crystal.

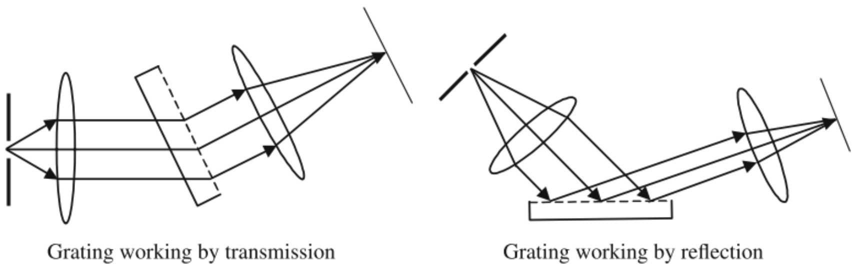
Diffraction gratings - 2



Comparison between diffusion and diffraction. For a periodic arrangement, the diffracted beams are observed only along specific directions called the orders of diffraction. Diffused light is emitted in all directions.

Some usual groove profiles. The periodicity ranges from $10\,\mu$m to $0.5\,\mu$m, i.e., 100 to 2000 grooves per millimeter.

Diffraction gratings - 3



A grating can work either by transmission or by reflection. The second arrangement is preferred since it is not concerned with a possible lack of homogeneity of the substrate that supports the grating.

- The directions of constructive interference are those for which the phase difference between the contributions of two neighboring grooves is very close to a multiple of 2π.
- For an infinite number of grooves the phase difference is strictly equal to a multiple of 2π (*phase matching condition*).

Diffraction gratings - 4

The Bragg formula in the optical case

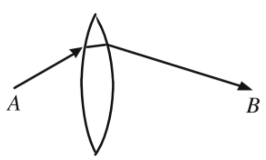
- The Bragg formula is a relationship between the angle of incidence and the angle of diffraction.
- It indicates the directions in which the different diffracted beams are emitted.
- It does not indicate how energy is shared among the diffracted beams.
- The shape of the grooves governs the partition rule.

$$(\sin\theta_{\mathrm{diff}} - \sin\theta_{\mathrm{inc}}) = p\frac{\lambda}{a} \quad \text{(same orientation for the two beams),}$$

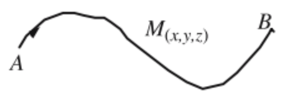
$$(\sin\theta_{\mathrm{diff}} + \sin\theta_{\mathrm{inc}}) = p\frac{\lambda}{a} \quad \text{(opposite orientation for the two beams)}$$

a is the periodicity, p is an integer (only such that $\sin\theta_{\mathrm{diff}} < 1$)

Inhomogeneous media - 1

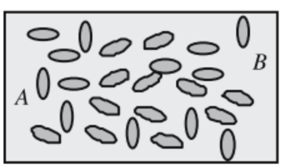
	*Homogeneous medium.* Same properties at any point. The speed of light and the refractive index have, respectively, the same value everywhere. To go from one point to another the light follows the path taking the shortest time, i.e., a straight line.
	*Discontinuously inhomogeneous medium.* Succession of different mediums, each of them being homogeneous. Two consecutive mediums are in contact along "surfaces of discontinuity." Inside a given medium the light follows a straight line, the light path is made of several rectilinear segments intersecting on discontinuity surfaces.

Inhomogeneous media - 2



Continuously inhomogeneous medium. Properties are not the same at the different points, however, their variations are continuous functions of the coordinates. The path actually followed by the light when going from a point A to a point B is no longer a straight line but a curve along which the transit time t_{AB} is the shortest possible one. $V_{(x,y,z)}$ being the local value of the light speed and s the abscissa along the curve t_{AB} is given by the integral: $t_{AB} = \int_A^B \frac{ds}{V_{(x,y,z)}}$.

Inhomogeneous media - 3



Discontinuously inhomogeneous mediums made of particles.
The medium is a collection of particles. If the size of the particles is small as compared to the wavelength, the medium can be considered as fully homogeneous with, however, the restriction that the particles' distribution should be homogeneous. If the particles are of the same order of magnitude as (or larger than) the wavelength, the situation is complicated, in most cases light is diffused in all directions.

Effect of thermal fields

Lorentz-Lorenz equation
(transparent media)

$$\frac{n^2-1}{n^2+2}\frac{1}{\rho}=\frac{r}{M_m}$$

r = Molar refractivity [m³/mol]
M_m = Molar mass [kg/mol]
ρ = Density [kg/m³]

For a gas:

$$n\cong 1 \quad\Rightarrow\quad \begin{array}{c} n^2-1=(n+1)(n-1)\cong 2(n-1) \\ n^2+1\cong 3 \end{array}$$

Gladstone-Dale equation
(gases)

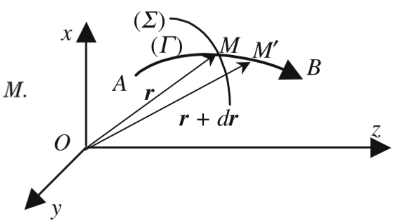
$$\frac{2(n-1)}{3\rho}=\frac{r}{M_m} \qquad n(T,p)=1+\frac{3r\rho(T,p)}{2M_m}$$

Ideal gas

$$n(T,p)=1+\frac{3pr}{2RT}$$

The ray equation - 1

- A light ray (Γ) going from point A to point B.
- Two points M and M' belonging to (Γ).
- The surface (Σ) of constant index containing point M.



Optical path: length that would be travelled in the same time by the light in a vacuum.

$$S(B)-S(A)=\int_{A(\Gamma)}^{B(\Gamma)} n\,ds$$

$$dS=\nabla S\cdot d\mathbf{r} \qquad \forall\, d\mathbf{r}$$

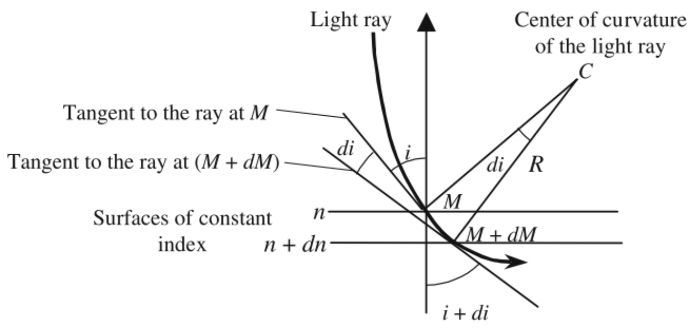
$$S=S(\mathbf{r}) \quad\Rightarrow\quad dS=\nabla S\cdot d\mathbf{r}$$

$$ds=\mathbf{u}\cdot d\mathbf{r} \qquad \text{vector } \mathbf{u}//\boldsymbol{\Gamma}$$

The variation dS of the optical path, when going from M to M', is given by $dS = \text{grad}\, S\, d\boldsymbol{r}\boldsymbol{u}$ ($\boldsymbol{u}$ is a unit vector perpendicular to (Σ) at point M). As $d\boldsymbol{r}$ may have any orientation, it is concluded that

$$\nabla S=n\mathbf{u} \quad\Rightarrow\quad |\nabla S|=n$$

The ray equation - 2



Propagation of a light ray in an inhomogeneous material. We have drawn the two planes that are tangent to the surfaces of constant index (respectively, n and $n + dn$). The light ray is bent and intersects the index surfaces at points M and M'. C and $R = CM$ are, respectively, the center and the radius of curvature of the ray at point M.

The ray equation - 4

$$\frac{d(n\boldsymbol{u})}{ds} = \frac{d}{ds}(\mathrm{grad}\, S) = \frac{d}{ds}\left(\frac{\partial S}{\partial x}\mathbf{x} + \frac{\partial S}{\partial y}\boldsymbol{y} + \frac{\partial S}{\partial z}\boldsymbol{z}\right),$$

$$d\left(\frac{\partial S}{\partial x}\right) = \mathrm{grad}\left(\frac{\partial S}{\partial x}\right)\boldsymbol{dr} \quad \rightarrow \quad \frac{d}{ds}\left(\frac{\partial S}{\partial x}\right) = \mathrm{grad}\left(\frac{\partial S}{\partial x}\right)\frac{\boldsymbol{dr}}{ds},$$

$$\frac{\boldsymbol{dr}}{ds} = \boldsymbol{u} \quad \rightarrow \quad \frac{d}{ds}\left(\frac{\partial S}{\partial x}\right) = \mathrm{grad}\left(\frac{\partial S}{\partial x}\right)\boldsymbol{u} = \mathrm{grad}\left(\frac{\partial S}{\partial x}\right)\left[\frac{1}{n}\mathrm{grad}\, S\right].$$

After development of the scalar product we obtain

$$\frac{d}{ds}\left(\frac{\partial S}{\partial x}\right) = \frac{1}{2n}\frac{\partial}{\partial x}(n^2) = \frac{\partial n}{\partial x} \quad \text{and similar expressions for } y \text{ and } z.$$

Finally, the differential equation of a light ray is

$$\frac{d}{ds}(n\boldsymbol{u}) = \left(\frac{\partial n}{\partial x}\boldsymbol{x} + \frac{\partial n}{\partial y}\boldsymbol{y} + \frac{\partial n}{\partial z}\boldsymbol{z}\right) = \mathrm{grad}(n).$$

The ray equation - 5

$$\frac{d\boldsymbol{u}}{ds}=\frac{\nu}{R} \quad (\nu \text{ is the unit vector of the principal normal})$$

$$\frac{dn}{ds}\boldsymbol{u}+\frac{n}{R}\nu=\mathrm{grad}(n).$$

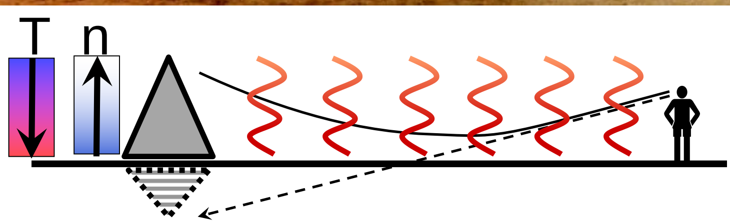
After scalar multiplication by ν, we have

$$\frac{dn}{ds}\frac{n}{R}\nu=\nu\,\mathrm{grad}(n).$$

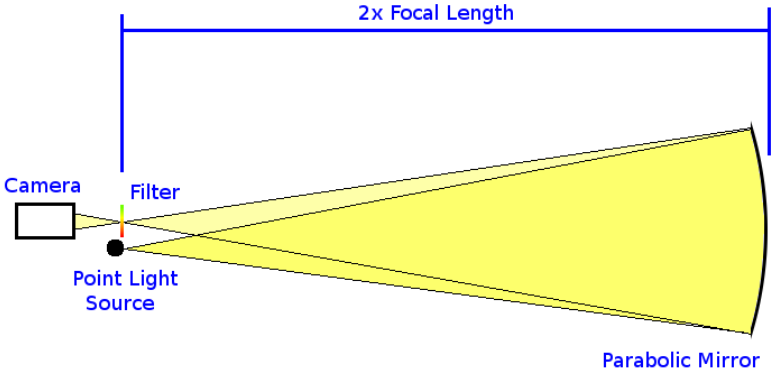
As a radius of curvature is always positive, the angle of the vector ν with the gradient is smaller than 90°, which implies that the concavity of a light ray is on the same side as the gradient. If i is the angle of incidence on the surface of constant index, the radius of curvature is given by

$$\frac{1}{R}=-\frac{1}{n}\frac{dn}{ds}\tan i.$$

The mirage

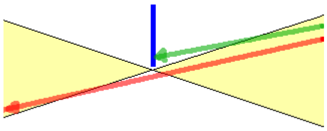


The Schlieren method - 1



A Schlieren Object, i.e. a variable refractive index field is placed between the source and the mirror

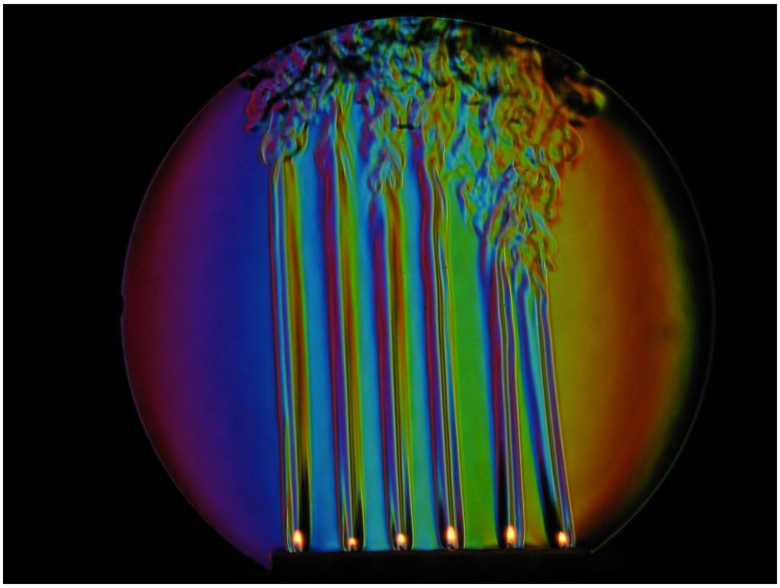
The Schlieren method - 2



Reflected rays after being refracted by the Schlieren Object do not converge to the focus

- By using a “knife edge” part of the refracted light can be cut off.
- The resulting image is characterized by an alternating bright and dark pattern.
- A colored filter works as well, coloring the intercepted light.

The Schlieren method - 3



The Schlieren method - 3