2d Lt James Marvin

HW 3/4

1. Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is one-to-one if
   a. $f(n) = n^3$

      It is one to one because if f(x)=f(y) -> x=y. You can see this because all n values produce unique f(n) values. Passes horizontal line test.

   b. $f(n) = \left\lceil \frac{n}{2} \right\rceil$

      It is not one to one because f(1)=1=f(2) and 2 is not equal to 1 and therefore violates the definition of a one-to-one function.

2. Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if
   a. $f(m, n) = m^2 - n^2$

      Not onto because you can't represent 2 as the difference between two squares.

   b. $f(m) = |m| - |n|$

      Onto, because you can get any nonnegative number setting m to that number and n to be 0. And you can get any negative number by setting m to be 0 and n to be that number.

3. Use the identities $\frac{1}{k\,(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers, to compute $\sum_{k=1}^{n} \frac{1}{k\,(k+1)}$.

   The goal is to get the equation $1/((k)(k+1))$ in the form of the difference between the term in a sequence and the previous term in the sequence so we can use the second identity in the problem. Given that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ we can rewrite the right side of the equality as $-\frac{1}{k+1} - (-\frac{1}{k})$. This allows us to use the second identity because it is the in the same form where $a_j = -1/(k+1)$ and $a_{j-1} = -1/k$. Using the second identity we can get $\sum_{j=1}^{n}(a_j - a_{j-1}) = \left(-\frac{1}{n+1}\right) - \left(-\frac{1}{0+1}\right) = 1 - \frac{1}{n+1}$ this gives us the answer $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit (show) a one-to-one correspondence between $\mathbb{Z}^+$ and that set.
   a. The set of integers not divisible by 3.

      The set is countable. You can map the set of integers to it by using mod 3. The set is composed by only integers that x mod 3 is 1 or 2. To build the set you can use the equation $x \equiv 1(mod\ 3), x \equiv 2(mod\ 3)$.

   b. The set of integers divisible by 5 but not by 7.

      Countable. You can build the set it by using mod 5 and mod 35. You include it if the mod 5 of that integer is 0 and if the integer mod 35 is not 0.

   c. The set of real numbers with decimal representations of all 1s or 9s.

      Uncountable

5. For each of the following properties, give an example of two uncountable sets A and B such that $A - B$ is a set having that property.
   a. Finite

      A= {0,1,[2,∞)}

      B={1,[2,∞)}

   b. Countably infinite

      A=Z

      B=(0,1) in R

   c. Uncountable

      A=(1,2) in R

      B=(0,1) in R

6. Find all solutions, if any, to the following system of congruences.

$$x \equiv 1 \pmod{2} x \equiv 2 \pmod{3} x \equiv 3 \pmod{5}$$

Set m1=2, m2=3, m3=5

Set a1=1, a2=2, a3=3

Set m=m1m2m3=2*3*5=30

M1=m/m1=30/2=15

M2=m/m2=30/3=10

M3=m/m3=30/5=6

Y1=M1 mod m1= 15 mod 2 = 1

Y2=M2 mod m2 = 10 mod 3 = 1

Y3= M3 mod m3 = 6 mod 5 = 1

X=a1M1Y1+a2M2Y2+a3M3Y3

X= 1(15)(1)+(2)(10)(1)+(3)(6)(1)= 53 ≡ 23 (mod 30)

7. Use Fermat's little theorem and the Chinese remainder theorem to find
   a. $3^{302} \bmod 5$

      302=4*75+2

      $(3^4)^{75} * 3^2 mod\ 5 = 4$

   b. $3^{302} \bmod 7$

      302=6(50)+2

      $(3^6)^{50} * 3^2 mod\ 7 = 2$

c. $3^{302} \bmod 11$

302=10(30)+2

$(3^{10})^{30} * 3^2 mod 11 = 9$

d. $3^{302} \bmod 385$

385=5*7*11

$x \equiv 4(mod\ 5), x \equiv 2(mod\ 7), x \equiv 9(mod\ 11)$

$a1 = 4, a2 = 2, a3 = 9, m1 = 5, m2 = 7, m3 = 9$

$M1 = 7 * 9 = 63, M2 = 5 * 9 = 45, M3 = 5 * 7 = 35$

$y1 = 63(mod\ 5) = 3, y2 = 45(mod\ 7) = 3, y3 = 35(mod\ 9) = 8$

$x = (4 * 63 * 3) + (2 * 45 * 3) + (9 * 35 * 8) = 3546 \equiv 81(mod\ 385)$

81

8. Use strong induction to prove that postage of $n \geq 18$ cents can be formed using just 4-cent stamps and 7-cent stamps.

   Let P(n) be the proposition “every postage of n>=18 can be formed using just 4-cent and 7-cent stamps”

   Base cases: P(18)=7+7+4, P(19)=7+4+4+4, P(20)=4+4+4+4+4, P(21)=7+7+7

   Inductive hypothesis: The inductive hypothesis is that P(j) is true for all integers with 18<=j<=k, that is, the assumption that amounts of j-cent postage can be formed using just 4-cent stamps and 7-cent stamps whenever j is an integer between 18 and not exceeding k.

   For k>=21 we can get to the k+1 postage by replacing a 7-cent stamp with two 4-cent stamps or five 4-cent stamps with three 7-cent stamps. This adds one because 7 becomes 8 in the first case or 20 becomes 21 in the second case.

We have shown that P(18),P(19), P(20), P(21) are true. We have also shown that for all j-cent postage between 18 and k $P(j) \to P(k+1)$. It follows from the principle of strong mathematical induction that P(n) is true for n>=18.

9. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two. *Hint: For the inductive step, consider even and odd numbers separately.*

Let P(n) be the proposition "every integer n can be formed using a sum of distinct powers of two".

Base cases: P(1) is true because you can represent it as 2^0. P(2) is true because you can represent it as 2^1.

Inductive hypothesis: The inductive hypothesis is that P(j) is true for all integers $1 \leq j \leq k$, that is the assumption that an integer j can be formed using a sum of distinct powers of two when j is at least 1 and not exceeding k.

Inductive step: to complete the inductive step, it must be shown that P(k+1) is true under the assumption of the inductive hypothesis, that is, that it is the case that an integer k+1 can be formed using a sum of powers of two.

To show the P(k+1) case is true we can split it into the case where k+1 is even or odd. If k+1 is even we know (k+1)/2 is also even. Because (k+1)/2 is less than or equal to k we know its true under the inductive hypothesis. We can get from the solution to (k+1)/2 to k+1 by multiplying the solution by 2. This raises every power of two by 1 and we still have an equation with distinct powers of 2. For when k+1 is odd we know that it is the same equation for k plus 2^0. We know that 2^0 is not already in the k equation because an even function represented as distinct powers of 2 does not have a term that represents 1.

We have shown that P(1) and P(2) are true. We have also shown that $2 \leq j \leq k$ $P(j)$ $\to P(k+1)$. It follows from the principle of mathematical induction that P(n) is true for all n>=2.

10. Suppose that $P(n)$ is a propositional function. Determine for which $n \in \mathbb{N}$ the statement $P(n)$ must be true if
    a. $P(0)$ is true and $P(n) \to P(n+2)$.

0 and all positive even numbers.

b. $P(0)$ is true and $P(n) \rightarrow P(n+3)$.

0 and all positive multiples of three.

c. $P(0)$ and $P(1)$ are true and $P(n) \wedge P(n+1) \rightarrow P(n+2)$.

0 and all positive integers.

d. $P(0)$ is true and $P(n) \rightarrow P(n+2) \wedge P(n+3)$.

0 and all integers greater than and equal to 2.

11. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for all $n \in \mathbb{Z}^+$ where f_n is the nth Fibonacci number

To prove this, we will look at the right side of the equation as a recurrence relationship. A Fibonacci number $f_{n+1} = f_n + f_{n-1}$. We can substitute this into the right side of the equation in our proposition $f_n f_{n+1} = f_n(f_n + f_{n-1})$ which can be rewritten as $f_n^2 + f_{n-1}f_n$. Recursing another step we get the equation $f_n^2 + f_{n-1}^2 + f_{n-1}f_{n-2}$. Going all the way down to n=2 we are left with the equation $f_n^2 + f_{n-1}^2 + \cdots + f_2^2 + f_2 f_1$ and because the first and second Fibonacci numbers are 1 we can rewrite this as $f_n^2 + f_{n-1}^2 + \cdots + f_2^2 + f_1^2$. This is equivalent to the left side of our original proposition. We have proved that the proposition defined in the question is true.