Adjusting the L Statistic when Self-Transitions are Excluded in Affect Dynamics

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Affect dynamics, the investigation of how student affect transitions from one state to another, is a popular area of research in adaptive learning environments. Recently, the commonly used transition metric L has come under critical examination when applied to data that exclude self-transitions (i.e., transitions where a student remains in the same affective state on consecutive observations); in this situation, recent work has shown that it potentially overestimates the significance of certain transitions. In order to deal with the unintended side effects of removing self-transitions, a solution was proposed that shifts the chance value of L from zero to a positive value that varies based on the number of affective states in the study. Although this treatment compensates for the aforementioned issues, it could lead to misinterpretation of the results due to the counterintuitive nonzero chance value.

Motivated by these issues, in this work we study a modified version of the L statistic, which we refer to as L^*, with an aim to shift the chance value back to zero when self-transitions are removed. We begin by studying the mathematical and statistical properties of L^*, and we also compare these properties to those of the L statistic. After analyzing the theoretical attributes of L^*, we then evaluate its performance when applied to data. In our first evaluation, we compute L^* values on simulated sequences of affective states, where the base rates of the states vary from being uniform to highly non-uniform. The results provide evidence that the L^* values at chance are not sensitive to unequal base rates, as the computed values from our experiments are centered closely around zero. Our second evaluation applies L^* to actual student data generated from students working in a digital learning environment; in this case, the use of L^* seemingly gives a more coherent picture in comparison to the values returned by the L statistic. Finally, along with these analyses, we also outline a recommended procedure for applying L^* to sequences of affective states.

Keywords: affect dynamics, L statistic, self-transitions, transition metric

1. Introduction

Affect is an important construct in artificially intelligent educational systems, as it has been shown to correlate closely with student learning, motivation and engagement (Bosch and D'Mello, 2017; Rodrigo et al., 2008). Accordingly, affect-sensitive interventions have been designed in virtual learning environments to improve students' learning (DeFalco et al., 2018) and overall experience (Karumbaiah et al., 2017). One commonly used approach to collect affect labels is through classroom field observations by certified expert coders. A frequently used technique for these observations is the Baker Rodrigo Ocumpaugh Monitoring Protocol (BROMP; Ocumpaugh et al. 2015), an affect coding protocol wherein students are observed by certified coders. These observations are made in a round-robin fashion, with each observation lasting up to 20 seconds. The affective state labels are boredom (BOR), confusion (CON), frustration (FRU), and engaged concentration (FLO). During coding, some observations are labeled NA, corresponding to the cases where (a) the student could not be observed, (b) the affective state was unclear to the observer, or (c) the student was in an affective state other than the states being coded. Some research studies have also used self-reporting methods, collecting affect data either through spontaneous in-system surveys (McQuiggan and Lester, 2009) or through the retrospective judgments of the participants themselves (D'Mello and Graesser, 2012). In addition, several research studies in the past decade have focused on building automated detectors of affect using physical and physiological sensors (Nye et al., 2018), as well as interaction log data (Botelho et al., 2017).

Affect dynamics in particular is a field of research that focuses on how affect develops and manifests over time (D'Mello and Graesser, 2012). Understanding affect dynamics is an important part of developing effective affect-sensitive interventions (Kuppens, 2015). Additionally, a large body of research has examined how students transition from one affective state to another while engaging in learning activities (see, e.g., the review in Karumbaiah et al. 2018). A popular transition metric used in this research is the L statistic, which was originally introduced by D'Mello et al. (2007). The L statistic calculates the relative likelihood that an affective state ($prev$) will transition to a subsequent state ($next$), given the base rate of the subsequent state.

Definition 1 (D'Mello's L)**.** For affective states A and B, let A_{prev} represent the occurrence of A as the previous state in a transition, and let B_{next} represent the occurrence of B as the next state in a transition. We then have

$$L(A_{prev} \to B_{next}) := \frac{P(B_{next} \mid A_{prev}) - P(B_{next})}{1 - P(B_{next})}. \tag{1.1}$$

A special case that is not fully discussed in most of the literature is that of *self-transitions*; these are simply transitions where the student remains in the same affective state for more than one step in a sequence. In many recent studies, researchers have removed self-transitions in the data preparation stage (see the review in Karumbaiah et al. 2018). Typically, this choice is based on the goal of the research study. If some affective states are particularly persistent (Baker et al., 2010), including self-transitions in the analysis helps the researcher obtain a better understanding of this behavior. However, as persistent affective states can suppress the apparent significance of transitions between different states, excluding self-transitions could be a better choice if the goal is to reveal a larger number of important affective patterns.

While this straightforward procedure of excluding self-transitions seems quite logical, it has unintended consequences when applying the L statistic. Specifically, if A_{prev} and B_{next} are independent then $P(B_{next} \mid A_{prev}) = P(B_{next})$, thus making the L value at chance equal to zero. However, as observed by Karumbaiah et al. (2019), removing self-transitions violates the assumption of independence between B_{next} and A_{prev}, as the next state can now only take on values other than A. Hence, when self-transitions are excluded, the at chance values of $P(B_{next} \mid A_{prev})$ and $P(B_{next})$ are not necessarily equal, and we can then no longer assume that zero represents the at chance value of L. Thus, in past studies that excluded self-transitions the L results are likely to have been misinterpreted. In light of these observations, Karumbaiah et al. (2019) showed that for a state space with n affective states ($n > 2$), the value of L at chance is

$$L(A_{prev} \to B_{next}) = \begin{cases} 0, & \text{if self-transitions are included} \\ \frac{1}{(n-1)^2}, & \text{if self-transitions are excluded.} \end{cases}$$

This finding showed that the L statistic must be interpreted differently depending on how many affective states are being observed. Although this solution offers a remedy for the issues encountered when applying the L statistic after self-transitions are removed, it increases the difficulty of interpreting the statistic. While running statistical tests or attempting to make sense of the transition patterns, the users have to be cautious about choosing the correct L value at chance; additionally, this choice also affects the interpretation of the directionality of the transitions. Thus, there is the danger of wrongly assuming that certain positive L values are significantly greater than the chance value, when in fact the opposite may be true. For instance, in a study with four affective states, all L values from 0 to 0.11 are smaller than the value at chance, which is counterintuitive and could lead to misinterpretation of the results.

Due to these issues, as an alternative to the procedure proposed in Karumbaiah et al. (2019), in this paper we instead study a modification to the L statistic, the goal of which is to center the chance value back at zero. We present a detailed account of the mathematical and statistical properties of this modified statistic, which we refer to as L^*, and we compare its properties to those of the original L statistic. After analyzing the theoretical attributes of L^*, we then evaluate its performance empirically by applying it to data. Our first evaluation applies L^* to simulated sequences of affective states, where the base rates of the states vary from being uniform to highly non-uniform. Then, our second evaluation applies L^* to actual student data generated from students working in a digital learning environment. Along the way, we also outline a recommended procedure for applying L^* to sequences of affective states.

2. MODIFYING THE L STATISTIC

In this section we define L^*, a modified version of D'Mello's L statistic. Assume that we are interested in studying transitions of the form $A_{prev} \to B_{next}$, and that we want to ignore transitions from A to itself. The L^* statistic is intended to be used for this situation, and it is designed so that the value at chance is zero.

Definition 2. Let A and B be two affective states, and let

$$T_{\overline{A}} = \{\text{transitions where } next \neq A\}. \tag{2.1}$$

That is, $T_{\overline{A}}$ consists of all transitions where the next affective state is not equal to A. We can then define

$$L^*(A_{prev} \to B_{next}) := \frac{P(B_{next} \mid A_{prev}, T_{\overline{A}}) - P(B_{next} \mid T_{\overline{A}})}{1 - P(B_{next} \mid T_{\overline{A}})}. \tag{2.2}$$

Note that the above formula is equivalent to applying (1.1) to the transitions in $T_{\overline{A}}$. Alternatively, the probabilities in (2.2) can also be computed as

$$P(B_{next} \mid T_{\overline{A}}) = \frac{P(B_{next})}{1 - P(A_{next})}, \tag{2.3}$$

and

$$P(B_{next} \mid A_{prev}, T_{\overline{A}}) = \frac{P(B_{next} \mid A_{prev})}{1 - P(A_{next} \mid A_{prev})}, \tag{2.4}$$

where the values on the right-hand sides of (2.3) and (2.4) are derived from the full set of transitions (i.e., transitions where $next = A$ are also included).

During the preparation of this manuscript, the authors were made aware of a similar statistic, λ, which appears in Bosch and Paquette (2020). Under the formulation given in Definition 2, the base rates can be computed either individually for each sequence, or averaged over a set of sequences (where the average is taken from the individual rates computed for each sequence). If the base rates are computed individually per sequence, L^* is equivalent to λ; on the other hand, it is distinct from λ if the base rates are averaged over an entire sample of sequences. Anecdotally, we know that some researchers prefer the latter procedure of averaging over the entire sample; however, it has also been argued that calculating the base rates individually per sequence is less likely to return extreme values (Karumbaiah et al., 2018). We compare and contrast these two approaches for computing the base rates in several places throughout the rest of this manuscript.

The intuition behind the definition of L^* and, in particular, the restriction to only the transitions in $T_{\overline{A}}$, is the following. Suppose we have a transition that begins in affective state A, and suppose also that, as has been done in many past studies (Bosch and D'Mello, 2013; Bosch and D'Mello, 2017; Botelho et al., 2018; D'Mello and Graesser, 2010; D'Mello and Graesser, 2012; D'Mello et al., 2007; D'Mello et al., 2009; Karumbaiah et al., 2018), we are excluding *all* self-transitions. With the L statistic, we want to evaluate $P(B_{next} \mid A_{prev})$ to get a measure of the association between affective states A and B; to control for the base rate of B, we do this by comparing the value of $P(B_{next} \mid A_{prev})$ with $P(B_{next})$. However, when self-transitions are excluded, A is not an option for the next affective state, which means we are not making a fair comparison. That is, $P(B_{next})$ is computed under the assumption that all states are possible, while $P(B_{next} \mid A_{prev})$ is computed with state A removed as a possible affective state to transition to; in most cases, this would serve to inflate the difference $P(B_{next} \mid A_{prev}) - P(B_{next})$. Further complicating the issue is that, with self-transitions removed, two consecutive $next$ states must be different; in such a case, $P(B_{next})$ cannot be much larger than 0.5, which could again inflate the difference $P(B_{next} \mid A_{prev}) - P(B_{next})$.

To compensate for these issues, $P(B_{next} \mid T_{\overline{A}})$ and $P(B_{next} \mid A_{prev}, T_{\overline{A}})$ are normalized values of $P(B_{next})$ and $P(B_{next} \mid A_{prev})$, respectively, where the influence of the previous state A has been removed. Thus, the goal of applying these normalizations is to remove any bias by computing the frequency of B only in relation to the other possible next states (i.e., all the states except for A). In order to formalize this last claim, however, we need to borrow the concept of *conditional independence*.

Definition 3. Let A and B be two affective states, and let $T_{\overline{A}}$ be defined as in (2.1). Then, we say that the events B_{next} and A_{prev} are conditionally independent given $T_{\overline{A}}$ if

$$P(B_{next} \cap A_{prev} \mid T_{\overline{A}}) = P(B_{next} \mid T_{\overline{A}}) \cdot P(A_{prev} \mid T_{\overline{A}}). \tag{2.5}$$

Note that the definition of conditional independence is very similar to the standard definition of independence, with the only difference being that each probability is conditioned on $T_{\overline{A}}$. Thus, if we restrict ourselves to looking only at transitions in $T_{\overline{A}}$, the definition of conditional independence simplifies to the standard definition of independence, giving us a straightforward way to check if (2.5) holds.

The motivation for using conditional independence is the following. Suppose we have a transition of the form $A_{prev} \to B_{next}$. As observed by Karumbaiah et al. (2019), when self-transitions are excluded there is no longer independence between the events A_{prev} and B_{next}, as the next state can now only take on values other than A; in other words, the set of possible values for the next state is explicitly dependent on the value of the previous state. As a direct consequence of this dependence between the events A_{prev} and B_{next}, the equality of $P(B_{next} \mid A_{prev})$ and $P(B_{next})$ is no longer guaranteed, and thus we cannot assume that $L = 0$ represents the value at chance.

As shown in Definition 2, we have two equivalent ways in which to compute L^* on a sequence of affective states. We can either modify our computations of the probabilities in (1.1), or we can modify the sequence itself and then apply the standard formula for the L statistic. In the latter case, removing transitions to A also has an effect on our concept of conditional independence; as mentioned previously, once transitions to A are removed, the formula for conditional independence (2.5) then simplifies to the formula for regular independence. Thus, if regular independence holds for the modified sequence with transitions to A removed, it follows that the L statistic (and, hence, L^*) should be equal to zero; that is, when transitions to A are excluded the concept of at chance is now captured by the conditional independence assumption. Our next theorem formalizes this result.

Theorem 1. Let A and B be affective states, and suppose that the conditional independence requirement given in (2.5) holds. Assume also that $P(B_{next} \mid T_{\overline{A}}) < 1$. Then,

$$L^*(A_{prev} \to B_{next}) = 0.$$

In other words, the value of L^* at chance is equal to zero.

Proof. By Definition 2 we have

$$L^*(A_{prev} \to B_{next}) = \frac{P(B_{next} \mid A_{prev}, T_{\overline{A}}) - P(B_{next} \mid T_{\overline{A}})}{1 - P(B_{next} \mid T_{\overline{A}})}.$$

Using the definition of conditional probability, we begin by expanding both sides of (2.5) to get the equality

$$\frac{P(B_{next} \cap A_{prev} \cap T_{\overline{A}})}{P(T_{\overline{A}})} = \frac{P(B_{next} \cap T_{\overline{A}})}{P(T_{\overline{A}})} \cdot \frac{P(A_{prev} \cap T_{\overline{A}})}{P(T_{\overline{A}})}.$$

Multiplying both sides of the above equation by $P(T_{\overline{A}})$, we then have

$$P(B_{next} \cap A_{prev} \cap T_{\overline{A}}) = \frac{P(B_{next} \cap T_{\overline{A}}) \cdot P(A_{prev} \cap T_{\overline{A}})}{P(T_{\overline{A}})}. \tag{2.6}$$

Applying (2.6), along with the definition of conditional probability once again, it follows that

$$\begin{aligned} P(B_{next} \mid A_{prev}, T_{\overline{A}}) &= \frac{P(B_{next} \cap A_{prev} \cap T_{\overline{A}})}{P(A_{prev} \cap T_{\overline{A}})} && \text{(definition of conditional probability)} \\ &= \frac{P(B_{next} \cap T_{\overline{A}}) \cdot P(A_{prev} \cap T_{\overline{A}})}{P(A_{prev} \cap T_{\overline{A}}) \cdot P(T_{\overline{A}})} && \text{(using (2.6))} \\ &= \frac{P(B_{next} \cap T_{\overline{A}})}{P(T_{\overline{A}})} \\ &= P(B_{next} \mid T_{\overline{A}}). \end{aligned}$$

Plugging this result into (2.2), it follows that $L^*(A_{prev} \to B_{next}) = 0$. □

3. EXAMPLES

Now that we have introduced L^*, we next look at specific examples to illustrate how it works. To simplify this discussion, in each example we compute the base rates of the states from each sample individually (as opposed to computing the average base rates from a sample of sequences).

Example 1. Consider the following sequence, which was first discussed in Karumbaiah et al. (2019).

$$ABBCAACCBA$$

In this sequence of transitions each affective state is followed by each of the other affective states equally (that is, A transitions once to each of B and C; B transitions once to each of A and C; etc.). Because of this, we would expect $L^*(A_{prev} \to B_{next})$ to be zero. We can verify this both directly and by applying the conditional independence condition. For the latter, we start by looking at the transitions in $T_{\overline{A}}$.

$$AB, \quad BB, \quad BC, \quad AC, \quad CC, \quad CB \tag{3.1}$$

Since there is one transition from A to B, we have

$$P(B_{next} \cap A_{prev} \,|\, T_{\overline{A}}) = \frac{1}{6}.$$

As B is the next state in three of the transitions, we have

$$P(B_{next} \,|\, T_{\overline{A}}) = \frac{3}{6} = \frac{1}{2},$$

and since A is the previous state in two of the transitions

$$P(A_{prev} \,|\, T_{\overline{A}}) = \frac{2}{6} = \frac{1}{3}.$$

We can then verify that the conditional independence condition given in (2.5) is satisfied:

$$\begin{aligned} P(B_{next} \cap A_{prev} \,|\, T_{\overline{A}}) &= \frac{1}{6} \\ &= \frac{1}{2} \cdot \frac{1}{3} \\ &= P(B_{next} \,|\, T_{\overline{A}}) \cdot P(A_{prev} \,|\, T_{\overline{A}}). \end{aligned}$$

Based on Theorem 1, we now know that $L^*(A_{prev} \to B_{next}) = 0$. To verify this result directly, we compute $L^*(A_{prev} \to B_{next})$ using (2.2). Once again using the transitions listed in (3.1), we note that there are two transitions with A as the previous state, one of which has B as the next state; thus, we have

$$P(B_{next} \,|\, A_{prev}, T_{\overline{A}}) = \frac{1}{2}.$$

Combining this with the previous computation for $P(B_{next} \,|\, T_{\overline{A}})$, it follows that

$$L^*(A_{prev} \to B_{next}) = \frac{\frac{1}{2} - \frac{1}{2}}{1 - \frac{1}{2}} = 0.$$

The previous computation demonstrates an example where L^* returns a value of zero when all transitions are made equally. However, as a consequence of the transitions being evenly distributed, the affective states all appear with equal frequencies (i.e., the base rates of the states are all equal). Our next example highlights a case when the affective states occur with different frequencies.

Example 2. Consider the following sequence of affective states.

$$ABABACAD$$

For this sequence, we have the following base rates.

$$P(A_{next}) = \frac{3}{7}$$
$$P(B_{next}) = \frac{2}{7}$$
$$P(C_{next}) = P(D_{next}) = \frac{1}{7}$$

Furthermore, we also have the following conditional rates.

$$P(B_{next} \,|\, A_{prev}) = \frac{1}{2}$$
$$P(C_{next} \,|\, A_{prev}) = P(D_{next} \,|\, A_{prev}) = \frac{1}{4}$$

So, whether or not we condition on A being the previous state, C and D each occur as the next state half as often as B. Put another way, the relative rates of B, C, and D are not affected when the previous state is A. Thus, based on this argument, we would expect that conditional independence is satisfied for B_{next} and A_{prev}.

To verify this, we begin by computing $P(B_{next} \cap A_{prev} \,|\, T_{\overline{A}})$. Note that there are four transitions in which A is not the next state; that is, there are four transitions that are contained in $T_{\overline{A}}$.

$$AB, \quad AB, \quad AC, \quad AD \tag{3.2}$$

Two of the transitions are from A to B; thus,

$$P(B_{next} \cap A_{prev} \,|\, T_{\overline{A}}) = \frac{2}{4} = \frac{1}{2}.$$

Since A is the previous state in all four of these transitions, we have

$$P(A_{prev} \,|\, T_{\overline{A}}) = \frac{4}{4} = 1,$$

and since two transitions are to B we have

$$P(B_{next} \,|\, T_{\overline{A}}) = \frac{2}{4} = \frac{1}{2}.$$

Putting this all together, we can see that the conditional independence condition given in (2.5) is satisfied:

$$\begin{aligned} P(B_{next} \cap A_{prev} \,|\, T_{\overline{A}}) &= \frac{1}{2} \\ &= \frac{1}{2} \cdot 1 \\ &= P(B_{next} \,|\, T_{\overline{A}}) \cdot P(A_{prev} \,|\, T_{\overline{A}}). \end{aligned}$$

So, Theorem 1 now tells us that $L^*(A_{prev} \to B_{next})$ is equal to zero. We next check this directly using the transitions in (3.2). We first note that, since A is the previous state in all four transitions, and since two of these transitions have B as the next state, we can compute

$$P(B_{next} \,|\, A_{prev}, T_{\overline{A}}) = \frac{2}{4} = \frac{1}{2}.$$

Combining this with our previous calculation for $P(B_{next} \,|\, T_{\overline{A}})$, we have

$$L^*(A_{prev} \to B_{next}) = \frac{\frac{1}{2} - \frac{1}{2}}{1 - \frac{1}{2}} = 0.$$

4. Properties of L and L^*

In Section 2 we showed that the value of L^* is zero when the assumption of conditional independence is satisfied; in other words, the value of L^* at chance is zero. In this section we look in more detail at the properties of both L^* and L. Throughout this discussion we are considering transitions of the form $A_{prev} \to B_{next}$, and when applying L we always assume that all self-transitions have been removed; on the other hand, when applying L^* we always begin with the full sequences containing all self-transitions.

4.1. Maximum values

We first note that the maximum value for both L and L^* is one. This maximum value holds under any of the possible scenarios we have considered thus far; for example, it holds if we remove all self-transitions and then compute L using (1.1), and it also holds if we look only at transitions in $T_{\overline{A}}$ and apply (2.2) to compute L^*. This can be seen by observing that both statistics have a value of one if (a) state A is always followed by state B, and (b) the base rate of B is less than one. To see this, consider the following example sequence.

$$ABCAB \tag{4.1}$$

Since there are no self-transitions (i.e., the sequence already has all self-transitions "removed"), we can directly compute the following values from (4.1).

$$\begin{aligned} P(B_{next} \,|\, A_{prev}) &= \frac{2}{2} = 1 \\ P(B_{next}) &= \frac{2}{4} = \frac{1}{2} \end{aligned}$$

Plugging these into (1.1) gives

$$L(A_{prev} \to B_{next}) = \frac{1 - \frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

Next, consider the transitions in $T_{\overline{A}}$.

$$AB, \quad BC, \quad AB$$

Based on these transitions, we have

$$P(B_{next} \,|\, A_{prev}, T_{\overline{A}}) = \frac{2}{2} = 1$$
$$P(B_{next} \,|\, T_{\overline{A}}) = \frac{2}{3}.$$

Applying (2.2), it follows that

$$L^*(A_{prev} \to B_{next}) = \frac{1 - \frac{2}{3}}{1 - \frac{2}{3}} = 1.$$

Note that if we had instead computed the base rate of B averaged over a set of sequences, the results of the above computations would be unchanged as long as the base rate is less than one (if the base rate is exactly equal to one, L^* is undefined). Thus, as a consequence of the arguments and example given in this section, we have the following result.

Theorem 2. Let K be a sequence of affective states, where K includes at least one transition with $prev = A$. We then have

$$L^*(A_{prev} \to B_{next}) \leq 1,$$

regardless of whether the base rates are computed individually per sequence, or averaged over all sequences. Furthermore, we also have

$$L(A_{prev} \to B_{next}) \leq 1,$$

regardless of whether the base rates are computed individually per sequence, or averaged over all sequences, and also irrespective of whether or not self-transitions have been removed.

4.2. Minimum values

Now that we have established the maximum values for L and L^*, we next look at the minimum values for the statistics. We first focus on the procedure that has historically been used when excluding self-transitions; specifically, the case when all self-transitions are removed, and then L is computed by applying (1.1) to the remaining transitions.

To that end, consider the following sequence.

$$CBACB \tag{4.2}$$

Note that the sequence does not contain any self-transitions. From this sequence we can compute the following values.

$$P(B_{next} \,|\, A_{prev}) = \frac{0}{1} = 0$$
$$P(B_{next}) = \frac{2}{4} = \frac{1}{2}$$

Applying (1.1), we have

$$L(A_{prev} \rightarrow B_{next}) = \frac{0 - \frac{1}{2}}{1 - \frac{1}{2}} = -1.$$

Assuming that self-transitions are completely removed, it turns out that this is the minimum value for L when the base rates are computed individually for each sequence.

Theorem 3. Let K be a sequence of affective states containing $k \geq 2$ transitions, where K includes at least one transition with $prev = A$. Assume also that K does not contain any self-transitions, and that all base rates are computed individually per sequence. We then have

$$L(A_{prev} \rightarrow B_{next}) \geq -1.$$

Proof. The main intuition behind the proof is the observation that, as we are completely excluding self-transitions, $P(B_{next})$ is bounded away from one (that is, with self-transitions excluded, it's not possible for B to be the next state in consecutive transitions). Thus, the denominator of (1.1) is then bounded away from zero, which prevents L from being large (in absolute value) and negative.

To begin, we note that since we are excluding self-transitions, each consecutive pair of states in K must be different. Since K contains k total transitions, it also contains k total $next$ states. Thus, the maximum number of times any one affective state B can appear in K as the $next$ state is given by

$$\max(B) = \begin{cases} \frac{k}{2} & \text{for } k \text{ even,} \\ \frac{k+1}{2} & \text{for } k \text{ odd.} \end{cases}$$

We first handle the case when k is odd and B appears as the next state $\frac{k+1}{2}$ times; this means that B appears as every second state. Thus, for every transition in which $prev = A$, we must also have $next = B$ (since B appears as every second state). As we are assuming there is at least one transition with $prev = A$, we must have $P(B_{next} \,|\, A_{prev}) = 1$, which then implies that $L(A_{prev} \rightarrow B_{next}) = 1 > -1$.

To handle the rest of the cases, we need to show that the rate of B is bounded above by 0.5. To that end, assume once again that k is odd, and suppose that B appears as the next state at most $\frac{k+1}{2} - 1$ times. We then have

$$P(B_{next}) \leq \frac{\frac{k+1}{2} - 1}{k} < 0.5.$$

Finally, when k is even we have the inequality

$$P(B_{next}) \leq \frac{\frac{k}{2}}{k} = 0.5.$$

So, we have now shown that either $L(A_{prev} \to B_{next}) = 1$, or $P(B_{next}) \leq 0.5$. Next, observe that L is strictly decreasing as a function of $P(B_{next})$; this can be seen by noting that the derivative of L with respect to $P(B_{next})$ is negative for any value of $P(B_{next})$ in $(0, 1)$. Thus, the minimum value of L occurs when $P(B_{next}) = 0.5$, from which it follows that

$$L \geq \frac{0 - 0.5}{1 - 0.5} = -1.$$

$\square$

The next result gives the lower bound for L when self-transitions are removed and the base rates are computed by averaging over all sequences.

Theorem 4. Let K be a sequence of affective states containing $k \geq 2$ transitions, where K includes at least one transition with $prev = A$. Assume also that K does not contain any self-transitions, and that all base rates are computed by averaging over all sequences. We then have

$$L(A_{prev} \to B_{next}) \geq -2.$$

Proof. As discussed in the proof of Theorem 3, for a given sequence K containing k transitions, the maximum number of times any one affective state B can appear in K as the $next$ state is given by

$$\max(B) = \begin{cases} \frac{k}{2} & \text{for } k \text{ even,} \\ \frac{k+1}{2} & \text{for } k \text{ odd.} \end{cases}$$

If k is even, then the base rate of B for any one sequence is bounded by

$$P(B_{next}) \leq \frac{\frac{k}{2}}{k} = \frac{1}{2}.$$

Next, if k is odd, we have

$$P(B_{next}) \leq \frac{\frac{k+1}{2}}{k} = \frac{1}{2} + \frac{1}{2k}.$$

The expression on the right side is a decreasing function of k; thus, since we are assuming that k is odd and that $k \geq 2$, the maximum value of the expression occurs at $k = 3$ and is given by $\frac{2}{3}$. Combining these results, we've now shown that for any individual sequence the base rate of B is bounded above by $\frac{2}{3}$. Plugging this into (1.1), and using the fact that $P(B_{next} \mid A_{prev}) \geq 0$, we have

$$\begin{aligned} \frac{P(B_{next} \mid A_{prev}) - P(B_{next})}{1 - P(B_{next})} &\geq \frac{0 - P(B_{next})}{1 - P(B_{next})} \\ &= \frac{-P(B_{next})}{1 - P(B_{next})}. \end{aligned} \tag{4.3}$$

Next, observe that (4.3) is strictly decreasing as a function of $P(B_{next})$ on the interval $(0, 1)$; this can be seen by noting that the derivative of (4.3) is always negative on $(0, 1)$. This means that the minimum value of (4.3) occurs at the largest possible value of $P(B_{next})$. Since $P(B_{next})$ is bounded above by $\frac{2}{3}$, we have

$$\begin{aligned} \frac{-P(B_{next})}{1 - P(B_{next})} &\geq \frac{-\frac{2}{3}}{1 - \frac{2}{3}} \\ &= -2. \end{aligned}$$

$\square$

Now that we've given lower bounds for L, we next turn to L^*. As mentioned at the start of the section, for our analysis of L^* we assume that we always begin with the full sequence that includes all self-transitions. From this full sequence, we then focus only on transitions in $T_{\overline{A}}$.

Theorem 5. Let K be a sequence of affective states with $k \geq 2$ transitions in $T_{\overline{A}}$. Assume further that at least one of these transitions has $prev = A$ and at least one has $next \neq B$. Then, if all base rates are computed individually per sequence, we have

$$L^*(A_{prev} \to B_{next}) \geq -k + 1.$$

The full proof of Theorem 5, which is not as intuitive as the proof of Theorem 3, is given in the appendix. Our next example shows that the bound given in Theorem 5 is optimal.

Example 3. Consider the following sequence of affective states, where we assume that all self-transitions have been included.

$$AC\underbrace{B\ldots B}_{k-1 \text{ times}}$$

This sequence contains k total transitions, all of which are in $T_{\overline{A}}$. We can compute the following probabilities from the sequence.

$$P(B_{next} \mid T_{\overline{A}}) = \frac{k-1}{k}$$
$$P(B_{next} \mid A_{prev}, T_{\overline{A}}) = \frac{0}{1} = 0$$

Thus, it follows that

$$L^*(A_{prev} \to B_{next}) = \frac{0 - \frac{k-1}{k}}{1 - \frac{k-1}{k}} = -(k-1) = -k + 1,$$

which is equal to the lower bound given in Theorem 5.

Next, suppose we are computing the base rate of B by averaging over all sequences. Since we are no longer following the procedure of removing all self-transitions, $P(B_{next})$ can, in theory, take on any value in $(0, 1)$ (across all the sequences, we are assuming that B appears as the *next* state at least once and that there is at least one other distinct affective state). More specifically, since the base rate of B can be arbitrarily close to one, $L^*(A_{prev} \to B_{next})$ has no finite lower bound.

Theorem 6. Let K be a set of transitions in $T_{\overline{A}}$, and assume that at least one of these transitions has $prev = A$. Then, $L^*(A_{prev} \to B_{next})$ has no finite lower bound.

4.3. Range of values for L (self-transitions removed) and L^* (transitions in $T_{\overline{A}}$)

We can summarize the results from Sections 4.1 and 4.2 as follows. (In the case of L^*, we assume that k is the number of transitions in $T_{\overline{A}}$; that is, k is the number of transitions remaining after removing all transitions to A.)

- L with all self-transitions removed and base rates computed individually per sequence:
  Range$= [-1, 1]$

- L with all self-transitions removed and base rates averaged over all sequences:
  Range$= [-2, 1]$

- L^* with transitions to A removed and base rates computed individually per sequence:
  Range$= [-k+1, 1]$

- L^* with transitions to A removed and base rates averaged over all sequences:
  Range$= (-\infty, 1]$

5. PROCEDURE FOR APPLYING L^*

In this section we outline the full procedure for applying L^* to a data set consisting of sequences of affect transitions. In what follows, assume that we are starting with full affect sequences that include all self-transitions, and assume also that we are interested in computing L^* for transitions of the form $A_{prev} \to B_{next}$.

We begin by considering the case when the base rates are computed individually for each sequence. As discussed in Section 2, L^* can be computed by looking only at transitions where $next \neq A$ (and in which case the formula for L^* reduces to the formula for L). Under this assumption, Theorem 5 applies when a sequence has at least two transitions in $T_{\overline{A}}$ *and* contains a transition where $prev = A$. On the other hand, suppose we have a sequence with at most one transition in $T_{\overline{A}}$. First, consider the situation when the sequence consists solely of the transition AB. In this case, $P(B_{next}) = P(B_{next} \,|\, A_{prev}) = 1$, which means that L^* is undefined. Next, since we are assuming the sequence consists of at most one transition in $T_{\overline{A}}$, in any other case either A does not occur as the previous state or B does not occur as the next state. When A does not occur as the previous state, we follow the recommendation from Karumbaiah et al. (2018) and consider L^* to be undefined; then, if B does not occur as the next state (but A does appear as the previous state at least once), (2.2) returns a value of zero for L^*. We can summarize these examples as follows.

(1) If the affect sequence consists solely of transitions of the form AB, we have

$$P(B_{next} \,|\, A_{prev}, T_{\overline{A}}) = P(B_{next} \,|\, T_{\overline{A}}) = 1.$$

In this case, both the numerator and denominator of (2.2) are zero and, thus, L^* is undefined.

(2) If no transitions from A occur, then we don't know what affective state would follow A. Thus, in this case L^* is undefined as well.

(3) If no transitions to B occur, but at least one transition from A occurs, we have

$$P(B_{next} \,|\, A_{prev}, T_{\overline{A}}) = P(B_{next} \,|\, T_{\overline{A}}) = 0.$$

Thus, $L^* = 0$.

Note that (1) – (3) cover any affect sequences with one or zero transitions in $T_{\overline{A}}$; additionally, a sequence with no transitions from A is covered by (2). Thus, any sequence that is not covered by (1) – (3) would then satisfy the conditions for Theorem 5. So, if a procedure for computing L^* were based on these rules, it would then follow that L^* is either (a) undefined or (b) has a value in $[-k+1, 1]$.

Next, consider the case when we compute the base rates by averaging over all of the sequences; that is, for each state we compute the base rates individually for each student, and then we take the averages of these to get the final base rate for the state. If we make the minor assumption that each base rate is less than one, things are simplified, as (1) no longer occurs; thus, the only situation in which L^* is undefined is when no transitions from A occur. Algorithm 1 describes a specific procedure for computing L^* based on the arguments in this section. In order to avoid any possible issues with undefined values or division by zero errors, the algorithm is structured so that the various special cases we have discussed are explicitly handled by the conditional statements; then, in the event that none of the conditions for these special cases are satisfied, the value of L^* is computed using equation (2.2).

Algorithm 1 Computing L^*

Inputs:
K, the sequence of affective states with transitions to A excluded
Transition pair of the form $A_{prev} \to B_{next}$

Procedure:
if base rates are computed individually **and** K consists only of transitions of the form $A_{prev} \to B_{next}$ **then**
  $L^*(A_{prev} \to B_{next})$ is undefined
else if transitions from A do not occur **then**
  $L^*(A_{prev} \to B_{next})$ is undefined
else if transitions to B do not occur **then**
  $L^*(A_{prev} \to B_{next}) = 0$
else
  Compute $L^*(A_{prev} \to B_{next})$ using (2.2)
end if

Output:
Value of $L^*(A_{prev} \to B_{next})$

After L^* is computed for each student using Algorithm 1, as recommended by Karumbaiah et al. (2018) a two-tailed *t*-test can then be used on the entire sample of L^* values to measure whether the average L^* value is significantly different from zero. Additionally, if multiple transition pairs are being studied, we also recommend using a Benjamini-Yekutieli post hoc correction procedure (Benjamini and Yekutieli, 2001), with an α value of 0.05, to control for false positives.[1]

6. Evaluating L^* with simulated affect sequences

In this section we report the results from a set of simulations that are designed to verify the at chance behavior of L^*.[2] Specifically, when transitions are chosen at random, with no dependence on the previous state, we want to check that the returned L^* estimates are close to zero. In all experiments we compute the rates both individually for each student (as was done in the examples from Section 3), and also using the population average base rates from our complete sample. As shown in Section 4.2, computing the base rates per sequence has a restricted range that is dependent on the length of the sequence; on the other hand, using the average base rates gives a range of possible values from $-\infty$ to 1. For our first set of simulations, we begin by generating sequences of affect transitions by randomly choosing the states A, B, C, and D with equal probability; that is, each state is chosen with probability 0.25. For each sequence we generate a total of 21 states, resulting in 20 total transitions, and we generate a total of 50,000 sequences; while this is an unrealistically large number, we do this to reduce as much as possible any statistical noise in our results.

For our next set of 50,000 sequences, rather than using equal base rates for each state, we add 0.03 to the base rate of A, so that A is now chosen with probability 0.28; the other states are then chosen equally, each with probability 0.24. We then repeat this procedure, adding 0.03 to the base rate of A, until we have 24 different sets of 50,000 sequences; over these sets of sequences, the base rate of A ranges from 0.25 to 0.94; on the other hand, the base rates for B, C, and D are always equal and range from 0.25 to 0.02. For each set of 50,000 sequences, the average value of $L^*(A_{prev} \to B_{next})$ is calculated using both the base rate of B computed individually from each sequence and averaged over all the sequences. Note that, as the states are being chosen randomly, we should expect a value of zero for L^*.

The results using the procedure outlined in Section 5 are shown in Figure 1, where we can see that the L^* values are indeed very close to, and centered at, zero. The maximum and minimum values are 0.002 and -0.002, respectively. Furthermore, the values from using the individual base rates or the mean base rates are essentially identical. Thus, these results seem to indicate that, at chance, L^* returns a value of zero even when one base rate is disproportionately high and dominates the other base rates.

Our next set of simulations investigates the situation when two base rates dominate. We again start by generating sequences of affect transitions by randomly choosing the states A, B, C, and D with equal probability, and as before we generate a total of 50,000 sequences, each with 21 states. After generating our first set of 50,000 sequences, we add 0.01 to the base rates

[1]Note that the Benjamini-Hochberg procedure (Benjamini and Hochberg, 1995) has appeared in previous studies of affect dynamics (Ocumpaugh et al., 2017; Karumbaiah et al., 2018). However, this procedure is only valid if the statistical tests are independent, or if they satisfy certain dependency conditions (Benjamini and Hochberg, 1995; Benjamini and Yekutieli, 2001). To the best of our knowledge, we do not know of any results that indicate our experimental setup satisfies these conditions. Thus, we instead recommend the Benjamini-Yekutieli procedure here, as it is more conservative and is valid under arbitrary dependence conditions between the statistical tests (Benjamini and Yekutieli, 2001).

[2]A Python module for performing these simulations and computing L and L^* values is available at `https://github.com/jmatayoshi/affect-transitions`.

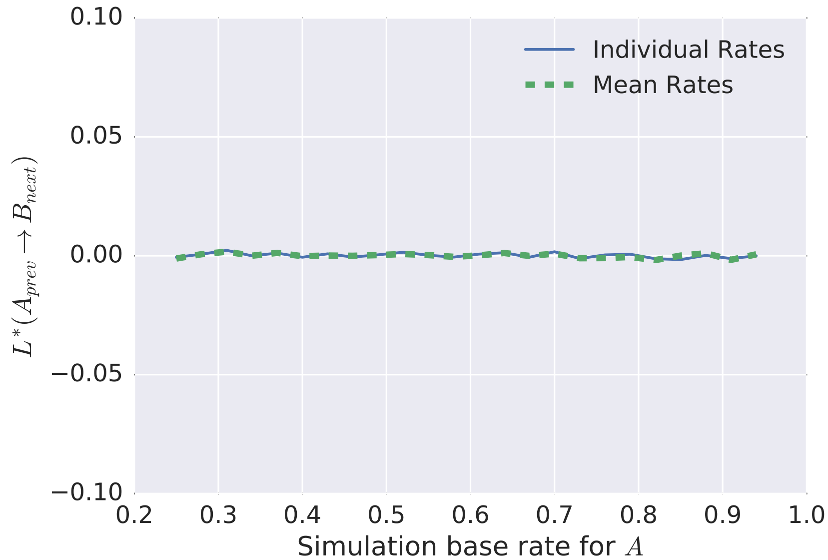

Figure 1: Values of $L^*(A_{prev} \to B_{next})$ for the first set of simulations. In these simulations, the base rate of A used while sampling the states varies from 0.25 to 0.94.

of both A and B, so that A and B are now each chosen with probability 0.26; the other states are then chosen equally, each with probability 0.24. We then repeat this procedure, adding 0.01 to the base rates of A and B, until we have 23 different sets of 50,000 sequences; over these sets of sequences, the base rates of A and B, which are always equal, range from 0.25 to 0.47; conversely, the base rates for C and D are always equal and range from 0.25 to 0.03.

For each set of sequences, the value of $L^*(A_{prev} \to B_{next})$ is plotted in Figure 2 using the base rate of B computed individually from each individual sequence, or averaged over all the sequences. As before, the L^* values are small in absolute value and centered at zero, with the maximum and minimum values given by 0.006 and -0.004, respectively. The two methods of computing the base rates again give essentially equivalent results.

7. AN APPLICATION OF L^* TO STUDENT DATA

Now that we have analyzed the behavior of the L^* statistic on simulated data, we next look in detail at the results of applying L^* to actual student data. The data set we consider consists of sequences of affective states from 180 high school students, with an average of 135 states being recorded per student (Andres et al., 2015). The recordings were made using BROMP while the students were working in the Physics Playground learning environment (Shute and Ventura, 2013).

We begin by focusing on the analysis of this data set from Karumbaiah et al. (2018), where all self-transitions are removed and the L statistic is applied to the remaining transitions, using the individual base rates from each sequence. In this analysis, the main states of interest are flow (FLO), confusion (CON), frustration (FRU), and boredom (BOR); additionally, the remaining states that are not in one of the four previous categories have all been merged into the dummy state NA. Table 1 has the results from replicating this procedure, and for completeness we also

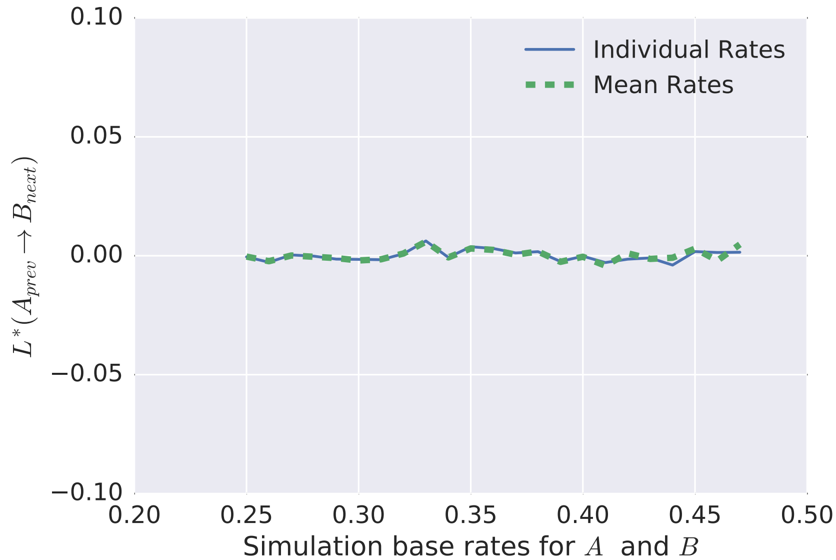

Figure 2: Values of $L^*(A_{prev} \rightarrow B_{next})$ for the second set of simulations. In these simulations, the base rates of A and B used while sampling the states are always equal and vary from 0.25 to 0.47.

include the mean and standard deviation values for transitions involving NA. To adjust for the number of statistical tests that are performed, we report in bold the transition pairs that are statistically significant after applying a Benjamini-Yekutieli post hoc correction, with an α value of 0.05. As shown, all transition pairs with *prev*=FLO have L values that are positive. However, note that the L statistic compares the two probabilities $P(B_{next})$ and $P(B_{next} \,|\, A_{prev})$; thus, this result implies that, for any state B, each conditional probability $P(B_{next} \,|\, \text{FLO}_{prev})$ is larger than the base probability $P(B_{next})$. In other words, these results suggest that the following inequalities hold.

$$P(\text{CON}_{next} \,|\, \text{FLO}_{prev}) > P(\text{CON}_{prev})$$
$$P(\text{FRU}_{next} \,|\, \text{FLO}_{prev}) > P(\text{FRU}_{prev})$$
$$P(\text{BOR}_{next} \,|\, \text{FLO}_{prev}) > P(\text{BOR}_{prev})$$
$$P(\text{NA}_{next} \,|\, \text{FLO}_{prev}) > P(\text{NA}_{prev})$$

As we are excluding self-transitions, the four terms on the left-hand side represent all the possible transitions from FLO, and so they must sum to one. Thus, the only way in which these (strict) inequalities can hold is if the rates on the right-hand side have a sum less than one (that is, if both sides sum to one, it's impossible for each term on the left-hand side to be strictly greater than each corresponding term on the right-hand side). As has been discussed previously, this is actually the case, as the application of L in this situation is not accounting for the fact that self-transitions from FLO are being excluded. Thus, the base rates on the right-hand side actually sum to $1 - P(\text{FLO}_{prev})$, thereby making it possible for each conditional rate to be strictly greater than its corresponding base rate.

For comparison, the results from using L^* are shown in Table 2. We can see that the number of significant transitions is three if the base rates are computed individually per sequence, while

prev	*next*	L: Individual Base Rates			
		Mean	SD	t	p-value
FLO	FLO	–	–	–	–
	CON	**0.15**	0.17	11.16	0.000
	FRU	**0.09**	0.13	9.57	0.000
	BOR	**0.03**	0.07	6.10	0.000
	NA	0.37	0.28	–	–
CON	FLO	**0.52**	0.48	13.12	0.000
	CON	–	–	–	–
	FRU	-0.01	0.22	-0.38	0.702
	BOR	**-0.02**	0.08	-2.77	0.006
	NA	-0.14	0.33	–	–
FRU	FLO	**0.47**	0.52	11.06	0.000
	CON	**-0.07**	0.16	-4.94	0.000
	FRU	–	–	–	–
	BOR	0.02	0.18	1.58	0.116
	NA	-0.12	0.34	–	–
BOR	FLO	**0.56**	0.49	9.94	0.000
	CON	**-0.10**	0.15	-5.61	0.000
	FRU	-0.01	0.17	-0.74	0.459
	BOR	–	–	–	–
	NA	-0.16	0.33	–	–
NA	FLO	0.79	0.25	–	–
	CON	-0.08	0.13	–	–
	FRU	-0.08	0.16	–	–
	BOR	-0.03	0.08	–	–
	NA	–	–	–	–

Table 1: Results from Karumbaiah et al. (2018) using L with all self-transitions removed. Significance tests are performed for all transition pairs not involving NA states, and L values that are significant after applying a Benjamini-Yekutieli post hoc correction procedure, with an α value of 0.05, are in bold.

there are none if the average base rates are used (where in both cases we applied a Benjamini-Yekutieli post hoc correction, with an α value of 0.05, before categorizing a result as significant). Thus, at least for this data set, using the average base rates gives more conservative results. Additionally, we can compare the results for transitions from FLO with the results in Table 1. Regardless of whether we use the individual base rates or the average base rates, the results with L^* give a mix of values both above and below zero. While all the transitions from FLO have L values that are positive and statistically significant, the only statistically significant L^* value is actually negative. Thus, this is seemingly evidence that the L statistic not only inflates the significance of these transitions, but in some of the cases it actually reverses the direction of the sign. In turn, this could easily mislead a researcher into overestimating the importance of starting in the state FLO.

8. Discussion

As discussed by the analyses presented in Karumbaiah et al. (2018), applying the L statistic as-is after removing all self-transitions can give misleading results. The solution proposed by Karumbaiah et al. (2019) attends to the primary issue with the L statistic, but leaves the interpretation of the statistic counterintuitive, in a way that could lead to inflated transition likelihood values. Motivated by these previous works, in this manuscript we study in detail the properties of a modification to the L statistic, where this modified statistic is intended to be applied specifically when self-transitions are excluded. We give a mathematical proof showing that this statistic, which we refer to as L^*, returns a value of zero when transitions happen at chance. We also derive maximum and minimum values for the L^* statistic, both when the base rates are computed individually per sequence and when they are averaged over all sequences; as a point of comparison, similar analyses are also performed for the L statistic when all self-transitions are removed. The results show that the minimum values differ greatly based on the statistic used (L or L^*), as well as the procedure used to compute the base rates (per sequence or over all sequences).

In addition to our theoretical analysis of L^*, we apply the statistic to both simulated student data and real student data. In the former case, the simulations give empirical evidence that the value of L^* at chance is zero even when the base rates are highly non-uniform. In the latter case, the results from using L^* seemingly give a more coherent picture in comparison to the values returned by the L statistic. For example, in one particular case an application of the L statistic leads to the counterintuitive result that all possible transitions from a certain state have positive values; in other words, without using the adjustment proposed in Karumbaiah et al. (2019), the resulting values seem to indicate that all the possible transitions are more likely than chance. On the other hand, applying L^* gives a mix of positive and negative values, with the only statistically significant result actually being in the negative direction. Finally, we also outline a step-by-step procedure for applying the L^* statistic, which we hope will be useful to other researchers.

While this work focuses on the removal of self-transitions from affect dynamics data, different scenarios may occur when dealing with other types of sequential data. That is, rather than considering only self-transition pairs for removal, it may be necessary to remove any particular type of transition pair, self-transition or not, from the analysis; this could be due to the goals of the study, the preferences of the researcher, or the simple fact that it's impossible for certain transitions to occur. As an illustration of this, some previous studies have analyzed sequences of student actions within digital learning systems (Biswas et al., 2010; Bosch and D'Mello, 2017).

prev	*next*	L^*: Individual Base Rates Mean	SD	t	p-value	L^*: Average Base Rates Mean	SD	t	p-value
FLO	FLO	–	–	–	–	–	–	–	–
	CON	0.02	0.12	2.15	0.033	0.02	0.28	0.89	0.375
	FRU	-0.03	0.15	-2.65	0.009	-0.01	0.22	-0.60	0.547
	BOR	**-0.05**	0.16	-4.00	0.000	-0.03	0.13	-2.58	0.011
	NA	0.02	0.24	–	–	0.03	0.53	–	–
CON	FLO	-0.7	5.12	-1.67	0.098	-0.21	1.22	-2.08	0.039
	CON	–	–	–	–	–	–	–	–
	FRU	**0.05**	0.20	3.07	0.003	0.05	0.22	2.68	0.008
	BOR	-0.01	0.09	-1.72	0.09	-0.01	0.088	-1.40	0.164
	NA	0.02	0.25	–	–	0.01	0.26	–	–
FRU	FLO	-0.32	1.64	-2.42	0.017	-0.29	1.25	-2.81	0.006
	CON	0.01	0.13	1.23	0.220	0.01	0.15	0.88	0.379
	FRU	–	–	–	–	–	–	–	–
	BOR	0.04	0.17	2.72	0.007	0.03	0.18	2.27	0.025
	NA	0.04	0.26	–	–	0.03	0.27	–	–
BOR	FLO	0.01	1.29	0.05	0.961	-0.04	1.17	-0.29	0.769
	CON	**-0.04**	0.10	-3.23	0.002	-0.03	0.09	-3.20	0.002
	FRU	0.03	0.15	1.48	0.143	0.02	0.17	1.20	0.234
	BOR	–	–	–	–	–	–	–	–
	NA	0.03	0.23	–	–	0.02	0.25	–	–
NA	FLO	0.26	0.94	–	–	0.24	0.97	–	–
	CON	-0.01	0.10	–	–	-0.01	0.11	–	–
	FRU	-0.02	0.11	–	–	-0.02	0.11	–	–
	BOR	-0.01	0.07	–	–	-0.01	0.07	–	–
	NA	–	–	–	–	–	–	–	–

Table 2: Results from applying L^* to the Physics Playground data set (Andres et al., 2015). Significance tests are performed for all transition pairs not involving NA states, and L^* values that are significant after applying a Benjamini-Yekutieli post hoc correction procedure, with an α value of 0.05, are in bold.

Depending on the design of the system, direct transitions may not exist for all of the possible pairs of action. For example, suppose the system requires the student to submit a correct answer before moving on to a new problem; in such a case, a direct transition from an incorrect answer to a new problem is not allowed, and this must be taken into account when analyzing the different transition pairs. As our current work does not apply in such situations, further research is needed to design and evaluate an alternative procedure that can be used for these more general scenarios.

9. APPENDIX

For convenience, we begin by restating the definition of L^*.

Definition 2. Let A and B be two affective states, and let

$$T_{\overline{A}} = \{\text{transitions where } next \neq A\}. \tag{2.1}$$

That is, $T_{\overline{A}}$ consists of all transitions where the next affective state is not equal to A. We can then define

$$L^*(A_{prev} \to B_{next}) := \frac{P(B_{next} \mid A_{prev}, T_{\overline{A}}) - P(B_{next} \mid T_{\overline{A}})}{1 - P(B_{next} \mid T_{\overline{A}})}. \tag{2.2}$$

Note that the above formula is equivalent to applying (1.1) to the transitions in $T_{\overline{A}}$. Alternatively, the probabilities in (2.2) can also be computed as

$$P(B_{next} \mid T_{\overline{A}}) = \frac{P(B_{next})}{1 - P(A_{next})}, \tag{2.3}$$

and

$$P(B_{next} \mid A_{prev}, T_{\overline{A}}) = \frac{P(B_{next} \mid A_{prev})}{1 - P(A_{next} \mid A_{prev})}, \tag{2.4}$$

where the values on the right-hand sides of (2.3) and (2.4) are derived from the full set of transitions (i.e., transitions where $next = A$ are also included).

Theorem 5. Let K be a sequence of affective states with $k \geq 2$ transitions in $T_{\overline{A}}$. Assume further that at least one of these transitions has $prev = A$ and at least one has $next \neq B$. Then, if all base rates are computed individually per sequence, we have

$$L^*(A_{prev} \to B_{next}) \geq -k + 1.$$

Proof. For any two affective states A and B, define the following variables (where $|\{\cdot\}|$ represents the cardinality, or size, of the set $\{\cdot\}$):

$$m_{\overline{A}_{prev}} = |\{\text{transitions where } next = B,\ prev \neq A\}| \tag{9.1}$$

$$n_{\overline{A}_{prev}} = |\{\text{transitions where } next \neq A,\ prev \neq A\}| \tag{9.2}$$

$$m_{A_{prev}} = |\{\text{transitions where } next = B,\ prev = A\}| \tag{9.3}$$

$$n_{A_{prev}} = |\{\text{transitions where } next \neq A,\ prev = A\}|. \tag{9.4}$$

We then have

$$P(B_{next} \mid A_{prev}, T_{\overline{A}}) = \frac{m_{A_{prev}}}{n_{A_{prev}}} \tag{9.5}$$

and

$$P(B_{next} \,|\, T_{\overline{A}}) = \frac{m_{A_{prev}} + m_{\overline{A}_{prev}}}{n_{A_{prev}} + n_{\overline{A}_{prev}}}. \tag{9.6}$$

Plugging (9.5) and (9.6) into (2.2) we get

$$L^*(A_{prev} \to B_{next}) = \frac{\frac{m_{A_{prev}}}{n_{A_{prev}}} - \frac{m_{A_{prev}}+m_{\overline{A}_{prev}}}{n_{A_{prev}}+n_{\overline{A}_{prev}}}}{1 - \frac{m_{A_{prev}}+m_{\overline{A}_{prev}}}{n_{A_{prev}}+n_{\overline{A}_{prev}}}}.$$

Finding the common denominators separately for the terms in the numerator and denominator of the right-hand side, we have

$$L^*(A_{prev} \to B_{next}) = \frac{\frac{m_{A_{prev}} \cdot (n_{A_{prev}}+n_{\overline{A}_{prev}}) - n_{A_{prev}} \cdot (m_{A_{prev}}+m_{\overline{A}_{prev}})}{n_{A_{prev}} \cdot (n_{A_{prev}}+n_{\overline{A}_{prev}})}}{\frac{n_{A_{prev}}+n_{\overline{A}_{prev}}-m_{A_{prev}}-m_{\overline{A}_{prev}}}{n_{A_{prev}}+n_{\overline{A}_{prev}}}}.$$

After some algebraic manipulations, we can rewrite this equation as

$$L^*(A_{prev} \to B_{next}) = \frac{m_{A_{prev}} \cdot n_{\overline{A}_{prev}} - m_{\overline{A}_{prev}} \cdot n_{A_{prev}}}{n_{A_{prev}} \cdot (n_{A_{prev}} + n_{\overline{A}_{prev}} - m_{A_{prev}} - m_{\overline{A}_{prev}})},$$

and if we then divide both the numerator and denominator by $n_{A_{prev}}$, we arrive at the equation

$$L^*(A_{prev} \to B_{next}) = \frac{\frac{m_{A_{prev}} \cdot n_{\overline{A}_{prev}}}{n_{A_{prev}}} - m_{\overline{A}_{prev}}}{n_{A_{prev}} + n_{\overline{A}_{prev}} - m_{A_{prev}} - m_{\overline{A}_{prev}}}.$$

Next, if we assume $n_{A_{prev}} > 0$ (i.e., that there is at least one transition with $prev = A$), we get the lower bound

$$\begin{aligned} L^*(A_{prev} \to B_{next}) &= \frac{\frac{m_{A_{prev}} \cdot n_{\overline{A}_{prev}}}{n_{A_{prev}}} - m_{\overline{A}_{prev}}}{n_{A_{prev}} + n_{\overline{A}_{prev}} - m_{A_{prev}} - m_{\overline{A}_{prev}}} \\ &\geq \frac{-m_{\overline{A}_{prev}}}{n_{A_{prev}} + n_{\overline{A}_{prev}} - m_{A_{prev}} - m_{\overline{A}_{prev}}}. \end{aligned} \tag{9.7}$$

Since we are assuming there is at least one transition of the form $next \neq B$, the denominator in (9.7) must be larger than or equal to one. Furthermore, this also implies that $m_{\overline{A}_{prev}} \leq k - 1$, as there are a total of k transitions, and in at least one of these transitions we have $next \neq B$. Combining these facts with (9.7), it follows that

$$\begin{aligned} L^*(A_{prev} \to B_{next}) &\geq \frac{-(k-1)}{n_{A_{prev}} + n_{\overline{A}_{prev}} - m_{A_{prev}} - m_{\overline{A}_{prev}}} \\ &\geq \frac{-(k-1)}{1} \\ &= -k + 1, \end{aligned}$$

as claimed. □

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REFERENCES

ANDRES, J. M. L., RODRIGO, M. M. T., SUGAY, J. O., BANAWAN, M. P., PAREDES, Y. V. M., CRUZ, J. S. D., AND PALAOAG, T. D. 2015. More fun in the Philippines? Factors affecting transfer of western field methods to one developing world context. In *AIED Workshops*.

BAKER, R. S., D'MELLO, S. K., RODRIGO, M. T., AND GRAESSER, A. C. 2010. Better to be frustrated than bored: The incidence, persistence, and impact of learners' cognitive-affective states during interactions with three different computer-based learning environments. *International Journal of Human-Computer Studies 68,* 4, 223–241.

BENJAMINI, Y. AND HOCHBERG, Y. 1995. Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the Royal Statistical Society: Series B (Methodological) 57,* 1, 289–300.

BENJAMINI, Y. AND YEKUTIELI, D. 2001. The control of the false discovery rate in multiple testing under dependency. *Annals of Statistics*, 1165–1188.

BISWAS, G., JEONG, H., KINNEBREW, J., SULCER, B., AND ROSCOE, R. D. 2010. Measuring self-regulated learning skills through social interactions in a teachable agent environment. *Res. Pract. Technol. Enhanc. Learn. 5*, 123–152.

BOSCH, N. AND D'MELLO, S. 2013. Sequential patterns of affective states of novice programmers. In *The First Workshop on AI-supported Education for Computer Science (AIEDCS 2013)*. 1–10.

BOSCH, N. AND D'MELLO, S. 2017. The affective experience of novice computer programmers. *International Journal of Artificial Intelligence in Education 27,* 1, 181–206.

BOSCH, N. AND PAQUETTE, L. 2020. What's next? Edge cases in measuring transitions between sequential states. Submitted for publication.

BOTELHO, A. F., BAKER, R. S., AND HEFFERNAN, N. T. 2017. Improving sensor-free affect detection using deep learning. In *International Conference on Artificial Intelligence in Education*. 40–51.

BOTELHO, A. F., BAKER, R. S., OCUMPAUGH, J., AND HEFFERNAN, N. T. 2018. Studying affect dynamics and chronometry using sensor-free detectors. In *Proceedings of the 11th International Conference on Educational Data Mining*. 157–166.

DEFALCO, J., ROWE, J., PAQUETTE, L., GEORGOULAS-SHERRY, V., BRAWNER, K., MOTT, B., BAKER, R., AND LESTER, J. 2018. Detecting and addressing frustration in a serious game for military training. *International Journal of Artificial Intelligence in Education 28*, 152–193.

D'MELLO, S. AND GRAESSER, A. 2010. Modeling cognitive-affective dynamics with hidden Markov models. In *Proceedings of the Annual Meeting of the Cognitive Science Society*. Vol. 32.

D'MELLO, S. AND GRAESSER, A. 2012. Dynamics of affective states during complex learning. *Learning and Instruction 22,* 2, 145–157.

D'MELLO, S., TAYLOR, R. S., AND GRAESSER, A. 2007. Monitoring affective trajectories during complex learning. In *Proceedings of the Annual Meeting of the Cognitive Science Society, 29 (29)*. 203–208.

D'MELLO, S. K., PERSON, N. K., AND LEHMAN, B. 2009. Antecedent-consequent relationships and cyclical patterns between affective states and problem solving outcomes. In *AIED*. 57–64.

KARUMBAIAH, S., ANDRES, J., BOTELHO, A. F., BAKER, R. S., AND OCUMPAUGH, J. S. 2018. The implications of a subtle difference in the calculation of affect dynamics. In *Proceedings of the 26th International Conference on Computers in Education*. 29–38.

KARUMBAIAH, S., BAKER, R. S., AND OCUMPAUGH, J. 2019. The case of self-transitions in affective dynamics. In *International Conference on Artificial Intelligence in Education*. 172–181.

KARUMBAIAH, S., LIZARRALDE, R., ALLESSIO, D., WOOLF, B., ARROYO, I., AND WIXON, N. 2017. Addressing student behavior and affect with empathy and growth mindset. In *Proceedings of the 10th International Conference on Educational Data Mining*. 96–103.

KUPPENS, P. 2015. It's about time: A special section on affect dynamics. *Emotion Review 7,* 4, 297–300.

MCQUIGGAN, S. W. AND LESTER, J. C. 2009. Modelling affect expression and recognition in an interactive learning environment. *International Journal of Learning Technology 4,* 3-4, 216–233.

NYE, B. D., KARUMBAIAH, S., TOKEL, S., CORE, M., STRATOU, G., AUERBACH, D., AND GEORGILA, K. 2018. Engaging with the scenario: Affect and facial patterns from a scenario-based intelligent tutoring system. In *International Conference on Artificial Intelligence in Education*. Springer, 352–366.

OCUMPAUGH, J., ANDRES, J. M., BAKER, R., DEFALCO, J., PAQUETTE, L., ROWE, J., MOTT, B., LESTER, J., GEORGOULAS, V., BRAWNER, K., ET AL. 2017. Affect dynamics in military trainees using vMedic: From engaged concentration to boredom to confusion. In *International Conference on Artificial Intelligence in Education*. Springer, 238–249.

OCUMPAUGH, J., BAKER, R. S., AND RODRIGO, M. M. T. 2015. Baker Rodrigo Ocumpaugh Monitoring Protocol (BROMP) 2.0 technical and training manual. *New York, NY and Manila, Philippines: Teachers College, Columbia University and Ateneo Laboratory for the Learning Sciences 60*.

RODRIGO, M., ANGLO, E., SUGAY, J., AND BAKER, R. 2008. Use of unsupervised clustering to characterize learner behaviors and affective states while using an intelligent tutoring system. In *International Conference on Computers in Education*. 57–64.

SHUTE, V. J. AND VENTURA, M. 2013. *Stealth assessment: Measuring and supporting learning in video games*. MIT Press.