Matrix Operations Notes

1 Matrix Addition

Matrix addition is performed element-wise. If A and B are two matrices of the same dimensions $m \times n$, their sum $C = A + B$ is defined as:

$$C_{ij} = A_{ij} + B_{ij}$$

where i and j are the row and column indices.

2 Matrix Subtraction

Matrix subtraction is similar to addition and is also performed element-wise. If A and B are two matrices of the same dimensions $m \times n$, their difference $C = A - B$ is defined as:

$$C_{ij} = A_{ij} - B_{ij}$$

3 Matrix Multiplication

Matrix multiplication is not element-wise. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, their product $C = A \cdot B$ is an $m \times p$ matrix defined as:

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

where i and j are the row and column indices, and k is the summation index.

4 Matrix Transposition

The transpose of a matrix A, denoted A^T, is obtained by swapping its rows and columns. If A is an $m \times n$ matrix, A^T is an $n \times m$ matrix defined as:

$$(A^T)_{ij} = A_{ji}$$

5 Determinants

The determinant of a square matrix A is a scalar value that provides important properties about the matrix, such as whether it is invertible. For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the determinant is calculated as:

$$\det(A) = ad - bc$$

For larger matrices, the determinant can be computed using various methods, such as cofactor expansion or row reduction.

6 Identity Matrix

The identity matrix I_n of size $n \times n$ is a square matrix with ones on the diagonal and zeros elsewhere. It serves as the multiplicative identity for square matrices of size $n \times n$ in matrix multiplication:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

7 Matrix Inversion

The inverse of a square matrix A, denoted A^{-1}, satisfies:

$$A \cdot A^{-1} = I$$

where I is the identity matrix. A matrix is invertible if and only if it is square and its determinant $\det(A) \neq 0$.

7.1 Example: Matrix Inversion

Consider a 2×2 matrix A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of A, denoted A^{-1}, is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\det(A) = ad - bc$ is the determinant of A. For A to be invertible, $\det(A) \neq 0$.

8 Proof: Inverse of Product of Two Matrices

Let A and B be invertible matrices of the same size. We aim to prove that:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof

By the definition of the inverse of a matrix, we know:

$$(AB)(AB)^{-1} = I$$

where I is the identity matrix.

Substitute $(AB)^{-1} = B^{-1}A^{-1}$ into the equation:

$$(AB)(B^{-1}A^{-1}) = I$$

Using the associative property of matrix multiplication:

$$A(BB^{-1})A^{-1} = I$$

Since $BB^{-1} = I$:

$$AIA^{-1} = I$$

And $AI = A$, so:

$$AA^{-1} = I$$

Finally, $AA^{-1} = I$ holds true, proving that:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$\square$

9 Proof: Inverse of Transpose of a Matrix

Let A be an invertible matrix. We aim to prove that A^T is also invertible and that:

$$(A^T)^{-1} = (A^{-1})^T$$

Proof

Since A is invertible, we know:

$$A \cdot A^{-1} = I$$

where I is the identity matrix.

Taking the transpose of both sides:

$$(A \cdot A^{-1})^T = I^T$$

Using the property of transposes that $(XY)^T = Y^T X^T$:

$$(A^{-1})^T \cdot A^T = I^T$$

Since $I^T = I$:

$$(A^{-1})^T \cdot A^T = I$$

By the definition of the inverse of a matrix, A^T is invertible, and its inverse is $(A^{-1})^T$. Thus:

$$(A^T)^{-1} = (A^{-1})^T$$

$\square$

10 Proof: Uniqueness of Matrix Inverse

Let A be an invertible matrix. We aim to prove that the inverse of A is unique.

Proof

Suppose B and C are both inverses of A. By the definition of the inverse, we have:

$$AB = I \quad \text{and} \quad AC = I$$

where I is the identity matrix.

Consider B multiplied by AC:

$$B(AC) = B \cdot I = B$$

Using the associative property of matrix multiplication:

$$(BA)C = B$$

Since $AB = I$, we substitute I for BA:

$$IC = B$$

And $IC = C$, so:

$$C = B$$

Thus, B and C are the same, proving that the inverse of A is unique.

$\square$