COMP3141

Software System Design and Implementation

More on the Curry Howard Isomorphism

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- Intuitionistic logic does not contain the axiom of excluded middle $p \vee \neg p$ or equivalently $\neg\neg p \rightarrow p$.
- In classical logic more can be proven but less can be expressed.
- Intuitionistic proof of an existence statement gives a witness for the statement.

Example of Existence in the Classical Sense

- Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{I}$ be the set of irrational numbers.
- Consider the statement $\exists x, y.(x \in \mathbb{I}) \wedge (y \in \mathbb{I}) \wedge (x^y \in \mathbb{Q})$.

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  2. Otherwise if ${\sqrt{2}}^{\sqrt{2}} \in \mathbb{I}$
     - Pick $x = {\sqrt{2}}^{\sqrt{2}}$ and $y = \sqrt{2}$
     - Then $x^y = ({\sqrt{2}}^{\sqrt{2}})^{\sqrt{2}} = 2$ so $x^y \in \mathbb{Q}$

Recall: The Curry-Howard Isomorphism

This correspondence goes by many names, but is usually attributed to Haskell Curry and William Howard.

It is a *very deep* result:

Logic	Programming
Propositions	Types
Proofs	Programs
Proof Simplification	Evaluation

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It turns out, no matter what logic you want to define, there is always a corresponding λ-calculus, and vice versa.

Constructive Logic	Typed λ-Calculus
Classical Logic	Continuations
Modal Logic	Monads
Linear Logic	Linear Types, Session Types
Separation Logic	Region Types

Translating

We can translate logical connectives to types and back:

Conjunction ($\wedge$)	Tuples
Disjunction ($\vee$)	`Either`
Implication	Functions
True	`()`
False	`Void`

We can also translate our *equational reasoning* on programs into *proof simplification* on proofs!

Examples

```
prop_or_false :: a -> (Either a void)
prop_or_false a = Left a

prop_or_true :: a -> (Either a ())
prop_or_true a = Right ()

prop_and_true :: a -> (a, ())
prop_and_true a = (a, ())

prop_double_neg_intro :: a -> (a -> void) -> void
prop_double_neg_intro a f = f a

prop_triple_neg_elim ::
  (((a-> void) -> void) -> void) -> a -> void
prop_triple_neg_elim f a = f (\g ->  g a)
```

Wrap-up

1. Assignment 2 is due in just over a week.
2. There is a quiz for this week, but no exercise.
3. Next week's lectures consist of a **guest lecture** on Tuesday, from **Dr. Hira Taqdees Syeda** of the Trustworthy Systems group at data61 (CSIRO), and a **revision lecture** on Wednesday.
4. Come up with **questions** to ask us for the revision lecture! It will be over very quickly otherwise.
5. If you enjoyed the course and want to do more in this direction, ask us for thesis topics, taste of research projects, and consider attending COMP3161 and COMP4161.
6. Fill in the myExperience reports, it is important for us to receive your feedback.