++ is assoc.

Base Case

$$([\,] ++ ys) ++ zs$$

$$= ys ++ zs \quad (1)$$

$$= ys ++ zs \quad (1)$$

$$= [\,] ++ (ys ++ zs)$$

Inductive Case

I.H $(xs' ++ ys) ++ zs = xs' ++ (ys ++ zs)$

$$((x : xs') ++ ys) ++ zs$$

$$= (x : (xs' ++ ys)) ++ zs \quad (2)$$

$$= (x : ((xs' ++ ys) ++ zs)) \quad (2)$$

$$= x : (xs' ++ (ys ++ zs)) \quad (\text{I.H})$$

$$= x : (xs' ++ (ys ++ zs)) \quad (2)$$

$$= (x : xs') ++ (ys ++ zs)$$

Right identity for lists

Base Case

$$[\,] +\!+ [\,]$$
$$= [\,] \quad (1)$$
$$= [\,]$$

Inductive Case

I.H. $xs' +\!+ [\,] = xs'$

$$x : xs' +\!+ [\,]$$
$$= x : (xs' +\!+ [\,]) \quad (2)$$
$$= x : xs' \quad (\text{I.H.})$$
$$= x : xs'$$

Reverse is an involution

Base Case

reverse (reverse [])

= reverse [] (A)

= [] (A)

Inductive Case

I.H. reverse(reverse ls') = ls'

reverse (reverse (x : ls'))

= reverse (reverse ls' ++ [x]) (B)

= ~~~ STUCK! Use lemma on next page

= x : reverse (reverse ls') (I.H)

= x : ls'

Base case

$$
\begin{aligned}
& reverse([\,] ++ [x]) \\
&= reverse\ [x] && (1) \\
&= (reverse\ [\,]) ++ [x] && (B) \\
&= [\,] ++ [x] && (A) \\
&= [x] && (1) \\
&= [x] && (A) \\
&= x : reverse\ [\,]
\end{aligned}
$$

Helper lemma

$[x] = x : [\,]$

Inductive case

I.H. $reverse\ (ys' ++ [x]) = x : reverse\ ys'$

$$
\begin{aligned}
& reverse\ ((y:ys') ++ [x]) \\
&= reverse(\ y : (ys' ++ [x])) && (2) \\
&= reverse\ (ys' ++ [x]) ++ [y] && (B) \\
&= (x : reverse\ ys') ++ [y] && (I.H.) \\
&= (x : reverse\ ys') ++ [y] && (B) \\
&= x : reverse\ (y : ys')
\end{aligned}
$$